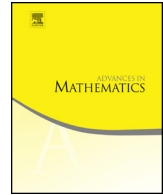




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Making the motivic group structure on the endomorphisms of the projective line explicit



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ABSTRACT

We construct a group structure on the set of pointed naive homotopy classes of scheme morphisms from the Jouanolou device to the projective line. The group operation is defined via matrix multiplication on generating sections of line bundles and only requires basic algebraic geometry. In particular, it is completely independent of the construction of the motivic homotopy category. We show that a particular scheme morphism, which exhibits the Jouanolou device as an affine torsor bundle over the projective line, induces a monoid morphism from Cazanave's monoid to this group. Moreover, we show that this monoid morphism is a group completion to a subgroup of the group of scheme morphisms from the Jouanolou device to the projective line. This subgroup is generated by a set of morphisms that are very simple to describe.

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1. Introduction

Let $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ denote the set of self-maps of the projective line in the pointed \mathbb{A}^1 -homotopy category over a field k introduced by Morel and Voevodsky [23]. The set $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ admits the structure of an abelian group and plays the role of the fundamental group of the circle in motivic homotopy theory.

We briefly recall how the group operation on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is defined. The standard covering of \mathbb{P}^1 by two affine lines with intersection \mathbb{G}_m yields an \mathbb{A}^1 -weak equivalence $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$. The simplicial circle S^1 (or some suitable homotopy equivalent model of it, like $\partial\Delta^2$) admits the structure of an h-cogroup. The h-cogroup structure on S^1 makes it possible to define a group operation on $[S^1 \wedge \mathbb{G}_m, \mathbb{P}^1]^{\mathbb{A}^1}$ in an analogous way that one defines the fundamental group of a topological space. Although the construction mimics the usual construction in algebraic topology, the set of \mathbb{A}^1 -homotopy classes of maps $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ is not simply the set of morphisms $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ modulo an equivalence relation. It is unsettling that such an important group does not arise in some elementary way as a set of morphisms up to a homotopy relation with some geometrically defined group operation. The purpose of the present paper is to remedy this defect.

Our results build on the work of Asok, Hoyois, and Wendt and the work of Cazanave. In [12] Cazanave defines an operation \oplus^N which turns the set $[\mathbb{P}^1, \mathbb{P}^1]^N$ of pointed naive¹ homotopy classes into a monoid and shows that the canonical map from naive to \mathbb{A}^1 -homotopy classes $\nu_{\mathbb{P}^1}: [\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group completion. However, this approach cannot yield candidates for scheme morphisms that represent inverses of \mathbb{A}^1 -homotopy classes. In [6] Asok, Hoyois, and Wendt show that the set $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is in bijection with an explicit set of maps modulo the naive homotopy relation by using the larger set $\text{Sm}_k(\mathcal{J}, \mathbb{P}^1)$ of morphisms of smooth k -schemes where \mathcal{J} denotes the Jouanolou device of \mathbb{P}^1 . Recall that the smooth affine scheme \mathcal{J} has the concrete description as the spectrum of the ring

$$R = \frac{k[x, y, z, w]}{(x + w - 1, xw - yz)}.$$

We consider \mathcal{J} equipped with a morphism $\pi: \mathcal{J} \rightarrow \mathbb{P}^1$ that exhibits \mathcal{J} as an affine bundle torsor.² More precisely, the work by Asok, Hoyois, and Wendt can be used³ to show that there is a bijection $\xi: [\mathcal{J}, \mathbb{P}^1]^N \xrightarrow{\cong} [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. The map ξ is the composite of the canonical map $\nu: [\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ and the inverse of the map $\pi_{\mathbb{A}^1}^*: [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ induced by a scheme morphism $\pi: \mathcal{J} \rightarrow \mathbb{P}^1$. The set $[\mathcal{J}, \mathbb{P}^1]^N$ is concrete in the following sense: it is the set of pointed scheme morphisms $\mathcal{J} \rightarrow \mathbb{P}^1$ modulo an equivalence relation generated by naive homotopies. This resolves the problem of a lack of candidates of morphisms which may represent inverses in $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. However, it is not clear at all how the group operation on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ or the operation \oplus^N of [12] may be lifted.

In the present paper we define an explicit group structure on the set $[\mathcal{J}, \mathbb{P}^1]^N$ of pointed naive homotopy classes. The construction of this group operation is independent of the general machinery of motivic homotopy theory and only uses basic algebraic geometry. We then show that the induced map $\pi_N^*: [\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^N$ is a morphism of monoids where $[\mathbb{P}^1, \mathbb{P}^1]^N$ has the monoid structure from Cazanave [12, §3]. Moreover, we show that π_N^* has image in a concrete subgroup \mathbf{G} and that the map $\pi_N^*: [\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow \mathbf{G}$ is a group completion. Hence there are canonical isomorphisms between \mathbf{G} and $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ which are compatible with π_N^* and $\nu_{\mathbb{P}^1}$. A key feature of the group \mathbf{G} is that it is defined by explicit generating scheme morphisms $\mathcal{J} \rightarrow \mathbb{P}^1$ that are defined in terms of very simple (2×2) -matrices. Hence our result provides a concrete and simple set of scheme morphisms whose images provide generators together with their inverses for the group $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$.

We will now describe our results in more detail. Recall that Cazanave shows in [12, Theorem 3.22] that there is an operation \oplus^N which provides $[\mathbb{P}^1, \mathbb{P}^1]^N$ with the structure

¹ A naive homotopy between two pointed morphisms $f, g: \mathcal{J} \rightarrow \mathbb{P}^1$ is given by a morphism $H: \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$ for which the evident restrictions satisfy $H_0 = f$ and $H_1 = g$. We note that H must be pointed in the sense that $* \times \mathbb{A}^1$ maps to the basepoint of \mathbb{P}^1 . For more details we refer to Section 2.6.

² See Definition 6 in Section 2.1 for the definition of π .

³ We explain in Appendix A how the unpointed results of [5] and [6] imply that the canonical map $[\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ between sets of homotopy classes of pointed morphisms is a bijection.

of a commutative monoid and that the canonical map $[\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group completion. We now state our first main result:

Theorem 1. *There is an operation \oplus which makes $([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$ an abelian group such that the morphism $\pi: \mathcal{J} \rightarrow \mathbb{P}^1$ induces a morphism of commutative monoids*

$$\pi_N^*: ([\mathbb{P}^1, \mathbb{P}^1]^N, \oplus^N) \rightarrow ([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$$

where the left-hand side denotes the monoid of [12, §3].

We will now outline the ideas that lead to the proof of Theorem 1. First we describe the construction of the explicit group structure \oplus on $[\mathcal{J}, \mathbb{P}^1]^N$. Recall that a morphism $f: \mathcal{J} \rightarrow \mathbb{P}^1$ is determined by an invertible sheaf \mathcal{L} over \mathcal{J} and a choice of two generating sections $s_0, s_1 \in \Gamma(\mathcal{L}, \mathcal{J})$. The invertible sheaf \mathcal{L} is the pullback $f^*\mathcal{O}(1)$. We say that a morphism $f: \mathcal{J} \rightarrow \mathbb{P}^1$ has degree 0 if $f^*\mathcal{O}(1)$ is the structure sheaf on \mathcal{J} . As we will show in Section 2.7, the maps $\mathcal{J} \rightarrow \mathbb{P}^1$ of degree 0 are exactly the maps which factor through the Hopf map $\eta: \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$. Let R denote the ring such that $\mathcal{J} = \text{Spec } R$. The set $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ has an apparent group structure: A morphism $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ is given by a unimodular row (A, B) in R^2 , i.e., there exist $U, V \in R$ for which $AU + BV = 1$. Any such unimodular row can be completed to a (2×2) -matrix over R , and the product of these matrices defines a group operation on $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$. We describe the details of this construction in Section 3.1. The subgroup of degree 0 maps $\mathcal{J} \rightarrow \mathbb{P}^1$ is quite large. In fact, the degree map fits into an exact sequence of pointed sets of the form

$$1 \rightarrow [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^N \xrightarrow{\text{deg}} \text{Pic}(\mathcal{J}) \rightarrow 1.$$

Hence, in order to turn $[\mathcal{J}, \mathbb{P}^1]^N$ into a group, it suffices to define an action of $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ on $[\mathcal{J}, \mathbb{P}^1]^N$. The key idea is that any morphism $f: \mathcal{J} \rightarrow \mathbb{P}^1$ is given by the choice of a line bundle together with two generating sections. We can then let a morphism $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ given by a (2×2) -matrix act on the sections via matrix multiplication. We explain the details of this operation in Section 3.2 and we complete the construction of the group structure on $[\mathcal{J}, \mathbb{P}^1]^N$ in Section 4, see Definition 73 and Theorem 80 where we show that there is an isomorphism of groups $[\mathcal{J}, \mathbb{P}^1]^N \xrightarrow{\varphi} [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$.

Next we describe the main idea for the proof that π_N^* is a morphism of monoids. Let $u \in k^\times$. As in [12] we identify a rational function X/u in the indeterminate X with the morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined by $[x_0 : x_1] \mapsto [x_0 : ux_1]$. For $u, v \in k^\times$, let $g_{u,v}: \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ denote the morphism given by the unimodular row $(x + \frac{v}{u}w, (u - v)y)$ in R^2 , where x, y, z , and w are the polynomial generators of the ring

$$R = \frac{k[x, y, z, w]}{(x + w - 1, xw - yz)}$$

defining $\mathcal{J} = \text{Spec } R$. For the rational functions X/u and X/v we then have the identity

$$g_{u,v} \oplus \pi_N^*(X/v) = \pi_N^*(X/u) \tag{2}$$

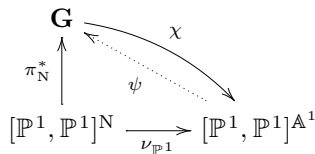
which we emphasize is an identity of morphisms not just homotopy classes. In the particular case where $v = 1$, we have $\pi_N^*(X/1) = \pi$ and Formula (2) reads $g_{u,1} \oplus \pi = \pi_N^*(X/u)$. This reduces computations for $\pi_N^*(X/u)$ to computations for $g_{u,1}$ and π . The key technical result needed to prove Theorem 1 is that, for every pointed morphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$, we have an explicit naive homotopy

$$\pi_N^*(X/u \oplus^N f) \simeq g_{u,1} \oplus (\pi_N^*(X/1 \oplus^N f)). \tag{3}$$

The construction of the concrete homotopy in Formula (3) is based on computations of the resultants of certain morphisms which we provide in Section 5.1 and Appendix B. Theorem 1 then follows from the fact that the set of homotopy classes $[X/u]$ for all $u \in k^\times$ generates $[\mathbb{P}^1, \mathbb{P}^1]^N$ and a successive application of Formula (3). The details of this argument are explained in Section 5, see Theorem 96.

Our second main result is then based on the observation that Identity (2) implies that the image of π_N^* is contained in the subgroup $\mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^N$ generated by the homotopy classes $[g_{u,v}]$ for all $u, v \in k^\times$ and $[\pi]$. Theorem 1 and the work of Cazanave [12, Theorem 3.22] then imply that there is a unique group homomorphism $\psi: [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow \mathbf{G}$ such that $\psi \circ \nu_{\mathbb{P}^1} = \pi_N^*$. In Section 6 we show the following key result, see Theorem 113:

Theorem 4. *The monoid morphism $\pi_N^*: [\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow \mathbf{G}$ is a group completion. There is a unique isomorphism $\chi: \mathbf{G} \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ such that the diagram below commutes, where χ and ψ are mutual inverses to each other.*



Theorem 4 gives a very concrete description of all pointed endomorphisms of \mathbb{P}^1 in the unstable \mathbb{A}^1 -homotopy category in the following sense: the group \mathbf{G} is given by a simple set of generating morphisms, and the group operation \oplus in \mathbf{G} inherited from $[\mathcal{J}, \mathbb{P}^1]^N$ is defined in basic algebro-geometric terms.

Finally, we note that the isomorphisms $\mathbf{G} \xrightarrow{\chi} [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xleftarrow{\psi} [\mathcal{J}, \mathbb{P}^1]^N$ do not imply that \mathbf{G} equals $[\mathcal{J}, \mathbb{P}^1]^N$. However, we conjecture that the inclusion $\mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^N$ is an equality and we show in Section 6.3 that this is true for all finite fields by computing the first Milnor–Witt K-theory group $K_1^{\text{MW}}(\mathbb{F}_q)$, which is isomorphic to $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ and to the subgroup generated by all classes $[g_{u,v}]$ in $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$.

Below we provide a list of frequently used and important notation together with a reference where the notation is first used after the introduction.

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List of important notation

Notation	Brief description	First discussed
k	a field	Section 2.1
\mathbb{P}^1	projective line over k pointed at ∞	Section 2.1
R	The ring $R := k[x, y, z, w]/(xw - yz, x + w - 1)$	Definition 5
\mathcal{J}	the Jouanolou device $\mathcal{J} = \text{Spec } R$	Definition 5
\mathbf{j}	basepoint $(x - 1, y, z, w) \subseteq R$ of \mathcal{J}	Definition 5
\mathbf{j}'	basepoint $(x - 1, y, z, w) \subseteq R[T]$ of $\mathcal{J} \times \mathbb{A}^1$	Remark 40
\mathcal{P}_1	line bundle over \mathcal{J} , image of $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$	Section 2.1
\mathcal{Q}_1	line bundle over \mathcal{J} , image of $\begin{pmatrix} x & z \\ y & w \end{pmatrix}$	Section 2.1
$[s_0, s_1]$	map $\mathcal{J} \rightarrow \mathbb{P}^1 \leftrightarrow$ line bundle \mathcal{P}_n with sections s_0, s_1	Construction 27
X/u	morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1, [x_0 : x_1] \mapsto [x_0 : ux_1]$	Section 5.1
$\text{res}(A, B)$	resultant of polynomials A, B	Proposition 22
$(a_0, a_1 : b_0, b_1)_n$	morphism $\mathcal{J} \rightarrow \mathbb{P}^1$	Definition 32
deg	degree of \mathcal{L} of a morphism $(\mathcal{L}, s_0, s_1): \mathcal{J} \rightarrow \mathbb{P}^1$	Section 2.7
η	the Hopf map $\eta: \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$	Section 2.7
SL_2	$\text{SL}_2 := \text{Spec}(k[a, b, c, d]/(ad - bc - 1))$	Section 2.7
M	an $\text{SL}_2(R)$ -matrix	Section 2.7
ϕ	first column morphism $\text{SL}_2 \rightarrow \mathbb{A}^2 \setminus \{0\}$	Definition 50
$g_{u,v}$	unimodular row $(x + \frac{u}{v}w, (u - v)y)$ in R	Definition 86
$m_{u,v}$	$\text{SL}_2(R)$ -matrix $\begin{pmatrix} x + \frac{u}{v}w & \frac{u-v}{uv}z \\ (u-v)y & x + \frac{u}{v}w \end{pmatrix}$	Definition 86
Sm_k	the category of smooth finite type k -schemes	Section 2.6
$\text{Sm}_k(X, Y)_*$	set of pointed morphisms in Sm_k	Section 2.6
\simeq	naive homotopy relation	Definition 39
$[X, Y]^N$	set of pointed naive homotopy classes	Definition 39
$[\mathcal{J}, \mathbb{P}^1]^N_n$	set of pointed naive homotopy classes of degree n	Section 2.7
$[X, Y]^{\mathbb{A}^1}$	set of pointed \mathbb{A}^1 -homotopy classes	Section 6
ν	canonical map $[\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$	Section 6
π_N^*	map $[\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^N$ induced by π	Section 5
$\pi_{\mathbb{A}^1}^*$	map $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ induced by π	Section 6
ξ	map $\xi = (\pi_{\mathbb{A}^1}^*)^{-1} \circ \nu: [\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$	Equation (8)
\oplus	group operation on $[\mathcal{J}, \mathbb{P}^1]^N$	Definition 77
\oplus^N	monoid operation on $[\mathbb{P}^1, \mathbb{P}^1]^N$	Section 5
$\oplus^{\mathbb{A}^1}$	group operation on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$	Section 6
φ	group isom. $([\mathcal{J}, \mathbb{P}^1]^N, \oplus) \xrightarrow{\cong} ([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1})$	Theorem 80
$\text{deg}^{\mathbb{A}^1}$	Morel's \mathbb{A}^1 -Brouwer degree $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow \text{GW}(k)$	Section 6.1

2. The Jouanolou device and morphisms to \mathbb{P}^1

In this section we work out the details needed about the Jouanolou device \mathcal{J} , morphisms $\mathcal{J} \rightarrow \mathbb{P}^1$, and the pointed naive homotopy relation. We keep the terminology as elementary as possible and hope that the details provided help make our approach accessible.

2.1. Definition of \mathcal{J} , π , and ξ

Throughout this paper k will always denote a field. All schemes are schemes over $\text{Spec } k$. We denote by \mathbb{P}^1 the projective line over k pointed at $\infty := [1 : 0] \in \mathbb{P}^1$. The letter R will always denote the following ring.

Definition 5. Let R denote the ring

$$R = \frac{k[x, y, z, w]}{(x + w - 1, xw - yz)}.$$

The Jouanolou device of \mathbb{P}^1 is the smooth affine k -scheme $\mathcal{J} = \text{Spec } R$. We consider \mathcal{J} to be pointed at $\mathbf{j} = (x - 1, y, z, w)$.

The ring R may be interpreted as the ring representing (2×2) -matrices with trace 1 and determinant 0. Namely, a ring homomorphism $R \rightarrow S$ is equivalent to a (2×2) -matrix over S with trace 1 and determinant 0.

While we will discuss morphisms $\mathcal{J} \rightarrow \mathbb{P}^1$ in more detail later, we point out that there are two evident morphisms that exhibit \mathcal{J} as an affine torsor bundle over \mathbb{P}^1 . The matrices

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \text{ and } \begin{pmatrix} x & z \\ y & w \end{pmatrix}$$

over R are idempotent. When viewed as linear transformations from R^2 to R^2 , the image of each matrix defines a projective module, denoted by \mathcal{P}_1 and \mathcal{Q}_1 respectively. Both \mathcal{P}_1 and \mathcal{Q}_1 have rank 1 and so they yield invertible sheaves on \mathcal{J} .

Definition 6. We define the morphism of schemes $\pi: \mathcal{J} \rightarrow \mathbb{P}^1$ by selecting the invertible sheaf associated to \mathcal{P}_1 and the generating sections

$$s_0 = \begin{pmatrix} x \\ z \end{pmatrix} \text{ and } s_1 = \begin{pmatrix} y \\ w \end{pmatrix}.$$

Similarly, we define the morphism of schemes $\tilde{\pi}: \mathcal{J} \rightarrow \mathbb{P}^1$ by using \mathcal{Q}_1 and the choice of generating sections

$$s_0 = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } s_1 = - \begin{pmatrix} z \\ w \end{pmatrix}.$$

We intuitively understand the map π as sending a point in \mathcal{J} corresponding to a matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ to either the point with homogeneous coordinates $[x : y]$ or $[z : w]$, depending on which is defined. When both points make sense in \mathbb{P}^1 , they agree, so the map is well-defined. A similar argument shows that $\tilde{\pi}$ is well-defined. Both π and $\tilde{\pi}$ exhibit \mathcal{J} as an affine torsor bundle over \mathbb{P}^1 , hence they are \mathbb{A}^1 -homotopy equivalences. It follows that

$$\pi_{\mathbb{A}^1}^* : [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1} \tag{7}$$

is a bijection. We show in Proposition 131 in Appendix A that the canonical map $\nu : [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}} \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ is a bijection because \mathcal{J} is affine and \mathbb{P}^1 is \mathbb{A}^1 -naive. Thus, the composition of the bijection ν and the inverse of $\pi_{\mathbb{A}^1}^*$ is a bijection

$$\xi : [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}} \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}. \tag{8}$$

This bijection may be described as follows. A naive pointed homotopy class of maps $[f]$ represented by the pointed scheme morphism $f : \mathcal{J} \rightarrow \mathbb{P}^1$ is sent to $\xi([f]) = [f \circ \pi^{-1}]^{\mathbb{A}^1}$, the pointed \mathbb{A}^1 -homotopy class of the zig-zag $\mathbb{P}^1 \xleftarrow{\pi} \mathcal{J} \xrightarrow{f} \mathbb{P}^1$.

In the following sections we will investigate the set $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$ of pointed naive homotopy classes of pointed morphisms $\mathcal{J} \rightarrow \mathbb{P}^1$.

2.2. Convenient coordinates for \mathcal{J}

The map $\pi : \mathcal{J} \rightarrow \mathbb{P}^1$ encourages the choice of a convenient set of coordinate charts for \mathcal{J} . For \mathbb{P}^1 , we use the standard notation $U_0 = \mathbb{P}^1 \setminus \{[0 : 1]\}$ and $U_1 = \mathbb{P}^1 \setminus \{[1 : 0]\}$. It is straightforward to verify that the preimages under π of U_0 and U_1 are $\pi^{-1}(U_0) = D(x) \cup D(z)$ and $\pi^{-1}(U_1) = D(y) \cup D(w)$. Both of these open sets are isomorphic to \mathbb{A}^2 under the following maps.

Lemma 9. *The open set $D(x) \cup D(z) \subseteq \mathcal{J}$ is isomorphic to $\text{Spec}(k[a, b])$ under the map $\Psi_0 : \mathbb{A}^2 \rightarrow \mathcal{J}$ given by $x \mapsto 1 - ab$, $y \mapsto a(1 - ab)$, $z \mapsto b$, and $w \mapsto ab$.*

Similarly, the open set $D(y) \cup D(w) \subseteq \mathcal{J}$ is isomorphic to $\text{Spec} k[c, d]$ under the map $\Psi_1 : \mathbb{A}^2 \rightarrow \mathcal{J}$ given by $x \mapsto cd$, $y \mapsto d$, $z \mapsto c(1 - cd)$, and $w \mapsto 1 - cd$.

Proof. The proof proceeds by studying the map locally. For instance, Ψ_0 induces an isomorphism of rings $k[a, b][[(1 - ab)^{-1}]] \rightarrow R[x^{-1}]$ and also of $k[a, b][[b^{-1}]] \rightarrow R[z^{-1}]$. The open sets $D(1 - ab)$ and $D(b)$ cover \mathbb{A}^2 , so it follows that Ψ_0 maps surjectively onto $D(x) \cup D(z)$. The inverse map is obtained by gluing the maps that are defined on $D(x)$ and $D(z)$, giving the result. A similar argument works for Ψ_1 . \square

Remark 10. The open affine subschemes $D(x) \cup D(z)$ and $D(y) \cup D(w)$ of $\text{Spec } R$ have the odd property that their ring of global sections is not a localization of R .

2.3. Invertible sheaves on \mathcal{J}

By [16, Theorem II.7.1], a morphism $\mathcal{J} \rightarrow \mathbb{P}^1$ is determined by an invertible sheaf \mathcal{L} on \mathcal{J} and two generating global sections of \mathcal{L} . We now take the time to study the invertible sheaves on \mathcal{J} to enable our study of the morphisms $\mathcal{J} \rightarrow \mathbb{P}^1$. We will assume familiarity with the basic terminology presented in both [25, Chapter 1] and [7].

Since $\mathcal{J} = \text{Spec } R$ is an irreducible affine scheme, the invertible sheaves on \mathcal{J} correspond to projective R -modules of rank 1. We have already seen the projective modules \mathcal{P}_1 and \mathcal{Q}_1 used to define π and $\tilde{\pi}$ above. Since the map $\pi: \mathcal{J} \rightarrow \mathbb{P}^1$ is an \mathbb{A}^1 -weak equivalence and the Picard group functor is homotopy invariant, the induced map on Picard groups is an isomorphism $\pi^*: \text{Pic}(\mathbb{P}^1) \rightarrow \text{Pic}(\mathcal{J})$. Since $\pi^*(\mathcal{O}(1)) = \mathcal{P}_1$ and $\text{Pic}(\mathcal{J}) \cong \mathbb{Z}$, it follows that \mathcal{P}_1 generates the Picard group of \mathcal{J} . For future reference, we state this as a lemma.

Lemma 11. *The Picard group of \mathcal{J} is isomorphic to \mathbb{Z} and \mathcal{P}_1 is a generator.*

Furthermore, $\mathcal{Q}_1 = -\mathcal{P}_1$ in $\text{Pic}(\mathcal{J})$ as the following proposition shows.

Proposition 12. *There is an isomorphism $\mathcal{P}_1 \otimes \mathcal{Q}_1 \cong R$.*

Proof. The R -module $\mathcal{P}_1 \otimes \mathcal{Q}_1$ is generated by

$$\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} z \\ w \end{bmatrix} \right\}.$$

Consider the module homomorphism $m: R^2 \otimes R^2 \rightarrow R^2$ induced by component-wise multiplication $m\left(\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} ac \\ bd \end{bmatrix}$. We restrict m to $\mathcal{P}_1 \otimes \mathcal{Q}_1$ and observe that the image of $\mathcal{P}_1 \otimes \mathcal{Q}_1$ under m is the submodule $\left\langle \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle \subseteq R^2$ (use $x + w = 1$). This is a free R -module of rank 1. As $m: \mathcal{P}_1 \otimes \mathcal{Q}_1 \rightarrow \left\langle \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle$ is surjective, it follows that it is locally an isomorphism at all maximal ideals $\mathfrak{m} \subseteq R$. Hence the map m itself restricted to $\mathcal{P}_1 \otimes \mathcal{Q}_1$ is an isomorphism onto its image. \square

We would like to understand the tensor powers of \mathcal{P}_1 and \mathcal{Q}_1 .

Definition 13. Let \mathcal{P}_n and \mathcal{Q}_n denote the submodules of R^2 generated by

$$\mathcal{P}_n = \left\langle \begin{bmatrix} x^n \\ z^n \end{bmatrix}, \begin{bmatrix} x^{n-1}y \\ z^{n-1}w \end{bmatrix}, \dots, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right\rangle \text{ and } \mathcal{Q}_n = \left\langle \begin{bmatrix} x^n \\ y^n \end{bmatrix}, \begin{bmatrix} x^{n-1}z \\ y^{n-1}w \end{bmatrix}, \dots, \begin{bmatrix} z^n \\ w^n \end{bmatrix} \right\rangle.$$

The following lemma is useful for simplifying proofs. It shows that what we prove about \mathcal{P}_n by symmetry holds for \mathcal{Q}_n .

Lemma 14. Define the involution $\tau: R \rightarrow R$ by

$$\begin{aligned} \tau(x) &= x & \tau(y) &= z \\ \tau(z) &= y & \tau(w) &= w. \end{aligned}$$

Pulling back along τ gives R -module isomorphisms $\tau^*\mathcal{P}_n \cong \mathcal{Q}_n$ and $\tau^*\mathcal{Q}_n \cong \mathcal{P}_n$.

Proof. To more easily distinguish between them, we give the domain and codomain of τ different names and write $\tau: R \rightarrow R'$. Pulling back the R' -module \mathcal{P}_n , we get the R -module $\tau^*\mathcal{P}_n$, where the multiplication is defined by $r \cdot_R p = \tau(r) \cdot_{R'} p$. The map $\tau^*\mathcal{P}_n \rightarrow \mathcal{Q}_n$ is defined on basis elements by

$$\begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} \mapsto \begin{bmatrix} x^{n-i}z^i \\ y^{n-i}w^i \end{bmatrix}.$$

It is easily checked that f is bijective and R -linear and hence an R -module isomorphism.

