Analysis and Mathematical Physics (2025) 15:2
https://doi.org/10.1007/s13324-024-00998-9

Old and new on the Peetre K-functional and its relations to real interpolation theory, quasi-monotone functions and wavelets

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Received: 8 August 2024 / Revised: 19 November 2024 / Accepted: 27 November 2024

Abstract
The Peetre K-functional is a key object in the development of the real method of interpolation. In this paper we point out a less known relation to wavelet theory and its applications to approximation theory and engineering applications. As a new basis for further development of these studies we present some known properties in the form appropriate for further applications and then derive new information and prove some new results concerning the K-functional and its close relation to (almost) quasi-monotone functions, various indices and interpolation theory. In particular, we extend and unify some known function parameter generalizations of the standard real interpolation spaces $(A_0, A_1)_{\theta,q}$.

Keywords Functions of real variables · Quasi-monotone function · Indices · K-functional · Real interpolation theory · Wavelets · Engineering mathematics

Mathematics Subject Classification Primary 26A48 · 46B70; Secondary 26A51 · 26D10 · 00A06 · 65T60

1 Introduction

One motivation for this investigation was the PhD thesis [9], where, in particular, a not so known relation between wavelet coefficients and the Peetre K-functional was

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Published online: 10 December 2024

pointed out and applied. The idea is now to go on with this research of interest for applications in engineering sciences. As the starting point in this paper we present some known properties in the form appropriate for further applications. Then we prove some new results concerning the close relation between (almost) quasi-monotone functions, various indices and interpolation theory, which is one of the main novelties in this paper.

The Peetre K-functional is defined as follows: For $a \in A_0 + A_1$ and $0 < t < \infty$ we define

$$K = K(t) = K(t, a; A_0, A_1) := \inf_{\substack{a=a_0+a_1 \\ a_0 \in A_0, a_1 \in A_1}} \left(\|a_0\|_{A_0} + t\|a_1\|_{A_1} \right), \tag{1.1}$$

where (A_0, A_1) is a compatible quasi-Banach couple (i.e. A_0 and A_1 are both embedded in a common Hausdorff topological vector space).

Directly from the definition (1.1) we obtain the following first interesting properties of the Peetre K-functional:

For each fixed t, the Peetre K-functional is an equivalent quasi-norm on the space $A_0 + A_1$. (1.2)

As a function of t, the K-functional is a concave function. (1.3)

$K(t)$ is a non-decreasing function. (1.4)

$\frac{K(t)}{t}$ is a non-increasing function. (1.5)

Later on in this paper we will point out some useful consequences of (1.3), (1.4) and (1.5). Moreover, our theoretical investigations will, in particular, imply several other new properties of the K-functional.

In Sect. 2 we discuss the important class of quasi-monotone functions, where the properties (1.4) and (1.5) are natural special cases. We also present some new facts for quasi-monotone functions and consider generalizations of functions based on the so-called almost monotonicity, which are important for our further developments in this paper. The main results in this section are Theorems 2.1 and 2.8.

In the light of the ideas of quasi-monotonicity exposed in Sect. 2, in Sect. 3 we consider a variety of indices related to quasi-monotone functions and sometimes generalize some known facts for them. As special cases we point out these indices for the Peetre K-functional. The main focus in this section is to investigate the close relation between quasi-monotone-type functions and various indices. The main results are given in Theorems 3.5, 3.10 and 3.18.

The most well-known application of the Peetre K-functional is that it can be used to define the real interpolation spaces $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, $q > 0$. See e.g. the book [3] (by Jaak Peetre's former students J. Bergh and J. Löfström). There are some generalizations of these interpolation spaces, e.g. interpolation with a parameter function by L. E. Persson (see [29]). Here the basic idea was that since the Peetre K-functional is quasi-monotone it is natural to replace the weight function t^θ in the definition by another quasi-monotone function $\varphi(t)$, which controls the growth of the weight corresponding to the condition $0 < \theta < 1$.

In Sect. 4 we suggest a generalization of the parameter method. One reason for this is that then it is easier to see that this method is more general than another generalization namely "the slowly varying method" for the case $0 < \theta < 1$. Main result is our introduction of new generalized real interpolation method [GP] which does not exclude the endpoint cases $\theta = 0$ and $\theta = 1$. For these cases we have not yet contributed with some our own results but refer to the new paper [13]. Our new result for the case $0 < \theta < 1$ is presented in Theorem 4.10, and also Proposition 4.12 illustrates how this idea can be used.

In the first part of Sect. 5 we briefly expose our present knowledge of the connection between the Peetre K-functional, interpolation theory and wavelets. We hope that the investigations in this paper may be useful to deepen the understanding and applicability of that connection.

Finally, the second part of Sect. 5 is reserved for some final remarks and results, e.g. connections between indices used in some other interpolation methods (Proposition 5.7) and new properties of the K-functional (Example 5.2, Remark 5.4 and Example 5.5).

The authors thank the referees for carefully reading the paper.

2 Basic properties of quasi-monotone functions

We say that a non-negative function on $\mathbb{R}_+ = (0, \infty)$ is *quasi-monotone* if $\varphi(x)x^{-a}$ is non-decreasing or non-increasing for some $a \in \mathbb{R}$.

Let a_0 and a_1 be real numbers, such that $a_0 \le a_1$. The class $Q[a_0, a_1]$ is defined as consisting of all quasi-monotone functions such that $\varphi(t)t^{-a_0}$ is non-decreasing and $\varphi(t)t^{-a_1}$ is non-increasing.

Observe that if $\varphi \in Q[a_0, a_1]$, then $\varphi(t_0) = 0$ for some t_0, implies $\varphi(t) \equiv 0$ on $\mathbb{R}_+$.

Note that

$$Q[a_0, a_1] \subseteq Q[b_0, b_1]$$

whenever $-\infty < b_0 \le a_0 \le a_1 \le b_1 < \infty$.

We define the class $Q(a_0, a_1)$ by the condition that for a function φ there exists $0 < \varepsilon < \frac{a_1 - a_0}{2}$ such that $\varphi \in Q[a_0 + \varepsilon, a_1 - \varepsilon]$, i.e.

$$Q(a_0, a_1) = \bigcup_{0 < \varepsilon < \frac{a_1 - a_0}{2}} Q[a_0 + \varepsilon, a_1 - \varepsilon].$$

We also permit hybrid cases $Q[a_0, b_0)$, $Q(a_0, b_0]$, $Q(a_0, -]$, $Q[-, -)$ etc. where e.g. $\varphi \in Q[a_0, -)$ means that $\varphi(t)t^{-a_0}$ is non-decreasing and $\varphi(t)t^{-b}$ is non-increasing for some $b > a_0$, i.e.

$$Q[a_0, -) = \bigcup_{b>a_0} Q[a_0, b].$$

The following theorem shows that, though functions in the classes $Q[a_0, a_1]$ are defined only by means of two monotonicity conditions, they automatically possess a variety of useful properties, in particular, they have certain smoothness properties.

Theorem 2.1 *Let $\varphi \in Q[a_0, a_1]$, $-\infty < a_0 \le a_1 < +\infty$. Then*

(a) $\varphi(t) = t^{a_0}\varphi_0(t^{a_1-a_0})$, *where* $\varphi_0 \in Q[0, 1]$;
(b) $\varphi(t^\alpha) \in Q[a_0\alpha, \alpha_1\alpha]$, $\alpha > 0$,
$\varphi(t^\alpha) \in Q[a_1\alpha, \alpha_0\alpha]$, $\alpha < 0$;
(c) the function φ is quasi-additive in the following sense:

$$\varphi(t_1 + t_2) \le c_1[\varphi(t_1) + \varphi(t_2)],\ t_1, t_2 \in \mathbb{R}_+ \text{ if } a_1 \ge 0; \tag{2.1}$$

and

$$\varphi(t_1 + t_2) \ge c_0[\varphi(t_1) + \varphi(t_2)],\ t_1, t_2 \in \mathbb{R}_+ \text{ if } a_0 \ge 0, \tag{2.2}$$

where $c_1 = \max\{1, 2^{a_1-1}\}$ *and* $c_0 = \min\{1, 2^{a_0-1}\}$;
(d) if $a_0 \ge 1$, *then the function φ is quasi-convex in the following sense:*

$$\varphi(\lambda t + (1-\lambda)s) \le 2^{a_1-1}\left[\lambda\varphi(t) + (1-\lambda)\varphi(s)\right], \tag{2.3}$$

$s, t \in \mathbb{R}_+$ *and* $\lambda \in (0, 1)$,
if $0 \le a_0 \le a_1 \le 1$, *then φ is quasi-concave in the following sense:*

$$\varphi(\lambda t + (1-\lambda)s) \ge 2^{a_0-1}\left[\lambda\varphi(t) + (1-\lambda)\varphi(s)\right], \tag{2.4}$$

$s, t \in \mathbb{R}_+$ *and* $\lambda \in (0, 1)$.
(e) φ is Lipschtzian on any subinterval $[\delta, N]$ *of* $\mathbb{R}_+$:

$$|\varphi(t) - \varphi(s)| \le C|t - s|,\ t, s \in [\delta, N], \tag{2.5}$$

where $C = C(\delta, N)$, $0 < \delta < N < \infty$;
(f) if $a_0 > 0$, *then*

$$|\varphi(t) - \varphi(s)| \le C|t - s|^{a_0}, \tag{2.6}$$

for all $t, s \in [0, N]$, $0 < N < \infty$ *and* $C = C(\varphi)$.
(g) if $\varphi(t) \ne 0$, *then the inverse* $\varphi^{-1}(t)$ *exists and* $\varphi^{-1}(t) \in [a_1^{-1}, a_0^{-1}]$, *whenever* $a_0 > 0$.

