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What are the characteristics of mathematical creativity?

An empirical and theoretical investigation of mathematical creativity?

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Preface

There are several people I would like to thank for support and help during these last four years I’ve been working on this dissertation. My main supervisors Anne Fyhn and Bharath Sriraman have been instrumental from beginning to end; offering advice, help and guidance whenever I needed it. Liv Sissel Grønmo and Ragnar Soleng were also helpful along the way.

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In 2011-2012 I spent 7 months as a visiting scholar at the University of California, Berkeley. I want to thank the faculty of science and technology at the University of Tromsø for providing me with the opportunity to visit UC Berkeley.

Finally, I want to thank my family who have always been supportive and believed in me. Thanks!

This dissertation consists of an introduction and four articles. The articles are:

Article 1:

Haavold, P. (2010). What characterises high achieving students' mathematical reasoning. In *The elements of creativity and giftedness in mathematics*. Sense Publishers.

Article 2:

Haavold, P. (under review). An empirical investigation of a theoretical model for mathematical creativity. *Educational Studies in Mathematics*.

Article 3:

Haavold, P. (under review). Differences in mathematical creativity between 14 year old and 17 year old students. *Journal of Mathematical Behaviour*.

Article 4:

Sriraman, B., Haavold, P. & Lee, K. (2013). Mathematical creativity and giftedness: a commentary on and review of theory, new operational views, and ways forward. *ZDM*, 45(2), 215-225.

Content

1. Personal and scientific motivation

The first scientific article I came across in the field of mathematics education that really caught my eye was a study by Johan Lithner (2004). The author investigated possible strategies for solving exercises in undergraduate calculus textbooks. The framework used to analyze the results resonated particularly well with my own experiences as a mathematics teacher in upper secondary school. After first defining reasoning according to a four-step structure, Lithner separated reasoning into two categories*: Mathematically well-founded reasoning* (or plausible reasoning) and *superficial reasoning*. Mathematically well-founded reasoning is: a) founded on intrinsic mathematical properties of the components involved in the reasoning and b) meant to guide towards what probably is the truth, without necessarily having to be complete or correct. Several examples of superficial reasoning were also given: keyword strategy, repeated algorithmic reasoning and reasoning based on established experiences. A key difference noted in the article between the two categories, plausible reasoning and superficial reasoning were whether or not the reasoning was based on intrinsic mathematical properties. An intrinsic property is central in a particular context and problematic situation. A surface property has no or little true relevance.

Lithner (2004) found that most exercises might be solved by mathematically superficial strategies, often with no concern for the core mathematics of the book section in question. In a related study Lithner (2003) discovered that undergraduate students' reasoning in mathematics was superficial of nature and the students paid little attention to intrinsic mathematical properties of the components involved in their work. As mentioned, the findings and framework used by Lithner (2004), helped me make sense of my own experiences a mathematics teacher. Students would often ask for which procedure or algorithm they were supposed to use solving a particular mathematical problem. By observing students working on mathematical tasks and problems and talking to them about what they were doing and thinking, it seemed as if the students were mostly interested in getting things "right" and not necessarily understand the mathematical concepts and procedures. Other mathematics teachers at this upper secondary school expressed similar suspicions. Using Lithners framework as a theoretical background, the students' reasoning appeared in hindsight to be mostly superficial and not based on intrinsic mathematical properties. Even high achievers in mathematics focused mostly on "decoding" mathematics and applying the correct, previously seen formula or procedure in order to solve a mathematical task or problem. These anecdotal observations made me ponder on the meaning of the concept of giftedness in mathematics and its relationship with mathematical attainments. If high achievers in mathematics resorted to superficial reasoning, even when it was inappropriate, could they also be classified as gifted in mathematics? This was the starting point, or tentative research question, for the research project.

The focus of the project was further refined when I came across another article by Johan Lithner (2008) in which he proposed a framework for *imitative* and *creative* reasoning. This article built on the earlier work on *mathematically well-founded* reasoning and *superficial* reasoning. Two types of imitative reasoning were proposed. The first was *memorized reasoning* which fulfilled the following conditions: a) the strategy choice is founded on recalling a complete answer and b) the strategy implementation consists only of writing it down. The second was *algorithmic reasoning* which fulfilled the following conditions: a) the strategy choice is to recall a solution algorithm. The predictive argumentation may be of different kinds, but there is no need to create a new solution. b) The remaining reasoning parts of the strategy implementation are trivial for the reasoner, only a careless mistake can prevent an answer from being reached.

Clearly distinguished from *imitative reasoning*, *creative reasoning* fulfilled the following criteria: a) Novelty. A new (to the reasoned) reasoning sequence is created, or a forgotten one is re-created. b) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible. c) Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. The concepts of *creative reasoning* and creativity became the central focal points for exploring giftedness and attainments in mathematics and the relationship between them. Both giftedness and attainments are complex and multidimensional concepts and, as such, any investigation would require a distinct and clear "lens". As a metaphor, the term lens can be defined as *"a channel through which something can be seen or understood;"* In other words, the idea is that by looking at *creative reasoning* in mathematics, the concepts of giftedness and attainments in mathematics could be further developed and understood.

Creative mathematical thinking is, according to several frameworks, mathematicians and researchers, an essential part of mathematics as a field, both at the professional level and at the K-12 level (Pekhonen, 1997; Sriraman, 2005; Lithner, 2008; Pelczer & Rodríguez, 2011). In school mathematics, attainment (attainment and achievement are used interchangeably in this text) is for the most part considered to be students' results and grades on tests and exams. High achievers in mathematics are students who get good grades in school mathematics. If we accept these premises as true, the implication is that high achievers in school mathematics should also be able to display creative mathematical thinking. In order to investigate this hypothesis, the general and larger research question of this research project is:

> "What are the characteristics of mathematical creativity?"

Creativity has during the last decade been increasingly emphasized as a key component in education by policy makers, media, funding bodies, scientists et cetera (Van Harpen & Sriraman, 2011). It has been investigated extensively in the general field of psychology for decades and creativity research is currently booming (Runco & Albert, 2010). However, until recently the published research on mathematical creativity was sparse and included only a handful of articles (Leikin, Berman & Koichu, 2010). Furthermore, the relationship between mathematical ability and attainment and mathematical creativity is ambiguous (Kattou et al., 2013). The larger research question of this research project therefore sets out to extend our understanding of mathematical creativity and its relationship to mathematical ability and attainments.

Characteristic is defined by thefreedictionary online as "*a distinguishing quality, attribute, or trait*" (Farlex, 2013) The research project, being a doctoral dissertation, is of course limited in scope, resources and time. A complete investigation of mathematical creativity and all its aspects and characteristics is not possible within the parameters of the project. Therefore, it was necessary to limit the project into more specific research questions in three different articles; each dealing with distinct aspects of mathematical creativity. The first article, published in "The Elements of Creativity and Giftedness" (Sriraman & Lee, 2011), asked "What characterizes high achieving students' mathematical reasoning?" The study investigated high achieving students' mathematical reasoning when faced with an unfamiliar trigonometric equation. The findings indicated that even high achievers' thinking is strongly linked with imitative reasoning and lacked flexibility. The second article, currently under review in Educational Studies in Mathematics, investigated whether a theoretical model for maximizing creativity in the classroom could predict mathematical creativity in lower secondary school students. It also set out to further shed light on the relationship between mathematical attainment and mathematical creativity. The third article, currently under review in Journal of Mathematical Behavior, explored differences in mathematical creativity between upper and lower secondary students. The fourth article is slightly different from the three other articles, in the sense that it is not original research by the author of this dissertation, but instead a commentary and synthesis of three other articles on the subject of giftedness and creativity in mathematics. The article was co-written with one of the author's supervisors, Bharath Sriraman, and Kyeonghwa Lee. The commentary is featured in issue four of ZDM this year.

Every investigation into complex phenomena such as mathematical cognitive activity has to be simplified in some ways (Niss, 1999). Although the research question asks "what are the characteristics of mathematical creativity", it does not set out to give a complete answer. Instead this research project focuses on certain key characteristics of mathematical creativity: the relationship between mathematical achievement and mathematical creativity and personality traits indicative of mathematical creativity. The research project was carried out in two phases. The first phase consisted of a qualitative study investigating the characteristics of high achieving students' mathematical reasoning. The second phase then looked at the relationship between mathematical achievement and mathematical creativity and traits indicative of mathematical creativity.

The intention behind the first, qualitative investigation was to: a) develop hypotheses on the relationship between school achievements in mathematics and mathematical creativity and b) identify key characteristics of the students' mathematical reasoning. The findings of the study then formed the basis of research questions that were investigated quantitatively. The results of the first study indicated that even high achieving students lacked flexibility in their mathematical reasoning and, consequently, high achievement in school mathematics did not necessarily imply mathematical creativity, as flexibility and overcoming fixations of thought is one of the key components of mathematical creativity (Haylock, 1987). The relationship between mathematical achievements (and mathematical knowledge) and mathematical creativity was then investigated quantitatively. In addition a theoretical model for optimizing mathematical creativity was operationalized with a questionnaire to see if certain characteristics, other than mathematical achievement, were statistically linked to mathematical creativity.

The rationale behind the research project is primarily descriptive, in the sense that it sets out to investigate what the characteristics of mathematical creativity are. Its purpose is to further the mathematics education community's knowledge about mathematical creativity. Both in regards to what it is, what it is not and why this matters. The findings of the project, interpreted in light of earlier findings, can increase the community's knowledge and understanding of mathematical creativity. Implicitly, however, the findings of this research project may serve to improve mathematics education itself. A better understanding of mathematical creativity and, in particular, its relationship with mathematical achievements may help teachers develop a more conscious awareness of its importance to school mathematics. In the following sections the research project is explained and presented. First, a short summary of each of the articles is given. The summaries are provided to give the reader a quick and basic understanding of how the project was organized. Second, a theoretical background and framework for the project is given. Then methodological issues of the project will be discussed. In the last sections the research project's main results and conclusions will be presented. Here, the findings of each article will be presented more in detail and common themes from all articles will be extracted and discussed.

However, before the theoretical framework and literature review is given, a short disclaimer is needed. In the field of mathematics (or even general psychology) terms like creativity, ability, achievement and giftedness are not unambiguous. There are many different definitions of these terms; Mann (2005) argues for instance that there are more than 100 definitions of creativity observed in the literature. As such one cannot assume the operationalization of concepts in this project is identical, or even similar, to other studies into mathematical giftedness, creativity, ability and achievements. Therefore giftedness, ability, achievements and creativity are all conceptualized and operationalized explicitly both in this text and in each article.

2. Summary of articles

1. Article one

The first article asked two research questions:

- Is it true that high achieving students display superficial reasoning when given an unfamiliar trigonometric equation?
- What characterizes the students' mathematical reasoning when given an unfamiliar trigonometric equation?

The underlying motivation for the study was to investigate the relationship between mathematical achievements in school and mathematical competence. Previous research have indicated that even high achievers in mathematics make use of superficial reasoning when faced with unfamiliar mathematical problems (see for instance Lithner, 2000; Lithner, 2003 and Lithner, 2008). Three high achieving students in upper secondary school were given an unfamiliar trigonometric equation and asked to solve it while "thinking aloud". Each clinical interview lasted for about 30 minutes. In the first part, the author stayed silent and only reminded the students to keep talking if they stayed quiet for some time. The last part of the interview was less structured and the author asked more explicit and probing questions, asking the students to explain their reasoning.

When the students were given the equation, they all tried to solve it using a familiar algorithm even though the algorithm was inappropriate. One possible explanation is that the students didn't focus on the intrinsic properties in the equation, but instead only considered the surface appearance of the equation. Only with guidance from the author were the students able to display creative and flexible reasoning. The author also concludes, based on the observations in the study, that the students had the sufficient mathematical knowledge to solve the equation, but were unable to break from established and fixed mental sets on their own. The capacity for flexible, creative reasoning is an important aspect of doing mathematics and mathematical competence (NCTM, 2000; Lithner, 2008). So the question becomes why weren't the three high achieving students' reasoning more flexible and creative? A possible explanation lies in that students learn what they are given an opportunity to learn (Hiebert, 2003). If imitative and algorithmic reasoning is rewarded in school and all that is needed to get good grades in mathematics, then imitative and algorithmic reasoning will also be prominent even in high achieving students.

2. Article two

In the second article, a theoretical model outlining five principles for optimizing creativity in school mathematics was investigated empirically using Analysis of Covariance (ANCOVA). The relationship between mathematical creativity and mathematical achievements was also investigated. Based on a synthesis of the literature, Sriraman (2005) proposed a model consisting of five principles to maximize creativity in a K-12 setting: the gestalt principle, the aesthetic principle, the free market principle, the uncertainty principle and the scholarly principle. The five principles were operationalized with a questionnaire. Mathematics achievement was defined as the students' final assessment grades for fall and they were classified by three categories: high, medium and low achievement. Mathematical creativity was measured using a creative ability in mathematics test based on Balka's Creative Ability in mathematics test (1974).