To see that $\tau^*\mathcal{Q}_n \cong \mathcal{P}_n$, we pull back the isomorphism $\mathcal{Q}_n \cong \tau^*\mathcal{P}_n$, which we just proved, along τ on both sides. Since $\tau \circ \tau = \text{id}$, this simplifies to $\tau^*\mathcal{Q}_n \cong \tau^*\tau^*\mathcal{P}_n = \mathcal{P}_n$. \square

Proposition 15. The R -modules \mathcal{P}_n and \mathcal{Q}_n are also generated in the following way

$$\mathcal{P}_n = \left\langle \begin{bmatrix} x^n \\ z^n \end{bmatrix}, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right\rangle \quad \text{and} \quad \mathcal{Q}_n = \left\langle \begin{bmatrix} x^n \\ y^n \end{bmatrix}, \begin{bmatrix} z^n \\ w^n \end{bmatrix} \right\rangle.$$

Proof. We only prove the claim for \mathcal{P}_n , as the proof for \mathcal{Q}_n is analogous by Lemma 14. Containment in one direction is clear by definition of \mathcal{P}_n . Now fix n and pick a number $0 \leq i \leq n$. We then have

$$\begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = (x + w)^n \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = \sum_{d=0}^n \binom{n}{d} x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix}.$$

For each d , one of the following holds:

$$x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = x^{n-i-d}y^i w^d \begin{bmatrix} x^n \\ z^n \end{bmatrix} \quad \text{if } i + d \leq n,$$

$$x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = x^{n-d}z^{n-i}w^{d+i-n} \begin{bmatrix} y^n \\ w^n \end{bmatrix} \quad \text{if } i + d > n.$$

This completes the proof. \square

We will now show that the R -modules \mathcal{P}_n and \mathcal{Q}_n are algebraic line bundles, that is, finitely generated R -modules of constant rank 1. We check locally and in the process give a description of \mathcal{P}_n and \mathcal{Q}_n with open patching data [25, 2.5 on page 14].

Proposition 16. *The R -modules \mathcal{P}_n and \mathcal{Q}_n are algebraic line bundles over R , that is, finitely generated R -modules of constant rank 1, and determine invertible sheaves on \mathcal{J} . They are described in terms of open patching data by*

$$\mathcal{P}_n \cong \left\{ (f_x, f_w) \in R[x^{-1}] \times R[w^{-1}] \mid f_w = \left(\frac{z}{x}\right)^n f_x \right\},$$

$$\mathcal{Q}_n \cong \left\{ (f_x, f_w) \in R[x^{-1}] \times R[w^{-1}] \mid f_w = \left(\frac{y}{x}\right)^n f_x \right\}.$$

Proof. The canonical projections $\mathcal{P}_n[x^{-1}] \rightarrow R[x^{-1}]$ and $\mathcal{P}_n[w^{-1}] \rightarrow R[w^{-1}]$ are isomorphisms. Since $D(x) \cup D(w) = \mathcal{J}$, we conclude \mathcal{P}_n is locally free of constant rank 1 and is thus an algebraic line bundle [25, Lemma 2.4]. For $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{P}_n[x^{-1}, w^{-1}]$, one checks that $(z/x)^n f = g$, which determines \mathcal{P}_n in terms of open patching data. A similar calculation works for \mathcal{Q}_n . \square

Remark 17. Proposition 16 shows us how to interpret an element

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \text{ with } a_0, a_1 \in R$$

that is a global section of the invertible sheaf associated to \mathcal{P}_n . Namely, the global section s_0 restricted to $D(x)$ is described by $a_0x^n + a_1y^n$, while on $D(w)$ the section is $a_0z^n + a_1w^n$. On the overlap, the two sections agree when compared using the appropriate transition functions.

Combining Propositions 15 and 16 it follows that any tuple $(f_x, f_w) \in R[x^{-1}] \times R[w^{-1}]$ satisfying $f_w = \left(\frac{z}{x}\right)^n f_x$ can be expressed as

$$\begin{bmatrix} f_x \\ f_w \end{bmatrix} = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \text{ for some } a_0, a_1 \in R.$$

The algebraic line bundles \mathcal{P}_n and \mathcal{Q}_n may also be described as the image of an idempotent (2×2) -matrix of rank 1. For $n \geq 1$, let $A = \sum_{i=0}^{n-1} \binom{2n-1}{i} x^{n-1-i} w^i$ and $B = \sum_{i=n}^{2n-1} \binom{2n-1}{i} x^{2n-1-i} w^{i-n}$. Then $x^n A + w^n B = (x + w)^{2n-1} = 1$. Define

$$M_n = \begin{pmatrix} x^n A & y^n B \\ z^n A & w^n B \end{pmatrix} \text{ and } M'_n = \begin{pmatrix} x^n A & z^n B \\ y^n A & w^n B \end{pmatrix}. \tag{18}$$

Proposition 19. *For every $n \geq 1$, the matrices M_n and M'_n are idempotent of rank 1. When viewed as linear transformations from R^2 to R^2 , the image of M_n and M'_n is \mathcal{P}_n and \mathcal{Q}_n respectively.*

Proof. It is straightforward to verify that M_n is idempotent using the relation $1 = x^n A + w^n B$ and that $\text{Im}(M_n) \subset \mathcal{P}_n$. Note that

$$x^n \begin{bmatrix} x^n A \\ z^n A \end{bmatrix} + z^n \begin{bmatrix} y^n B \\ w^n B \end{bmatrix} = (x^n A + w^n B) \begin{bmatrix} x^n \\ z^n \end{bmatrix} = \begin{bmatrix} x^n \\ z^n \end{bmatrix}$$

and similarly,

$$y^n \begin{bmatrix} x^n A \\ z^n A \end{bmatrix} + w^n \begin{bmatrix} y^n B \\ w^n B \end{bmatrix} = (x^n A + w^n B) \begin{bmatrix} y^n \\ w^n \end{bmatrix} = \begin{bmatrix} y^n \\ w^n \end{bmatrix}.$$

So $\mathcal{P}_n \subset \text{Im}(M_n)$, and the image is equal to \mathcal{P}_n . The argument for M'_n and \mathcal{Q}_n is similar. \square

Proposition 20. *For every $n \geq 1$, the morphisms $\mathcal{P}_n \otimes \mathcal{P}_1 \rightarrow \mathcal{P}_{n+1}$ and $\mathcal{P}_1^{\otimes n} \rightarrow \mathcal{P}_n$ obtained from component-wise multiplication are isomorphisms. A similar statement holds for \mathcal{Q}_n .*

Proof. Consider the R -module map $m: R^2 \otimes R^2 \rightarrow R^2$ induced by component-wise multiplication. By the description of the generators of the modules \mathcal{P}_n , it is clear that m restricts to a map $m: \mathcal{P}_n \otimes \mathcal{P}_1 \rightarrow \mathcal{P}_{n+1}$ and this map is surjective. As both $\mathcal{P}_n \otimes \mathcal{P}_1$ and \mathcal{P}_{n+1} are algebraic line bundles, the map m is surjective locally at every maximal ideal $\mathfrak{m} \subseteq R$ and hence an isomorphism. Thus $m: \mathcal{P}_n \otimes \mathcal{P}_1 \rightarrow \mathcal{P}_{n+1}$ is itself an isomorphism by [7, Proposition 3.9]. This proves the first claim. The second claim now follows by induction. \square

We now have a complete description of the isomorphism $\mathbb{Z} \cong \text{Pic}(\mathcal{J})$.

Proposition 21. *Under the isomorphism $\text{Pic}(\mathcal{J}) \cong \text{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ arising from the \mathbb{A}^1 -homotopy equivalence $\pi: \mathcal{J} \rightarrow \mathbb{P}^1$, the modules \mathcal{P}_n and \mathcal{Q}_n correspond to n and $-n$, respectively, while the trivial invertible sheaf \mathcal{O} corresponds to 0.*

Proof. By Lemma 11, $\text{Pic}(\mathcal{J}) = \mathbb{Z}$, and \mathcal{P}_1 generates the Picard group. By Proposition 12, the inverse of \mathcal{P}_1 is \mathcal{Q}_1 . By Proposition 20, the modules \mathcal{P}_n and \mathcal{Q}_n correspond to n and $-n$ in the Picard group. \square

2.4. Pointed morphisms $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $\mathcal{J} \rightarrow \mathbb{P}^1$

We will now study morphisms to \mathbb{P}^1 in more detail. By [16, Theorem II.7.1], for a smooth k -scheme X , the data needed to give a morphism $f: X \rightarrow \mathbb{P}^1$ are an invertible sheaf \mathcal{L} over X and the choice of two global sections $s_0, s_1 \in \Gamma(X, \mathcal{L})$ that generate the invertible sheaf \mathcal{L} . That is, at every point $p \in X$, the stalks of the sections $(s_0)_p$ and $(s_1)_p$ generate the local ring \mathcal{L}_p . We then write $[s_0, s_1]$ for the morphism $X \rightarrow \mathbb{P}^1$ given

by the data above, where we usually omit the invertible sheaf \mathcal{L} from the notation. We note that throughout the paper we use the terms *morphism* and *map* interchangeably.

The scheme \mathbb{P}^1 is pointed at $\infty = [1 : 0]$. A pointed morphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by definition is a morphism satisfying $f(\infty) = \infty$. A pointed morphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by the invertible sheaf $\mathcal{O}(n)$ on \mathbb{P}^1 with two generating sections $\sigma_0, \sigma_1 \in k[x_0, x_1]_{(n)}$ has the following special form by work of Cazanave [12].

Proposition 22. [12, Proposition 2.3] *A pointed k -scheme morphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ corresponds uniquely to the data of a natural number n and a choice of two polynomials, $A = \sum_{i=0}^n a_i X^i$ and $B = \sum_{i=0}^{n-1} b_i X^i$ in $k[X]$ for which $a_n = 1$ and the resultant $\text{res}(A, B)$ is non-zero. The integer n is called the degree of f and is denoted $\text{deg}(f)$; the scalar $\text{res}(f) = \text{res}(A, B) \in k^\times$ is called the resultant of f . We recall that $\text{res}(A, B) = \det(\text{Syl}(A, B)) \in k$, where $\text{Syl}(A, B)$ is the Sylvester matrix of the pair of polynomials (A, B) which we recall in Definition 132.*

Remark 23. One easily translates from the morphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by n, A , and B in Proposition 22 to the morphism given by the invertible sheaf $\mathcal{O}(n)$ and the choice of global sections $\sigma_0 = \sum_{i=0}^n a_i x_0^i x_1^{n-i}$ and $\sigma_1 = \sum_{i=0}^{n-1} b_i x_0^i x_1^{n-i}$ where we understand $b_n = 0$. The resultant condition guarantees that these global sections generate $\mathcal{O}(n)$. The condition $a_n = 1$ is a normalizing condition to give a bijective correspondence between morphisms and the data n, A , and B . We will find it more convenient to use the data $[\sigma_0, \sigma_1]: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $\mathcal{O}(n)$ to describe a pointed map in what follows.

We will now explain in detail how we can use this perspective to describe morphisms via line bundles and generating sections in the special case $\mathcal{J} \rightarrow \mathbb{P}^1$.

Proposition 24. *Consider a pointed map $[\sigma_0, \sigma_1]: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ with invertible sheaf $\mathcal{O}(n)$, $\sigma_0 = \sum_{i=0}^n a_i x_0^i x_1^{n-i}$ and $\sigma_1 = \sum_{i=0}^{n-1} b_i x_0^i x_1^{n-i}$. The composition $[\sigma_0, \sigma_1] \circ \pi$ is the map $[s_0, s_1]: \mathcal{J} \rightarrow \mathbb{P}^1$ with invertible sheaf \mathcal{P}_n and global sections*

$$s_0 = \sum_{i=0}^n a_i \begin{bmatrix} x^i y^{n-i} \\ z^i w^{n-i} \end{bmatrix} \text{ and } s_1 = \sum_{i=0}^{n-1} b_i \begin{bmatrix} x^n y^{n-i} \\ z^i w^{n-i} \end{bmatrix}. \tag{25}$$

Proof. This is a straightforward calculation. The condition on the resultant ensures that the sections s_0 and s_1 generate \mathcal{P}_n . \square

Remark 26. We note that the difference between a general map $[s_0, s_1]: \mathcal{J} \rightarrow \mathbb{P}^1$ with invertible sheaf \mathcal{P}_n and a map $\mathcal{J} \rightarrow \mathbb{P}^1$ which factors as $f \circ \pi$ with $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is that the coefficients a_i and b_i in the expressions of the sections in Equation (25) are in the field k when the map factors, but in general the coefficients are in R .

We now look at the data needed to describe a general morphism $\mathcal{J} \rightarrow \mathbb{P}^1$ and also see what condition pointedness imposes.

Construction 27. A morphism $f: \mathcal{J} \rightarrow \mathbb{P}^1$ is determined by the following data: an invertible sheaf \mathcal{L} on \mathcal{J} and the choice of two global sections $s_0, s_1 \in \Gamma(\mathcal{J}, \mathcal{L})$ that generate \mathcal{L} [16, Theorem II.7.1]. Since $\text{Pic}(\mathcal{J}) \cong \mathbb{Z}$, the invertible sheaf \mathcal{L} may be chosen to be either $\mathcal{P}_n, \mathcal{Q}_n$, or \mathcal{O} . We call the integer corresponding to the class of \mathcal{L} in $\text{Pic}(\mathcal{J}) \cong \mathbb{Z}$ the degree of f .

We will now make the assignment $(\mathcal{L}, s_0, s_1) \mapsto f$ explicit. We will study only the case of \mathcal{P}_n , as \mathcal{Q}_n is handled in the same way by Proposition 15. The case of \mathcal{O} is discussed later in Section 2.7.

For $\mathcal{L} = \mathcal{P}_n$, two generating sections $s_0, s_1 \in \Gamma(\mathcal{J}, \mathcal{P}_n)$ may be chosen to be of the form

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix} \quad s_1 = b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}.$$

Define $D(s_i) = \{p \in \mathcal{J} \mid (s_i)_p \notin \mathfrak{m}_p(\mathcal{P}_n)_p\}$. The map $[s_0, s_1]$ is defined on the open set $D(s_i)$ to map into $U_i = \{[x_0, x_1] \mid x_i \neq 0\}$. Here $U_0 \cong \text{Spec } k[y_1]$ and $U_1 \cong \text{Spec } k[y_0]$, where $y_0 = x_0/x_1$ and $y_1 = x_1/x_0$. The map $D(s_i) \rightarrow U_i$ is given by the corresponding map of rings $k[y_j] \rightarrow \mathcal{P}_n[s_i^{-1}]$ determined by $y_j \mapsto s_j/s_i$. This requires some explanation due to the description of the sheaf \mathcal{P}_n . Proposition 16 shows that the components of each section s_i describe the section on the open sets $D(x)$ and $D(w)$. Hence there are four cases to consider to get a description of the map in concrete terms of affine open sets.

- (1) $D(x) \cap D(s_0)$: Here s_0 is described by $a_0x^n + a_1y^n$ in the ring $R[x^{-1}]$ and s_1 is given by $b_0x^n + b_1y^n$ in the ring $R[x^{-1}]$. Hence on $D(s_0)$ the corresponding ring map $k[y_1] \rightarrow R[x^{-1}, (a_0x^n + a_1y^n)^{-1}]$ is given by $y_1 \mapsto \frac{b_0x^n + b_1y^n}{a_0x^n + a_1y^n}$.
- (2) $D(x) \cap D(s_1)$: Here s_0 is described by $a_0x^n + a_1y^n$ in the ring $R[x^{-1}]$ and s_1 is given by $b_0x^n + b_1y^n$ in the ring $R[x^{-1}]$. Hence on $D(s_1)$ the corresponding ring map $k[y_0] \rightarrow R[x^{-1}, (b_0x^n + b_1y^n)^{-1}]$ is given by $y_0 \mapsto \frac{a_0x^n + a_1y^n}{b_0x^n + b_1y^n}$.
- (3) $D(w) \cap D(s_0)$: Here s_0 is described by $a_0z^n + a_1w^n$ in the ring $R[w^{-1}]$ and s_1 is given by $b_0z^n + b_1w^n$ in the ring $R[w^{-1}]$. Hence on $D(s_0)$ the corresponding ring map $k[y_1] \rightarrow R[w^{-1}, (a_0z^n + a_1w^n)^{-1}]$ is given by $y_1 \mapsto \frac{b_0z^n + b_1w^n}{a_0z^n + a_1w^n}$.
- (4) $D(w) \cap D(s_1)$: Here s_0 is described by $a_0z^n + a_1w^n$ in the ring $R[w^{-1}]$ and s_1 is given by $b_0z^n + b_1w^n$ in the ring $R[w^{-1}]$. Hence on $D(s_1)$ the corresponding ring map $k[y_0] \rightarrow R[w^{-1}, (b_0z^n + b_1w^n)^{-1}]$ is given by $y_0 \mapsto \frac{a_0z^n + a_1w^n}{b_0z^n + b_1w^n}$.

This information can be consolidated into the two maps $D(x) \rightarrow \mathbb{P}^1$ and $D(w) \rightarrow \mathbb{P}^1$ given in terms of the pair of sections $[a_0x^n + a_1y^n, b_0x^n + b_1y^n]$ and $[a_0z^n + a_1w^n, b_0z^n + b_1w^n]$ respectively. Written in this form, we see that a map $\mathcal{J} \rightarrow \mathbb{P}^1$ given by the invertible sheaf \mathcal{P}_n with two generating sections s_0, s_1 should be interpreted as giving a map to \mathbb{P}^1 on the open sets $D(x)$ and $D(w)$ according to the first component of the sections s_0, s_1 on $D(x)$ and according to the second component of the sections s_0, s_1 on $D(w)$.

Remark 28. Recall that \mathcal{J} is pointed at $\mathbf{j} = (x - 1, y, z, w)$ and \mathbb{P}^1 is pointed at $\infty = [1 : 0]$. A map $f: \mathcal{J} \rightarrow \mathbb{P}^1$ is pointed if $f(\mathbf{j}) = \infty$. If $f = [s_0, s_1]$ with line bundle \mathcal{L} and generating sections s_0, s_1 , pointedness can be verified by checking that the stalk $s_1(\mathbf{j})$ satisfies $s_1(\mathbf{j}) = 0$ in the local ring $\mathcal{L}_{\mathbf{j}}$. For us, it suffices to work on $D(x)$ where our line bundles are trivial, and verify that modulo \mathbf{j} the section s_1 vanishes.

We give a concrete criterion for checking pointedness of a map $f: \mathcal{J} \rightarrow \mathbb{P}^1$ with line bundle \mathcal{P}_n . The case of \mathcal{Q}_n is similar.

Proposition 29. *A map $[s_0, s_1]: \mathcal{J} \rightarrow \mathbb{P}^1$ with invertible sheaf \mathcal{P}_n and generating sections*

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \quad s_1 = b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}$$

is pointed if and only if $b_0 \in \mathbf{j}$, i.e., $b_0(\mathbf{j}) = 0$.

Proof. First, assume the map $[s_0, s_1]$ is pointed. Construction 27 gives a description of the map in local coordinates. Note that for \mathbf{j} to map to $\infty \in U_0$, it is necessary that $\mathbf{j} \in D(s_0)$. Since $\mathbf{j} \in D(x) \cap D(s_0)$, the map in local coordinates is obtained by taking Spec of the ring map $g: k[y_1] \rightarrow R[x^{-1}, (a_0x^n + a_1y^n)^{-1}]$ which is given by $g(y_1) = \frac{b_0x^n + b_1y^n}{a_0x^n + a_1y^n}$. The condition for pointedness is then that the preimage of \mathbf{j} under g is the maximal ideal (y_1) . This is equivalent to the condition that y_1 maps into the ideal $(x - 1, y, z, w) \subseteq R[x^{-1}, (a_0x^n + a_1y^n)^{-1}]$. By the definition of g , the requirement is that $\frac{b_0x^n + b_1y^n}{a_0x^n + a_1y^n} \in (x - 1, y, z, w)$, which is equivalent to $b_0x^n + b_1y^n \in (x - 1, y, z, w)$. Since $y \in (x - 1, y, z, w)$ and x is invertible, this condition is met when $b_0 \in \mathbf{j}$. Thus when $[s_0, s_1]$ is pointed, $\mathbf{j} \in D(s_0)$ and $b_0 \in \mathbf{j}$.

Now assume that $b_0 \in \mathbf{j}$. This implies $\mathbf{j} \in D(s_0)$, since the sections s_0, s_1 generate $(\mathcal{P}_n)_{\mathbf{j}}$, and $b_0 \in \mathbf{j}$ implies $s_1(\mathbf{j}) = 0$. Here we can use the same construction above, since $\mathbf{j} \in D(x) \cap D(s_0)$. The algebra above shows that when $b_0 \in \mathbf{j}$ the preimage of \mathbf{j} under s is (y_1) , i.e., the map $[s_0, s_1]$ is pointed. \square

Proposition 30. *Let $f = [s_0, s_1]: \mathcal{J} \rightarrow \mathbb{P}^1$ be a pointed map with invertible sheaf \mathcal{P}_n . If $r = s_0(\mathbf{j})$, then $[\frac{s_0}{r}, \frac{s_1}{r}]: \mathcal{J} \rightarrow \mathbb{P}^1$ is a pointed map that is equal to f . Thus any pointed map with line bundle \mathcal{P}_n may be represented by a pair of generating global sections $[s_0, s_1]$ where $s_0(\mathbf{j}) = 1$ and $s_1(\mathbf{j}) = 0$.*

Proof. Proposition 29 has established that $s_1(\mathbf{j}) = 0$ and $s_0(\mathbf{j}) = r$ is a unit. We verify that the maps $[s_0, s_1]$ and $[\frac{s_0}{r}, \frac{s_1}{r}]$ are equal in local coordinates by Construction 27, where the constants $\frac{1}{r}$ cancel out in every local coordinate chart. \square

Proposition 31. *Let s_0 and s_1 be the following sections in \mathcal{P}_n*

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \quad s_1 = b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}.$$

The sections s_0, s_1 generate \mathcal{P}_n if and only if there exist $U_x, V_x, U_w, V_w \in R$ such that

$$U_x(x^n a_0 + y^n a_1) + V_x(x^n b_0 + y^n b_1) + U_w(z^n a_0 + w^n a_1) + V_w(z^n b_0 + w^n b_1) = 1.$$

Employing similar notation, sections s_0, s_1 generate \mathcal{Q}_n if and only if there exist $U_x, V_x, U_w, V_w \in R$ such that

$$U_x(x^n a_0 + z^n a_1) + V_x(x^n b_0 + z^n b_1) + U_w(y^n a_0 + w^n a_1) + V_w(y^n b_0 + w^n b_1) = 1.$$

Proof. By Lemma 14, it suffices to prove this for \mathcal{P}_n . Assume s_0, s_1 generate \mathcal{P}_n . Then there exist U, V such that $U s_0 + V s_1 = \begin{bmatrix} x^n \\ z^n \end{bmatrix}$. The first component of this identity gives

$$(U(x^n a_0 + y^n a_1) + V(x^n b_0 + y^n b_1)) = x^n.$$

Similarly, there exist U', V' such that $U' s_0 + V' s_1 = \begin{bmatrix} y^n \\ w^n \end{bmatrix}$. This gives the relation

$$(U'(z^n a_0 + w^n a_1) + V'(z^n b_0 + w^n b_1)) = w^n.$$

We need to show that the ideal (x^n, w^n) is the unit ideal. However, the equation $1 = (x + w)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^{2n-i} w^i$ demonstrates that 1 can be expressed as a linear combination of x^n and w^n over R , since each summand $\binom{2n}{i} x^{2n-i} w^i$ can be written as $r_i x^n$ with $r_i = \binom{2n}{i} x^{n-i} w^i \in R$ or $r'_i w^n$ with $r'_i = \binom{2n}{n+i} x^{n-i} w^i \in R$ for $i = 0, 1, \dots, n$. Thus, (x^n, w^n) is the unit ideal. Now we assume that there exist elements $U_x, V_x, U_w, V_w \in R$ such that

$$U_x(x^n a_0 + y^n a_1) + V_x(x^n b_0 + y^n b_1) + U_w(z^n a_0 + w^n a_1) + V_w(z^n b_0 + w^n b_1) = 1.$$

A straight forward computation yields $(U_x x^n + U_w z^n) s_0 + (V_x x^n + V_w z^n) s_1 = \begin{bmatrix} x^n \\ z^n \end{bmatrix}$, and $(U_x y^n + U_w w^n) s_0 + (V_x y^n + V_w w^n) s_1 = \begin{bmatrix} y^n \\ w^n \end{bmatrix}$. These two elements generate \mathcal{P}_n , thus $[s_0, s_1]$ do as well. \square

For brevity, we write maps $\mathcal{J} \rightarrow \mathbb{P}^1$ of nonzero degree using the following notation.

Definition 32. Let n be a positive integer. Let $(a_0, a_1 : b_0, b_1)_n$ denote the map $\mathcal{J} \rightarrow \mathbb{P}^1$ with invertible sheaf \mathcal{P}_n and generating sections

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \quad s_1 = b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix} \quad \text{with } a_0, a_1, b_0, b_1 \in R.$$

Similarly, let $(a_0, a_1 : b_0, b_1)_{-n}$ denote the map $\mathcal{J} \rightarrow \mathbb{P}^1$ with invertible sheaf \mathcal{Q}_n and generating sections

$$s_0 = a_0 \begin{bmatrix} x^n \\ y^n \end{bmatrix} + a_1 \begin{bmatrix} z^n \\ w^n \end{bmatrix}, \quad s_1 = b_0 \begin{bmatrix} x^n \\ y^n \end{bmatrix} + b_1 \begin{bmatrix} z^n \\ w^n \end{bmatrix} \quad \text{with } a_0, a_1, b_0, b_1 \in R.$$

2.5. Detecting morphisms $\mathcal{J} \rightarrow \mathbb{P}^1$ via resultants

For later purposes, we extend the definition of the resultant to homogeneous polynomials in two variables. We collect some further facts about resultants in Appendix B. The main goal of this subsection is to prove Proposition 35. Motivated by the observation of Remark 23 make the following definition.

Definition 33. Let $R[\alpha, \beta]$ denote the polynomial ring over R in variables α and β , and let $R[\alpha, \beta]_{(n)}$ denote the R -submodule of homogeneous polynomials of degree n . For every $n \geq 1$, the map $s: R[\alpha, \beta]_{(n)} \rightarrow \mathcal{P}_n$, defined by $s(\alpha^i \beta^{n-i}) = \begin{bmatrix} x^i y^{n-i} \\ z^i w^{n-i} \end{bmatrix}$ for all $0 \leq i \leq n$ is a surjective morphism of R -modules.