Proof The proof of the properties (a) and (b) are straightforward.

To prove the property (c), we use the elementary inequality $(t_1+t_2)^{a_1} \le c_1(t_1^{a_1} + t_2^{a_1})$ and get

$$\varphi(t_1+t_2) \le c_1\left[\frac{\varphi(t_1+t_2)}{(t_1+t_2)^{a_1}}t_1^{a_1} + \frac{\varphi(t_1+t_2)}{(t_1+t_2)^{a_1}}t_2^{a_1}\right] \le c_1\left[\varphi(t_1)+\varphi(t_2)\right],$$

so (2.1) is proved. Similarly, the inequality (2.2), can be proved with the use of the inequality $(t_1+t_2)^{a_0} \ge c_0(t_1^{a_0}+t_2^{a_0})$.

To prove the property (d) we use the property (c) and obtain

$$\varphi(\lambda t + (1-\lambda)s) \le 2^{a_1-1}\left[\varphi(\lambda t) + \varphi((1-\lambda)s)\right].$$

Since $\frac{\varphi(t)}{t^{a_0}}$ is non-decreasing, we get $\varphi(\lambda t) \le \lambda^{a_0}\varphi(t) \le \lambda\varphi(t)$ and $\varphi((1-\lambda)t) \le (1-\lambda)^{a_0}\varphi(s) \le (1-\lambda)\varphi(s)$, which proves (2.3). The inequlity (2.4) can be proved in a similar way.

To prove the properties (e) and (f) we use the following estimates:

$$(t^{a_1}-s^{a_1})\frac{\varphi(s)}{s^{a_1}} \le \varphi(t)-\varphi(s) \le (t^{a_0}-s^{a_0})\frac{\varphi(s)}{s^{a_0}},\ 0<t<s, \tag{2.7}$$

and

$$(t^{a_0}-s^{a_0})\frac{\varphi(s)}{s^{a_0}} \le \varphi(t)-\varphi(s) \le (t^{a_1}-s^{a_1})\frac{\varphi(s)}{s^{a_1}},\ 0<s<t. \tag{2.8}$$

To prove (2.7) by the definition of the class $Q[a_0,a_1]$, we have $\varphi(t) \le \frac{t^{a_0}}{s^{a_0}}\varphi(s)$ and $\varphi(t) \ge \frac{t^{a_1}}{s^{a_1}}\varphi(s)$. Hence

$$\varphi(t)-\varphi(s) \le (t^{a_0}-s^{a_0})\frac{\varphi(s)}{s^{a_0}} \text{ and } \varphi(t)-\varphi(s) \ge (t^{a_1}-s^{a_1})\frac{\varphi(s)}{s^{a_1}},$$

from which (2.7) follows. Similarly, (2.8) can be proved.

In particular, for $a_0=0$ and $a_1=1$ from (2.7) and (2.8) we have

$$|\varphi(t)-\varphi(s)| \le |t-s|\frac{\varphi(s)}{s}.$$

Hence, (2.5) follows for the case $a_0=0$ and $a_1=1$. The general case for (2.5) follows from the formula $\varphi(t)=t^{a_0}\varphi_0(t^{a_1-a_0})$, where $\varphi_0 \in Q[0,1]$.

Let now $a_0>0$. To prove (2.6), suppose for definiteness that $t \le s$. From the estimate (2.7) we get

$$|\varphi(t)-\varphi(s)| \le \frac{s^{a_1}-t^{a_1}}{s^{a_1}}\varphi(s),\ t,s \in \mathbb{R}_+.$$

We have $\frac{s^{a_1}-t^{a_1}}{s^{a_1}} \le C\frac{(s-t)^{a_1}}{s^{a_1}} = c(1-\frac{t}{s})^{a_1} \le c(1-\frac{t}{s})^{a_0}$. Hence

$$|\varphi(t)-\varphi(s)| \le |t-s|^{a_0}\frac{\varphi(s)}{s^{a_0}} \le c(s-t)^{a_0}$$

for all $s, t \in [0, N]$, with $c = c(N)$ in general.

For the property (g) it is easy to derive from the property (a) that φ is strictly increasing and the range of φ over $\mathbb{R}_+$ is $\mathbb{R}_+$. From the property (e) we also see that φ is continuous. Consequently, this implies the existence of the inverse function. The proof of the rest in the property (g) is straightforward. □

Remark 2.2 The properties $(a) - (g)$ of Theorem 2.1 hold also for functions in the classes $Q(a_0, a_1)$ or any of the hybrid cases under appropriate reformulations of these properties.

Example 2.3 Let $K = K(t)$ be the Peetre K-functional (for any quasi-Banach couple (A_0, A_1). Then, the function $t^{a_0}K(t^{a_1-a_0}) \in Q[a_0, a_1]$, $a_0, a_1 \in \mathbb{R}$, $a_0 \le a_1$. Moreover, in view of Theorem 2.1 this function has a number of properties which are not explicitly pointed out in literature, e.g. it is quasi-additive and Lipschitzian on any subinterval $[\delta, N]$ on $\mathbb{R}_+$ etc.

In view of Remark 2.2, by means of the properties (a) and (b) of Theorem 2.1 the following information about the involution function $\varphi^*(t) = t\varphi\left(\frac{1}{t}\right)$, important in interpolation theory, is derived.

Example 2.4 $\varphi \in Q(0, 1)$ if and only if $\varphi^* \in Q(0, 1)$.

For our discussions about indices and interpolation spaces in our next sections it is important to extend the definition of quasi-monotone functions. This extension uses the notion of almost monotonicity which traces back to S. Bernstein (see [4] and c.f. also [5]).

We say that a non-negative function ω on $\mathbb{R}_+ = (0, \infty)$ is almost increasing, if $\omega(t) \le c_0\omega(s), t < s$ and almost decreasing, if $\omega(s) \le c_1\omega(t), t < s$, for some $a \in \mathbb{R}$ and $c_0, c_1 \ge 1$.

We say that a non-negative function on $\mathbb{R}_+$ is *almost quasi-monotone* if $\varphi(x)x^{-a}$ is almost increasing or almost decreasing for some $a \in \mathbb{R}$.

Next, we say that the non-negative functions $\varphi(t)$ and $\psi(t)$ on $\mathbb{R}_+$ are equivalent (written $\varphi(t) \approx \psi(t)$) if there are positive constants $c_0 \le 1$ and $c_1 \ge 1$ such that $c_0\psi(t) \le \varphi(t) \le c_1\psi(t)$ for all $t \in \mathbb{R}_+$.

In the lemma below we use the following non-decreasing majorant

$$\varphi^*(t) = \sup_{0<s\le t} \varphi(s)$$

and the non-increasing minorant

$$\varphi_*(t) = \inf_{0<s\le t} \varphi(s)$$

of a non-negative function φ.