190 grade eight students from two lower secondary schools in a medium sized city in Norway participated in the study. Intrinsic motivation and an aesthetic sense of mathematics were found have a significant and low-to-medium effect on mathematical creativity, controlled for mathematical achievement. The results also indicated that there is a strong relationship between mathematical achievement and mathematical creativity. However, there were several exceptions, showing that mathematical achievement may be a necessary, but not sufficient condition for mathematical creativity to manifest.

3. Article three

Article three was based on data collected from the same study as article two. In this study 190 grade eight students and 118 grade eleven participated. The students were given a creative mathematics test in order to evaluate whether or not there are differences in mathematical creativity between the two groups. The students were selected from two lower secondary schools and three upper secondary schools in a medium sized city in Norway. The schools were selected because they appeared to be typical or normal. By that the author means that the selected schools were public schools with a homogenous student population located in urban, middle class areas. The schools were not exceptional, one way or the other, in terms of academic performance either. Mathematical creativity was measured by a creative ability test in mathematics based on Balka's CAMT (1974).

T-tests show that the older students score higher on the creative mathematics test. The 11^{th} grade students scored higher on each of the tasks on the creative mathematics test and on each of the three categories that make up the creativity score: fluency, flexibility and originality. However, only one of the tasks and the originality category indicated significantly different scores in favor of the 11^{th} grade mathematics students. This means that the older students did not provide significantly more responses or significantly more response categories. They did however provide statistically more unique or original responses. A possible explanation for this is that the older students have three more years of schooling. Furthermore, the older students may have developed a stronger connectedness of their mathematical knowledge base. Although the younger students have been exposed to the mathematics necessary for providing original solutions to the task, the older students have seen the same mathematical topics expanded upon and studied them deeper during their three extra years of schooling. They have also, to a greater degree, had an opportunity to see more connections made between different mathematical topics. As Haylock (1987), Krutetskii (1976) and Sheffield (2009) points out, mathematical creativity includes the ability to overcome fixations and connect seemingly unrelated ideas. With a deeper and more matured knowledge base, it is plausible that the older students are able to make connections between ideas that the younger students don't or deem inappropriate.

4. Article four

The fourth article, unlike the three other articles in this dissertation, was not based on original research by the author. Instead it was a co-written commentary with Bharath Sriraman and Kyeonghwa Lee for a special issue of ZDM (issue 4, 2013) on mathematical creativity. In the commentary the authors critique and synthesize three articles ((Leikin, & Lev, 2013; Kattou, Kontoyianni, Pitta-Pantazi, & Christou, 2013; Pitta-Pantazi, Sophocleous, & Christou, 2013) by addressing the theory that bridges the constructs of mathematical giftedness and mathematical creativity. Although the commentary discusses important issues like problem sequencing and the need for a reliable metric to assess problem difficulty, the primary relevance for this dissertation lies in the synthesis of the relationships between ability, giftedness and creativity in mathematics.

The main purpose of all three articles synthesized in the commentary was to investigate the relationship between mathematical giftedness and mathematical creativity. The synthesis has three different themes: conceptual relationships, what characterizes the mathematically creative and implications for teaching mathematics. The two most pertinent themes being conceptual relationships and what characterizes the mathematically creative. The three articles all focus on slightly different, but related characteristics of mathematical creativity. Nevertheless there are certain similarities that might be inferred on a structural level. High IQ, spatial cognitive style and general mathematical ability are all linked to mathematical creativity. As a concept, mathematical creativity does not exist in a vacuum, features and factors are required for mathematical creativity to arise. If we assume that all three articles operationalize mathematical creativity similarly, it becomes clear that general giftedness (high IQ), mathematical ability and spatial cognitive style are all linked with mathematical creativity.

All three studies can be said to cluster individuals according to their level of mathematical creativity. In the three studies mathematically able students, students with a preference for spatial cognitive style and gifted students (high IQ students) were all mathematically creative. Or correspondingly mathematically creative students are characterized by high IQ, a preference for spatial cognitive style and a high mathematical ability. There are certain inconsistencies in the studies synthesized. Kattou et al. (2013) report that mathematically able students are also mathematically creative students. Leikin & Lev (2013), on the other hand, found that gifted (or high IQ) was a much stronger predictor of mathematical creativity than level of instruction. In Leikin & Lev the high level instruction group of students can be classified as mathematically able students. It would have been interesting to further examine the mathematically able students in the study by Kattou et al. (2013). Haylock (1997), for instance, claims that within the group of high achievers in mathematics there are both low-creative and high-creative students.

3. Theoretical framework

1. Creativity

To fully understand creativity, a multitude of approaches is needed. It is a multifaceted and multidimensional concept; individual, situational, social and cultural factors all work together to influence the probability and the magnitude of a creative outcome (Ward & Kolomyts, 2010). Timing is also an essential part of creativity. Rembrandt was for instance not all that well known in his own time, Van Gogh died a pauper and Mendel's theories were not influential until 50 years later. Conceptions of creativity have changed throughout history. The ancient Greeks had no explicit terms that reflected the ability to create. Except for poetry, the Greeks looked at art as a discipline that was subjected to rules and laws. To them, nature was perfect and as such man should try to discover its laws and rules. Not seek freedom and deviate from them. The artist discovered things; he did not invent things (Runco & Albert, 2010).

It is commonly understood that the idea of the creative act and the notion of creativity originated in the western culture through Christianity. Creativity was in the early Western history confined to the realm of God. Only God had the ability to create something new. It was not until the Renaissance that the individual was seen as capable of creative acts Creativity was no longer seen as just a conduit for the divine, but abilities from "great men". However, our modern understanding of creativity was no developed until the age of enlightenment. During the 1700s and 1800s the idea of creativity was separated from genius, talent and formal education. Four distinctions, which are the basis of our current understanding of creativity, were develop: a) genius was separated from the divine; b) genius was a potential in every individual; c) talent and "original" genius was distinguished from each other; and d) potential and exercise of genius depend on the political atmosphere at the time (Runco & Albert, 2010).

The modern understanding of creativity is however not unequivocal or clear (Parkhurst, 1999; Runco, 2004). There are many definitions and understandings of the concept and many researchers simply avoid defining the term at all (Plucker & Makel, 2010). This lack of definition partially explains the conflicting results seen in the field. Despite the conceptual fragmentation, there are certain common themes and characteristics that seem to have gained acceptance in the field. Barron (1955) first introduced new and useful as key descriptors of creativity. This conceptualization of creativity has since then been modified and specified. Parkhurst (1999) proposed the following definition: "*the ability or quality displayed when solving hitherto unsolved problems, when developing novel solutions to problems others have solved differently, or when developing original and novel (at the least to the originator) products*." (p. 18) However, this did not take into account the social appraisal of creativity. Recently Plucker & Beghetto (2004) offered a an empirical definition of creativity based on a literature review of earlier research: "*the interplay between ability and process by which an individual or group produces an outcome or product that is both novel and useful as defined within some social context*" (p.156) This definition will be applied to the concept of creativity in this dissertation.

1. Four p's of Creativity

When attempting to understand the field of creativity, it is important to take into account which aspect of creativity the research emphasizes. Traditionally, research into creativity has focused on four different variables, referred to as "the four P's": person, process, products and press (Davis, 2004). A longstanding perspective on creativity has been on the creative person or personality. With a research focus on person or personality, traits that were indicative or contraindicative of creativity were often investigated. Several traits have been found to correlate with creativity across domains: intrinsic motivation, wide interests, openness to experience, autonomy et cetera (Barron, 1995). There are also more domain specific personality traits seen in either artistic domains or scientific domains (Feist, 1999). According to Kozbel, Beghetto & Runco (2010) the environment often influences the expression of personality. Press factors refer to interactions between persons and environments. There are individual differences in terms of preferences, but in general creativity tends to flourish where there are opportunities for exploration, independent work and when originality is encouraged and appreciated (Amabile, 1990).

Research that focuses on the creative process, aims to explore the nature of the mental mechanisms that occur during a creative activity or creative thinking (Kozbelt et al., 2010). Process research usually specifies different stages of the creative process or particular mental mechanisms of creative thought (Simonton, 1984; Ward, Smith & Finke, 1999; Mumford, Baughman, Maher, Costanza & Supinski, 1997). One of the more popular stage theories is Wallas' (1926) four stage model of problem solving: preparation, incubation, illumination and verification. The first stage of preparation refers to an initial period of working on a problem using logic, different strategies and reasoning. If a solution is not reached the problem solver stops working on the problem. This marks the beginning of the incubation period. The incubation period can last from a few minutes to years. During this period the attention of the problem solver is diverted from the problem, either taking a break or focusing on other problems. The third stage, illumination, is when the solution to the problem suddenly appears, sometimes while the problem solver is engaged in unrelated activities. The final stage is when the problem solver goes back to the problem and verifies that the solution is correct. Key issues in the study of creative processes include the different cognitive mechanisms involved in creative and non-creative thought, the roles of conscious and unconscious processes and the role of chance versus more controlled processes (Kozbelt et al., 2010).

Product refers to the study of what characterizes creative products: works of art, inventions, publications, music and so on (Kozbelt et al., 2010). There is a certain quantitative objectivity to investigations of creative products as they can be counted. Creative products can also often be viewed and judged directly, so inter-rater reliability can readily be determined. The down side is that little can be said about the creative process which brought forth the creative product. Furthermore, creative products are usually constructed by creative individuals, so although it may be possible to make inferences about the creative process of creative individuals, little can be said about a person's unfulfilled creative potential (Runco, 1996).

2. Big C and little C

Another useful distinction when researching creativity is the different levels of creative magnitude; the social aspect and appraisal of creativity has to be taken into account. The most common distinction is between Big C and little c. Most research of creativity takes one of the two directions (Kaufmann & Beghetto, 2009). The first direction, Big C, focus on eminent creativity. The goals are often to explore creative genius and creative works that may last and change a field forever. Big C has often been researched by looking at the lives of renowned geniuses and interviews of people who excel at high levels of creativity within one particular field. In contrast to eminent (or extraordinary) creativity, the other predominant direction of research focuses on everyday creativity or little c. Everyday creativity refers to experiences and expressions accessible to most anyone. In other words, creative actions that non-experts participate in each day (Richards, Kinney, Benet & Merzel, 1988). Most of the studies that investigate little c use students or children as participants.

However, this dichotomy between Big C and little c lack nuance. In the field of mathematics, a student who comes up with an original solution to a difficult problem would be classified as little c or everyday creativity. A graduate student who was able to prove some minor conjecture, could also be categorized as little c. Each situation is an example of little c and not Big C, but they are qualitatively different levels of everyday creativity. The distinction between Big C and little c helps classify creativity, but at the same time the categories risk becoming too inclusive (Kozbelt et al., 2010). Kaufman and Beghetto (2009) proposed to additional categories in an attempt to address this weakness. Mini c helps separate subjective and objective forms of little c creativity. The mini c category emphasizes the personal and developmental aspects of creativity. Pro c allows the distinction between professional level creativity, like professional mathematicians or doctoral students, who haven't yet reached eminent creativity status, but are well beyond little c status (Kozbelt et al., 2010). In this research project the participants were students in lower and upper secondary school and, as such, the focus was little c creativity in mathematics.

3. Creativity as domain specific or domain general

One of the more prominent debates within the field of creativity research, that is also relevant for this dissertation, is the question whether creativity is domain specific or domain general. Is creativity a general, domain-transcending set of skills, aptitudes, traits and motivations that can be applied in any domain or are they limited to specific domains? Common usage of the word creativity suggests that most people think of creativity as a domain general skill. Creativity is seen in many ways as intelligence, a general ability that will affect performance in almost any field (Baer, 2010). More recently this perception has been challenged. Feist (2004) wrote that: "this is a rather naïve and ultimately false position and that creative talent is in fact domain specific" (p.57). Feist's position is relatively new, but it has a growing number of supporters in the field of psychology (Baer, 2010).

The debate is primarily centered on the kind of evidence that is emphasized. Unlike proponents of domain specificity who look at creative performances, proponents of domain generality typically focus on psychometric and personality data. Plucker (1998) note that: "The conclusions of researchers using the CAT are almost always that creativity is predominantly task or content specific… [but] researchers utilizing traditional psychometric methods usually conclude that creativity is predominantly domain general" (p.181). According to Baer (2010, the most likely solution to the debate is some form of hierarchical model that takes into account both domain generality and domain specificity. The key traits of such a model would be that *"talents, knowledge, skills, motivations, traits, propensities, and so forth that underlie creative performance a) vary depending on the kind of work one is undertaking b) are similar across related field or kinds of creative work, and c) become progressively dissimilar as one moves to increasingly disparate fields of endeavor*" (p.338).

