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Modelling laser-matter interactions using resonant states

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Abstract

Studying how light interacts with materials has become important for many technological applications from optical communication to developing of new materials. Therefore scientists have always tried to improve their understanding of these effects. The primary goal has always been to microscopically describe the pertinent processes. This paper provides a brief introduction into the interactions of atoms with laser fields. Precisely this interaction, photoelectric effect and the blackbody radiation were those findings which started off the development of quantum mechanics. This theory allowed better description of atoms and it will be used in this work to handle the problem we are confronting. We will consider two of the simplest potentials and let the atom interact with a strong laser pulse in these potentials. From this interaction the so called resonant states will arise. The goal of this thesis is to investigate to what extent and in what meaning these resonant states form a complete set of functions and consequently can be used for expansion of atomic states of physical importance.

Sammendrag

Å observere hvordan lys samhandler med materialer har blitt viktig for mange teknologiske anvendelser fra optisk kommunikasjon til utvikling av nye materialer. Forskere har derfor prøvd å forbedre deres forståelse av disse effektene. Hovedmålet har alltid vært å beskrive de relevante prosessene på en mikroskopisk måte. Hensikten med denne masteroppgaven er å tilby en kort innledning til samspill mellom atomer og laserfelt. Det var akkurat funnene som denne interakjsonen, fotoelektrisk effekt og svart legeme radiasjon som satt utviklingen av kvantemekanikk i gang. Denne toerien tillot en bedre beskrivning av atomer og kommer til å bli brukt i denne oppgaven til å håndtere problemet vårt. Vi skal anta to av de enkleste potensialene og la atomen samhandle med sterk laser påvirkning sammed med disse potensialene. Som løsning til denne interaksjonen, skal de såkalte resonante tilstandene oppstå. Hensikten med denne oppgaven er å undersøke på hvilken måte danner disse resonante tilstandene et komplett funksjonssett og dermed kan de brukes til å ekspandere atomiske tilstander av fysisk betydning.

Forord

En stor takk rettes til veileder Per Kristen Jakobsen for hjelp, inspirasjon, bemerkninger, gode råd og inspirerende samtaler gjennom hele læringsprosessen i denne masteroppgaven.

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List of symbols

Symbol	Description	Page
H	Hamiltonian of the Schrödinger equation	1
$\Psi(x,t)$	solution to the time-dependent Schrödinger equation	2
ω_j	energy eigenvalues	14
A	area of the Dirac delta potential	7
ε	strength of the laser	7
$\delta(x)$	Dirac delta function	7
$y(x)$	variable transformation $y(x) = -\alpha(2\varepsilon x + 2\omega)$	10
α	constant $\alpha = (2\varepsilon)^{-\frac{2}{3}}$	10
$\mathrm{Ai}(x)$	Airy function $\mathrm{Ai}(x)$	11
$\mathrm{Bi}(x)$	Airy function $\mathrm{Bi}(x)$	11
$\mathrm{Ci}^{\pm}(x)$	$\mathrm{Bi}(x) \pm i\mathrm{Ai}(x)$	11
θ	angle of the complex line,	
	in Appendices appearing as a variable in $\omega = Re^{i\theta}$	15
$\Phi(z(x))$	solution of the time-independent Schrödinger equation on a complex line	15
$\tilde{y}(t)$	variable transformation on the complex line	16
$\tilde{\alpha}$	constant $\tilde{\alpha} = \left(4\varepsilon^2 e^{i6\theta}\right)^{-\frac{1}{3}}$	16
$\mathcal{V}_{1,2}$	vector spaces	24
$\mathbf{M}(y_0)$	continuity matrix for the Dirac delta potential	29
$\psi_\omega(x)$	scattering form of the resonant states for the Dirac delta potential	29
χ	normalisation constant for both potentials	29
Γ_R	complex contour $\Gamma_R = C_R \cup [-R, R]$	30
ρ	constant $\rho = 2\alpha\varepsilon$	37
$V(x)$	square well potential	47
V_0	depth of the well potential	47
d	width of the well potential	47
$y_1(x)$	variable transformation $y_1(x) = y(x) = -2\alpha(\varepsilon x + \omega)$	52
$y_2(x)$	variable transformation $y_2(x) = -2\alpha_2(\omega + V_0 + \varepsilon x)$	53
A_0	$\mathrm{Ai}(y_1(-d))$	54
A_1	$\mathrm{Ai}(y_2(-d))$	54
B_1	$\mathrm{Bi}(y_2(-d))$	54
A_2	$\mathrm{Ai}(y_2(d))$	54

