HOMOGENIZATION OF NONLOCAL SPECTRAL PROBLEMS[‡]

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Abstract. We consider a spectral problem for convolution type operators in environments with locally periodic microstructure, and study the asymptotic behavior of the bottom of the spectrum. We show that the bottom point of the spectrum converges as the microstructure period tends to zero, and identify the limit in terms of an additive eigenvalue problem for effective Hamilton-Jacobi equation. In the periodic case, we establish a more accurate two-term asymptotic formula.

Key words. Spectral problems, homogenization, nonlocal operators

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1. Introduction.

This work deals with the homogenization of spectral problems for nonlocal convolution type operators in environments with locally periodic microstructure. In a bounded C^1-domain $\Omega \subset \mathbb{R}^d$ we consider the spectral problem

$$\mathcal{L}_\varepsilon \rho_\varepsilon = \lambda_\varepsilon \rho_\varepsilon(x) \quad \text{in } \Omega \tag{1.1}$$

for the operator

$$\mathcal{L}_\varepsilon \rho = -\frac{1}{\varepsilon^d}\int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big)\kappa\Big(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\rho(y)dy + a\Big(x,\frac{x}{\varepsilon}\Big)\rho(x), \tag{1.2}$$

with a small parameter $\varepsilon > 0$ that characterizes the microscopic length scale of the medium. Under natural positiveness and periodicity conditions as well as fast decay of J at infinity we study the bottom of the spectrum of this problem. The main focus of this work is on the asymptotic behavior as $\varepsilon \to 0$ of the point of the spectrum with the smallest real part.

Many models in mathematical biology and population dynamics take into account nonlocal interactions in the studied systems. These interactions are described by convolution-type integral operators with integrable kernels. More specifically, the simplest nonlocal model of population dynamics reads (see, e.g., [14]), [20])

$$\partial_t \rho(x,t) - \int_\Omega J(x-y)\rho(y,t)dy + \int_{\mathbb{R}^d} J(y-x)dy\,\rho(x,t) = 0 \quad \text{in } \Omega, \quad \rho = 0 \quad \text{on } \mathbb{R}^d \setminus \Omega,$$

where ρ denotes population density, $J(x-y) \geq 0$ is a dispersal kernel that describes the rate of jumps from the location y to the location x and the above equation defines nonlocal transport in Ω, while the Dirichlet condition $\rho = 0$ (imposed everywhere on $\mathbb{R}^d \setminus \Omega$) represents the case of a hostile exterior domain. In the case when the growth of the population is taken into account the above equation is also supplemented by an additional KPP type (local) nonlinear term (see, e.g., [11], [13]). Then the large time behavior of $\rho(x,t)$ can be qualitatively characterized by linearizing the problem and

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studying the bottom part of the spectrum of the corresponding operator, in particular, an optimal persistence criterion is formulated in terms of the bottom point of the spectrum [2], [6], [8], [12], [16], [26].

To model nonlocal diffusion in strongly inhomogeneous media, in [24], [25] evolution problems with dispersal kernels of the form $J(x-y)\kappa(x,y)$ with integrable $J\geq 0$ and positive periodic κ were considered in the parabolic scaling $t\to t/\varepsilon^2$, $x\to x/\varepsilon$. It was shown in [24] that the asymptotic behavior of solutions is described by a local effective diffusion problem in the symmetric case (when $J(z)=J(-z)$ and $\kappa(x,y)=\kappa(y,x)$), while [25] revealed a large effective drift appearing in asymmetric case, and the corresponding homogenization result was established in rapidly moving coordinates. The approach in [25] was developed for problems stated in the whole space $\mathbb{R}^d$, and it fails to work in the case of a bounded domain because of the presence of large effective drift. To overcome this difficulty one can combine the study of the evolution problem with the spectral analysis of problem (1.1).

Peculiar features of the spectral problem (1.1) can be seen in the case of nonlocal diffusion operator in $\mathbb{R}^d$

$$\hat{\mathcal{L}}_\varepsilon\rho=-\frac{1}{\varepsilon^d}\int_{\mathbb{R}^d}J\Big(\frac{x-y}{\varepsilon}\Big)\kappa\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\,(\rho(y)-\rho(x))\,dy \tag{1.3}$$

with a periodic coefficient κ. One can formally assign to $\hat{\mathcal{L}}_\varepsilon$ the singularly perturbed differential operator $\hat{\mathcal{D}}_\varepsilon$ given by the two-term asymptotic formula $\hat{\mathcal{D}}_\varepsilon=-\varepsilon^2\mathrm{div}(A\nabla\cdot)+\varepsilon B\cdot\nabla$ with a constant matrix $A>0$ and a constant vector B. Indeed, as shown in [25] the semigroups $e^{-t\hat{\mathcal{L}}_\varepsilon/\varepsilon^2}$ and $e^{-t\hat{\mathcal{D}}_\varepsilon/\varepsilon^2}$ generated by operators $\frac{1}{\varepsilon^2}\hat{\mathcal{L}}_\varepsilon$ and $\frac{1}{\varepsilon^2}\hat{\mathcal{D}}_\varepsilon$ are close as $\varepsilon\to 0$ (uniformly on finite time intervals). Therefore, it can be expected that the asymptotic behavior of the bottom part of the spectrum of the operator $\mathcal{L}_\varepsilon$ is also somehow similar to that of singularly perturbed elliptic differential operators.

It should be noted that, in contrast with differential operators, the point of the spectrum of $\mathcal{L}_\varepsilon$ with the smallest real part need not be a principal eigenvalue, it might lie on the edge of the essential spectrum. However, we show that the limit of this point as $\varepsilon\to 0$ can be specified in terms of an additive eigenvalue for an effective Hamilton-Jacobi equation, like in the case of principal eigenvalues of singularly perturbed elliptic differential operators. Despite this similarity of the results, related to the fact that the operator $\mathcal{L}_\varepsilon$ is becoming more and more localized for small ε, many technical aspects in proofs in local and nonlocal cases are quite different.

The asymptotic behavior of the principal eigenpair of singularly perturbed local (differential) convection-diffusion operators with oscillating coefficients was studied in [23] (see also [22]), where the Cole–Hopf transformation $e^{-W_\varepsilon(x)/\varepsilon}$ of the first eigenfunction was used, yielding a perturbed Hamilton-Jacobi equation. The latter was naturally dealt with by means of the vanishing viscosity techniques. In the case of nonlocal operators the Cole–Hopf transformation does not lead to a perturbed Hamilton-Jacobi equation, and we rather exploit the monotonicity of the map $W_\varepsilon\mapsto e^{-W_\varepsilon/\varepsilon}$ to devise a version of perturbed test functions method [10] for nonlocal operators. The main difficulty however is in finding relevant uniform bounds for the function W_ε. In the case of local operators, appropriate tools are Bernstein's estimates and Harnack's inequality. Bernstein's method is developed for local operators and it is not clear how to adapt it in the nonlocal case, while the known results [7] on the Harnack type inequalities are established for dispersal kernels with a finite support and are not scaling invariant. We prove instead uniform estimates in Lemma 4.2 to establish the existence of half-relaxed limits of W_ε.

When the functions κ and a in (1.2) does not depend on the slow variables (periodic case) the solution of the effective Hamilton-Jacobi problem is a linear function. In this case, using a factorization trick inspired by asymptotic analysis of differential operators with periodic coefficients [4], we establish a more accurate two-term asymptotic formula for eigenvalues (if exist) in the bottom part of the spectrum. This, in particular, generalizes an asymptotic result of [3] obtained for symmetric operators with homogeneous dispersal kernels (remark however that in this special case, the analysis in [3] covers also unbounded domains that are outside of the scope of the present work).

It is known that operator (1.3) is the generator of a jump Markov process in $\mathbb{R}^d$. Therefore, one of the ways of obtaining homogenization results for problems involving operator (1.3) is based on probabilistic interpretation of solutions to these problems and on the limit theorems for the corresponding jump Markov processes. In particular, an alternative approach of studying the principal eigenpair of problem (1.1)–(1.2) could rely on a probabilistic interpretation of operator (1.2). Since this operator is the generator of a jump Markov process with birth and death, one can try to exploit the large deviation principle for this process to investigate the asymptotic properties of problem (1.1)–(1.2). In the case of operators defined in (1.3) and similar locally periodic operators the large deviation result for the mentioned jump processes was obtained in [21].

The paper is organized as follows. In Section 2 we state Theorem 2.1 describing the limit as $\varepsilon \to 0$ of the bottom point of the spectrum of the operator (1.2) in the general locally periodic case. Section 3 is devoted to establishing more precise asymptotics of eigenvalues in the case of periodic environments. Finally, Section 4 contains the proof of Theorem 2.1.