4. Divergent thinking

Within the field of creativity there is probably as much research on divergent thinking (DT) as any other single topic (Runco, 2010). Divergent thinking refers in a broader sense to the concept of ideation, which can be defined as generation of ideas, judgments, evaluations and decisions (Runco, 2010). Many of our actions are based on routine and often mindless (Langer, 1989). The fact that can rely on routine or habit provides us with capacity to focus when we need to and relax when we do not. When we cope with and process new information mindfully, we also are most likely to produce original and useful ideas. Divergent thinking research is one of the more fruitful ways to study ideation and, thus, potential for creativity and problem solving (Runco, 2010). Although divergent thinking is not synonymous to creativity, it represents "estimates of the potential for creative thinking and problem solving." (Runco, 2010. p. 424). Within psychometrics divergent thinking is the most promising and most used candidate for the foundation of creative ability (Plucker & Renzulli, 1999).

Divergent thinking is assessed with divergent thinking tasks, in which participants generate ideas in response to verbal and figural prompts (Kim, 2006). In order to define divergent thinking, it is best to contrast it with convergent thinking. Divergent thinking moves in several, different directions, while convergent thinking moves towards one or a very few correct answers. Convergent thinking is judged by correctness while divergent thinking is slightly more difficult to evaluate. In the tradition of Guilford and Torrance, it is judged on fluency, flexibility and originality (Runco, 2008). Fluency is the number of relevant ideas and it shows the ability to produce several different responses (Torrance, 1967). Usually it is simply the number of relevant responses to a task. It also relates to the continuity of ideas, flow of associations and use of basic and universal knowledge (Leikin, 2009). Flexibility is generally based on the number of categories or classes represented in a respondent's pool of ideas/responses (Torrance, 1967). It is associated with changing ideas in producing a variety of solutions (Leikin, 2009). Originality is usually defined as statistical infrequency (Torrance, 1967). It is characterized by a unique way of thinking and unique products of mental activity (Leikin, 2009).

2. Giftedness

As with the concept of creativity, there is no unified and clear definition of giftedness. For the most part of the previous century researchers and psychologists described giftedness mainly in terms of intelligence (Coleman & Cross, 2005). The popularity of intelligence and cognitive ability tests is to a large extent explained by the desire for objectivity. Numbers provide comfort and tidiness for administrators and policy makers. However, people closest to direct services, such as teachers, have often questioned the validity of "objective" tests (Brown, Renzulli, Gubbins, Siegle, Zhan & Chen, 2005). Success in life depends on a broader range of abilities than what conventional tests measure. This observation has also been empirically investigated. Terman and his research team for instance analyzed the accomplishments of 1528 geniuses and discovered that early intelligence tests were not necessarily the best predictor of adult accomplishments (Oden, 1968). That is not to say that intelligence is not an important component of giftedness, but over the past couple of decades the concept of giftedness has evolved to include a more multifaceted approach. Motivation, self-concept and creativity are just some of the qualities included in more contemporary theoretical models of giftedness (Cramond, 2004). In this section a few theoretical models of giftedness will be presented to show how giftedness is considered a multidimensional concept and how it is linked with creativity. Though before looking at specific models of giftedness, it should also be noted that even within theories of general intelligence, it is widely accepted that abilities are hierarchically differentiated (e.g., Carrol, 1993; Cattel, 1971).

One of the more well-known models for giftedness is Renzulli's *Three-ring Model* (1986). This model portrays giftedness as an interaction of three components: above average ability, task commitment and creativity. The model presents the three components as three overlapping circles and giftedness is thought to be found in the middle, where all three circles overlap each other. Above average ability, task commitment and creativity must all be present for giftedness can arise. Above average ability refers to both general abilities and specific abilities. General abilities consist of traits that can be applied across domains (e.g. general intelligence) they include the capacity to process information, integrate experiences and abstract thinking. Specific abilities consist of the capacity to acquire knowledge, skill or ability to perform in a specialized kind of activity. Task commitment refers to perseverance, hard work, endurance, self-confidence and a special fascination with a specific subject. The third cluster consists of factors associated with creativity, such as flexibility, originality of though, fluency, willingness to take risks and openness to experience.

Another well-known model that also includes environmental factors and chance is Abraham Tannenbaum's (2003) *Star Model*. In this model giftedness is defined as the ability to produce thought or tangibles or perform artistry or human services that are proficient or creative. It consists of five elements arranged in a star pattern: a) superior general intellect, b) distinctive special aptitudes, c) nonintellective requisites, d) environmental support and e) chance. The model separates what is traditionally considered general intelligence, with the inclusion of superior general intellect, and more domain specific abilities, here referred to as distinctive special aptitudes. Unlike Renzulli's Three-ring Model, the Star Model also recognized external factors that influence giftedness through environmental support and chance. The final element, nonintellective requisites, refer to creativity, motivation, self-concept and other individual characteristics related to giftedness. Both the Star model and the Three-ring Model identify giftedness as an interaction between a multitude of elements and traits. They both exemplify how the concept of giftedness has gone from being looked at as intelligence, to more complex models where intelligence is but one part of giftedness.

3. Mathematical creativity

As with general creativity arguably the main challenge in investigating mathematical creativity is the lack of a clear and accepted definition of the term mathematical. Previous examinations of the literature have concluded that there is no universally accepted definition of mathematical creativity (Sriraman 2005; Mann, 2005). Treffinger, Young, Selby & Shepardson (2002) claims for instance that there are more than 100 contemporary definitions of mathematical creativity. Nevertheless, Plucker and Beghetto (2004) and Mayer (1999) note that there are two key features of creativity that is seen throughout the literature: originality and usefulness. Plucker and Beghetto (2004) go on to note how both originality and usefulness is judged on the basis of a certain social context. In the previous section the distinction between Big-C creativity and little-c creativity was explained and an outline of the debate regarding domain specificity and domain generality of creativity was given. Both of which can shed some light on what is meant by social context.

At the K-12 level one does not expect works of extraordinary creativity, as opposed to the professional level of some field. For instance, Andrew Wiles' proof of Fermat's Last Theorem can only be judged by a handful of professional mathematicians and is considered to be extraordinary creativity. However, as pointed out by Sriraman (2005), it is possible to see unusual/insightful solutions to problems, formulation of new problems and/or old problems regarded from new angles in a K-12 setting. Within the field of mathematics there is both Big-C creativity seen in the professional level and little-c creativity seen in school (and elsewhere). The domain itself also influences how creativity is evaluated and assessed. What is considered a creative (novel and useful) process or product in mathematics will often differ significantly from other fields such as for example literature and poetry. In other words creativity depends on the context and on the user. Building on Plucker and Beghettos (2004) definition of creativity, and taking into account the social context of creativity, Sriraman (2005) proposes the following operational definition of mathematical creativity in a K-12 level: *"a) the process that results in unusual (novel) and/or insightful solution(s) to a given problem or analogous problems, and/or b) the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle requiring imagination."* Using Sriraman's definition as a starting point, this research project employs two frameworks in order to investigate mathematical creativity in K-12 setting.

1. Mathematical creativity as divergent thinking

Haylock (1987), in summarizing research on mathematical creativity and proposing a framework for investigating mathematical creativity, found that two key investigative models emerge: the ability to overcome fixations in mathematical problem solving and the ability for divergent production within mathematical situations. Creativity as divergent thinking comes primarily from the research of Guilford and Torrance (Runco, 1999). Divergent thinking is not synonymous with creativity, but it is proven to be a good estimate for potential for creative thinking (Runco, 1999). Divergent thinking production can also be applied to the field of mathematics, usually through divergent thinking tests (Haylock, 1987; Silver, 1997; Leikin, 2009). Performance on divergent thinking tests in mathematics appear to be unrelated to general divergent thinking tests (Dirkes, 1974; Haylock, 1978). The common theme of all such tests is problems and situations with many possible responses. As opposed to convergent thinking where the subject must seek one, and only one, solution, divergent thinking tasks open up for many possible solutions (Haylock, 1987).

Divergent thinking tests are usually scored in terms of fluency, flexibility and originality. According to Leikin (2009) fluency in mathematics refers to the ability to produce many ideas, flexibility is the number of approaches that are seen in a solution and originality is the possibility of extraordinary, unique and new ideas. Another underlying criterion for scoring divergent thinking tasks in mathematics is appropriateness. A response might be highly unusual, and thus original, but it must also be mathematically correct. For example $\sqrt{8}$ as a question generating the response 4 is highly original, but also mathematically incorrect (Haylock, 1997). This example illustrates some of the difference between general divergent thinking tests and divergent thinking tests in mathematics. In the latter there are clearly defined rules that determine whether a response is appropriate or not.

Haylock (1987) identified three categories of divergent production tasks for use in divergent thinking tests: a) *problem solving*. Here the subject is given a problem that has many possible solutions. The students are asked come up with as many different, unusual and interesting solutions as they can. b) *problem posing*. Here the subject is given situation and asked to make up as many interesting mathematical problems and questions as possible that can be answered given the information available. c) *redefinition*. Here the subject is asked to continuously redefine the elements of a situation in terms of their mathematical properties. The categories are not completely disjoint or mutually exclusive (there is some overlap), but serves as a framework for generating and classifying divergent production tasks in mathematics (Haylock, 1997).

2. Mathematical creativity as overcoming fixations

The other investigative model proposed by Haylock (1987) focuses on the process of mathematical creativity and the importance of overcoming fixations. Creative thinking is closely related to flexibility of thought (Haylock, 1997). The opposite of flexibility is rigidity of thought and the ability to break from established mental sets is an important aspect of the creative process. In order to investigate students' creative thinking process, Lithner's (2008) framework for creative and imitative reasoning was utilized. Reasoning, according to Lithner (2008), can be seen as a thinking process with the following four steps: a problematic situation is met, a strategy choice is made, the strategy is implemented and finally a conclusion is obtained. In the literature the term reasoning often refer to some kind of high-quality line of thought, similar to proofs and deductive reasoning. Lithner (2008) defines reasoning as any way of thinking that concerns task solving. It does not have to be based on formal, deductive logic. There are two basic types of reasoning in the framework: *creative mathematically founded reasoning* and *imitative reasoning*.

Creative mathematically founded reasoning fulfils the following conditions:

- *Novelty*. A new reasoning sequence (to the subject) is created.
- *Plausibility*. The strategy choice and implementation is supported by arguments as to why the conclusion is true or plausible.
- *Mathematical foundation*. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

Plausibility in Lithner's (2008) framework does not necessarily imply deductive reasoning a la proofs, but rather reasoning that is supported by arguments. The quality of the reasoning is connected to the context. A lower secondary student that argues for equality can be said to carry out high quality reasoning by producing several numerical examples, but the same reasoning produced by a university student would be considered poor reasoning (Bergquist, 2007). The framework is inspired by Polya (1954) who wrote that: *"In strict reasoning the principal thing is to distinguish a proof from a guess, […] In plausible reasoning the principal thing is to distinguish a guess from a guess, a more reasonable guess from a less reasonable guess."* The stronger the logical value of the reasoning, the more plausible it is. Furthermore, the arguments must be based on intrinsic mathematical properties of the components involved in the reasoning. In the framework components are objects such as e.g. numbers, functions, matrices et cetera. An intrinsic mathematical property of a component is central to the problematic situation, while a surface property has little or no relevance.

The other line of reasoning described in the framework is called *imitative reasoning* and it is built on copying task solutions or through remembering an algorithm or answer. An answer is defined as "a sufficient description of the properties asked for in the task" and a solution is an answer together with arguments supporting the correctness of the answer. Two main classes of imitative reasoning is presented in the framework: *memorized reasoning* and *algorithmic reasoning*. Memorized reasoning satisfies two conditions:

- The strategy choice is based on recalling a complete answer by memory.
- The strategy implementation consists only of writing down the answer.

An example of memorized reasoning would be to recall the exact steps of an advanced proof of a theorem. An important interpretation of the definition of memorized reasoning is that the different steps of the proof could be written down in the wrong order since the different steps do not depend on each other.

According to Lithner (2008) an algorithm is a “set of rules that if followed will solve a particular task type.” The standard formula for solving quadratic equations is an example of a well-known algorithm. The difference between memorized reasoning and algorithmic reasoning is first and foremost that memorized reasoning involves a completely memorized solution while algorithmic reasoning involves memorizing the difficult steps of a procedure and then performing the easy steps. The order of the steps is also important to algorithmic reasoning, unlike memorized reasoning where the order of the steps could mistakenly be written down in the wrong order. Algorithmic reasoning fulfills two conditions:

- The strategy choice is based on recalling a set of rules that will guarantee that a correct solution can be reached.
- The strategy implementation consists of carrying out trivial (to the reasoned) calculations or actions by following the rules.