Symbol	Description	Page
B_2	$\mathrm{Bi}(y_2(d))$	54
C_3	$\mathrm{Ci}^+(y_1(d))$	54
D_3	$\mathrm{Ci}^-(y_1(d))$	63
$\mathbf{M}(\omega)$	continuity matrix for the square well potential	54
$\psi_\omega(x)$	scattering form of the resonant states	
	for the square well potential	64
$Q(x,x')$	$\int_{-\infty}^{\infty}\tilde{\psi}_\omega(x)\tilde{\psi}_\omega(x')\mathrm{d}\omega$	101
$z(x), z_{1,2}(x)$	this quantity appears as $-y(x)$ resp. $-y_{1,2}(x)$	102,109,120,144
β	$2\alpha\varepsilon$ in Appendix A, $2\alpha\varepsilon d$ in Appendix C and D	102,121,144
γ	2α	102
ω	$Re^{i\theta}$, where θ is a variable	109
κ	$2\alpha e^{i\theta}$	109
ζ_0	$\frac{2}{3}(z_0)^{\frac{3}{2}}$	110
ϑ	$\frac{2}{3}(2\alpha)^{\frac{3}{2}}\cos\left(\frac{3}{2}\theta\right)+i\frac{2}{3}(2\alpha)^{\frac{3}{2}}\sin\left(\frac{3}{2}\theta\right)$	110
ζ	$\frac{2}{3}(z(x))^{\frac{3}{2}}=\vartheta R^{\frac{3}{2}}-i\varpi xR^{\frac{1}{2}}$	111
ϖ	$-\varepsilon(2\alpha)^{\frac{3}{2}}\sin\left(\frac{1}{2}\theta\right)+i\varepsilon(2\alpha)^{\frac{3}{2}}\cos\left(\frac{1}{2}\theta\right)$	111
$\tilde{\kappa}$	$2\alpha e^{i(\theta+\pi)}$	113
$\tilde{\vartheta}$	$\frac{2}{3}(2\alpha)^{\frac{3}{2}}\cos\left(\frac{3}{2}(\theta+\pi)\right)+i\frac{2}{3}(2\alpha)^{\frac{3}{2}}\sin\left(\frac{3}{2}(\theta+\pi)\right)$	114
$\tilde{\varpi}$	$\varepsilon\,(2\alpha)^{\frac{3}{2}}\cos\left(\frac{1}{2}(\theta+3\pi)\right)+i\varepsilon\,(2\alpha)^{\frac{3}{2}}\sin\left(\frac{1}{2}(\theta+3\pi)\right)$	116
σ	$2\alpha V_0$	121
$\zeta_1^\pm$	$\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}\pm\beta(\gamma\omega)^{\frac{1}{2}}$	121,145
$\zeta_2^\pm$	$\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma\pm\beta)(\gamma\omega)^{\frac{1}{2}}$	121,145
μ	$2\alpha\varepsilon$	127
ϱ	$(2\alpha)^{\frac{1}{2}}\cos\left(\frac{1}{2}\theta\right)+i(2\alpha)^{\frac{1}{2}}\sin\left(\frac{1}{2}\theta\right)$	145
ζ_1	$\frac{2}{3}(z_1(x))^{\frac{3}{2}}=\vartheta R^{\frac{3}{2}}-i\varpi xR^{\frac{1}{2}}$	148
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$\tilde{\zeta}_1^\pm$	$\frac{2}{3}(y_1(\pm d))^{\frac{3}{2}}=\tilde{\vartheta}R^{\frac{3}{2}}\mp\beta\tilde{\varrho}R^{\frac{1}{2}}$	154
$\tilde{\zeta}_2^\pm$	$\frac{2}{3}(y_2(\pm d))^{\frac{3}{2}}=\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varrho}R^{\frac{1}{2}}(\sigma\pm\beta)$	154
$\tilde{\varrho}$	$(2\alpha)^{\frac{1}{2}}\cos\left(\frac{1}{2}(\theta+\pi)\right)+i(2\alpha)^{\frac{1}{2}}\sin\left(\frac{1}{2}(\theta+\pi)\right)$	154
$\tilde{\zeta}_1$	$\frac{2}{3}(y_1(x))^{\frac{3}{2}}=\tilde{\vartheta}R^{\frac{3}{2}}+\tilde{\varpi}xR^{\frac{1}{2}}$	158
$\tilde{\zeta}_2$	$\frac{2}{3}(y_2(x))^{\frac{3}{2}}=\tilde{\vartheta}R^{\frac{3}{2}}+\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)$	159

1 Introduction

Thanks to the modern laser technology, the electric field of the laser radiation can be significantly stronger than fields inside the atom. For a long time, the description of laser-matter interactions was identical with the usual framework of perturbation analysis. After the development of quantum mechanics, this description was replaced from microscopic theory in terms of oscillating dipoles to quantum mechanical systems which is still the most frequently used description.

The qualitative change of physical situation in the last decades has brought a development of intense pulsed lasers as a light source as a new tool to investigate these interactions much deeper. The most detailed investigations of these processes focus on atoms which are the basic essentials of matter. The radiation involved in these processes is fundamentally light, that means a radiation whose wavelength is in the range from few decades of micrometers to few micrometers. In our computation, this electric field will be represented by the time dependent function $\varepsilon(t)$.

The quantum mechanical picture tells us, many times in a quizzacious way, that depending on the energy of the system, the atom can in one of infinite number of states represented mathematically as the eigenstates of the Hamiltonian H of the system. In quantum mechanics, the description of how the quantum state changes with time is reached by the Schrödinger equation. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.

The time dependent Schrödinger equation has the form

$$i\hbar\partial_t\Psi(\vec{x},t) = H\Psi(\vec{x},t) \tag{1.1}$$

where $\hbar$ is the Planck's constant divided by 2π and the Hamiltonian H representing the overall energy of the system is

$$H = K + V(\vec{x}) \tag{1.2}$$

consisting of the kinetic energy $K = -\frac{\hbar^2}{2\mu}\nabla^2$, where μ is the reduced mass of the atom. The constants $\hbar$, μ, however, will be omitted from mathematical expressions of physical laws during the whole work by setting $\hbar = \mu = 1$. This assumption has the apparent advantage of simplicity.

The potential energy $V(\vec{x})$ describes all interactions on the system which can be

$$V(\vec{x}) = V_{nuc}(\vec{x}) + V_{ee}(\vec{x}) + V_{ext}(\vec{x}) \tag{1.3}$$

where $V_{nuc}(\vec{x})$ is the electron-nucleus interaction potential which in our paper will be a Dirac delta function and a square well. $V_{ee}(\vec{x})$ is the interaction potential between electrons which we will not consider. Moreover, we will assume only one electron in play, because in many situations only a few electrons are taking part of dynamical processes. As a matter of fact, a good approximation is to assume one electron. These kinds of atoms are called Hydrogen-like atoms.