2. Problem setup. Convergence result for the bottom point of the spectrum. We begin with specifying assumptions on the functions J, κ and a appearing in the definition (1.2) of $\mathcal{L}_\varepsilon$. We assume that J satisfies

$$J \in C(\mathbb{R}^d),\ J(0) > 0 \quad \text{and} \quad 0 \le J(z) \le Ce^{-|z|^{1+\beta}}\ \forall z \in \mathbb{R}^d, \tag{2.1}$$

for some $C, \beta > 0$, besides,

$$\kappa > 0 \quad \text{and} \quad \kappa \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{T}^d \times \mathbb{T}^d), \quad a \in C(\overline{\Omega} \times \mathbb{T}^d), \tag{2.2}$$

where $\mathbb{T}^d$ denotes the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and we identify periodic functions in $\mathbb{R}^d$ with functions defined on $\mathbb{T}^d$. Thus coefficients κ and a in (1.2) are periodic functions of the fast variables $\xi = x/\varepsilon$ and $\eta = y/\varepsilon$, so that operator (1.2) correspond to a locally periodic environment.

We are interested in the asymptotic behavior as $\varepsilon \to 0$ of the following quantity introduced in [6],

$$\lambda_\varepsilon = \sup\Big\{\lambda \,\big|\, \exists v \in C(\overline{\Omega}),\, v > 0 \text{ such that } \mathcal{L}_\varepsilon v \ge \lambda v \text{ in } \Omega\Big\}. \tag{2.3}$$

It is known [18] (Theorem 2.2) that λ_ε belongs to the spectrum $\sigma(\mathcal{L}_\varepsilon)$ of the operator $\mathcal{L}_\varepsilon$ (considered in $L^2(\Omega)$ or $C(\overline{\Omega})$) and $\lambda_\varepsilon = \inf\{\mathrm{Re}\lambda \,|\, \lambda \in \sigma(\mathcal{L}_\varepsilon)\}$, i.e. λ_ε is the bottom point of the spectrum. Typically λ_ε is the principal eigenvalue of the operator $\mathcal{L}_\varepsilon$, i.e. an isolated simple eigenvalue (with minimal real part) whose corresponding eigenfunction can be chosen strictly positive, as in the case of elliptic differential operators. However if $\lambda_\varepsilon = \min_{\overline{\Omega}} a(x, \frac{x}{\varepsilon})$ rather than $\lambda_\varepsilon < \min_{\overline{\Omega}} a(x, \frac{x}{\varepsilon})$, the principal eigenvalue does not exist and λ_ε is the bottom point of the essential spectrum of $\mathcal{L}_\varepsilon$. In both cases the sign of λ_ε is crucial for stability of the corresponding evolution semigroup $e^{-\mathcal{L}_\varepsilon t}$ or for the maximum principle to hold, see Theorem 2.3 in [18].

To state the main result on the asymptotic behavior of λ_ε for locally periodic environments, introduce the following function

$$H(p,x) = \sup\Big\{ \lambda \,\Big|\, \exists \varphi \in C(\mathbb{T}^d),\ \varphi > 0,\ \text{such that} \\ -\int_{\mathbb{R}^d} J(z) e^{p\cdot z} \kappa(x,x,\xi,\xi - z)\varphi(\xi - z)dz + a(x,\xi)\varphi(\xi) \geq \lambda\varphi(\xi) \text{ on } \mathbb{T}^d \Big\}. \tag{2.4}$$

Theorem 2.1. *Suppose that J satisfies* (2.1) *and κ, a satisfy* (2.2). *Then*

$$\lambda_\varepsilon \to -\Lambda \quad \text{as } \varepsilon \to 0, \tag{2.5}$$

where Λ is a unique additive eigenvalue of the problem

$$-H(\nabla W(x), x) = \Lambda \quad \text{in } \Omega, \quad -H(\nabla W(x), x) \geq \Lambda \quad \text{on } \partial\Omega. \tag{2.6}$$

Both the equation and the boundary condition in (2.6) are understood in the viscosity sense, see, e.g., [9]. It follows from the definition (2.4) that function $H(p,x)$ is continuous, $H \in C(\mathbb{R}^d \times \overline{\Omega})$, concave in p and $H(p,x) \to -\infty$ uniformly in $x \in \overline{\Omega}$ as $|p| \to \infty$. Then (see, e.g., [5]) there is a unique Λ such that problem (2.6) has a continuous viscosity solution. Moreover, the additive eigenvalue Λ can be calculated by the following formula

$$\Lambda = \inf_{W \in C^1(\overline{\Omega})} \max_{x \in \overline{\Omega}} -H(\nabla W(x), x). \tag{2.7}$$

Yet another representation for Λ is given by minimization of action functional for the Lagrangian $L(q,x) = \max_{p\in\mathbb{R}^d}(q\cdot p + H(p,x))$,

$$\Lambda = -\inf\Big\{ \frac{1}{T}\int_0^T L(\dot\xi(t), \xi(t))dt \,\Big|\, T > 0, \xi \in W^{1,\infty}(0,T;\overline{\Omega}) \Big\}. \tag{2.8}$$

In the periodic case $H(p,x)$ is independent of x, $H(p,x) = H(p)$, and one can show that the additive eigenvalue Λ in this case is given by $\Lambda = -\max H(p)$ while solutions $W(x)$ of (2.6) are linear functions $W(x) = p\cdot x$ with p solving $H(p) = -\Lambda$. This suggests the representation (3.10) for eigenfunctions with the exponential factor $e^{-p\cdot x/\varepsilon}$. In this case however we find much more accurate asymptotic formulas for eigenvalues (see Theorem 3.1) and eigenfunctions (see Remark 3.7).

3. Periodic case. In this section we consider a particular case of problem (1.1) when functions κ and a do not depend on the slow variables x and y, so that the operator $\mathcal{L}_\varepsilon$ has the following form

$$\mathcal{L}_\varepsilon \rho(x) = -\frac{1}{\varepsilon^d}\int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big)\kappa\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\rho(y)dy + a\Big(\frac{x}{\varepsilon}\Big)\rho(x) \tag{3.1}$$

with

$$\kappa \in C(\mathbb{T}^d \times \mathbb{T}^d),\ \kappa > 0,\ a \in C(\mathbb{T}^d), \tag{3.2}$$

and J satisfying (2.1).

Let us introduce the notation

$$m = \min a(x), \quad M = \max a(x), \tag{3.3}$$

and for any $p \in \mathbb{R}^d$ define

$$\begin{aligned} H(p) = \sup\Big\{ \lambda \,\Big|\, & \exists \varphi \in C(\mathbb{T}^d),\ \varphi > 0, \text{ such that} \\ & -\int_{\mathbb{R}^d} J(\xi-\eta) e^{p\cdot(\xi-\eta)} \kappa(\xi,\eta)\varphi(\eta)d\eta + a(\xi)\varphi(\xi) \ge \lambda\varphi(\xi) \text{ on } \mathbb{T}^d \Big\}. \end{aligned} \tag{3.4}$$

It follows from (3.4) that $H(p)$ is a continuous concave function, taking finite values for all $p \in \mathbb{R}^d$ and such that $H(p) \to -\infty$ as $|p| \to \infty$. Also, by Theorem 2.2 in [18] one has $H(p) \le m$. Let p_0 be a maximum point of $H(p)$,

$$H(p_0) = \max H(p). \tag{3.5}$$

The asymptotic behavior of the bottom part of the spectrum of the operator $\mathcal{L}_\varepsilon$ in the periodic case is described in the following

THEOREM 3.1. *Assume that conditions* (2.1), (3.2) *are fulfilled. Let λ_ε be the point of the spectrum of $\mathcal{L}_\varepsilon$ with the minimal real part. Then λ_ε belongs to the essential spectrum of $\mathcal{L}_\varepsilon$ for sufficiently small ε in the case $H(p_0) = m$, or λ_ε is the principal eigenvalue of $\mathcal{L}_\varepsilon$ in the case $H(p_0) < m$, and*

(i) If $H(p_0) = m$ then $\lambda_\varepsilon = H(p_0)$ for sufficiently small ε;

(ii) If $H(p_0) < m$ then

$$\lambda_\varepsilon = H(p_0) + \Lambda_1\varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \tag{3.6}$$

where Λ_1 is the principal eigenvalue of the operator

$$\mathcal{L}_0 v = -\mathrm{div}\big(A\nabla v\big) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{3.7}$$

whose matrix of coefficients A has entries

$$A_{ij} = -\frac{1}{2}\partial^2_{p_i p_j} H(p_0). \tag{3.8}$$

Moreover, in the case $H(p_0) < m$ the operator $\mathcal{L}_\varepsilon$ has a large or infinite number of other eigenvalues $\lambda_\varepsilon^{(j)}$ for small ε. Assuming that eigenvalues of both $\mathcal{L}_\varepsilon$ and $\mathcal{L}_0$ are arranged by their increasing real parts (and repeated according to their multiplicities), we have

$$\lambda_\varepsilon^{(j)} = H(p_0) + \Lambda_j\varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \tag{3.9}$$

where Λ_j are eigenvalues of the operator $\mathcal{L}_0$.