Algorithmic reasoning is appropriate and effective in situations of routine task solving. The problem is that students also use algorithmic reasoning in problem solving situations (Lithner, 2003; Lithner, 2004; Haavold, 2010).

There are three variants of algorithmic reasoning: *familiar algorithmic reasoning*, *delimiting algorithmic reasoning* and *guided algorithmic reasoning*. Familiar algorithmic reasoning is when:

- The reason for the strategy choice is that the task is seen as being of a familiar type that can be solved by a corresponding known algorithm.
- The algorithm is implemented.

Delimiting algorithmic reasoning is slightly different from familiar algorithmic reasoning. Here:

- An algorithm is chosen from a set that is delimited by the reasoner through the algorithms’ surface relations to the task. The outcome is not predicted.

- The verificative argumentation is based on surface considerations that are related only to the reasoner's expectations of the requested answer or solution. If the implementation does not lead to a (to the reasoner) reasonable conclusion it is simply terminated without evaluation and another algorithm may be chosen from the delimited set. (Lithner, 2008)

As a last resort, when familiar or delimiting algorithmic reasoning don't work, one might aim for external guidance. In guided algorithmic reasoning the following conditions apply:

- The strategy choice is based on identifying surface similarities in the task and an example, definition, theorem et cetera that can be found in a text form.
- The algorithm is implemented without any sort of verificative argumentation.

3. DNR Framework

In the first article high achieving students' mathematical reasoning was investigated. Harel (2008) defines mathematical activity as a triad of concepts: mental act, way of understanding and way of thinking:

"A person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's way of understanding associated with that mental act. Repeated observations of one's way of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a way of thinking associated with that mental act".

Close examination of the students' arguments, solutions and written work, may reveal a pattern characterizing their work. According to Harel (2008) these are characteristics of their way of understanding and might give some insight into their way of thinking. There is an important difference between behavior and cognition. We can only observe the former. The product of the thinking processes (i.e. the way of understanding) we can observe as behavior, but whatever inferences we make regarding the underlying cognitive processes, will still be, to some degree, speculative. In the first study, mathematical reasoning was classified as a mental act and the purpose was to investigate the students' way of thinking. The frameworks of Lithner (2008) and Harel (2008) were combined. In the first study mathematical reasoning was classified as a mental act and the students' written work and statements made during the interviews were classified as their way of understanding. Using Lithner's (2008) framework for classifying mathematical reasoning as imitative or creative, inferences on the students' way of thinking could be made. Or in other words, by looking characteristics of the students' mathematical reasoning, certain implications regarding their mathematical understanding could be hypothesized.

According to Harel (2008) knowledge consists of many mental acts such as interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, defining and problem solving. The DNR framework proposed by Harel focuses primarily on the products and characteristics of these mental acts. Product is a particular outcome of a mental act and character is a particular feature of that product. They are defined as *way of understanding* and *way of thinking* respectively. In order to explain the difference between the two categories, Harel (2008) gives the following example: "Two first graders Aaron and Betty solve the problem 3+4=?" Based on conversations with the two students it is inferred that Aaron views the problem simply as a command – add 3 and 4 and write the answer where the question mark is. Betty on the other hand interprets the problem as equality between two quantities. These different interpretations are products of the two students' mental acts of interpreting or in other words their way of understanding. On the basis of a multitude of such observations infer what the two students' characters, or way of thinking, of their interpreting mental acts are.

4. Mathematical giftedness

Usiskin's (2000) hierarchy of giftedness may shed some light on not only giftedness in mathematics, but also the relationship between giftedness and creativity in mathematics. In this hierarchy mathematical talent ranges from level 0 to level 7. The first level, level 0, represents adults who know very little mathematics. Level 1 represents adults who have a basic number sense as a function of their cultural usage and is comparable to students in grad six to nine. It is obvious that the first two levels include a large portion of the total population. Level 2 represents the honor students in mathematics who have the potential of majoring in mathematics and may end up eventually teaching secondary mathematics. Level 3 is the terrific student who has the potential to do beginning graduate work in mathematics. Level 4 is the exceptional student. These are students who receive admission to math academies, are able to converse with mathematicians about mathematics and construct proofs. Level 5, although a bit vaguely described, represents the productive mathematician who is capable of publishing in the field of mathematics. Level 6 represents the exceptional mathematicians whose work moved the field forward and are the best in their age group in the country. The final level, level 7, represents the all-time greats such as Gauss, Hilbert, Ramanujan and so on.

In Usiskin's hierarchy the professional mathematician is found at level 5, while the creative mathematician is found at level 6 and 7. Implying that creativity in the field of mathematics includes giftedness, but giftedness does not necessarily imply creativity. However, the concepts of giftedness, talent, creativity et cetera or the relationship between them are not unequivocally agreed upon. Krutetiskii (1976) saw giftedness and creativity as essentially the same in mathematics. He identified that the abilities to grasp formal structures, think logically in spatial, numeric and symbolic relationships, generalize rapidly and broadly, be flexible with mental processes, appreciate clarity, simplicity and rationality, switch from direct to reverse trains of thought and memorize mathematical objects, schemes and relationships, characterize mathematically able students.

5. Relationship between mathematical creativity and mathematical attainment

In this section the relationship between mathematical creativity and several aspects of and related aspects of mathematical attainment will be given. Often, mathematical ability has been seen as equivalent to mathematical attainment and to some degree, there is some truth to that notion. There is a statistical relationship between academic attainment in mathematics and high mathematical ability (Benbow & Arjmand, 1990). However, Ching (1997) discovered that hidden talent go largely unnoticed in typical classrooms and Kim, Cho & Ahn (2003) state that traditional tests rarely identify mathematical creativity. Investigations into the relationship between mathematical creativity and mathematical ability have as such varied in conceptualization; sometime focusing on current mathematical ability, sometimes focusing on mathematical attainment and sometimes focusing on mathematical knowledge. In the online thefreedictionary.com (Farlex, 2013), ability is defined as *"the quality of being able to do something, especially the physical, mental, financial, or legal power to accomplish something."* Attainment is defined as "*Something, such as an accomplishment or achievement, that is attained*." The key difference is that ability points to a potential to do something, while attainment refers to something that has been accomplished.

Sriraman (2005) claims that mathematical creativity in K-12 setting is seen on the fringes of giftedness. Mathematical knowledge/achievement is a necessary but not sufficient requirement for mathematical creativity. This hypothesis is strengthened further as traditional mathematics teaching emphasizes procedures, computation and algorithms. There is little attention to developing conceptual ideas, mathematical reasoning and problem solving activities. The result is that in general students' mathematical knowledge is without much depth and conceptual understanding (Hiebert, 2003). This is also seen in Selden, Selden &

Mason’s (1994) study where students with grades A and B struggle with non-routine problems. Selden et al. concluded that the students possessed a sufficient knowledge base of calculus skill and that the students' problem solving difficulties was often not caused by a lack of basic resources. Instead, they say, traditional teaching does not prepare students for the use of calculus creatively. Many students, even high achieving students, try to solve problems using superficial reasoning. An explanation is offered by Cox (1994), where the author argues that first year students in universities are able to get good grades by focusing on certain topics at a superficial level, rather than develop a deep understanding. It is conceivable that the same is seen in K-12 settings. In Haavold (2010) for instance, three high achieving students in mathematics were given an unusual trigonometric problem and only with some guidance were they able to display flexible and creative reasoning.

Pekhonen (1997) acknowledges the theory of functional asymmetry in the human brain as a vital contribution to understanding mathematical creativity and its relationship to divergent thinking. According to the theory the left hemisphere is connected with logical thinking and the right hemisphere helps with visual thinking. There isn’t a clear dichotomy here, but rather a continuous interval. Nevertheless verbal processing is mainly the domain of the left hemisphere while nonverbal processing takes place in the right hemisphere. The point Pekhonen makes is that mathematical creativity is dependent on both hemispheres. Many weaknesses observed in students’ problem solving skills and other high level thinking might be a result of excessive left hemisphere activity. The constant emphasis on rules and procedures may prevent development of creativity, problem solving and spatial ability. In successful problem solving and during the creative process in mathematics, both hemispheres are needed.

Creativity and knowledge

Knowledge itself cannot be directly observed, it has to be inferred from external observations in the form of tests, interviews, conversations and so on. There are many aspects and discussions surrounding the concept of knowledge. From epistemological issues of what is knowledge to more psychological questions issues that deal with the human brain and mind. However, for the purpose of this text a more pragmatic approach is taken. Knowledge is here defined, as it conventionally has been, as beliefs that are both true and justified. Thefreedictionary (Farlex, 2013) defines knowledge similarly online as: *“the facts, feelings or experiences known by a person or group of people.”* The concepts attainment, ability and knowledge are to some extent overlapping especially in terms of operationalization, but they are not synonymous. The relationship between mathematical creativity and mathematical knowledge, mathematical ability and mathematical attainment respectively, will therefore be dealt with individually.

There are two contesting theoretical perspectives on the relationship between creativity and knowledge: the *tension view* and the *foundation view* (Weissberg, 1999). The tension view posits that the relationship is shaped like an inverted U where the potential for creativity is at its highest point in a middle range of knowledge. Knowledge in a field is essential to create something novel and useful within said field. However an "excess" of knowledge might create a mental barrier. This is similar to a model proposed by Plucker and Beghetto (2004) in which levels of creativity flows from superficiality, via generality and domain specificity, to a fixed perspective as high levels of experience is gained. If an individual is well trained in a particular field he might be unable to come up with new ideas beyond what is common and expected (Sternberg, 2006; Weissberg, 1999). Overcoming fixations is necessary for creativity to emerge (Krutetskii, 1976). Cunningham (as cited in Haylock, 1987) asserted that drill and the learning of fixed procedures, which is common for many in school mathematics, may contribute to a rigidity or fixation of thought. So on one hand a solid knowledge base is necessary for mathematical creativity to emerge, but on the other too much drill can hinder mathematical creativity.

The foundation view suggests a positive relationship between knowledge and creativity. Since a knowledgeable individual knows what is done within a field, he can move forward and come up with new and useful ideas (Weissberg, 1999). Deep knowledge within a field is essential to the creative process. Instead of breaking from a set of traditions, creative thinking builds on knowledge (Weissberg, 1999). The relationship between mathematical knowledge and mathematical creativity is, as the relationship between knowledge and creativity in general, uncertain. Sak & Maker (2006) gave 841 1st through 5th grade students four types of math problems ranging from closed to open ended. The students' answers were compared on the basis of fluency, flexibility, originality and elaboration. They concluded that content knowledge was the strongest predictor variable of students' mathematical creativity. Kattou et al. (2013) investigated the relationship between mathematical ability and mathematical creativity by giving 359 elementary school students a mathematical ability and a mathematical creativity test. Although not defined as mathematical knowledge, the mathematical ability was considered a multidimensional construct including quantitative ability, causal ability, spatial ability, qualitative ability and inductive/deductive ability. In many ways representing a mathematical knowledge base. Mathematical creativity was measured with five open ended multiple solution tasks that were assessed on the basis of fluency, flexibility and originality. There was a strong correlational relationship between mathematical creativity and mathematical ability.

Leikin & Kloss (2011) asked 158 8^{th} graders and 108 10^{th} graders to solve four multiple solution tasks. The students' problem solving performance was compared on the basis of correctness, fluency, flexibility and originality. The 10^{th} grade students scored significantly higher than the 8^{th} grade students on correctness and fluency (number of solutions) on each of the four tasks. However in terms of originality (relatively unusual solutions) and flexibility (number of different categories) the results were task dependent. With 8^{th} graders scoring higher on some tasks and lower on other tasks. Tabach and Friedlander (2013) gave 76 students, ranging from 4^{th} grade to 9^{th} grade, three mathematical problems which were scored according to fluency, flexibility and originality. The findings indicate that mathematical creativity increased with age and that an increase in mathematical knowledge has the potential to raise the level in creativity as well.

According to Meissner (2000) and Sheffield (2009) mathematical knowledge is a vital prerequisite for mathematical creativity. Solid content knowledge is required for individuals to make connections between different concepts and types of information. Feldhausen and Westby (2003) assert that an individual's knowledge base is the fundamental source of their creative thought. Previous research seems to indicate a strong support for the relationship between mathematical knowledge and mathematical creativity. Older mathematics students would be, on average, expected to have acquired a greater knowledge base and therefore also be more mathematically creative than younger mathematics students. This also fits within Piaget's cognitive development theory in which children goes through different stages of cognitive development. Gradually organizing knowledge in increasingly complex structures (Ginsburg & Opper, 1988). There are of course individual differences in terms of cognitive development at any stage during childhood and adolescence, but on average older children should be able to organize knowledge in more complex structures than younger children.

