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Counting and Equidistribution of Reciprocal Geodesics and Dihedral Groups

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We study the growth of the number of conjugacy classes of infinite dihedral subgroups of lattices in $\mathrm{PSL}_2\,\mathbb{R}$, generalizing the earlier work of Sarnak [9] and Bourgain–Kontorovich [4] on the growth of the number of reciprocal geodesics on the modular surface. We also prove that reciprocal geodesics are equidistributed in the unit tangent bundle.

1

In this note we are interested in counting conjugacy classes of *infinite dihedral subgroups*, that is subgroups isomorphic to $(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/2\mathbb{Z})$, of lattices $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$. Evidently, we will only care about lattices Γ with 2-torsion, that is lattices that have elements of order two—otherwise Γ has no infinite dihedral subgroups. With the action of Γ on the hyperbolic plane $\mathbb{H}^2$ in mind, we refer to the elements of order two as *involutions*.

Discrete infinite dihedral subgroup of $\mathrm{PSL}_2\,\mathbb{R}$, for example those that arise as subgroups of a lattice, preserve a unique geodesic $\mathcal{A}_D$ in $\mathbb{H}^2$, the *axis* of D. We will refer to the length $\ell(D)$ of the quotient $D\backslash\mathcal{A}_D$ as the *length* of D. Our first goal is to study the asymptotic behaviour of the number of conjugacy classes of dihedral subgroups of Γ of at most length L, but before stating a precise result we need some notation that will be

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used throughout the paper. We will denote by

$$\mathcal{I}_\Gamma = \{\gamma \in \Gamma \setminus \mathrm{Id},\ \gamma^2 = \mathrm{Id}\}$$

the set of involutions in Γ, by

$$\mathcal{D}_\Gamma = \{\text{subgroups} D \subset \Gamma \text{ isomorphic to } (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})\}$$

the set of all infinite dihedral subgroups of Γ, and by

$$\mathcal{D}_\Gamma(L) = \{D \in \mathcal{D}_\Gamma \text{ with } \ell(D) \leqslant L\}$$

that consisting of infinite dihedral subgroups of length at most L. Note that the action of Γ on itself by conjugation induces actions on $\mathcal{I}_\Gamma$, $\mathcal{D}_\Gamma$, and $\mathcal{D}_\Gamma(L)$—normalizers $\mathcal{N}_\Gamma(\cdot)$ become then stabilizers.

Theorem 1.1. For every lattice $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ with 2-torsion we have

$$|\Gamma\backslash\mathcal{D}_\Gamma(L)| \sim \frac{C(\Gamma)}{|\chi^{or}(\Gamma\backslash\mathbb{H}^2)|} \cdot e^L$$

where $\chi^{or}(\Gamma\backslash\mathbb{H}^2)$ is the Euler characteristic of the orbifold $\Gamma\backslash\mathbb{H}^2$, where

$$C(\Gamma) = \frac{1}{4} \cdot \left(\sum_{\sigma \in \Gamma\backslash\mathcal{I}_\Gamma} \frac{1}{|\mathcal{N}_\Gamma(\sigma)|} \right)^2$$

and $\sim$ means that the ratio between the two quantities tends to 1 when $L \to \infty$.

Theorem 1.1 generalizes a result of Sarnak. Recall namely that a hyperbolic element γ in $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ is *reciprocal* if it is conjugated to its inverse, that is if there is $\sigma \in \Gamma$ with $\gamma^{-1} = \sigma^{-1}\gamma\sigma$. An unoriented closed geodesic in the orbifold $\Gamma\backslash\mathbb{H}^2$ is *reciprocal* if its free homotopy class is represented by a reciprocal element in Γ. Now, as already pointed out by Fricke and Klein [7] there is a bijection between (maximal) infinite dihedral subgroups and (primitive) reciprocal geodesics: the (unoriented) reciprocal geodesic in $\Gamma\backslash\mathbb{H}^2$ corresponding to the infinite dihedral group D is the quotient $\gamma_D = T_D\backslash\mathcal{A}_D$ where T_D is the index two subgroup of D consisting of hyperbolic elements and where $\mathcal{A}_D$ is, as above, the axis of D. The length of the dihedral group and the trace of the associated reciprocal geodesic are related by

$$\mathrm{Tr}(\gamma_D) = 2 \cdot \cosh(2^{-1}\ell(\gamma_D)) = 2 \cdot \cosh(\ell(D)), \tag{1.1}$$

and since $2 \cdot \cosh(\ell) \sim e^{\ell}$ for large ℓ we get from Theorem 1.1 the following:

Corollary 1.2. For every lattice $\Gamma \subset \mathrm{PSL}_2 \mathbb{R}$ with 2-torsion we have

$$|\{\gamma \text{ reciprocal geodesics in } \Gamma\backslash\mathbb{H}^2 \text{ with } \mathrm{Tr}(\gamma) \leqslant X\}| \sim \frac{C(\Gamma)}{|\chi^{or}(\Gamma\backslash\mathbb{H}^2)|} \cdot X$$

as $X \to \infty$. Here notation is as in Theorem 1.1.

In the particular case that $\Gamma = \mathrm{PSL}_2 \mathbb{Z}$ we have $C(\mathrm{PSL}_2 \mathbb{Z}) = \frac{1}{16}$ and $\chi^{or}(\mathrm{PSL}_2 \mathbb{Z}) = \frac{-1}{6}$, meaning that in the modular surface we have

$$|\{\gamma \text{ reciprocal geodesics in } \mathrm{PSL}_2 \mathbb{Z}\backslash\mathbb{H}^2 \text{ with } \mathrm{Tr}(\gamma) \leqslant X\}| \sim \frac{3}{8} \cdot X. \tag{1.2}$$

This asymptotic was first obtained, among other results, by Sarnak in [9] and it was Sarnak's paper what got us interested in these matters.

Another paper that motivated us was one by Bourgain and Kontorovich [4] giving lower bounds for the number of "low-lying" reciprocal geodesics in the modular surface. More precisely they proved that for every $\delta > 0$ there is a compact subset $K_\delta \subset \mathrm{PSL}_2 \mathbb{Z}\backslash\mathbb{H}^2$ with

$$|\{\gamma \text{ reciprocal geodesics in } K_\delta \text{ with } \mathrm{Tr}(\gamma) \leqslant X\}| > X^{1-\delta} \tag{1.3}$$

for all $X > 0$ large enough. Again we get a generalization of this theorem:

Theorem 1.3. Let Γ be a lattice with 2-torsion. For every $\delta > 0$ there is a compact set $K_\delta \subset \Gamma\backslash\mathbb{H}^2$ such that

$$|\Gamma\backslash\{D \in \mathcal{D}_\Gamma(L) \text{ with } D\backslash\mathcal{A}_D \subset K_\delta\}| > e^{(1-\delta)\cdot L}$$

for all $L > 0$ large enough.

Remark. We note that the actual theorem of Bourgain–Kontorovich in [4] is stronger than what we state here. They consider only *fundamental* reciprocal geodesics, that is, those arising from elements γ satisfying $\mathrm{Tr}^2(\gamma) - 4$ being square free. This number theoretical condition has no obvious analogue for general lattices.

While Theorem 1.1 and Theorem 1.3 are close to plain vanilla generalizations of Sarnak's (1.2) and Bourgain–Kontorovich's (1.3), what is somewhat different are the proofs. Or at least the point of view. Indeed, dropping all number theory from the picture and considering it all just as a geometric problem we reduce both theorems to very classical results on counting lattice points in the hyperbolic plane.