The external potential $V_{ext}(\vec{x})$ represents interaction between electrons and some external electromagnetic field caused by for example a strong laser. For lasers in the visible part of the spectrum, the electric field varies on a length scale of microns ($10^{-6}m$). On the other hand, most atoms are of a size between 0.3 and 3 Ångström or around 1000 time smaller than the length scale on which the laser field varies. Therefore, it is a very good approximation to assume that the electric field does not vary across the atom. This electric field can be represented by the function $\varepsilon(t)$. As we know, the relation between a force $\vec{F}(\vec{x},t)$ acting on the system and the potential energy is $\vec{F}(\vec{x},t) = -\nabla V_{ext}(\vec{x})$, thus the external potential will look like $-\varepsilon(t)\vec{x}$ and the whole Schrödinger equation is

$$i\hbar\partial_t\Psi(\vec{x},t) = -\frac{\hbar^2}{2\mu}\nabla^2\Psi(\vec{x},t) - V_{nuc}(\vec{x}) - \varepsilon(t)\vec{x} \tag{1.4}$$

Let us restrict the system into one dimension. A common way to calculate the Schrödinger equation is to represent the solution, also called wave function, in the form

$$\Psi(x,t) = \psi(x)e^{-i\omega t} \tag{1.5}$$

where $\omega = \frac{E}{\hbar}$ is the angular frequency of the eigenstate $\psi(x)$ with energy E. As we know, the frequencies or the energy eigenvalues are discrete numbers ω_i each of them corresponding to an eigenstate $\psi_i(x)$, therefore we can approximate the solution (1.5) with a truncated sum

$$\Psi(x,t) \approx \sum_{j=1}^{N} c_j\psi_j(x)e^{-i\omega_j t} \tag{1.6}$$

where frequencies are $\omega_i = \frac{E_i}{\hbar}$ and the energy eigenstates ψ_i with the energy E_i are solutions to the eigenvalue problem

$$H\psi_i = E_i\psi_i \tag{1.7}$$

In quantum mechanics one first learns how to compute energy eigenstates, the so-called bound states. They satisfy the eigenvalue problem (1.7) and are assumed to decay at infinity so they can be normalized. Thanks to these conditions, the Hamiltonian operator H is self-adjoint which makes the energy eigenvalues ω_j real and to form a discrete set.

A more advanced solutions to the Schrödinger equation are scattering states which represent an electron approaching the potential from infinity, interacting with the potential and then returning back to infinity. In this case the energy eigenstates are real because the Hamiltonian stays self-adjoint but they typically form a continuum. They cannot be therefore normalized but they have physical importance since they describe a realistic experimental situation.

The wave function $\Psi(x,t)$ associated with a particle has a statistical interpretation. If the particle is described by the wave function normalised to unity, then the probability $\mathcal{P}(x,t)$ of finding the particle at time t in a finite interval $[a,b]$ is

$$\int_a^b \mathcal{P}(x,t)\mathrm{d}x = \int_a^b |\Psi(x,t)|^2 \,\mathrm{d}x \tag{1.8}$$

One consequence of this is the following. An atom gets under the influence of a strong electric field excited which basically means that its electrons acquire a higher energy state. If the strength of the field is big enough, the outermost electron receives such amount of energy that its connection to the nucleus disappears and becomes no longer a part of the atom. The atom became ionized. In this case, the corresponding eigenstates are called **resonant states** and they have a common feature that they grow endlessly as x approaches infinity. Resonant states are defined by assuming that there are only outgoing waves at infinity. These boundary conditions causes the wave functions to increase exponentially at infinity and can not be therefore normalized. As a direct consequence of the exponential growth of the wave functions is the fact that the resonant states are decaying exponentially in time. That is because exponential decay in time indicates that it is exponentially more likely that the electron was released earlier than later which means that is exponentially more probable to find the electron far away from the nucleus than closer to the nucleus. Thus the wave function grows exponentially. For this reason numerical calculations of resonant wave functions can not be considered. During the calculations in our paper, we are going to solve this problem by solving the Schrödinger equation on a complex line, where this difficulty vanishes.

The resonant states represents a situation where there is an outgoing non-zero flux of electrons across any surface surrounding the atom. These states thus represent an atom that is about to being ionized. They were first introduced by Gamow [3] in 1928 in the context of nuclear physics, they are therefore also called Gamow states. The problem was also solved independently by Ronald W. Gurney and Edward U. Condon [4]. However, Gurney and Condon did not achieve the quantitative results achieved by Gamow. But this was not the first time decaying eigenstates were used in physics. The decaying phenomena in electromagnetism was also described by J. J. Thomson [5]. Siegert [6] introduced the definition of a resonant state as a solution of the time-independent Schrödinger equation with purely out-going waves at large distances. Humblet [7], Peierls [8] and Couteur [9] formulated and developed rigorous dispersion theories for elastic and inelastic scattering characterising the nuclear scattering matrix. The decaying states characterized by Siegert became known as resonant states. Much work was done by Berggren [10] to find a theory which is capable of describing the resonant behaviour (compound nucleus processes) as well as the non-resonant behaviour (direct reactions). He investigated the orthogonality properties of the resonant states, and derived the appropriate completeness relation. However, the question of completeness and asymptotic series based on resonant states has been of continuing interest for many years.

Resonant states are not stationary states for the Schrödinger equation. The reason for this is clear since atoms that are ionizing, are loosing their electron and then the probability of finding the electron in a bounded region centred on the atom must decrease. The mathematical challenge is to understand it what sense the resonant states form a complete set and to develop analytical and numerical approximation methods for finding the resonant states for Hydrogen-like atoms and beyond.

What we are looking for is to represent, using resonant states, wave-functions that are solutions of the Schrödinger equation, starting from the ground state and where the time dependent potential corresponds to the passing of a laser pulse.

As a start in chapter 2 we begin with a zero range potential case, then we proceed to a square well potential in chapter 3. In both cases we calculate the ground states and the resonant states. We use numerical approximations to work out the time dependent case of the electric field.

At the end of the chapters for both potentials, we investigate to what extent expansions using resonant states can be used to represent a wave function or any other functions with a compact support. It means, we will look into the completeness of the resonant states whether they can form a linear basis to represent functions $f(x)$ through the expansion

$$f(x) = \sum_{\omega_i} c_i \psi_{\omega_i}(x) \tag{1.9}$$

where $\psi_{\omega_i}(x)$ are the resonant states. This computation will be done in two different proofs, which provide weaker and stronger results. In chapter 2.4 are the mentioned proofs for the Dirac delta potential. In the weaker version of the proof we find out, that the function $f(x)$ we are expanding has to have its support on the negative real axis, for the expansion to converge. The result in the stronger proof tells us we get the same convergence under the condition $x < 0$, which means that the convergence depends on x, not the location of the support itself.