Creativity and attainment

Several studies have found a significant correlational relationship between mathematical creativity and mathematical attainment in various forms (see for instance Kaltsounis & Stephens, 1973; Mccabe, 1991; Ganihar et.al. 2009; Kadir & Maker, 2011). Prouse (1967) investigated mathematical creativity in seventh graders in 14 classrooms in 5 schools in Iowa. The reported correlation was r=.53 between a mathematical creativity test and the *Iowa Tests of Basic Skills*. Jensen (1973) studied the relationship between mathematical creativity, numerical aptitude and mathematical achievement in 232 6th grade students. Mathematical creativity was operationalized as the ability to produce numerous different and applicable responses when presented with a mathematical situation. Numerical aptitude was measured with the California Tests of Maturity and mathematical achievement was measured using the computation section of the Metropolitan Achievement test. Jensen found weak correlations between mathematical creativity and both numerical aptitude and mathematical achievement. She went on to caution against the use of traditional achievement tests as there were high achievers with low creativity scores and low achievers with high creativity scores in her study. Mann (2005) gave 89 7th graders the *Connecticut Mastery Tests* and the divergent production items from the *Creative Ability in Mathematics Test* developed by Balka (1974). The correlation between the Connecticut Mastery Tests and the Creative Ability in Mathematics test was reported to be significant at $p<.01$ with r=.48. Using the same creativity test developed by Balka (1974) Walia (2012) examined the relationship between mathematical creativity and mathematical achievement in 8th grade students (N=180). Mathematical achievement was measured by the students' latest sessional assessment in mathematics. As Mann (2005), Walia (2012) reported a strong correlation between mathematical achievement and mathematical creativity (r=.725).

Other studies have focused on the distinction between academically gifted students in mathematics and creatively gifted students in mathematics. Hong and Aqui (2004) compared cognitive and motivational characteristics of high school students who were academically gifted in math, creatively talented in math and non-gifted. The two groups of gifted students scored higher than the non-gifted students in every category investigated. The authors did not find any difference between the academically gifted and creatively talented students in terms of ability, value or self-efficacy. They did, however, note that the creatively talented students used more cognitive strategies than the academically gifted students. Livne and Milgram (2006) provide further evidence to support the distinction between academically gifted students and creative gifted students in mathematics. The authors goes on to conclude that general academic ability predicted academic, but not creative, ability in mathematics, while creative thinking predicted creative, but not academic, ability in mathematics. High achieving students may therefore not be a homogeneous group in terms of mathematical knowledge, skill and understanding. This conjecture is plausible as Ching (1997) discovered that hidden talent go largely unnoticed in typical classrooms and Kim et al. (2003) state that traditional tests rarely identify mathematical creativity.

Haylock (1997) points out that mathematical attainment limits the student's performance on overcoming fixation and divergent production problems, but does not determine it. Low attaining students do not have the sufficient mathematical knowledge and skills to demonstrate creative thinking in mathematics. High achieving students in mathematics are usually also the most creative students in mathematics, but there are significant differences within the group of high achieving students. Within the group of high achievers in mathematics there are both low-creative and high-creative students. Separating students according to creativity and IQ, Cleanthous, Pitta-Pantazi, Christou, Kontoyianni, & Kattou (2010) explored differences of mathematical abilities between high IQ and low-creative students, low IQ and high-creative students and high IQ and high-creative students. The results indicated that high IQ and high-creative students had consistently higher scores than the other two groups on a mathematics ability test.

Further support for the heterogeneity of high achieving students can be found in Leikin and Lev (2007). Building on the work of Torrance they investigated mathematical creativity through the use of multiple solution tasks. In the study the authors employ both a regular procedural task, a system of linear equations, and a nonconventional mathematical problem. The responses were judged on the basis of flexibility, fluency and originality. The sample was separated into three different ability groups – gifted, proficient and non-gifted. Gifted students were those identified with high IQ and high achievements in mathematics. Proficient students were those who were not classified as gifted, but still high achievers in high level mathematics. Non-gifted students were regular students who performed well in regular mathematics. In order to reduce knowledge differences between the three groups, the first two groups were 10th graders and the last group consisted of 11th graders. Leikin & Lev (2007) conclude that the differences between the three groups are task dependent, as they were greater for the nonconventional task than the conventional task. Indicating that nonconventional mathematical problems may allow identification of creative and (probably) gifted students in mathematics.

Creativity and problem posing

In the research literature problem posing is seen as a vital part of creativity. Jay and Perkins (1997) state that "*the act of finding and formulating a problem is a key aspect of creative thinking and creative performance in many fields, an act that is distinct from and perhaps more important than problem solving*" (p. 257). Silver (1997) claims that both problem solving and problem posing are important aspects of mathematical creativity. However, problem posing is also the least understood and most overlooked part of mathematical creativity. Only a handful of studies have investigated the relationship between mathematical creativity, in the form of problem posing, and mathematical achievement, ability and/or knowledge (Van Harpen and Sriraman, 2012).

Van Harpen & Sriraman (2012) explored high school students' creativity by analysing their problem solving abilities in geometric scenarios. In their study the authors explored high school students' mathematical creativity in the USA and China by analyzing their problem posing abilities in geometric scenarios. The students were from one location in the USA and two locations in China. The authors conclude that even mathematically high achievers had difficulties in posing good and/or novel mathematical problems. Furthermore they claim that the differences in the three groups of students' performances can be, at least in part, by differences in the mathematics content that they have learned. The group with the strongest mathematical knowledge also performed better than the other two groups on the problem posing test. Shifting focus from the relationship between general mathematical ability and problem posing, English (1997) looked at the relationship between children's number sense and the development of problem posing abilities. The author notes that a strong number sense seems to play an important role in this development, more so than novel problem solving competence. Most studies exploring the relationship between mathematical creativity and mathematical ability have been on school students. Leung and Silver (1997) examined the relationship between arithmetic problem posing and mathematical knowledge of 63 prospective elementary school teachers. Problem posing was significantly related to mathematical knowledge, lending further support to the notion that knowledge in a field is important for creativity in that field to manifest.

4. Methodology

In this section the two studies, the qualitative and the quantitative, will be discussed in terms of methodological issues. First, in the following section, certain theoretical foundations regarding methods will be discussed. Then, in the last section, specific issues of this research project will be discussed in light of the theoretical foundations.

1. Pragmatic approach

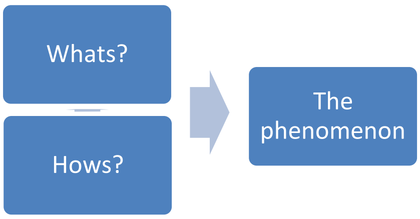
This research project uses both qualitative and quantitative methods, so before discussing the specific methodological issues related to each study, this section is focused on the debate and distinction between qualitative and quantitative methods. Cobb (2007) argues that mathematics education "can be productively viewed as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes" (Cobb, p. 7). He goes on to write that instead of making forced choices between theoretical perspectives, we should adopt a pragmatic approach that combines several perspectives that are useful for research and instruction. Silverman (2011) adopts a similar position and claims that the choice between different research methods should depend upon what you're trying to find out. The key to evaluate research lies in validity and reliability, regardless of the methods chosen.

However, that does not mean all choices of methods are equally sound, but instead that certain types of phenomenon and research questions are best investigated through the use of one particular method and perspective. The first article adopts a perspective Cobb (2007) calls cognitive psychology which focuses on how students think and work. The research question was "what characterizes high achieving students' mathematical reasoning?" A quantitative method could have been chosen to investigate said research question, but it would not have provided the detailed and rich data provided by the clinical interview. The author intentionally chose clinical interviews to investigate the students' reasoning process, as the focus of the study was: what are the characteristics of the students' reasoning and how do they reason. A purely quantitative perspective would be more generalizable, but also hide much of the specific details in the students' reasoning process. For instance, by observing the students' work directly, the process itself could be investigated and not just the final product. Each step of the students' reasoning could be observed (albeit not directly). That is not to say a quantitative perspective could not give some insight into the process, but much of the details in the data may have been lost and the resources necessary to carry out such a project would be greater than what was available to the author.

After the qualitative study, where a hypothesis was generated, the author wanted to explore the validity and generalizability of the hypothesis; was it a wide spread phenomenon or simply located to the initial few cases. In the qualitative study, it was found that even high achieving students may have lacked a mental flexibility when faced with an unusual trigonometric problem. The tentative hypothesis generated from the study was: even high achieving students might lack mathematical creativity and flexibility when faced with mathematical problems. A second, quantitative study was then carried out to investigate to what extent this was the case.

Silverman (2011) claims that the main strength of qualitative research is the ability to study phenomena which are simply unavailable elsewhere. While quantitative research can tell us about statistical relationships and inputs and outputs to some phenomenon, it cannot describe how the phenomenon is locally constituted. Qualitative data can, on the other hand, use naturally occurring data to find the sequences ("how") in which participants' meanings ("what") are deployed (Silverman, 2011). Silverman illustrates how quantitative and qualitative research complements each other with two figures:

Figur 1



Figur 2

Figure 1 and 2 illustrates how Silverman (2011) summarizes the strengths and weaknesses of qualitative and quantitative research. In figure 1, quantitative research, the phenomenon is defined operationally at the outset. This aids measurements, but simultaneously some of the particular context that makes the phenomenon what it is may be lost. Qualitative research, as seen in figure 2, is more contextual sensitive and researchers can see how a phenomenon is actually put together by the participants.

2. Validity

Shadish, Cook and Campbell (2002) claim that the term validity refers to the approximate truth of an inference. It is a property of inferences, not a property of design or methods. Different circumstances might contribute to more or less validity for the same research design. When something is valid, a judgment about whether or not the evidence supports the inference is made. However, we can never be sure whether an inference is true or false. That is why validity judgments are not absolute; various degrees of validity can be invoked. No method guarantees the validity of an inference (Shadish, Cook & Campbell, 2002). As a result the use of one particular method may affect different types of validity differently. One of the best examples is the randomized and controlled experiment that helps internal validity, but hurts external validity. Shadish et al. (2002) defines four types of validity: *Statistical Conclusion Validity*, *Internal Validity*, *Construct validity* and *External validity*. The four types of validity correspond, respectively, to four major questions when interpreting causal studies: 1) how large and reliable is the covariation between the presumed cause and effect? 2) Is the covariation causal, or would the same covariation have been obtained without the treatments? 3) Which general constructs are involved in the persons, observations and settings of the experiment? 4) How generalizable is the locally embedded causal relationships over varied persons, treatments and observations?

1. Statistical Conclusion validity

Shadish et al. (2002) defines statistical conclusion validity as "The validity of inferences about the correlation between treatment and outcome", which is essentially unchanged from Cook and Campbell (1979). In other words it is the degree to which our conclusions about associations in our data are reasonable. Statistical conclusion variety concerns primarily two questions: a) do the variables covary? and b) how strongly do they covary? The first question is related to erroneous conclusions. Either conclude that there is a correlation between two variables, when there isn't (Type 1 error) or conclude that there isn't a correlation when there is (Type 2 error). For the second question, the results might be exaggerated or understated depending on research design and implementation. Shadish et al. lists nine different threats to statistical conclusion validity, that may explain why inferences about covariation between two variables may be incorrect: low statistical power, violated assumptions of statistical tests, fishing and the error rate problem, unreliability of measures, restriction of range, unreliability of treatment implementation, extraneous variance in the experimental setting, heterogeneity of units and inaccurate effect size estimation.

Reliability refers to the reproducibility or consistency of scores from one assessment to another (American Educational Research Association, American Psychological Association, and National Council on Measurement in Education, 1999). It is a necessary, but not sufficient component of validity and is closely related to statistical conclusion validity. In Shadish et al.'s (2002) list of threats to statistical conclusion validity, unreliability of measures, unreliability of treatment implementation and extraneous variance are in particular relevant vis-à-vis statistical reliability. Unreliability of measures refers to measurement error and how it weakens the relationship between two variables or makes the relationship between three or more variables more unpredictable. Some of the more common tools to improve reliability of measurements are using more items or raters or better items or training of raters. Estimates of inter-rater reliability and test-retest reliability are often employed to determine if different raters and items are consistent. Unreliability of treatment implementation refers to how the treatment is implemented and if it is not implemented in a standardized manner for all participants, effects may be underestimated or inaccurate. Similarly, extraneous variance in the experimental setting refers to how external factors (for instance noise, heat, administrative differences et cetera) may influence the conclusions about covariance. In both cases, an effort has to be made to ensure a standardized setting and procedure for collecting data and measure variables.

