Estimation of statistics for single point measurements in the Scrape Off Layer

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Time series of plasma particle density by single point measurements, as Langmuir probes or gas-puff imaging, in the scrape-off layer of magnetically confined plasmas universally feature intermittent burst events from blobs. Conditional averaging reveals that these burst events feature a steep front and a trailing wake as well as exponentially distributed waiting times between single burst events[1]. Recent work proposed to model particle density time series by a shot noise process. Assuming that the burst amplitudes are also exponentially distributed, it was shown that the resulting PDF of the time series is Gamma distributed[2]. The resulting PDF is in good agreement with time series obtained by gas-puff imaging in the scrape-off layer of Alcator C-Mod[3] and other devices[4].
From the shot noise model proposed in [2], we derive expressions for the mean-squared error on the estimators of mean and variance as a function of sample length and sampling frequency. These errors are propagated on errors on estimators of skewness and kurtosis, which are frequently used statistics to describe the deviation of particle density time series from normality.

Stochastic model

Model time series as a shot noise process

$$\Phi(t) = \sum A_k \psi(t - t_k)$$

Conditional averaging suggest universal burst shape

$$\psi(t) = A \exp\left(-\frac{t}{\tau_d}\right)\,\Theta(t)$$

Uniformly distributed burst arrival times imply exponentially distributed waiting times

$$P(t_k) = \frac{1}{T}$$

Exponentially distributed burst amplitudes

$$P(A_k) = \frac{1}{\langle A \rangle} \exp\left(-\frac{A_k}{\langle A \rangle}\right)$$

Resulting PDF for the time series is a Gamma distribution with shape parameter gamma

$$P(\Phi) = \frac{1}{\Gamma(\gamma)} \left(\frac{\gamma}{\langle \Phi \rangle}\right)^{\gamma} \Phi^{\gamma - 1} \exp\left(-\frac{\gamma \Phi}{\langle \Phi \rangle}\right)$$

Shape parameter also can be expressed by the ratio of burst decay time to waiting time

$$\gamma = \frac{\tau_d}{\tau_w}$$

Mean-square errors on estimators on statistics

Evaluate correlation functions to find uncertainty and correlation function on estimators

$$\mathrm{MSE}\left(\widehat{\mu}\right) = \frac{1}{N} \langle A \rangle^2 \frac{\tau_d}{\tau_w} \left(1 + \frac{2}{N} \frac{\alpha N + e^{-\alpha N} - 1}{\alpha^2}\right) \qquad \text{where} \qquad \alpha = \frac{\triangle_t}{\tau_d}$$

$$\mathrm{MSE}\left(\widehat{\sigma^2}\right) = \langle A \rangle^4 \left[\left(\frac{\tau_d}{\tau_w}\right)^2 \left(\frac{2}{N\alpha} + \frac{1}{N^2} \frac{-5 - 8e^{-N\alpha} + 2e^{-2N\alpha}}{\alpha^2}\right) + \left(\frac{\tau_d}{\tau_w}\right)\left(\frac{6}{N\alpha} + \frac{1}{N^2}\frac{-27 + 3e^{-2N\alpha}}{\alpha^2}\right)\right] + \mathcal{O}\left(N^{-3}\right)$$

$$\mathrm{COV}\left(\widehat{\mu}, \widehat{\sigma^2}\right) = \langle A \rangle^3 \left[\left(\frac{\tau_d}{\tau_w}\right)^2 \frac{4\left(1 - e^{-\alpha N}\right)}{N^2 \alpha^2} + \left(\frac{\tau_d}{\tau_w}\right)\left(\frac{3}{N\alpha} + \frac{-17 + 4e^{-\alpha N}}{2N^2\alpha^2} + \frac{9 - 12e^{-\alpha N} + 3e^{-2\alpha N}}{N^3 \alpha^3}\right)\right]$$

Propagate uncertainty on $\widehat{\mu}$ and $\widehat{\sigma^2}$ using $\widehat{S_\Gamma} = 2\frac{\widehat{\sigma^2}}{\widehat{\mu}}$ $\quad \widehat{K_\Gamma} = 6\frac{\widehat{\sigma^2}}{\widehat{\mu}^2}$