Remark 3.2. The following example inspired by [6] shows that the case $H(p_0) = m$ and $H(p_0) < m$ do occur. Assume that $d \ge 3$, $\kappa = 1$ and consider $J(z)$ given by

$$J(z) = \frac{\mu e^{-|z|^2}}{\sum_{l\in\mathbb{Z}^d} e^{-|z+l|^2}},$$

where $\mu > 0$. Then $\int_{\mathbb{R}^d} J(x-y)\varphi(y)dy = \mu \int_{\mathbb{T}^d} \varphi(y)dy$ for every $\varphi \in C(\mathbb{T}^d)$. Let $a(x)$ be a smooth periodic function strictly positive in $\mathbb{R}^d \setminus \mathbb{Z}^d$ and such that $a(0) = 0$ and $a(x) > \tau|x|^2$ ($\tau > 0$) in a neighborhood of zero. For such J, κ and a, we have $m = 0$, while $H(0) = 0$ if μ is small and $H(0) < 0$ if μ is sufficiently large. Indeed, according to [18] it always holds that $H(0) \le 0$ and $H(0) < 0$ iff $\exists\, \varphi \in C(\mathbb{T}^d)$, $\varphi > 0$ satisfying $-\mu \int_{\mathbb{T}^d} \varphi(y)dy + a(x)\varphi = H(0)\varphi$, i.e. $\varphi = \frac{1}{a(x)-H(0)}$ (up to multiplication by a positive constant) and $\int_{\mathbb{T}^d} \frac{\mu}{a(y)-H(0)}dy = 1$. Thus $H(0) < 0$ iff $\int_{\mathbb{T}^d} \frac{\mu}{a(x)}dx > 1$.

3.1. Problem reduction. Similarly to spectral problems for differential operators with periodic oscillating coefficients [4], problem (1.1) can be transformed via a factorization trick to a form more convenient for the asymptotic analysis. First we set

$$\rho_\varepsilon(x) = e^{-p\cdot x/\varepsilon} u_\varepsilon(x) \quad \text{in } \Omega, \tag{3.10}$$

so that the new unknown $u_\varepsilon(x)$ satisfies

$$-\frac{1}{\varepsilon^d}\int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big) e^{\frac{1}{\varepsilon}p\cdot(x-y)}\kappa\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)u_\varepsilon(y)dy + a\Big(\frac{x}{\varepsilon}\Big)u_\varepsilon(x) = \lambda_\varepsilon u_\varepsilon(x) \quad \text{in } \Omega. \tag{3.11}$$

Then consider a periodic counterpart of (3.11) in the rescaled variables $\xi = x/\varepsilon$, $\eta = y/\varepsilon$,

$$-\int_{\mathbb{R}^d} J(\xi-\eta)e^{p\cdot(\xi-\eta)}\kappa(\xi,\eta)\varphi(\eta)d\eta + a(\xi)\varphi(\xi) = H(p)\varphi(\xi) \quad \text{in } \mathbb{T}^d. \tag{3.12}$$

Specifically, we are interested in the principal eigenvalue $H(p)$ (with the minimal real part), which, if exists, is real and simple, its corresponding eigenfunction is sign preserving and thus can be chosen strictly positive, $\varphi > 0$. It is known [18] that $H(p)$ given by (3.4) is always the principal eigenvalue of the problem (3.12) provided that $H(p) < m$ (otherwise $H(p) = m$, it lies at the bottom point of the essential spectrum and principal eigenvalue does not exist). In the case $H(p) < m$ the adjoint problem

$$-\int_{\mathbb{R}^d} J(\eta-\xi)e^{p\cdot(\eta-\xi)}\kappa(\eta,\xi)\varphi^*(\eta)d\eta + a(\xi)\varphi^*(\xi) = H(p)\varphi^*(\xi) \quad \text{in } \mathbb{T}^d \tag{3.13}$$

has the same principal eigenvalue $H(p)$ and there also is a positive eigenfunction φ^*.

Now we perform another change of the unknown

$$u_\varepsilon(x) = \varphi\Big(\frac{x}{\varepsilon}\Big)v_\varepsilon(x), \tag{3.14}$$

where φ is a positive solution of (3.12), and introduce an affine change of the spectral parameter

$$\mu_\varepsilon = \frac{1}{\varepsilon^2}\left(\lambda_\varepsilon - H(p)\right) \tag{3.15}$$

to transform (1.1) to the spectral problem

$$\tilde{\mathcal{L}}_\varepsilon v_\varepsilon = \mu_\varepsilon v_\varepsilon \quad \text{in } \Omega, \tag{3.16}$$

where

$$\tilde{\mathcal{L}}_\varepsilon v = -\frac{1}{\varepsilon^{d+2}}\int_\Omega K\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)v(y)dy + \frac{1}{\varepsilon^{d+2}}\int_{\mathbb{R}^d} K\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)dy\, v(x), \tag{3.17}$$

and

$$K(x,y) = \frac{1}{\varphi(x)} J(x-y) e^{p\cdot(x-y)} \kappa(x,y)\varphi(y). \tag{3.18}$$

In what follows we will also deal with the kernel

$$Q(x,y) = \varphi^*(x)\varphi(x)K(x,y), \tag{3.19}$$

by virtue of (3.12)–(3.13) this kernel satisfies the following important property

$$\int_{\mathbb{R}^d} Q(x,y)dy = \int_{\mathbb{R}^d} Q(y,x)dy \quad \forall x \in \mathbb{R}^d. \tag{3.20}$$

Since the operators in problems (3.12) and (3.13) analytically depend on p_i, and $H(p)$ is a simple isolated eigenvalue if $H(p) < m$, then by perturbation theory [17] $H(p)$ is an analytic function of p_i and eigenfunctions φ, φ^* can also be chosen analytic in p_i, $i = 1, \dots, d$.

PROPOSITION 3.3. *Let p_0 be the maximum point of $H(p)$, and assume that $H(p_0) < m$. Then the function $\chi_i^* := \partial_{p_i} log\varphi^*\big|_{p=p_0}$ satisfies*

$$\int_{\mathbb{R}^d} Q\Big(\xi+z,\xi\Big)\Big(z_i + \chi_i^*(\xi+z) - \chi_i^*(\xi)\Big)dz = 0 \quad \text{in } \mathbb{T}^d,\ i = 1,\dots,d, \tag{3.21}$$

and

$$\partial^2_{p_ip_j} H(p_0) = -\frac{1}{\int_{\mathbb{T}^d}\varphi(\xi)\varphi^*(\xi)d\xi}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q\Big(\xi+z,\xi\Big)\Big(z_iz_j + 2\chi_i^*(\xi+z)z_j\Big)dzd\xi. \tag{3.22}$$

Proof. To derive (3.21) we differentiate (3.13) with respect to p_i at $p = p_0$, then multiply by $\varphi(\xi)$ and change the variables in the integral by setting $z = \eta - \xi$. Similarly, (3.22) is obtained by taking second derivatives of (3.13) and integrating the result over $\mathbb{T}^d$ with the weight $\varphi(\xi)$. □

LEMMA 3.4. *Let p_0 be the maximum point of $H(p)$, and assume that $H(p_0) < m$. Then*

$$\partial^2_{p_ip_j} H(p_0) q_i q_j < 0 \quad \forall q \in \mathbb{R}^d \setminus \{0\}. \tag{3.23}$$

Hereafter we assume summation over repeated indices.