Lemma 2.5 *For each almost increasing or almost decreasing function $\varphi(t)$ there exist non-decreasing and non-increasing functions $\varphi_0(t)$ and $\varphi_1(t)$, respectively, such that $\varphi(t) \approx \varphi_0(t) \approx \varphi_1(t)$, and one may take $\varphi_0 = \varphi^*(t)$ and $\varphi_1 = \varphi_*(t)$.*

Proof It is easy to see that

$$\varphi(t) \leq \varphi^*(t) \leq c\varphi(t)$$

when φ is almost increasing and

$$\frac{1}{c}\varphi(t) \leq \varphi_*(t) \leq \varphi(t)$$

when φ is almost decreasing, where $c > 1$. This completes the proof. □

Next we extend the definition of the class $Q[a_0, a_1]$ to the more general class $Q^*[a_0, a_1]$ by just replacing the conditions that the functions $\varphi(t)t^{-a_0}$ and $\varphi(t)t^{-a_1}$ are non-decreasing and non-increasing by that they are almost non-decreasing and almost non-increasing, respectively.

Similar generalizations can be done to define the related classes $Q^*(a_0, a_1)$, $Q^*(a_0, a_1]$, $Q^*(-, a_1)$ etc.

Example 2.6 Let $\alpha \in \mathbb{R},\ \beta \in \mathbb{R}_+$. Denote $\widetilde{\ln}\frac{e}{t} = \begin{cases} \ln\frac{e}{t}, 0 < t \leq 1, \\ 1, t > 1. \end{cases}$

The function $\omega(t) = t^\alpha \widetilde{\ln}\frac{e}{t}$ belongs to $Q^*[a_0, a_1]$ whenever $a_0 < \alpha$ and $a_1 \geq \alpha$, but it does not belong to any class $Q[a_0, a_1]$ with such values of a_0 and a_1.

Next we state the following result we need later on.

Proposition 2.7 *Let $\psi \in Q^*[a_0, a_1]$. Then there exists a constant $c = c(\psi) > 0$ such that*

$$\frac{1}{c}\min(s^{a_0}, s^{a_1})\psi(t) \leq \psi(st) \leq c\max(s^{a_0}, s^{a_1})\psi(t) \text{ for } s > 0 \text{ and } t > 0 \quad (2.9)$$

Proof The condition $\psi(t) \in Q^*[a_0, a_1]$ implies that

$$\psi(st)(st)^{-a_0} \leq c\psi(t)t^{-a_0} \quad \text{for } 0 < s \leq 1 \quad (2.10)$$
$$c\psi(st)(st)^{-a_0} \geq \psi(t)t^{-a_0} \quad \text{for } s \geq 1 \quad (2.11)$$
$$\psi(st)(st)^{-a_1} \leq c\psi(t)t^{-a_1} \quad \text{for } s \geq 1 \quad (2.12)$$
$$c\psi(st)(st)^{-a_1} \geq \psi(t)t^{-a_1} \quad \text{for } 0 < s \leq 1 \quad (2.13)$$

The right hand side inequality in (2.9) follows by just combining (2.10) with (2.12) and the left hand side inequality in (2.9) is a similar consequence of (2.11) and (2.13), which completes the proof. □

Our final result in this section reads:

Theorem 2.8 *For every function* $\omega \in Q^*[a_0, a_1]$ *there exist functions* $\varphi_0(t) \approx \omega(t)$ *and* $\varphi_1(t) \approx \omega(t)$, *such that* $\varphi_0(t)t^{-a_0}$ *is non-decreasing and* $\varphi_1(t)t^{-a_1}$ *is non-increasing. One can write explicit expressions for the functions* φ_0 *and* φ_1 :

$$\varphi_0(t) = t^{a_0} \left(\frac{\omega(t)}{t^{a_0}}\right)^* = t^{a_0} \sup_{0<s<t} \frac{\omega(s)}{s^{a_0}} \tag{2.14}$$

and

$$\varphi_1(t) = t^{a_1} \left(\frac{\omega(t)}{t^{a_1}}\right)_* = t^{a_1} \inf_{0<s<t} \frac{\omega(s)}{s^{a_1}}. \tag{2.15}$$

Proof By Lemma 2.5, there exists a non-decreasing function $\psi_0(t)$ such that $\omega(t)t^{-a_0} \approx \psi_0(t) = \left(t^{-a_0}\omega(t)\right)^*$. Re-denote $\psi_0(t) = \varphi_0(t)t^{-a_0}$, so that

$$\omega(t) \approx \varphi_0(t) \text{ and } \varphi_0(t)t^{-a_0} \text{ is non} - \text{decreasing.}$$

Similarly, by Lemma 2.5 there exists a non-increasing function $\psi_1(t) = \left(t^{-a_1}\omega(t)\right)_*$, such that $\psi_1(t) \approx \left(t^{-a_1}\omega(t)\right)$. Re-denote $\psi_1(t) = \varphi_1(t)t^{-a_1}$, so that

$$\omega(t) \approx \varphi_1(t) \text{ and } \varphi_1(t)t^{-a_1} \text{ is non} - \text{increasing.}$$

This completes the proof. □

3 Almost quasi-monotone functions vis-a-vis various index conditions

Let $\psi(t)$ be a quasi-monotone function. In the paper [16] the following “index function” was introduced:

$$\bar{\alpha}_\psi(t) := \sup_{s>0} \frac{\psi(st)}{\psi(s)}. \tag{3.1}$$

Moreover, the important class $\mathfrak{P}^{+-}$ was defined as consisting of all functions $\psi(t)$ in $Q[0, 1]$ such that

$$\bar{\alpha}_\psi(t) = o(\max(1, t)) \text{ as } t \to 0^+ \text{ or } t \to \infty. \tag{3.2}$$

In this paper we complement the definition (3.1) by introducing the following related index function

$$\bar{\beta}_\psi(t) := \inf_{s>0} \frac{\psi(st)}{\psi(s)}. \tag{3.3}$$

The reason for this is that then we have the following useful result.

Theorem 3.1 *Let $\psi(t)$ be a quasi-monotone function. If $\psi \in Q[a_0, a_1]$, then*

$$\min(t^{a_0}, t^{a_1}) \leq \bar{\beta}_\psi(t) \leq \bar{\alpha}_\psi(t) \leq \max(t^{a_0}, t^{a_1}), \;\; t > 0. \tag{3.4}$$

where $\bar{\alpha}_\psi(t)$ and $\bar{\beta}_\psi(t)$ are defined by (3.1) and (3.3), respectively.

Proof The condition $\psi \in Q[a_0, a_1]$ means, in particular, that

$$\frac{\psi(st)}{\psi(s)} \leq t^{a_0} \quad \text{for } 0 < t \leq 1, \tag{3.5}$$

$$\frac{\psi(st)}{\psi(s)} \geq t^{a_0} \quad \text{for } t \geq 1, \tag{3.6}$$

$$\frac{\psi(st)}{\psi(s)} \leq t^{a_1} \quad \text{for } t \geq 1 \tag{3.7}$$

and

$$\frac{\psi(st)}{\psi(s)} \geq t^{a_1} \quad \text{for } 0 < t \leq 1. \tag{3.8}$$

By combining (3.5) with (3.7), we obtain the right hand side inequality in (3.4). The left hand side inequality in (3.4) follows similarly from (3.6) and (3.8). The proof is complete. □

Corollary 3.2 *If $\psi(t) \in Q(0, 1)$, then $\psi(t) \in \mathfrak{P}^{+-}$.*

Proof Let $\psi(t) \in Q(0, 1)$. Apply the right hand side inequality in (3.4), we get $\bar{\alpha}_\psi(t) \leq \max(t^{\varepsilon}, t^{-1+\varepsilon})$ for some $\varepsilon > 0$, so that (3.2) is satisfied. □

Example 3.3 Let $K = K(t)$ be the Peetre K-functional (for any quasi-Banach couple). Since $K(t) \in Q[0, 1]$, we have that, for all $t > 0$,

$$\min(1, t) \leq \bar{\beta}_K(t) \leq \bar{\alpha}_K(t) \leq \max(1, t).$$

For the case when the quasi-monotone function $\psi(t)$ is differentiable, the following indices were first introduced in [18] (see also [15]):

$$\beta_\psi = \inf_{t>0} \frac{t\psi'(t)}{\psi(t)}, \quad \alpha_\psi = \sup_{t>0} \frac{t\psi'(t)}{\psi(t)}.$$

Note that quasi-monotone functions are differentiable almost everywhere in view of the property (e) of Theorem 2.1. Indices of such a type but in the form

$$\liminf_{t\to\infty} \frac{t\psi'(t)}{\psi(t)} \quad \text{and} \quad \limsup_{t\to\infty} \frac{t\psi'(t)}{\psi(t)}$$

were earlier introduced in [38] in the study of interpolation in Orlicz spaces.