2. Internal validity

Internal validity refers to inferences about whether observed covaration between A and B also implies a causal relationship between A and B. To establish such an inference, it must be demonstrated that A preceded B in time, that A covariates with B and that no other explanations for the relationship are plausible (Shadish et al., 2002). However, often there are numerous variables and circumstances involved, in particular in social science, that are uncontrollable. These may lead to alternative explanations for the effects found and the magnitude of the effects found. That is why internal validity is often more of a question of degree rather than a question of either-or. In this research project no claims of causality are made, therefor the matter of internal validity is less relevant than the other three categories of validity discussed here.

3. Construct validity

Construct validity refers to the validity of inferences about the higher order constructs that represent sampling particulars (Shadish et al., 2002). In other words, it is the extent we measure what we claim to measure. In this project, does for instance a mathematical creativity test actually measure mathematical creativity? The scale, in this case mathematical creativity, attempts to operationalize the theoretical concept of mathematical creativity by measuring observable phenomena. Some constructs, such as human height, are easily measured, while others such as psychological constructs like vocabulary or creativity are more problematic. Naming things is a challenge in all of science, as names reflect categories that in turn have relationships to other categories and theories. Shadish et al. (2002) state that it is never possible to establish a one-to-one relationship between operationalized scales in a study and the corresponding theoretical constructs. Good construct explication is vital to construct validity, but equally important is good assessment of the sampling particulars in a study. So that the researchers can assess the match between the assessments and the construct itself.

Threats to construct validity usually concern either explication of the constructs or the measurement design. The operationalization of constructs might not incorporate all the necessary characteristics or they may contain superfluous characteristics (Shadish et al., 2002). Shadish et al. (2002) goes on to list 14 specific treats to construct validity. Some of them are: inadequate explication of constructs, construct confounding, mono-operation bias, reactive self-report changes and novelty and disruption effect. The specific meaning and importance of each treat will be discussed in more detail in the next section, where the methods employed in this research project will be analyzed.

4. External validity

The last category of validity mentioned by Shadish et al. (2002) is external validity. External validity refers to the validity of generalization of inferences. In other words, it is a matter of to what extent the results of a study with a particular local sample can be generalized to other situations and to other participants. Shadish et al. (2002) gives five different targets of generalizations. The first is *narrow to broad* and refers to generalizations from smaller samples to larger populations. The second is *broad to narrow* which is from the study sample to a smaller group or even one individual. The third target *is at a similar* level which means that there is a generalization from a sample to another sample at the same level of aggregation. The fourth kind is generalization *to a different kind of sample*. The fifth and last type is from *random sample to population members*.

3. Methodological issues

In this section, specific methodological issues relating to the studies included in this dissertation will be discussed. The section is divided into subsections, where different themes are discussed in terms of internal validity, external validity, statistical conclusion validity and construct validity. Each of the theoretical constructs used in article two and three are discussed in terms of construct validity, then the statistical relationships between the constructs and other issues relevant to article two and three will be discussed in terms of external validity, internal validity and statistical conclusion validity. First, however, methodological issues relating to article one is discussed separately as it is based on qualitative methods.

1. Qualitative methodology

Unlike articles two and three, which are based on a quantitative study, article one is based on a qualitative study. For this reason issues regarding validity and reliability of article one will be discussed separately in this section. Reliability in quantitative research usually refers to the extent which an experiment, test or measurement produces the same result or consistent measurements on repeated trials (Silverman, 2011). In qualitative research it is however difficult to estimate reliability. Moisander and Valtonen (2006) suggest two manners in which reliability can be improved in non-quantitative research:

- By making the research process transparent by describing the research strategy and the data analysis in a detailed way.
- By paying attention to theoretical transparency. This is accomplished by making the theoretical framework and how the data is interpreted on the basis of the theoretical framework explicit.

It should be noted that some social researchers reject the notion of reliability in qualitative research. The author of this dissertation will not go into the discussion surrounding reliability in social research, but rather simply acknowledge that there is a discussion. Silverman (2011) claims that low-inference descriptors are central to high reliability in qualitative research. This means that observations are recorded as concrete as possible. To improve reliability of the qualitative research in article one, the author explicitly described theory, methods, data, interpretations and conclusions in detail in article one.

Data cannot be valid or invalid. Instead the question of validity applies to inferences drawn from the data and how the researcher's presence might have affected them (Hammersley & Atkinson, 2007). However, the concept of validity in qualitative studies have been defined and described in numerous ways (Golafshani, 2003). Some researchers even go so far as to claim that validity is not applicable to qualitative research. For instance, Lincoln and Guba (1985) proposed credibility, transferability and dependability explicitly as alternatives to the more traditional quantitative oriented criteria of internal validity, external validity and reliability respectively. However, Silverman (2011) claims that regardless of one's theoretical orientation, there is some need for some form of qualifying check or measure of the research. Triangulation and respondent validation are two widely used methods for validation of qualitative research. Triangulation usually refers to combining multiple theories, methods and data to produce a more accurate and objective representation of the object of study. Respondent validation means that the researcher attempts to validate their findings by taking them back to the people they studied to see whether the findings correspond with their own "experience". Silverman (2011) claims that both of these methods are usually inappropriate in qualitative research. Instead, Silverman offers five methods for validating studies largely or entirely based on qualitative data: analytic induction, constant comparative method, deviant-case analysis, comprehensive data treatment and using appropriate tabulations.

In the qualitative study, aspects of analytic induction, constant comparative method and comprehensive data treatment are used to strengthen validity. Analytic induction seeks to investigate some phenomenon and to generate a provisional hypothesis by identifying essential characteristics of the phenomenon studied in a limited set of cases. The constant comparative method involves inspecting and comparing all the data fragments that arise in a single case, while comprehensive data treatment refers to how "all cases of data… are incorporated in the analysis" in qualitative research (Mehan, 1979). In article one, a theoretical framework based on research by Lithner (2003) and Harel (2008) was constructed and used to analyze the data. During the process of analyzing the data, aspects of analytical induction, constant comparative method and comprehensive data treatment were employed.

In this particular study, differences and similarities between high achieving students' mathematical reasoning when faced with an unusual trigonometric problem was investigated. As reported in article one, the students' mathematical reasoning was to a large extent imitative and based on superficial properties. This observation led to the hypothesis that even high achieving students' lacked flexibility in their mathematical reasoning. Constant comparison and comprehensive data treatment were the tools used to further strengthen said hypothesis. Silverman (2011) claims that it is necessary to include all relevant data in a comprehensive data treatment to avoid anecdotalism and that new ideas have to be constantly checked, adjusted and refined through constant comparison with both other cases and other data in the same case. As such, each of the three cases investigated in this study was thoroughly investigated. The entire interview was transcribed and all the data was analyzed. The different cases were compared to each other several times and different episodes in each single case were analyzed and compared others in the same case. All in an attempt to understand and incorporate all relevant data, including deviant cases and data.

2. Mathematical achievements

In the second article, the intention was to investigate the relationship between mathematical achievements and mathematical creativity. Not necessarily the relationship between mathematical ability and mathematical creativity. Mathematical achievement was measured by the students' final assessment grades for the fall semester. A further description and explication of classification of students is found in article two. Construct validity, as mentioned earlier, is concerned with the extent to which we measure what we claim to measure. It is impossible to create a one to one relationship between the operations of a study and their corresponding constructs, in this case mathematical achievements. If the half year grades had been assessed by different teachers, the results might have been different. However, given that the teachers are experienced mathematics teachers and that they reported no anomalies, it is unlikely that for instance high achievers were labelled low achievers and vice versa. The final assessment grades for the fall semester measured what the teachers considered to be mathematical performance or attainment. As such, for the purpose of both this study and school performance itself, grades five and six are considered high achievement, grades three and four are considered medium achievement and grades one and two are considered low achievement.

Mathematical achievement was theoretically defined as the students' performance in mathematics; therefore the final assessment grades for the fall semester serve as a straightforward operationalization of the theoretical construct. The students' final assessment grades were to a large extent per definition also their mathematical achievement in school, as the final assessment grades for the fall semester 8th grade is the first major assessment for students in the Norwegian school system. The fact that the data for mathematical achievements was collected in a natural setting further strengthens construct validity. Shadish et al. (2002) points out several threats to construct validity in terms of the experimental situation itself. Participant responses reflect not just measures, but also the participants' expectations and perceptions of the experimental situation. Experimenter expectancies can influence participant responses by conveying certain expectations about desirable responses. Participants may also respond unusually to novel situations. These and similar factors constitute a threat to construct validity. However, since mathematical achievement was operationalized as the students' final assessment grades; many of these threats were circumvented.

3. Mathematical creativity

Mathematical creativity is a particularly difficult concept to operationalize. Treffinger et al. (2002) claim that there are more than 100 contemporary definitions of mathematical creativity. Even though two key features – originality and usefulness – are seen throughout the literature (Mayer, 1999; Plucker and Beghetto, 2004), any inference regarding mathematical creativity is problematic and influenced by subjective choices by the researcher. Sriraman (2005) proposed: *"a) the process that results in unusual (novel) and/or insightful solution(s) to a given problem or analogous problems, and/or b) the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle requiring imagination."* as a definition of mathematical creativity in a K-12 setting. The definition also served as a theoretical starting point for this study. The theoretical construct of mathematical creativity was operationalized using items from Balka's (1974) Creative Ability in Mathematics Test (CAMT). The test is described and explicated in detail in both article two and article three and the full test is added as an appendix to article two.

The reasons why items from the CAMT was chosen for measuring mathematical creativity is given in detail in article two and article three. Specific reasons as well as a more general discussion of divergent production as an operationalization of mathematical creativity is also provided. Sriraman's (2005) definition was chosen as a starting point for the investigation of mathematical creativity, as it is both sufficiently specific and based on an extensive synthesis and review of the literature. Haylock's (1987) framework was then used as a theoretical framework, placing the construct in a greater, theoretical context. Finally items from Balka's CAMT (1974) was used as measurement inputs; operationalizing the theoretical construct of mathematical creativity.

4. Creativity model

Operationalizing Sriraman's (2005) theoretical model for optimizing creativity in the classroom was more of a challenge than the other constructs used in this study. Even though there is no agreed upon definition of mathematical creativity and mathematical achievements, there is an extensive research literature on both constructs. Sriraman's model, on the other hand, is more recent and has not yet been investigated empirically. The five constructs (principles) - gestalt principle, aesthetic principle, uncertainty principle, scholarly principle and free market principle – have yet to be operationalized prior to this study. In this study the creativity model was operationalized using a questionnaire consisting of 36 items. The questionnaire is described and explicated in detail in article two and article three, with examples of items given in the appendix of article two. In this section the focus is therefore more on general issues regarding development of questionnaires as a method for operationalizing theoretical concepts and the author's rationale behind each step in the process.

Shadish et al. (2002) point out that a mismatch between operations and constructs can arise from inadequate analysis of a construct under study. Each of the five principles proposed in Sriraman's (2005) model had to be carefully analyzed and explored, before they could be operationalized. However, because the model had yet to be empirically investigated and each of the five principles had only been approximately defined, the author of this dissertation had to first conceptualize each of the five principles. Construct validity refers to, in a lay man sense, whether we measure what we claim to measure. Human height is for instance a construct that is easily explained and measure. The five principles in Sriraman's (2005) model are not easily measured. Not only have they yet to be empirically investigated, meaning that there is no natural or agreed upon measurement available, but they have also yet to be clearly defined. The author therefore had to first conceptualize the five principles, before operationalizing the principles by developing an appropriate instrument for measuring the constructs. This process is described in article two, but in short this was done by delineating the dimensions of the principles and then develop indicators that sufficiently incorporated the characteristics of the constructs.

Due to the fact that the five principles had yet to be conceptualized and empirically investigated, prior to this study, any investigation would be based on several subjective decisions by the researcher. Here, the uncertainty principle is used as an example to demonstrate how the construct was conceptualized and then operationalized. In his description of the uncertainty principle, Sriraman (2005) highlights the uncertainty and ambiguity that is found in professional mathematics. Creating, as opposed to learning, requires the students to be exposed to uncertainty and the difficulty of creating mathematics. Sriraman further states that it is important that teachers provide affective support when students experience frustration over being unable to solve a difficult problem. Although the principle is described fairly detailed in Sriraman's model, it is not defined sufficiently specific. There are many possible and plausible ways of empirically testing and measure the principle. According to Sriraman's (2005) model and related literature closely related to tolerance of ambiguity. Tolerance of ambiguity is required for sustained effort when an idea is forming, but not yet formed and there is pressure to prematurely finish a creative problem solving process.