Proof. By (3.22) we have to show positive difiniteness of the matrix A^* with entries

$$A^*_{ij} = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q\Big(\xi+z,\xi\Big)\Big(\frac{1}{2}z_iz_j + \chi_i^*(\xi+z)z_j\Big)dzd\xi, \tag{3.24}$$

where $\chi_i^* \in L^2(\mathbb{T}^d)$ are solutions of problems (3.21). To this end for any $q \in \mathbb{R}^d$ we write, using (3.21),

$$\begin{aligned} 2A^*_{ij}q_iq_j = &\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\xi+z,\xi)\left(q_iz_iq_jz_j + 2q_i(\chi_i^*(\xi+z) - \chi_i^*(\xi))q_jz_j\right)dzd\xi \\ &- 2q_iq_j\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\eta,\xi)\chi_i^*(\xi)(\chi_j^*(\eta) - \chi_j^*(\xi))d\eta d\xi. \end{aligned}$$

Thanks to (3.20) we have

$$
\begin{aligned}
&-\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\eta,\xi)\left(\chi_i^*(\xi)(\chi_j^*(\eta)-\chi_j^*(\xi))+\chi_j^*(\xi)(\chi_i^*(\eta)-\chi_i^*(\xi))\right)d\eta d\xi = \\
&\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\eta,\xi)\chi_i^*(\xi)\chi_j^*(\xi)d\eta d\xi + \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\xi,\eta)\chi_i^*(\xi)\chi_j^*(\xi)d\eta d\xi \\
&\qquad -\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\eta,\xi)\left(\chi_i^*(\xi)\chi_j^*(\eta)+\chi_i^*(\eta)\chi_j^*(\xi)\right)d\eta d\xi,
\end{aligned}
\tag{3.25}
$$

also

$$
\begin{aligned}
\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\xi,\eta)\chi_i^*(\xi)\chi_j^*(\xi)d\eta d\xi &= \sum_{l\in\mathbb{Z}^d}\int_{\mathbb{T}^d\times\mathbb{T}^d} Q(\xi,\eta+l)\chi_i^*(\xi)\chi_j^*(\xi)d\eta d\xi \\
&= \sum_{l\in\mathbb{Z}^d}\int_{\mathbb{T}^d\times\mathbb{T}^d} Q(\xi-l,\eta)\chi_i^*(\xi-l)\chi_j^*(\xi-l)d\eta d\xi \\
&= \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\eta,\xi)\chi_i^*(\eta)\chi_j^*(\eta)d\eta d\xi.
\end{aligned}
$$

Therefore

$$
2A_{ij}^* q_i q_j = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(\xi+z,\xi)q_i(z_i+\chi_i^*(\xi+z)-\chi_i^*(\xi))q_j(z_j+\chi_j^*(\xi+z)-\chi_j^*(\xi))dz d\xi \geq 0.
$$

The inequality is strict unless $q=0$, otherwise $q_i\chi_i^*(x)$ is a linear function whose gradient equals $-q$ and hence $q_i\chi_i^*(x)$ cannot be periodic if $q\neq 0$. ∎

From now on we will assume that $p=p_0$, in particular, this assumption will always be tacitly made when we refer to (3.12)–(3.13) and (3.15)–(3.21).

3.2. Resolvent convergence. Assume that $H(p_0)<m$, and consider, for a given $f_\varepsilon\in L^2(\Omega)$ the following problem

$$
\tilde{\mathcal{L}}_\varepsilon v_\varepsilon + v_\varepsilon = f_\varepsilon \quad \text{in } \Omega, \qquad v_\varepsilon = 0 \quad \text{in } \mathbb{R}^d\setminus\Omega,
\tag{3.26}
$$

where $\tilde{\mathcal{L}}_\varepsilon$ is given by (3.17). Since $v_\varepsilon=0$ in $\mathbb{R}^d\setminus\Omega$ we can rewrite

$$
\tilde{\mathcal{L}}_\varepsilon v_\varepsilon = -\frac{1}{\varepsilon^{d+2}}\int_{\mathbb{R}^d} K\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\Big(v_\varepsilon(y)-v_\varepsilon(x)\Big)dy.
\tag{3.27}
$$

THEOREM 3.5. *There is a unique solution $v_\varepsilon(x)$ of the problem* (3.26) *in $L^2(\mathbb{R}^d)$ for any $f_\varepsilon\in L^2(\Omega)$. If $\|f_\varepsilon\|_{L^2(\Omega)}\leq C$ with a constant C independent of ε then the sequence of solutions v_ε contains a subsequence converging strongly in $L^2(\mathbb{R}^d)$ as $\varepsilon\to 0$. If additionally $f_\varepsilon\to f$ strongly in $L^2(\Omega)$ then the whole sequence of solutions v_ε converges to the unique solution of the problem*

$$
-A_{ij}\partial^2_{x_ix_j}v(x)+v(x)=f(x) \quad \textit{in } \Omega,
\tag{3.28}
$$

$$
v(x)=0 \qquad \textit{on } \partial\Omega,
\tag{3.29}
$$

extended by setting $v=0$ in $\mathbb{R}^d\setminus\Omega$.

Proof. It is convenient to extend $f_\varepsilon(x)$ by zero into $\mathbb{R}^d\setminus\Omega$. Observe that the Fredholm alternative applies to the problem (3.26) since the operator on the left hand side is represented as the sum of a compact operator and an invertible one. To show that there is a solution of (3.26) and to derive an a priori estimate multiply (3.26) by $\varphi(\frac{x}{\varepsilon})\varphi^*(\frac{x}{\varepsilon})v_\varepsilon(x)$ and integrate over $\mathbb{R}^d$. Using (3.20) we obtain

$$\begin{aligned}\frac{1}{2\varepsilon^{d+2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Q\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\,|v_\varepsilon(y)-v_\varepsilon(x)|^2\,dydx \\ +\int_{\mathbb{R}^d}\Big(|v_\varepsilon(x)|^2 - f_\varepsilon(x)v_\varepsilon(x)\Big)\,\varphi\Big(\frac{x}{\varepsilon}\Big)\varphi^*\Big(\frac{x}{\varepsilon}\Big)dx = 0\end{aligned} \tag{3.30}$$

It follows that problem (3.26) cannot have nonzero solution for $f_\varepsilon = 0$. Thus (3.26) has a unique solution and using the Cauchy-Schwartz inequality we get

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Q\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\,|v_\varepsilon(y)-v_\varepsilon(x)|^2\,dydx \le C\varepsilon^{d+2}, \quad \|v_\varepsilon\|_{L^2(\mathbb{R}^d)} \le C, \tag{3.31}$$

with a constant C independant of ε. Due to the fact that $Q(\xi,\eta) = \varphi^*(\xi)J(\xi-\eta)e^{p_0\cdot(\xi-\eta)}\varphi(\eta)$ and $J(0) > 0$, we then have

$$\int_{\mathbb{R}^d} dx \int_{|z|\le r_0\varepsilon} |v_\varepsilon(x+z)-v_\varepsilon(x)|^2 dz \le C\varepsilon^{d+2} \tag{3.32}$$

for some $r_0 > 0$ independent of ε.

Lemma 3.6. *Let $v_\varepsilon \in L^2(\mathbb{R}^d)$ be a sequence of functions satisfying* (3.32) *and such that $v_\varepsilon = 0$ in $\mathbb{R}^d \setminus \Omega$. Then, up to extracting a subsequence, functions v_ε converge strongly in $L^2(\mathbb{R}^d)$ to some limit v as $\varepsilon \to 0$. Moreover $v \in H^1(\mathbb{R}^d)$ and $v = 0$ in $\mathbb{R}^d \setminus \Omega$.*

Proof. Without loss of generality we can assume that $v_\varepsilon \in C_0^\infty(\mathbb{R}^d)$. By Fubini's theorem

$$\int_0^{r_0\varepsilon} dr \int_{\mathbb{R}^d} dx \int_{|z|=r} |v_\varepsilon(x+z)-v_\varepsilon(x)|^2 dS \le C\varepsilon^{d+2},$$

therefore there exists r_ε such that $r_0\varepsilon/2 \le r_\varepsilon \le r_0\varepsilon$ and

$$\int_{\mathbb{R}^d} dx \int_{|z|=r_\varepsilon} |v_\varepsilon(x+z)-v_\varepsilon(x)|^2 dS \le 2C\varepsilon^{d+1}/r_0. \tag{3.33}$$

Consider functions

$$\overline{v}_\varepsilon(x) = \frac{1}{|B_1|\,r_\varepsilon^d}\int_{|z|\le r_\varepsilon} v_\varepsilon(x+z)dz.$$

where $|B_1| = \frac{\Gamma(\frac{d}{2}+1)}{\pi^{d/2}}$ is the volume of the unit ball in $\mathbb{R}^d$. From (3.32) using Jensen's inequality we get

$$\int_{\mathbb{R}^d} |\overline{v}_\varepsilon(x) - v_\varepsilon(x)|^2 dx \le C\varepsilon^2. \tag{3.34}$$

Next observe that $\forall\ i = 1,\dots,d$,

$$\partial_{x_i}\overline{v}_\varepsilon(x) = \frac{1}{|B_1|r_\varepsilon^d}\int_{|z|=r_\varepsilon} v_\varepsilon(x+z)\nu_i dS = \frac{1}{2|B_1|r_\varepsilon^d}\int_{|z|=r_\varepsilon}(v_\varepsilon(x+z)-v_\varepsilon(x-z))\nu_i dS,$$

where $\nu_i = \nu_i(z) = z_i/|z|$ denotes the i-th component of the unite outward pointing normal to the $(d-1)$-sphere $|z| = r_\varepsilon$. Hence, using the Cauchy-Schwarz inequality we obtain

$$|\partial_{x_i}\overline{v}_\varepsilon(x)|^2 \le \frac{C}{\varepsilon^{d+1}}\int_{|z|=r_\varepsilon} |v_\varepsilon(x+z)-v_\varepsilon(x)|^2\,dS.$$