Definition 34. The resultant of a pair of homogeneous polynomials

$$(F_0, F_1) = \left(\sum_{i=0}^n a_i \alpha^i \beta^{n-i}, \sum_{i=0}^n b_i \alpha^i \beta^{n-i} \right) \in (R[\alpha, \beta]_{(n)})^2$$

is defined to be the resultant of the associated univariate polynomials $(\mathcal{F}_0, \mathcal{F}_1) = (\sum_{i=0}^n a_i X^i, \sum_{i=0}^n b_i X^i)$ in the indeterminate X . That is,

$$\text{res}(F_0, F_1) := \text{res}(\mathcal{F}_0, \mathcal{F}_1) = \text{res} \left(\sum_{i=0}^n a_i X^i, \sum_{i=0}^n b_i X^i \right) = \det(\text{Syl}(\mathcal{F}_0, \mathcal{F}_1)),$$

where $\text{Syl}(\mathcal{F}_0, \mathcal{F}_1)$ is the Sylvester matrix of the pair of polynomials $(\mathcal{F}_0, \mathcal{F}_1)$ in $R[X]$. See Definition 132 for a definition of the Sylvester matrix.

Proposition 35. Consider a pair (F_0, F_1) of homogeneous polynomials of degree $n \geq 1$ in $R[\alpha, \beta]_{(n)}$. If $\text{res}(F_0, F_1)$ is a unit, then the pair of sections $(s(F_0), s(F_1))$ generates \mathcal{P}_n and defines a morphism $[s(F_0), s(F_1)]: \mathcal{J} \rightarrow \mathbb{P}^1$.

Proof. Consider $(F_0, F_1) = (\sum_{i=0}^n a_i x^i y^{n-i}, \sum_{i=0}^n b_i x^i y^{n-i})$ with unit resultant. It suffices to show that $(s_0, s_1) = (s(F_0), s(F_1))$ generate \mathcal{P}_n on the open patches $D(x)$ and $D(w)$. On $D(x)$, this requires showing that the ideal $(\sum_{i=0}^n a_i x^i y^{n-i}, \sum_{i=0}^n b_i x^i y^{n-i})$ is the unit ideal in $R[x^{-1}]$. The ideal is the same as the ideal $(\sum_{i=0}^n a_i (\frac{y}{x})^{n-i}, \sum_{i=0}^n b_i (\frac{y}{x})^{n-i})$ which corresponds to a pair of polynomials of degree n in the variable $\frac{y}{x}$. By Lemma 134, this pair of polynomials has unit resultant. Since the resultant is a unit, there exists $U_x, V_x \in R[x^{-1}]$ by Lemma 133 giving a Bézout relation $U_x \sum_{i=0}^n a_i (\frac{y}{x})^{n-i} + V_x \sum_{i=0}^n b_i (\frac{y}{x})^{n-i} = 1$. On $D(w)$ we need to prove that the ideal $(\sum_{i=0}^n a_i z^i w^{n-i}, \sum_{i=0}^n b_i z^i w^{n-i})$ is the unit ideal in $R[w^{-1}]$. The ideal is equal to

$(\sum_{i=0}^n a_i (\frac{z}{w})^i, \sum_{i=0}^n b_i (\frac{z}{w})^i)$. This pair of polynomials has unit resultant by assumption. By Lemma 133, unit resultant implies existence of a Bézout relation $U_w \sum_{i=0}^n a_i (\frac{z}{w})^i + V_w \sum_{i=0}^n b_i (\frac{z}{w})^i = 1$ in $R[w^{-1}]$. This proves that $[s_0, s_1]$ defines a morphism $\mathcal{J} \rightarrow \mathbb{P}^1$. \square

Remark 36. Let $[\sigma_0, \sigma_1]: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a pointed map given by invertible sheaf $\mathcal{O}(n)$, and sections $\sigma_0 = \sum_{i=0}^n a_i x_0^i x_1^{n-i}$ and $\sigma_1 = \sum_{i=0}^n b_i x_0^i x_1^{n-i}$. Then the pair of homogeneous polynomials $F_0 = \sum_{i=0}^n a_i \alpha^{n-i} \beta^i$ and $F_1 = \sum_{i=0}^n b_i \alpha^{n-i} \beta^i$ in $R[\alpha, \beta]_{(n)}$ has unit resultant. By Proposition 35, the pair $(s(F_0), s(F_1))$ defines a morphism $[s(F_0), s(F_1)]: \mathcal{J} \rightarrow \mathbb{P}^1$. This morphism is equal to the morphism $[s_0, s_1]$, constructed from $[\sigma_0, \sigma_1]$ in Proposition 24.

Remark 37. We note that there exist pairs of polynomials $(F_0, F_1), (F'_0, F'_1)$ such that $(s(F_0), s(F_1)) = (s(F'_0), s(F'_1))$, while $\text{res}(F_0, F_1) \neq \text{res}(F'_0, F'_1)$. An example is given by $(x\alpha + z\beta, \beta)$ and (α, β) . We calculate

$$(s(x\alpha + z\beta), s(\beta)) = \left(x \begin{bmatrix} x \\ z \end{bmatrix} + z \begin{bmatrix} y \\ w \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \right) = \left(\begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \right) = (s(\alpha), s(\beta)).$$

The resultants are

$$\text{res}(x\alpha + z\beta, \beta) = x \neq 1 = \text{res}(\alpha, \beta).$$

2.6. The pointed naive homotopy relation

Naive homotopy theory for schemes is a generalization of the homotopy theory of rings in classical algebra, see [15] for a definition. Naive homotopy classes of maps between schemes do not generally have the good properties one expects from a homotopy theory. In our case, however, the work of Asok, Hoyois, and Wendt in [6] shows that naive homotopy classes behave sufficiently well. We denote by Sm_k the category of smooth finite type k -schemes. We denote the set of morphisms between objects $X, Y \in \text{Sm}_k$ by $\text{Sm}_k(X, Y)$. If X and Y are pointed, we denote by $\text{Sm}_k(X, Y)_*$ the set of *pointed* morphisms in Sm_k .

Definition 38. Let X and Y be smooth schemes finite type k -schemes. For $a \in k$, let $i_a = \text{id}_X \times a$ be the map obtained by taking the Cartesian product of id_X and the inclusion map $a: \text{Spec } k \rightarrow \mathbb{A}^1$ given by the ring map $k[t] \rightarrow k$ sending t to a . An elementary homotopy between two morphisms $f: X \rightarrow Y$ and $g: X \rightarrow Y$ is given by a morphism $H(T): X \times \mathbb{A}^1 \rightarrow Y$ satisfying $H(0) = f$ and $H(1) = g$, i.e., $H(0) = H(T) \circ i_0$ and $H(1) = H(T) \circ i_1$. We say that f and g are elementarily homotopic and write $f \sim g$.

The relation of morphisms being elementarily homotopic is symmetric and reflexive, but not transitive. To obtain an equivalence relation on the set of morphisms $\text{Sm}_k(X, Y)$, we take the transitive closure of \sim . That is, we define two morphisms $f, g \in \text{Sm}_k(X, Y)$ to be naively homotopic if there is a finite sequence of elementary homotopies $H_i(T): X \times \mathbb{A}^1 \rightarrow Y$, for $0 \leq i \leq n$ with $H_0(0) = f$, $H_n(1) = g$, and for all $0 \leq i < n$ $H_i(1) =$

$H_{i-1}(0)$. We write $f \simeq g$ in this case. The relation \simeq is now an equivalence relation on $\text{Sm}_k(X, Y)$, so we can study the set of naive homotopy classes of morphisms from X to Y .

For our constructions, we will work with pointed maps and pointed naive homotopies. We define the latter next.

Definition 39. If X and Y are smooth k -schemes, pointed at k -points x and y respectively, we say that an elementary homotopy $H(T): X \times \mathbb{A}^1 \rightarrow Y$ is pointed if the generic point of $\{x\} \times \mathbb{A}^1$ maps to y . Said another way, the points x and y correspond to morphisms $x: \text{Spec } k \rightarrow X$ and $y: \text{Spec } k \rightarrow Y$, and we require that $H(T) \circ (x \times \text{id}_{\mathbb{A}^1}) = y \circ p_1$ where $p_1: \text{Spec } k \times \mathbb{A}^1 \rightarrow \text{Spec } k$ is the projection onto the first factor.

As in the unpointed case, the relation on the set of pointed morphisms $\text{Sm}_k(X, Y)_*$ given by pointed elementary homotopies is not an equivalence relation. We say that pointed morphisms $f, g \in \text{Sm}_k(X, Y)_*$ are naively homotopic, and write $f \simeq g$, if there is a chain of pointed elementary homotopies from f to g . The naive homotopy relation is an equivalence relation on pointed morphisms. We write $[X, Y]^N = \text{Sm}_k(X, Y)_*/ \simeq$ for the set of equivalence classes.

Remark 40. For us, the most important case is when $X = \mathcal{J} = \text{Spec } R$ with basepoint $\mathbf{j} = (x - 1, y, z, w)$. This ideal extends to $\mathbf{j}' = (x - 1, y, z, w) \subseteq R[T]$. The condition that a homotopy $H(T): \mathcal{J} \times \mathbb{A}^1 \rightarrow Y$ be pointed is simply that $H(T)(\mathbf{j}') = y$, where y is the basepoint of Y .

We now formulate a criterion which will help us to construct homotopies of the form $\mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$. Let $p_1: \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathcal{J}$ denote the projection to the first factor. Similar to Definition 33, we will use the following notation. Let $(R[T])[\alpha, \beta]$ be the polynomial ring over $R[T]$ in variables α and β , and let $(R[T])[\alpha, \beta]_{(n)}$ denote the $R[T]$ -submodule of homogeneous polynomials of degree n . For every $n \geq 1$, we consider the map $s: (R[T])[\alpha, \beta]_{(n)} \rightarrow p_1^* \mathcal{P}_n$, defined by $s(\alpha^i \beta^{n-i}) = \begin{bmatrix} x^i y^{n-i} \\ z^i w^{n-i} \end{bmatrix}$ for all $0 \leq i \leq n$.

Proposition 41. Let $F_0, F_1 \in (R[T])[\alpha, \beta]_{(n)}$ be a pair of homogeneous polynomials of degree $n \geq 1$ over the ring $R[T]$. If $\text{res}(F_0, F_1)$ is a unit, then the pair of sections $(s(F_0), s(F_1))$ generates the line bundle $p_1^* \mathcal{P}_n$ and defines a morphism $[s(F_0), s(F_1)]: \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$.

Proof. The proof is analogous to the proof of Proposition 35 after replacing the ring R with $R[T]$. \square

2.7. Morphisms $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$

We write $\text{deg}: [\mathcal{J}, \mathbb{P}^1]^N \rightarrow \text{Pic}(\mathcal{J}) \cong \mathbb{Z}$ for the map that sends a map f to $f^* \mathcal{O}(1)$. Our choices thus far set $\text{deg}(\pi) = 1$ and $\text{deg}(\tilde{\pi}) = -1$. Write $[\mathcal{J}, \mathbb{P}^1]^N_n$ for the set

of naive homotopy classes of maps $\mathcal{J} \rightarrow \mathbb{P}^1$ with degree n . Our goal for this section is to describe the maps $\mathcal{J} \rightarrow \mathbb{P}^1$ of degree 0. We consider the scheme $\mathbb{A}^2 \setminus \{0\} = \text{Spec}(k[t_0, t_1]) \setminus \{(t_0, t_1)\}$ to be pointed at $(t_0 - 1, t_1)$ and write $\eta: \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ for the Hopf map given by the trivial algebraic line bundle $\mathcal{O}_{\mathbb{A}^2 \setminus \{0\}}$ with the choice of sections $\eta_0 = t_0, \eta_1 = t_1$. A scheme morphism $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ is given by a morphism $\mathcal{J} \rightarrow \mathbb{A}^2$ that does not have $\{0\}$ in the image. Thus, a morphism $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ is given by a pair $(s_0, s_1) \in R^2$ such that the ideal (s_0, s_1) generates R , i.e., there are $U, V \in R$ for which $s_0U + s_1V = 1$. In other words, a morphism $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ is given by unimodular row (s_0, s_1) in R^2 .

Proposition 42. *Consider a map $f: \mathcal{J} \rightarrow \mathbb{P}^1$. Then we have $\text{deg}(f) = 0$ if and only if f factors through the Hopf map $\eta: \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$.*

Proof. First, since $\text{Pic}(\mathbb{A}^2 \setminus \{0\}) = 0$, it follows that any map that factors as $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\} \xrightarrow{\eta} \mathbb{P}^1$ has degree 0. Second, assume that the morphism $f: \mathcal{J} \rightarrow \mathbb{P}^1$ satisfies $\text{deg}(f) = 0$, i.e., $f^*\mathcal{O}(1) = \mathcal{O}_R$. We recall from Construction 27 that then f is given by global sections $s_0, s_1 \in R = \Gamma(\mathcal{J}, \mathcal{O}_R)$ that generate \mathcal{O}_R , i.e., the ideal (s_0, s_1) generates R . As we explained above, this shows that (s_0, s_1) determines a morphism $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$. Since η is given by the trivial bundle, the composition $\mathcal{J} \xrightarrow{(s_0, s_1)} \mathbb{A}^2 \setminus \{0\} \xrightarrow{\eta} \mathbb{P}^1$ corresponds to the trivial algebraic line bundle \mathcal{O}_R on \mathcal{J} with global sections s_0, s_1 . Thus, the composition $(s_0, s_1) \circ \eta$ equals f which finishes the proof. \square

Corollary 43. *Let $f: \mathcal{J} \rightarrow \mathbb{P}^1$ be a pointed map of degree 0. Then there exists a unique pointed map $f': \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ such that $f = f' \circ \eta$.*

Proof. Let $(s_0, s_1): \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ be a factorization of f through the Hopf map. Note that $r = s_0(\mathbf{j})$ need not be 1, although r is a unit of k . Instead, the map $f' = (\frac{1}{r}s_0, \frac{1}{r}s_1)$ is pointed and satisfies $f = f' \circ \eta$.

To show uniqueness, let $(s'_0, s'_1): \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ be another pointed map that factors f through η . That is, we assume $[s_0, s_1] = [s'_0, s'_1]$. Note that in this case, $D(s_0) = D(s'_0) = D(f^*x_0)$ and $D(s_1) = D(s'_1) = D(f^*x_1)$. Working locally in $D(s_1) = D(s'_1)$, we have $s_0/s_1 = s'_0/s'_1$ in $R[s_1^{-1}]$ by construction. We may write $s'_0 = c_0s_0$ for $c_0 = s'_1/s_1 \in R[s_1^{-1}]$. Similarly, in $D(s_0) = D(s'_0)$, we obtain $s'_1 = c_1s_1$ for $c_1 = s'_0/s_0$. In the intersection $D(s_0) \cap D(s_1)$ we have $s'_0/s'_1 = s_0/s_1$, which implies $s'_0/s_0 = s'_1/s_1$. This is exactly the equation $c_0 = c_1$. The elements $c_1 \in R[s_0^{-1}]$ and $c_0 \in R[s_1^{-1}]$ therefore glue together to an element $c \in R$. Hence c satisfies $s'_0 = cs_0$ and $s'_1 = cs_1$. Observe that $c(s_0u' + s_1v') = 1$, that is, $c \in R^\times = k^\times$. The pointedness assumption forces $c(\mathbf{j}) = 1$, hence, $c = 1$ with which we conclude $(s_0, s_1) = (s'_0, s'_1)$. \square

Remark 44. The previous proposition says, in other words, that a map $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ is equivalent to a unimodular row (A, B) of length two in R . Furthermore, a pointed map

$\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ is equivalent to a unimodular row (A, B) of length two in R that also satisfies $A(\mathbf{j}) = 1$ and $B(\mathbf{j}) = 0$.

Pointed elementary homotopies between maps of degree 0 can also be lifted to a pointed elementary homotopy of maps $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$.

Proposition 45. *Let $H(T) = [s_0(T), s_1(T)]: \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$ be a pointed elementary homotopy between maps $H(0)$ and $H(1)$ which have degree 0. There is a pointed elementary homotopy $H'(T): \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{A}^2 \setminus \{0\}$ between the lifts $H'(0)$ and $H'(1)$.*

Proof. Since $H(0)$ and $H(1)$ have degree 0, the homotopy $H(T)$ is degree 0 too, that is, the line bundle it determines is the trivial one $\mathcal{O}_{\mathcal{J} \times \mathbb{A}^1}$. We can use the two generating global sections $s_0(T), s_1(T) \in R[T]$ to build a map $(s_0(T), s_1(T)): \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{A}^2 \setminus \{0\}$. Note that since $s_0(T)$ and $s_1(T)$ generate $R[T]$, there are $u(T), v(T) \in R[T]$ for which $s_0(T)u(T) + s_1(T)v(T) = 1$. Since $H(T)$ is pointed, $s_1(T)(\mathbf{j}') = 0$ in $R[T]/\mathbf{j}'$. This implies that $s_0(T)(\mathbf{j}')u(T)(\mathbf{j}') = 1$ in $R[T]/\mathbf{j}'$. The ring $R[T]/\mathbf{j}'$ is easily seen to be isomorphic to $k[T]$. Hence $r = s_0(T)(\mathbf{j}')$ is a unit of $k[T]$, and the units of $k[T]$ are exactly the units of k . Thi shows that the map $(\frac{1}{r}s_0(T), \frac{1}{r}s_1(T)) : \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{A}^2 \setminus \{0\}$ is a pointed homotopy between $H'(0)$ and $H'(1)$. \square

Let SL_2 denote the affine scheme $\mathrm{Spec}(k[a, b, c, d]/(ad - bc - 1))$ pointed at the ideal $(a - 1, b, c, d - 1)$. Intuitively, this is the scheme of (2×2) -matrices with determinant 1, pointed at the identity matrix. Let (A, B) be a unimodular row in R . That is, there exist $U, V \in R$ for which $AU + BV = 1$. Thus the data of a map $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ can be used to produce a matrix $\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in \mathrm{SL}_2(R)$, in other words, a map $\mathcal{J} \rightarrow \mathrm{SL}_2$.

Lemma 46. *A pointed map $(A, B): \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ can be lifted to a pointed map $\begin{pmatrix} A & -V \\ B & U \end{pmatrix} : \mathcal{J} \rightarrow \mathrm{SL}_2$.*

Proof. Let $\begin{pmatrix} A & -V_1 \\ B & U_1 \end{pmatrix}$ be an arbitrary lift of (A, B) . Note that $A(\mathbf{j}) = 1, B(\mathbf{j}) = 0$, and $U_1(\mathbf{j}) = 1$, but $V_1(\mathbf{j}) = v$ for some $v \in k$. For any $d \in R$, we can construct a different lift by setting $U_2 = U_1 + Bd$ and $V_2 = V_1 - Ad$. Set $d = v$. Then $U_2(\mathbf{j}) = 1$, and $V_2(\mathbf{j}) = 0$, so $\begin{pmatrix} A & -V_2 \\ B & U_2 \end{pmatrix}$ is pointed. \square

Remark 47. The pointed lift of Lemma 46 is not unique in general. For example, the unimodular row $(1, 0)$ lifts to the pointed maps

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

However, we now show that any two pointed lifts of a pointed unimodular row (A, B) are naively homotopic.

Lemma 48. *Let $(A, B): \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ be a pointed map. Any two lifts of (A, B) to a pointed map $\mathcal{J} \rightarrow \text{SL}_2$ are naively homotopic.*

Proof. Let

$$\tilde{f}_i = \begin{pmatrix} A & -V_i \\ B & U_i \end{pmatrix}$$

for $i \in \{1, 2\}$ be two pointed lifts of (A, B) . A pointed elementary homotopy between \tilde{f}_1 and \tilde{f}_2 is given by

$$\tilde{f}_t = \begin{pmatrix} A & -(TV_1 + (1 - T)V_2) \\ B & TU_1 + (1 - T)U_2 \end{pmatrix}$$

which proves the claim. \square

Proposition 49. *Every pointed elementary homotopy*

$$H(T) = (s_0(T), s_1(T)): \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{A}^2 \setminus \{0\}$$

can be lifted to a pointed elementary homotopy

$$\begin{pmatrix} s_0(T) & -V(T) \\ s_1(T) & U(T) \end{pmatrix}: \mathcal{J} \times \mathbb{A}^1 \rightarrow \text{SL}_2.$$

Proof. Recall $\mathbf{j}' = (x - 1, y, z, w)$ must map to the basepoint for the homotopy $H(T)$ to be pointed. The sections $s_0(T)$ and $s_1(T)$ generate the unit ideal, hence there exist $u(T)$ and $v(T)$ in $R[T]$ for which $s_0(T)u(T) + s_1(T)v(T) = 1$. The pointedness assumption gives the relation among ideals $(s_0(T) - 1, s_1(T)) \subseteq \mathbf{j}' \subseteq R[T]$.

With these data, we construct the matrix

$$\begin{pmatrix} s_0(T) & -v(T) \\ s_1(T) & u(T) \end{pmatrix} \in \text{SL}_2(R[T]).$$

This matrix determines a map $\mathcal{J} \times \mathbb{A}^1 \rightarrow \text{SL}_2$ that lifts the unimodular row $(s_0(T), s_1(T))$ we started with. This homotopy need not be a pointed homotopy. However, for any choice of $d(T) \in R[T]$, the matrix

$$\begin{pmatrix} s_0(T) & -v(T) + s_0(T)d(T) \\ s_1(T) & u(T) + s_1(T)d(T) \end{pmatrix}$$

is also a lift of $(s_0(T), s_1(T))$. We will now show that, for $d(T) = v(T)$, the map this matrix determines is a pointed homotopy. Write $u_2(T) = u(T) + s_1(T)v(T)$ and

$v_2(T) = v(T) - s_0(T)v(T)$. Our assumption that $(s_0(T), s_1(T))$ is pointed gives us $(s_0(T) - 1, s_1(T)) \subseteq \mathbf{j}' \subseteq R[T]$. We must show that $(v_2(T), u_2(T) - 1) \subseteq \mathbf{j}'$ too. Since $(s_0(T) - 1) \in \mathbf{j}'$, we have $v_2(T) = -v(T)(s_0(T) - 1) \in \mathbf{j}'$. Observe that $u_2(T) - 1 \in \mathbf{j}'$ if $u(T) - 1 \in \mathbf{j}'$ since $s_1(T) \in \mathbf{j}'$. Since $s_0(T)u(T) + s_1(T)v(T) = 1$, it follows that $s_0(T)u(T) - 1 \in \mathbf{j}'$. This can be rewritten as $s_0(T)u(T) - 1 = (s_0(T) - 1)u(T) + u(T) - 1$. Since $s_0(T) - 1 \in \mathbf{j}'$ it follows that $u(T) - 1 \in \mathbf{j}'$ too. \square

Definition 50. Let $\phi: \text{SL}_2 \rightarrow \mathbb{A}^2 \setminus \{0\}$ be the morphism determined by the ring map $f: k[t_0, t_1] \rightarrow k[a, b, c, d]/(ad - bc - 1)$ given by $f(t_0) = a$ and $f(t_1) = c$. Intuitively, this is the morphism that extracts the first column from a matrix in SL_2 . As given, this map has codomain \mathbb{A}^2 , but it is clear from the relation $ad - bc = 1$ that ϕ maps into $\mathbb{A}^2 \setminus \{0\}$.

Proposition 51. *The maps $\phi: \text{SL}_2 \rightarrow \mathbb{A}^2 \setminus \{0\}$ and $\eta: \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ induce bijections of naive homotopy classes of pointed maps*

$$[\mathcal{J}, \text{SL}_2]^N \xrightarrow{\phi_*} [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \xrightarrow{\eta_*} [\mathcal{J}, \mathbb{P}^1_0]^N.$$

Proof. The map ϕ_* is surjective by Lemma 46, and ϕ_* is injective by Lemma 48. Corollary 43 shows that η_* is bijective on the level of pointed morphisms. This shows that η_* is surjective. To show that η_* is injective, it suffices to show that a pointed elementary homotopy $H(T): \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$ between degree 0 maps lifts to a pointed elementary homotopy $H'(T): \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{A}^2 \setminus \{0\}$, which is done in Proposition 45. \square

3. Operations on naive homotopy classes of morphisms

3.1. Group structure on maps of degree 0

We may now define a binary operation on naive homotopy classes of morphisms $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$. This is analogous to Cazanave’s naive sum of pointed rational functions. The group structure is obtained by lifting maps $f, g: \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ to $\tilde{f}, \tilde{g}: \mathcal{J} \rightarrow \text{SL}_2$, multiplying the two resulting maps using the group structure on SL_2 , then mapping back down to $\mathbb{A}^2 \setminus \{0\}$ via the map $\phi: \text{SL}_2 \rightarrow \mathbb{A}^2 \setminus \{0\}$.

A morphism $f: \mathcal{J} \rightarrow \text{SL}_2$ is equivalent to the data of a matrix $M \in \text{SL}_2(R)$. A matrix $M \in \text{SL}_2(R)$ corresponds to a pointed morphism if upon evaluation at \mathbf{j} , the resulting matrix is the identity matrix. The set of pointed maps corresponds to a subgroup of $\text{SL}_2(R)$. The operation of matrix multiplication respects the naive homotopy relation for pointed maps and therefore defines a group operation on $[\mathcal{J}, \text{SL}_2]^N$, the set of pointed naive homotopy classes of morphisms. It suffices to prove the following proposition, given that the naive homotopy relation is the transitive closure of pointed elementary homotopies.

Proposition 52. *Let $M(T) \in \text{SL}_2(R[T])$ be a pointed elementary homotopy between the matrices $M_0 = M(0) \in \text{SL}_2(R)$ and $M_1 = M(1) \in \text{SL}_2(R)$ corresponding to pointed*

morphisms. Let $M'(T) \in \text{SL}_2(R[T])$ be another elementary homotopy where similar notation is employed. The pointed morphisms corresponding to $M_0 \cdot M'_0$ and $M_1 \cdot M'_1$ are elementarily homotopic.

Proof. All that needs to be verified is that the morphism corresponding to the matrix product $M(T) \cdot M'(T)$ is pointed. Since both $M(T)$ and $M'(T)$ are pointed, evaluation at \mathbf{j}' gives the identity matrix in $\text{SL}_2(k[T])$. It's clear then that the product $M(T) \cdot M'(T)$ will evaluate to the identity matrix at \mathbf{j}' too. \square

Definition 53. Consider two pointed naive homotopy classes $[(A_i, B_i): \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}]$ for $i = 1, 2$ represented by the unimodular rows $(A_i, B_i) \in R^2$. Pick completions of the unimodular rows to matrices corresponding to pointed maps, as guaranteed by Lemma 46:

$$\begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix}, \begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix} \in \text{SL}_2(R).$$

We define $[(A_1, B_1)] \oplus [(A_2, B_2)]$ to be the naive homotopy class $[(A_3, B_3)]$ where (A_3, B_3) is the unimodular row obtained from the matrix product

$$\begin{pmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{pmatrix} = \begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix}.$$

Proposition 54. The operation \oplus of Definition 53 is well-defined and gives the set $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ the structure of a group.