Tolerance of ambiguity, or uncertainty, as a construct has attracted research in various branches of psychology and education since the 1940s (Furnham & Ribchester, 1995). Most researchers base their definition on Frenkel-Brunswik's (1949) research. She argued that tolerance of ambiguity generalizes to everything from cognitive functioning to social behaviour. However, in this research project the author was only interested in tolerance of ambiguity in a particular setting; school mathematics and its relationship with mathematical creativity. Therefore the students were asked about tolerance of ambiguity and uncertainty primarily in two areas: when they work on mathematical problems themselves and when the teacher presents new mathematical ideas. Questionnaires that measured the construct of ambiguity tolerance were explored and investigated prior to the development of the scale used to measure the uncertainty principle in this study. Then, items which were deemed relevant to the setting of this study and were found to be appropriate for children aged 14-17 were chosen and adjusted to be more specific to school mathematics. For instance, two of the items in the questionnaire were "I get frustrated if I don't understand a math problem" and "I become annoyed if I am not able to solve a math problem." The two items seem similar and are both related to the problem solving process. However, they reflect two different aspects of the problem solving process. The first statement refers to interpreting and understanding what is meant by a math problem and what it is asking the solver to do. The second statement refers to actually solving the problem, by carrying out a chosen strategy.

5. Statistical relationships

No claim of causality is made in this study. For that reason internal validity, or inferences about causality, is not the most pertinent issue in this research project. However, the issue of internal validity is still mentioned in this section as it is one of the four most known aspects of validity (Shadish et al., 2002). For this reason this section, on the relationships between the constructs investigated in this research project, is a discussion on issues relating to statistical conclusion validity, reliability and external validity.

Statistical conclusion validity concern the degree our conclusions about data are reasonable. In particular do the variables covary and how strongly do they covary. Many of the specific issues relating to threats to statistical conclusion validity are discussed in detail in article two and article three. For the exact numerical values and theoretical discussion, see article two and three. Article two and three do not go into a detailed discussion of external validity. Instead the sample chosen is described in detail and the reader can make their own inferences and evaluation of external validity. The author will however emphasize that the sample chosen consist of typical Norwegian students from typical secondary schools in Norway. As such, the author contends that the results of this study can be generalized to fairly numerous other, similar situations and students.

Reliability is concerned with consistency or reproducibility of results, inferences and interpretations (Crocker & Algina, 1986; Shadish et al., 2002; Silverman, 2011). Does a questionnaire, test, observation or any other measurement procedure produce the same results over time and across raters or are the results based, to some degree, on coincidences? The researcher must provide arguments or evidence supporting the claim that the findings are not just a result of chance. Crocker and Algina (1986) make a distinction between systematic and random errors of measurement. The former are errors that consistently affect an individual's score. For example if an item on a test is unintentionally ambiguous, an individual may interpret it wrongly and this error would most likely be repeated the next time as well. The latter, random errors are errors that affect results as a result of pure chance. To reduce systematic errors both the questionnaire and creative mathematics test were given to a group of 40 students in a pilot study. Items which were ambiguous, unclear, difficult to understand or otherwise inappropriate were replaced, removed or altered. To reduce random errors all students received the same standard information prior to the data collection, the data were collected at the same time of the day and the test was administered by the author of this dissertation. Furthermore, to strengthen reliability of the creative mathematics test, ten cases were randomly chosen and given to a different researcher who scored the tests separately. Afterwards the different scores for the ten tests were compared pairwise. In three of the cases there were some minor differences. The differences were discussed, resolved and the rest of the sample was scored using the additional guidelines.

5. Ethical considerations

Hammersley and Atkinson (2007) notes that there are five main areas of ethical considerations: informed consent, privacy, harm, exploitation and consequences for future research. This view is also in line with the guidelines from The National committee on research ethics in humanities and social sciences (NESH, 2010).

The first area of ethical considerations, informed consent, refers to the fact that research should only be conducted on persons that has given a free and informed consent. The consent is considered free if the consent has been given without any sort of external force or limitation of personal freedom. The consent is considered informed if the research subject has been informed about his or her role or participation in the research project in a clear and understandable manner. In this project school administration, teachers and students were involved at different levels of the project as participants and research subjects. First, the school administration was contacted via email, informing them in general about my project and research questions; the school administration was also asked if they were interested in participating in the project. When the schools emailed back saying they were interested in participating in the project, the author met with members of the school administration and explained in more detail the project and how and why the school could participate. Second, the teachers were asked in person if they wanted their students to participate in the project. What I, the researcher, needed to collect the necessary data and how the data would be analyzed and published was explained in detail to the teachers. Third, both the students' parents and the students themselves were informed about the project and what the students would be participating in. The parents were given information through a letter two weeks prior to the data collection sessions. The students were first given information by their teacher two weeks prior to the data collection and additional information were given by the researcher prior to each data collection session. Each student could opt out of the project at any time.

Privacy, the second area, refers to the participants' rights to be anonymous and keep private information away from the public (Hammersley & Atkinson, 2007). All names of students, teachers, schools and school officials were anonymized, with only the researcher knowing the real names of the participants. The third area of ethical concerns, harm, usually refers to threats to participants' health and safety, but also includes research that might provoke stress and anxiety in participants. In this research project, the most likely way to harm the participants would be a breach of privacy or negatively describe the teachers or students. The author cannot conclusively say that no harm, in any imaginable way, came to any of the participants, but the study was descriptive and not normative in nature. Furthermore, the author explicitly explained to all the participants both the motivation for the research and how the data would be used and published. The author also made sure to refrain from commenting on teachers', students' or school's performance on the tests or questionnaire.

Exploitation concerns the fact that researchers exploit participants, by demanding time, information and/or extra work, without giving anything back (Hammersley & Atkinson, 2007). In this study both the teachers and students gave up part of their time during school hours, without asking for anything in return. However, after the data had been analyzed and published, the teachers were informed on the results and the author's conclusions and interpretation of the results. The teachers themselves seemed both interested and grateful for the updates and expressed that the results would be useful for their teaching. The last area of ethical concerns refers to the researcher's responsibility to colleges and future research (Hammersley & Atkinson, 2007); by some way exploit or harm the participants or misrepresent the results, affecting the participants' willingness to participate in future research. The researcher tackled this challenge by being open and honest about the project, treating the data with respect and presenting the results in an, as best as it could be done, unbiased and fair manner.

6. Findings

The doctoral research project set out to answer the research question:

> "What are the characteristics of mathematical creativity?"

More specifically the project investigated the relationship between mathematical achievement and mathematical creativity and personality traits indicative of mathematical creativity. In this section the results from each of the articles will be presented and then common features of the four articles will be presented, in an attempt to answer the main research question of the doctoral thesis. To avoid simply repeating the result section of each article, the focus here will be the results most pertinent for the overall research question and how they fit into the process of the doctoral research project.

1. Article one

The data from study one was collected in the spring of 2010 from a local upper secondary school in a city in northern Norway. Three high achieving students were then given a trigonometric problem during a clinical interview. The exact process of the data collection is given in article one. One key aspect of the study is that the author did not want to limit or guide the teacher's selection of "high achieving students". The teacher was simply asked to pick between 2-4 students who she considered to be "high achieving students". In some studies the researchers themselves define high mathematical ability according to certain criteria. Often in an attempt to investigate relationships between theoretical constructs of mathematical ability and for instance mathematical creativity. In article one, however, the author wanted to investigate how high achieving students reasoned when faced with an unfamiliar problem. Achievements and ability are closely related. However, certain findings (see Benbow and Arjmand, 1990; Ching, 1997; Kim et al., 2003; Hong and Aqui, 2004; et cetera) indicate that mathematical ability and mathematical achievement in a K-12 setting are not necessarily synonymous. The rationale for investigating high achieving students' reasoning, and not high ability students' reasoning, was to investigate the quality of the high achieving students' mathematical reasoning. Do high achieving students resort to low quality reasoning and if so, why? Low quality reasoning was here defined as imitative reasoning, while high quality reasoning was defined as creative reasoning (Lithner, 2008).

Based on an extensive analysis of the data collected during the three interviews, several interesting findings were reported. All three students, when first given the equation, attempted to solve it by applying known algorithms they had recently learned to solve superficially similar equations. On the surface, the equation was similar to equations they had recently worked on in their textbook. However, on a structural level, the equation was very different. One of the students even explicitly expressed her desire to find "the right algorithm" when she said that she didn't remember how to solve the equation. All three students were able to use flexible mathematical reasoning only when given some form of help, either in the form of a prompt during the interview or explicit guidance from the interviewer. Based on the observations made in the study, the author concludes that the students possess the necessary domain knowledge to solve the equation, but their mathematical reasoning was to a large extent imitative and based on superficial properties.

The findings resulted in the generation of a hypothesis: even high achieving students may lack flexibility and creativity when faced with unusual mathematical problems. In order to further explore this hypothesis, the author wanted to investigate to what extent high achieving students were also mathematically creative students and under which conditions. A quantitative study, which resulted in two separate articles, was designed and implemented on this basis.

2. Article two

Article two set out to 1) explore the relationship between attainment in school mathematics and mathematical creativity and 2) investigate the characteristics of mathematically creative students. These two goals were formulated as two research question:

"To what extent does Sriraman's theoretical model of mathematical creativity predict mathematical creativity?"

and:

"T*o what extent does mathematical achievement predict mathematical creativity?"*

To investigate the characteristics of mathematically creative students, a model developed by Sriraman (2005) for optimizing creativity in the classroom was chosen as a theoretical foundation. Previous research has linked both mathematical creativity and creativity to numerous motivational factors, beliefs and abilities (Vlahovic-Stetic, 1999; Hong & Aqui, 2004; Mann, 2005; Sternberg, 2006; Sriraman, 2009 et cetera). Furthermore, there are several theoretical models describing the relationship between giftedness, abilities, creativity and other factors (see for instance Sternberg, 2003; Tannenbaum, 2003; Babaeva, 1999 et cetera). However, Sriraman's model is, to the author's knowledge, the only model that attempts to connect mathematical creativity to distinct principles (or factors). Through an extensive review and synthesis of existing literature on mathematical creativity, Sriraman (2005) constructed a model with five principles for optimizing mathematical creativity in the classroom.

Although Sriraman's (2005) model provided a theoretical foundation for the study, it had yet to be sufficiently conceptualized. Each principle had only been given a short description in the original article (Sriraman, 2005). In order to operationalize the model and empirically test it, the model had to first be conceptualized and linked to previous and relevant research. This provided a clearer and more specific definition of each of the five principles outlined by Sriraman. After conceptualizing each of the five principles, they were operationalized into six scales measured by with a questionnaire consisting of 35 items. The questionnaire and how it was developed is described in detail in article two. Mathematical creativity was measured by a creative mathematics test based on Balka's Creative Ability in Mathematics Test (1974). Both the test itself and a methodological discussion of its validity and reliability are found in article two.

The research questions were analysed using analysis of covariance (ANCOVA). The dependent variable was the score on the creative mathematics test and the independent variables were mathematical achievement and each of the six scales – motivation, freedom, aesthetic, scholarly, free market and uncertainty - measured by the questionnaire. The results indicated that motivation and aesthetic significantly predicted mathematical creativity controlled for mathematical achievement level. Motivation accounted for 4.6% of the variance in mathematical creativity and aesthetic accounted for 4.8% of the variance in mathematical creativity, measured by eta squared. Which, according to Cohen (1988), constitutes a small, but almost medium, effect. The data also revealed a strong relationship between mathematical achievements and mathematical creativity. However, the relationship between mathematical achievement and mathematical creativity appear to be asymmetrical. Mathematical achievement seems to be a necessary, but not sufficient requirement for mathematical creativity. This was based on the observation that the students who scored well on the mathematical creativity test were almost exclusively also high achievers, but simultaneously there were several high achievers who did not score well on the mathematical creativity test. The results further strengthens the notion of heterogeneity among high achieving students (see Haylock, 1997; Hong and Aqui, 2004; Livne and Milgram, 2006; Leikin and Lev, 2007 et cetera). A closer investigation, either using qualitative methods or employing a larger sample of high achieving students, may be required in order to discover what characterises high achievers who are also mathematically creative students.

3. Article three

Article three investigated the relationship between mathematical knowledge and mathematical creativity along a different dimension than article two. Article two focused on the relationship between mathematical achievements and mathematical creativity. In article three, a different perspective was chosen. In this study, the research question was:

> *Are there differences in mathematical creativity between 13 year old and 17 year old mathematics students? If so, what are the differences?*

17 year old students have three more years of formal mathematical schooling than 14 year olds. If the basic premise, that there is a strong relationship between mathematical knowledge and mathematical creativity, holds ground then it would be expected to see the older students outperform the younger students on a creative mathematics test. However, there are those who claim that schools stifle creativity (Azzam, 2009). Certain studies have also demonstrated a negative relationship between age and creativity (Rosenblatt & Winner, 1988; Runco, 1989; Runco, 1991). To investigate the research question 190 8th grade students and 118 11th grade students were given the same test for measuring mathematical creativity as used in article two.