Thus, thanks to (3.33) we have

$$\int_{\mathbb{R}^d} |\nabla \overline{v}_\varepsilon(x)|^2\, dx \le C, \tag{3.35}$$

and since functions v_ε vanish in $\mathbb{R}^d \setminus \Omega$ it holds that $\overline{v}_\varepsilon = 0$ in $\mathbb{R}^d \setminus \Omega'$ for sufficiently small ε, where Ω' is any bounded domain containing $\overline{\Omega}$. Then it follows from (3.35) that, up to extracting a subsequence, functions $\overline{v}_\varepsilon$ converge weakly in $H^1_0(\Omega')$ to a function $v \in H^1(\mathbb{R}^d)$ vanishing in $\mathbb{R}^d \setminus \overline{\Omega}$. Thanks to the compactness of the embedding $H^1_0(\Omega') \subset L^2(\Omega')$ and (3.34) we also have the strong L^2-convergence of functions v_ε to v. Finally, since $v = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$ and $\partial\Omega$ is C^1-smooth, $v = 0$ on $\partial\Omega$. Lemma 3.6 is proved. ∎

We continue the proof of Theorem 3.5. By Lemma 3.6 we can extract a subsequence of functions v_ε converging strongly to some function $v \in H^1(\mathbb{R}^d)$ such that $v = 0$ in $\mathbb{R}^d \setminus \Omega$. Thus to complete the proof, it suffices to show that (3.28) is satisfied in the sense of distributions. To this end consider an arbitrary $\phi \in C_0^\infty(\Omega)$ ($\phi = 0$ in $\mathbb{R}^d \setminus \Omega$) and construct test functions $\phi_\varepsilon(x)$ such that, as $\varepsilon \to 0$,

$$-\frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} Q\Big(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\Big) \left(\phi_\varepsilon(y) - \phi_\varepsilon(x)\right) dy \rightharpoonup -A^*_{ij} \partial^2_{x_i x_j} \phi(x) \ \text{ weakly in } L^2(\Omega), \tag{3.36}$$

$$\phi_\varepsilon(x) \to \phi(x) \ \text{ strongly in } L^2(\Omega), \tag{3.37}$$

where A^*_{ij} are given by (3.24). We set

$$\phi_\varepsilon(x) = \phi(x) + \varepsilon \partial_{x_i} \phi(x) \chi^*_i(x/\varepsilon), \tag{3.38}$$

where χ^*_i are solutions of (3.21). Then it is straightforward to see that (3.37) holds. To check (3.36) perform changes of variables $x/\varepsilon = \xi$, $y = x + \varepsilon z$,

$$\frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} Q\Big(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\Big) \left(\phi_\varepsilon(y) - \phi_\varepsilon(x)\right) dy = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} Q\Big(\xi + z, \xi\Big) \left(\phi_\varepsilon(x + \varepsilon z) - \phi_\varepsilon(x)\right) dz,$$

and substitute the expansion

$$\begin{aligned}\phi_\varepsilon(x + \varepsilon z) = \phi_\varepsilon(x) &+ \varepsilon(z_i + \chi^*_i(\xi + z) - \chi^*_i(\xi))\partial_{x_i}\phi(x) + \frac{\varepsilon^2}{2} \partial^2_{x_i x_j}\phi(x) z_i z_j \\ &+ \varepsilon^2 \chi^*_i(\xi + z) \Big(z_j \partial^2_{x_i x_j}\phi(x) + O(\varepsilon |z|^2)\Big) + O(\varepsilon^3 |z|^3).\end{aligned}$$

Taking into account (3.21) we find that

$$\begin{aligned}\frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} Q\Big(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\Big) \big(\phi_\varepsilon(y) - \phi_\varepsilon(x)\big) dy = &\int_{\mathbb{R}^d} Q(\xi + z, \xi) \partial^2_{x_i x_j}\phi(x) \big(\tfrac{z_i}{2} + \chi^*_i(\xi + z)\big) z_j dz \\ &+ O(\varepsilon).\end{aligned}$$

Since functions $a^*_{ij}(\xi) = \int_{\mathbb{R}^d} Q(\xi + z, \xi) \left(\frac{1}{2} z_j + \chi^*_i(\xi + z) z_j\right) dz$ are periodic, we have

$$a^*_{ij}(x/\varepsilon) \partial^2_{x_i x_j}\phi(x) \rightharpoonup \partial^2_{x_i x_j}\phi(x) \int_{\mathbb{T}^d} a^*_{ij}(\xi) d\xi \quad \text{weakly in } L^2(\Omega),$$

so that (3.36) is also proved.

Now we can use $\varphi(x/\varepsilon)\varphi^*(x/\varepsilon)\phi_\varepsilon(x)$ as a test function in (3.26) and pass to the limit as $\varepsilon \to 0$. We have

$$\frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} v_\varepsilon(x) \int_{\mathbb{R}^d\times\mathbb{R}^d} Q\Big(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\Big)\big(\phi_\varepsilon(y) - \phi_\varepsilon(x)\big)dydx$$
$$- \int_{\mathbb{R}^d} \varphi\Big(\frac{x}{\varepsilon}\Big)\varphi^*\Big(\frac{x}{\varepsilon}\Big)\big(v_\varepsilon(x) - f_\varepsilon(x)\big)\phi_\varepsilon(x)dx = 0,$$

whence we find in the limit $\varepsilon \to 0$,

$$-\int_{\mathbb{R}^d} v(x)A^*_{ij}\partial^2_{x_ix_j}\phi(x)dx = \int_{\mathbb{T}^d}\varphi(\xi)\varphi^*(\xi)d\xi \int_{\mathbb{R}^d}\big(f(x) - v(x)\big)\phi(x)dx.$$

Thus, v is a solution of the problem (3.28)–(3.29), and thanks to its uniqueness the whole sequence of functions v_ε converge to v strongly in $L^2(\mathbb{R}^d)$ as $\varepsilon \to 0$. Theorem 3.5 is proved. □

3.3. Proof of Theorem 3.1. Consider first the case $H(p_0) = m$. By (3.4) we have, $\forall\delta > 0$ there is a function $\phi \in C(\mathbb{T}^d)$ such that

$$-\int_{\mathbb{R}^d} J(\xi - \eta)e^{p_0\cdot\xi-\eta)}\kappa(\xi, \eta)\phi(\eta)d\eta + a(\xi)\phi(\xi) \le (m - \delta)\phi(\xi).$$

On the other hand

$$\lambda_\varepsilon = \sup\Big\{\lambda \,\Big|\, \exists\rho \in C(\overline{\Omega}),\ \rho > 0,\ \text{such that}$$
$$-\frac{1}{\varepsilon^d}\int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big)\kappa\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\rho(y)dy + a\Big(\frac{x}{\varepsilon}\Big)\rho(x) \ge \lambda\rho(x) \text{ in } \overline{\Omega}\Big\},$$

then taking $\rho(x) = e^{p_0\cdot x/\varepsilon}\phi(x/\varepsilon)$ we see that $\lambda_\varepsilon \ge m - \delta$. Thus $\lambda_\varepsilon \ge m$ and if $\min_{x\in\overline{\Omega}} a(x/\varepsilon) = m$ (that is always true for sufficiently small ε) then $\lambda_\varepsilon = m$ and it belongs to the essential spectrum of $\mathcal{L}_\varepsilon$.

Now consider the case $H(p_0) < m$. Let v_ε be an eigenfunction corresponding to an arbitrary (not necessarily principal) eigenvalue μ_ε. First we show that if the real part of μ_ε is bounded then $|\mu_\varepsilon|$ is bounded. To this end multiply (3.16) by $\varphi(\frac{x}{\varepsilon})\varphi^*(\frac{x}{\varepsilon})\overline{v}_\varepsilon(x)$, where $\overline{v}_\varepsilon$ denotes the complex conjugate, take real part and integrate over $\mathbb{R}^d$. Then using (3.20) we obtain the following equality

(3.39)
$$\frac{1}{2\varepsilon^{d+2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Q\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\left|v_\varepsilon(y) - v_\varepsilon(x)\right|^2 dydx = \mathrm{Re}\mu_\varepsilon \int_{\mathbb{R}^d}|v_\varepsilon(x)|^2\varphi\Big(\frac{x}{\varepsilon}\Big)\varphi^*\Big(\frac{x}{\varepsilon}\Big)dx$$

Therefore if $\|v_\varepsilon\|_{L^2(\Omega)} = 1$ then (3.32) holds and by Lemma 3.6 functions converge strongly in $L^2(\Omega)$ to a (nonzero) function v as $\varepsilon \to 0$ along a subsequence. Then introducing $\tilde{v}_\varepsilon = \frac{1}{1+\mu_\varepsilon}v_\varepsilon$ and passing to the limit in the equality $\tilde{\mathcal{L}}_\varepsilon\tilde{v}_\varepsilon + \tilde{v}_\varepsilon = v_\varepsilon$ we obtain by virtue of Theorem 3.5 that functions $\tilde{v}_\varepsilon$ converge to a solution $\tilde{v}$ of the problem $\mathcal{L}_0\tilde{v} = v$. One the other hand $\tilde{v}_\varepsilon \to 0$ (since $|\mu_\varepsilon| \to \infty$), thus $v = 0$, a contradiction.