Remark 3.4 The indices β_ψ and α_ψ were used in [15] to define the class B_ψ (also related to real interpolation) defined as all continuously differentiable functions $\psi = \psi(t)$ such that $0 < \beta_\psi \le \alpha_\psi < 1$.

Theorem 3.5 *Let $\psi(t)$ be a continuously differentiable function. Then the following conditions are equivalent:*

(a) $\psi(t) \in Q[a_0, a_1]$.
(b) $a_0 \le \beta_\psi \le \alpha_\psi \le a_1$.

Proof Assume that (a) holds. Let $f(t) := \psi(t)t^{-a_0}$ be non-decreasing. Then we have that, for all $t > 0$,

$$\begin{aligned} f'(t) &= \psi(t)(-a_0)t^{-a_0-1} + \psi'(t)t^{-a_0} \\ &= t^{-a_0-1}\psi(t)\left(\frac{t\psi'(t)}{\psi(t)} - a_0\right) \ge 0 \quad \text{for all } t > 0, \end{aligned}$$

which implies that $\beta_\psi \ge a_0$. Similarly, we find that if $f(t) := \psi(t)t^{-a_1}$ is non-increasing,

$$f'(t) = t^{-a_1-1}\psi(t)\left(\frac{t\psi'(t)}{\psi(t)} - a_1\right) \le 0 \quad \text{for all } t > 0,$$

so that $\alpha_\psi \le a_1$. Hence we have proved that $(a) \Rightarrow (b)$.

Assume now that (b) holds i.e. that $\beta_\psi \ge a_0$ and $\alpha_\psi \le a_1$. Then, for all t, with $f(t) = \psi(t)t^{-a_0}$ we find that

$$f'(t) = t^{-a_0-1}\psi(t)\left(\frac{t\psi'(t)}{\psi(t)} - a_0\right) \ge t^{-a_0-1}\psi(t)\left(\beta_\psi - a_0\right) \ge 0 \text{ for all } t > 0,$$

i.e. $\psi(t)t^{-a_0}$ is non-decreasing. Similarly with $f(t) = \psi(t)t^{-a_1}$ we have that

$$f'(t) = t^{-a_1-1}\psi(t)\left(\frac{t\psi'(t)}{\psi(t)} - a_1\right) \le t^{-a_1-1}\psi(t)\left(\beta_\psi - a_1\right) \le 0 \text{ for all } t > 0,$$

i.e. $\psi(t)t^{-a_1}$ is non-increasing. Hence also the implication $(b) \Rightarrow (a)$ is proved so the proof is complete. □

From Theorem 3.5 the following example can be derived.

Example 3.6 Let $\psi(t)$ be continuously differentiable. Then $\psi \in B_\psi$ if and only if $\psi \in Q(0, 1)$.

Example 3.7 Let $K = K(t)$ be the Peetre K-functional (for any quasi-Banach couple). Since $K(t) \in Q[0, 1]$ we have that

$$0 \le \beta_K \le \alpha_K \le 1.$$

Next we introduce the indices $m(\psi)$ and $M(\psi)$. We shall consider functions ψ positive on $\mathbb{R}_+$. More precisely, since we do not suppose here that the functions ψ are continuous, we assume that

$$0 < \inf_{\delta<t<N} \psi(t) \leq \sup_{\delta<t<N} \psi(t) < \infty \tag{3.9}$$

for all $0 < \delta < N < \infty$. The indices $m(\psi)$ and $M(\psi)$ of functions ψ, satisfying the assumption (3.9), are defined as follows:

$$m(\psi) := \lim_{t\to 0} \frac{\ln\left(\limsup_{s\to 0} \frac{\psi(st)}{\psi(s)}\right)}{\ln t} = \sup_{0<t<1} \frac{\ln\left(\limsup_{s\to 0} \frac{\psi(st)}{\psi(s)}\right)}{\ln t} \tag{3.10}$$

and

$$M(\psi) := \lim_{t\to \infty} \frac{\ln\left(\limsup_{s\to 0} \frac{\psi(st)}{\psi(s)}\right)}{\ln t} = \inf_{t>1} \frac{\ln\left(\limsup_{s\to 0} \frac{\psi(st)}{\psi(s)}\right)}{\ln t}. \tag{3.11}$$

The coincidence of the two expressions in (3.10) as well as in (3.11) follows from the fact that the function $\Omega(t) = \limsup_{s\to 0} \frac{\psi(st)}{\psi(s)}$ or $\Omega(t) = \limsup_{s\to \infty} \frac{\psi(st)}{\psi(s)}$ is sub-multiplicative, i.e. $\Omega(t_1, t_2) \leq \Omega(t_1)\Omega(t_2)$, and it is known that $\sup_{0<t<1} \frac{\ln \Omega(t)}{\ln t} = \lim_{t\to 0} \frac{\ln \Omega(t)}{\ln t}$ and $\inf_{t>1} \frac{\ln \Omega(t)}{\ln t} = \lim_{t\to\infty} \frac{\ln \Omega(t)}{\ln t}$ for sub-multiplicative functions, see [23] or [20].

These indices were defined in [23] in the study of Young functions defining Orlicz spaces. They were independently defined in [33] in a more general setting of weight functions (see also [34]) and used to obtain numerical characteristics for description of the so-called Bary-Stechkin classes, which goes back to the paper [2]. An overview of various properties of these indices may be found in [36, Section 6]. We use this opportunity to note a typo there: $\sup_{x>1}$ in the formula (6.2) in [36] should be replaced by $\inf_{x>1}$.

Remark 3.8 Given a function ψ, in order to calculate its indices $m(\psi)$ and $M(\psi)$ it is sufficient to know the values of ψ only near the origin, in view of the presence $\limsup_{s\to 0}$ in (3.10) and (3.11). In other words, these indices remain unchanged if we arbitrary change the values of $\psi(t)$ for $t > \delta$.

By $\Psi^{\uparrow}$ ($\Psi_{\downarrow}$, respectively) we denote the class of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying the condition (3.9) for which there exists a number $\xi = \xi(\psi)$ such that the function

$$\frac{\psi(t)}{t^{\xi}} \text{ is almost increasing (almost decreasing, respectively),} \tag{3.12}$$

so that

$$\Psi^{\uparrow} \bigcap \Psi^{\downarrow} = \bigcup_{-\infty < a_0 \le a_1 < \infty} Q^*[a_0, a_1].$$

Evidently, if $\psi \in \Psi^{\uparrow}$ ($\Psi_{\downarrow}$, resp.), then $\frac{\psi(t)}{t^a} \in \Psi^{\uparrow}$ ($\Psi_{\downarrow}$, resp.), $a \in \mathbb{R}$.

In the sequel we use the abbreviations a.i. and a.d. for *almost increasing* and *almost decreasing*, respectively.

In the following lemma we consider equivalence of functions in $\Psi^{\uparrow} \bigcap \Psi^{\downarrow}$ to continuous functions.

Lemma 3.9 *Let* $\frac{\psi(t)}{t^{\xi}}$ *be a.i. and* $\frac{\psi(t)}{t^{\eta}}$ *be a.d.,* $-\infty < \xi < \eta < \infty$. *Then, for every* $\nu < \xi$, *the function* $\psi_\nu(t) := \frac{\psi(t)}{t^{\nu}}$ *is equivalent to a continuous function* $\widetilde{\psi_\nu}$ *with* $\widetilde{\psi_\nu}(0) = 0$.

Proof We show that $\widetilde{\psi_\nu}(t)$ may be taken as $\widetilde{\psi_\nu}(t) = \int\limits_0^t \frac{\psi(s)}{s^{\nu+1}} ds$. To this end, we obtain

$$\widetilde{\psi_\nu}(t) = \int\limits_0^t \frac{\psi(s)}{s^{\nu+1}} ds = \int\limits_0^t \frac{\psi(s)}{s^{\xi}} s^{\xi-\nu-1} ds \le C \frac{\psi(t)}{t^{\xi}} \int\limits_0^t s^{\xi-\nu-1} ds = \frac{C}{\xi-\nu} \frac{\psi(t)}{t^{\nu}}.$$

On the other hand,

$$\psi_\nu(t) = (\eta-\nu)\frac{\psi(t)}{t^{\eta}} \int\limits_0^t s^{\eta-\nu-1} ds \le (\eta-\nu) C \int\limits_0^t \frac{\psi(s)}{s^{\nu+1}} ds = (\eta-\nu) C \widetilde{\psi_\nu}(t),$$

which completes the proof. □

In [33], it was shown that, given a function ψ, its indices m_ψ and M_ψ are sharp upper and lower bounds of the exponents ξ admissible for (3.12), i.e.

$$\begin{aligned} m(\varphi) &= \sup\left\{\xi \in \mathbb{R} : \frac{\psi(t)}{t^{\xi}} \text{ is almost increasing}\right\} \\ &= \sup\left\{\xi \in \mathbb{R} : \psi(st) \le c s^{\xi} \psi(t),\ 0 < s < 1\right\} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} M(\varphi) &= \inf\left\{\eta \in \mathbb{R} : \frac{\psi(t)}{t^{\eta}} \text{ is almost increasing}\right\} \\ &= \inf\left\{\eta \in \mathbb{R} : \psi(st) \le c s^{\eta} \psi(t),\ s > 1\right\}, \end{aligned} \tag{3.14}$$

under some a priori assumptions on the function ψ. Under weaker assumptions it was proved in [19, Theorem 3.6], (see also [36, Theorem 6.2]). We formulate the corresponding result later under improved assumptions (see Corollary 3.18).