In chapter 3.3 we provide the same proofs adapted to the square well potential respectively. Here, the results are very similar only the boundary value for the convergence is shifted to $x < d$. Also, the two kinds of proofs offer the same explanation of the results as in the Dirac delta case. There is however one aspect of the result that needs to be mentioned. In particular, that the convergence does not depend on the depth of the well V_0. This leads to the assumption that if we had a general potential which is non-zero on the whole space, we would get a convergence in (1.9) for all x. This is an interesting observation and certainly gives space to further assumptions. One can claim, that this assumption can be viewed as an opinion or judgement based on inconclusive or incomplete evidence, which is a definition of a conjecture. Thus it can be considered as a highlight of this work.

2 Dirac delta potential

In this section we consider the potential energy $V_{nuc}(x) = V(x) = -A\delta(x)$ with a constant A so the Hamiltonian becomes

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - A\delta(x) \tag{2.1}$$

which represents the ground state since $\varepsilon = 0$. This Hamiltonian is also self-adjoint that can be easily proved as follows. For any functions $\phi(x), \psi(x)$ that go to zero as $x \to \pm\infty$ we have

$$\begin{aligned}(\phi, H\psi) &= \int_{-\infty}^{\infty} \phi\left(-\frac{1}{2}\psi'' - A\delta\psi\right)\mathrm{d}x \\ &= -\frac{1}{2}\phi\psi'\Big|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty}\phi'\psi'\mathrm{d}x - A\int_{-\infty}^{\infty}\psi\phi\mathrm{d}x \\ &= \frac{1}{2}\phi'\psi\Big|_{-\infty}^{\infty} - \frac{1}{2}\int_{-\infty}^{\infty}\phi''\psi\mathrm{d}x - A\int_{-\infty}^{\infty}\psi\phi\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\left(-\frac{1}{2}\phi'' - A\delta\phi\right)\psi\mathrm{d}x = (H\phi, \psi)\end{aligned} \tag{2.2}$$

This section will also include the calculation of the ground and resonant states of the Schrödinger equation. At the end of this section we investigate the completeness of the resonant states in two different proofs.

2.1 Ground state

We start computing first the ground states, where $\varepsilon = 0$. The form of solution to (1.1), we are looking for, is

$$\Psi(x,t) = \psi(x)e^{-i\omega t} \tag{2.3}$$

Substituting it into (1.1) considering the assumption on page 1 we get

$$\begin{aligned}i(-i)\omega\psi(x)e^{-i\omega t} &= -\frac{1}{2}\psi''(x)e^{-i\omega t} - A\delta(x)\psi(x)e^{-i\omega t} \\ \frac{1}{2}\psi''(x) &= -\omega\psi(x) - A\delta(x)\psi(x)\end{aligned} \tag{2.4}$$

This equation has the same form both for $x < 0$ and $x > 0$. In this region it gives us

$$\psi''(x) = -2\omega\psi(x) \tag{2.5}$$

which has a solution for $\omega = -\alpha^2, \alpha > 0$

$$\psi(x) = a_1 e^{kx} + a_2 e^{-kx} \tag{2.6}$$

where $k = \sqrt{2}\alpha$. We demand from our solution to decay as x is approaching $\pm\infty$. That is why the solutions $\psi_1(x)$ for $x < 0$ and $\psi_2(x)$ for $x > 0$ are

$$\psi_1(x) = a_1 e^{kx}, \quad x < 0 \tag{2.7}$$

$$\psi_2(x) = a_2 e^{-kx}, \quad x > 0 \tag{2.8}$$

We have 2 unknown coefficients left, so we need 2 conditions. The only undefined point in (2.4) is at $x = 0$. Since the solution should be continuous everywhere we require

$$\psi_1(0) = \psi_2(0) \tag{2.9}$$

Integrating (2.4) over the interval $(-\epsilon, \epsilon)$ and taking the limit $\epsilon \to 0$ we receive the second condition at this point.

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \frac{1}{2}\psi''(x)\mathrm{d}x = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} -\omega\psi(x)\mathrm{d}x - \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} A\delta(x)\psi(x)\mathrm{d}x \\
&\frac{1}{2}\psi'(\epsilon) - \frac{1}{2}\psi'(-\epsilon) = -A\psi(0) \\
&\quad \psi_2'(0) - \psi_1'(0) = -2A\psi_1(0) \qquad (2.10)
\end{aligned}$$

where we have used the condition (2.9) and the property of Dirac delta function $\int \delta(x - x_0) f(x)\mathrm{d}x = f(x_0)$. Applying the conditions at $x = 0$ for our solutions ψ_1, ψ_2 we get a system

$$a_1 - a_2 = 0 \tag{2.11}$$

$$-ka_2 - ka_1 + 2Aa_1 = 0 \tag{2.12}$$

which can be written also in a matrix form as

$$\begin{pmatrix} 1 & -1 \\ 2A - k & -k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.13}$$

This system has a zero solution that is not interesting. It has a non-zero solution only if the determinant of the system is zero.

$$\begin{aligned} 2A - 2k = 0 \\ 2A = 2\sqrt{2}\alpha \\ \alpha^2 = \frac{A^2}{2} \Rightarrow \omega = -\frac{A^2}{2} \end{aligned} \tag{2.14}$$

To compute the coefficients a_1, a_2 we use the ground state eigenvalue (2.14) in (2.13).

$$\begin{pmatrix} 1 & -1 \\ 2A - \sqrt{2}\frac{A}{\sqrt{2}} & -\sqrt{2}\frac{A}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ A & -A \end{pmatrix} \tag{2.15}$$

The null space of this system is $(1, 1)^T$, that means $a_1 = a_2$ where one can freely choose one coefficient. For $a_2 = 1$, the solution to (2.5) is

$$\psi(x) = \begin{cases} e^{Ax} & x < 0 \\ e^{-Ax} & x > 0 \end{cases} \tag{2.16}$$