Proof. We first show that the operation does not depend on the particular completion to a matrix in $\text{SL}_2(R)$. Let M_1 and M'_1 be two pointed completions of (A_1, B_1) , and similarly let M_2 and M'_2 be two pointed completions of (A_2, B_2) . There are two representatives for the product $[(A_1, B_1)] \oplus [(A_2, B_2)]$ from these choices. They are (A_3, B_3) , taken from the first column of $M_1 \cdot M_2$ and (A'_3, B'_3) , the first column of $M'_1 \cdot M'_2$. Any two completions of a unimodular row to a matrix in $\text{SL}_2(R)$ are homotopic by Lemma 48, hence $[M_1] = [M'_1]$ and $[M_2] = [M'_2]$ in $[\mathcal{J}, \text{SL}_2]^N$. By Proposition 52, the products $M_1 \cdot M_2$ and $M'_1 \cdot M'_2$ are homotopic as maps $\mathcal{J} \rightarrow \text{SL}_2$. Extracting the first column of this homotopy gives a homotopy between the resulting unimodular rows defining the resulting map.

We now show that the operation does not depend on the representative of the naive homotopy class chosen. Let $(A_1(T), B_1(T))$ and $(A_2(T), B_2(T))$ be pointed elementary homotopies. These can be completed to matrices $M_1(T) \in \text{SL}_2(R[T])$ and $M_2(T) \in \text{SL}_2(R[T])$ by Proposition 49. The first column of the product $M_1(T) \cdot M_2(T)$ provides the homotopy between the two possible representations of the product. We conclude that the operation is well-defined on the set $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ of pointed naive homotopy classes.

The identity for the operation is given by the unimodular row $(1, 0): \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$. Associativity of \oplus follows from the associativity of matrix multiplication. Finally, let $(A, B): \mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ be given by the unimodular row $(A, B) \in R^2$ and complete it to a

matrix $\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in \mathrm{SL}_2(R)$ giving a pointed map. The inverse of this matrix in $\mathrm{SL}_2(R)$ is the matrix $\begin{pmatrix} U & V \\ -B & A \end{pmatrix}$, and the first column of this matrix represents the inverse of (A, B) in $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ for \oplus . That is, $-[(A, B)] = [(U, -B)]$. \square

Lemma 55. *The map $\phi: \mathrm{SL}_2 \rightarrow \mathbb{A}^2 \setminus \{0\}$ induces an isomorphism of groups*

$$\phi_*: [\mathcal{J}, \mathrm{SL}_2]^N \xrightarrow{\cong} [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N.$$

Proof. The map ϕ_* is a group homomorphism by our definition of \oplus on $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ in terms of matrix multiplication. We have shown in Proposition 51 that ϕ_* is bijective, hence the result. \square

For the next result, we recall that the cogroup structure of $\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$ in the pointed \mathbb{A}^1 -homotopy category endows $[\mathbb{P}^1, X]^{\mathbb{A}^1}$ with a group operation for any motivic space X . It is a simple exercise to produce an \mathbb{A}^1 -weak equivalence $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ using the standard covering of \mathbb{P}^1 by two affine lines with intersection \mathbb{G}_m . The simplicial circle S^1 (or some suitable homotopy equivalent model of it, like $\partial\Delta^2$) admits the structure of an h-cogroup, or just a cogroup in the homotopy category. Explicitly, the pointed simplicial set $\partial\Delta^2 \simeq S^1$ admits two maps: a pinch map $p: S^1 \rightarrow S^1 \vee S^1$ and an inverse map $S^1 \rightarrow S^1$. These operations fit into homotopy commutative diagrams that give the expected algebraic properties, like associativity and the definition of the inverse [24, Chapter 2]. These two observations together allow us to define a group operation on $[S^1 \wedge \mathbb{G}_m, \mathbb{P}^1]^{\mathbb{A}^1}$ as follows. Given two maps $f, g: S^1 \wedge \mathbb{G}_m \rightarrow \mathbb{P}^1$ in the \mathbb{A}^1 -homotopy category, the composition below represents the sum $f \oplus^{\mathbb{A}^1} g$ of the maps f and g .

$$S^1 \wedge \mathbb{G}_m \xrightarrow{p \wedge 1} (S^1 \vee S^1) \wedge \mathbb{G}_m \xrightarrow{\cong} (S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m) \xrightarrow{f \vee g} \mathbb{P}^1 \quad (56)$$

Note that the morphism $f \vee g$ exists by the universal property of wedge sums. One must take the time to verify that the operation defined above does indeed make $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ into a group, but the pleasant properties of the \mathbb{A}^1 -homotopy category make this doable. We refer to the operation $\oplus^{\mathbb{A}^1}$ as the *conventional group structure*.

Definition 57. Let $\xi_0: [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \rightarrow [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ denote the composition of the natural map $\nu_0: [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \rightarrow [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ and the bijection $(\pi_{\mathbb{A}^1}^*)^{-1}: [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} \rightarrow [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ that is given by the inverse of the bijection $\pi_{\mathbb{A}^1}^*$. We note that ν_0 is a bijection by Proposition 128 since $\mathbb{A}^2 \setminus \{0\}$ is \mathbb{A}^1 -naive.

Theorem 58. *The map ξ_0 is an isomorphism of groups between the group $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ with operation \oplus and the group $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ with the conventional group operation.*

Proof. Let $\xi'_0: [\mathcal{J}, \mathrm{SL}_2]^N \rightarrow [\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}$ denote the composition of the canonical map $[\mathcal{J}, \mathrm{SL}_2]^N \rightarrow [\mathcal{J}, \mathrm{SL}_2]^{\mathbb{A}^1}$ and the bijection $[\mathcal{J}, \mathrm{SL}_2]^{\mathbb{A}^1} \rightarrow [\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}$ which is given by

the inverse of the bijection $\pi_{\mathbb{A}^1}^*$. Since both groups $[\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}$ and $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ inherit their operation from the cogroup structure of \mathbb{P}^1 , the \mathbb{A}^1 -weak equivalence $\mathrm{SL}_2 \xrightarrow{\phi} \mathbb{A}^2 \setminus \{0\}$ induces an isomorphism of groups $\phi_*: [\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1} \rightarrow [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$. By Lemma 55 the map $\phi_*: [\mathcal{J}, \mathrm{SL}_2]^N \rightarrow [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ is a group isomorphism. We then have the following commutative diagram.

$$\begin{array}{ccc}
 [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N & \xrightarrow{\xi_0} & [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} \\
 \phi_* \uparrow \cong & & \cong \uparrow \phi_* \\
 [\mathcal{J}, \mathrm{SL}_2]^N & \xrightarrow{\xi'_0} & [\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}
 \end{array}$$

Hence, in order to establish that ξ_0 is a group isomorphism, it suffices to show that ξ'_0 is a group isomorphism. Since \mathcal{J} is affine and SL_2 is \mathbb{A}^1 -naive by [6, Theorem 4.2.1], we know that ξ'_0 is a bijection by Proposition 128. Hence it suffices to show that ξ'_0 is a group homomorphism.

Again, because SL_2 is \mathbb{A}^1 -naive, the canonical map $[\mathcal{J}, \mathrm{SL}_2]^N \rightarrow [\mathcal{J}, \mathrm{SL}_2]^{\mathbb{A}^1}$ is a bijection by Proposition 128. This bijection is a group isomorphism because the operation on both sets is defined using the same construction, that is, the sum of two maps is given by

$$\mathcal{J} \xrightarrow{\Delta} \mathcal{J} \times \mathcal{J} \xrightarrow{f \times g} \mathrm{SL}_2 \times \mathrm{SL}_2 \xrightarrow{m} \mathrm{SL}_2,$$

where $m: \mathrm{SL}_2 \times \mathrm{SL}_2 \rightarrow \mathrm{SL}_2$ is the multiplication on SL_2 . In other words, the group structure is induced by the group object structure on SL_2 .

Similarly, the set $[\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}$ also obtains the structure of a group using that SL_2 is a group object in the pointed \mathbb{A}^1 -homotopy category. The Eckmann–Hilton argument given in [24, Proposition 2.25] can be applied in this scenario to show that this group structure coincides with the conventional group structure, see also [2, Proposition 2.2.12]. Hence we may assume that the group operation on $[\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}$ is induced by the group object structure on SL_2 . Combining these observations shows that the composition

$$[\mathcal{J}, \mathrm{SL}_2]^N \rightarrow [\mathcal{J}, \mathrm{SL}_2]^{\mathbb{A}^1} \rightarrow [\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}$$

is a group homomorphism. This is the map ξ'_0 which proves the assertion. \square

Corollary 59. *The group $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ is abelian.*

Proof. Since $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ is isomorphic to $[\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}$, the Eckmann–Hilton argument shows that this group is abelian. \square

Remark 60. Morel shows in [22, §7.3] that the group $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ is isomorphic to $K_1^{\mathrm{MW}}(k)$, the first Milnor–Witt K -theory group of the field k (see Definition 118 below).

In short, the computation [22, Theorem 7.13] and the \mathbb{A}^1 -weak equivalence between SL_2 and $\mathbb{A}^2 \setminus \{0\}$ gives $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\}) \cong \underline{\mathbf{K}}_2^{\mathrm{MW}}$. The contraction of this sheaf evaluated at $\mathrm{Spec} k$ then computes $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$:

$$[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} \cong \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\})_{-1}(\mathrm{Spec} k) \cong (\underline{\mathbf{K}}_2^{\mathrm{MW}})_{-1}(\mathrm{Spec} k) \cong K_1^{\mathrm{MW}}(k).$$

Hence our results show that there is an isomorphism $[\mathcal{J}, \mathrm{SL}_2]^{\mathbb{N}} \cong K_1^{\mathrm{MW}}(k)$. We make this isomorphism explicit in Section 6.3.

Remark 61. For any two pointed matrices $M, M' \in \mathrm{SL}_2(R)$, which represent pointed morphisms $\mathcal{J} \rightarrow \mathrm{SL}_2$, there is a chain of elementary homotopies connecting $M \cdot M'$ and $M' \cdot M$. We do not know of a general algorithm to construct this chain of homotopies explicitly.

The following explicit naive homotopies will be used in the later sections.

Lemma 62. Consider a matrix $M = \begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in \mathrm{SL}_2(R)$. Then M and $(M^{-1})^T$ are naively homotopic. Thus, the unimodular rows (A, B) and (U, V) are naively homotopic.

Proof. Consider the matrix $H = \begin{pmatrix} 1 - T^2 & -T \\ T(2 - T^2) & 1 - T^2 \end{pmatrix} \in \mathrm{SL}_2(R[T])$. The matrix H defines an unpointed homotopy from the identity matrix to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is straightforward to verify that the product HMH^{-1} is a pointed homotopy between M and $(M^{-1})^T$ as claimed. \square

Lemma 63. Consider a matrix $\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in \mathrm{SL}_2(R)$ and let $u \in k^\times$. Then there is an elementary homotopy

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \simeq \begin{pmatrix} A & -\frac{1}{u^2}V \\ u^2B & U \end{pmatrix}. \tag{64}$$

Thus, the unimodular row (A, B) is naively homotopic to the unimodular row (A, u^2B) .

Proof. The matrix on the right-hand side of Equation (64) can be written as the following product

$$\begin{pmatrix} A & -\frac{1}{u^2}V \\ u^2B & U \end{pmatrix} = \begin{pmatrix} \frac{1}{u} & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} A & -V \\ B & U \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}.$$

The diagonal matrices can be decomposed to a product of elementary matrices, which are all homotopic to the identity. \square

3.2. Action of degree 0 maps on degree n maps

Recall that we write $[\mathcal{J}, \mathbb{P}^1]_n^N$ for the set of naive homotopy classes of maps $\mathcal{J} \rightarrow \mathbb{P}^1$ with degree n . We define a group action of $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \cong [\mathcal{J}, \text{SL}_2]^N \cong [\mathcal{J}, \mathbb{P}^1]_0^N$ on $[\mathcal{J}, \mathbb{P}^1]_n^N$ for all $n \neq 0$. We start by first defining an operation on actual morphisms, and then show that the operation respects the naive homotopy equivalence relation.

Definition 65. Let $M: \mathcal{J} \rightarrow \text{SL}_2$ be a morphism with corresponding matrix

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix}$$

and consider a map $[s_0, s_1]: \mathcal{J} \rightarrow \mathbb{P}^1$ determined by $n \in \mathbb{N}$, the algebraic line bundle \mathcal{P}_n or \mathcal{Q}_n , and generating global sections s_0, s_1 .

We define $M \oplus [s_0, s_1]: \mathcal{J} \rightarrow \mathbb{P}^1$ to be the morphism determined by the same algebraic line bundle with the generating global sections $M \oplus [s_0, s_1] = [As_0 - Vs_1, Bs_0 + Us_1]$ which are obtained from the following matrix multiplication

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} = \begin{pmatrix} As_0 - Vs_1 \\ Bs_0 + Us_1 \end{pmatrix}.$$

Proposition 66. Given a map $[s_0, s_1]: \mathcal{J} \rightarrow \mathbb{P}^1$ with algebraic line bundle \mathcal{L} (either \mathcal{P}_n or \mathcal{Q}_n) and a map $M: \mathcal{J} \rightarrow \text{SL}_2$, the construction $M \oplus [s_0, s_1]$ is a morphism from \mathcal{J} to \mathbb{P}^1 . If both maps are pointed, the result is also pointed. Furthermore, the operation of Definition 65 defines a left group action of $[\mathcal{J}, \text{SL}_2]^N$ on the set $\text{Sm}_k(\mathcal{J}, \mathbb{P}^1)$.

Proof. The morphism $M: \mathcal{J} \rightarrow \text{SL}_2$ is described by a matrix

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in \text{SL}_2(R).$$

We observe that $U(As_0 - Vs_1) + V(Bs_0 + Us_1) = s_0$ and $-B(As_0 - Vs_1) + A(Bs_0 + Us_1) = s_1$. By assumption, the sections s_0, s_1 generate the algebraic line bundle \mathcal{L} . Hence the pair of sections $As_0 - Vs_1, Bs_0 + Us_1$ generate \mathcal{L} as well. This proves the first assertion.

That the map $[s_0, s_1]$ is pointed means that $s_1 \in \mathfrak{j} \subseteq R$, or equivalently, $s_1(\mathfrak{j}) = 0$ in R/\mathfrak{j} . That M is pointed means $M(\mathfrak{j})$ is the identity matrix. To verify $M \oplus [s_0, s_1]$ is pointed, we must check that $B(\mathfrak{j})s_0(\mathfrak{j}) + U(\mathfrak{j})s_1(\mathfrak{j}) = 0$, but this is clear as $B(\mathfrak{j}) = 0$ and $s_1(\mathfrak{j}) = 0$ from our assumptions.

The fact that the operation is a left group action follows from the associativity of matrix multiplication and the definition of the group structure on maps $\mathcal{J} \rightarrow \text{SL}_2$. \square

The next theorem employs the notation of Definition 32 for morphisms $\mathcal{J} \rightarrow \mathbb{P}^1$.

Theorem 67. Let $f: \mathcal{J} \rightarrow \mathbb{P}^1$ be a map of degree n . Then there exists a matrix $M \in \text{SL}_2(R)$ such that $f = M \oplus (1, 0 : 0, 1)_n$.

Proof. We prove the assertion for $n > 0$. The proof for $n < 0$ is similar. Let $f = (a_0, a_1 : b_0, b_1)_n$ with the notation introduced in Definition 32. For $c, c', d, d' \in R$, consider the matrix

$$M = \begin{pmatrix} a_0 + y^n c + w^n c' & a_1 - x^n c - z^n c' \\ b_0 - y^n d - w^n d' & b_1 + x^n d + z^n d' \end{pmatrix}. \tag{68}$$

The matrix M can be written as the sum

$$M = \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} + \begin{pmatrix} y^n c + w^n c' & -x^n c - z^n c' \\ -y^n d - w^n d' & x^n d + z^n d' \end{pmatrix}.$$

By definition of \oplus and the notation in Definition 32 we compute

$$\begin{aligned} & \begin{pmatrix} y^n c + w^n c' & -x^n c - z^n c' \\ -y^n d - w^n d' & x^n d + z^n d' \end{pmatrix} \oplus (1, 0 : 0, 1)_n \\ &= \begin{pmatrix} y^n c + w^n c' & -x^n c - z^n c' \\ -y^n d - w^n d' & x^n d + z^n d' \end{pmatrix} \begin{pmatrix} \begin{bmatrix} x^n \\ z^n \end{bmatrix} \\ \begin{bmatrix} y^n \\ w^n \end{bmatrix} \end{pmatrix} \\ &= \begin{bmatrix} (y^n c + w^n c') \begin{bmatrix} x^n \\ z^n \end{bmatrix} + (-x^n c - z^n c') \begin{bmatrix} y^n \\ w^n \end{bmatrix} \\ (-y^n d - w^n d') \begin{bmatrix} x^n \\ z^n \end{bmatrix} + (x^n d + z^n d') \begin{bmatrix} y^n \\ w^n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

The last equality follows from the relations $y^n \begin{bmatrix} x^n \\ z^n \end{bmatrix} = x^n \begin{bmatrix} y^n \\ w^n \end{bmatrix}$ and $w^n \begin{bmatrix} x^n \\ z^n \end{bmatrix} = z^n \begin{bmatrix} y^n \\ w^n \end{bmatrix}$. Hence, for any choice of $c, c', d, d' \in R$, we have

$$\begin{aligned} M \oplus (1, 0 : 0, 1)_n &= M \oplus \left[\begin{bmatrix} x^n \\ z^n \end{bmatrix}, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right] = \left[a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right] \\ &= (a_0, a_1 : b_0, b_1)_n. \end{aligned}$$

We now show that there always exist c, c', d, d' such that $M \in SL_2(R)$. The determinant of M is given by the formula

$$\det(M) = a_0 b_1 - a_1 b_0 + c(x^n b_0 + y^n b_1) + c'(z^n b_0 + w^n b_1) + d(x^n a_0 + y^n a_1) + d'(z^n a_0 + w^n a_1).$$

Since $(a_0, a_1 : b_0, b_1)_n$ determines a morphism of schemes, it follows from Proposition 31 that the ideal $I := (x^n a_0 + y^n a_1, z^n a_0 + w^n a_1, x^n b_0 + y^n b_1, z^n b_0 + w^n b_1)$ is the unit ideal. Thus $1 - a_0 b_1 - a_1 b_0$ is in I , and there exist elements c, c', d, d' such that $\det(M) = 1$. This shows there exists $M \in SL_2(R)$ satisfying $f = M \oplus (1, 0 : 0, 1)_n$ as desired. \square

Theorem 67 implies that the action \oplus is transitive. More concretely, the theorem has the following consequence:

Corollary 69. *Let $f, g: \mathcal{J} \rightarrow \mathbb{P}^1$ be two morphisms of degree n . Then there exists a matrix $M \in \text{SL}_2(R)$ such that $M \oplus f = g$.*

Proof. By Theorem 67, there exist M' and M'' such that $M' \oplus (1, 0 : 0, 1)_n = f$ and $M'' \oplus (1, 0 : 0, 1)_n = g$. The desired matrix is given by $M = M'' \cdot (M')^{-1}$. \square

Remark 70. The $\text{SL}_2(R)$ -matrix M constructed in the proof of Theorem 67 is not always pointed, even if the map f we started with is pointed. For example, following the construction for the map $f = (1, 1 : 0, 1)_1$ yields the matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which is not pointed, since $M(\mathbf{j})$ is not the identity matrix.

Remark 70 shows that we have to improve our argument in order to get an action on pointed homotopy classes. We will now prove the necessary adjustments.

Proposition 71. *Let $f: \mathcal{J} \rightarrow \mathbb{P}^1$ be a pointed map of degree $n \neq 0$. Then there is a pointed naive homotopy between f and a map of the form $M \oplus (1, 0 : 0, 1)_n$ for some pointed matrix $M \in \text{SL}_2(R)$.*

Proof. Let $f = (a_0, a_1 : b_0, b_1)_n$, where we may assume $a_0(\mathbf{j}) = 1$ by Proposition 30. By Theorem 67 we can find a matrix $M' \in \text{SL}_2(R)$ such that $M' \oplus (1, 0 : 0, 1)_n = f$. However, M' may not be pointed. We can replace M' with a pointed map M as follows. Assuming M' is of the form given in Equation (68) we get $b_1(\mathbf{j}) + d(\mathbf{j}) = 1$. Moreover, this implies that there is an element $e \in k$ such that $M'(\mathbf{j}) = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ and $e = a_1(\mathbf{j}) - c(\mathbf{j})$. Define M to be $M = \begin{pmatrix} 1 & -e \\ 0 & 1 \end{pmatrix} M'$. We compute $M \oplus (1, 0 : 0, 1)_n = (a_0 - eb_0, a_1 - eb_1 : b_0, b_1)_n$. The assertion now follows from the fact that the morphism $(a_0 - Teb_0, a_1 - Teb_1 : b_0, b_1)_n$ is a pointed homotopy between $M \oplus (1, 0 : 0, 1)_n$ and f , where T denotes the parameter for the homotopy. \square

Corollary 72. *Let $f, g: \mathcal{J} \rightarrow \mathbb{P}^1$ be two pointed morphisms of degree n . There exists a pointed map $M: \mathcal{J} \rightarrow \text{SL}_2$ such that $M \oplus f$ is pointed naively homotopic to g .*

Since the line bundle corresponding to the morphisms $M \oplus f$ and f are the same by definition of \oplus , it is clear that \oplus preserves degrees of morphisms. Hence we make the following definition.

Definition 73. Let $[(A, B)] \in [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N \cong [\mathcal{J}, \mathbb{P}^1]_0^N$ be a pointed naive homotopy class represented by the map with unimodular row (A, B) in R . Let $[f] \in [\mathcal{J}, \mathbb{P}^1]_n^N$ be a pointed naive homotopy class of degree n with $n \neq 0$ represented by a pointed morphism

$f: \mathcal{J} \rightarrow \mathbb{P}^1$. We define $[(A, B)] \oplus [f] := [M \oplus f]$ where M is a completion of (A, B) to a matrix in $\text{SL}_2(R)$ corresponding to a pointed map.

It remains to show that \oplus respects naive homotopy classes.

Theorem 74. *The operation of Definition 73 is well-defined and for each $n \in \mathbb{Z}$ provides the set $[\mathcal{J}, \mathbb{P}^1]_n^N$ with a left-action by the group $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$.*

Proof. First, consider a pointed map $f = [s_0, s_1]$. We show that $[(A, B)] \oplus [f]$ is independent of the choice of completion of (A, B) to a matrix in $\text{SL}_2(R)$. So let

$$M = \begin{pmatrix} A & -V \\ B & U \end{pmatrix}, \quad M' = \begin{pmatrix} A & -V' \\ B & U' \end{pmatrix}$$

be two completions to matrices in $\text{SL}_2(R)$ which correspond to pointed maps. Then the naive homotopy $H(T) = \begin{pmatrix} A & -(TV + (1 - T)V') \\ B & TU + (1 - T)U' \end{pmatrix}$ is pointed independently of T , and $H(T) \oplus f$ is a pointed homotopy between $M \oplus f$ and $M' \oplus f$.

Now we show that $[(A, B)] \oplus [f]$ is independent of the choice of the representing unimodular row (A, B) . Suppose we have a pointed elementary homotopy $(A(T), B(T))$ between two unimodular rows. Proposition 49 shows that we can lift it to a pointed elementary homotopy $M(T) = \begin{pmatrix} A(T) & -V(T) \\ B(T) & U(T) \end{pmatrix} \in \text{SL}_2(R[T])$. Then $M(T) \oplus f$ is a pointed homotopy between $(A, B) \oplus f$ and $(A', B') \oplus f$. Now we consider a unimodular row (A, B) and let $M = \begin{pmatrix} A & -V \\ B & U \end{pmatrix}$ be a lift to a matrix in $\text{SL}_2(R)$. Let $f_0, f_1: \mathcal{J} \rightarrow \mathbb{P}^1$ be two pointed morphisms which are homotopic via a pointed naive homotopy. Let $f(T): \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$ be a pointed naive homotopy. We let \mathcal{L}' denote the line bundle $\mathcal{J} \times \mathbb{A}^1$ which corresponds to the morphism $f(T)$. We define the map $H(T) := M \oplus f(T): \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$ with the same algebraic line bundle \mathcal{L}' on $\mathcal{J} \times \mathbb{A}^1$ and global sections

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \cdot \begin{pmatrix} s_0(T) \\ s_1(T) \end{pmatrix} = \begin{pmatrix} As_0(T) - Vs_1(T) \\ Bs_0(T) + Us_1(T) \end{pmatrix}.$$

We note that $H(T)$ thus defined is in fact a morphism $\mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$, since we have $U(As_0(T) - Vs_1(T)) + V(Bs_0(T) + Us_1(T)) = s_0(T)$, and $-B(As_0(T) - Vs_1(T)) + A(Bs_0(T) + Us_1(T)) = s_1(T)$. By assumption, the sections $s_0(T), s_1(T)$ generate the line bundle \mathcal{L}' . Hence $(As_0(T) - Vs_1(T), Bs_0(T) + Us_1(T))$ generate \mathcal{L}' as well. This shows that $H(T)$ defines a morphism. We now verify that $H(T)$ is pointed by showing $Bs_0(T) + Us_1(T) \in \mathfrak{j}'$. Pointedness of $[s_0(T), s_1(T)]$ means that $s_1(T)(\mathfrak{j}') = 0$ in $R[T]/\mathfrak{j}'$. Pointedness of (A, B) means $M(\mathfrak{j})$ is the identity matrix. We calculate

$$B(\mathfrak{j}')s_0(T)(\mathfrak{j}') + U(\mathfrak{j}')s_1(T)(\mathfrak{j}') = 0 \cdot s_0(T)(\mathfrak{j}') + 1 \cdot 0 = 0$$

which completes the verification. This shows that \oplus is independent of the choice of representatives in both naive homotopy classes and completes the proof of the first assertion. The second assertion then follows from Proposition 66. \square

Remark 75. There are several variations to the operation given in Definition 73 that produce valid group actions. For $M \in \text{SL}_2(R)$, the operation in Definition 65 is given by the matrix multiplication $M \cdot (s_0 \ s_1)^T$. We could have taken equally well either $M^T \cdot (s_0 \ s_1)^T$ or $M^{-1} \cdot (s_0 \ s_1)^T$, although this would give a right-action rather than a left-action on maps. Up to homotopy, the latter two choices in fact agree, since M^T is homotopic to M^{-1} by Lemma 62. Thus there are two natural choices for this action, one of which applies the inverse operation to the morphism in $\text{SL}_2(R)$ before acting. In Appendix C we will use real realization to check which of these operations can represent the group operation on $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ induced from Morel’s group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ via $\pi_{\mathbb{A}^1}^*$. In fact, in Examples 149 and 150 we show that only the choice of Definitions 65 and 73 can be compatible.

4. The group structure on $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$

4.1. The definition of the group structure

In this section we define an explicit group structure on $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$. We will then discuss some alternative approaches and open questions.

Definition 76. Let $-[\text{id}]$ denote the additive inverse of $[\text{id}: \mathbb{P}^1 \rightarrow \mathbb{P}^1]$ under the conventional group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. Define $-\pi: \mathcal{J} \rightarrow \mathbb{P}^1$ to be a morphism which represents the \mathbb{A}^1 -homotopy class $-[\text{id}: \mathbb{P}^1 \rightarrow \mathbb{P}^1] \in [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ under the bijection $\xi: [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}} \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ of Equation (8). More generally, for any integer n , let $n\pi$ denote a morphism $n\pi: \mathcal{J} \rightarrow \mathbb{P}^1$ which represents the \mathbb{A}^1 -homotopy class $n[\text{id}: \mathbb{P}^1 \rightarrow \mathbb{P}^1]$ under the bijection $\xi: [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}} \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$.