In the following theorem we provide a simple proof for the bounds $\xi \leq m(\psi)$ and $\eta \geq M(\psi)$ not touching the issue of sharpness, just to show that for this fact we need no assumption on the function ψ.

Theorem 3.10 *If $\frac{\psi(t)}{t^\xi}$ is a.i. for some $\xi \in \mathbb{R}$, then $\xi \leq m(\psi)$.*

If $\frac{\psi(t)}{t^\xi}$ is a.d. for some $\eta \in \mathbb{R}$, then $\eta \leq M(\psi)$.

Proof We rewrite the formula (3.10), via the changes of variables $t = \frac{1}{\tau}$ and $s = \sigma\tau$, in the form

$$m(\psi) = \sup_{\tau>1} \frac{\ln\left(\liminf_{\sigma\to 0} \frac{\psi(\sigma\tau)}{\psi(\sigma)}\right)}{\ln\tau},$$

so that

$$m(\psi) \geq \frac{\ln\left(\liminf_{\sigma\to 0} \frac{\psi(\sigma\tau)}{\psi(\sigma)}\right)}{\ln\tau}$$

for all $\tau > 1$. Since $\frac{\psi(\sigma)}{\sigma^\xi}$ is a.i., we have $\frac{\psi(\sigma\tau)}{\psi(\sigma)} \geq c\tau^\xi$. Hence,

$$\frac{\ln\left(\liminf_{\sigma\to 0} \frac{\psi(\sigma\tau)}{\psi(\sigma)}\right)}{\ln\tau} \geq \xi + \frac{\ln c}{\ln\tau}$$

and then $m(\psi) \geq \xi + \frac{\ln c}{\ln\tau}$. Passing to the limit as $\tau \to \infty$, we obtain $m(\psi) \geq \xi$.

The second statement of the theorem is similarly obtained from (3.11), in this case no change of variables is needed. The proof is complete. □

Corollary 3.11 *Every function $\psi \in \Psi^{\uparrow} \bigcap \Psi^{\downarrow}$ has finite indices $m(\psi)$ and $M(\psi)$.*

Corollary 3.12 *Let $\psi \in Q^*[a_0, a_1]$. Then $a_0 \leq m(\psi) \leq M(\psi) \leq a_1$.*

Example 3.13 Let $K = K(t)$ denote the Peetre functional (for any quasi-Banach couple). Then

$$0 \leq m(K) \leq M(K) \leq 1.$$

As mentioned, in terms of the indices $m(\psi)$ and $M(\psi)$ in [33] and [34], there were given numerical characteristics for the description of the Bary-Stechkin class. This class, denoted by Φ, was introduced in [2] in the study of the smoothness of conjugate functions, or in other words, boundedness of the singular operator along a unit circle.

More generally than before, now we consider functions on $(0, \ell)$, $0 < \ell \leq \infty$. In the case $\ell < \infty$ we interpret the condition (3.9) taking $\delta, N \in (0, \ell)$.

We say that a function ψ belongs to the class Φ if

$$\int_0^s \frac{\psi(t)}{t}dt \le C\psi(s) \text{ and } \int_s^\ell \frac{\psi(t)}{t^2}dt \le C\frac{\psi(s)}{s}, \tag{3.15}$$

where $C = C(\psi) > 0$ does not depend on s. The inequalities in (3.15) are known as Zygmund conditions.

The Bari-Stechkin class Φ and the corresponding Zygmund conditions were generalized, in relation with various applications, in [19] and [35]. Following these papers we define the Zygmund classes $\mathbb{Z}^\beta$ and $\mathbb{Z}_\gamma$ by the conditions

$$\int_0^s \frac{\psi(t)}{t^{1+\beta}}dt \le C\frac{\psi(s)}{s^\beta} \text{ and } \int_s^\ell \frac{\psi(t)}{t^{1+\gamma}}dt \le C\frac{\psi(s)}{s^\gamma}, \tag{3.16}$$

respectively, and define

$$\Phi_\gamma^\beta = \mathbb{Z}^\beta \bigcap \mathbb{Z}_\gamma,$$

so that $\Phi = \Phi_1^0$.

The following propositions, proved in [19, Theorems 3.1 and 3.2] (see also [34, Theorem 7.9] for the case $\beta = \gamma = 0$), contain a characterization of the classes $\mathbb{Z}^\beta$ and $\mathbb{Z}_\gamma$ under some à priory assumption on the functions ψ, namely, there is assumed that $\psi \in W$, where the class W is defined by the conditions: 1) (3.9) holds, 2) ψ is continuous near the origin and $\psi(0) = 0$, 3) ψ is a.i.

Remark 3.14 Lemma 3.9 states, in other words, that if $\psi \in Q^*[\xi, \eta]$, then for every $\nu < \xi$ the function $\frac{\psi(t)}{t^\nu}$ is equivalent to a function in W.

The following propositions provide certain characterization of the classes $\mathbb{Z}^\beta$ in terms of indices.

Proposition 3.15 *Let $\psi \in W$ and $\beta \in \mathbb{R}$. The following statements are equivalent:*

(1) $\psi \in \mathbb{Z}^\beta$,
(2) $\frac{\psi(t)}{t^{\beta+\delta}}$ *is a.i. for some* $\delta > 0$,
(3) $m(\psi) > \beta$.

If one of the conditions 1), 2) or 3) holds, then 2) holds with any $\delta \in (0, m(\psi) - \beta)$

Proposition 3.16 *Let $\psi \in W$ and $\gamma \in \mathbb{R}$. The following statements are equivalent:*

(1) $\psi \in \mathbb{Z}_\gamma$,
(2) $\frac{\psi(t)}{t^{\gamma-\delta}}$ *is a.d. for some* $\delta > 0$,
(3) $M(\psi) < \gamma$.

If one of the conditions 1), 2) or 3) holds, then 2) holds with any $\delta \in (0, M(\psi) - \gamma)$

Remark 3.17 Theorems 3.1 and 3.2 in [19] were formulated for $\beta \geq 0$ and $\gamma > 0$ because this is the most interesting case. As for the case of $\beta < 0$ and $\gamma \leq 0$, the validity of these theorems in this case follows from the validity for positive values of β and γ, since $\psi \in W \Rightarrow t^{-\beta}\psi(t) \in W$ and $\psi \in W \Rightarrow t^{-\gamma}\psi(t) \in W$ when $\beta < 0$ and $\gamma \leq 0$.

Our next result concerns the sharpness of the bounds in (3.13) and (3.14).

Theorem 3.18 *The formulas* (3.13) *and* (3.14) *hold true for every function* $\psi \in \Psi^{\uparrow} \bigcap \Psi_{\downarrow}$.

Moreover, if $\psi \in \Psi^{\uparrow} \bigcap \Psi_{\downarrow}$, *then*

$$\psi \in \mathbb{Z}^{\beta} \iff m(\psi) > \beta \tag{3.17}$$

and

$$\psi \in \mathbb{Z}_{\gamma} \iff M(\psi) < \gamma. \tag{3.18}$$

Proof By assumption there exist numbers ξ and η such that $\frac{\psi(t)}{t^{\xi}}$ is a.i. and $\frac{\psi(t)}{t^{\eta}}$ is a.d. Then $\frac{\psi(t)}{t^{\xi-\varepsilon}}$ is equivalent to a function in W, see Remark 3.14. So we may consider the function $\varphi(t) := \frac{\psi(t)}{t^{\xi-\varepsilon}}$ itself as belonging to W, since the statements (1), (2) and (3) of Proposition 3.15 are invariant with respect to replacement of the function ψ by any equivalent one. We then apply Proposition 3.15 with $\beta = 0$ to the function φ and observe that the condition 2) from Proposition 3.15 for this function means that $\frac{\varphi(t)}{t^{\delta}} = \frac{\psi(t)}{t^{\xi}}$ is a.i. under the choice $\delta = \varepsilon$. Then, by Proposition 3.15, the function $\frac{\varphi(t)}{t^{\delta}}$ is a.i. for every $\delta < m(\varphi)$, i.e. $\frac{\psi(t)}{t^{\xi+\delta-\varepsilon}}$ is a.i. for $\xi + \delta - \varepsilon < m(\psi)$, which proves the validity of the formula (3.13).