Next we show that for μ from every compact subset M of $\mathbb{C}\setminus\cup_{k=1}^\infty\{\Lambda_k\}$ and sufficiently small ε the operator $\big(\mu I - \tilde{\mathcal{L}}_\varepsilon\big)^{-1} : L^2(\Omega) \to L^2(\Omega)$ exists and its operator norm is uniformly bounded. Indeed, otherwise there exist functions v_ε with $\|v_\varepsilon\|_{L^2(\Omega)} = 1$

and numbers $\mu_\varepsilon \in M$ such that $\tilde{\mathcal{L}}_\varepsilon v_\varepsilon - \mu_\varepsilon v_\varepsilon \to 0$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$ (along a subsequence). Then, by Theorem 3.5 one can extract a subsequence of functions v_ε converging to a nontrivial solution v of the equation $\mathcal{L}_0 v - \mu v = 0$ for some $\mu \in M$, that is impossible since $M \cap \cup_{k=1}^{\infty}\{\Lambda_k\} = \emptyset$. Moreover, applying Theorem 3.5 we conclude that $\forall f \in L^2(\Omega)$ and $\mu \in M$, $\big(\mu I - \tilde{\mathcal{L}}_\varepsilon\big)^{-1} f \to \big(\mu I - \mathcal{L}_0\big)^{-1} f$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$, therefore spectral projectors $\Pi_\varepsilon(\omega) = \frac{1}{2\pi i}\int_{\partial\omega}\big(\mu I - \tilde{\mathcal{L}}_\varepsilon\big)^{-1} d\mu$ converge strongly to the projector $\Pi_0(\omega) = \frac{1}{2\pi i}\int_{\partial\omega}\big(\mu I - \mathcal{L}_0\big)^{-1} d\mu$ for any bounded open set $\omega \subset \mathbb{C}$ whose boundary is smooth and does not contain eigenvalues Λ_k. In fact, there is the compact convergence of projectors, i.e. additionally to the strong convergence it holds that for any sequence of functions f_ε bounded in $L^2(\Omega)$ the sequence of projections $v_\varepsilon = \Pi_\varepsilon(\omega) f_\varepsilon$ contains a strongly converging subsequence. Indeed, observe that v_ε satisfy

$$\tilde{\mathcal{L}}_\varepsilon v_\varepsilon + v_\varepsilon = \frac{1}{2\pi i}\int_{\partial\omega}(1+\mu)\big(\mu I - \tilde{\mathcal{L}}_\varepsilon\big)^{-1} f_\varepsilon d\mu$$

and thanks to the uniform boundedness of $\big(\mu I - \tilde{\mathcal{L}}_\varepsilon\big)^{-1}$ Theorem 3.5 guarantees that the sequence of functions v_ε does contain a strongly converging (as $\varepsilon \to 0$) subsequence. The compact convergence of spectral projectors in turn implies that the dimensions of the subspaces $\Pi_\varepsilon(\omega)L^2(\Omega)$ and $\Pi_0(\omega)L^2(\Omega)$ coincide as ε is sufficiently small, i.e. operators $\tilde{\mathcal{L}}_\varepsilon$ and $\mathcal{L}_0$ have the same number of eigenvalues (counting multiplicities) in the domain ω. This means, in particular, that there is an eigenvalue of $\tilde{\mathcal{L}}_\varepsilon$ converging to Λ_1 as $\varepsilon \to 0$. Therefore the principal eigenvalue of $\tilde{\mathcal{L}}_\varepsilon$ exists for sufficiently small ε. It remains bounded as $\varepsilon \to 0$ since its real part remains bounded, and it converges (up to a subsequence) to an eigenvalue of $\mathcal{L}_0$ (any compact subset of $\mathbb{C} \setminus \cup_{k=1}^{\infty}\{\Lambda_k\}$ belongs to the resolvent set of $\tilde{\mathcal{L}}_\varepsilon$ for sufficiently small ε). Thus the principal eigenvalue of $\tilde{\mathcal{L}}_\varepsilon$ converges to Λ_1. Other eigenvalues can be treated similarly. Thus Theorem 3.1 is proved.

Remark 3.7. It follows from the above proof of Theorem 3.1 that, in the case $H(p_0) < m$, the j-th eigenfunction $\rho_\varepsilon^{(j)}$ of $\mathcal{L}_\varepsilon$ can be represented as

$$\rho_\varepsilon^{(j)}(x) = e^{-p_0\cdot x/\varepsilon}\big(v_\varepsilon^{(j)}(x) + o(1)\big),$$

where $v_\varepsilon^{(j)}$ is a j-th eigenfunction of (3.7) with unit $L^2(\Omega)$-norm, and $o(1)$ stands for a function whose norm in $L^2(\Omega)$ tends to zero as $\varepsilon \to 0$.

4. Locally periodic case. This section is devoted to the proof of Theorem 2.1, i.e. we study operator $\mathcal{L}_\varepsilon$ given by (1.2) with generic functions κ and a satisfying (2.2). Introduce the notation

$$E_\varepsilon(x,y) = \frac{1}{\varepsilon^d} J\Big(\frac{x-y}{\varepsilon}\Big)\kappa\Big(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big),$$

then $\mathcal{L}_\varepsilon$ writes as

$$\mathcal{L}_\varepsilon \rho_\varepsilon = -\int_\Omega E_\varepsilon(x,y)\rho_\varepsilon(y)dy + a\Big(x, \frac{x}{\varepsilon}\Big)\rho_\varepsilon(x), \tag{4.1}$$

and

$$\lambda_\varepsilon = \sup\Big\{\lambda \,\Big|\, \exists v \in C(\overline{\Omega}),\ v > 0 \text{ such that } -\int_\Omega E_\varepsilon(x,y)\frac{v(y)}{v(x)}dy + a\Big(x, \frac{x}{\varepsilon}\Big) \geq \lambda \text{ in } \Omega\Big\}. \tag{4.2}$$

Assume first that $\lambda_\varepsilon < \min_{\overline{\Omega}} a(x, x/\varepsilon)$, then λ_ε is the principal eigenvalue of $\mathcal{L}_\varepsilon$. Therefore the corresponding eigenfunction can be written as $\rho_\varepsilon = e^{-\frac{1}{\varepsilon}W_\varepsilon(x)}$. We consider the following ansatz for W_ε, $W_\varepsilon(x) = W(x) + \varepsilon w(x, x/\varepsilon) + \dots$, where $w(x,\xi)$ is periodic in ξ. Together with the fast variable $\xi = x/\varepsilon$ we also introduce $\eta = y/\varepsilon$ and regard these variables as independent of the slow ones, x and y. We also hypothesize that λ_ε converges to a finite number $-\Lambda$. Then, for fixed $x \in \Omega$ we expand

$$W(y) = W(x + \varepsilon(\eta - \xi)) = W(x) + \varepsilon \nabla W(x) \cdot (\eta - \xi) + \dots$$

and formally obtain in the leading term of the eigenvalue equation $\mathcal{L}_\varepsilon \rho_\varepsilon = \lambda_\varepsilon \rho_\varepsilon$ that $\Lambda = -H(\nabla W(x), x)$, $H(p,x)$ being the principal eigenvalue of the cell problem

(4.3)
$$-\int_{\mathbb{R}^d} J(\xi-\eta) e^{p\cdot(\xi-\eta)} \kappa(x,x,\xi,\eta)\varphi(\eta,p,x)d\eta + a(x,\xi)\varphi(\xi,p,x) = H(p,x)\varphi(\xi,p,x) \text{ on } \mathbb{T}^d$$

depending on the parameters $p \in \mathbb{R}^d$ and $x \in \overline{\Omega}$, while $w(x,\xi) = -\log\varphi(\xi, \nabla W(x), x)$. Adopting the normalization condition $\int_{\mathbb{T}^n} \varphi(\xi,p,x)d\xi = 1$ we obtain a function $\varphi(\xi,p,x)$ continuous in all their arguments (ξ, p and x), provided that the principal eigenvalue exists. Notice that the principal eigenvalue $H(p,x)$ is given by (2.4) and always satisfies $H(p,x) < \min_{\xi\in\mathbb{T}^d} a(x,\xi)$, moreover the latter inequality is sufficient and necessary for existence of a principal eigenvalue of (4.3). We show below that if

$$\lambda_\varepsilon < \min_{\overline{\Omega}} a(x, x/\varepsilon) \text{ and } H(p,x) < \min_{\xi\in\mathbb{T}^d} a(x,\xi) \tag{4.4}$$

then $\lambda_\varepsilon \to -\Lambda$ as $\varepsilon \to 0$, where Λ is in fact the minimal eigenvalue of the problem $-H(\nabla W(x), x) = \Lambda$ in Ω or, equivalently,

$$\Lambda = \min\big\{\tilde{\Lambda} \,|\, \exists \text{ a viscosity subsolution of } -H(\nabla W(x),x) \le \tilde{\Lambda} \text{ in } \Omega\big\}.$$

As known, see, e.g., [19] this formula (along with (2.7) and (2.8)) determines the unique additive eigenvalue Λ of problem (2.6).