2.2 Scattering states

With considering the possibility of the wave to scatter from the potential barrier, we assume the solution to (2.5) to have the form

$$\psi_1(x) = \begin{cases} a_1 e^{ikx} + a_2 e^{-ikx} & x < 0 \\ a_4 e^{ikx} + a_3 e^{-ikx} & x > 0 \end{cases} \tag{2.17}$$

but this time with $k = \sqrt{2\omega}$. We assume that the particle is coming from the left, so we put $a_4 = 0$. We use the coefficients of scattering states and the conditions (2.9) and (2.10) to calculate the transmission coefficient $T = \frac{|a_3|^2}{|a_1|^2}$ and the reflection coefficient $R = \frac{|a_2|^2}{|a_1|^2}$ which are probabilities of the particle to reflect from the barrier, respectively to come out on the other side of the barrier. From the conditions we have

$$a_1 + a_2 = a_3 \tag{2.18}$$

$$-a_3 ik - a_1 ik + a_2 ik = -2A(a_1 + a_2) \tag{2.19}$$

We solve this system with respect to the unknowns a_1, a_2.

$$a_1 = \frac{A}{ik} a_3, \quad a_2 = \frac{ik - A}{ik} a_3 \tag{2.20}$$

According to the form of the reflection coefficient R mentioned before, we get

$$R = \frac{|a_2|^2}{|a_1|^2} = \frac{\frac{|ik-A|^2}{|k|^2} a_3^2}{\frac{A^2}{|k|^2} a_3^2} = \frac{|ik - A|^2}{A^2} \tag{2.21}$$

These two coefficients R, T are probabilities of the particle to penetrate or reflect from the barrier. Because of this interpretation we have $T + R = 1$ and $R \leq 1$.

2.3 Resonant states

Resonant states occur when $\varepsilon \neq 0$ in the total energy (1.2). Let us for now consider $\varepsilon > 0$, so the Schrödinger equation (1.1) becomes

$$i\partial_t \Psi(x, t) = \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} - A\delta(x) - \varepsilon x \right) \Psi(x, t) \tag{2.22}$$

The form of the solution stays the same as in (2.3). After substituting it into (2.22) we get

$$\psi'' + (2\varepsilon x + 2\omega)\, \psi = 0 \tag{2.23}$$

for $x \neq 0$. We introduce a transformation for the variable x in the following way

$$y(x) = -\alpha(2\varepsilon x + 2\omega) \tag{2.24}$$

and the equation (2.23) can be rewritten as

$$4\alpha^2 \varepsilon^2 \psi''(y(x)) - \frac{y(x)}{\alpha} \psi(y(x)) = 0$$

$$\psi''(y) - \frac{y}{4\alpha^3 \varepsilon^2} \psi(y) = 0 \Rightarrow \alpha = (4\varepsilon^2)^{-\frac{1}{3}} \tag{2.25}$$

$$\psi''(y) - y\psi(y) = 0 \tag{2.26}$$

The free constant α was chosen so the equation has the correct form to solve it with Airy functions. Airy functions $\mathrm{Ai}(y), \mathrm{Bi}(y)$ are two independent solutions to ODE of the form (2.26).

We can see their form on Figure (2.3-1).

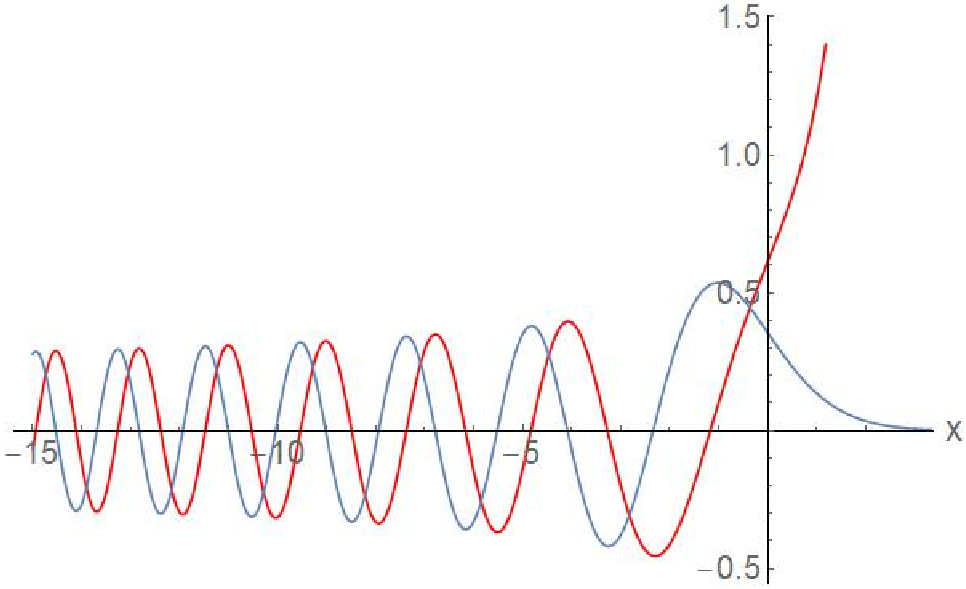


Figure 2.3-1: Plot of Airy functions $\mathrm{Ai}(x)$(blue) and $\mathrm{Bi}(x)$(red).

Judging by their behaviour at $\pm\infty$, the correct combination of these function is chosen for $x < 0$ and $x > 0$ respectively. According to the transformation (2.24) if $x \to \pm\infty$ then $y \to \mp\infty$. That is for $\psi_1(x)$, which represents the solution in $x < 0$, we should choose $\mathrm{Ai}(x)$. For $\psi_2(x)$, that is in the region $x > 0$, both of the functions seem to be doing well so let us choose the following combination

$$\psi_1(x) = a_1 \mathrm{Ai}(y(x)) \tag{2.27}$$

$$\psi_2(x) = a_2 \left[\mathrm{Bi}(y(x)) \pm i\mathrm{Ai}(y(x))\right] = a_2 \mathrm{Ci}^{\pm}(y(x)) \tag{2.28}$$

where we have denoted $\mathrm{Ci}^{\pm}(x) = \mathrm{Bi}(x) \pm i\mathrm{Ai}(x)$. We investigate the behaviour of $\mathrm{Ci}^{\pm}(x)$ for large negative x with help of [1] formulas (10.4.60) and (10.4.64) and decide which sign should we choose.