This simplified framework also helps to study how reciprocal geodesics are distributed in the unit tangent bundle $T^1\Gamma\backslash\mathbb{H}^2$. Again we think of them in terms of infinite dihedral subgroups. Although it is unoriented, the reciprocal geodesic γ_D associate to the infinite dihedral group $D \in \mathcal{D}_\Gamma$ corresponds to a unique geodesic flow orbit. We denote by $\vec{\gamma}_D$ the measure on $T^1\Gamma\backslash\mathbb{H}^2$ given by integrating along this geodesic flow orbit, normalized to have total mass equal to the length of the geodesic. In other words $\vec{\gamma}_D$ has total measure twice the length $\ell(D)$ of the infinite dihedral group D itself. The behavior of the measures

$$\mu_L = \sum_{D \in \mathcal{D}_\Gamma(L)} \vec{\gamma} \tag{1.4}$$

was already consider by Sarnak in [9], where he proved that there is a constant c such that for every compact set $\Omega \subset T^1 \,\mathrm{PSL}_2\,\mathbb{Z}\backslash\mathbb{H}^2$ one has

$$\liminf_{L\to\infty} \frac{1}{\|\mu_L\|}\mu_L(\Omega) \geqslant c \cdot \mathrm{vol}(\Omega)$$

where vol is the probability Liouville measure on the unit tangent bundle, that is the probability measure induced by the Haar measure via the identification $T^1\Gamma\backslash\mathbb{H}^2 = \Gamma\backslash \,\mathrm{PSL}_2\,\mathbb{R}$.

Sarnak also conjectures in [9] that, after normalization, the measures μ_L converge to vol when $L \to \infty$. This is the statement of the following result:

Theorem 1.4. If $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ is a lattice that has 2-torsion then the measures μ_L as in (1.4) converge projectively to the Liouville probability measure vol. More precisely we have

$$\lim_{L\to\infty} \frac{1}{\|\mu_L\|} \int f \, d\mu_L = \int f \, d\,\mathrm{vol}$$

for every compactly supported continuous function on $T^1\Gamma\backslash\mathbb{H}^2$.

Remark. It would be reasonable for the reader to just care about maximal dihedral subgroups, or about primitive reciprocal geodesics, or they might want to replace in (1.4) the sum over infinite dihedral subgroups by a sum over reciprocal geodesics. The results stated above remain valid in all those settings because, as we will see below, the proportion of non-maximal dihedral subgroups (resp. non-primitive reciprocal geodesics) among all dihedral subgroups (reciprocal geodesics) with at most length L tends exponentially fast to 0.

Let us now breeze over the organization of the paper. In section 2 we recall a few facts about dihedral subgroups of Fuchsian groups, analyzing with some care how the set of conjugacy classes of such subgroups are parametrized by conjugacy classes of pairs of involutions. It follows that to count conjugacy classes of dihedral groups it suffices to count involutions, or rather their fixed points. This is used in section 3 to deduce Theorem 1.1 from Delsarte's classical orbit points counting result and in section 4 to get Theorem 1.3 from the fact that lattices in $\mathrm{PSL}_2\,\mathbb{R}$ have convex cocompact subgroups with critical exponent close to 1. Still working under the same framework, we prove Theorem 1.4 in section 5, modulo another equidistribution result whose proof we defer to section 6, but which experts will probably consider evident.

Before moving on we should mention another paper that got us interested in reciprocal geodesics: in [1] Basmajian–Suzzi Valli prove versions of Sarnak's and Bourgain–Kontorovich's results where trace is replaced by word length with respect to the standard generating set of the fundamental group $\mathrm{PSL}_2\,\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ of the modular surface. Although we have not pursued this direction, it might well be that the methods we use here can also be used to recover the Basmajian–Suzzi Valli theorems.

Actually, after completion of this paper, we learned from Ara Basmajian and Robert Suzzi Valli that they were aware that one could recover Sarnak's theorem using an argument similar to the one we use. Although nothing appeared in print, Ara gave talks outlining the argument. We were not aware of it.

2

Let $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ be a discrete, non-elementary, finitely generated subgroup that has 2-torsion. With $\mathcal{D}_\Gamma$ and $\mathcal{D}_\Gamma(L)$ as above, let $\mathcal{D}_\Gamma^{\max} \subset \mathcal{D}_\Gamma$ and $\mathcal{D}_\Gamma^{\max}(L) \subset \mathcal{D}_\Gamma(L)$ be the corresponding sets of maximal infinite dihedral subgroups of Γ. The goal of this section is to prove the following:

Proposition 2.1. Let $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ be a discrete finitely generated subgroup. Suppose that the set $\mathcal{I}_\Gamma$ of order 2 elements in Γ is non-empty, denote by p_σ the unique fixed points of $\sigma \in \mathcal{I}_\Gamma$, and let $\mathcal{J} \subset \mathcal{I}_\Gamma$ be a set of representatives of the set $\Gamma\backslash\mathcal{I}_\Gamma$ of all Γ-conjugacy classes. The map

$$\begin{aligned} \pi_L &: \bigsqcup_{(\sigma,\bar\sigma)\in\mathcal{J}\times\mathcal{J}} \left(\Gamma\cdot p_{\bar\sigma} \cap B^*(p_\sigma, L)\right) \to \Gamma\backslash\mathcal{D}_\Gamma(L) \\ \pi_L &: (\sigma,\bar\sigma,\gamma\cdot p_{\bar\sigma}) \mapsto \Gamma\text{-conjugacy class of} \langle\sigma, \gamma\bar\sigma\gamma^{-1}\rangle \end{aligned} \tag{2.1}$$

is surjective for all $L > 0$. Moreover, for $D = \pi_L(\sigma,\bar\sigma,\gamma\cdot p_{\bar\sigma})$ we have

$$|\pi_L^{-1}(D)| \leqslant |\mathcal{N}_\Gamma(\sigma)| + |\mathcal{N}_\Gamma(\bar\sigma)|$$

with equality if $D \in \mathcal{D}_\Gamma^{\max}$. Here $B^*(p,L) = \{q \in \mathbb{H}^2 \text{ with } 0 < d_{\mathbb{H}^2}(p,q) \leqslant L\}$ is the punctured ball of radius L and centered at p.

The interest of Proposition 2.1 is that, as we will exploit in the next sections, it reduces counting dihedral subgroups to counting lattice points. Indeed, the following is an immediate corollary:

Corollary 2.2. Let $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ be a discrete finitely generated subgroup. Suppose that the set $\mathcal{I}_\Gamma$ of order 2 elements in Γ is non-empty, denote by p_σ the unique fixed points of $\sigma \in \mathcal{I}_\Gamma$, and let $\mathcal{J} \subset \mathcal{I}_\Gamma$ be a set of representatives of $\Gamma\backslash\mathcal{I}_\Gamma$. Then we have

$$\begin{aligned} |\Gamma\backslash\mathcal{D}_\Gamma(L)| &\leqslant \sum_{(\sigma,\bar\sigma)\in\mathcal{J}\times\mathcal{J}} |\Gamma\cdot p_{\bar\sigma} \cap B^*(p_\sigma, L)| \\ |\Gamma\backslash\mathcal{D}_\Gamma(L)| &\geqslant \sum_{(\sigma,\bar\sigma)\in\mathcal{J}\times\mathcal{J}} \frac{|\Gamma\cdot p_{\bar\sigma} \cap B^*(p_\sigma, L)|}{|\mathcal{N}_\Gamma(\sigma)| + |\mathcal{N}_\Gamma(\bar\sigma)|} \\ |\Gamma\backslash\mathcal{D}_\Gamma^{\max}(L)| &\leqslant \sum_{(\sigma,\bar\sigma)\in\mathcal{J}\times\mathcal{J}} \frac{|\Gamma\cdot p_{\bar\sigma} \cap B^*(p_\sigma, L)|}{|\mathcal{N}_\Gamma(\sigma)| + |\mathcal{N}_\Gamma(\bar\sigma)|} \end{aligned}$$

for all L.

