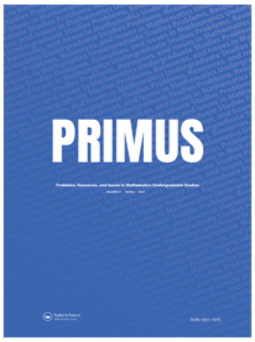

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To cite this article: Helge Fredriksen, Josef Rebenda, Ragnhild Johanne Rensaa & Petter Pettersen (17 Jul 2024): Inquiry-Based Linear Algebra Teaching and Learning in a Flipped Classroom Framework: A Case Study, PRIMUS, DOI: 10.1080/10511970.2024.2375712

To link to this article: https://doi.org/10.1080/10511970.2024.2375712

Published online: 17 Jul 2024.

PRIMUS
https://doi.org/10.1080/10511970.2024.2375712

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CASE REPORT

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Inquiry-Based Linear Algebra Teaching and Learning in a Flipped Classroom Framework: A Case Study

Helge Fredriksen, Josef Rebenda, Ragnhild Johanne Rensaa, and Petter Pettersen

ABSTRACT
Flipped Classroom (FC) approaches, which utilize video distribution via modern internet platforms, have recently gained interest as a pedagogical framework. Inquiry Based Mathematics Education (IBME) has proven to be a valid form of task design to motivate active learning and enhance classroom interactivity. This article presents a practical combination of introductory videos and inquiry-based class activities adoptable in a basic linear algebra course for stimulating students' exploration of the underlying mathematics. Teachers' and students' work addressed in the article was realized in two case studies in engineering programs in Norway and the Czech Republic. The learning objective was to connect different interpretations of the matrix equation Ax = b, which is often perceived as challenging for engineering students. Feedback from classroom sessions, interviews, and questionnaires encourage further research and inspired us as teachers to closely examine the mathematics behind the task design.

KEYWORDS
Inquiry-based mathematics education; flipped classroom; mathematics for engineering students

1. INTRODUCTION

Traditional lecture-based teaching is still the most common way to organize mathematics education at the university level. However, evidence is in favor of promoting various active learning styles [6,12,14]. Active learning in mathematics has many facets, but the main goal for students is to achieve deeper reflection about the topics at hand, and as such increase conceptual mathematical understanding. Moreover, a turn towards such student-centered learning and teaching approaches is seen to increase student engagement and motivation [17]. The main motivation for setting up a joint Czech-Norwegian project was the need to develop mixed/blended learning activities for students at tertiary education level, especially in response to the recent Covid pandemic in Norway and the Czech Republic. The initiative was intended to contribute to better learning of mathematics among students in engineering study programs in any environment under any circumstances including unexpected situations such as the Covid pandemic. However, the overall objective of the project was not to create an entire course in linear algebra, but rather to

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This article has been corrected with minor changes. These changes do not impact the academic content of the article.

investigate the effect of combining inquiry based linear algebra teaching with the flipped classroom framework. That is, exploring possible tailored teaching designs consisting of out-of-class video preparation and in-class inquiry, implemented in university-level mathematics education.

2. INSIGHTS FROM MATHEMATICS EDUCATION THEORIES

Inquiry-based mathematics education (IBME) is one of many frameworks considered to be an active learning style of teaching mathematics [8]. Evidence from many studies indicates that an inquiry-based approach to instruction leads to improved student performance [15]. Moreover, an interesting effect of IBME seems to be a significant improvement in the learning experience among low-achieving students [9].

Flipped Classroom (FC) as a way of organizing the teaching can be an effective vehicle for promoting IBME. The main principles are those of (1) direct computer-aided individual instruction outside class through video lectures followed by (2) interactive group learning activities in the classroom. Thus, the key affordance of the FC is to facilitate an enhanced quality of teaching in the classroom due to the possibility of exploring mathematical topics at greater depth than through lecturing [5]. The combined efforts of FC video preparations and IBME classroom activities can be thought of as a Flipped Inquiry Based Mathematics Education design (FIBME).

Linear algebra is an important topic in undergraduate mathematics for engineering students, and it also seems to be a good candidate topic for the employment of IBME [1,3,13]. In our study of inquiry based linear algebra, we chose to introduce three interpretations of the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. In students' curricula on introductory linear algebra courses, focus is often on the solution techniques including the Gauss-Jordan method, finding the inverse of matrix A, and sometimes Cramer's method utilizing determinants. However, little attention is directed towards interpretations beyond the solution to a set of linear equations. Larson and Zandieh [10] extended the interpretation of $A\boldsymbol{x} = \boldsymbol{b}$ to also include linear transformations and linear combination as a richer example space for the students to explore. A linear combination problem would be the problem of calculating weights $x_1, x_2, \ldots, x_n$ in the linear combination $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_n\boldsymbol{a}_n = \boldsymbol{b}$, while a linear transformation problem $T : \boldsymbol{x} \to \boldsymbol{b}, T(\boldsymbol{x}) = A\boldsymbol{x}$ often involves determining the transformation matrix A based on certain desired properties of the transform.

Indeed, there have been previous studies of the fruitful interplay between FC and IBME [2,11,18]. However, in this study we also found the design process to be of great importance, that is, how the dialogue between mathematicians and didacticians aids in forming the tasks and videos for the students [16].

3. METHODS

This section reflects on the research design and presents how it was conducted, including the data collection and the realization of two different teaching case studies based on the material we developed. Moreover, we consider the rationale behind the design of out-of-class videos and in-class tasks for the students.

Our design of FIBME sessions consisting of videos and tasks was implemented at both a Czech and a Norwegian university, making it possible to draw conclusions based on a wider cultural variety than previous studies. The first and second author of this paper acted as teachers in both case studies. One of the task sessions was conducted in a virtual classroom in Microsoft Teams due to the Covid pandemic, a setup different from most of the previous research on IBME. We divided the learning material into two sessions: (a) introduction to theory on the three different interpretations and (b) examples of applications of these three interpretations.