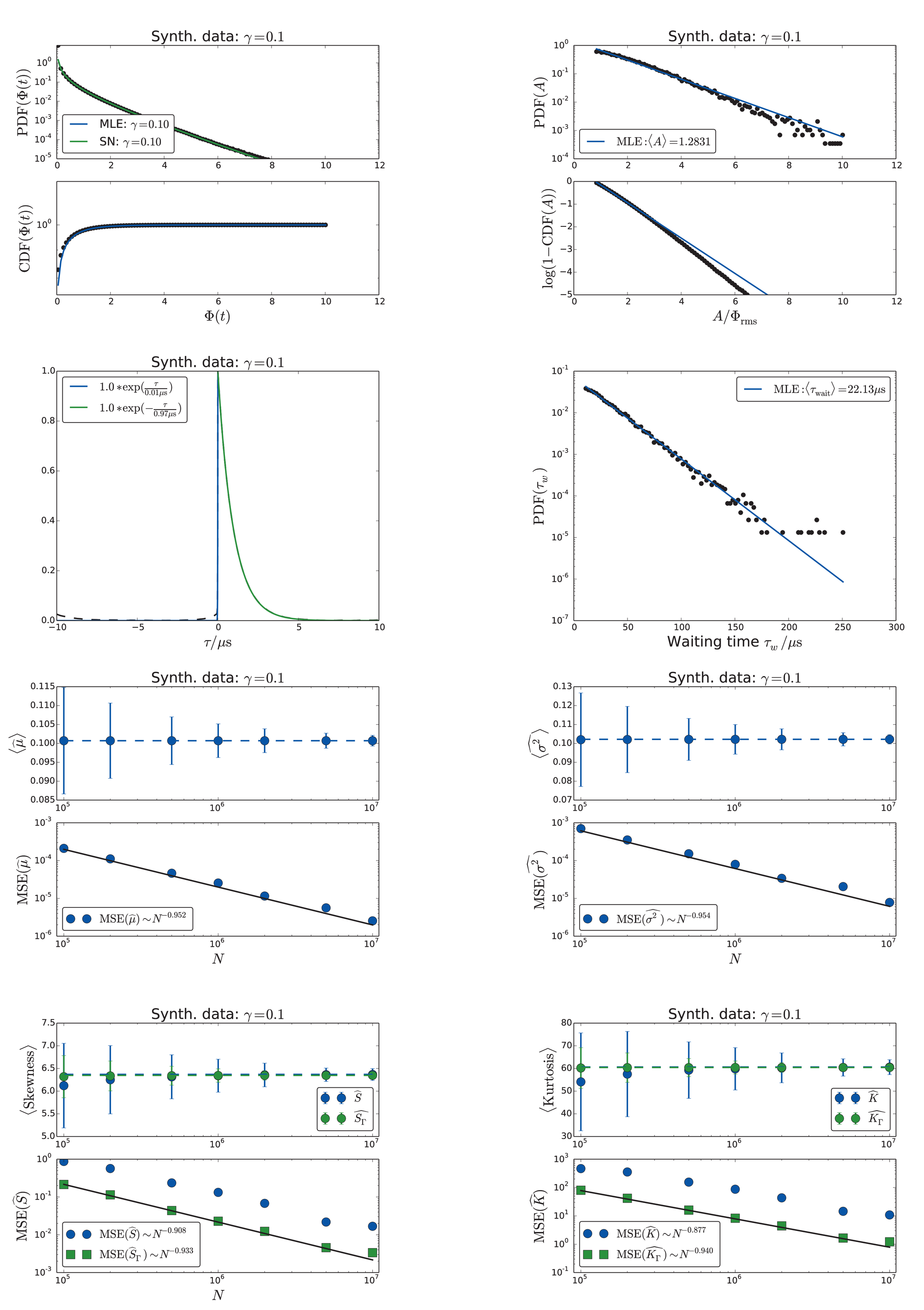
$$\mathrm{MSE}\left(\widehat{S_\Gamma}\right) = 4\frac{\widehat{\sigma^2}}{\widehat{\mu}^4}\mathrm{MSE}\left(\widehat{\mu}\right) + \frac{1}{\widehat{\sigma^2}\widehat{\mu}^2}\mathrm{MSE}\left(\widehat{\sigma^2}\right) - 4\frac{1}{\widehat{\mu}^3}\mathrm{COV}\left(\widehat{\mu}, \widehat{\sigma^2}\right)$$

$$\mathrm{MSE}\left(\widehat{K_\Gamma}\right) = 144\frac{\widehat{\sigma^2}^2}{\widehat{\mu}^6}\mathrm{MSE}\left(\widehat{\mu}\right) + 36\frac{1}{\widehat{\mu}^4}\mathrm{MSE}\left(\widehat{\sigma^2}\right) - 144\frac{\widehat{\sigma^2}}{\widehat{\mu}^5}\mathrm{COV}\left(\widehat{\mu}, \widehat{\sigma^2}\right)$$