We are now ready to define a group operation on $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$.

Definition 77. Let $f: \mathcal{J} \rightarrow \mathbb{P}^1$ and $g: \mathcal{J} \rightarrow \mathbb{P}^1$ be morphisms of degrees n and m respectively. By Corollary 72 there are degree 0 maps $f_0: \mathcal{J} \rightarrow \mathbb{P}^1$ and $g_0: \mathcal{J} \rightarrow \mathbb{P}^1$ for which $f \simeq f_0 \oplus n\pi$ and $g \simeq g_0 \oplus m\pi$. We define the sum of $[f]$ and $[g]$ to be

$$\begin{aligned} [f] \oplus [g] &= ([f_0] \oplus [n\pi]) \oplus ([g_0] \oplus [m\pi]) \\ &= ([f_0] \oplus [g_0]) \oplus [(n + m)\pi]. \end{aligned}$$

The term $[f_0] \oplus [g_0]$ is calculated by matrix multiplication via Definition 53. The group action of Definition 73 is used to compute $(f_0 \oplus g_0) \oplus (n + m)\pi$.

Remark 78. It follows from Theorem 74 that the operation \oplus of Definition 77 is well-defined. We also note that, for $n > 0$, the proofs of Theorem 67 and Proposition 71 may be used to write down a concrete algorithm to find a map f_0 such that $f \simeq f_0 \oplus n\pi$ for any degree n map f .

Remark 79. For $n > 0$, we may construct morphisms $n\pi$ by using Cazanave’s group operation on morphisms $[\mathbb{P}^1, \mathbb{P}^1]^N$ and lift it to an element in $[\mathcal{J}, \mathbb{P}^1]^N$. A recursive description of the maps $n\pi$ for $n > 0$ can be given as follows: Set $G_0 = 1$ and $G_1 = \begin{bmatrix} x \\ z \end{bmatrix}$. For $n > 0$, we define G_{n+1} recursively by setting

$$G_{n+1} = \begin{bmatrix} x \\ z \end{bmatrix} \cdot G_n - \begin{bmatrix} y^2 \\ w^2 \end{bmatrix} \cdot G_{n-1}$$

where we recall that multiplication of sections is induced by component-wise multiplication in R . For $n > 0$, the morphism $n\pi$ is given by sections

$$\begin{bmatrix} G_n, \begin{bmatrix} y \\ w \end{bmatrix} \cdot G_{n-1} \end{bmatrix}.$$

We note that $[(1, 0 : 0, 1)_n]$ is in general not equal to $[n\pi]$ for $n > 1$. We will explain this observation in Remark 104 using Morel’s motivic Brouwer degree and the work of Kass and Wickelgren.

We are now ready to prove the following important result.

Theorem 80. *The operation \oplus turns the set $[\mathcal{J}, \mathbb{P}^1]^N$ into an abelian group. Moreover, there is an isomorphism of groups $\varphi: ([\mathcal{J}, \mathbb{P}^1]^N, \oplus) \xrightarrow{\cong} ([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1})$.*

Proof. We observe that the set $\{[n\pi] : n \in \mathbb{Z}\}$ inherits the structure of an abelian group from \mathbb{Z} . In Definition 77 we construct the group $([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$ as the direct product of the two groups $\{[n\pi] : n \in \mathbb{Z}\}$ and $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$. Both are abelian by Corollary 59. This implies the first assertion.

By definition of the operation \oplus , the group $([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$ fits into the short exact sequence displayed in the top row of Diagram (81) below. By the work of Morel in [22, §7.3], the group $([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1})$ fits into the short exact sequence displayed in the bottom row.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N & \longrightarrow & [\mathcal{J}, \mathbb{P}^1]^N & \xrightarrow{\text{deg}} & \mathbb{Z} \longrightarrow 1 \\
 & & \xi_0 \downarrow \cong & & \varphi \downarrow & & \downarrow \cong \\
 1 & \longrightarrow & [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} & \longrightarrow & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} & \xrightarrow{\text{deg}} & \mathbb{Z} \longrightarrow 1
 \end{array} \tag{81}$$

By Theorem 58, the vertical map ξ_0 on the left-hand side is an isomorphism. The vertical map q on the right-hand side is an isomorphism as well. We define φ to be the unique group homomorphism satisfying $\varphi([\pi]) = [\text{id}]$ and $\varphi([f_0]) = \xi_0([f_0])$ for all $[f_0] \in [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$. The diagram commutes by our definition of φ . Since ξ_0 and the right-hand vertical map in Diagram (81) are isomorphisms, φ is an isomorphism of groups as well by the five-lemma. \square

4.2. Open questions and potential alternative approaches

Recall the map $\xi: [\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ which is the composite of the canonical map $\nu: [\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ and the inverse of the induced map $\pi_{\mathbb{A}^1}^*: [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$. Unfortunately, Theorem 80 does not imply that the bijection ξ is a group isomorphism. However, since ξ restricts to an isomorphism on the subgroups $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ and $\{[n\pi] \mid n \in \mathbb{Z}\}$, we do believe that the bijection ξ is a group isomorphism, which we state as a conjecture below.

Conjecture 82. *The bijection $\xi: [\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group isomorphism and equals φ .*

One obstacle to prove Conjecture 82 is that, for $n < 0$, we do not know which morphism $\mathcal{J} \rightarrow \mathbb{P}^1$ is sent to $n[\text{id}]$ under ξ . In particular, we do not know which morphism $\mathcal{J} \rightarrow \mathbb{P}^1$ is mapped to the motivic homotopy class $-\text{id}: \mathbb{P}^1 \rightarrow \mathbb{P}^1$. A potential candidate for $-\pi$ may be the map $\tilde{\pi} = (1, 0 : 0, -1)_{-1}$ determined by the line bundle \mathcal{Q}_1 and generating sections

$$s_0 = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } s_1 = - \begin{pmatrix} z \\ w \end{pmatrix}.$$

Question 83. Is $\tilde{\pi}$ the inverse of π for \oplus , i.e., is $\tilde{\pi}$ naively homotopic to $-\pi$?

In Appendix C we present further evidence for Conjecture 82. We use the real realization functor for fields $k \subset \mathbb{R}$ and Morel’s theorem which states that the signature of the motivic Brouwer degree equals the topological Brouwer degree under real realization. This provides a potential obstruction to the compatibility of $\oplus^{\mathbb{A}^1}$ and the action of $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ on $[\mathcal{J}, \mathbb{P}^1]^N$. We then compute concrete examples and show that other choices for the action of $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ on $[\mathcal{J}, \mathbb{P}^1]^N$ are not compatible with $\oplus^{\mathbb{A}^1}$, while our choice of operation in Definition 73 is compatible with $\oplus^{\mathbb{A}^1}$ after real realization in the chosen examples.

In Proposition 103 we show that the naive homotopy class of π is mapped to the class $(\langle 1 \rangle, 1)$ in $\text{GW}(k) \times_{k \times /_k \times 2} k^\times$ as expected if ξ is a group homomorphism. Based on the computations in Appendix C we prove in Theorem 105 that the image of $[\tilde{\pi}]$ under the motivic Brouwer degree is the class $-\langle 1 \rangle$ in $\text{GW}(k)$. This brings us very close to a

positive answer to Question 83. We are, however, not able to compute the resultant, i.e., the image of $\tilde{\pi}$ in k^\times .

We end this section with comments on potential alternative approaches:

Remark 84. Since π is an \mathbb{A}^1 -weak equivalence, it induces a bijection $\pi_*: [\mathcal{J}, \mathcal{J}]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^N$. Hence there is a bijection between $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ and the set of pointed naive homotopy classes $[\mathcal{J}, \mathcal{J}]^N$. In [11, page 31] Cazanave speculates whether $[\mathcal{J}, \mathcal{J}]^N$ can be used to study the group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. A morphism $\mathcal{J} \rightarrow \mathcal{J}$ corresponds to a ring homomorphism $R \rightarrow R$, or equivalently, the data of a (2×2) -matrix with entries in R and with trace 1 and determinant 0. For every map $f: \mathcal{J} \rightarrow \mathbb{P}^1$ we can find a map $F: \mathcal{J} \rightarrow \mathcal{J}$ such that $f = \pi \circ F$. We will refer to such a map F as a lift of f . There is a particularly nice way to construct a lift in the case $f: \mathcal{J} \rightarrow \mathbb{P}^1$ has degree 0. Assume that f is given by a unimodular row (A, B) . Let $U, V \in R$ be such that $\begin{pmatrix} A & -V \\ B & U \end{pmatrix}$ has determinant 1. Then F is given by the matrix $\begin{pmatrix} AU & BU \\ AV & BV \end{pmatrix}$ which has trace 1 and determinant 0. Composing the map with π yields the $\mathcal{J} \rightarrow \mathbb{P}^1$ map given by either $[AU : BU]$ or $[AV : BV]$, whenever they are defined, which coincides with the map corresponding to the unimodular row (A, B) . If f has non-zero degree, there is also a concrete procedure to find a lift of f , which we leave to the reader.

Since morphisms $\mathcal{J} \rightarrow \mathcal{J}$ can be represented by matrices, it may seem plausible that one can find a suitable operation on $[\mathcal{J}, \mathcal{J}]^N$ which may help to describe the group $([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1})$. However, neither addition nor multiplication of matrices equip the set $[\mathcal{J}, \mathcal{J}]^N$ with an operation which is compatible with the conventional group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. We have verified in examples that composition of maps in $[\mathcal{J}, \mathcal{J}]^N$ descends to the operation \circ on $[\mathbb{P}^1, \mathbb{P}^1]^N$ of [12, Definition 4.5]. As pointed out in [12, Remark 4.7] the latter does not distribute over the conventional group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. We were not able to make a reasonable guess which other operation on $[\mathcal{J}, \mathcal{J}]^N$ might work. We have therefore not pursued this path further.

Remark 85. An alternative approach to construct a group structure on $[\mathcal{J}, \mathcal{J}]^N$ may be the following. One can hope to construct a cogroup structure on \mathcal{J} . However, this is not so easy, even though Asok and Fasel have done much of the work to make this possible. In [3], Asok and Fasel give an explicit construction of a smooth scheme $\widetilde{\mathcal{J} \vee \mathcal{J}}$ that is \mathbb{A}^1 -weak equivalent to the wedge sum $\mathcal{J} \vee \mathcal{J}$. We have constructed an explicit map $\mathcal{J} \rightarrow \widetilde{\mathcal{J} \vee \mathcal{J}}$ that conjecturally represents the pinch map $\mathbb{P}^1 \rightarrow \mathbb{P}^1 \vee \mathbb{P}^1$. We also have a candidate for a map representing the inverse map $\mathcal{J} \rightarrow \mathbb{P}^1$, but unfortunately both of these claims have proven too difficult to verify. We therefore decided not to include this construction in this paper.

5. Compatibility with Cazanave’s monoid structure

The goal of this section is to show that the map

$$\pi_N^* : ([\mathbb{P}^1, \mathbb{P}^1]^N, \oplus^N) \rightarrow ([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$$

is a morphism of monoids, where \oplus^N denotes the monoid operation defined by Cazanave in [12]. We will achieve this goal in Theorem 96.

5.1. Compatibility with certain degree 0 maps

We first study an important family of degree 0 morphisms and their compatibility with \oplus , \oplus^N , and π_N^* .

Definition 86. For $u, v \in k^\times$, we write $g_{u,v}$ for the pointed morphism $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$ given by the unimodular row $(x + \frac{v}{u}w, (u - v)y)$ in R . This unimodular row can be completed to the $SL_2(R)$ -matrix

$$m_{u,v} = \begin{pmatrix} x + \frac{v}{u}w & \frac{u-v}{uv}z \\ (u - v)y & x + \frac{u}{v}w \end{pmatrix}.$$

We now prove some basic properties of the maps $g_{u,v}$ which will be necessary to show that π_N^* is a monoid morphism.

Lemma 87. For all $u, v, s \in k^\times$, we have the identity $g_{u,v} \oplus g_{v,s} = g_{u,s}$. In particular, we have $g_{u,v} \oplus g_{v,u} = (1, 0)$ and $g_{u,v} \oplus g_{v,1} = g_{u,1}$.

Proof. A direct computation, using $xw = yz$, shows

$$m_{u,v} \cdot m_{v,s} = \begin{pmatrix} x^2 + \frac{uv+vs}{uv}xw + \frac{s}{u}w^2 & \frac{uv-sv}{ysv}xz + \frac{-vs+uv}{usv}zw \\ (u - s)xy + (u - s)wy & x^2 + \frac{uv+sv}{sv}xw + \frac{u}{s}w^2 \end{pmatrix},$$

and since $x + w = 1$, this simplifies to the matrix $m_{u,s}$. Then $g_{u,v} \oplus g_{v,u} = g_{u,u} = (1, 0)$ and $g_{u,v} \oplus g_{v,1} = g_{u,1}$ are special cases for respectively $s = u$ and $s = 1$. \square

Lemma 88. Let $u, v, c \in k^\times$. Then $[g_{u,v}] = [gc^2u, c^2v]$.

Proof. By Lemma 63, we have

$$g_{u,v} = \left(x + \frac{v}{u}w, (u - v)y\right) \simeq \left(x + \frac{v}{u}w, c^2(u - v)y\right) = gc^2u, c^2v. \quad \square$$

Lemma 89. Let $u, v \in k^\times$ and let v be a square. Then we have $[g_{u,1}] \oplus [g_{v,1}] = [g_{uv,1}]$.

Proof. Lemma 88 and Lemma 87 imply

$$g_{u,1} \oplus g_{v,1} \simeq g_{u,1} \oplus g_{1,1/v} = g_{u,1/v} \simeq g_{uv,1},$$

and hence the claim. \square

Now we study the relationship of the maps $g_{u,v}$ with \oplus and π_N^* . We adopt the following notation from Cazanave [12]: For $u \in k^\times$, we identify a rational function X/u in the indeterminate X with the morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined by $[x_0 : x_1] \mapsto [x_0 : ux_1]$. We then have

$$\pi = \pi_N^* \left(\frac{X}{1} \right) = \left[\begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \right] \quad \text{and} \quad \pi_N^* \left(\frac{X}{u} \right) = \left[\begin{bmatrix} x \\ z \end{bmatrix}, u \begin{bmatrix} y \\ w \end{bmatrix} \right]$$

as maps $\mathcal{J} \rightarrow \mathbb{P}^1$. Our next goal is to prove Proposition 92. To do so, we need some preparation.

Lemma 90. *For every $u \in k^\times$, we have $g_{u,1} \oplus \pi = \pi_N^* \left(\frac{X}{u} \right)$.*

Proof. A direct computation using the facts that $x \begin{bmatrix} y \\ w \end{bmatrix} = y \begin{bmatrix} x \\ z \end{bmatrix}$ and $z \begin{bmatrix} y \\ w \end{bmatrix} = w \begin{bmatrix} x \\ z \end{bmatrix}$ shows

$$\begin{aligned} g_{u,1} \oplus \pi &= \left(\begin{matrix} x + \frac{1}{u}w & \frac{u-1}{u}z \\ (u-1)y & x + uw \end{matrix} \right) \oplus \left[\begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \right] \\ &= \left[\left(x + \frac{1}{u}w \right) \begin{bmatrix} x \\ z \end{bmatrix} + \left(\frac{u-1}{u}z \right) \begin{bmatrix} y \\ w \end{bmatrix}, (u-1)y \begin{bmatrix} x \\ z \end{bmatrix} + (x + uw) \begin{bmatrix} y \\ w \end{bmatrix} \right] \\ &= \left[\begin{bmatrix} x \\ z \end{bmatrix}, u \begin{bmatrix} y \\ w \end{bmatrix} \right] \end{aligned}$$

and hence the result by definition of the maps involved. \square

Lemma 91. *For all $u, v \in k^\times$, we have the identity*

$$g_{u,v} \oplus \pi_N^* \left(\frac{X}{v} \right) = \pi_N^* \left(\frac{X}{u} \right).$$

Proof. Using Lemmas 87, 90, and Definition 77 we get

$$g_{u,v} \oplus \pi_N^* \left(\frac{X}{v} \right) = (g_{u,v} \oplus g_{v,1}) \oplus \pi = g_{u,1} \oplus \pi = \pi_N^* \left(\frac{X}{u} \right)$$

and hence the result. \square

We are now ready to prove a key result for the compatibility of π_N^* with the monoid operations.

Proposition 92. *Let $u \in k^\times$ and $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a pointed morphism. Then there is a pointed naive homotopy*

$$\pi_N^* \left(\frac{X}{u} \oplus^N f \right) \simeq g_{u,1} \oplus \left(\pi_N^* \left(\frac{X}{1} \oplus^N f \right) \right).$$

Proof. We begin by clearly describing the maps under consideration before defining our naive homotopy. A pointed morphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree n can be expressed as a rational function $\frac{A}{B} = \frac{X^n + a_{n-1}X^{n-1} + \dots + a_0}{b_{n-1}X^{n-1} + \dots + b_0}$ where $\text{res}(A, B)$ is a unit by Proposition 22. If we define

$$(F_0, F_1) = (\alpha^n + a_{n-1}\alpha^{n-1}\beta + \dots + a_0\beta^n, b_{n-1}\alpha^{n-1}\beta + \dots + b_0\beta^n) \in (R[\alpha, \beta]_{(n)})^2,$$

then $\pi_N^*(f) = [s(F_0), s(F_1)]$ with respect to the invertible sheaf \mathcal{P}_n by Proposition 35 and Remark 36. Note $\text{res}(F_0, F_1) = \text{res}(A, B)$ is a unit. Write $f_0 = s(F_0)$ and $f_1 = s(F_1)$, so that $\pi_N^*(f) = [f_0, f_1]$ to simplify the notation.

The map $\pi_N^*(\frac{X}{u} \oplus^N f)$ is described as the following matrix product by the calculation of Cazanave [12, Example 3.3] when interpreted in the notation of Remark 36:

$$\pi_N^*\left(\frac{X}{u} \oplus^N f\right) = \begin{pmatrix} \begin{bmatrix} x \\ z \\ y \\ w \end{bmatrix} & -\frac{1}{u} \begin{bmatrix} y \\ w \end{bmatrix} \\ u & 0 \end{pmatrix} \cdot \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

Note that here we use the isomorphism $\mathcal{P}_n \otimes \mathcal{P}_1 \xrightarrow{\cong} \mathcal{P}_{n+1}$ of Proposition 20 to identify the product of a pair of column vectors with its image in \mathcal{P}_{n+1} , where we recall that multiplication of sections is induced by component-wise multiplication of elements in R . The pair of generating sections that determine the morphism $g_{u,1} \oplus (\pi^*(\frac{X}{1} \oplus^N f))$ is given by the matrix product

$$g_{u,1} \oplus \left(\pi_N^*\left(\frac{X}{1} \oplus^N f\right)\right) = \begin{pmatrix} x + \frac{1}{u}w & \frac{u-1}{u}z \\ (u-1)y & x + uw \end{pmatrix} \cdot \begin{pmatrix} \begin{bmatrix} x \\ z \\ y \\ w \end{bmatrix} & -\begin{bmatrix} y \\ w \end{bmatrix} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

Note that we have the following equality of matrix products

$$\begin{aligned} \begin{pmatrix} x + \frac{1}{u}w & \frac{u-1}{u}z \\ (u-1)y & x + uw \end{pmatrix} \cdot \begin{pmatrix} \begin{bmatrix} x \\ z \\ y \\ w \end{bmatrix} & -\begin{bmatrix} y \\ w \end{bmatrix} \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} \begin{bmatrix} x \\ z \\ y \\ w \end{bmatrix} & -(x + \frac{1}{u}) \begin{bmatrix} y \\ w \end{bmatrix} \\ u & (1-u)y \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} x \\ z \\ y \\ w \end{bmatrix} & -\frac{1}{u} \begin{bmatrix} y \\ w \end{bmatrix} \\ u & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{u-1}{u}y \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This motivates the definition of the matrix $h(T)$ defined by the following product.

$$h(T) = \begin{pmatrix} \begin{bmatrix} x \\ z \\ y \\ w \end{bmatrix} & -\frac{1}{u} \begin{bmatrix} y \\ w \end{bmatrix} \\ u & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{u-1}{u}yT \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

$$= \begin{pmatrix} f_0 \begin{bmatrix} x \\ z \end{bmatrix} + \frac{u-1}{u} yTf_1 \begin{bmatrix} x \\ z \end{bmatrix} - \frac{1}{u} f_1 \begin{bmatrix} y \\ w \end{bmatrix} \\ uf_0 \begin{bmatrix} y \\ w \end{bmatrix} + (u-1)yTf_1 \begin{bmatrix} y \\ w \end{bmatrix} \end{pmatrix}$$

We claim that the rows of the matrix $h(T)$ provide generating sections of the line bundle $p_1^* \mathcal{P}_{n+1}$ on $\mathcal{J} \times \mathbb{A}^1$, where $p_1: \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathcal{J}$ is the projection to the first factor. This implies that h defines a morphism $h: \mathcal{J} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$, and that h is a pointed homotopy from $h(0) = \pi_N^* \left(\frac{X}{u} \oplus^N f \right)$ to $h(1) = g_{u,1} \oplus \left(\pi_N^* \left(\frac{X}{1} \oplus^N f \right) \right)$.

Define $H(T)$ by

$$\begin{aligned} H(T) &= \begin{pmatrix} \alpha & -\frac{1}{u}\beta \\ u\beta & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{u-1}{u}yT \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} \\ &= \begin{pmatrix} F_0\alpha + \frac{u-1}{u}yTF_1\alpha - \frac{1}{u}F_1\alpha \\ uF_0\beta + (u-1)yTF_1\beta \end{pmatrix}. \end{aligned}$$

Since $\text{res}(F_0, F_1)$ is a unit, Lemma 135 implies that the pair of polynomials $(F_0 + \frac{u-1}{u}yTF_1, F_1)$ over the ring $R[T]$ has unit resultant. Lemma 136 then implies that the resultant

$$\text{res} \left(F_0\alpha + \frac{u-1}{u}yTF_1\alpha - \frac{1}{u}F_1\alpha, uF_0\beta + (u-1)yTF_1\beta \right)$$

is a unit as well. Finally, Proposition 41 shows that h is a morphism and thus the desired pointed naive homotopy. \square

To give a concrete example of the homotopy constructed in the proof of Proposition 92, we look at the special case $f = X/1$:

Example 93. For every $u \in k^\times$, the morphism H defined by

$$\begin{aligned} H &= \left[\begin{bmatrix} x^2 \\ z^2 \end{bmatrix} + T \frac{u-1}{u} y \begin{bmatrix} xy \\ zw \end{bmatrix} - \left(x + \frac{1}{u}w \right) \begin{bmatrix} y^2 \\ w^2 \end{bmatrix}, u \begin{bmatrix} xy \\ zw \end{bmatrix} \right] \\ &\quad + \left[(T(u-1) - (u-1)y) \begin{bmatrix} y^2 \\ w^2 \end{bmatrix} \right] \end{aligned}$$

is a homotopy between $H(0) = g_{u,1} \oplus \pi_N^* \left(\frac{X}{1} \oplus^N \frac{X}{1} \right)$ and $H(1) = \pi_N^* \left(\frac{X}{u} \oplus^N \frac{X}{1} \right)$.

5.2. The map π_N^* is a monoid morphism

We will now prove that the map $\pi_N^*: ([\mathbb{P}^1, \mathbb{P}^1]^N, \oplus^N) \rightarrow ([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$ induced by π is a morphism of monoids.

Lemma 94. We have

$$\pi_N^* \left(\frac{X}{1} \oplus^N \dots \oplus^N \frac{X}{1} \right) \simeq \pi \oplus \dots \oplus \pi,$$

where there are n summands on both sides.

Proof. Since $\nu_{\mathbb{P}^1} \left(\frac{X}{1} \right) = \text{id}_{\mathbb{P}^1}$ and $\nu_{\mathbb{P}^1}$ is a morphism of monoids by [12, Proposition 3.23], we have the equality $n[\text{id}] = \left[\frac{X}{1} \oplus^N \dots \oplus^N \frac{X}{1} \right]$ in $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. Thus, $\pi_N^* \left(\frac{X}{1} \oplus^N \dots \oplus^N \frac{X}{1} \right)$ is naively homotopic to $n\pi$. By definition, $[\pi] \oplus \dots \oplus [\pi] = [n\pi]$, hence the result follows. \square

Proposition 95. For $u_1, \dots, u_n \in k^\times$ we have

$$\pi_N^* \left(\frac{X}{u_1} \oplus^N \frac{X}{u_2} \oplus^N \dots \oplus^N \frac{X}{u_n} \right) \simeq \pi_N^* \left(\frac{X}{u_1} \right) \oplus \pi_N^* \left(\frac{X}{u_2} \right) \oplus \dots \oplus \pi_N^* \left(\frac{X}{u_n} \right).$$

Proof. Since both \oplus^N and \oplus are commutative and \oplus is associative, we may apply Proposition 92 and Lemma 94 to compute

$$\begin{aligned} \pi_N^* \left(\frac{X}{u_1} \oplus^N \dots \oplus^N \frac{X}{u_n} \right) &\simeq g_{u_1,1} \oplus \pi_N^* \left(\frac{X}{1} \oplus^N \frac{X}{u_2} \oplus^N \dots \oplus^N \frac{X}{u_n} \right) \\ &\simeq g_{u_1,1} \oplus g_{u_2,1} \oplus \pi_N^* \left(\frac{X}{1} \oplus^N \frac{X}{1} \oplus^N \frac{X}{u_3} \oplus^N \dots \oplus^N \frac{X}{u_n} \right) \\ &\simeq g_{u_1,1} \oplus \dots \oplus g_{u_n,1} \oplus n\pi \\ &\simeq (g_{u_1,1} \oplus \pi) \oplus (g_{u_2,1} \oplus \pi) \oplus \dots \oplus (g_{u_n,1} \oplus \pi) \\ &\simeq \pi_N^* \left(\frac{X}{u_1} \right) \oplus \dots \oplus \pi_N^* \left(\frac{X}{u_n} \right). \end{aligned}$$

The final step follows from Lemma 90. \square

Theorem 96. The map $\pi_N^*: ([\mathbb{P}^1, \mathbb{P}^1]^N, \oplus^N) \rightarrow ([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$ induced by π is a morphism of monoids.

Proof. Let $f, g: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be two pointed morphisms. By [12, Lemma 3.13], $[\mathbb{P}^1, \mathbb{P}^1]^N$ is generated by elements in degree 1. Hence we can assume $f \simeq \frac{X}{u_1} \oplus^N \frac{X}{u_2} \oplus^N \dots \oplus^N \frac{X}{u_n}$ and $g \simeq \frac{X}{v_1} \oplus^N \frac{X}{v_2} \oplus^N \dots \oplus^N \frac{X}{v_m}$ for some $u_1, \dots, u_n, v_1, \dots, v_m \in k^\times$. Then Proposition 95 implies the identity

$$\pi_N^* ([f] \oplus^N [g]) = [\pi_N^*(f)] \oplus [\pi_N^*(g)]$$

and hence the result. \square

6. Group completion

The morphism $\pi : \mathcal{J} \rightarrow \mathbb{P}^1$ induces the following commutative diagram of solid arrows.

$$\begin{array}{ccc}
 [\mathcal{J}, \mathbb{P}^1]^N & \xrightarrow{\nu_{\mathcal{J}}} & [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1} \\
 \pi_N^* \uparrow & \swarrow \psi & \uparrow \pi_{\mathbb{A}^1}^* \downarrow (\pi_{\mathbb{A}^1}^*)^{-1} \\
 [\mathbb{P}^1, \mathbb{P}^1]^N & \xrightarrow{\nu_{\mathbb{P}^1}} & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}
 \end{array} \tag{97}$$

Recall that $\xi = (\pi_{\mathbb{A}^1}^*)^{-1} \circ \nu_{\mathcal{J}}$ denotes the bijection $\xi : [\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ of Equation (8).