The videos for the first session, Session 1, contained a formal presentation of the three different interpretations of the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. First, we reviewed the classical interpretation most students encounter in linear algebra, the system of linear equations. Second, we presented the problem of linear combination of n-dimensional vectors, and how this could be formulated as a solution to the corresponding matrix equation. Third, the idea of linear transformations of vectors was introduced, where the mapping of vectors from a domain to a co-domain was presented as an example of the function concept which might be experienced as novel to students. The videos for the second session, Session 2, were based on applications of the three interpretations. The system of equations approach was exemplified by modeling population dynamics using left stochastic (Markov) matrices including a discussion of equilibrium solutions. Secondly, we considered the case of landing a vessel on Mars using three rockets aimed in three linearly independent directions. This was used to illustrate the concept of linear combination, and the concept of span in $\mathbb{R}^3$. This example was inspired by the realistic mathematics task of movement in the plane, utilizing two modes of transportation presented in Wawro et al. [19] but this time redesigned into a three-dimensional case. Finally, an application of linear transformations was illustrated by scaling and rotations of vectors in two dimensions. The videos may be viewed on the YouTube playlist “LINALG 2021 (EEA Grants)” on this link https://www.youtube.com/playlist?list=PLTDpZy64tCLUJIZLXqzQlNedWJU8lzP_U.

The tasks for the two FIBME sessions were designed to allow students to make use of the information they had gained in the videos. In the three tasks in Session 1, the students could use any method they liked to accomplish the task’s objective. The tasks focused on applications following the topics of the videos for Session 2. To apply the theory on systems of linear equations, Markov chains were used to find equilibria of a chemical system consisting of two compounds. The idea was based on the paper written by Fahidy [4]. The topic of linear transformations was exemplified by a transformation of the letter “N” to an italicized and enlarged version of the same letter presented in Andrews-Larson et al. [1], but generalized to other letters whose shape cannot be described by linear elements only. Mars landing seemed to be an interesting topic, so we kept it for the third task and asked more challenging questions, with the intent of promoting students’ inquiry towards the concept of linear span.

In the process of designing the tasks, an inquiry approach was used. The collaboration among researchers was set up as a conversation in MS OneNote to form a dialogue-based reflection on how students could interpret and work on the tasks in terms of IBME. A complete list of the tasks and videos produced for the two FIBME sessions, in addition to the questionnaire form and interview guide, can be found at https://linalg.ceitec.cz/educational-material-outputs. The tasks are also available in the Appendix.

3.1. Data Collection

Data from the case studies was collected in four different ways. First, the teachers took notes from the classroom activity in the form of written accounts mainly of the group collaboration (Bodø, Norway) and individual work (Brno, Czech Republic). During the two sessions in Bodø, one of the teachers observed and guided the students' activity in both breakout rooms, while the other observed the students' activity in one of the breakout rooms for the whole session. Second, we collected students' written notes from working with the tasks from the two Bodø sessions. Third, we collected participating students' opinions in an anonymous questionnaire. Finally, we selected two of the students for a semi-structured interview. One of them participated in the two sessions in Bodø, and the other in the session in Brno. Due to the relatively few numbers of students participating in the designed additional teaching sessions, it was easy to send follow-up reminders. Thus, we obtained questionnaire data on all students. The students were asked 10 questions on four main topics: the group work on the tasks, the use of digital tools, the educational videos, and the general impression about the teaching session. We made a separate guide for the interview, where the main purpose was to attain qualitative, in-depth knowledge about the students' impressions. Two invited students were chosen based on their previously shown ability to supply critical and informative comments.

4. RESULTS

This section reports on observational notes from classroom activities including some reflections from the observer related to these activities. Then we present some analyses from the interviews, followed by the findings from the survey to which students responded after the classroom sessions.

4.1. Observational Results from the Classroom

The first case study was held with the Norwegian students during two days in April 2021. Both sessions were held digitally in Teams meetings. Before Session 1 and 2 took place, the students had been told to prepare by watching the videos. Statistics from the Learning Management System showed that all students had prepared for the sessions. Four Bodø students participated in these sessions and the two groups formed from these consisted of the same 2 students in each session. After having

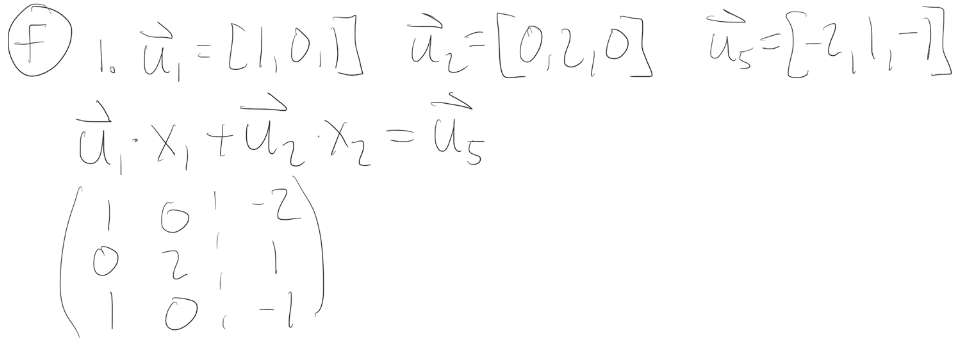

Figure 1. Group 2 working with a system having no solution, emerging from an attempt to express one of three linearly independent vectors as a sum of the two others.

introduced the topic of the day, the students were presented with the tasks associated with the teaching session. There were three different tasks for each of the two sessions, related to the three different themes mentioned earlier, and students were told to choose one of these for their work in the session. The students were divided into two different groups, each group choosing a separate task, and gathered in separate breakout rooms to work on the tasks. One of the authors was an observer in one of the groups, and one was moving between the two groups.

In the beginning, both groups struggled to understand the tasks, and they needed to consult with the teacher. However, after having received the necessary teacher guidance, both groups were able to progress with the tasks. Below we show two examples of the students' written material from these sessions.

The notes in Figure 1 were taken from the attempt of Group 2 about how to use matrix notation to find a solution to the linear combination problem of two vectors in $\mathbb{R}^3$ (see Appendix, Session 1, Theme 3, page 3). The hidden challenge for the students was the empty solution space, since the resulting third vector was linearly independent of the two vectors students were supposed to utilize to express it. Although the students ended up with the correct augmented matrix, they struggled with the interpretation of the first and third rows in this matrix representing a logical contradiction. However, this provided space for valuable discussion with the teacher about the topic.

Group 1 worked on the interpretation of the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ as a system of linear equations (see Appendix, Session 1, Theme 1, page 1). The inquiry-based framing of the task was based on the geometrical representation by lines in the plane, and from that going further with planes in the 3-dimensional space. During the videos, the analogy of lines illustrating equations was presented, and the students seemed to be able to build on this visual interpretation in their work on the task as illustrated in Figure 2.