As regards the statement (3.17), to derive it from Proposition 3.15 avoiding the assumption $\psi \in W$, we use the property $\psi \in \mathbb{Z}^{\beta} \iff t^{a}\psi(t) \in \mathbb{Z}^{\beta+a}$ for any a and note that the function $t^{a}\psi(t)$ is equivalent to a function in W for sufficiently big a, according to the arguments used above.

In a similar way, one can justify the validity of the formula (3.14) by means of Proposition 3.16 and obtain the equivalence (3.18). □

Corollary 3.19 *Let* $\psi \in \Psi^{\uparrow} \bigcap \Psi_{\downarrow}$. *Then* $\psi \in \Phi^{\beta}_{\gamma}$, $-\infty < \beta < \gamma < \infty$, *if and only if*

$$\beta < m(\psi) \leq M(\psi) < \gamma.$$

Corollary 3.20 *Let* $\psi \in \Psi^{\uparrow} \bigcap \Psi_{\downarrow}$. *Then, for every* $\varepsilon > 0$,

$$c_1 \left(\frac{t}{s}\right)^{M(\psi)+\varepsilon} \leq \frac{\psi(t)}{\psi(s)} \leq c_2 \left(\frac{t}{s}\right)^{m(\psi)-\varepsilon}, \quad 0 < t < s < \ell, \tag{3.19}$$

where c_1 *and* c_2 *may depend on* ε, *but do not depend on* t *and* s.

Proof By Theorem 3.18, the function $\frac{\psi(t)}{t^{m(\psi)-\varepsilon}}$ is a.i. and $\frac{\psi(t)}{t^{M(\psi)+\varepsilon}}$ is a.d., which yields (3.19). □

From Corollaries 3.19 and 3.20 we have the following examples.

Example 3.21 If $\psi(t) \in Q(0,1)$, then $\psi(t) \in \Phi$ and, for all $\varepsilon \in (0, \frac{1}{2})$, there exist positive constants $C_1 = C_1(\psi, \varepsilon)$ and $C_2 = C_2(\psi, \varepsilon)$ such that

$$C_1 t^{1-\varepsilon} \leq \psi(t) \leq C_2 t^{\varepsilon}, t < 1 \text{ and } C_1 t^{\varepsilon} \leq \psi(t) \leq C_2 t^{1-\varepsilon}, t > 1. \tag{3.20}$$

Example 3.22 Since the Peetre K-functional $K = K(t)$ belongs to the class $Q[0, 1]$ and thus to the wider class $Q^*[0, 1]$, it has the following property:

$$0 \leq m(K) \leq M(K) \leq 1.$$

We hope that the studies in this section may be useful for investigation of, for instance, optimality problems in various function spaces. For the optimality problems in Orlicz spaces we refer to the recent paper [24].

4 Some generalizations of the real interpolation spaces via the Peetre K-functional

Let (A_0, A_1) be a quasi-Banach couple and let $a \in A_0 + A_1$. We consider the following scale of equivalent quasi-norms on (A_0, A_1), namely the Peetre K-functional:

$$K(t) = K(t, a) = K(t, a; A_0, A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

The classical real interpolation space $(A_0, A_1)_{\theta,q}, 0 < \theta < 1, 0 < q \leq \infty$ consists of all $a \in A_0 + A_1$ such that

$$\|a\|_{\theta,q;K} := \left(\int_0^\infty \left(\frac{K(t,a)}{t^\theta}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty, \tag{4.1}$$

with the usual supremum interpretation of the integral for $q = \infty$. See e.g. the book [3] by Jaak Peetre's students J. Bergh and J. Löfström.

There are some popular generalizations, where we mention the following ones:

[P] The real interpolation space with a parameter function, see [29]. Here the function t^θ is replaced by a "parameter function" $\varphi(t) \in Q(0, 1)$ to obtain the real parameter function spaces $(A_0, A_1)_{\varphi,q}$ with the formula (4.1) replaced by

$$\|a\|_{\varphi,q} := \left(\int_0^\infty \left(\frac{K(t,a)}{\varphi(t)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty. \tag{4.2}$$

We say that a Lebesgue measurable function $b(t) : [1, \infty) \to \mathbb{R}_+$ is a slowly varying function if for any $\varepsilon > 0$ the function $t^{\varepsilon} b(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function. Moreover, we define

$$\gamma_b(t) = b\big(\max(t, t^{-1})\big), t > 0. \tag{4.3}$$

[S] The real interpolation method with a slowly varying function, see e.g. [14]. Here the function t^{θ} is replaced by the function $\frac{t^{\theta}}{\gamma_b(t)}$, where $\gamma_b(t)$ is defined by (4.3), to define the real interpolation spaces $(A_0, A_1)_{\theta,q,b}$ with the formula

$$\|a\|_{(A_0,A_1)_{\theta,q,b}} = \left(\int_0^{\infty} \left(t^{-\theta}\gamma_b(t)K(t,a)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty. \tag{4.4}$$

Remark 4.1 In this method also the endpoint cases $\theta = 0$ and $\theta = 1$ are included so it is only assumed that $0 \le \theta \le 1$. We are not sure who first introduced these spaces but at least in the recent paper [17] the authors referred to the paper [14]. For the end point cases $\theta = 0$ and $\theta = 1$ we especially refer to the recent paper [13].

Our aim is to compare these generalized methods. First we suggest the following modification of the generalization [P]:

[P*] Here the function t^{θ} is replaced by a function $\bar{\varphi}(t)$ which is equivalent to a function $\varphi(t) \in Q(0, 1)$ to obtain the real parameter spaces $(A_0, A_1)_{\bar{\varphi}(t),q}$ with (4.1) replaced by (4.2) but now with φ replaced by $\bar{\varphi}$.

Remark 4.2 In the paper [29] it was proved that several of the most important results in the classical real interpolation theory can be generalized to the case with interpolation with a parameter function, e.g. the interpolation theorem, the equivalence theorem (with the corresponding real interpolation space with the J-functional $\sup\left(\|a_0\|_{A_0}, t\|a_1\|_{A_1}\right)$ involved), the duality theorem, the reiteration theorem, Holmstedt's formula, Wolff's theorem, etc. From the arguments, used in [29], combined with investigations in Sects. 2 and 3, we conclude that all this holds also in the case of the generalized real interpolation parameter function space $(A_0, A_1)_{\bar{\varphi},q}$.

Remark 4.3 For simplicity we have avoided the case $q = \infty$ in the descriptions. However, in this case all the generalized real interpolation spaces [P], [S] and [P*] can be also defined, handled and applied by just replacing the integrals with the standard supremum interpretations of the integrals when $q = \infty$.

The most important fact is:

Remark 4.4 For the case $0 < \theta < 1$ the method [P*] is, of course, more general than the method [S] since the function $t^{-\theta}\gamma_b(t)$ is required to be equivalent to a function $\varphi(t) \in Q(-\varepsilon + \theta, \varepsilon + \theta)$, with $0 < \varepsilon < \min(\theta, 1 - \theta)$.

Example 4.5 In the case [S] we must have functions equivalent to functions varying slowly not far from t^{θ}, $0 < \theta < 1$, e.g. $\varphi(t) = t^{\theta}s(t)$ with $s(t) = 1 + \log t$, $s(t) = (\varepsilon + \log t)^{\alpha}\left(\log(\varepsilon + \log t)\right)^{\beta}$, $(\alpha, \beta \in \mathbb{R})$ and $s(t) = \exp\left(\sqrt{\log t}\right)$. See [17, Example

2.2]. Contrary in the case [P*] (and [P]) we can use functions $\bar{\varphi}$ (and φ) which can have much more general freedom of variation e.g. that, for some positive constants C_0 and C_1 and each ε, $0 < \varepsilon < \frac{1}{2}$,

$$C_1 t^{\varepsilon} \leq \varphi(t) \leq C_2 t^{1-\varepsilon}.$$

This is obvious but see also our Example 3.21.