The goal for the rest of this section is to prove Proposition 2.1. We start by going over a few facts about infinite dihedral subgroups of our Fuchsian group Γ. An infinite dihedral subgroup D contains a unique index two infinite cyclic subgroup T_D. Since the normalizer in $\mathrm{PSL}_2\,\mathbb{R}$ of a parabolic subgroup is torsion free, we get that the infinite cyclic subgroup of any infinite dihedral subgroup D of Γ is hyperbolic. It follows that D acts on a geodesic $\mathcal{A}_D \subset \mathbb{H}^2$—the action $D \curvearrowright \mathcal{A}_D$ is conjugated to the standard action of the infinite dihedral subgroup on the real line: the length of $D\backslash\mathcal{A}_D$ is the length $\ell(D)$ of the dihedral group and the geodesic $\gamma_D = T_D\backslash\mathcal{A}_D$ is the reciprocal geodesic associated to D. The stabilizer $\mathrm{Stab}_\Gamma(\mathcal{A}_D)$ of the axis is also an infinite dihedral group—it is in fact the unique maximal dihedral subgroup of Γ containing D.

Note now that if $D \notin \mathcal{D}_\Gamma^{\max}$ then $\ell(\mathrm{Stab}_\Gamma(\mathcal{A}_D)) \leqslant \frac{1}{2}\ell(D)$. Note also that an infinite dihedral group contains exactly $\left\lfloor \frac{3\cdot k}{2} \right\rfloor$ conjugacy classes of dihedral subgroups of index at most k. In plain language this means that every non-maximal infinite dihedral subgroup is the child of a dihedral subgroup of at most half the length, and that dihedral subgroups don't have may kids. Out of these two observations we get bounds for the number of maximal infinite dihedral subgroups of bounded length:

Lemma 2.3. If $\varepsilon_0 < \ell(D)$ for every $D \in \mathcal{D}_\Gamma$ then we have

$$|\Gamma\backslash\mathcal{D}_\Gamma(L)| \geqslant |\Gamma\backslash\mathcal{D}_\Gamma^{\max}(L)| \geqslant |\Gamma\backslash\mathcal{D}_\Gamma(L)| - \frac{3\cdot L}{2\cdot\varepsilon_0}\cdot|\Gamma\backslash\mathcal{D}_\Gamma(2^{-1}\cdot L)|$$

for all $L > 0$.

Continuing with generalities about infinite dihedral subgroups note that any such $D \subset \Gamma$ is generated by two distinct involutions σ and $\bar{\sigma}$ fixing the axis $\mathcal{A}_D$. In fact, there are precisely two D-conjugacy classes of ordered pairs of involutions generating D, namely $(\sigma, \bar{\sigma})$ and $(\bar{\sigma}, \sigma)$.

In the opposite direction suppose that $\sigma \neq \bar{\sigma} \in \Gamma$ are distinct involutions. Then the group $D = \langle\sigma, \bar{\sigma}\rangle$ they generate is infinite dihedral and $\mathcal{A}_D$ is the infinite geodesic passing through the unique fixed points p_σ and $p_{\bar{\sigma}}$ of σ and $\bar{\sigma}$, respectively. In those terms, the length of the dihedral group is given by

$$\ell(\langle\sigma, \bar{\sigma}\rangle) = d_{\mathbb{H}^2}(p_\sigma, p_{\bar{\sigma}}). \tag{2.2}$$

All of this gives us a way to parametrize the set of all infinite dihedral subgroups of Γ. As all along let $\mathcal{I}_\Gamma \subset \Gamma$ be the set of all involutions in Γ, that is of all elements of order two. From the discussion above we get surjectivity of the map

$$\mathcal{I}_\Gamma \times \mathcal{I}_\Gamma \setminus \Delta \to \mathcal{D}_\Gamma, \quad (\sigma, \bar{\sigma}) \mapsto \langle\sigma, \bar{\sigma}\rangle \tag{2.3}$$

where Δ is the diagonal in $\mathcal{I}_\Gamma \times \mathcal{I}_\Gamma$. The group Γ acts on $\mathcal{I}_\Gamma$ by conjugation. The map (2.3) is equivariant under this action and the induced map

$$\Gamma\backslash(\mathcal{I}_\Gamma \times \mathcal{I}_\Gamma \setminus \Delta) \to \Gamma\backslash\mathcal{D}_\Gamma \tag{2.4}$$

is surjective. Recall that, as we pointed out earlier, every ordered pair of involutions generating the infinite dihedral group $D = \langle \sigma, \bar{\sigma} \rangle$ is conjugated, within D, to either $(\sigma, \bar{\sigma})$ or $(\bar{\sigma}, \sigma)$. It follows that the map (2.4) is at worst 2-to-1 and that it is exactly 2-to-1 over the set of self-normalizing infinite dihedral subgroups. We record these facts for later use:

Lemma 2.4. The map (2.4) is at most 2-to-1. Moreover, conjugacy classes of maximal infinite dihedral subgroups have exactly two preimages.

Recall now that we are assuming that Γ is finitely generated. This implies that it has only finitely many conjugacy classes of finite order elements and hence that the set $\Gamma\backslash\mathcal{I}_\Gamma$ is finite. Let $\mathcal{J} \subset \mathcal{I}_\Gamma$ be a subset consisting of one representative of every Γ-conjugacy class. The map

$$\bigsqcup_{\sigma \in \mathcal{J}} \big(\{\sigma\} \times (\mathcal{I}_\Gamma \setminus \{\sigma\})\big) \to \Gamma\backslash(\mathcal{I}_\Gamma \times \mathcal{I}_\Gamma \setminus \Delta) \tag{2.5}$$

sending $(\sigma, \bar{\sigma})$ to its conjugacy class is surjective and its restriction to the set $\{\sigma\} \times (\mathcal{I}_\Gamma \setminus \{\sigma\})$ has fibers of cardinality equal to that of the normalizer $\mathcal{N}_\Gamma(\sigma)$ of σ in Γ.