Both groups were able to finish about half of the tasks. Group 1, working on the "Linear equations" task, managed to work through the questions in parts (a), (b), (c), and (e) successfully, while Group 2 working on the "Linear combination" task processed the questions from the beginning to the first question in part (f).

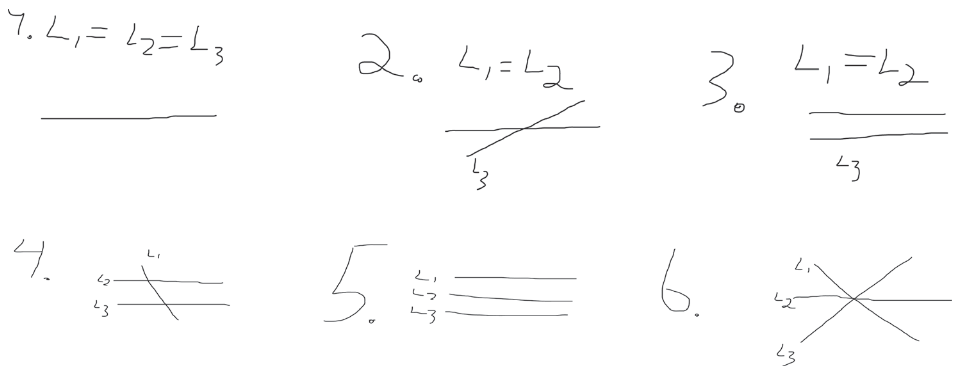

Figure 2. Group 1 worked on illustrating how three different lines relate geometrically to the various number of solutions to systems of linear equations.

As previously stated, Session 2 aimed towards the application of the theoretical foundations that the students had worked with in Session 1. Group 1 chose to work with the task on the Mars landing (see Appendix, Session 2, Theme 3, page 8), where they considered the situation of a possible failure of one of four rockets used for steering the vessel. In one of the tasks, the rockets had directions given by $\vec{u}_1 = [1, 0, 1]$, $\vec{u}_2 = [0, 2, 0]$, $\vec{u}_3 = [-1, 1, 0]$ and $\vec{u}_4 = [0, 1, 1]$, where we also allowed the hypothetical use of a reverse operation of the rockets. The students utilized GeoGebra for exploring the setup as can be seen in Figure 3. Based on the graphical display, they were able to conclude that by removing $\vec{u}_2$, the other three vectors were linearly dependent and spanned the depicted plane for the vessel to move in. However, the students did not have time left to work on part (3) and (4) of the task sheet.

Regarding considering systems of linear equations, we chose the Markov model as an illustration. The video on this topic included an example about the development of populations involving movement patterns between two cities, while the task for the students was to model a chemical reaction between two compounds (see Appendix, Session 2, Theme 1, page 5). Having seen the videos beforehand, the students in Group 2 were able to progress to the end of the task and plot the values of the first compound in GeoGebra. From this plot, they continued further and experimented with polynomial curve fitting. The students even tried exponential curve fitting but failed in this attempt due to the lack of a baseline parameter in the GeoGebra modelling tool. The discussion arising from this inquiry stimulated a further exploration of the problem of predicting intermediate values between discrete time steps in the model by the authors. This led to a separate follow-up meeting with one of the students interested in the idea of exponential modeling on the system of difference equations arising from this task.

In the second teaching case study conducted in person at the Brno University of Technology in October 2021, there were two international students working with the same set of videos and tasks as the Norwegian students but limited to Session 1. There was a short introduction to the session at the meeting with students in

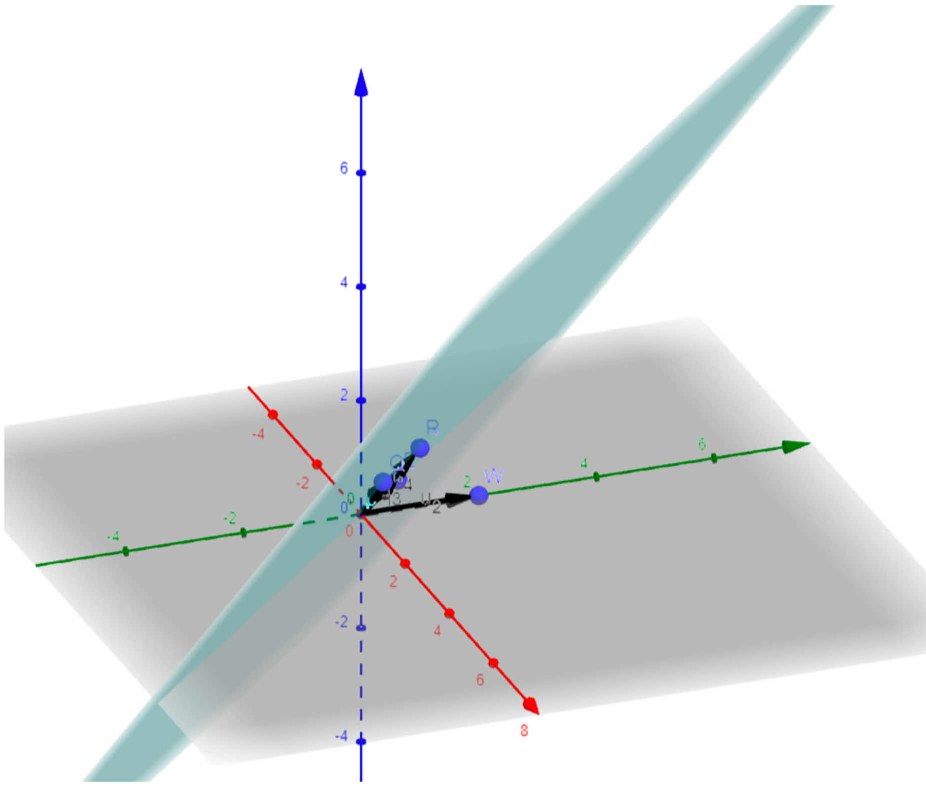

Figure 3. Student inquiry into the Mars landing task, showing the plane of possible movement upon failure of one of the rockets. Note: This is an original screenshot from one of the students' screens, regrettably at an angle of sight not optimal for visualizing the vectors placement in three dimensions.

the classroom. The concepts of inquiry-based teaching and flipped classroom were explained to them, including that the videos were part of this scheme. Both students had prepared by watching the videos. The students tried to collaborate at the beginning of the session, but it soon became apparent that communication problems hindered collaborative work. Thus, the teachers took the role of peers in the communication with the students. The first teacher supported one of the students in her calculations, while the second teacher discussed the topic with the other one on the whiteboard. The first student needed some guidance to get started with the task. The discussion was on the use of the matrix system to solve the problem of crossing lines. The student was not familiar with the Gauss-Jordan process, thus much of the time was devoted to working on an operational level with this process, but at the end of the session she was able to conclude the task. The second student felt more confident about the task and was interested in discussing her own understanding of the topic rather than performing calculations by hand. The discussion continued on the whiteboard with a focus on understanding how the number and the structure of solutions to a system of linear equations are related to the row echelon form of a matrix.