Remark 4.6 We conjecture that in many of the papers where the [S]-method have been used for the case $0 < \theta < 1$ more general results could have been obtained by instead using the [P] or [P*] method. (See e.g. [17]). One reason for this is that the used discretization (e.g. in dyadic blocks $\sum_{2^k}^{2^{k+1}}$) works perfectly since both the K-functional and the weights are quasi-monotone functions with good control both up and down. Indeed, this was one motivation to develop this type of generalization in [29]. It is also worth to be mentioned that this author(L.E.Persson) studied and used quasi-monotone functions already in PhD thesis [32] from 1974.

Remark 4.7 Instead of replacing the condition "$\varphi \in Q(0, 1)$" in the [P]-method by the condition "$\bar{\varphi} \in Q(0, 1)$" in the [P*]-method we could have replaced by the condition "$\varphi \in Q^*(0, 1)$" and the outcome should have been more or less equivalent.

Next we propose a further generalization of the classical real interpolation method for the endpoint cases this is just the [S] method, see [13, 14], which contains and unifies all methods described above (e.g. [P], [P*] and [S]). We call it the generalized parameter function method.

[GP] Let $b(t)$ be a slowly varying function and $\gamma_b(t)$ defined by (4.3). Moreover, let $\varphi(t) \in Q^*(0, 1)$. We define the generalized parameter function $\psi = \psi_\theta(t)$, $0 \leq \theta \leq 1$, as follows:

$$\psi_\theta(t) = \begin{cases} \varphi(t), & 0 < \theta < 1, \\ \frac{1}{\gamma_b(t)}, & \theta = 0, \\ \frac{t}{\gamma_b(t)}, & \theta = 1. \end{cases}$$

The real interpolation space with the generalized parameter function $\psi_\theta(t)$ is defined as all $a \in A_0 + A_1$ satisfying

$$\|a\|_{\psi_\theta, q} := \left(\int_0^\infty \left(\frac{K(t, a)}{\psi_\theta(t)} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty, \quad 0 < q \leq \infty, \; 0 \leq \theta \leq 1,$$

(with the usual supremum interpretation of the integral when $q = \infty$).

Remark 4.8 From the discussions and motivations above it is clear that at least for the case $0 < \theta < 1$ all classical real interpolation results mentioned in Remark 4.2 hold also in this more general case. If some of the classical function spaces involving parameters are generalized by replacing the corresponding parameter with the parameter function we think that [GP] method may be useful. This suggestion is inspired and supported by the recent paper [17]. See also the book [8] and Remark 5.9.

Remark 4.9 For the endpoint cases $\theta = 0$ and $\theta = 1$, concerning the reiteration we refer to the paper [13]. However, for the case $0 < \theta < 1$ (like in the classical case) a powerful technique to prove reiteration theorems is to use Holmstedt-type formulas and Hardy-type inequalities.

Next we state such a new result for the current case [GP].

Theorem 4.10 *Let $\overline{A} = (A_0, A_1)$ be a quasi-Banach couple. Let $\omega_{\theta,0}$ and $\omega_{\theta,1}$, $0 < \theta < 1$, be generalized parameter functions as defined in [GP] and $\tau_\theta(t) = \frac{\omega_{\theta,1}}{\omega_{\theta,0}}$. If the inverse of $\tau_\theta(t)$ exists, then the following Holmstedt-type formula holds:*

$$K(t, f, \overline{A}_{\omega_{\theta,0},q_0}, \overline{A}_{\omega_{\theta,1},q_1}) \approx \left(\int_0^{\tau_0^{-1}(t)} \left(\frac{K(s, f, \overline{A})}{\omega_{\theta,0}(s)} \right)^{q_0} \frac{ds}{s} \right)^{1/q_0} + t \left(\int_{\tau_0^{-1}(t)}^{\infty} \left(\frac{K(s, f, \overline{A})}{\omega_{\theta,1}(s)} \right)^{q_1} \frac{ds}{s} \right)^{1/q_1},$$

where $0 < q_0, q_1 < \infty$.

Proof The proof consists of step-by-step following the proof of P. Nilsson [26, pp.310-311], for the classical case $\omega_{\theta,i}(t) = t^{\theta_i}$, $i = 0, 1$, so we omit the details. See also [29, p.210]. □

Remark 4.11 According to our investigations in Sect. 3, it is possible to instead of using parameter functions (or generalized parameter functions) in real interpolation theory, we could have formulated the results in terms of indices. We do not go on further in this direction but just do the following reformulation of the duality theorem in [29, Theorem 2.4].

Proposition 4.12 *Let $\overline{A} = (A_0, A_1)$ be a Banach couple such that $\Delta\overline{A}$ is dense in both A_0 and A_1 and let ψ satisfy the following condition $0 < m(\psi) \leq M(\psi) < 1$. Then, for $1 \leq q < \infty$,*

$$(A_0, A_1)'_{\psi,q} = (A'_0), A'_1)_{\psi_1,q_1},$$

where $\psi_1(t) = 1/\psi(\frac{1}{t})$ and $\frac{1}{q} + \frac{1}{q'} = 1$.
(As usual, (A'_i) denotes the dual space of A_i, $i = 0, 1$.)

Proof Use Corollary 3.12 and follow the classical proof in [3, p.54] step-by-step. See also [29, p.205]. □

Remark 4.13 Nowadays there also are known more general real interpolation spaces defined by the Peetre K-functional. For example, we have the more general K-interpolation space $\overline{A}_\Phi = (A_0, A_1)_\Phi$ consisting of those $f \in A_0 + A_1$ for which the quasi-norm

$$\|f\|_{\overline{A}_\Phi} = \|K(t, f)\|_\Phi$$

is finite, see [6]. Here Φ is a quasi-norm space of Lebesgue measurable functions defined on $(0, \infty)$ with monotone quasi-norm. However, the applications of this most general theory are not yet fully developed, so it is of interest to prove such results separately.

Our final remark is related to Theorem 4.10.

Remark 4.14 In the classical situation Holmstedt's formula can be used to prove the reiteration theorem. This is true also in the first parameter method [P] (see [29], pp. 208-212) and, thus, as motivated in this paper, for our more general real interpolation method [GP] for the case $0 < \theta < 1$. Recently in [1] a Holmstedt-type formula was proved also for the general case described in Remark 4.13. However, in this case it is less obvious how this Holmstedt-type formula implies come corresponding reiteration theorem in this generality. It is also worth to mention that the proof in [1] is surprisingly simple and that a crucial ingredient is that (in our terminology) the K-functional $K(t, f) \in Q[0, 1]$ (see (1.4) and (1.5)).

5 Final considerations

5.1 On the relations to wavelets

In this subsection we shortly describe such relations both from an engineering and a mathematical point of view. Concerning the first aspect we refer to the PhD thesis [9], which was the starting point and motivation for this paper. In both aspects the standard Besov spaces B^s_{pq}, $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ are central. There are some different, but of course equivalent, definitions of these spaces, see e.g. the books [3, 27, 40] and the paper [39].

First we give a short description from [10]. The wavelet expansion of $f(x)$ is

$$f(x) = \sum_{k\in\mathbb{Z}} \alpha_{j_0k}\phi_{j_0k}(x) + \sum_{j=j_0}^{\infty}\sum_{k\in\mathbb{Z}} \beta_{jk}\psi_{jk}(x),$$

where $j_0 \in \mathbb{Z}$, ϕ_{j_0k} and ψ_{jk}, $j = j_0, j_0 + 1, \ldots$ form an orthonormal basis of $L_2(\mathbb{R})$. Convergence of the wavelet series is described using the quasi-norm topology of the Besov spaces B^s_{pq}, $0 < p \leq \infty$, $0 < q \leq \infty$. All algebraic polynomials P with $\deg P \leq [r]$ and $\phi \in B^r_{\infty\infty}$ are contained in the linear span of $\phi(. - k)$, $k \in \mathbb{Z}$, for $r > 0$. Moreover, ϕ_{j_0k}, ψ_{jk} form a Riesz basis for all B^s_{pq}, $0 < p \leq \infty$, $0 < q \leq \infty$, $\max\left(0, \frac{1}{p-1}\right) < s < r$. See e.g. [37].