Composing the maps (2.4) and (2.5) we get thus a surjective map

$$\pi : \bigsqcup_{\sigma \in \mathcal{J}} \big(\{\sigma\} \times (\mathcal{I}_\Gamma \setminus \{\sigma\})\big) \to \Gamma\backslash\mathcal{D}_\Gamma. \tag{2.6}$$

Let us recap what we can say about the cardinality of the fibers of (2.6). First, the preimage of $D \in \Gamma\backslash\mathcal{D}_\Gamma$ under (2.4) has at most two points $(\sigma, \bar{\sigma})$ and $(\bar{\sigma}, \sigma)$, with equality if D is maximal. Now, the conjugacy class of $(\sigma, \bar{\sigma})$ has $|\mathcal{N}_\Gamma(\sigma)|$ preimages under (2.5), and that of $(\bar{\sigma}, \sigma)$ has $|\mathcal{N}_\Gamma(\bar{\sigma})|$ preimages. Altogether we get that:

Lemma 2.5. For every conjugacy class of infinite dihedral subgroups $D = \langle \sigma, \bar{\sigma} \rangle \in \mathcal{D}_\Gamma$ of Γ we have

$$|\pi^{-1}(\langle \sigma, \bar{\sigma} \rangle)| \leqslant |\mathcal{N}_\Gamma(\sigma)| + |\mathcal{N}_\Gamma(\bar{\sigma})|$$

with equality if $D \in \mathcal{D}_\Gamma^{\max}$ is maximal.

We are now ready to prove Proposition 2.1:

Proof of Proposition 2.1. The basic observation needed to relate the statement of Proposition 2.1 with what we have been discussing so far is that each involution $\sigma \in \mathcal{I}_\Gamma$ is uniquely determined by its fixed points $p_\sigma \in \mathbb{H}^2$. From this point of view, the map (2.6) can be rewritten as

$$\begin{aligned} \pi : & \bigsqcup_{(\sigma,\bar{\sigma})\in\mathcal{J}\times\mathcal{J}} (\Gamma \cdot p_{\bar{\sigma}} \setminus \{p_\sigma\}) \to \mathcal{D}_\Gamma(L) \\ \pi : & (\sigma, \bar{\sigma}, \gamma \cdot p_{\bar{\sigma}}) \mapsto \Gamma\text{-conjugacy class of} \langle \sigma, \gamma\bar{\sigma}\gamma^{-1} \rangle. \end{aligned} \tag{2.7}$$

The map π_L in (2.1), in the statement of the proposition, is just the restriction of this map to the set $\bigsqcup_{(\sigma,\bar{\sigma})\in\mathcal{J}\times\mathcal{J}} \left(\Gamma \cdot p_{\bar{\sigma}} \cap B^*(p_\sigma, L)\right)$. Now we get from (2.2) that (groups in the conjugacy class of) the dihedral group $\pi(\sigma, \bar{\sigma}, \gamma p_{\bar{\sigma}})$ have length $d_{\mathbb{H}^2}(p_\sigma, \gamma p_{\bar{\sigma}})$. It follows that the map π_L in (2.1) takes values in the desired set, and surjectivity follows from the surjectivity of π. The final claim of the proposition follows also automatically from Lemma 2.5. ∎

3

In this section we prove Theorem 1.1 from the introduction. We restate it here for the convenience of the reader:

Theorem 1.1 . For every lattice $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ with 2-torsion we have

$$|\Gamma\backslash\mathcal{D}_\Gamma(L)| \sim \frac{C(\Gamma)}{|\chi^{or}(\Gamma\backslash\mathbb{H}^2)|} \cdot e^L$$

where $\chi^{or}(\Gamma\backslash\mathbb{H}^2)$ is the Euler characteristic of the orbifold $\Gamma\backslash\mathbb{H}^2$, where

$$C(\Gamma) = \frac{1}{4} \cdot \left(\sum_{\sigma\in\Gamma\backslash\mathcal{I}_\Gamma} \frac{1}{|\mathcal{N}_\Gamma(\sigma)|} \right)^2$$

and $\sim$ means that the ratio between both quantities tends to 1 when $L \to \infty$.

Proof. The key fact we will need is Delsarte's classical result [6] that

$$|\Gamma \cdot y \cap B(x, R)| \sim \frac{\mathrm{vol}(B(x, R))}{|\,\mathrm{Stab}_\Gamma(y)| \cdot \mathrm{vol}(\Gamma\backslash\mathbb{H}^2)} \tag{3.1}$$

when $R \to \infty$. Here $\mathrm{vol}(\Gamma\backslash\mathbb{H}^2)$ is the volume of the given orbifold. Via Gauß–Bonnet we can restate this in terms of the orbifold Euler-characteristic

$$|\Gamma \cdot y \cap B(x,R)| \sim \frac{e^R}{2 \cdot |\,\mathrm{Stab}_\Gamma(y)| \cdot |\chi^{or}(\Gamma\backslash\mathbb{H}^2)|},$$

where we have used that $\mathrm{vol}(B(x,R)) = 2\pi(\cosh(R) - 1) \sim \pi \cdot e^R$. Plugging this into Corollary 2.2 and noting that $\mathrm{Stab}_\Gamma(p_\sigma) = \mathcal{N}_\Gamma(\sigma)$ for every involution σ we get

$$|\Gamma\backslash\mathcal{D}_\Gamma(L)| \lesssim c \cdot \frac{e^L}{|\chi^{or}(\Gamma\backslash\mathbb{H}^2)|} \qquad \text{for some } c > 0, \tag{3.2}$$

$$|\Gamma\backslash\mathcal{D}_\Gamma(L)| \gtrsim C(\Gamma) \cdot \frac{e^L}{|\chi^{or}(\Gamma\backslash\mathbb{H}^2)|}, \qquad \text{and} \tag{3.3}$$

$$|\Gamma\backslash\mathcal{D}_\Gamma^{\max}(L)| \lesssim C(\Gamma) \cdot \frac{e^L}{|\chi^{or}(\Gamma\backslash\mathbb{H}^2)|} \tag{3.4}$$

where

$$C(\Gamma) = \frac{1}{2} \cdot \sum_{(\sigma,\bar\sigma)\in\mathcal{J}\times\mathcal{J}} \frac{1}{|\mathcal{N}_\Gamma(\bar\sigma)| \cdot (|\mathcal{N}_\Gamma(\sigma)| + |\mathcal{N}_\Gamma(\bar\sigma)|)} \tag{3.5}$$

Here $\lesssim$ and $\gtrsim$ mean that the inequalities hold asymptotically when $L \to \infty$. Anyways, from (3.2) and (3.3) we get that $|\Gamma\backslash\mathcal{D}_\Gamma(L)|$ grows coarsely as e^L. It thus follows from Lemma 2.3 that

$$|\Gamma\backslash\mathcal{D}_\Gamma(L)| \sim |\Gamma\backslash\mathcal{D}_\Gamma^{\max}(L)| \tag{3.6}$$

From (3.3) and (3.4) we get a lower bound for the left side and and upper bound for the right side by the same quantity. We thus get

$$|\Gamma\backslash\mathcal{D}_\Gamma(L)| \sim C(\Gamma) \cdot \frac{e^L}{|\chi^{or}(\Gamma\backslash\mathbb{H}^2)|}$$

To conclude, elementary algebra yields that $C(\Gamma)$ as defined in (3.5) can be rewritten as in the statement of the theorem. ■

Remark. Since it is going to be of some importance, we want to emphasize that Lemma 2.3 together with the exponential growth of the number of infinite dihedral subgroups implies that the proportion of non-maximal elements in $\Gamma\backslash\mathcal{D}_\Gamma(L)$ decreases exponentially when $L \to \infty$.