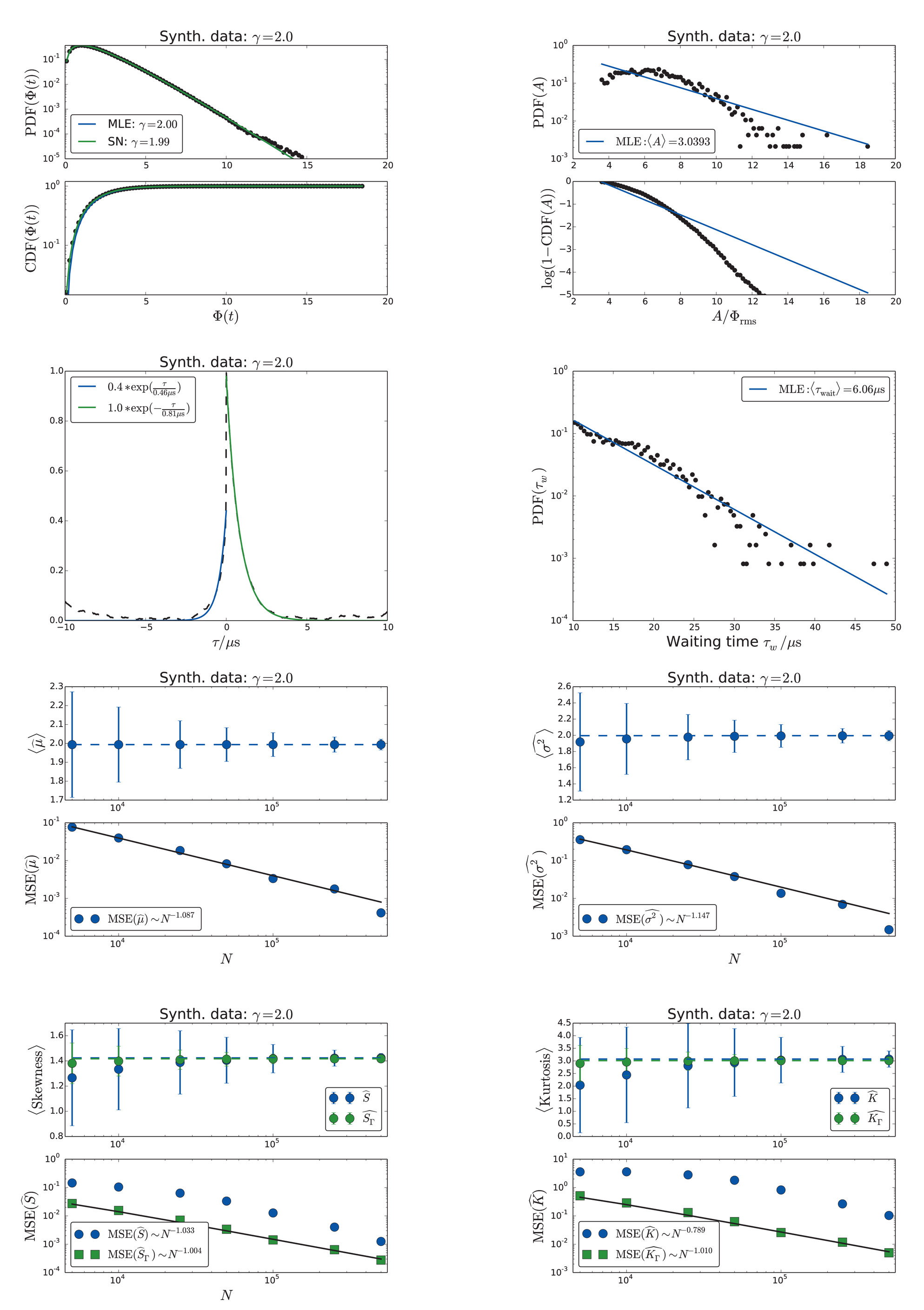
Time series analysis

Synthetic time series of a shot noise process with 100,000 bursts, $\langle A \rangle$ = 1.0 and varying γ with τ_d = 1.0 are generated. Using conditional averaging, we compute the conditionally averaged burst shape as well as the amplitude and waiting time distribution. Estimators of skewness and kurtosis for a Gamma distribution are compared to estimators based on the method of moments, $\widehat{S}$ and $\widehat{K}$. As a third example, we use the same analysis methods for a time series obtained by gas-puff imaging in the scrape-off layer of Alcator C-Mod for an ohmic L-mode plasma.