4.2. Results from the Interviews

We interviewed one student from each of the two case studies. The student in the first case study in Bodø, called Jim, was interviewed by the first author of the article. His personal profile was that of an active and ambitious student.

At the beginning of each interview, the students were asked about their attitude towards mathematics in general. Jim explained that he understood the necessity of learning mathematics, if for example "you want to progress to like artificial intelligence and machine learning." He seemed to have a clear understanding of the need to attain in-depth knowledge about the underlying mathematics and statistics in machine learning to be able to understand how to tune these models properly.

When asked about the role of the videos, the student stressed that learning mathematics only through watching videos would be useless. The necessity of solving tasks related to the videos, either while watching them or later in an exercise session, was considered vital for understanding. However, the videos were seen as an important instrument for discourse with other students and the teacher. The student remembered earlier flipped classroom sessions, where the video preparation stage had been neglected and stated that "we would spend the time basically trying to learn what we were supposed to know already and not being able to do the tasks."

Since the in-class part of the work was done online in Teams, collaboration was considered somewhat problematic due to the necessity of working together on the shared whiteboard. Most students did not possess convenient equipment for handwriting mathematical expressions on a PC. The shared whiteboard in Teams was also not considered very user-friendly. However, having the opportunity for individual group guidance by the teacher moving back and forth between breakout rooms was seen as very helpful. This was considered important since the inquiry-based tasks had a certain threshold in complexity. Jim phrased it like this: "People will get stuck right, and if you don't get help then the session is over and, yeah, you do a summary at the end but, yeah … " and "I kind of needed a bit push there I remembered in some cases to get in the right direction." About the inquiry-related nature of the classroom sessions, Jim told us that it "helps a lot to get inputs from others so I would say if I was sitting by myself here in school doing it I probably would have solved it over time but it would probably take much more time I think, … , so being stuck in a room like that even though it's digital it still kind of puts you on the spot somehow."

The other interviewed student called Julia, was an international student from North Macedonia, studying electrical engineering in Brno. Upon being asked about the inquiry-based learning approach, she commented on the process as it is

> something that tells you how you can get to that solution or gives you an inclination as to where you should keep solving (…) you kind of intuitively make your own solution as you do the mental steps and create your own algorithm.

Julia had many ideas on how educational videos could be utilized. "I played the videos while I was cooking or while I was going grocery shopping" she told us, and additionally "I can immediately write notes on my phone and when I go to the lecture, I can just directly ask it." So, she enjoyed the enhanced flexibility and variability of ways to initially encounter the presented mathematical topics. She also appreciated "the practical examples not just theoretical ones" as a way to understand "why I'm actually learning it and how I can apply it later on." Moreover, she considered videos to provide tools for communication ability: "You can understand the materials either on a surface level or fully and you can still communicate with the professor." When being asked if she got the help she needed if she was stuck in the process, she claimed that "it was very interactive, and I didn't feel pressure of any kind." Following up on this she also stated that she "… was very confident to make an experiment with solutions and problem solving which is something that I didn't get during high school."

4.3. Results from the Survey

As previously stated, the survey was completed by all six participants of the two case studies. Considering the responses from the Bodø students, feedback on the quality of the videos and the group work was quite positive. Out of four students, three reported that they very much enjoyed working online in breakout rooms with mathematics, and that the discussions in the groups were very effective according to how well they worked with the mathematical tasks. However, when being asked if they would like to work more with inquiry-based tasks like this, three said "some," and one person answered "no." None of the students answered "yes" to this. To the question about what could have been done to increase the quality of learning, the students answered that they would have liked to have larger groups, a smaller number of sub-tasks and a physical presence in a classroom when working in an inquiry-based way.

The Brno students reported that the content of both videos and tasks was clear and that they learned mathematics effectively working with the tasks. These students were on the opposite side of the scale compared to the Bodø students in the question about if they wanted more time to work on inquiry-based tasks. Both students answered yes to this question. The students seemed very satisfied with the teacher's guidance. One of them confirmed in the interview the impression about the low-pressure and down-to-earth focus of the work with the teacher. Upon being asked about what could have been done better, one of the students said that she was the one who needed to improve, especially her own level of preparation before coming to class.

5. DISCUSSION

During the initial case study in Bodø, we observed students working with a variety of approaches to the linear algebra problem at hand. Although everything was performed online via Teams, observations from the work on the tasks and the interview with one of the students afterwards indicated engagement and motivation to learn mathematics. An interesting statement from the interview with Jim was his impression of the inquiry-based work to be that of "being put on spot," as he phrased it. We interpret this as a statement illustrating how the collaborative situation somehow enforces a more focused and intense attitude towards the work on tasks, compared to working with the tasks individually. This was emphasized by his reflection of the efficiency of the group work on more "open-ended" tasks like the one presented in the classroom sessions. Moreover, both students being interviewed had a clear view of the videos as an important baseline for working with inquiry-based tasks. This was also supported by the survey results.

However, we cannot make a conclusion about the popularity of the tasks among students. None of the students in the Bodø case study stated a "yes" when being asked if they wanted more such tasks. Also, we noticed that one of the students in Brno achieved little progress with the tasks during the in-class session. We believe that employment of such tasks suits the FC framework and the engineering students very well, but care needs to be taken on equal involvement of all student groups. This includes students who do not manage to prepare in-depth through the videos or those being too "introverted" to be able to communicate efficiently with their peers.

An interesting, unexpected effect of the work with the students on Markov chain modeling was the follow-up inquiry it triggered. The failure of exponential curve fitting in GeoGebra raised the question about transition from the discrete model to a continuous model that allows data interpolation over an interval. This student inquiry led to a more in-depth mathematical investigation by one of the teachers. During this exploration and further discussion, both teachers widened their understanding of such systems, thus demonstrating the applicability of open inquiry (Heck and Másilko [7]) in their own work. It suggests that inquiry-based mathematics education can promote inquiry activity in different stages and on different levels, for example in the creation of the tasks, in the students' collaboration on the tasks, and in the interpretation of students' work.