The fact that most infinite dihedral subgroups are maximal comes in handy now. Indeed, recall that we can associate to the conjugacy class of an infinite dihedral subgroup D a reciprocal geodesic γ_D, that this map is surjective, and that it is in fact a bijection from the set of conjugacy classes of maximal infinite dihedral subgroups to the set of primitive reciprocal geodesics. Since most infinite dihedral subgroups are maximal we get that counting reciprocal geodesics is asymptotically equivalent to counting conjugacy classes of infinite dihedral subgroups. Corollary 1.2 from the introduction follows then immediately from Theorem 1.1 together with the relation (1.1) between lengths of infinite dihedral subgroups and lengths of the associated reciprocal geodesics.

Corollary 1.2 . For every lattice $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ with 2-torsion we have

$$|\{\gamma \text{ reciprocal geodesics in } \Gamma\backslash\mathbb{H}^2 \text{ with } \operatorname{Tr}(\gamma) \leqslant X\}| \sim \frac{C(\Gamma)}{|\chi^{or}(\Gamma\backslash\mathbb{H}^2)|}\cdot X$$

as $X \to \infty$. Here notation is as in Theorem 1.1.

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Let us now turn our attention to Theorem 1.3, which we also restate here:

Theorem 1.3 . Let Γ be a lattice with 2-torsion. For every $\delta > 0$ there is a compact set $K_\delta \subset \Gamma\backslash\mathbb{H}^2$ such that

$$|\Gamma\backslash\{D \in \mathcal{D}_\Gamma(L) \text{ with } D\backslash\mathcal{A}_D \subset K_\delta\}| > e^{(1-\delta)\cdot L}$$

for all $L > 0$ large enough.

We will reduce this theorem to the fact that the lattice $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ has, for any $\delta < 1$, a convex cocompact subgroup Γ_0 with critical exponent

$$\delta(\Gamma_0) = \lim_{L\to\infty} \frac{1}{L}\cdot \log|\{y \in \Gamma_0 \cdot x_0 \text{ with } d_{\mathbb{H}^2}(x_0, y) \leqslant L\}| > \delta$$

Indeed the following is true:

Lemma 4.1. Every lattice $\Gamma \subset \mathrm{PSL}_2 \mathbb{R}$ has a sequence of finitely generated subgroups $\Gamma_k \subset \Gamma$ without parabolic elements and with

$$\lim_{k \to \infty} \delta(\Gamma_k) = 1$$

If Γ has 2-torsion then Γ_k can be chosen to also have 2-torsion for all k.

Example 1. How do these groups look like for the modular group $\mathrm{PSL}_2 \mathbb{Z}$? Well, in this case one can take Γ_k to be the subgroup generated by the set $\{\eta^i \sigma \eta^{-i}$ with $i = -k, \ldots, k\}$ where $\eta, \sigma \in \mathrm{PSL}_2 \mathbb{Z}$ correspond to the Möbius transformations $\eta(z) = z + 2$ and $\sigma(z) = -z^{-1}$.

Lemma 4.1 will not surprise anybody and we suspect that in one way or the other it might well have appeared already in the literature. We prove it below using some amount of (very classical) technology but it can be done using elementary means and we encourage the reader to try to do it by themselves. Anyways, before going any further let us use this lemma to settle Theorem 1.3:

Proof of Theorem 1.3. We get from Lemma 4.1 a finitely generated subgroup $\Gamma' \subset \Gamma$, with 2-torsion, without parabolic elements, and with $\delta(\Gamma') > 1 - \delta$. Finite generation and lack of parabolics imply that Γ' is convex cocompact and hence that there is a compact subset $K \subset \Gamma \backslash \mathbb{H}^2$, which contains $D \backslash \mathcal{A}_D$ for every infinite dihedral group whose conjugacy class admits a representative contained in Γ'.

Fix now an involution $\sigma \in \Gamma' \subset \Gamma$, choose the set $\mathcal{J}$ of representatives of $\Gamma \backslash \mathcal{I}_\Gamma$ in such a way that $\sigma \in \mathcal{J}$, and restrict the map π_L in Proposition 2.1 to the set $\{(\sigma, \sigma)\} \times (\Gamma' \cdot p_\sigma)$. From the proposition we get that this map is at most $2 \cdot |\mathcal{N}_\Gamma(\sigma)|$-to-1. This means that at least $\frac{1}{2 \cdot |\mathcal{N}_\Gamma(\sigma)|} |\Gamma' \cdot p_\sigma \cap B^*(p_\sigma, L)|$ elements in $\Gamma \backslash \mathcal{D}_\Gamma(L)$ have representatives contained in Γ'. From the very definition of the critical exponent and from the bound $\delta(\Gamma') > 1 - \delta$ we get that the cardinality of $\Gamma' \cdot p_\sigma \cap B^*(p_\sigma, L)$ grows faster than $e^{(1-\delta) \cdot L}$. Altogether we get that, for large L, there are at least $e^{(1-\delta) \cdot L}$ elements D in $\Gamma \backslash \mathcal{D}_\Gamma(L)$ with $D \backslash \mathcal{A}_D$ contained in K. We are done. ■

Now, let us prove Lemma 4.1:

Proof of Lemma 4.1. The proof has two different steps. In a first algebraic/topological step we give a sequence of groups Γ_k. Then we use the relation between bottom of the spectrum and critical exponent to show that the groups Γ_k have critical exponent tending to 1.

Anyways, let us start. Since this is the case we are interested in we are going to assume that Γ has both 2-torsion and parabolic elements. This assumption implies that Γ contains a subgroup H isomorphic to $\mathbb{Z}*\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$ where the two first free factors correspond to maximal parabolic subgroups of Γ. Now, H is the intersection of the finite index subgroups of Γ containing it [10]. It follows thus that Γ has a finite index subgroup Γ' such that the associated orbifold $\Gamma'\backslash\mathbb{H}^2$ has at least two cusps and a cone point of order 2. We will find our subgroups inside Γ'. ■

Remark. In the case of the modular group $\mathrm{PSL}_2\,\mathbb{Z}$ the subgroup Γ' can be taken to be the group generated by η and σ from Example 1.

Let now $c_0, \ldots, c_k$ be the cusps of $\Gamma'\backslash\mathbb{H}^2$ and let $\eta = \eta_1 \cup \cdots \cup \eta_k \subset \Gamma'\backslash\mathbb{H}^2$ be a simple arc system contained in the regular part of our orbifold, with η_i joining c_0 and c_i. We orient all those arcs in such a way that c_0 is always the origin and denote by

$$\alpha : \Gamma' \to \mathbb{Z}$$

the homomorphism given as follows: represent $\gamma \in \Gamma' = \pi_1(\Gamma'\backslash\mathbb{H}^2)$ by an oriented loop in the regular part of $\Gamma'\backslash\mathbb{H}^2$ and let $\alpha(\gamma)$ be the algebraic intersection number of that loop with the arc system $\eta = \eta_1 \cup\cdots\cup \eta_k$. It is a surjective homomorphism and by construction no element in $\Gamma'' = \ker(\alpha)$ is parabolic and Γ'' has 2-torsion—for what it is worth, note that Γ'' is infinitely generated.

Remark. In the case of $\mathrm{PSL}_2\,\mathbb{Z}$ and Γ' as in the previous remark, the homomorphism α is given by $\alpha(\eta) = 1$ and $\alpha(\sigma) = 0$.

Let now $\Gamma_k \subset \Gamma''$ be any sequence of finitely generated subgroups with $\Gamma_k \subset \Gamma_{k+1}$ for all k and with $\Gamma'' = \cup_k \Gamma_k$. These are our groups and all that is left to argue is that $\delta(\Gamma_k) \to 1$ when k grows.