According to [10], for penalized wavelet estimation, let $A_1 = B^s_{pq}$, $A_0 = B^\sigma_{\pi u}$, for a fixed value of t, then the estimated function $\tilde{f}(x)$ is expected to be in B^s_{pq}, and the loss functional is the quasi-norm in $B^\sigma_{\pi u}$. For the Hilbert case $\pi = u = p = q = 2$, $\sigma = 0$, the estimator

$$\tilde{f}(x) = \sum_{k\in\mathbb{Z}} \tilde{\alpha}_{j_0k}\phi_{j_0k}(x) + \sum_{j=j_0}^{j_1}\sum_{k\in\mathbb{Z}} \tilde{\beta}_{jk}\psi_{jk}(x)$$

is linear in the data for any fixed t in the K-functional. As we have noted before this is just an equivalent norm on the sum space $A_0 + A_1$.

Furthermore, according to [10, Lemma 1], for fixed $s > 0$ and $t > 0$, the quality of fit can be measured by the K-functional $K_2(t, \hat{f}, B^0_{2,2}, B^s_{2,2})$. Then the K-functional is attained with coefficients $\tilde{\alpha}_{j_0k} = \hat{\alpha}_{j_0k}$, $\tilde{\beta} = \frac{\hat{\beta}_{jk}}{1+t^2 2^{2js}}$, $j = j_0, \ldots, j_1$, where $\hat{\alpha}_{j_0k}$ and $\hat{\beta}_{jk}$ are empirical wavelet coefficients, and in this case the resulting estimator is level-dependent and of non-threshold shrinkage type. This fact is of great interest for some engineering applications (see [9]), but also for more theoretical developments e.g. those presented in this paper.

The more theoretical interest is when we try to overcome exact descriptions of the real interpolation spaces

$$A = \left(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}\right)_{\theta,q}$$

for all involved parameters. In some cases we fall inside the Besov scale of spaces e.g. in the so-called diagonal case ($s^* = (1-\theta)s_0 + s_1$, $\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$) we have that

$$A = B^{s^*}_{p^*q^*}.$$

But in some other cases this is a more difficult problem and as far as we know not all cases are fully solved.

It seems that this problem was first observed by J.Peetre in his monograph [27]. After that several developments were given, for instance, in [11, 12], see also the books [39, 40]. As a recent result, we mention that in the paper [22] there was proved an exact description of the spaces $\left(B^s_{p_0,q}, B^s_{p_1,q}\right)_{\theta,r}$ by using so-called Meyer wavelets. Moreover, for the case $r = q$ this description falls inside the scale of Besov-Lorentz spaces. This is a typical result since many such new results are using the fact that one way to describe Besov type spaces is via suitable wavelet theory.

Hence we can claim that the relation between wavelets, Besov spaces and the K-functional / real interpolation is important both from theoretical and engineering points of view.

5.2 Concluding remarks and results

In all of this subsection we assume that as before $K = K(t) = K(t, a; A_0, A_1)$ denotes the Peetre K-functional for any quasi-Banach couple (A_0, A_1).

Remark 5.1 The fact that $K(t)$ is concave, i.e. $-K(t)$ is convex implies automatically a lot of interesting properties and applications. See e.g. the book [25] and the references therein. In particular, it is known that the concept of convexity implies most of the classical inequalities see e.g. [30] and [21]. Just as one simple example, we think has not been pointed out in the interpolation literature before, is the following consequence of the reversed Hardy-Knopp inequality:

Example 5.2 Let $f(t)$ be a measurable function on $\mathbb{R}_+$. Then, by Jensen's reversed inequality we have that

$$K\left(\frac{1}{x}\int_0^x f(t)dt\right) \geq \frac{1}{x}\int_0^x K\big(f(t)\big)dt$$

so that, by Fubini's theorem,

$$\begin{aligned}\int_0^\infty \frac{1}{x^\theta} K\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} &\geq \int_0^\infty \frac{1}{x^{2+\theta}}\int_0^x K(f(t))dtdx \\ &= \int_0^\infty K(f(t))\int_t^\infty \frac{1}{x^{2+\theta}}dxdt \\ &= \frac{1}{1+\theta}\int_0^\infty \frac{(Kf(t))}{t^\theta}\frac{dt}{t},\end{aligned}$$

i.e. that

$$(1+\theta)\left\|\frac{1}{x}\int_0^x f(t)dt\right\|_{\theta,1} \geq \|f(x)\|_{\theta,1}.$$

Remark 5.3 Note that the arithmetic mean operator $H : Hf(t) := \frac{1}{t}\int_0^t f(x)dx$ is the so-called Hardy operator which is fundamental in the theory of Hardy-type inequalities, see the book [21] and [31] for some results related to this paper. Hence, Example 5.2 shows a not so pronounced relation between Hardy operators and K-functional / Interpolation theory.

Remark 5.4 In Sect. 3 of this paper we have pointed out several both old and new conditions concerning the quasi-monotone class $Q[a_0, a_1]$ and index conditions. Since $K(t) \in Q[0, 1]$, many of these conditions is inherited by $K(t)$. Next, we sum up some of these conditions which has not been explicitly pointed out in the interpolation literature before.

Example 5.5 Let $\psi(t)$ be a quasi-monotone function. Let $\bar{\beta}_\psi(t)$ and $\bar{\alpha}_\psi(t)$ be the index functions defined in Sect. 3. Moreover, let β_ψ, α_ψ, $m(\psi)$, $M(\psi)$, $\alpha^*(\psi)$ and $\beta^*(\psi)$ be the indices defined in the same Section. Then, in particular,

$$\begin{aligned}\min(1,t) \leq \bar{\beta}_K(t) \leq \bar{\alpha}_K(t) \leq \max(1,t), \\ 0 \leq \beta_K \leq \alpha_K \leq 1, \\ 0 \leq m_K \leq M_K \leq 1, \\ 0 \leq \alpha^*(K) \leq \beta^*(K) \leq 1.\end{aligned}$$

Remark 5.6 Another consequence of our investigations in Sect. 3 is the close connection between the parameter classes $Q(0, 1)$ and various index conditions. For example in the generalization [P] everything holds if the condition "$\varphi(t) \in Q(0, 1)$" is replaced by the index condition "$0 < \beta_\psi \leq \alpha_\psi < 1$". See also Remark 4.11 and also Proposition 4.12.

Finally, we sum up and complement the close relations between the parameter functions classes $Q(0, 1)$, $\mathfrak{P}^{+-}$ and B_ψ, all related to interpolation theory defined in Sect. 3.

Proposition 5.7 *Let $\psi(t)$ be a continuously differentiable quasi-monotone function. Then*

(*a*) $B_\psi \subset Q(0, 1) \subset \mathfrak{P}^{+-}$

(*b*) *If $\varphi(t) \in \mathfrak{P}^{+-}$, then there exists a function $\psi(t) \in B_\psi$ such that $\varphi(t) \approx \psi(t)$*

Proof The inclusions in (a) follows by just combining Corollary 3.2, Remark 3.4 and Theorem 3.5. The proof of (b) is implicitly done in [15], so we owe this argument to J. Gustavsson, see also [29, p.208]. The key is that the equivalent function ψ is defined by the formula

$$\psi(s) = \int_0^\infty \min\left(1, \frac{s}{t}\right) \varphi(t) \frac{dt}{t}.$$

Indeed by making some calculations (see [15, p.293]) we find that $\psi(t) \in B_\psi$. □

Remark 5.8 In particular, Proposition 5.7 gives new possibilities to replace the condition $\varphi(t) \in Q(0, 1)$ in [P] and [P*] by further index or index function conditions. See also Remark 4.11.

Remark 5.9 In connection to interpolation with a parameter function (see [29] and our generalization [GP] in Sect. 3 of this paper) it is natural to do a similar parameter function generalization of the usual parameter in the definition of the involved interpolation spaces. For some developments in this direction in the Hardy-Sobolev case we refer to the book [8] by F. Cobos and D. Fernandez.

Remark 5.10 Concerning the difficulties in real interpolation in off-diagonal cases when the spaces are fairly close to each other we refer to the paper [28] by L.E. Persson.

Remark 5.11 As described in this paper and all books we refer to, in the standard real interpolation theory we interpolate between two Banach spaces A_0 and A_1. However, there also are less known studies of real interpolation theory concerning real interpolation between finite or infinite many Banach spaces, even so-called families of Banach spaces. For such studies we refer to the paper [7].

Author Contributions All co-authors equally contributed in joint discussions and preparation of the manuscript, and approved the version to be published.

Funding Open access funding provided by UiT The Arctic University of Norway (incl University Hospital of North Norway).

Data availibility No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

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