6. CONCLUSION

The case studies with the students in Bodø (Norway) and the international students in Brno (Czech Republic) showed a great variety of student perception of the idea of inquiry-based teaching and learning. Most of the students adhered to the ideas of Flipped Classroom and seemed to have prepared well for the lessons. Also, their engagement with the tasks was high, even though the idea of inquiry-based working was somewhat unfamiliar territory for most of them. The concept of combining Flipped Classroom and IBME seems to be fruitful since it made the students reflect deeper on mathematics and provided the possibility of enhanced guidance from the teachers. All students managed to work successfully within the mathematical context of linear algebra during the classroom sessions, but with various extent of support from the teachers. However, the mathematical learning objectives, i.e., exploring and discovering links between three interpretations of matrix equation A$\boldsymbol{x} = \boldsymbol{b}$, were not achieved. The main reason for this was a lack of time; two learning sessions were barely enough to explore one interpretation only. Based on this experience, we suggest that up to six learning sessions might be needed to achieve these objectives.

DISCLOSURE STATEMENT

No potential conflict of interest was reported by the author(s).

FUNDING

Research material for the work on this paper was funded through the project No. EHP-BFNU-OVNKM-3-073-2021 titled "Joint Czech-Norwegian initiative for the development of a modern mathematics education for engineers" supported by the Bilateral Fund within the framework of the EEA and Norway Grants 2014-2021. This support is gratefully acknowledged.

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APPENDIX

Session 1

Theme 1: Systems of equations

In this task we will study the connection between lines in two dimensions and planes in three dimensions.

Imagine we have two lines in the plane expressed as the following:

$$L_1 : 3x + 2y = 4$$

$$L_2 : x - 2y = -1$$

a) Find a proper way to visualize the lines, for example in GeoGebra. What is the connection between the lines?
b) How would you find this connection without visualization? What method for calculating this would you use that you just learned about?

A set of two new lines are given as

$$L_3 : x + 2y = 2$$

$$L_4 : -x - 2y = -3$$

c) What is the connection between these?
d) Can we generalize our findings to the general case of

$$L_a : a_1x + a_2y = a_3 \quad L_b : b_1x + b_2y = b_3$$

Where a_1, a_2, a_3, b_1, b_2 and b_3 are coefficients that can be considered constant.
(You may skip this task if it takes too much time.)

e) Now imagine that we have three lines in the plane. Does it change anything? How would we describe the (general) situation mathematically? How would we visualize the possibilities? (may also be done in written form).
f) Could this problem be formulated in matrix form?

Imagine we have **two** planes in space.

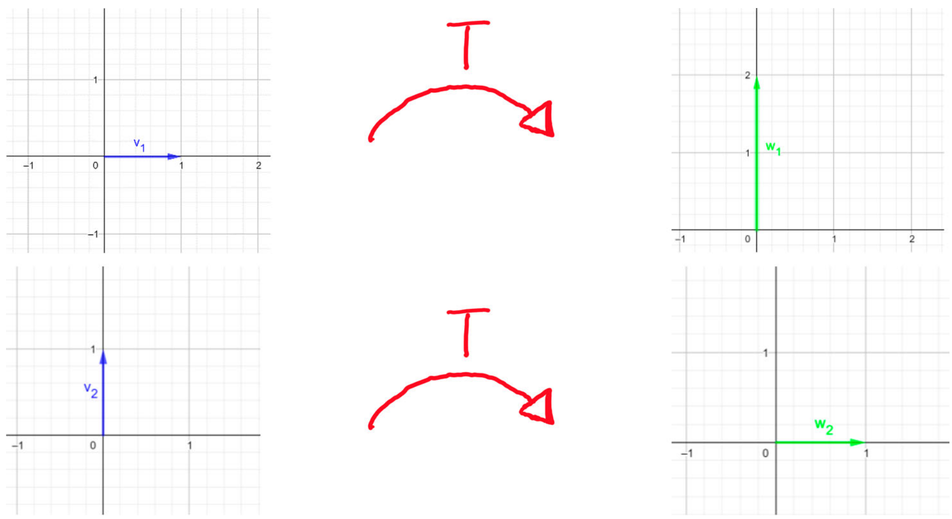
g) What are the possible relationships between the planes?
h) If we extend this to **three** planes, which possibilities exist then? Discuss and try out some visualization.
i) Could we make a concrete example and "calculate" the relationship?
j) How many equations do we have, and how many variables are there for the two cases (two and three planes)?

Now we try to see these results in a more general light.

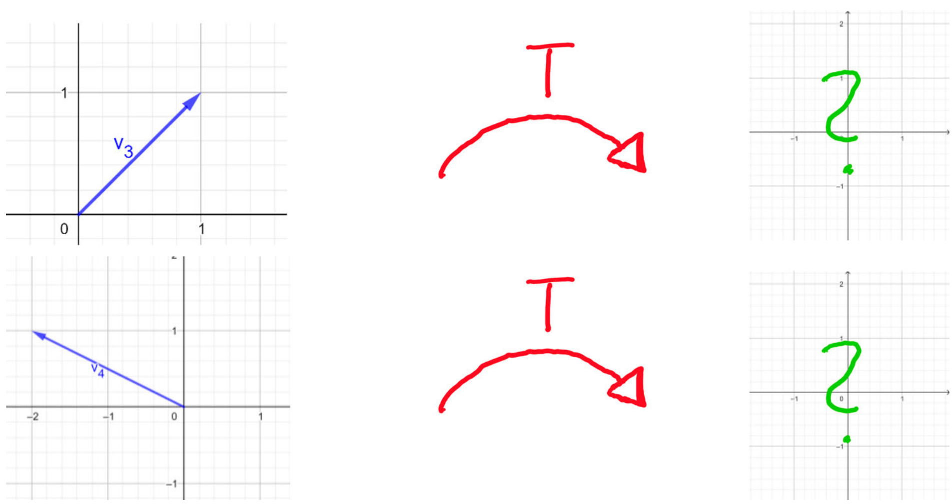
k) Can you see a connection between the number of equations, variables, and number of solutions?
   a. For example: What is the difference between three lines in the plane and two planes in space? How will the matrix system for the two cases look like?
   b. In which cases is there a chance for infinitely many solutions? And for no solutions?