Claim 1. $\lim_{k\to\infty} \delta(\Gamma_k) = 1$.

To establish this claim we will make use of a result of Patterson [8] and Sullivan [11] asserting that for any discrete group $G \subset \mathrm{PSL}_2\,\mathbb{R}$ with $\lambda_0(G\backslash\mathbb{H}^2) < \frac{1}{4}$ the critical exponent is given by the formula:

$$\delta(G) = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_0(G\backslash\mathbb{H}^2)}. \tag{4.1}$$

Note that $\lambda_0(\Gamma'\backslash\mathbb{H}^2)=0$ because Γ' is a lattice. Now, since $\Gamma'' \trianglelefteq \Gamma'$ is normal with $\Gamma'/\Gamma'' \simeq \mathbb{Z}$ amenable we get from [5], or rather from [2], that $\lambda_0(\Gamma''\backslash\mathbb{H}^2) = \lambda_0(\Gamma'\backslash\mathbb{H}^2) = 0$. This means that for all $\varepsilon > 0$ there is a compactly supported function $f \in C_c^\infty(\Gamma''\backslash\mathbb{H}^2)$ with Rayleigh quotient $\mathcal{R}(f) \leqslant \varepsilon$. Now, if Σ is any compact connected subsurface containing the support of f there is k_0 such that Γ_k is contained in $\pi_1(\Sigma)$ for all $k \geqslant k_0$. This means that the surface Σ lifts under the cover $\Gamma_k\backslash\mathbb{H}^2 \to \Gamma''\backslash\mathbb{H}^2$. Lifting the function f we get a function on $\Gamma_k\backslash\mathbb{H}^2$, which still has Rayleigh quotient less than ε. In particular,

$$\lambda_0(\Gamma_k\backslash\mathbb{H}^2) \leqslant \varepsilon$$

for all $k \geqslant k_0$. Claim 1 follows now from (4.1).

5

In this section we prove Theorem 1.4, that is the equidistribution of reciprocal geodesics, making use of Proposition 5.1 below, a different equidistribution result which no expert will find surprising and which we prove in the next section. Consider namely for $x, y \in \mathbb{H}^2$ the measures

$$\tilde{\mu}_L^{x,y} = \sum_{z \in \Gamma\cdot x \cap B^*(y,L)} \overrightarrow{yz} \tag{5.1}$$

where $\overrightarrow{yz}$ is the measure on $T^1\mathbb{H}^2$ obtained by integrating, with respect to arc length, along the lift to the unit tangent bundle of the geodesic arc from y to z. In the next section we will prove:

Proposition 5.1. Let $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ be a lattice and $\tilde{\mu}_L^{x,y}$ as in (5.1). For any compactly supported function $f \in C_c(T^1\Gamma\backslash\mathbb{H}^2)$ and any two $x, y \in \mathbb{H}^2$ we have

$$\lim_{L\to\infty} \frac{1}{\|\tilde{\mu}_L^{x,y}\|} \int \tilde{f}\, \mathrm{d}\tilde{\mu}_L^{x,y} = \int f\, \mathrm{d}\,\mathrm{vol}$$

where $\tilde{f}$ is the lift of f to $T^1\mathbb{H}^2$.

Assuming Proposition 5.1 for the time being, we prove Theorem 1.4:

Theorem 1.4 . If $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ is a lattice that has 2-torsion then the measures μ_L as in (1.4) converge projectively to the Liouville probability measure vol. More precisely we have

$$\lim_{L\to\infty} \frac{1}{\|\mu_L\|} \int f \,\mathrm{d}\mu_L = \int f \,\mathrm{d}\,\mathrm{vol}$$

for every compactly supported continuous function on $T^1\Gamma\backslash\mathbb{H}^2$.

Proof. The measure μ_L is the measure on $T^1\Gamma\backslash\mathbb{H}^2$ obtained by integrating over the reciprocal geodesics in $\Gamma\backslash\mathbb{H}^2$ associated to infinite dihedral groups with length at most L. By Proposition 2.1 we have a map

$$\pi_L : \bigsqcup_{(\sigma,\bar\sigma)\in\mathcal{J}\times\mathcal{J}} \big(\Gamma\cdot p_{\bar\sigma} \cap B^*(p_\sigma, L)\big) \to \Gamma\backslash\mathcal{D}_\Gamma(L)$$

$$\pi_L : (\sigma,\bar\sigma,\gamma\cdot p_{\bar\sigma}) \mapsto \Gamma\text{-conjugacy class of } \langle\sigma,\gamma\bar\sigma\gamma^{-1}\rangle$$

of which we think as being a "parametrization" of the set of conjugacy classes of infinite dihedral groups. Recall that here $\mathcal{J}\subset\mathcal{I}_\Gamma$ is a set of representatives of $\Gamma\backslash\mathcal{I}_\Gamma$, the set of conjugacy classes on involutions in Γ, and that p_σ is the unique fixed point of the involution σ. With this notation consider the measure

$$\hat\mu_L = \sum_{(\sigma,\bar\sigma)\in\mathcal{J}\times\mathcal{J}} \left[\frac{1}{|\mathcal{N}_\Gamma(\sigma)|+|\mathcal{N}_\Gamma(\bar\sigma)|} \sum_{q\in\Gamma\cdot p_{\bar\sigma}\cap B^*(p_\sigma,L)} \vec\gamma_{\pi_L(\sigma,\bar\sigma,q)}\right].$$

The measures μ_L and $\hat\mu_L$ are supported by exactly the same set of orbits of the geodesic flow by surjectivity of the map π_L. Moreover, on each component the two measures are multiples of each other, the multiple given by the cardinality of the fibers of π_L. In fact, the bound for the cardinality of the fibers in Proposition 2.1 implies that these multiples are at most 1, with equality whenever the dihedral group $\pi_L(\sigma,\bar\sigma,\gamma\cdot p_{\bar\sigma})$ is maximal and say has length at least $\frac{1}{2}L$. Since the proportion of non-maximal dihedral groups of length at most L is exponentially small (compare with (3.6) and with the comment following the proof of Theorem 1.1) we deduce that to prove that the measures $\frac{1}{\|\mu_L\|}\mu_L$ converge to vol it suffices to show that

$$\lim_{L\to\infty}\frac{1}{\|\hat\mu_L\|}\hat\mu_L = \mathrm{vol}\,.$$

Note now that the segment with end points p_σ and $(\gamma\bar{\sigma}\gamma^{-1})(p_\sigma)$ projects to the geodesic $\gamma_{\pi_L(\sigma,\bar{\sigma},\gamma\cdot p_{\bar{\sigma}})}$ under the covering map $\mathbb{H}^2 \to \Gamma\backslash\mathbb{H}^2$ and that the point $\gamma \cdot p_{\bar{\sigma}}$ is the midpoint of this segment. This means that the measure $\vec{\gamma}_{\pi_L(\sigma,\bar{\sigma},\gamma\cdot p_{\bar{\sigma}})}$ is the projection to $T^1\Gamma\backslash\mathbb{H}^2$ of the measure $\overrightarrow{p_\sigma(\gamma p_{\bar{\sigma}})} + \overrightarrow{(\gamma p_{\bar{\sigma}})((\gamma\bar{\sigma}\gamma^{-1})(p_\sigma))}$ where, as earlier, $\overrightarrow{xy}$ is the measure on $T^1\mathbb{H}^2$ obtained by integrating with respect to arc length along the lift to the unit tangent bundle of the geodesic arc from x to y. Translating the second measure by γ^{-1} we then get that the measure $\vec{\gamma}_{\pi_L(\sigma,\bar{\sigma},\gamma\cdot p_{\bar{\sigma}})}$ is also the projection of the measure $\overrightarrow{p_\sigma(\gamma p_{\bar{\sigma}})} + \overrightarrow{p_{\bar{\sigma}}((\bar{\sigma}\gamma^{-1})(p_\sigma))}$. Altogether we get that $\hat{\mu}_L$ is the projection to $T^1\Gamma\backslash\mathbb{H}^2$ of the measure