$$\text{Ai}(-x) \approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \left[\sin\left(\zeta + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k c_{2k} \zeta^{-2k} - \cos\left(\zeta + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \tag{2.29}$$

$$\text{Bi}(-x) \approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \left[\cos\left(\zeta + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k c_{2k} \zeta^{-2k} + \sin\left(\zeta + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \tag{2.30}$$

where $\zeta = \frac{2}{3} x^{\frac{3}{2}}$ and c_k are some constants. The series in the expressions contain exponentially decaying terms, so for simplicity only the first term can be taken. Then the function $\text{Ci}^{\pm}(x)$ for large negative x is

$$\begin{aligned}
\text{Ci}^{\pm}(-x) &\approx \text{Bi}(-x) \pm i\text{Ai}(-x) \approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \cos\left(\zeta + \frac{\pi}{4}\right) c_0 \\
&+ \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \sin\left(\zeta + \frac{\pi}{4}\right) c_1 \zeta^{-1} \\
&\pm i \left(\pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \sin\left(\zeta + \frac{\pi}{4}\right) c_0 - \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \cos\left(\zeta + \frac{\pi}{4}\right) c_1 \zeta^{-1} \right) \\
&\approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} c_0 \cos\left(\zeta + \frac{\pi}{4}\right) - i^2 \frac{3}{2} \pi^{-\frac{1}{2}} x^{-\frac{7}{4}} c_1 \sin\left(\zeta + \frac{\pi}{4}\right) \\
&\pm i \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} c_0 \sin\left(\zeta + \frac{\pi}{4}\right) \mp i \frac{3}{2} \pi^{-\frac{1}{2}} x^{-\frac{7}{4}} c_1 \cos\left(\zeta + \frac{\pi}{4}\right) \\
&\approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} c_0 e^{\pm i\left(\zeta + \frac{\pi}{4}\right)} - i \frac{3}{2} \pi^{-\frac{1}{2}} x^{-\frac{7}{4}} c_1 e^{\pm i\left(\zeta + \frac{\pi}{4}\right)} \\
&\approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{\pm i\left(\zeta + \frac{\pi}{4}\right)}
\end{aligned} \tag{2.31}$$

where $c_0 = 1$. In the last line dropped the second term, because it is decaying much faster than the first one. Since we consider the particle to "fly" out in the positive x direction, we wish to have a wave moving to the positive x direction and that wave is represented by $\text{Ci}^{+}(x)$. Now that we have the functions in both regions, we can apply the conditions at $x = 0$, (2.9) and (2.10). Note, that they stay the same in this case $\varepsilon > 0$ as well.

$$a_1 \text{Ai}(y(0)) = a_2 \text{Ci}^{+}(y(0)) \tag{2.32}$$

$$a_2 y'(0) \text{Ci}'^{+}(y(0)) - a_1 y'(0) \text{Ai}'(y(0)) = -2A a_1 \text{Ai}(y(0)) \tag{2.33}$$

what can be also written as a system

$$\mathbf{M}_0 = \begin{pmatrix} \mathrm{Ai}(y(0)) & -\mathrm{Ci}^+(y(0)) \\ -\frac{A}{\alpha\varepsilon}\mathrm{Ai}(y(0)) - \mathrm{Ai}'(y(0)) & \mathrm{Ci}'^+(y(0)) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.34}$$

This system has a non zero solution if the determinant of the matrix is zero.

$$-\frac{A}{\alpha\varepsilon}\mathrm{Ai}(-2\alpha\omega)\mathrm{Ci}^+(-2\alpha\omega) + \mathrm{Ai}(y(0))\mathrm{Bi}'(y(0)) - \mathrm{Ai}'(y(0))\mathrm{Bi}(y(0)) = 0$$

$$\det \mathbf{M}_0(\omega) = -\frac{A}{\alpha\varepsilon}\mathrm{Ai}(-2\alpha\omega)\mathrm{Ci}^+(-2\alpha\omega) + \frac{1}{\pi} = 0 \tag{2.35}$$

$$\mathrm{Ai}(-2\alpha\omega)\mathrm{Ci}^+(-2\alpha\omega) - \frac{\alpha\varepsilon}{A\pi} = 0 \tag{2.36}$$

where we have used the formula (10.4.10) from [1]. From this equation (2.36) we compute ω which are the energy eigenvalues of the resonant states. To cover all the possibilities for ω that satisfies this equation, let us consider $\omega = \omega_R + i\omega_I$ as a complex number. These solutions can be found using numerical methods for example Newton's method, where the starting points can be determined in the following way. Since the constant $\frac{\alpha\varepsilon}{A\pi}$ is a small number, we set it to 0 and examine the zeros of $\mathrm{Ai}(-2\alpha\omega)$ and $\mathrm{Ci}^+(-2\alpha\omega)$. From [1] (10.4.94) we get the zeros ω_j^A of $\mathrm{Ai}(x)$. In our case the zeros will be $-\frac{\omega_j^A}{2\alpha}$. The zeros ω_j^C of $\mathrm{Ci}^+(x)$ are computed from (10.4.9) in [1].

$$\mathrm{Ai}\left(xe^{\pm\frac{2\pi}{3}i}\right) = \frac{1}{2}e^{\pm\frac{\pi}{3}i}\left[\mathrm{Ai}(x) \mp i\mathrm{Bi}(x)\right] = -\frac{1}{2}ie^{\pm\frac{\pi}{3}i}\left[\pm\mathrm{Bi}(x) + i\mathrm{Ai}(x)\right]$$

$$\mathrm{Ai}\left(xe^{\frac{2\pi}{3}i}\right) = -\frac{1}{2}ie^{\frac{\pi}{3}i}\mathrm{Ci}^+(x)$$

$$2ie^{-\frac{\pi}{3}i}\mathrm{Ai}\left(xe^{\frac{2\pi}{3}i}\right) = \mathrm{Ci}^+(x) \tag{2.37}$$