Theme 2: Linear transformations and mappings

We have two vectors in the plane $\vec{v}_1$and $\vec{v}_2$. We call these input-vectors, or test-vectors. An unknown "machine" which we call *the linear transformation T*, maps the vectors $\vec{v}_1$and $\vec{v}_2$ to new vectors $\vec{w}_1$and $\vec{w}_2$ as follows:



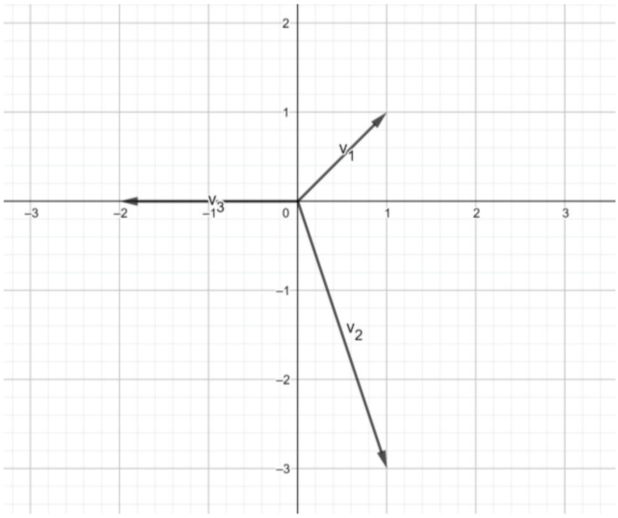
a) What will the transformation T do with the vectors $\vec{v}_3$and $\vec{v}_4$ as described in the figures below?



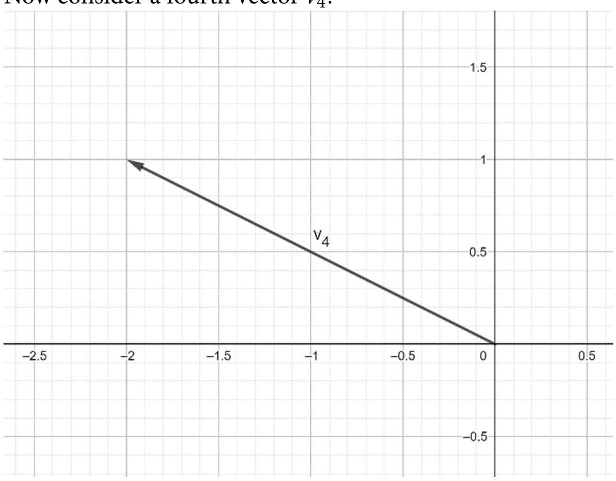
b) How about a general two-dimensional vector $\vec{v}_5 = [x, y]$. How will the mapping transform such a vector?
c) If the two test vectors $\vec{v}_1$and $\vec{v}_2$ were colinear/parallel, how would the mappings $\vec{w}_1$ and $\vec{w}_2$ relate to each other? What if the mapped vectors $\vec{w}_1$ and $\vec{w}_2$ were colinear, how do the test vectors $\vec{v}_1$and $\vec{v}_2$ relate?
d) What would happen in $\mathbb{R}^3$? How many vectors would be needed to determine the linear transformation T? How would the mathematical description of T look like?
e) What about linear mappings M between different dimensions? Say between $\mathbb{R}^3$ and $\mathbb{R}^2$? How could we visualize these? How could we examine the effect of such a transformation?
f) How many vectors would be needed to decide a linear mapping M between $\mathbb{R}^3$ and $\mathbb{R}^2$?
g) How many vectors would be needed to decide a linear mapping M between $\mathbb{R}^2$ and $\mathbb{R}^3$?
h) How would the mathematical description of these two mappings look like? What types of vectors do we need to decide M? And what types to decide N?

Theme 3: Linear combinations of vectors

Consider three vectors in the plane $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$:



Now consider a fourth vector $\vec{v}_4$:



a) Would it be possible to express $\vec{v}_4$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$? What about $\vec{v}_1$and $\vec{v}_3$? And $\vec{v}_2$ and $\vec{v}_3$? (Hint: it is not necessary to find the numbers to express these combinations)
b) Could we express $\vec{v}_4$ as a linear combination of **all three** vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$? How could we write that? Try to find such a linear combination.
c) In general: What is the chance that we may express a fourth vector as a combination of the three others?

Suppose we now move to $\mathbb{R}^3$. And suppose we have the vectors $\vec{u}_1 = [1, 0, 1]$, $\vec{u}_2 = [0, 2, 0]$, $\vec{u}_3 = [-1, 1, 0]$ and $\vec{u}_4 = [0, 1, 1]$. Also consider a vector $\vec{u}_5 = [-2, 1, -1]$.

d) Is it possible to express $\vec{u}_5$ as a linear combination of $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$? What about $\vec{u}_2$, $\vec{u}_3$and $\vec{u}_4$?
e) Is it possible to express $\vec{u}_5$ as a linear combination of all the vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ and $\vec{u}_4$? How would you express that?

f) Is it possible to express $\vec{u}_5$ as a linear combination of only two of the vectors?
   1. $\vec{u}_1$ and $\vec{u}_2$?
   2. What about $\vec{u}_1$ and $\vec{u}_3$?
   3. $\vec{u}_2$ and $\vec{u}_3$?
   4. $\vec{u}_2$ and $\vec{u}_4$?
   5. $\vec{u}_3$ and $\vec{u}_4$?

g) Now have a look at the vector $\vec{u}_6 = [-2, 0, 2]$? Is it possible to express this vector as a linear combination of
   1. $\vec{u}_1$ and $\vec{u}_2$?
   2. $\vec{u}_1$ and $\vec{u}_3$?
   3. $\vec{u}_2$ and $\vec{u}_3$?
   4. $\vec{u}_2$ and $\vec{u}_4$?

h) Now you can try to guess: What is the chance that we may express a $\mathbb{R}^3$ vector $\vec{w}$ as a linear combination of two other $\mathbb{R}^3$ vectors $\vec{u}$ and $\vec{v}$? Is there a "hint" that we may use from geometry that could come to our aid here?

Session 2

Theme 1: Markov model

Markov chains can be used to solve problems in chemistry, as here in kinetic chemistry.

Part I

Suppose we have a homogenous mixture of two chemical components A and B reacting with each other. At the start of the reaction, there is 1.0 mole of component A and 0.2 moles of component B.

Each minute 75% of moles of A are converted to B, and 5% of B are converted to A. We would like to know the following ***(PS: Do not start until you read the rest of the task description!)***

a) How many moles of A and B are present in the mixture at one minute after the process has started?
b) And how much after two minutes?
c) And after 10 minutes?
d) What would be the final composition/stable state/stable equilibrium?