$$\tilde{\mu}_L = 2 \cdot \sum_{(\sigma,\bar{\sigma})\in\mathcal{J}\times\mathcal{J}} \left[\frac{1}{|\mathcal{N}_\Gamma(\sigma)| + |\mathcal{N}_\Gamma(\bar{\sigma})|} \sum_{q\in\Gamma\cdot p_{\bar{\sigma}}\cap B^*(p_\sigma,L)} \overrightarrow{p_\sigma q} \right]$$

on $T^1\mathbb{H}^2$. With the same notation as in (5.1) we can rewrite this as

$$\tilde{\mu}_L = 2 \cdot \sum_{(\sigma,\bar{\sigma})\in\mathcal{J}\times\mathcal{J}} \left[\frac{1}{|\mathcal{N}_\Gamma(\sigma)| + |\mathcal{N}_\Gamma(\bar{\sigma})|} \cdot \tilde{\mu}_L^{p_{\bar{\sigma}},p_\sigma} \right].$$

Proposition 5.1 asserts that the projections of the measures $\tilde{\mu}_L^{p_{\bar{\sigma}},p_\sigma}$ converge projectively to vol when $L \to \infty$. It follows that the same is true for $\hat{\mu}_L$, the projection of $\tilde{\mu}_L$. We are done. ■

It remains to prove Proposition 5.1.

6

In this section we prove Proposition 5.1. Let us fix from now on $x, y \in \mathbb{H}^2$ and $f \in C_c(T^1\Gamma\backslash\mathbb{H}^2)$, say with $\|f\|_\infty \leqslant 1$, and note that it suffices to prove that for all $\delta > 0$ there is L_0 with

$$\left| \frac{1}{\|\tilde{\mu}_L^{x,y}\|} \int \tilde{f}\, \mathrm{d}\tilde{\mu}_L^{x,y} - \int f\, \mathrm{d}\,\mathrm{vol} \right| < \delta$$

for all $L \geqslant L_0$. Given such a δ we choose $\varepsilon > 0$ with $|\tilde{f}(p) - \tilde{f}(q)| \leqslant \delta$ for all $p, q \in \mathbb{H}^2$ wich are at most at distance 10ε of each other—this is possible because $\tilde{f}$ is the lift of the compactly supported function f. Note also that since $\varepsilon > 0$ can be reduced as much as we want we can think of $L = k \cdot \varepsilon$ being an integer multiple of ε. This means that it suffices to prove that

$$\limsup_{k\to\infty}\left|\frac{1}{\|\tilde{\mu}^{x,y}_{k\cdot\varepsilon}\|}\int\tilde{f}\,\mathrm{d}\tilde{\mu}^{x,y}_{k\cdot\varepsilon}-\int f\,\mathrm{d}\,\mathrm{vol}\right|<\delta \tag{6.1}$$

Now, slicing the ball of radius $k\cdot\varepsilon$ as the union

$$B(y,k\cdot\varepsilon)=\cup_{r=0}^{k-1}A(y,r\cdot\varepsilon,\varepsilon)$$

of concentric annuli $A(y,r\cdot\varepsilon,\varepsilon)=B(y,(r+1)\cdot\varepsilon)\setminus B(y,r\cdot\varepsilon)$, we write our measure as

$$\tilde{\mu}^{x,y}_{k\cdot\varepsilon}=\sum_{r=0}^{k-1}\tilde{\nu}^{x,y}_r \quad \text{where} \quad \tilde{\nu}^{x,y}_r=\sum_{z\in\Gamma\cdot x\cap A(y,r\cdot\varepsilon,\varepsilon)}\overrightarrow{yz}.$$

Evidently, (6.1) follows if we prove that

$$\limsup_{k\to\infty}\left|\frac{1}{\|\tilde{\nu}^{x,y}_{k}\|}\int\tilde{f}\,\mathrm{d}\tilde{\nu}^{x,y}_{k}-\int f\,\mathrm{d}\,\mathrm{vol}\right|<\delta. \tag{6.2}$$

We are not done yet decomposing our measures. Consider namely

$$\tilde{\nu}^{x,y}_k=\sum_{r=0}^{k}\tilde{\lambda}^{x,y}_{k,r} \quad \text{where} \quad \tilde{\lambda}^{x,y}_{k,r}=\tilde{\nu}^{x,y}_k|_{A(y,r\cdot\varepsilon,\varepsilon)}$$

is for $r=0,\dots,k$ the restriction of $\tilde{\nu}^{x,y}_k$ to the annulus $A(y,r\cdot\varepsilon,\varepsilon)$. For fixed k, all of the measures $\tilde{\lambda}^{x,y}_{k,r}$ (save possibly the one with $r=k$, where it might be smaller) have the same total measure

$$\|\tilde{\lambda}^{x,y}_{k,r}\|=\varepsilon\cdot|\Gamma\cdot x\cap A(y,k\cdot\varepsilon,\varepsilon)|\sim\frac{\varepsilon\cdot\mathrm{vol}(A(y,k\cdot\varepsilon,\varepsilon))}{|\operatorname{Stab}_\Gamma(x)|\cdot\mathrm{vol}(\Sigma)} \tag{6.3}$$

where the claim about asymptotics follows for example from Delsarte's orbit counting result (3.1). Anyways, the point is that to prove that $\tilde{\nu}^{x,y}_k$ satisfies (6.2) it suffices to prove that the measures $\tilde{\lambda}^{x,y}_{k,r}$ satisfy the analogue claim for most r. More concretely, (6.2), and hence (6.1) and thus Proposition 5.1, follows once we establish the following:

Claim 2. There are k_1 and k_2 such that for all $k > k_1+k_2$ and all choices of $r_k \in [k_1, k-k_2]$ we have

$$\limsup_{k\to\infty}\left|\frac{1}{\|\tilde{\lambda}^{x,y}_{k,r_k}\|}\int \tilde{f}\,\mathrm{d}\tilde{\lambda}^{x,y}_{k,r_k} - \int f\,\mathrm{d}\,\mathrm{vol}\right| < \delta.$$

The role of k_1 is to ensure that the spheres of radius $r_k\cdot\varepsilon$ around x are well-mixed. Recall indeed that we can identify $T^1\mathbb{H}^2$ with $\mathrm{PSL}_2\,\mathbb{R}$ and that when doing so the geodesic flow becomes right multiplication by diagonal matrices $g_t \in \mathrm{SL}_2\,\mathbb{R}$ with entries $e^{\pm\frac{1}{2}t}$. Mixing of the geodesic flow of $\Gamma\backslash\mathbb{H}^2$ (see [3, III.2.3]) implies that the projection to $T^1\Gamma\backslash\mathbb{H}^2$ of (the outer normal of) the spheres $S_t(y) = (T^1_y\mathbb{H}^2)\cdot g_t$ gets equidistributed in $T^1\Gamma\backslash\mathbb{H}^2$ (see [3, III.3.3]). It follows that there is some k_1 with

$$\left|\int_{S_{r\cdot\varepsilon}(y)}\tilde{f} - \int f\,\mathrm{d}\,\mathrm{vol}\right| < \delta \quad \text{for all } r > k_1. \tag{6.4}$$