The results are given in article three, but in short the 17 year old mathematics students scored significantly higher than 14 year old students on the creative mathematics test. The 17 year old students scored higher on each of the three tasks and on each of the three categories (originality, flexibility and fluency) that make up the creativity score. However, only one of the tasks and originality was statistically significant in favor of the 17 year old mathematics students. This means that the older students did not provide significantly more responses or significantly more response categories. They did however provide statistically more unique responses. Therefore, the article concludes that the 17 year old students are more mathematically creative than the 14 year old students and the primary cause is a higher score on originality.

In the article the author pursues three different lines of reasoning that could explain the observed results. First, the scoring guide itself was investigated to see if originality was relative to the sample. A new weighting of originality based on the sample in this study, as opposed to the original sample used by Balka (1974) was created. However, the results were similar with both scoring guides. A second line of reasoning that could explain the results was that originality is different mathematics that the younger students have yet to learn. However, an analysis of the solution categories provided by the students revealed that none of them were outside the younger students' curriculum. The older students did not score higher on the originality category because they had an opportunity to learn mathematics that the younger students had yet to be exposed to. Instead, the author argues, the more original solution classes are less obvious than the more common solution classes. The less original solution classes are easily and directly observable, while the more original solution classes require some form of advanced inductive or deductive reasoning based on the mathematical properties in the problem situation.

4. Article four

Article four was, unlike the other three, not based on original research or study by the author. Instead, it was a synthesis of recent research on the relationship between mathematical creativity and ability. In article four the authors synthesize and critique three articles in a recent special issue of ZDM. The article also distill features of problem solving, problem posing, problem sequencing and provides a general discussion of creativity, ability and giftedness. However, in this section the author will focus on the synthesis of the three articles in ZDM (Leikin and Lev, 2013; Kattou, Kontoyianni, Pitta-Pantazi, and Christou, 2013; Pitta-Pantazi, Sophocleous and Christou, 2013). The reason is that the synthesis of the three articles provides an empirical context in which the other three articles can be further analyzed and common themes can be extracted from the doctoral research project.

All three studies investigated the relationship between mathematical creativity, ability and giftedness empirically. In Kattou et al. (2013) the authors looked at the relationship between mathematical ability and mathematical creativity in 359 elementary school students. Pitta et al. (2013) explored the relationship between mathematical creativity and cognitive styles in 96 prospective primary school teachers. In the final article, Leikin and Lev (2013) investigated the relationship between mathematical ability, general giftedness and mathematical creativity in 51 11^{th} and 12^{th} grade students. Although the three articles focused on different aspects of the relationship between mathematical creativity, giftedness and ability, certain structural similarities were found in the synthesis.

Certain cognitive styles, mathematical ability and general giftedness were all found correlate with mathematical creativity. High IQ, spatial cognitive style and general mathematical ability were all linked to mathematical creativity. As a concept, mathematical creativity does not exist in a vacuum. Certain features and factors are required for mathematical creativity to arise. Furthermore, the articles synthesized in article four all point out that certain individuals are more mathematically creative than others. In the three studies mathematically able students, students with a preference for spatial cognitive style and gifted students (high IQ students) were all mathematically creative. Or correspondingly mathematically creative students are characterized by high IQ, a preference for spatial cognitive style and a high mathematical ability.

5. Common theme

In this section the author will attempt to draw some common themes out of the four articles that form the basis for the doctoral thesis and provide some answers to the main research question:

> “What are the characteristics of mathematical creativity?”

To answer the main research question, this section will present and discuss the results in two stages. First, the author will mention and comment on some general themes that stood out in the four articles. Second, the findings of this dissertation will be discussed in light of previous research on the topic of creativity and mathematical creativity.

The main finding of this research project is the strong relationship between mathematical attainment and mathematical creativity. Article two indicates that high achievers are significantly more creative in mathematics than low- and medium achievers in mathematics. Article three indicates that older students, who have had three more years of schooling, are more creative in mathematics than younger students. Taken together, it is reasonable to assume that one of the main characteristics of mathematical creativity is mathematical attainment. Attainment is not necessarily only measured by grades within one year of school mathematics, but also the number of years of school mathematics a student has completed. However, although attainment is a strong predictor and necessary requirement of mathematical creativity, it seems not to be a sufficient requirement. This is seen in the heterogeneity in mathematical creativity among high achievers in article one, two and three (see also Haylock, 1997; Hong and Aqui, 2004; Livne and Milgram, 2006; Leikin and Lev, 2007 et cetera).

Article one, two and four all shed some light on this apparent paradox; that only some of the high achieving students are also mathematical creative students. In article one all three high achieving students first looked for an algorithm to solve the problem. Only with some help were they able to solve the problem, even though they all had the necessary mathematical knowledge. However, the boy was able to solve the problem with only a minimum of help. Unlike the two girls, he was able to make the necessary flexible generalization when the prompt was introduced. In article two, both motivation and a sense of the aesthetic qualities of mathematics significantly predicted mathematical creativity controlled for mathematical achievement. Finally, in the review article, the authors showed that mathematical creativity is linked to several other qualities such as general giftedness (high IQ) and a preference for certain cognitive styles (spatial style). These findings, taken together, indicate that mathematical creativity, as a concept, is linked to other concepts, especially at the higher echelons of mathematical attainment.

There are several theoretical perspectives on creativity and its relationship to other concepts. In this section the author will attempt to look the findings in this study in the context of larger, theoretical perspectives. The two primary views of the relationship between creativity and knowledge are the tension view and the foundation view (Weissberg, 1999). The tension view theorises that knowledge is essential to creativity, but at some point an excess of knowledge can creative a mental barrier. This view is in many ways analogous to the threshold theory (Sternberg, 2005), that posits that creativity and intelligence are correlated up to a threshold (an intelligence quotient of ca. 120) and after that they tend to vary independently. The foundation view, on the other hand, suggests a positive relationship between knowledge and creativity. Instead of breaking from a set of traditions, creative thinking builds on deep knowledge within a field (Weissberg, 1999). Knowledge, intelligence and attainment are not synonymous; however, said theoretical perspectives may shed some further light on the findings in this research project. Mathematical attainment was found to be a strong predictor of and a necessity for mathematical creativity in this project, but high attainment in mathematics did not, however, guarantee mathematical creativity. An observation analogous to both the tension view (Weissberg, 1999) and the threshold theory (Sternberg, 2005). It appears that, as with the relationship between intelligence and creativity and knowledge and creativity, mathematical attainment is strongly linked to mathematical attainment up to a certain point. However, among high achievers, mathematical creativity varies greatly. That is not to say that an excess of attainment, as knowledge, create a mental barrier, but rather that other factors attainment account for variation in mathematical creativity. Based on the research presented here, it is conceivable that some of these factors include intrinsic motivation, a sense the aesthetic of mathematics, intelligence and preference for a spatial cognitive style.

Renzulli's Three ring model (1986) and Tannenbaum's star model (2003) can also shed some light on the findings presented here. Both the Star model and the Three-ring Model see giftedness as an interaction between a multitude of elements and traits. They both illustrate how giftedness has gone from being looked at as intelligence, to more complex models where intelligence is but one part of giftedness. In both models, creativity is seen as a part of giftedness. The heterogeneity in terms of creativity among high achievers observed in this project, lend further support to the more complex models of giftedness. Not all high achieving students in mathematics are mathematically creative students, indicating that not all high achieving students can be classified as gifted students. The causal relationships between giftedness, attainment, creativity and other concepts becomes then an interesting aspect of these observations. Why not all high achievers are also gifted students? Leikin and Lev (2013), for instance, state that all students can develop flexibility and fluency over time in the right environment, but originality is more of a gift reserved for certain individuals. Such a view essentially recognizes a genetic or biological component to creativity and, thus, giftedness. The view that giftedness has a certain genetic component is generally agreed upon within the literature, but the exact nature of the relationship is unclear (Simonton, 2008).

Article two of this research project indicates that the 17 year old mathematics students score significantly higher on originality than the 14 year old mathematics students. The two most immediate conclusions are that: a) the samples are somehow different and that is why the originality scores differ significantly or b) the older students are more original than the younger students and originality in mathematics can be developed and is not necessarily only a gift exclusive to certain individuals. Similarly, the findings of this project lead to other interesting questions regarding causal relationships between mathematical creativity and other concepts. Are mathematically creative students more motivated, are motivated students mathematically more creative or is there a more complex, interdependent relationship between the two concepts? Do mathematically creative students appreciate the beauty of mathematics more than other students, or is it their sense of aesthetic in mathematics that help develop mathematical creativity? In order to further investigate these and other questions relating to the characteristics of mathematical creativity and its relationship with other concepts, different study designs are required.

The author wants to propose three future research designs that can further shed some light on the issues outlined in this section. One of the more ideal designs, however costly in terms of resources, planning and time, is a longitudinal study in which students can be tracked over an extended period of time. By tracking the same students over time, observing the same variables over time, it may be possible to say something about how mathematical creativity and related concepts develop over time in a particular setting. A second alternative would be a design similar to the quantitative study carried out in this project, but with a purposeful sampling procedure targeting gifted or high achieving students. This would allow the researcher to more specifically investigate the relationship between mathematical creativity and other concepts in a group of gifted or high achieving students.

6. Final remarks

In the beginning section of this text I, the author, outlined a personal and scientific motivation for research project. It provided added transparency and context to the studies carried out by me and the rationale for the research project. In this final section of this text, I will again add a slightly more personal and informal touch to the research project, by commenting on the results and placing them in a larger, theoretical perspective. More specifically, I will touch on the difference between the concepts of ability, giftedness and creativity within the field of mathematics. Previously in this text, I defined all three concepts. All three concepts are ambiguous in the sense that there is no agree upon definition in the literature or the scientific community of them. However, it is possible, based on the findings presented in this text and other studies, to identify differences between the three concepts and say something about the relationship between them.

In previous sections a theoretical exposition of ability, giftedness and creativity was provided; therefore I will not repeat earlier theoretical arguments, but instead offer my own opinion and assessment. Achievement is not the equivalent of ability, but achievement, as defined in this research project, is an assessment of the student's mathematical ability. The assessment may not be entirely valid or reliable, for numerous reasons, but it is nevertheless a measurement of mathematical ability. If we accept the premise that achievement is a measurement of ability in mathematics, then it becomes clear that ability is a necessary but not sufficient requirement for mathematical creativity. Accepting this premise is, however, not unproblematic. Within the research literature, the relationship between mathematical ability and mathematical creativity is ambigious (Kattou et al., 2013). Is mathematical ability an aspect of mathematical creativity, is mathematical creativity an aspect of mathematical ability or are they linked in some other way. Where does giftedness fit in all of this and is it domain general or domain specific?

Much of the uncertainty and confusion regarding these questions is a direct consequence of the lack of agreed upon and unambiguous definitions of ability, giftedness and creativity. A lack of a clear definition of the concepts also leads to different operationalizations of the concepts. In the dictionary, ability is simply defined as being able to do something (Farlex, 2013), thus within the field of mathematics, mathematical ability would imply the ability to do mathematics. However, "doing mathematics" in itself implies ambiguity. What is mathematics and, in particular, what is mathematics in a K-12 setting? Any answers to these questions would be normative by nature. However, that does not mean that any answer is equally valid. Rational and thoughtful arguments could provide some insight into these questions.

This text began with a presentation of a personal and scientific motivation for this research project. Just as Lithner's (2008) framework provided a framework for my own experiences, it, along with Sriraman's (2005) definition of mathematical creativity, can also shed some light on the relationship between creativity, giftedness and ability. If we accept Lithner's and Sriraman's definitions of creativity, creativity in mathematics would then imply that one is able to use mathematics insightfully and plausibly in novel situations; i.e. use mathematics independently. Mathematical creativity then becomes an aspect of mathematical ability. It becomes the ability to do mathematics in a very particular manner. As this research project and other studies have demonstrated, mathematical ability and mathematical achievement do not necessarily imply mathematical creativity; ability and/or achievement is a necessary, but not sufficient requirement. Therefore, one could argue that giftedness in mathematics requires both mathematical ability and mathematical creativity. A position similar to many of the more recent models of giftedness in general psychology (Renzulli, 1986; Tannenbaum, 2003), in which giftedness is a multifaceted concept. Many students do well in school mathematics, but only the truly gifted students are mathematically creative students as well.

7. References

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Paper 1

Paper 2

Paper 3

Paper 4

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