When you solve the task, please consider the following:

A. How to write/draw the model on paper using mathematics/figures? Is there any way to make use of vectors and matrices?
B. How do you code/write the model using Matlab or other tools like Octave online?
C. How many iterations did we perform before we thought we could guess the final state/stable solution/equilibrium?

Part II

A. In what other way(s) could we calculate the equilibrium? Is there any way to make use of vectors and matrices?
B. Let's have a look at the values we obtained doing iterations. We shall notice that the distances between the states in neighboring minutes differ, that is, are decreasing in time.

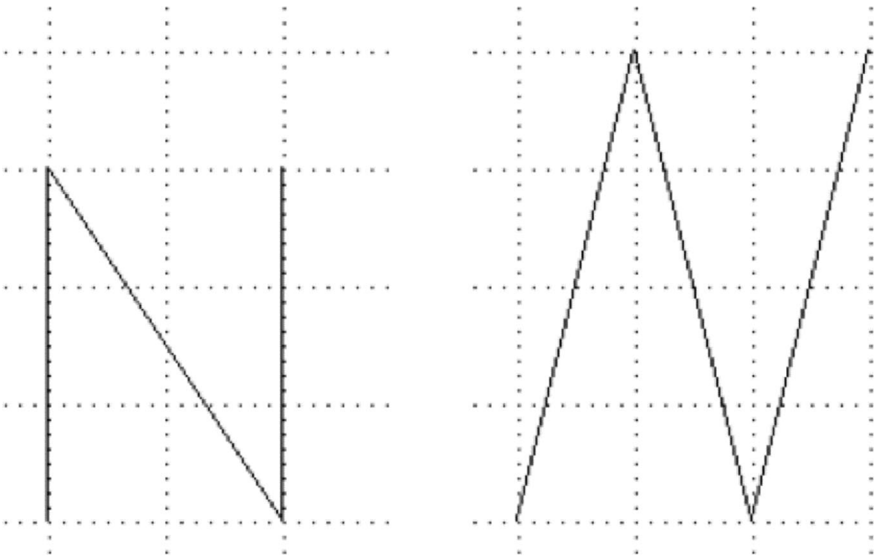
Now imagine that we would like to know how many moles of A and B are present in the mixture after, say, 90 or 150 seconds (1.5 and 2.5 minutes). How could we calculate the state at any time "between the whole minutes"?

Tool Tip: Matlab Utilities

Please use http://octave-online.net if you don't have Matlab installed on your computer
Example of a 2x2 matrix in Matlab syntax: `A = [1 2; 3 4]`
Example of a column vector: `x = [3;4]`

Theme 2: Font-transformations

Task 1)



Suppose that the letter N on the left-hand side is given in a 12-point format, and that the version on the right-hand side is in italic 16-point format.

Will there be a linear transformation that maps the N on the left to the *N* on the right? How can we go about to find this? How would the transformation look like? How could we write this mathematically? What objects/concepts can we use to express this transformation?

Task 2)

How would the letter N over (the version on the left-hand side) be transformed if you use these transformation matrices:

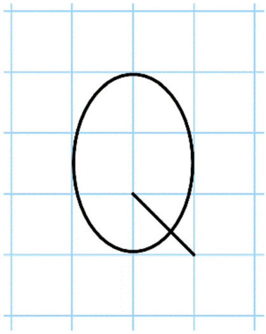
$$A = \begin{pmatrix} -3/2 & 0 \\ 0 & 5/3 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & -1/3 \\ 0 & -1 \end{pmatrix}$$

$$C = \begin{pmatrix} -1/2 & 1 \\ -1 & 0 \end{pmatrix}$$

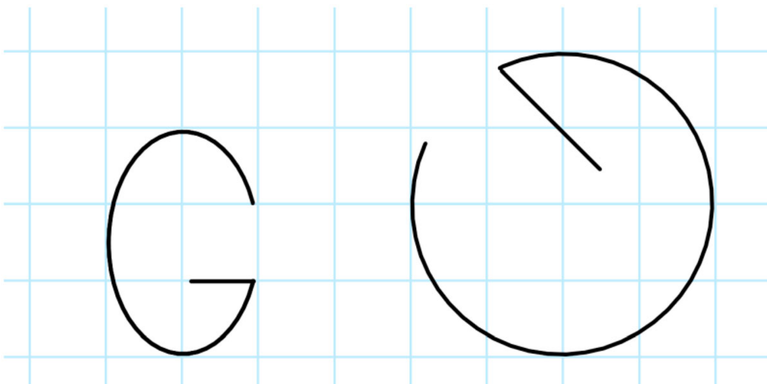
Task 3)

What would the transformation matrices in task 2 do to the letter Q?



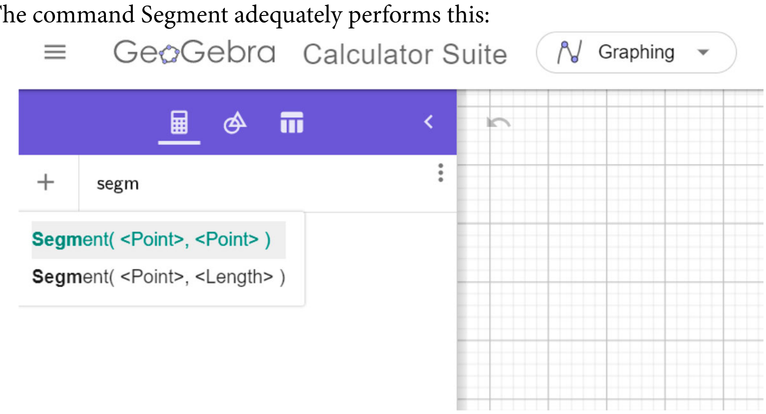
Task 4)

What linear transformation maps the letter G on the left-hand side to the version you see on the right-hand side below?