Suppose from now on that we fix some $r > k_1$ and cut the sphere $S_{r\cdot\varepsilon}(y)$ into segments $I_1, I_2, \ldots, I_N$ of length $\ell(I_i) \in [\varepsilon, 2\varepsilon]$ for all i. Denote then by

$$U_i = \cup_{t\in[0,\varepsilon]} I_i\cdot g_t$$

the little surface area obtained by pushing I_i via the geodesic flow for time ε. By the choice of ε we have that

$$\sup_i \sup_{p,q\in U_i} |f(p) - f(q)| < \delta.$$

This means that, choosing for all i some point $p_i \in I_i$, we have

$$\left|\int_{S_{r\cdot\varepsilon}(y)}\tilde{f} - \frac{1}{\ell(S_{r\cdot\varepsilon}(y))}\sum_i f(p_i)\cdot\ell(I_i)\right| < \delta$$

and similarly

$$\left|\frac{1}{\|\tilde{\lambda}^{x,y}_{k,r}\|}\int\tilde{f}\,\mathrm{d}\tilde{\lambda}^{x,y}_{k,r} - \frac{1}{\|\tilde{\lambda}^{x,y}_{k,r}\|}\sum_i f(p_i)\cdot\tilde{\lambda}^{x,y}_{k,r}(U_i)\right| < \delta.$$

These two bounds, together with (6.4) and the assumption that $\|f\|_\infty \leqslant 1$, imply that

$$\begin{aligned} \left| \frac{1}{\|\tilde{\lambda}^{x,y}_{k,r}\|} \int \tilde{f}\, \mathrm{d}\tilde{\lambda}^{x,y}_{k,r} - \int f \,\mathrm{d}\,\mathrm{vol} \right| \leqslant & \\ & \leqslant \delta + \left| \frac{1}{\|\tilde{\lambda}^{x,y}_{k,r}\|} \int \tilde{f}\, \mathrm{d}\tilde{\lambda}^{x,y}_{k,r} - \int_{S_{r\cdot\varepsilon}(y)} \tilde{f} \right| \\ & < 3\delta + \sum_{\substack{i \in \{1,\dots,N\} \text{ with} \\ U_i \cap \mathrm{Supp}(f) \neq \varnothing}} \left| \frac{\ell(I_i)}{\ell(S_{r\cdot\varepsilon}(y))} - \frac{\tilde{\lambda}^{x,y}_{k,r}(U_i)}{\|\tilde{\lambda}^{x,y}_{k,r}\|} \right|. \end{aligned}$$

Note that the proportion of $\lambda^{x,y}_{k,r}$ in U_i is given by

$$\frac{\tilde{\lambda}^{x,y}_{k,r}(U_i)}{\|\tilde{\lambda}^{x,y}_{k,r}\|} = \frac{|\Gamma \cdot x \cap [U_i \cdot g_{(k-r)\varepsilon}]|}{|\Gamma \cdot x \cap A(y, k\cdot\varepsilon, \varepsilon)|}$$

where $[V] \subset \mathbb{H}^2$ denotes the image of $V \subset T^1\mathbb{H}^2$ under the standard projection.

Now, the exact same argument used to prove the equidistribution of spheres [3, III.3.3] shows that the spherical segments $I_i \cdot g_{(k-r)\varepsilon}$ also get equidistributed when $(k-r) \to \infty$. Repeating word by word the argument that shows that equidistribution of spheres implies Delsarte's asymptotics (3.1) (see [3, III.3.5]) we get that

$$|\Gamma \cdot x \cap [U_i \cdot g_{(k-r)\varepsilon}]| \sim \frac{\mathrm{vol}([U_i \cdot g_{(k-r)\varepsilon}])}{|\,\mathrm{Stab}_\Gamma(x)| \cdot \mathrm{vol}(\Sigma)} \tag{6.5}$$

when $(k-r) \to \infty$. It follows thus from (6.3) and (6.5) that

$$\frac{\tilde{\lambda}^{x,y}_{k,r}(U_i)}{\|\tilde{\lambda}^{x,y}_{k,r}\|} \sim \frac{\mathrm{vol}([U_i \cdot g_{(k-r)\varepsilon}])}{\mathrm{vol}(A(y, k\cdot\varepsilon, \varepsilon))}$$

Now, since f has compact support and since the lengths of the segments I_i are pinched between two positive constants, we get that the last asymptotic statement holds uniformly for all i with $I_i \cap \mathrm{Supp}(\tilde{f})$, meaning that there is k_2 with

$$\frac{(1-\delta)\cdot \mathrm{vol}([U_i \cdot g_{(k-r)\varepsilon}])}{\mathrm{vol}(A(y, k\cdot\varepsilon, \varepsilon))} \leqslant \frac{\tilde{\lambda}^{x,y}_{k,r}(U_i)}{\|\tilde{\lambda}^{x,y}_{k,r}\|} \leqslant \frac{(1+\delta)\cdot \mathrm{vol}([U_i \cdot g_{(k-r)\varepsilon}])}{\mathrm{vol}(A(y, k\cdot\varepsilon, \varepsilon))}$$

for all i such that $U_i \cap \mathrm{Supp}(f) \neq \varnothing$ and all $k - r \geq k_2$. Given that ε is fixed (and very small) we get that the ratio between volumes is comparable to the ratio between lengths. Also, the length of I_i and the length of $S_{r\cdot\varepsilon}(y)$ grow at exactly the same rate when we apply the geodesic flow. Combining these two facts we get

$$\frac{(1-\delta-2\varepsilon)\cdot\ell(I_i)}{\ell(S_{r\cdot\varepsilon}(y))}\leqslant\frac{\tilde{\lambda}^{x,y}_{k,r}(U_i)}{\|\tilde{\lambda}^{x,y}_{k,r}\|}\leqslant\frac{(1+\delta+2\varepsilon)\cdot\ell(I_i)}{\ell(S_{r\cdot\varepsilon}(y))}$$

for all i such that $U_i\cap\mathrm{Supp}(f)\neq\varnothing$ and all $k\geqslant k_2$. This implies then that

$$\sum_{\substack{i\in\{1,\ldots,N\}\text{ with}\\ U_i\cap\mathrm{Supp}(f)\neq\varnothing}}\left|\frac{\ell(I_i)}{\ell(S_{r\cdot\varepsilon}(y))}-\frac{\tilde{\lambda}^{x,y}_{k,r}(U_i)}{\|\tilde{\lambda}^{x,y}_{k,r}\|}\right|\leqslant\delta+2\varepsilon$$

and hence that

$$\left|\int f\,\mathrm{d\,vol}-\frac{1}{\|\lambda^{x,y}_{k,r}\|}\int\tilde{f}\,\mathrm{d}\tilde{\lambda}^{x,y}_{k,r}\right|<4\delta+2\varepsilon$$

meaning that, up to changing one δ for another and after choosing ε really really small, we have proved Claim 2. Having proved the claim we have also proved Proposition 5.1.

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