Lemma 98. *We have the identity of morphisms $\xi \circ \pi_N^* = \nu_{\mathbb{P}^1}$.*

Proof. Since the outer square in Diagram (97) commutes, we have $(\pi_{\mathbb{A}^1}^*)^{-1} \circ \nu_{\mathcal{J}} \circ \pi_N^* = \nu_{\mathbb{P}^1}$. Since $\xi = (\pi_{\mathbb{A}^1}^*)^{-1} \circ \nu_{\mathcal{J}}$ by definition, this shows

$$\xi \circ \pi_N^* = \nu_{\mathbb{P}^1}$$

as desired. \square

In [12, Theorem 3.22] Cazanave proves that the canonical map $\nu_{\mathbb{P}^1} : ([\mathbb{P}^1, \mathbb{P}^1]^N, \oplus^N) \rightarrow ([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1})$ is a group completion. Hence there exists a unique group homomorphism

$$\psi : ([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}) \rightarrow ([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$$

making the lower triangle in Diagram (97) commute.

We will show in this section that π_N^* has image in a certain subgroup and induces a group completion. Together with Cazanave’s result this implies that we have a canonical isomorphism between the two group completions induced by π_N^* and $\nu_{\mathbb{P}^1}$, respectively. The main result is proven in Theorem 113.

6.1. Motivic Brouwer degree

In [21] Morel describes the analog of the topological Brouwer degree map in \mathbb{A}^1 -homotopy theory. For pointed endomorphisms of \mathbb{P}^1 it defines a homomorphism

$$\text{deg}^{\mathbb{A}^1} : [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow \text{GW}(k).$$

We recall that by the work of Cazanave [12, Corollary 3.10] and Morel [22, Theorem 7.36] the map given by

$$f \mapsto \left(\text{deg}^{\mathbb{A}^1}(f), \text{res}(f) \right),$$

where $\text{res}(f)$ denotes the resultant of f as in [12], which we recalled in Proposition 22, induces an isomorphism of groups

$$\rho: [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{\cong} \text{GW}(k) \times_{k^\times/k^{\times 2}} k^\times. \tag{99}$$

Since our definition of deg is compatible with the notion of degree of a rational function used by Cazanave in [12], the work of Cazanave and Morel implies that, for every pointed morphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ we have

$$\text{deg}([f]) = \text{rank} \left(\text{deg}^{\mathbb{A}^1}([f]) \right),$$

where rank denotes the homomorphism $\text{GW}(k) \rightarrow \mathbb{Z}$ induced by the rank of a quadratic form. For a pointed morphism $g: \mathcal{J} \rightarrow \mathbb{P}^1$ with $\xi([g]) = [f]$, we have

$$\text{deg}([g]) = \text{rank} \left(\text{deg}^{\mathbb{A}^1}(\xi([g])) \right).$$

Hence we have the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}} & \longrightarrow & [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}} & \xrightarrow{\text{deg}} & \text{Pic}(\mathcal{J}) \longrightarrow 1 \\
 & & \xi_0 \downarrow \cong & & \xi \downarrow \text{bijection} & & \downarrow \cong \\
 1 & \longrightarrow & [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} & \longrightarrow & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} & \xrightarrow{\text{deg}} & \text{Pic}(\mathbb{P}^1) \longrightarrow 1 \\
 & & \downarrow \cong & & \rho \downarrow \cong & & \downarrow \cong \\
 1 & \longrightarrow & \text{GW}(k)_0 \times_{k^\times/k^{\times 2}} k^\times & \longrightarrow & \text{GW}(k) \times_{k^\times/k^{\times 2}} k^\times & \xrightarrow{\text{rank}} & \mathbb{Z} \longrightarrow 1
 \end{array} \tag{100}$$

where $\text{GW}(k)_0 \times_{k^\times/k^{\times 2}} k^\times$ denotes the kernel of the rank homomorphism.

Proposition 101. *The map $\pi_{\mathbb{N}}^*$ is injective.*

Proof. By Lemma 98 we know $\xi \circ \pi_{\mathbb{N}}^* = \nu_{\mathbb{P}^1}$. Since ξ is a bijection, it suffices to show that $\nu_{\mathbb{P}^1}$ is injective. The isomorphism ρ fits into the commutative diagram

$$\begin{array}{ccc}
 [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{N}} & \xrightarrow{\nu_{\mathbb{P}^1}} & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \\
 \downarrow \cong & & \cong \downarrow \rho \\
 \text{MW}^s(k) \times_{k^\times/k^{\times 2}} k^\times & \xrightarrow{\hat{\nu}} & \text{GW}(k) \times_{k^\times/k^{\times 2}} k^\times
 \end{array}$$

where $MW^s(k)$ denotes the stable monoid of symmetric bilinear forms as in [12, Definition 3.8] and \hat{v} is the group completion induced by the group completion $v: MW^s(k) \rightarrow GW(k)$. Since the vertical maps are isomorphisms by [12, Corollary 3.10] and [22, §7.3], $\nu_{\mathbb{P}^1}$ is injective if and only if \hat{v} is injective. To show that \hat{v} is injective, it suffices to show that the group completion $v: MW^s(k) \rightarrow GW(k)$ is injective. Since $MW^s(k)$ satisfies the cancellation property by the definition of $MW^s(k)$ in [12, Definition 3.8], respectively by Witt’s cancellation theorem, the map v is indeed injective. This proves the assertion. \square

Proposition 102. *Let $u, v \in k^\times$. If $u \neq v$, then $[g_{u,1}] \neq [g_{v,1}]$.*

Proof. Assume that $u \neq v \in k^\times$. By Cazanave’s work [12, Corollary 3.10], this implies $[X/u] \neq [X/v]$. By Proposition 101 this implies $\pi_N^*([X/u]) \neq \pi_N^*([X/v])$. By Proposition 92 we have $\pi_N^*([X/u]) = [g_{u,1}] \oplus \pi$ and $\pi_N^*([X/v]) = [g_{v,1}] \oplus \pi$. Since $[\mathcal{J}, \mathbb{P}^1]^N$ is a group, this implies $[g_{u,1}] \neq [g_{v,1}]$. \square

Proposition 103. *For every $u \in k^\times$ we have*

$$\left(\deg^{\mathbb{A}^1}(\xi[\pi_N^*(X/u)]), \text{res}(\xi[\pi_N^*(X/u)]) \right) = (\langle u \rangle, u) \text{ in } GW(k) \times_{k^\times/k^{\times 2}} k^\times.$$

In particular, for $\pi_N^(X/1) = \pi$, we get*

$$\left(\deg^{\mathbb{A}^1}(\xi[\pi]), \text{res}(\xi[\pi]) \right) = (\langle 1 \rangle, 1) \text{ in } GW(k) \times_{k^\times/k^{\times 2}} k^\times.$$

Proof. By Lemma 98 we know $\xi[\pi_N^*(X/u)] = \nu_{\mathbb{P}^1}([X/u])$. In [12, 3.4] Cazanave shows that the image of $\nu_{\mathbb{P}^1}([X/u])$ in $GW(k) \times_{k^\times/k^{\times 2}} k^\times$ is $(\langle u \rangle, u)$ by assigning it to the rank 1 symmetric matrix $[u]$, which has determinant u and corresponds to the quadratic form $\langle u \rangle$. \square

Remark 104. We note that, since $[(1, 0 : 0, u)_n] = X^n/u$, the work of Kass and Wickelgren in [18, Lemma 5] implies that we have $\deg^{\mathbb{A}^1}(\xi([(1, 0 : 0, u)_n])) = \langle u \rangle + \frac{n-1}{2}\langle 1, -1 \rangle$ for n odd, and $\deg^{\mathbb{A}^1}(\xi([(1, 0 : 0, u)_n])) = \frac{n}{2}\langle 1, -1 \rangle$ for n even. On the other hand, by the choice of the morphism $n\pi$ in Definition 76 we have $\deg^{\mathbb{A}^1}(\xi([n\pi])) = \deg^{\mathbb{A}^1}(n[\text{id}]) = n\langle 1 \rangle$. In particular, this implies that $[(1, 0 : 0, 1)_n]$ and $[n\pi]$ are in general not equal for $n > 1$ which explains our comment at the end of Remark 79.

In light of Question 83 we would like to show that $\rho(\xi[\tilde{\pi}])$ is the class $(-\langle 1 \rangle, 1)$ in $GW(k) \times_{k^\times/k^{\times 2}} k^\times$. We are not able to confirm this yet, since we do not know how to compute the resultant of $\xi[\tilde{\pi}]$. We can, however, show the following fact based on the computations of topological degrees in Appendix C. We thank Kirsten Wickelgren for mentioning to us the idea to use the arguments of [8] and [9] to reduce the computation to the Grothendieck–Witt group of the integers.

Theorem 105. *For every field k , we have*

$$\text{deg}^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1 \rangle \text{ in } \text{GW}(k).$$

Proof. First we assume $k = \mathbb{F}_2$. By [12, Lemma 3.13], $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{N}}$ is generated by elements in degree 1, i.e., the class of $X/1$ generates $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{N}}$. Since $\nu_{\mathbb{P}^1} : [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{N}} \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group completion, this implies that $\text{deg}_{\mathbb{F}_2}^{\mathbb{A}^1} : [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{\cong} \text{GW}(\mathbb{F}_2) = \mathbb{Z}$ is an isomorphism, where we refer to [20] for the Grothendieck group $\text{GW}(\mathbb{F}_2)$ of symmetric bilinear forms over \mathbb{F}_2 and the isomorphism $\text{GW}(\mathbb{F}_2) = \mathbb{Z}$ (see [9, Lemma B.5], [20, III Remark (3.4)]). Since the right-hand side of Diagram (100) commutes, the fact that we have $\text{deg}([\tilde{\pi}]) = -1$ implies $\text{deg}^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1 \rangle$ in $\text{GW}(k)$.

Next we let k be a field of characteristic 2. Then the canonical morphism $\text{Spec } k \rightarrow \text{Spec } \mathbb{F}_2$ induces a commutative diagram of group homomorphisms

$$\begin{CD} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{F}_2}^{\mathbb{A}^1} @>>> [\mathbb{P}^1, \mathbb{P}^1]_k^{\mathbb{A}^1} \\ @V \text{deg}_{\mathbb{F}_2}^{\mathbb{A}^1} VV @VV \text{deg}_k^{\mathbb{A}^1} V \\ \text{GW}(\mathbb{F}_2) @>>> \text{GW}(k). \end{CD}$$

Since $\tilde{\pi}$ is defined over \mathbb{F}_2 and $\text{deg}_{\mathbb{F}_2}^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1 \rangle$ by the first case, this implies $\text{deg}_k^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1 \rangle$ in $\text{GW}(k)$.

Now we assume that k is a field of characteristic $\neq 2$. The proof for this case is also based on the fact that $\tilde{\pi}$ is already defined over \mathbb{Z} and not just k . To make the argument work, however, requires a bit more effort. For a ring S , let $\mathcal{SH}(S)$ denote the stable motivic homotopy category over $\text{Spec } S$. Let $KO_k \in \mathcal{SH}(k)$ denote the motivic spectrum over $\text{Spec } k$ which represents Hermitian K-theory. It is equipped with a unit morphism $\varepsilon_k : \mathbb{1}_k \rightarrow KO_k$ in $\mathcal{SH}(k)$. Let $\mathfrak{s} : [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow \mathbb{1}_k^{0,0}(\text{Spec } k)$ denote the homomorphism defined by stabilization and note that there is a canonical isomorphism $KO_k^{0,0}(\text{Spec } k) \cong \text{GW}(k)$. We then define the homomorphism δ as the following composition:

$$\delta : [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{\mathfrak{s}} \mathbb{1}_k^{0,0}(\text{Spec } k) \xrightarrow{\varepsilon_k} KO_k^{0,0}(\text{Spec } k) \cong \text{GW}(k).$$

We claim that the homomorphism δ can be identified with the motivic Brouwer degree $\text{deg}_k^{\mathbb{A}^1}$ over k . To prove the claim we follow the argument of Levine and Raksit in [19, proof of Theorem 8.6, page 1845]. By Morel’s computation [22, Theorem 6.40], the isomorphism $\text{GW}(k) \cong \mathbb{1}_k^{0,0}(\text{Spec } k)$ sends $\langle u \rangle \in \text{GW}(k)$, for $u \in k^\times$, to $\mathfrak{s}(\nu_{\mathbb{P}^1}[X/u])$, the image of the class of $X/u : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1, [x_0 : x_1] \mapsto [x_0 : ux_1]$, in $\mathbb{1}_k^{0,0}(\text{Spec } k)$. Hence the classes $\mathfrak{s}(\nu_{\mathbb{P}^1}[X/u])$ for all $u \in k^\times$ generate $\mathbb{1}_k^{0,0}(\text{Spec } k)$. Thus, in order to prove the claim it suffices to show that $\delta(\nu_{\mathbb{P}^1}[X/u]) = \langle u \rangle$ in $\text{GW}(k)$, since $\text{deg}_k^{\mathbb{A}^1}([X/u]) = \langle u \rangle \in \text{GW}(k)$. That is, we need to show $\varepsilon_k(\mathfrak{s}(\nu_{\mathbb{P}^1}[X/u])) = \langle u \rangle$. This follows from [1, Corollary 6.2] which proves the claim.

In [8, §3.8.3] Bachmann and Hopkins construct a motivic spectrum $KO'_\mathbb{Z} \in \mathcal{SH}(\mathbb{Z})$ with a unit morphism $\varepsilon'_\mathbb{Z}: \mathbb{1}_\mathbb{Z} \rightarrow KO'_\mathbb{Z}$, and write KO'_k and ε'_k for the pullback of $KO'_\mathbb{Z}$ and $\varepsilon'_\mathbb{Z}$ to $\mathcal{SH}(k)$ along the canonical morphism $\text{Spec } k \rightarrow \text{Spec } \mathbb{Z}$. Since the characteristic of k is not 2, there is an equivalence of ring spectra $KO'_k \simeq KO_k$ by [8, Lemma 3.38 (3)], which induces an isomorphism $(KO'_k)^{0,0}(\text{Spec } k) \cong KO_k^{0,0}(\text{Spec } k)$. Thus, there is an isomorphism $(KO'_k)^{0,0}(\text{Spec } k) \cong KO_k^{0,0}(\text{Spec } k)$ which fits into the following commutative diagram.

$$\begin{array}{ccc}
 \mathbb{1}_k^{0,0}(\text{Spec } k) & \xrightarrow{\varepsilon'_k} & (KO'_k)^{0,0}(\text{Spec } k) \\
 & \searrow \varepsilon_k & \downarrow \cong \\
 & & KO_k^{0,0}(\text{Spec } k)
 \end{array}$$

By the above, we may therefore identify $\text{deg}_k^{\mathbb{A}^1}$ over k with the composed homomorphism

$$[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{s} \mathbb{1}_k^{0,0}(\text{Spec } k) \xrightarrow{\varepsilon'_k} (KO'_k)^{0,0}(\text{Spec } k) \cong KO_k^{0,0}(\text{Spec } k) \cong \text{GW}(k).$$

Furthermore, by [8, Lemma 3.38 (2)], there is an isomorphism $\pi_{0,0}(KO'_\mathbb{Z}) \cong \text{GW}(\mathbb{Z})$, where $\text{GW}(\mathbb{Z})$ denotes the Grothendieck–Witt group over \mathbb{Z} defined in [20, Chapter II]. Let $[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1}$ denote the set of endomorphisms of \mathbb{P}^1 in the pointed unstable \mathbb{A}^1 -homotopy category over $\text{Spec } \mathbb{Z}$. We now define the homomorphism $\text{deg}_{\mathbb{Z}}^{\mathbb{A}^1}: [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1} \rightarrow \text{GW}(\mathbb{Z})$ as the composition

$$\text{deg}_{\mathbb{Z}}^{\mathbb{A}^1}: [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1} \xrightarrow{s_{\mathbb{Z}}} \mathbb{1}_{\mathbb{Z}}^{0,0}(\text{Spec } \mathbb{Z}) \xrightarrow{\varepsilon'_{\mathbb{Z}}} (KO'_\mathbb{Z})^{0,0}(\text{Spec } \mathbb{Z}) \cong \text{GW}(\mathbb{Z}).$$

The canonical homomorphism $\mathbb{Z} \rightarrow k$ then induces the following commutative square

$$\begin{array}{ccc}
 [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1} & \xrightarrow{\text{deg}_{\mathbb{Z}}^{\mathbb{A}^1}} & \text{GW}(\mathbb{Z}) \\
 \downarrow & & \downarrow \mathfrak{b}_k \\
 [\mathbb{P}^1, \mathbb{P}^1]_k^{\mathbb{A}^1} & \xrightarrow{\text{deg}_k^{\mathbb{A}^1}} & \text{GW}(k)
 \end{array}$$

where $\mathfrak{b}_k: \text{GW}(\mathbb{Z}) \rightarrow \text{GW}(k)$ denotes the change of coefficients homomorphism. As a consequence we see that if $[f] \in [\mathbb{P}^1, \mathbb{P}^1]_k^{\mathbb{A}^1}$ is in the image of the homomorphism $[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1} \rightarrow [\mathbb{P}^1, \mathbb{P}^1]_k^{\mathbb{A}^1}$, then

$$\text{deg}_k^{\mathbb{A}^1}([f]) = \mathfrak{b}_k(\text{deg}_{\mathbb{Z}}^{\mathbb{A}^1}([f])). \tag{106}$$

By [9, Lemma 5.6] (see also [20, Theorem II.4.3]), $\text{GW}(\mathbb{Z})$ is generated over \mathbb{Z} by the classes $\langle 1 \rangle$ and $\langle -1 \rangle$. For a class $\omega \in \text{GW}(\mathbb{Z})$, let $\omega_{\mathbb{C}}$ and $\omega_{\mathbb{R}}$ denote the images of

ω in $\text{GW}(\mathbb{C})$ and $\text{GW}(\mathbb{R})$, respectively. Then $\omega \in \text{GW}(\mathbb{Z})$ is uniquely determined by the integers $\text{rank}(\omega_{\mathbb{C}})$ and $\text{sgn}(\omega_{\mathbb{R}})$, given by the rank and signature of $\omega_{\mathbb{C}}$ and $\omega_{\mathbb{R}}$, respectively, via the formula

$$\omega = \frac{\text{rank}(\omega_{\mathbb{C}}) + \text{sgn}(\omega_{\mathbb{R}})}{2} \langle 1 \rangle + \frac{\text{rank}(\omega_{\mathbb{C}}) - \text{sgn}(\omega_{\mathbb{R}})}{2} \langle -1 \rangle \in \text{GW}(\mathbb{Z}). \tag{107}$$

Since \mathcal{J} and both morphisms π and $\tilde{\pi}$ are defined over $\text{Spec } \mathbb{Z}$, we can now apply the above observations to prove the assertion of the proposition. Since π is an \mathbb{A}^1 -weak equivalence over $\text{Spec } \mathbb{Z}$ as well, we can form the pointed \mathbb{A}^1 -homotopy class $\xi_{\mathbb{Z}}([\tilde{\pi}]) := [\tilde{\pi} \circ \pi^{-1}]^{\mathbb{A}^1} \in [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1}$ defined by the zig-zag $\mathbb{P}^1_{\mathbb{Z}} \xleftarrow{\pi} \mathcal{J}_{\mathbb{Z}} \xrightarrow{\tilde{\pi}} \mathbb{P}^1_{\mathbb{Z}}$. The class $\xi_{\mathbb{Z}}([\tilde{\pi}])$ is sent to $\xi([\tilde{\pi}])$ under base change. Thus, by the above arguments, to determine $\text{deg}_{\mathbb{C}}^{\mathbb{A}^1}(\xi[\tilde{\pi}])$ in $\text{GW}(k)$ it suffices to compute the rank and signature of $\text{deg}_{\mathbb{Z}}^{\mathbb{A}^1}(\xi_{\mathbb{Z}}[\tilde{\pi}])$ after base change to \mathbb{C} and \mathbb{R} , respectively. Since the right-hand side of Diagram (100) commutes, the fact that we have $\text{deg}([\tilde{\pi}]) = -1$ implies $\text{rank}(\text{deg}_{\mathbb{C}}^{\mathbb{A}^1}(\xi[\tilde{\pi}])) = -1$. In Appendix C and Example 148 we show that the signature of $\text{deg}_{\mathbb{R}}^{\mathbb{A}^1}(\xi[\tilde{\pi}])$ over \mathbb{R} is -1 . Thus, by Formula (107), we get $\text{deg}_{\mathbb{Z}}^{\mathbb{A}^1}(\xi_{\mathbb{Z}}[\tilde{\pi}]) = -\langle 1 \rangle$ in $\text{GW}(\mathbb{Z})$. By Equation (106) we can therefore conclude that $\text{deg}_k^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1 \rangle$ in $\text{GW}(k)$. \square

6.2. Group completion of naive homotopy classes

We will now describe the homomorphism $\psi: ([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}) \rightarrow ([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$ induced by the universal property of the group completion $\nu_{\mathbb{P}^1}$ in more detail. By [12, Lemma 3.13], $[\mathbb{P}^1, \mathbb{P}^1]^N$ is generated by elements in degree 1, i.e., it is generated by the set of classes $[X/u]$ for all $u \in k^\times$. Hence, since $\nu_{\mathbb{P}^1}$ is a group completion, every element in $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ of degree 0 can be written as a sum of the differences $\gamma_{u,v} := \nu_{\mathbb{P}^1}([X/u]) - \nu_{\mathbb{P}^1}([X/v])$ for suitable $u, v \in k^\times$. Thus the set of classes $\gamma_{u,v}$ for all $u, v \in k^\times$ generates the subgroup $[\mathbb{P}^1, \mathbb{P}^1]_0^{\mathbb{A}^1}$ of degree 0 elements. Because of this we would like to understand the image of the $\gamma_{u,v}$ under ψ . Since ψ is a group homomorphism, we know $\psi(\gamma_{u,v}) \oplus \psi(\nu_{\mathbb{P}^1}([X/v])) = \psi(\nu_{\mathbb{P}^1}([X/u]))$. Since $\psi \circ \nu_{\mathbb{P}^1} = \pi_N^*$, this implies $\psi(\gamma_{u,v}) \oplus \pi_N^*([X/v]) = \pi_N^*([X/u])$. By Lemma 91, the map $g_{u,v}$ satisfies $[g_{u,v}] \oplus \pi_N^*([X/v]) = \pi_N^*([X/u])$. Hence, since $[\mathcal{J}, \mathbb{P}^1]^N$ is a group, we get

$$\psi(\gamma_{u,v}) = [g_{u,v}] \text{ in } [\mathcal{J}, \mathbb{P}^1]^N.$$

This motivates the following definition of the subgroup $\mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^N$.

Definition 108. Let $\mathbf{G}_0 := \langle [g_{u,v}] \mid u, v \in k^\times \rangle \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ denote the subgroup generated by the homotopy classes of the maps $g_{u,v}$. Let $\mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^N$ be the subgroup generated by \mathbf{G}_0 and $[\pi]$.

We note that by Lemma 87, we have $-[g_{u,v}] = [g_{v,u}]$, while $[g_{u,u}]$ is the neutral element, and we therefore have $-([g_{u,v}] \oplus \pm n[\pi]) = [g_{v,u}] \oplus \mp n[\pi]$ in \mathbf{G} .

Remark 109. By Lemma 87, we have $[g_{u,v}] \oplus [g_{v,1}] = [g_{u,1}]$ for all $u, v \in k^\times$. Thus, every element in \mathbf{G}_0 is a sum of differences of elements in the submonoid of $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ generated by the set of homotopy classes $[g_{u,1}]$ for all $u \in k^\times$. This implies that the inclusion of the submonoid of $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ generated by the set of homotopy classes $[g_{u,1}]$ for all $u \in k^\times$ into \mathbf{G}_0 is a group completion.

Lemma 110. *The morphism of monoids $\pi_N^*: [\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^N$ has image in \mathbf{G} .*

Proof. By [12, Lemma 3.13], $[\mathbb{P}^1, \mathbb{P}^1]^N$ is generated by elements in degree 1, i.e., it is generated by the set of classes $[X/u]$ for all $u \in k^\times$. Hence it suffices to show that $\pi_N^*([X/u])$ is contained in \mathbf{G} . This follows from Lemma 90. \square

Proposition 111. *The morphism of monoids $\pi_N^*: [\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow \mathbf{G}$ is a group completion.*

Proof. Let H be an abelian group and $\mu: [\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow H$ be a morphism of monoids. We will show that there is a unique homomorphism of groups $\tilde{\mu}: \mathbf{G} \rightarrow H$ such that $\tilde{\mu} \circ \pi_N^* = \mu$.

We set $\tilde{\mu}([\pi]) := \mu([X/1])$. By Lemma 91, we have $[g_{u,v}] \oplus \pi_N^*([X/v]) = \pi_N^*([X/u])$ in $\mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^N$. Hence compatibility with μ forces the definition

$$\tilde{\mu}([g_{u,v}]) := \mu([X/u]) - \mu([X/v]).$$

By definition of \mathbf{G} this induces a unique group homomorphism $\tilde{\mu}$, once we have shown that it is well-defined.

Now we show that $\tilde{\mu}$ is well-defined. Because $\mathbf{G} \cong \mathbf{G}_0 \oplus \mathbb{Z}$, all relations in \mathbf{G} amongst the generators arise from relations of the classes $[g_{u,v}]$. Consider a relation of the form

$$[g_{u_1, v_1}] \oplus \dots \oplus [g_{u_s, v_s}] = 0. \tag{112}$$

We must then show that $\sum_i \tilde{\mu}([g_{u_i, v_i}]) = 0$ in H . Since \mathbf{G} is a group and by Lemma 91, we have

$$\sum_i g_{u_i, v_i} \oplus \pi_N^*([X/v_i]) = \sum_i \pi_N^*([X/u_i]).$$

Since \mathbf{G} is abelian, this implies

$$\sum_i [g_{u_i, v_i}] = \sum_i \pi_N^*([X/u_i]) - \pi_N^*([X/v_i]) = \sum_i \pi_N^*([X/u_i]) - \sum_i \pi_N^*([X/v_i]) = 0.$$

Hence

$$\sum_i \pi_N^*([X/u_i]) = \sum_i \pi_N^*([X/v_i])$$

in $[\mathbb{P}^1, \mathbb{P}^1]^N$. Since π_N^* is an injective monoid morphism, in $[\mathbb{P}^1, \mathbb{P}^1]^N$ we have the equation

$$\sum_i [X/u_i] = \sum_i [X/v_i].$$

It thus follows that

$$\mu \left(\sum_i [X/u_i] \right) = \mu \left(\sum_i [X/v_i] \right) \text{ in } H.$$

We calculate

$$\tilde{\mu} \left(\sum_i [g_{u_i, v_i}] \right) = \sum_i \mu([X/u_i]) - \mu([X/v_i]) = 0,$$

as desired. This shows that $\tilde{\mu}$ is well-defined.