The zeros of $\mathrm{Ci}(x)^+$ are then

$$\mathrm{Ci}^+\left(\omega_j^C\right) = 2ie^{-\frac{\pi}{3}i}\mathrm{Ai}\left(\omega_j^C e^{\frac{2\pi}{3}i}\right) = 0 \Rightarrow \omega_j^C e^{\frac{2\pi}{3}i} = \omega_j^A$$

$$\omega_j^C = \omega_j^A e^{-\frac{2\pi}{3}i} \tag{2.38}$$

We sum now up all the zeros of $\mathrm{Ai}(-2\alpha\omega)$ and $\mathrm{Ci}(-2\alpha\omega)$ which are in fact the starting points for numerical method to compute the real solutions of (2.36).

$$\text{zeros of } \mathrm{Ai}(-2\alpha\omega) = -\frac{\omega_j^A}{2\alpha} \tag{2.39}$$

$$\text{zeros of } \mathrm{Ci}^{+}(-2\alpha\omega) = -\frac{\omega_j^A}{2\alpha} e^{-\frac{2\pi}{3}i} \tag{2.40}$$

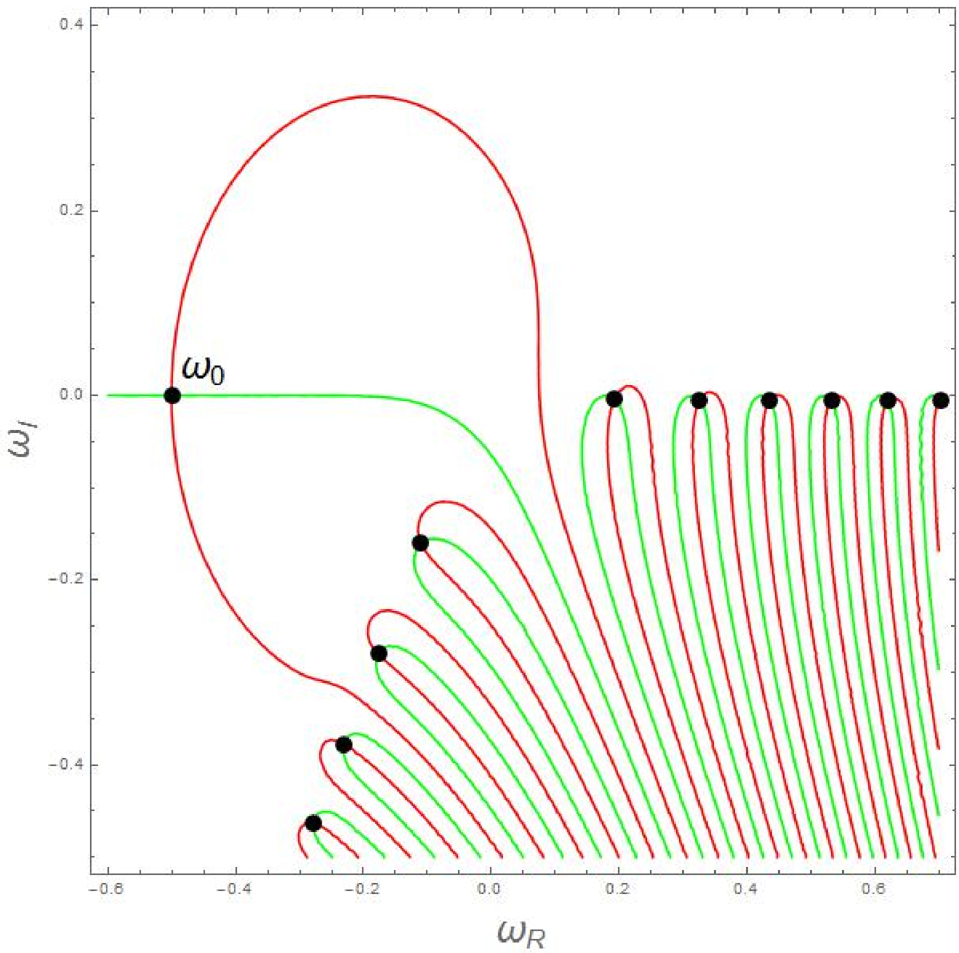
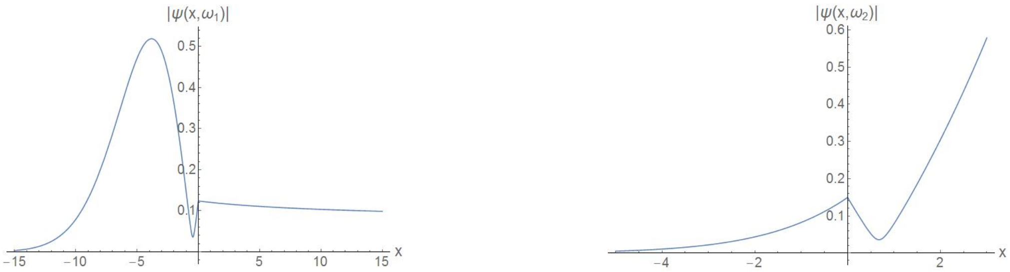


Figure 2.3-2: Plot of the zero contours of $\det \mathbf{M}_0(\omega_R + i\omega_I)$. The real part of the determinant $Re\left[\det \mathbf{M}_0(\omega_R + i\omega_I)\right]$ is in red, the imaginary part $Im\left[\det \mathbf{M}_0(\omega_R + i\omega_I)\right]$ is in green and the eigenvalues ω_j (black dots).

Figure (2.3-2) shows us the zero contours of the determinant (2.36). The ground state energy ω_0 (2.14) is also shown. For this figure was chosen $A = 1$ and $\varepsilon = 0.03$. We can see that all the resonant state energies for $\varepsilon > 0$ are located on the lower half plane of the complex space.