The additional technical assumptions (4.4) will then be eliminated by devising small deformations of $a(x,\xi)$ regularizing eigenvalue problems, and in this way we will get the proof of Theorem 2.1.

THEOREM 4.1. *Suppose that J satisfies* (2.1) *and κ, a satisfy* (2.2). *Assume also that λ_ε (given by* (2.3)*) and $H(p,x)$ (given by* (2.4)*) satisfy* (4.4). *Then $\lambda_\varepsilon \to -\Lambda$ as $\varepsilon \to 0$, where Λ is a unique additive eigenvalue of problem* (2.6).

Proof. We begin with the following lower bound, obtained by using the test function

$$v_\varepsilon(x) = e^{-\frac{1}{\varepsilon}W(x)}\varphi(x/\varepsilon, \nabla W(x), x)$$

in (4.2),

$$\lambda_\varepsilon \ge \min_{x\in\overline{\Omega}}\Big\{-\int_\Omega E_\varepsilon(x,y)\frac{v_\varepsilon(y)}{v_\varepsilon(x)}dy + a\Big(x, \frac{x}{\varepsilon}\Big)\Big\},$$

where W is an arbitrary function of the class $C_0^\infty(\mathbb{R}^d)$. Notice that uniformly in $x \in \Omega$,

$$\begin{aligned} -\int_\Omega E_\varepsilon(x,y)\frac{v_\varepsilon(y)}{v_\varepsilon(x)}dy + a\Big(x,\frac{x}{\varepsilon}\Big) &= a\Big(x,\frac{x}{\varepsilon}\Big) + o(1) \\ &- \int_{\frac{x}{\varepsilon} - \frac{1}{\varepsilon}\Omega} J(z)\kappa\Big(x,x,\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z\Big)e^{\frac{1}{\varepsilon}(W(x)-W(x-\varepsilon z))}\frac{\varphi(x/\varepsilon - z, \nabla W(x), x)}{\varphi(x/\varepsilon, \nabla W(x), x)}dz. \end{aligned}$$

Expanding $W(x-\varepsilon z) = W(x) - \varepsilon\nabla W(x)\cdot z + O(\varepsilon^2|z|^2)$, using (4.3) and taking into account (2.1) we get

$$\liminf_{\varepsilon\to 0}\lambda_\varepsilon \geq \min_{x\in\overline{\Omega}} H(\nabla W(x), x). \tag{4.5}$$

Therefore by density of functions $W|_{\overline{\Omega}}$, $W\in C_0^\infty(\mathbb{R}^d)$ in $C^1(\overline{\Omega})$ we have

$$\liminf_{\varepsilon\to 0}\lambda_\varepsilon \geq -\Lambda, \quad \text{where } \Lambda = \inf_{W\in C^1(\overline{\Omega})}\max_{x\in\overline{\Omega}} -H(\nabla W(x), x). \tag{4.6}$$

Next, considering a partial limit λ of λ_ε as $\varepsilon\to 0$, we use the techniques of half-relaxed limits (introduced in [1]) to show that there is a viscosity subsolution $W^*(x)$ of

$$-H(\nabla W^*(x), x) \leq -\lambda \quad \text{in } \Omega. \tag{4.7}$$

Specifically, let $e^{-\frac{1}{\varepsilon}W_\varepsilon(x)}$ be the eigenfunction of $\mathcal{L}_\varepsilon$ corresponding to the eigenvalue λ_ε and assume that this function satisfies the following normalization condition

$$\int_{\Omega'} W_\varepsilon(x)dx = 0, \tag{4.8}$$

where Ω' is a domain such that $\overline{\Omega'}\subset\Omega$. Since $\lambda_\varepsilon < \min_{x\in\overline{\Omega}} a(x, x/\varepsilon)$ and (4.6) holds, we can assume, after passing to a subsequence that $\lambda_\varepsilon\to\lambda$ as $\varepsilon\to 0$. Then we consider the half-relaxed limit

$$W^*(x) = \lim_{r\to 0}\limsup_{\varepsilon\to 0}\sup\{W_\varepsilon(\xi)\,|\,\xi\in B_r(x)\cap\Omega\}. \tag{4.9}$$

Lemma 4.2. *Assume that functions $W_\varepsilon(x)$ satisfy* (4.8). *Then $W^*(x)$ given by* (4.9) *is a bounded function in Ω.*

Proof. As $J(0)>0$ and J is continuous, there is $\hat r_0>0$ such that $\inf_{|z|\leq\hat r_0} J(z) > 0$. Furthermore, since $\partial\Omega$ is C^1-smooth, there is $\tilde r_\varepsilon \geq c\varepsilon$ (with $c>0$ independent of ε) such that any ball $B_{\hat r_0\varepsilon}(x)$ centered at a point $x\in\overline{\Omega}$ contains a ball $B_{\tilde r_\varepsilon}(\xi)$ that is also contained in Ω, i.e. $B_{\tilde r_\varepsilon}(\xi)\subset B_{\hat r_0\varepsilon}(x)\cap\Omega$.

We argue as in Lemma 3.6. Thanks to (4.6) eigenvalues λ_ε are uniformly bounded from below and we have

$$\int_\Omega\int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big)\kappa\Big(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)e^{\frac{1}{\varepsilon}(W_\varepsilon(x)-W_\varepsilon(y))}dxdy \leq C\varepsilon^d,$$

therefore

$$\int_{\Omega_\varepsilon}\int_{|z|<\tilde r_\varepsilon} e^{\frac{1}{\varepsilon}|W_\varepsilon(x+z)-W_\varepsilon(x)|}dzdx \leq C\varepsilon^d,$$

where $\Omega_\varepsilon = \{x\in\Omega\,|\,\mathrm{dist}(x,\partial\Omega) > \tilde r_\varepsilon\}$. In particular,

$$\int_{\Omega_\varepsilon}\int_{|z|<\tilde r_\varepsilon} |W_\varepsilon(x+z)-W_\varepsilon(x)|^{d+1}dzdx \leq C\varepsilon^{2d+1}. \tag{4.10}$$

It follows that there is some r_ε, $\tilde r_\varepsilon/2\leq r_\varepsilon\leq\tilde r_\varepsilon$ such that

$$\int_{\Omega_\varepsilon}\int_{|z|=r_\varepsilon} |W_\varepsilon(x+z)-W_\varepsilon(x)|^{d+1}dSdx \leq C\varepsilon^{2d}.$$

Set

$$\overline{W}_\varepsilon(x) = \frac{1}{|B_1| r_\varepsilon^d} \int_{|z|<r_\varepsilon} W_\varepsilon(x+z)dz.$$

Using Jensen's inequality we get

$$\int_{\Omega_\varepsilon} |\overline{W}_\varepsilon(x) - W_\varepsilon(x)|^{d+1} dx \le C\varepsilon^{d+1}. \tag{4.11}$$

Then, arguing as in Lemma 3.6 we derive

$$\int_{\Omega_\varepsilon} |\nabla \overline{W}_\varepsilon(x)|^{d+1} dx \le C. \tag{4.12}$$

Now, taking into account (4.8), (4.11) we can apply the Poincaré inequality to conclude that $\int_{\Omega_\varepsilon} |\overline{W}_\varepsilon(x)|^{d+1}dx \le C$ for small ε. Then by the compactness of the embedding $W^{1,d+1}(\Omega_\varepsilon) \subset C(\overline{\Omega}_\varepsilon)$ (Morrey's theorem) we derive that $|\overline{W}_\varepsilon(x)| \le C$ on Ω_ε with C independent of ε. Combining this with (4.11) we infer that $W^*(x)$ is bounded from below.