It remains to show $\tilde{\mu} \circ \pi_N^* = \mu$. By [12, Lemma 3.13], $[\mathbb{P}^1, \mathbb{P}^1]^N$ is generated by the set of classes $[X/u]$ for all $u \in k^\times$. Hence μ is completely determined by the images of $[X/u]$ for all $u \in k^\times$. Thus, in order to show $\tilde{\mu} \circ \pi_N^* = \mu$, it suffices to show $\tilde{\mu}(\pi_N^*([X/u])) = \mu([X/u])$ for every $u \in k^\times$. This is now immediate from the definition of $\tilde{\mu}$ and Lemma 91:

$$\begin{aligned} \tilde{\mu}(\pi_N^*([X/u])) &= \tilde{\mu}([g_{u,1}] \oplus \pi_N^*([X/1])) \\ &= \tilde{\mu}([g_{u,1}]) + \tilde{\mu}([\pi]) \\ &= \mu([X/u]) - \mu([X/1]) + \mu([X/1]) \\ &= \mu([X/u]). \end{aligned}$$

This shows that $\pi_N^*: [\mathbb{P}^1, \mathbb{P}^1]^N \rightarrow \mathbf{G}$ has the universal property of a group completion and finishes the proof. \square

Theorem 113. *There is a unique isomorphism of groups*

$$\chi: \mathbf{G} \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$$

such that $\chi \circ \pi_N^* = \nu_{\mathbb{P}^1}$. The homomorphism χ sends $[g_{u,v}]$ to the unique element $\gamma_{u,v}$ that satisfies $\nu_{\mathbb{P}^1}^*([X/u]) = \gamma_{u,v} \oplus^{\mathbb{A}^1} \nu_{\mathbb{P}^1}([X/v])$ and $[\pi]$ to $[\text{id}]$. Moreover, χ and the homomorphism

$$\psi: [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow \mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^N$$

are mutual inverses to each other.

Proof. The existence of χ and its definition is a consequence of Proposition 111 and its proof. The assertion that χ is the inverse of ψ follows from the fact that $\nu_{\mathbb{P}^1}$ is a

group completion proven by Cazanave in [12, Theorem 3.22] and the universal property of group completion. \square

As a particular consequence of Theorem 113 we get the following result.

Proposition 114. *The restriction χ_0 of the homomorphism χ to \mathbf{G}_0 defines an isomorphism of groups*

$$\chi_0: \mathbf{G}_0 \xrightarrow{\cong} [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}.$$

Proof. The elements $\gamma_{u,v}$ are of degree 0, and hence they lie in the subgroup $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$. As a consequence, the homomorphism χ and its restriction χ_0 fit in the following commutative diagram of abelian groups.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{G}_0 & \longrightarrow & \mathbf{G} & \longrightarrow & \mathbb{Z} \longrightarrow 1 \\ & & \chi_0 \downarrow & & \chi \downarrow \cong & & \downarrow \cong \\ 1 & \longrightarrow & [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} & \longrightarrow & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} & \longrightarrow & \mathbb{Z} \longrightarrow 1 \end{array}$$

Since the middle and right-most maps are isomorphisms, the assertion follows from the five-lemma. \square

The existence of the isomorphisms $\mathbf{G} \xrightarrow{\chi} [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xleftarrow{\varphi} [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$ does not imply that \mathbf{G} equals $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$. However, we make the following conjecture on the a priori subgroups \mathbf{G}_0 and \mathbf{G} .

Conjecture 115. *The inclusions $\mathbf{G}_0 \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}}$ and $\mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$ are equalities.*

We will show in Theorem 123 in Section 6.3 that Conjecture 115 is true whenever $k = \mathbb{F}_q$ is a finite field. This follows from an explicit computation of \mathbf{G}_0 and $K_1^{\text{MW}}(\mathbb{F}_q)$, the first Milnor–Witt K-theory of \mathbb{F}_q .

Remark 116. It follows from the structure of the group \mathbf{G} as a product of \mathbf{G}_0 and $\{n[\pi] \mid n \in \mathbb{Z}\}$ that in order to prove Conjecture 115 it suffices to show that the inclusion $\mathbf{G}_0 \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}}$ is an equality, i.e., that the set of homotopy classes $[g_{u,v}]$ for all $u, v \in k^\times$ generates the group $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}}$.

Remark 117. If Conjecture 115 is true, then the group homomorphism $\psi: [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$, induced by the fact that $\nu_{\mathbb{P}^1}$ is a group completion, is an isomorphism and it agrees with φ^{-1} , the inverse of the isomorphism of Theorem 80. We note, however, that this does not yet imply that the bijection $\xi = (\pi_{\mathbb{A}^1}^*)^{-1} \circ \nu: [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}} \rightarrow [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group homomorphism.

6.3. Milnor–Witt K-theory and morphisms in degree 0

Our final goal is to prove Conjecture 115 for finite fields. For the proof we use the Milnor–Witt K-theory of a field which we now recall from [22, Definition 3.1].

Definition 118. The Milnor–Witt K-theory of the field k , denoted $K_*^{\text{MW}}(k)$, is the graded associative ring generated by symbols $[u]$ in degree 1 for $u \in k^\times$ and the symbol η in degree -1 subject to the following relations:

- (1) For each $u \in k^\times \setminus \{1\}$, $[u].[1 - u] = 0$.
- (2) For each pair $u, v \in (k^\times)^2$, $[uv] = [u] + [v] + \eta.[u].[v]$.
- (3) For each $u \in k^\times$, $\eta.[u] = [u].\eta$.
- (4) Let $h := \eta.[-1] + 2$. Then $\eta.h = 0$.

Remark 119. It follows directly from the defining relations for $K_*^{\text{MW}}(k)$ that $[1] = 0$ and $\eta.[u^2] = 0$ for each $u \in k^\times$. See [22, §3.1] for a proof and other basic properties of Milnor–Witt K-theory.

Recall that $\mathbf{G}_0 := \langle [g_{u,v}] \mid u, v \in k^\times \rangle \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}}$ denotes the subgroup generated by the homotopy classes of the maps $g_{u,v}$. In this subsection we write $\mathbf{G}_0(k)$ and $\mathbf{G}(k)$ for the groups \mathbf{G}_0 and \mathbf{G} , respectively, to emphasize the dependency of the base field k .

Proposition 120. For every field k , there is an isomorphism $\mathbf{G}_0(k) \cong K_1^{\text{MW}}(k)$.

Proof. By Proposition 114, we have an isomorphism $\chi_0: \mathbf{G}_0(k) \xrightarrow{\cong} [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$. As recalled in Remark 60, the work of Morel in [22, §7.3] implies that there is an isomorphism of groups $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} \cong K_1^{\text{MW}}(k)$. The composition of these isomorphisms yields the assertion. \square

Lemma 121. Let $k = \mathbb{F}_q$ be a finite field of odd characteristic. Let v_1, v_2 be non-squares in \mathbb{F}_q^\times . Then we have $[v_1v_2] = [v_1] + [v_2]$ in $K_1^{\text{MW}}(\mathbb{F}_q)$.

Proof. Since q is odd, the kernel of the squaring homomorphism has two elements, -1 and 1 , i.e., $\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2} \cong \mathbb{Z}/2\mathbb{Z}$. Because 1 is a square, the set $\mathbb{F}_q \setminus \{0, 1\}$ contains more non-squares than squares. Construct pairs $(t, 1 - t)$ from elements $t \in \mathbb{F}_q \setminus \{0, 1\}$, and observe that there must exist at least one non-square t such that $1 - t$ is also a non-square. For the rest of the proof we fix t to be one such non-square. By relation (1) in $K_1^{\text{MW}}(\mathbb{F}_q)$, we have $[t].[1 - t] = 0$. Let v_1, v_2 be non-squares in \mathbb{F}_q^\times . By relation (2) of Definition 118 we have

$$[v_1v_2] = [v_1] + [v_2] + \eta.[v_1].[v_2]$$

Hence to prove the assertion of the lemma it suffices to prove $\eta.[v_1].[v_2] = 0$ in $K_1^{\text{MW}}(\mathbb{F}_q)$. Since $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2} \cong \mathbb{Z}/2\mathbb{Z}$, there exist units c_1 and c_2 such that $c_1^2 t = v_1$ and $c_2^2(1-t) = v_2$. Hence we get

$$\begin{aligned} \eta.[v_1].[v_2] &= \eta.[c_1^2 t].[c_2^2(1-t)] \\ &= \eta.([c_1^2] + [t]).([c_2^2] + [1-t]) \end{aligned}$$

where the second equality follows from the fact, for every non-square v and every square c^2 , we have $[vc^2] = [v] + [c^2]$. Since we have $\eta.[c_i^2] = 0$ by Remark 119 and $[t].[1-t] = 0$, we can conclude $\eta.[v_1].[v_2] = 0$ which proves the lemma. \square

Proposition 122. *Let $k = \mathbb{F}_q$ be a finite field. Then $K_1^{\text{MW}}(\mathbb{F}_q)$ is a finite cyclic group of order $q - 1$.*

Proof. First we assume that q is even. Then the squaring homomorphism is surjective, and hence every unit is a square. Fix u to be a multiplicative generator of \mathbb{F}_q^\times . It follows from [22, Lemma 3.6 (1)] that $K_1^{\text{MW}}(\mathbb{F}_q)$ is generated by the elements $[v]$ for $v \in k^\times$, which are subject to the relation $[vv'] = [v] + [v']$ for all $v, v' \in \mathbb{F}_q^\times$. The fact that $u^{q-1} = 1$ yields the result that $K_1^{\text{MW}}(\mathbb{F}_q)$ is cyclic of order $q - 1$ generated by the symbol $[u]$.

Now we assume that q is odd. Let u be a multiplicative generator of \mathbb{F}_q^\times . Using induction and Lemma 121 we get $[u^n] = n[u]$ for all $n \geq 1$. Since \mathbb{F}_q^\times is cyclic of order $q - 1$, this shows that $K_1^{\text{MW}}(\mathbb{F}_q)$ is cyclic of order $q - 1$ generated by the symbol $[u]$. \square

Theorem 123. *Let $k = \mathbb{F}_q$ be a finite field. Then Conjecture 115 is true, i.e., the inclusions $\mathbf{G}_0(\mathbb{F}_q) \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}}$ and $\mathbf{G}(\mathbb{F}_q) \subseteq [\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$ are equalities.*

Proof. By Remark 116 it suffices to prove the assertion for $\mathbf{G}_0(\mathbb{F}_q)$. By Propositions 120 and 122, both $\mathbf{G}_0(\mathbb{F}_q)$ and $K_1^{\text{MW}}(\mathbb{F}_q)$ are finite groups of the same cardinality. Since $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ and $K_1^{\text{MW}}(\mathbb{F}_q)$ are isomorphic and since $\xi_0: [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}} \rightarrow [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ is an isomorphism, $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}}$ is a finite group of the same cardinality as $\mathbf{G}_0(\mathbb{F}_q)$ as well. Hence $\mathbf{G}_0(\mathbb{F}_q) \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}}$ is an inclusion of finite groups of the same cardinality. This implies that the inclusion $\mathbf{G}_0(\mathbb{F}_q) \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{N}}$ is an equality. \square

We conclude this section with the following observation. While Proposition 120 shows that there is an isomorphism between $\mathbf{G}_0(k)$ and $K_1^{\text{MW}}(k)$, the proof of Proposition 122 suggests that the following map provides a concrete isomorphism. We consider this an interesting observation about $K_1^{\text{MW}}(k)$ that arises from our work on maps $\mathcal{J} \rightarrow \mathbb{A}^2 \setminus \{0\}$.

Proposition 124. *Let k be one of the following fields: a quadratically closed field, a finite field, or \mathbb{R} . Then the assignment $[u] \mapsto [g_{u,1}]$ defines an isomorphism $\kappa: K_1^{\text{MW}}(k) \rightarrow \mathbf{G}_0(k)$.*

Proof. We will show that $K_1^{\text{MW}}(k)$ and $\mathbf{G}_0(k)$ are generated by the classes $[u]$ and $[g_{u,1}]$, respectively, and that these generators satisfy exactly the same type of relations. This implies that κ is both a well-defined homomorphism and an isomorphism. We will prove the claim by looking at each type of field separately.

First we assume that k is a quadratically closed field, i.e., a field where every unit is a square. The group $K_1^{\text{MW}}(k)$ is generated by the elements $[u]$ and the relation $[uv] = [u] + [v]$ for all $u, v \in k^\times$. To prove that κ is an isomorphism we need to show that $\mathbf{G}_0(k)$ is generated by the classes $[g_{u,1}]$ subject to the relation $[g_{uv,1}] = [g_{u,1}] \oplus [g_{v,1}]$. Since every unit in k is a square, we get $[g_{u,v}] = [g_{u/v,1}]$ by Lemma 88. Hence $\mathbf{G}_0(k)$ is generated by elements $[g_{u,1}]$. By Proposition 102 we know that $[g_{u,1}] \neq [g_{v,1}]$ for $u \neq v \in k^\times$. By Lemma 89 we get the relation $[g_{u,1}] \oplus [g_{v,1}] = [g_{uv,1}]$. Thus, the map sending $[u]$ to $[g_{u,1}]$ induces a homomorphism which is surjective and injective. Hence κ is an isomorphism.

For $k = \mathbb{F}_q$, Proposition 122 shows that $K_1^{\text{MW}}(\mathbb{F}_q)$ is generated by the symbol $[u]$ for a multiplicative generator $u \in \mathbb{F}_q^\times$. When q is even, every unit is a square, and in this case κ is an isomorphism. So we assume that q is odd, and will now show that every element in $\mathbf{G}_0(\mathbb{F}_q)$ can be written in the form $m[g_{u,1}] \oplus [g_{u^{2m'},1}]$ for some $m, m' \in \mathbb{Z}$. We will use that every square in \mathbb{F}_q is equal to an even power of the generator $u \in \mathbb{F}_q^\times$, and distinguish three cases: Assume first $v_1, v_2 \in \mathbb{F}_q^\times$ are squares. By Lemma 88 we then have $[g_{v_1, v_2}] = [g_{v_1/v_2, 1}] = [g_{u^{2m}, 1}]$ for some m . Second, if v_1 is not a square and v_2 is a square, then $v_1/v_2 = u^{2m+1}$ for some $m \in \mathbb{Z}$. Then by Lemma 88 and 89 we know $[g_{v_1, v_2}] = [g_{u^{2m+1}, 1}] = [g_{u, 1}] \oplus [g_{u^{2m}, 1}]$. Note that $[g_{v_2, v_1}] = -[g_{v_1, v_2}] = -[g_{u, 1}] \oplus [g_{u^{-2m}, 1}]$. Third, assume that both v_1 and v_2 are non-squares in \mathbb{F}_q^\times . Since $\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2} \cong \mathbb{Z}/2\mathbb{Z}$, we can find an $m \in \mathbb{Z}$ such that $v_1/v_2 = u^{2m}$. We have $[g_{v_1, v_2}] = [g_{u \cdot u^{2m}, u}]$ by Lemma 88 and scaling by the square u/v_2 . We can now apply Lemma 87 and then Lemma 89 to get

$$[g_{u \cdot u^{2m}, u}] = [g_{u \cdot u^{2m}, 1}] \oplus [g_{1, u}] = [g_{u, 1}] \oplus [g_{u^{2m}, 1}] \oplus [g_{1, u}] = [g_{u^{2m}, 1}].$$

To conclude the argument we note that, for $v_1, v_2 \in \mathbb{F}_q^\times$ with $v_1 + v_2 \neq 1$, there is the relation $\langle v_1 \rangle + \langle v_2 \rangle = \langle v_1 + v_2 \rangle + \langle (v_1 + v_2)v_1v_2 \rangle$ in $\text{GW}(\mathbb{F}_q)$. For s and $1 - s$ in \mathbb{F}_q^\times , this gives $\langle s \rangle + \langle 1 - s \rangle = \langle 1 \rangle + \langle s(1 - s) \rangle = \langle 1, 1 \rangle$. In particular, since u, s , and $1 - s$ all differ by squares and hence $\langle u \rangle = \langle s \rangle = \langle 1 - s \rangle$ in $\text{GW}(\mathbb{F}_q)$, we have $\langle u \rangle + \langle u \rangle = \langle 1, 1 \rangle = \langle u^2, 1 \rangle$ in $\text{GW}(\mathbb{F}_q)$. By [12, Corollary 3.10] this relation implies $[X/u] \oplus^{\mathbb{N}} [X/u] = [X/u^2] \oplus^{\mathbb{N}} [X/1]$ in $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{N}}$. By Proposition 95 and Lemma 91, this implies the equality $[g_{u,1}] \oplus [g_{u,1}] = [g_{u^2,1}]$ in $\mathbf{G}_0(\mathbb{F}_q)$. Iterating this argument, we get $(q - 1)[g_{u,1}] = [g_{u^{q-1},1}] = [g_{1,1}]$. Since $[g_{v_1,1}] \neq [g_{v_2,1}]$ for $v_1 \neq v_2 \in \mathbb{F}_q^\times$ by Proposition 102, this implies that $\mathbf{G}_0(\mathbb{F}_q)$ is cyclic of order $q - 1$ generated by $[g_{u,1}]$. Hence the map $[u] \mapsto [g_{u,1}]$ is a well-defined homomorphism which is surjective and injective. Thus κ is an isomorphism in this case as well.

Finally, we assume $k = \mathbb{R}$. First we determine the generators and relations for $K_1^{\text{MW}}(\mathbb{R})$. For $u > 0$, we have $[-u] = [-1] + [u] + \eta \cdot [-1] \cdot [u] = [-1] + [u]$ and $-[u] = [1/u]$. Thus every element in $K_1^{\text{MW}}(\mathbb{R})$ can be written as $n[-1] + [u]$ with $u > 0$ for some $n \in \mathbb{Z}$

subject to the relation $(n[-1] + [u]) + (m[-1] + [v]) = (n + m)[-1] + [uv]$. Next we show that $\mathbf{G}_0(\mathbb{R})$ has analogous generators and relations. Assume $u, v > 0$. Since v is a square, Lemmas 88 and 89 imply the identities

$$[g_{u,v}] = [g_{u/v,1}], \text{ and } [g_{-u,v}] = [g_{-u/v,1}] = [g_{-1,1}] \oplus [g_{u/v,1}].$$

Using that v is a square, Lemma 88 yields the following identity

$$[g_{u,-v}] = [g_{u/v,-1}] = [g_{u/v,1}] \oplus [g_{1,-1}] = -[g_{-1,1}] \oplus [g_{u/v,1}]$$

where we have $[g_{1,-1}] = -[g_{-1,1}]$ by Lemma 87. Finally, using Lemma 88 and 89 we get

$$[g_{-u,-v}] = [g_{-u/v,-1}] = [g_{-u/v,1}] \oplus [g_{1,-1}] = [g_{-1,1}] \oplus [g_{u/v,1}] \oplus [g_{1,-1}] = [g_{u/v,1}].$$

This implies every element of $\mathbf{G}_0(\mathbb{R})$ can be written as a sum $n[g_{-1,1}] \oplus [g_{u,1}]$ with $u > 0$ and $n \in \mathbb{Z}$. By Proposition 102 we know that $[g_{u,1}] \neq [g_{v,1}]$ for $u \neq v \in \mathbb{R}^\times$. By Lemma 89 we get the relation $(n[g_{-1,1}] \oplus [g_{u,1}]) \oplus (m[g_{-1,1}] \oplus [g_{v,1}]) = (n + m)[g_{-1,1}] \oplus [g_{uv,1}]$ when $u, v > 0$. Hence the map $[u] \mapsto [g_{u,1}]$ is a well-defined homomorphism which is surjective and injective. Thus κ is an isomorphism in this case. This finishes the proof. \square

Appendix A. Affine representability for pointed spaces and homotopies

In this section, we discuss the results of Asok, Hoyois, and Wendt in [5], [6], and how we apply them. While the definition of our proposed group operation on $[\mathcal{J}, \mathbb{P}^1]^N$ in Definition 77 is independent of motivic homotopy theory and the results of [6], we use the affine representability results of [6] to compare our group operation with the conventional group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. A minor technical point to overcome is that Asok, Hoyois, and Wendt work in the unpointed motivic homotopy category, whereas we need the analogous results in the pointed setting. The purpose of this appendix is to explain how the pointed analogs can be deduced. To keep the presentation brief, we use the conventions and notation of the papers [5] and [6]. We thank Marc Hoyois for helpful comments.

Recall that we denote by Sm_k the category of smooth finite type k -schemes. Let $\text{sPre}(\text{Sm}_k)$ denote the category of simplicial presheaves on Sm_k . Let $\text{sPre}_*(\text{Sm}_k)$ denote the category of pointed simplicial presheaves on Sm_k . We refer to an object in $\text{sPre}(\text{Sm}_k)$ (respectively in $\text{sPre}_*(\text{Sm}_k)$) as a (pointed) motivic space. We equip with the Nisnevich-local \mathbb{A}^1 -model structure as in [6]. For a motivic space \mathcal{Y} , let $\text{Sing}^{\mathbb{A}^1} \mathcal{Y}$ denote the singular functor defined in [5, §4.1], see also [23, page 88]. If \mathcal{Y} is pointed by a morphism $y: \text{Spec } k \rightarrow \mathcal{Y}$, then the pointed singular functor $\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y}$ is defined as the fiber over y . More precisely, let X be a pointed smooth k -scheme pointed by the morphism $x: \text{Spec } k \rightarrow X$, then $\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y}$ is determined by the pullback square of simplicial sets

$$\begin{array}{ccc}
 (\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X) & \longrightarrow & (\text{Sing}^{\mathbb{A}^1} \mathcal{Y})(X) \\
 \downarrow & & \downarrow x^* \\
 \text{point} = (\text{Sing}^{\mathbb{A}^1} \text{Spec } k)(\text{Spec } k) & \xrightarrow{y_*} & (\text{Sing}^{\mathbb{A}^1} \mathcal{Y})(\text{Spec } k).
 \end{array} \tag{125}$$

We note that, on 0-simplices, x induces a map of sets $x^* : \text{sPre}(\text{Sm}_k)(X, \mathcal{Y}) \rightarrow \text{sPre}(\text{Sm}_k)(\text{Spec } k, \mathcal{Y})$. Hence, $(\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X)_0$ is the set $\text{sPre}_*(\text{Sm}_k)(X, \mathcal{Y})$ of pointed morphisms $X \rightarrow \mathcal{Y}$. On 1-simplices, x induces a map of sets $x^* : \text{sPre}(\text{Sm}_k)(X \times \mathbb{A}^1, \mathcal{Y}) \rightarrow \text{sPre}(\text{Sm}_k)(\mathbb{A}^1, \mathcal{Y})$. Hence, $(\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X)_1$ is the set of pointed naive \mathbb{A}^1 -homotopies of pointed morphisms $X \rightarrow \mathcal{Y}$.

Remark 126. In particular, if $\mathcal{Y} = Y$ is represented by a pointed smooth finite type k -scheme Y , the set $\pi_0((\text{Sing}_*^{\mathbb{A}^1} Y)(X))$ is the set of pointed naive homotopy classes of pointed morphisms $X \rightarrow Y$ described in Section 2.6, that is,

$$\pi_0((\text{Sing}_*^{\mathbb{A}^1} Y)(X)) = [X, Y]^N.$$

We recall the following definition from [6]:

Definition 127. [6, Definition 2.1.1] Let $\mathcal{F} \in \text{sPre}(\text{Sm}_k)$ and let $\mathcal{F} \rightarrow \tilde{\mathcal{F}}$ be a fibrant replacement in the \mathbb{A}^1 -model structure on $\text{sPre}(\text{Sm}_k)$. There is a canonical map $\text{Sing}^{\mathbb{A}^1} \mathcal{F} \rightarrow \tilde{\mathcal{F}}$ that is well-defined up to simplicial homotopy equivalence. Then $\mathcal{F} \in \text{sPre}(\text{Sm}_k)$ is called \mathbb{A}^1 -naive if the map $(\text{Sing}^{\mathbb{A}^1} \mathcal{F})(X) \rightarrow \tilde{\mathcal{F}}(X)$ is a weak equivalence of simplicial sets for every affine smooth finitely presented k -scheme X .

We will now show how the unpointed notion of \mathbb{A}^1 -naivity of Definition 127 translates to the pointed setting.

Proposition 128. Let $\mathcal{Y} \in \text{sPre}_*(\text{Sm}_k)$ be a pointed motivic space. Assume that the underlying unpointed motivic space \mathcal{Y} is \mathbb{A}^1 -naive. Then, for every affine pointed smooth finitely presented k -scheme X , the canonical map $\pi_0((\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X)) \xrightarrow{\cong} [X, \mathcal{Y}]^{\mathbb{A}^1}$ is a bijection.

Proof. Let (X, x) be a pointed smooth k -scheme, and let $p : X \rightarrow \text{Spec } k$ denote the canonical morphism. Then p induces a map $p^* : (\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(\text{Spec } k) \rightarrow (\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X)$ of simplicial sets such that $x^* \circ p^*$ is the identity on $(\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(\text{Spec } k)$. This shows that the map x^* is a Kan fibration. Since the Kan–Quillen model structure on simplicial sets is right proper, this implies that $(\text{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X)$ is the homotopy fiber of x^* .