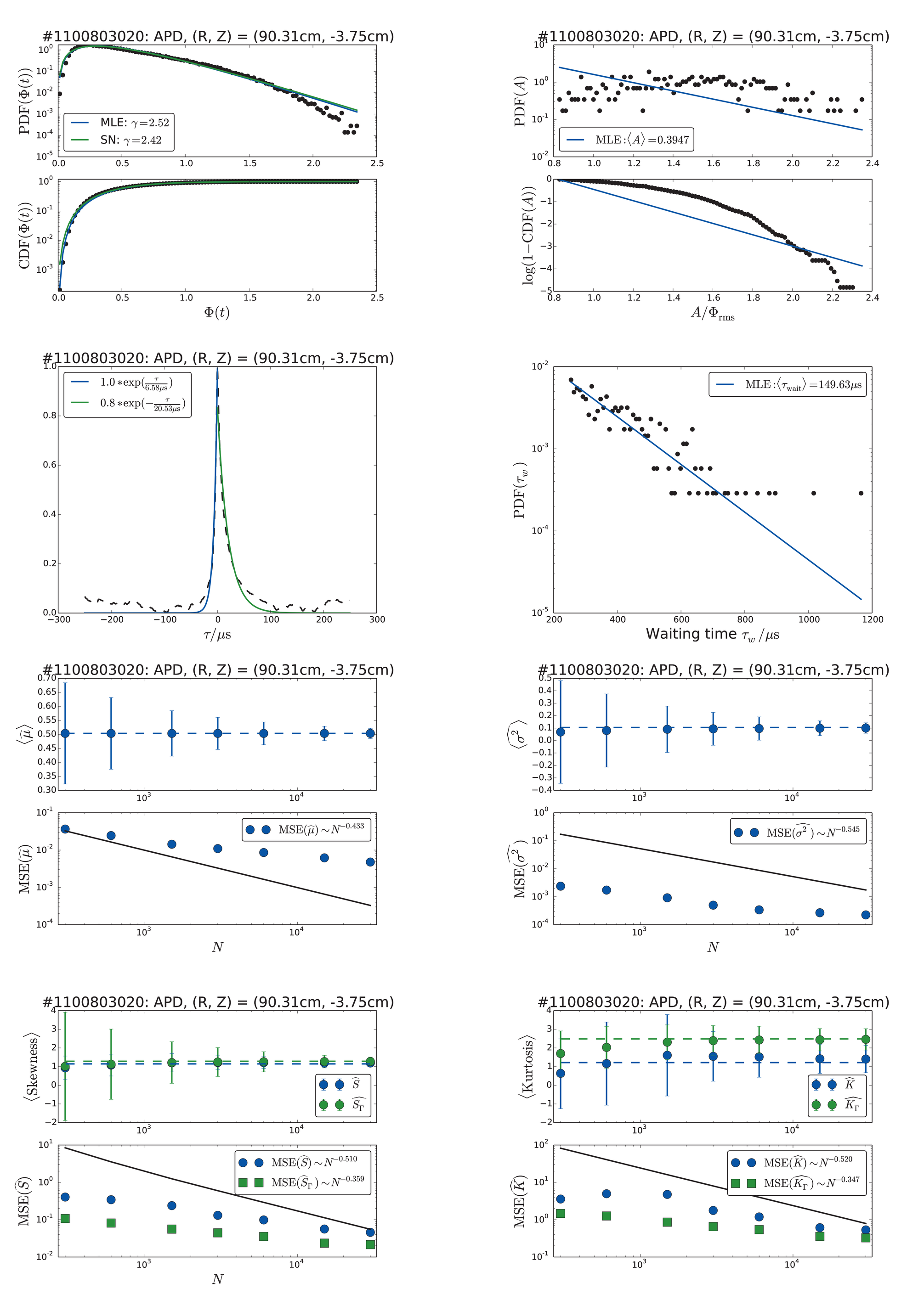
Synthetic time series



Synthetic time series



Gas-Puff imaging, Alcator C-Mod



Conclusions

For an increasing γ, the conditionally averaged burst shape smears out, as to include a smoother rise and a less steep wake. Also, the sampled burst amplitude histogram deviates increasingly from an exponential distribution since the burst overlap increases with γ. $\mathrm{MSE}(\widehat{\mu})$ and $\mathrm{MSE}(\widehat{\sigma^2})$ describe the observed variance of the estimators precisely. Applying the proposed methods to experimental time series shows that while the PDF is well described by a gamma distribution, the burst amplitude distribution cannot be determined. The convergence of the estimators with sample length shows the same trend as for the synthetic time series with $\widehat{S_\Gamma}$ being more precise than $\widehat{S}$. Analytic expressions for all estimators are sensitive to <A> and resemble the rate of convergence for both estimators.

References [1] D.I.D'Ippolito et al. Phys. Plasmas 18, 060501 (2011)
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