Repeating the above reasonings for the positive part $W_\varepsilon^+(x)$ of $W_\varepsilon(x)$ (notice that the inequality (4.10) is also valid for $W_\varepsilon^+(x)$) we get that

$$\overline{W}_\varepsilon^+(x) = \frac{1}{|B_1| r_\varepsilon^d} \int_{|z|<r_\varepsilon} W_\varepsilon^+(x+z)dz$$

satisfies $\int_{\Omega_\varepsilon} |\nabla \overline{W}_\varepsilon^+(x)|^{d+1}dx \le C$. Besides, using (4.11) one sees that $\int_{\Omega'} \overline{W}_\varepsilon^+(x)dx \le C$. Then applying the Poincaré inequality and exploiting the compactness of the embedding $W^{1,d+1}(\Omega_\varepsilon) \subset C(\overline{\Omega}_\varepsilon)$ we obtain that $\overline{W}_\varepsilon^+(x) \le C$ on Ω_ε with C independent of ε.

Taking log of

$$\frac{1}{\varepsilon^d} \int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big) \kappa\Big(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big) e^{-\frac{1}{\varepsilon} W_\varepsilon(y)} dy = (a(x) - \lambda_\varepsilon) e^{-\frac{1}{\varepsilon} W_\varepsilon(x)}$$

and using Jensen's inequality we get

$$W_\varepsilon(x) \le \frac{1}{\int_\Omega J\left(\frac{x-y}{\varepsilon}\right) \kappa(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}) dy} \int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big) \kappa\Big(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big) W_\varepsilon(y) dy + \varepsilon R_\varepsilon(x),$$

where

$$R_\varepsilon(x) = \log(a(x) - \lambda_\varepsilon) - \log\Big(\frac{1}{\varepsilon^d} \int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big) \kappa\Big(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big) dy\Big).$$

Let $x_\varepsilon \in \overline{\Omega}$ be a maximum point of W_ε. Choose a ball $B_{\tilde{r}_\varepsilon}(\xi_\varepsilon)$ contained in $B_{r_0\varepsilon}(x_\varepsilon) \cap \Omega$. Then $\xi_\varepsilon \in \Omega_\varepsilon$, $B_{r_\varepsilon}(\xi_\varepsilon) \subset B_{r_0\varepsilon}(x_\varepsilon) \cap \Omega$ and we have

$$\begin{aligned} W_\varepsilon(x_\varepsilon) &\le \frac{\int_{B_{r_\varepsilon}(\xi_\varepsilon)} J\Big(\frac{x_\varepsilon - y}{\varepsilon}\Big) \kappa\Big(x_\varepsilon, y, \frac{x_\varepsilon}{\varepsilon}, \frac{y}{\varepsilon}\Big) W_\varepsilon(y) dy}{\int_{B_{r_\varepsilon}(\xi_\varepsilon)} J\left(\frac{x_\varepsilon - y}{\varepsilon}\right) \kappa(x_\varepsilon, y, \frac{x_\varepsilon}{\varepsilon}, \frac{y}{\varepsilon}) dy} + C\varepsilon \\ &\le \frac{C_1 \int_{B_{r_\varepsilon}(\xi_\varepsilon)} W_\varepsilon^+(y) dy}{\int_{B_{r_\varepsilon}(\xi_\varepsilon)} J\left(\frac{x_\varepsilon - y}{\varepsilon}\right) \kappa(x_\varepsilon, y, \frac{x_\varepsilon}{\varepsilon}, \frac{y}{\varepsilon}) dy} + C\varepsilon \le C_2 \overline{W_\varepsilon}^+(\xi_\varepsilon) + C\varepsilon \le C_3. \end{aligned}$$

Lemma 4.2 is proved. □

To show that W^* is a subsolution of (4.7) consider an arbitrary test function $\Phi \in C_0^\infty(\mathbb{R}^d)$ and assume that a $\max_{x\in\overline{\Omega}}(W(x)-\Phi(x))$ is attained at a point $x_0 \in \Omega$, and this maximum is strict. Then we can extract a subsequence such that the maximum points x_ε of

$$\Psi_\varepsilon(x) = W_\varepsilon(x) - \Phi(x) + \varepsilon \log \varphi(x/\varepsilon, \nabla\Phi(x), x)$$

converge to x_0. We have $\Psi_\varepsilon(x_\varepsilon) - \Psi_\varepsilon(y) \geq 0$ for $y \in \overline{\Omega}$, or

$$W_\varepsilon(x_\varepsilon) - W_\varepsilon(y) \geq \Phi(x_\varepsilon) - \Phi(y) + \varepsilon \log \frac{\varphi(y/\varepsilon, \nabla\Phi(x), x)}{\varphi(x_\varepsilon/\varepsilon, \nabla\Phi(x_\varepsilon), x_\varepsilon)},$$

therefore

$$\int_\Omega E_\varepsilon(x_\varepsilon, y) e^{\frac{1}{\varepsilon}(W_\varepsilon(x_\varepsilon)-W_\varepsilon(y))} dy \geq \int_\Omega E_\varepsilon(x_\varepsilon, y) e^{\frac{1}{\varepsilon}(\Phi(x_\varepsilon)-\Phi(y))} \frac{\varphi(y/\varepsilon, \nabla\Phi(y), y)}{\varphi(x_\varepsilon/\varepsilon, \nabla\Phi(x_\varepsilon), x_\varepsilon)} dy.$$

Then

$$\begin{aligned} -\lambda_\varepsilon &= \int_\Omega E_\varepsilon(x_\varepsilon, y) e^{\frac{1}{\varepsilon}(W_\varepsilon(x_\varepsilon)-W_\varepsilon(y))} dy - a\Big(x_\varepsilon, \frac{x_\varepsilon}{\varepsilon}\Big) \\ &\geq \int_\Omega E_\varepsilon(x_\varepsilon, y) e^{\frac{1}{\varepsilon}(\Phi(x_\varepsilon)-\Phi(y))} \frac{\varphi(y/\varepsilon, \nabla\Phi(y), y)}{\varphi(x_\varepsilon/\varepsilon, \nabla\Phi(x_\varepsilon), x_\varepsilon)} dy - a\Big(x_\varepsilon, \frac{x_\varepsilon}{\varepsilon}\Big), \end{aligned}$$

and passing to the limit in this inequality as $\varepsilon \to 0$ we derive $-H(\nabla\Phi(x_0), x_0) \leq -\lambda$. Thus $W^*(x)$ is indeed an upper semicontinuous subsolution of (4.7), being a subsolution of (4.7) function $W^*(x)$ is in fact Lipschitz continuous on Ω (see, e.g., Appendix A.3 in [15]). Consequently $-\lambda \geq \Lambda$. Theorem 4.1 is proved. ∎

Proof of Theorem 2.1. For sufficiently small $\delta > 0$ set

$$\hat{a}(x) = \min_{\xi\in\mathbb{T}^d} a(x,\xi), \quad \hat{a}^{(\delta)}(x) = \max\Big\{\hat{a}(x), \min_{y\in\overline{\Omega}} \hat{a}(y) + \delta/2\Big\},$$

and

$$a^{(\delta)}(x,\xi) = \max\Big\{a(x,\xi), \hat{a}^{(\delta)}(x) + \delta/2\Big\}.$$

Then for any $x \in \overline{\Omega}$ the function $a^{(\delta)}(x,\xi)$ attains its minimum over $\xi \in \mathbb{T}^d$ on a set of positive measure, and $a^{(\delta)}(x, \frac{x}{\varepsilon})$ attains its minimum over $x \in \overline{\Omega}$ on a set of positive measure for sufficiently small ε. Hence we can replace a with $a^{(\delta)}$ to modify spectral problems (1.1) and (4.3) such that they do have some principal eigenvalues $\lambda_\varepsilon^{(\delta)}$ and $H^{(\delta)}(p,x)$ by Theorem 2.1 in [18]. Then applying Theorem 4.1 we get that $\lambda_\varepsilon^{(\delta)} \to -\Lambda^{(\delta)}$ as $\varepsilon \to 0$, where $\Lambda^{(\delta)} = \inf_{W\in C^1(\overline{\Omega})} \max_{x\in\overline{\Omega}} -H^{(\delta)}(\nabla W(x), x)$. On the other hand $|a^{(\delta)}(x,\xi) - a(x,\xi)| \leq \delta$ and therefore $|\lambda_\varepsilon^{(\delta)} - \lambda_\varepsilon| \leq \delta$, $|\Lambda - \Lambda^{(\delta)}| \leq \delta$. Thus, letting $\delta \to 0$ we obtain that $\lambda_\varepsilon \to -\Lambda$, Theorem 2.1 is proved.

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