Let $\mathcal{Y} \rightarrow R_{\mathbb{A}^1} \mathcal{Y}$ be a fibrant replacement of \mathcal{Y} in the \mathbb{A}^1 -model structure on $\text{sPre}_*(\text{Sm}_k)$. After forgetting the basepoint, $R_{\mathbb{A}^1} \mathcal{Y}$ is fibrant in the \mathbb{A}^1 -model structure on the category $\text{sPre}(\text{Sm}_k)$ of unpointed motivic spaces. Since the singular functor preserves \mathbb{A}^1 -fibrations, $\text{Sing}^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y}$ is fibrant and we may assume $\tilde{\mathcal{Y}} = \text{Sing}^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y}$. Moreover,

we get the following commutative diagram of simplicial sets which, by the above argument, is a morphism of homotopy fiber sequences for every pointed smooth k -scheme (X, x) .

$$\begin{array}{ccc}
 (\mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X) & \longrightarrow & (\mathrm{Sing}_*^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y})(X) \\
 \downarrow & & \downarrow \\
 (\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y})(X) & \longrightarrow & (\mathrm{Sing}^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y})(X) \\
 \downarrow x^* & & \downarrow x^* \\
 (\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y})(\mathrm{Spec} k) & \longrightarrow & (\mathrm{Sing}^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y})(\mathrm{Spec} k)
 \end{array} \tag{129}$$

Now we assume that the underlying simplicial presheaf of \mathcal{Y} is \mathbb{A}^1 -naive and that X is affine. Since \mathcal{Y} is \mathbb{A}^1 -naive and both X and $\mathrm{Spec} k$ are affine, the horizontal maps in the middle and at the bottom are weak equivalences of simplicial sets. Thus, since Diagram (129) is a morphism of homotopy fiber sequences, the top horizontal map is a weak equivalence of simplicial sets. Hence it induces a bijection on π_0 . Since $\pi_0((\mathrm{Sing}_*^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y})(X)) = [X, \mathcal{Y}]^{\mathbb{A}^1}$, this proves the assertion. \square

Lemma 130. *The smooth k -schemes \mathcal{J} and \mathbb{P}^1 are \mathbb{A}^1 -naive.*

Proof. Let Q_2 be the smooth affine quadric over \mathbb{Z} defined by $xy = z(z + 1)$. By [6, Theorem 4.2.2], Q_2 is \mathbb{A}^1 -naive. The scheme endomorphism of $\mathrm{Spec} \mathbb{Z}[x, y, z]$ given by the ring homomorphism defined by sending $x \mapsto z, y \mapsto -y, z \mapsto -x$ induces an isomorphism $Q_2 \cong \mathcal{J}$. Hence \mathcal{J} is \mathbb{A}^1 -naive. By [6, Lemma 4.2.4] an affine torsor bundle over a base space is \mathbb{A}^1 -naive if and only if the base space is \mathbb{A}^1 -naive. Since \mathcal{J} is \mathbb{A}^1 -naive and an affine torsor bundle over \mathbb{P}^1 , it follows that \mathbb{P}^1 is \mathbb{A}^1 -naive. \square

Proposition 131. *The canonical map $\nu: [X, \mathbb{P}^1]^{\mathbb{N}} \xrightarrow{\cong} [X, \mathbb{P}^1]^{\mathbb{A}^1}$ is a bijection for every affine pointed finitely presented smooth k -scheme X .*

Proof. The proposition follows from Remark 126 and Proposition 128, since Lemma 130 shows \mathbb{P}^1 is \mathbb{A}^1 -naive. \square

For $X = \mathcal{J}$, the Proposition 131 yields the comparison of the sets $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{N}}$ and $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ of pointed homotopy classes that we wanted.

Appendix B. Facts about the resultant

In Sections 2.4 and 5.1 we used the following facts about the resultant for which we now provide references or proofs. All results can be found or deduced from [10, Chapter IV].

Throughout this section we let S be an integral domain and let $A = a_nX^n + a_{n-1}X^{n-1} + \dots + a_0$ and $B = b_nX^n + b_{n-1}X^{n-1} + \dots + b_0$ be polynomials over S in the indeterminate X .

Definition 132. The Sylvester matrix $\text{Syl}(A, B)$ is defined as follows

$$\begin{pmatrix} a_n & 0 & \dots & 0 & b_n & 0 & \dots & 0 \\ a_{n-1} & a_n & & \vdots & b_{n-1} & b_n & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\ a_1 & a_2 & \dots & a_n & b_1 & b_2 & \dots & b_n \\ a_0 & a_1 & \dots & a_{n-1} & b_0 & b_1 & \dots & b_{n-1} \\ 0 & a_0 & \dots & a_{n-2} & 0 & b_0 & \dots & b_{n-2} \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_0 & 0 & 0 & \dots & b_0 \end{pmatrix}.$$

and we define $\text{res}(A, B) := \det(\text{Syl}(A, B))$.

Lemma 133 (Remark 4 on page IV.76 in [10]). Assume $\text{res}(A, B) \in S^\times$. Then there exist polynomials $U, V \in S[X]$ such that $AU + BV = 1$.

Lemma 134 (Remark 1 on page IV.76 in [10]). Let $\tilde{A} = a_n + a_{n-1}X + \dots + a_0X^n$ and $\tilde{B} = b_n + b_{n-1}X + \dots + b_0X^n$ be the reversed polynomials of A and B . If $\text{res}(A, B) \in S^\times$, then $\text{res}(\tilde{A}, \tilde{B}) = (-1)^n \text{res}(A, B) \in S^\times$.

Lemma 135 (Remark 5 on page IV.77 in [10]). Assume $\text{res}(A, B) \in S^\times$. Let $C \in S[X]$ be a polynomial such that $\deg(A) \geq \deg(BC)$. Then we have $\text{res}(A + BC, B) = \text{res}(A, B)$.

Lemma 136. Assume $\text{res}(A, B) \in S^\times$ and that A is monic. Then we have

$$\text{res}\left(AX - \frac{1}{u}B, uA\right) = -u \cdot \text{res}(A, B) \text{ for all } u \in S^\times.$$

Proof. The strategy of the proof is as follows: We determine the Sylvester matrix for the pair $(AX - \frac{1}{u}B, uA)$ and will then use elementary row and column operations to confirm that it has the determinant claimed.

First note that $\text{res}(AX - \frac{1}{u}B, uA) = u^{n+1} \text{res}(AX - \frac{1}{u}B, A)$. Let $A = \sum_{i=0}^n a_iX^i$, and $B = \sum_{i=0}^n b_iX^i$. Let $c_i = a_{i-1} - \frac{1}{u}b_i$ for $0 \leq i \leq n+1$ and set $a_{-1} = b_{n+1} = 0$. Then $AX - \frac{1}{u}B = \sum_{i=0}^{n+1} c_iX^i$. The Sylvester matrix for the pair $(AX - \frac{1}{u}B, A)$ is

$$\begin{pmatrix} c_{n+1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ c_n & c_{n+1} & & \vdots & a_n & 0 & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \\ c_1 & c_2 & \dots & c_{n+1} & a_1 & \dots & a_n & 0 \\ c_0 & c_1 & \dots & c_n & a_0 & \dots & a_{n-1} & a_n \\ 0 & c_0 & \dots & c_{n-1} & 0 & a_0 & \dots & a_{n-1} \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & c_0 & 0 & 0 & \dots & a_0 \end{pmatrix}.$$

Since $c_{n+1} = 1$, we can remove the first row and first column to obtain a submatrix with the same determinant, namely

$$\begin{pmatrix} c_{n+1} & 0 & \dots & 0 & a_n & 0 & \dots & 0 & 0 \\ c_n & c_{n+1} & & \vdots & a_{n-1} & a_n & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \\ c_2 & c_3 & \dots & c_{n+1} & a_1 & a_2 & \dots & a_n & 0 \\ c_1 & c_2 & \dots & c_n & a_0 & a_1 & \dots & a_{n-1} & a_n \\ c_0 & c_1 & \dots & c_{n-1} & 0 & a_0 & \dots & a_{n-2} & a_{n-1} \\ \vdots & c_0 & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & c_1 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & \dots & c_0 & 0 & 0 & \dots & 0 & a_0 \end{pmatrix}.$$

Subtracting column 1 from column $n + 1$ yields

$$\begin{pmatrix} c_{n+1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ c_n & c_{n+1} & & \vdots & \frac{1}{u}b_n & a_n & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & & \\ c_2 & c_3 & \dots & c_{n+1} & \frac{1}{u}b_2 & a_2 & \dots & a_n & 0 \\ c_1 & c_2 & \dots & c_n & \frac{1}{u}b_1 & a_1 & \dots & a_{n-1} & a_n \\ c_0 & c_1 & \dots & c_{n-1} & \frac{1}{u}b_0 & a_0 & \dots & a_{n-2} & a_{n-1} \\ \vdots & c_0 & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & c_1 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & \dots & c_0 & 0 & 0 & \dots & 0 & a_0 \end{pmatrix}.$$

Once again, the determinant of this matrix is the same as that of the submatrix where the first row and first and column removed. We remove them and obtain

$$\begin{pmatrix} c_{n+1} & 0 & \dots & 0 & \frac{1}{u}b_n & a_n & 0 & \dots & 0 & 0 \\ c_n & c_{n+1} & & \vdots & \frac{1}{u}b_{n-1} & a_{n-1} & a_n & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \\ c_3 & c_4 & \dots & c_{n+1} & \frac{1}{u}b_2 & a_2 & a_3 & \dots & a_n & 0 \\ c_2 & c_3 & \dots & c_n & \frac{1}{u}b_1 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ c_1 & c_2 & \dots & c_{n-1} & \frac{1}{u}b_0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ c_0 & c_1 & \dots & c_{n-2} & 0 & 0 & a_0 & \dots & a_{n-3} & a_{n-2} \\ \vdots & c_0 & \ddots & \vdots & \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & c_1 & 0 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & \dots & c_0 & 0 & 0 & 0 & \dots & 0 & a_0 \end{pmatrix}.$$

We subtract column $n + i$ from column i for each $i < n$, and the result is that at each entry c_i , we get instead $c_i - a_{i-1} = -\frac{1}{u}b_i$.

$$\begin{pmatrix} -\frac{1}{u}b_{n+1} & 0 & \dots & 0 & \frac{1}{u}b_n & a_n & 0 & \dots & 0 & 0 \\ -\frac{1}{u}b_n & -\frac{1}{u}b_{n+1} & & \vdots & \frac{1}{u}b_{n-1} & a_{n-1} & a_n & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \\ -\frac{1}{u}b_3 & -\frac{1}{u}b_4 & \dots & -\frac{1}{u}b_{n+1} & \frac{1}{u}b_2 & a_2 & a_3 & \dots & a_n & 0 \\ -\frac{1}{u}b_2 & -\frac{1}{u}b_3 & \dots & -\frac{1}{u}b_n & \frac{1}{u}b_1 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -\frac{1}{u}b_1 & -\frac{1}{u}b_2 & \dots & -\frac{1}{u}b_{n-1} & \frac{1}{u}b_0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ -\frac{1}{u}b_0 & -\frac{1}{u}b_1 & \dots & -\frac{1}{u}b_{n-2} & 0 & 0 & a_0 & \dots & a_{n-3} & a_{n-2} \\ \vdots & -\frac{1}{u}b_0 & \ddots & \vdots & \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & -\frac{1}{u}b_1 & 0 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & \dots & -\frac{1}{u}b_0 & 0 & 0 & 0 & \dots & 0 & a_0 \end{pmatrix}$$

Then multiplying the first n columns by u and applying a cyclic permutation of the n first columns yields the Sylvester matrix of the pair (B, A) . The sign of the permutation is $(-1)^{n-1}$. Interchanging column i with $n + i$ for all $i \leq n$ yields (A, B) , and this needed another permutation of sign $(-1)^{n-1}$, so the signs cancel out. \square

Appendix C. Testing compatibility via real realization and signatures

Now we provide the additional evidence for Conjecture 82 and a positive answer to Question 83 referred to in Section 4.

We assume that k is a subfield of \mathbb{R} . Let $\mathcal{H}_*(k)$ denote the homotopy category of pointed smooth k -schemes and let \mathcal{H}_* be the homotopy category of pointed topological spaces. By [23] sending a smooth k -scheme X to the topological space $X(\mathbb{R})$ equipped

with its usual structure of a smooth manifold extends to a functor $\mathfrak{R}: \mathcal{H}_*(k) \rightarrow \mathcal{H}_*$, see also [4, page 14] and [13, Section 5.3].

For a smooth map f between oriented compact smooth manifolds of the same dimension, let $\deg^{\text{top}}(f) \in \mathbb{Z}$ denote the topological Brouwer degree of f . In [21] Morel describes the analog of the topological degree map in \mathbb{A}^1 -homotopy theory. For endomorphisms of \mathbb{P}^1 it defines a homomorphism

$$\deg^{\mathbb{A}^1}: [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \rightarrow \text{GW}(k).$$

Let $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a morphism. Since we assume that k is a subfield of \mathbb{R} , we can form the real realization $\mathfrak{R}(f): \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}^1(\mathbb{R})$. Following Morel, the signature, denoted sgn , of the quadratic form given by the \mathbb{A}^1 -Brouwer degree of f equals the topological Brouwer degree of $\mathfrak{R}(f)$, i.e.,

$$\text{sgn} \left(\deg^{\mathbb{A}^1}(f) \right) = \deg^{\text{top}}(\mathfrak{R}(f)). \tag{137}$$

We note that, in some form, this was also shown by Eisenbud, Levine, and Teissier in [14, Theorem 1.2] for the local degree of maps between real affine spaces. The latter approach has been incorporated into the motivic theory by Kass and Wickelgren [17].

The motivic Brouwer degree map $\deg^{\mathbb{A}^1}$ is a homomorphism for the conventional group structure $\oplus^{\mathbb{A}^1}$ on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$, and the signature is additive. Hence, for morphisms $f, g: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and their sum $f \oplus^{\mathbb{A}^1} g$ in $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$, Equation (137) implies

$$\begin{aligned} \deg^{\text{top}} \left(\mathfrak{R} \left(f \oplus^{\mathbb{A}^1} g \right) \right) &= \text{sgn} \left(\deg^{\mathbb{A}^1} \left(f \oplus^{\mathbb{A}^1} g \right) \right) \\ &= \deg^{\text{top}}(\mathfrak{R}(f)) + \deg^{\text{top}}(\mathfrak{R}(g)). \end{aligned} \tag{138}$$

We will now use this fact to test the compatibility of the action of Definition 73 and thereby of Definition 77 with the conventional group structure in the following way.

The real points $\mathcal{J}(\mathbb{R})$ of \mathcal{J} form a surface in \mathbb{R}^3 given by the equation $x(1-x) - yz = 0$. The intersection with the plane defined by $y = z$ is the circle given by the set of points satisfying $x(1-x) - y^2 = 0$. Its center is the point $(1/2, 0, 0) \in \mathbb{R}^3$. We parameterize this circle via the map $\gamma: \mathbb{S}^1 \rightarrow \mathcal{J}(\mathbb{R})$ given by

$$\gamma: \theta \mapsto \left(1/2 + \cos(\theta)/2, \sin(\theta)/2, \sin(\theta)/2 \right).$$

The real realization of \mathbb{P}^1 is the topological real projective line $\mathbb{R}\mathbb{P}^1$. Hence, for a morphism $f: \mathcal{J} \rightarrow \mathbb{P}^1$, we may form the composition $\mathfrak{R}(f) \circ \gamma$ which is a smooth map $\mathbb{S}^1 \rightarrow \mathbb{R}\mathbb{P}^1$. We can then apply the topological Brouwer degree to the composition $\mathfrak{R}(f) \circ \gamma$. Since the real realization of a naive homotopy induces a homotopy of maps between topological spaces, this induces a well-defined map

$$\begin{aligned} \text{deg}^{\text{top}}(\mathfrak{R}(-) \circ \gamma) : [\mathcal{J}, \mathbb{P}^1]^N &\longrightarrow \mathbb{Z} \\ f &\longmapsto \text{deg}^{\text{top}}(\mathfrak{R}(f) \circ \gamma). \end{aligned}$$

Lemma 139. *The following diagram commutes.*

$$\begin{array}{ccc} [\mathcal{J}, \mathbb{P}^1]^N & \xrightarrow{\text{deg}^{\text{top}}(\mathfrak{R}(-) \circ \gamma)} & \mathbb{Z} \\ \downarrow \nu & \searrow \text{deg}^{\text{top}}(\mathfrak{R}(-) \circ \gamma) & \\ \xi \quad [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1} & \xrightarrow{\text{deg}^{\text{top}}(\mathfrak{R}(-) \circ \gamma)} & \mathbb{Z} \\ \uparrow \pi^* \quad \downarrow (\pi^*)^{-1} & \nearrow \text{deg}^{\text{top}}(\mathfrak{R}(-)) & \\ [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} & & \end{array} \tag{140}$$

Proof. To prove the assertion it suffices to show that both parts of the diagram commute. The functor \mathfrak{R} commutes with the canonical map $\nu : [\mathcal{J}, \mathbb{P}^1]^N \rightarrow [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$. This implies that the upper part commutes. We verify in Example 146 that for $\pi : \mathcal{J} \rightarrow \mathbb{P}^1$ the composite map $\mathfrak{R}(\pi) \circ \gamma : \mathbb{S}^1 \rightarrow \mathbb{R}\mathbb{P}^1$ is an orientation preserving diffeomorphism. Now let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a morphism. Since the composition with $\mathfrak{R}(\pi) \circ \gamma$ preserves degrees, we obtain the identity

$$\text{deg}^{\text{top}}(\mathfrak{R}(f \circ \pi) \circ \gamma) = \text{deg}^{\text{top}}(\mathfrak{R}(f) \circ \mathfrak{R}(\pi) \circ \gamma) = \text{deg}^{\text{top}}(\mathfrak{R}(f)).$$

This implies that the lower part of Diagram (140) commutes and finishes the proof. \square

This implies the following necessary condition for the compatibility of the operations \oplus and $\oplus^{\mathbb{A}^1}$:

Proposition 141. *Assume that ξ is a group homomorphism. Then we have*

$$\text{deg}^{\text{top}}(\mathfrak{R}(\xi(f \oplus g))) = \text{deg}^{\text{top}}(\mathfrak{R}(f) \circ \gamma) + \text{deg}^{\text{top}}(\mathfrak{R}(g) \circ \gamma).$$

Proof. The assumption that ξ is a group homomorphism implies

$$\text{deg}^{\text{top}}(\mathfrak{R}(\xi(f \oplus g))) = \text{deg}^{\text{top}}(\mathfrak{R}(\xi(f) \oplus^{\mathbb{A}^1} \xi(g))).$$

Identity (138) implies

$$\text{deg}^{\text{top}}(\mathfrak{R}(\xi(f) \oplus^{\mathbb{A}^1} \xi(g))) = \text{deg}^{\text{top}}(\mathfrak{R}(\xi(f))) + \text{deg}^{\text{top}}(\mathfrak{R}(\xi(g))).$$

Commutativity of Diagram (140) implies

$$\text{deg}^{\text{top}}(\mathfrak{R}(\xi(f))) + \text{deg}^{\text{top}}(\mathfrak{R}(\xi(g))) = \text{deg}^{\text{top}}(\mathfrak{R}(f) \circ \gamma) + \text{deg}^{\text{top}}(\mathfrak{R}(g) \circ \gamma).$$

Putting these identities together yields the assertion. \square

As a special case, we get the following necessary condition for the compatibility of \oplus with the conventional group structure:

Corollary 142. *Let $F: \mathcal{J} \rightarrow \mathbb{P}^1$ be a pointed morphism. Then, if ξ is a group homomorphism, we must have*

$$\text{deg}^{\text{top}}(\mathfrak{R}(F \oplus \pi) \circ \gamma) = \text{deg}^{\text{top}}(\mathfrak{R}(F) \circ \gamma) + 1.$$

In the following section we will apply Corollary 142 in a concrete case in Example 147.

Moreover, we exclude a potential alternative to the operation \oplus of Definition 73 in Example 150.

Remark 143. Let $F: \mathcal{J} \rightarrow \mathbb{P}^1$ again be a pointed morphism. Assume that Question 83 has a positive answer, i.e., assume that $\tilde{\pi}$ is naively homotopic to $-\pi$. Then Proposition 141 shows that, if ξ is a group homomorphism, then we must expect to get

$$\text{deg}^{\text{top}}(\mathfrak{R}(F \oplus \tilde{\pi}) \circ \gamma) = \text{deg}^{\text{top}}(\mathfrak{R}(F) \circ \gamma) - 1. \tag{144}$$

Note that, since we do not know whether $\tilde{\pi}$ is naively homotopic to $-\pi$, Equation (144) may fail to hold for some F even though ξ is a group homomorphism.

However, we confirm Formula (144) in a concrete case in Example 149.

We will now compute the topological degrees and thereby the signatures of several maps and apply the previous observations. For the following computations we will often identify the ring R with the ring $k[x, y, z]/(x(1-x) - yz)$ where it is convenient.

Example 145. Consider the morphism $g_{1,-1}: \mathcal{J} \rightarrow \mathbb{P}^1$ defined by the unimodular row $(2x - 1, 2y)$. Its real realization is the map $\mathfrak{R}(g_{1,-1}): \mathfrak{R}(\mathcal{J}) \rightarrow \mathfrak{R}(\mathbb{P}^1)$ defined by

$$\mathfrak{R}(g_{1,-1}): (x, y, z) \mapsto [2x - 1 : 2y].$$

Precomposing with γ gives

$$\mathfrak{R}(g_{1,-1}) \circ \gamma: \theta \mapsto [\cos(\theta) : \sin(\theta)],$$

which is the usual double cover of $\mathbb{R}\mathbb{P}^1$ by \mathbb{S}^1 and has topological Brouwer degree 2.

As explained in Section 2.4, a morphism $f: \mathcal{J} \rightarrow \mathbb{P}^1$ may be described by gluing together partially defined maps on open subsets. In the following examples we will define a morphism $\mathfrak{R}(f) \circ \gamma: \mathbb{S}^1 \rightarrow \mathbb{R}\mathbb{P}^1$ by gluing $\mathfrak{R}(f|_{D(x)}) \circ \gamma: \gamma^{-1}(\mathfrak{R}(D(x))) \rightarrow \mathbb{R}\mathbb{P}^1$ and $\mathfrak{R}(f|_{D(1-x)}) \circ \gamma: \gamma^{-1}(\mathfrak{R}(D(1-x))) \rightarrow \mathbb{R}\mathbb{P}^1$ on their overlaps in the respective domains.

Example 146. The real realization of $\pi: \mathcal{J} \rightarrow \mathbb{P}^1$ is defined on $\Re(D(x))$ by

$$\Re(\pi|_{D(x)}): (x, y, z) \mapsto [x : y],$$

and on $\Re(D(1-x))$ by

$$\Re(\pi|_{D(1-x)}): (x, y, z) \mapsto [z : 1-x].$$

Precomposing with γ gives

$$\begin{aligned} \Re(\pi|_{D(x)} \circ \gamma): \theta &\mapsto [1/2 + \cos(\theta)/2 : \sin(\theta)/2], \\ \Re(\pi|_{D(1-x)} \circ \gamma): \theta &\mapsto [\sin(\theta)/2 : 1/2 - \cos(\theta)/2], \end{aligned}$$

which glue together to give a map of degree 1:

$$(\Re(\pi) \circ \gamma)(\theta) = \begin{cases} [1 + \cos(\theta) : \sin(\theta)] & \theta \neq \pi, \\ [0 : 1] & \theta = \pi. \end{cases}$$

This shows that $\Re(\pi) \circ \gamma$ is an orientation preserving diffeomorphism and has topological Brouwer degree 1.

In the following example we test the necessary condition of Corollary 142 in a concrete case.

Example 147. Recall that the unimodular row $g_{1,-1} = (2x - 1, 2y)$ can be augmented to the following matrix with determinant 1:

$$m_{1,-1} = \begin{pmatrix} 2x - 1 & -2z \\ 2y & 2x - 1 \end{pmatrix}.$$

The group action of Definition 73 yields the map $F := g_{1,-1} \oplus \pi = (2x - 1, -2z : 2y, 2x - 1)_1$.

Taking real realization and precomposing with γ yields the map $\Re(F) \circ \gamma: \mathbb{S}^1 \rightarrow \mathbb{RP}^1$ given by

$$(\Re(F) \circ \gamma)(\theta) = \begin{cases} [\cos(\theta) + \cos(2\theta) : \sin(\theta) + \sin(2\theta)] & \theta \neq \pi, \\ [0 : 1] & \theta = \pi. \end{cases}$$

The topological degree of this map is 3.

Hence our computation confirms that

$$\deg^{\text{top}}(\Re(F) \circ \gamma) = 3 = 2 + 1 = \deg^{\text{top}}(\Re(g_{-1,1}) \circ \gamma) + \deg^{\text{top}}(\Re(\pi) \circ \gamma),$$

as required for the compatibility of \oplus with $\oplus^{\mathbb{A}^1}$.

The next example confirms that the signature of the motivic Brouwer degree of $\tilde{\pi}$ has the value -1 as expected if Question 83 has a positive answer.

Example 148. The real realization of the morphism $\tilde{\pi} = (1, 0 : 0, -1)_{-1} : \mathcal{J} \rightarrow \mathbb{P}^1$ is defined on $\Re(D(x))$ and $\Re(D(1-x))$ respectively by $\Re(D(x))$ by

$$\begin{aligned} \Re(\tilde{\pi}|_{D(x)}): (x, y, z) &\mapsto [x : -z], \\ \Re(\tilde{\pi}|_{D(1-x)}): (x, y, z) &\mapsto [y : -1 + x]. \end{aligned}$$

Precomposing with γ gives

$$\begin{aligned} \Re(\tilde{\pi}|_{D(x)}) \circ \gamma: \theta &\mapsto [1/2 + \cos(\theta)/2 : -\sin(\theta)/2], \\ \Re(\tilde{\pi}|_{D(1-x)}) \circ \gamma: \theta &\mapsto [\sin(\theta)/2 : -1/2 + \cos(\theta)/2], \end{aligned}$$

which glue together to give a map of topological Brouwer degree -1 .

Now we confirm that Identity (144) of Remark 143 does hold in an example.

Example 149. Consider the unimodular row $F = (2x - 1, 2z)$ which can be augmented to the following matrix

$$M = \begin{pmatrix} 2x - 1 & -2y \\ 2z & 2x - 1 \end{pmatrix}$$

with determinant 1. Note that F is homotopic to $g_{1,-1}$ by Lemma 62. We let F act on $\tilde{\pi}$ via the action of Definition 73. This yields the map $L := F \oplus \tilde{\pi} = (2x - 1, 2y : 2z, -2x + 1)_{-1}$. Precomposing its real realization with γ yields the same map as in Example 146 where we showed it has topological Brouwer degree 1.

Hence our computation confirms

$$\deg^{\text{top}}(\Re(L) \circ \gamma) = 1 = 2 - 1 = \deg^{\text{top}}(\Re(F) \circ \gamma) + \deg^{\text{top}}(\Re(\tilde{\pi}) \circ \gamma),$$

as required in Remark 143.

Example 150. Consider now an alternative action \boxplus of $[\mathcal{J}, \mathbb{P}^1]_0^N$ on $[\mathcal{J}, \mathbb{P}^1]^N$ defined as follows. For $[(A, B)] \in [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^N$ and $[s_0, s_1] \in [\mathcal{J}, \mathbb{P}^1]_n^N$, extend the unimodular row (A, B) to a matrix M in $\text{SL}_2(R)$ and define

$$[(A, B)] \boxplus [s_0, s_1] := [M^T \cdot (s_0, s_1)^T].$$

Again we look at the unimodular row $F = (A, B) = (2x - 1, 2z)$ and the matrix M of Example 149. The action \boxplus of F on π yields the morphism $H := F \boxplus \pi = (2x - 1, 2z :$

$-2y, 2x - 1)_1 = (1, 0 : 0, -1)_1$. Taking real realization and precomposing with γ yields the map $\Re(H) \circ \gamma: \mathbb{S}^1 \rightarrow \mathbb{RP}^1$ given by

$$(\Re(H) \circ \gamma)(\theta) = \begin{cases} [1 + \cos(\theta) : -\sin(\theta)] & \theta \neq \pi, \\ [0 : 1] & \theta = \pi. \end{cases}$$

This map has topological Brouwer degree -1 . Hence our computation shows

$$\deg^{\text{top}}(\Re(F \boxplus \pi) \circ \gamma) = -1 \neq 3 = \deg^{\text{top}}(\Re(F) \circ \gamma) + \deg^{\text{top}}(\Re(\pi) \circ \gamma).$$

Thus, by the analogous statement of Corollary 142 for \boxplus , we see that \boxplus cannot be used to define an operation compatible with the conventional group structure.

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