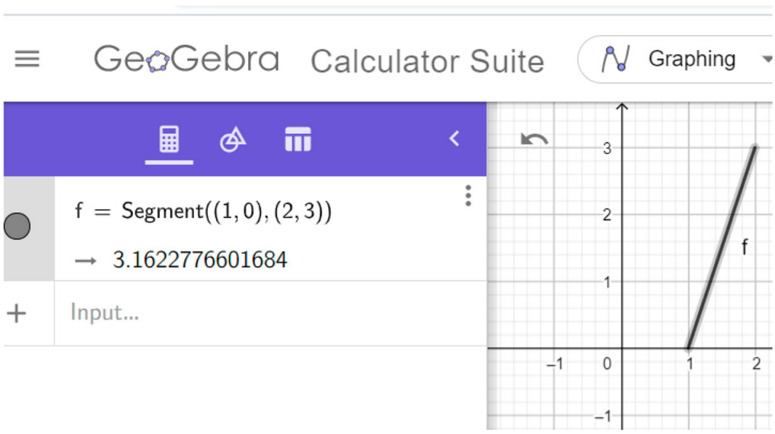


Tool Tip: Visualizing 2d line segments in GeoGebra and a bit about Matlab/Octave syntax

The command Segment adequately performs this:



Remember to surround a point in GeoGebra by parentheses like this:



You may use http://octave-online.net if you do not have Matlab installed on your computers.
This is an example of a matrix in Matlab: `A = [1 2; 3 4]`
And this is how a column vector is written: `x = [3;4]`

Theme 3: Mars landing

Task 1)

This task refers to the landing procedure on Mars in one of the videos, only now we assume that only two of the motors are working. These points in the directions $\vec{v}_1 = [1, 1, -1]$ and $\vec{v}_2 = [1, -1, 1]$. We have three opportunities for landing that need to be examined. For all three possibilities we consider the starting point to be the origan $(0, 0, 0)$ and the destination to be P (given in a) b) and c) below). On all occasions we ask the questions:

1. Could we get there with the two engines that are now working? How much does each vector need to be scaled?
2. How do we interpret these scaling factors in terms of space/time?
3. If we cannot get there, how do we fix it? That is, if we need to add another motor, which direction will it need to point in?

The points are

h) $P = (3, 0, 0)$
i) $P = (-3, 0, 0)$
j) $P = (-3, -3, -3)$

Task 2)

Now assume that our landing vessel is equipped with 4 rocket engines that may move the vessel in these four directions:

$$\vec{u}_1 = [1, 0, 1],\ \vec{u}_2 = [0, 2, 0],\ \vec{u}_3 = [-1, 1, 0],\text{ and } \vec{u}_4 = [0, 1, 1]$$

Where in space can we get to with these four vectors? (assume that you may move back and forth with each rocket). What happens if one of the rockets gets damaged, where can we get to then? Does this make a difference? Have a look at the vectors in a visualization tool like https://academo.org/demos/3d-vector-plotter or GeoGebra if you are stuck.

Task 3)

Finally, let's assume you have another set of rockets that can move the vessel freely in these four directions:

$$\vec{v}_1 = [1, 0, -1],\ \vec{v}_2 = [1, 0, 1],\ \vec{v}_3 = [0, 1, 0] \text{ and } \vec{v}_4 = [0, -1, 1]$$

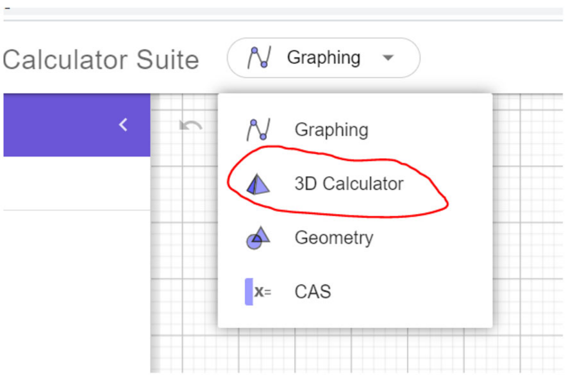
Where in space can we get with these vectors? What happens if one of the vectors gets damaged? Where can we get then? Does this make a difference?

Task 4)

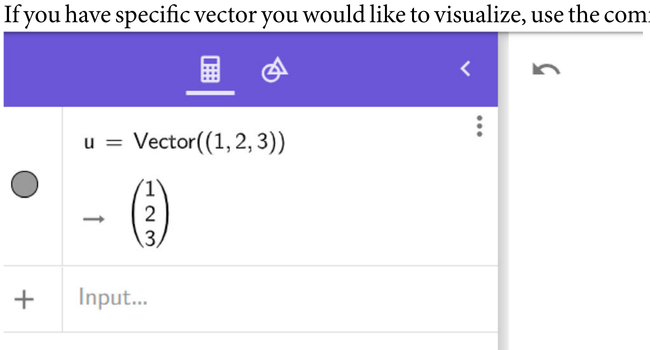
Comparing the two configurations in task 2 and 3, what can we say about the domain/area, also called Span, that the vessel can move in? What is the best configuration considering that one of the thrusters may malfunction? Can we understand this by using vectors/matrices/lines/planes etc.?

Tool Tip: About visualizing 3d vectors in GeoGebra

When you open GeoGebra (from geogebra.org), choose 3d Calculator:



If you have specific vector you would like to visualize, use the command `Vector(<point>)`.



NB: Notice that there need to be a double set of parentheses in the call to the Vector() function, since the vector need a Point as argument, which also is specified by

BIOGRAPHICAL SKETCHES

Helge Fredriksen received his PhD in mathematics education at the University of Agder 2020 and holds a position as Associate Professor at The Arctic University of Norway, Department of Computer Science. Fredriksen has research interests in active learning strategies in science-related subjects, in addition to various applications of machine learning in medicine and industry.

He teaches machine learning subjects at the master level and supervises students in computer science subjects.

Josef Rebenda is a researcher at the Central European Institute of Technology of Brno University of Technology, and an assistant professor at the Department of Mathematics of the Faculty of Electrical Engineering and Communication of Brno University of Technology. He earned his Ph.D. in mathematical analysis and worked as a data analyst in a private company for 5 years. His previous works include publications on differential equations, numerical analysis, and inquiry-based mathematics education. He participated in two research projects and has coordinated six educational projects in mathematics.

Ragnhild Johanne Rensaa received her PhD in mathematics from the Norwegian University of Science and Technology. She is Professor of Mathematics at the Arctic University of Norway, Department of Electrical Engineering at the Faculty of Engineering Science and Technology and holds a particular research interest in mathematics education. Her research areas are mathematics teaching and learning of engineering students but also mathematical content and gender perspectives in mathematics.

Petter Pettersen has since 1998 worked as a lecturer at Nord University Bodø. He has taught mathematics, mathematics didactics, statistics and coding to students enrolled in the university's programs for business students, student teachers, and engineering students. He has also published numerous textbooks and been awarded best lecturer for his work.