To compute the coefficients a_1, a_2 in (2.34) we substitute the computed eigenvalues ω_j into the matrix (2.34) and compute its null-space. But since these values are numerical, the determinant is not exactly zero, but a value very close to zero. We can solve this problem by computing the eigenvalues and eigenvectors of this matrix and pick that eigenvector which belongs to the least eigenvalue. This eigenvalue should be very close to zero as is the determinant. Thus, the picked eigenvector contains the coefficients a_1, a_2. These values depend on what ω_j we choose. The eigenstates then are

$$\psi(x,\omega_j) = \begin{cases} a_1\text{Ai}(-\alpha(2\varepsilon x + 2\omega_j)) & x < 0 \\ a_2\text{Ci}^+(-\alpha(2\varepsilon x + 2\omega_j)) & x > 0 \end{cases} \tag{2.41}$$



(a) $\omega_1 = 0.192757 - 0.00407091i$ (b) $\omega_2 = -0.109864 - 0.160202i$

Figure 2.3-3: Plot of two different resonant states for the parameters $A = 1, \varepsilon = 0.03$.

On Figure (2.3-3) we can see two resonant states that corresponds to two different eigenvalues ω_1, ω_2 for the parameters $A = 1, \varepsilon = 0.03$. Both of them grow to infinity in the positive x-direction, where the right one (2.3-3b) grows faster than the other. The starting point for the numerical method to compute the eigenvalue ω_2 was chosen to be the zero of $\text{Ci}^+(-2\alpha\omega)$ that corresponds to $j = 1$ (2.40). Let us call these zeros as the zeros from the C-series. Whereas the starting point for ω_1 was the zero of $\text{Ai}(-2\alpha\omega)$ again for $j = 1$ (2.39). These zeros can be called zeros from the A-series.

2.3.1 Resonant states on a complex line

The resonant states computed in the previous section grow endlessly in the positive x-direction. In this section we will fix this by solving the Schrödinger equation (2.22) on a complex line which we defined to be

$$z(x) = \begin{cases} x & x < x_c \\ x_c + e^{i\theta}(x - x_c) & x > x_c \end{cases} \tag{2.42}$$

where θ is the angle of the complex line after the breaking point x_c. This complex line is a straight line consisting of a real and a complex part. The solution we are looking for then is

$$\Psi(x,t) = \psi(x)e^{-i\omega t} \to \Phi(z(x))e^{-i\omega t} \tag{2.43}$$

To apply this change the Hamiltonian in (2.22) should be accordingly changed. The second derivative will then be

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) &= e^{i2\theta}\partial_{zz}\Phi(z(x))\\ e^{-i2\theta}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) &= \partial_{zz}\Phi(z(x)) \quad x > x_c\end{aligned} \tag{2.44}$$

and for $x < x_c$ the derivative stays the same. The modified Hamiltonian has the form

$$H = D - A\delta(x) - \varepsilon z(x), \quad D = \begin{cases} -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} & x < x_c \\ -\frac{1}{2}e^{-i2\theta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} & x > x_c \end{cases} \tag{2.45}$$

We are now ready to substitute the proposed solution (2.43) into (2.22) using (2.45) and get an ODE for $\psi(x)$. For $x < x_c$ everything remains the same as before, since there is no change in the evaluation line. We are going to focus mainly on $x \geq x_c$ where the solution is $\psi_3(x)$. The following computations will take place on $x \geq x_c$ unless it is told otherwise. After some easy operations similar to (2.4) we end up with

$$\begin{aligned}-\frac{1}{2}e^{-i2\theta}\psi_3''(x) - \varepsilon\left[x_c + e^{i\theta}(x - x_c)\right]\psi_3(x) - \omega\psi_3(x) = 0\\ \psi_3''(x) + \left[2\varepsilon x_c e^{i2\theta} + 2\varepsilon e^{i3\theta}(x - x_c) + 2\omega e^{i2\theta}\right]\psi_3(x) = 0\end{aligned} \tag{2.46}$$

Once again we introduce a transformation so that we get the right form of equation to solve with Airy functions.

$$\tilde{y}(x) = -\tilde{\alpha}\left[2\varepsilon x_c e^{i2\theta} + 2\varepsilon e^{i3\theta}(x - x_c) + 2\omega e^{i2\theta}\right] \tag{2.47}$$

The equation (2.46) then takes the form

$$\begin{aligned}\tilde{\alpha}^2 4\varepsilon^2 e^{i6\theta}\psi_3''(x) - \frac{\tilde{y}(x)}{\tilde{\alpha}}\psi_3(x) = 0\\ \psi_3''(x) - \frac{\tilde{y}(x)}{\tilde{\alpha}^3 4\varepsilon^2 e^{i6\theta}}\psi_3(x) = 0 \Rightarrow \tilde{\alpha} = \left(4\varepsilon^2 e^{i6\theta}\right)^{-\frac{1}{3}}\end{aligned} \tag{2.48}$$

$$\psi_3''(x) - \tilde{y}(x)\psi_3(x) = 0 \tag{2.49}$$

In the two other regions $x < 0$ and $0 < x < x_c$ we are solving the same equation as before in (2.26). Hence, the general solution for the whole space is

$$\psi(x) = \begin{cases} \psi_1(x) = a_1 \mathrm{Ai}(y(x)) & x < 0 \\ \psi_2(x) = a_2 \mathrm{Ai}(y(x)) + a_3 \mathrm{Bi}(y(x)) & 0 < x < x_c \\ \psi_3(x) = a_4 \mathrm{Ci}^+(\tilde{y}(x)) & x_c < x \end{cases} \tag{2.50}$$

The goal is to determine the coefficients a_2 and a_3. At the point x_c we require from the solution that it should be continuous and also its derivative should be continuous. The derivative of $\psi_3(x)$ at this point is however different and we will see that the derivative can not be continuous at this point after all. Let us apply these conditions to determine the coefficients in $\psi_2(x)$. We must not forget that in the region $x > x_c$ the derivative of $\Phi(z(x))$ is $\partial_z \Phi(z(x)) = e^{-i\theta} \frac{\mathrm{d}}{\mathrm{d}x} \psi(x)$ (2.44).

$$a_2 \mathrm{Ai}(y(x_c)) + a_3 \mathrm{Bi}(y(x_c)) = a_4 \mathrm{Ci}^+(\tilde{y}(x_c)) \tag{2.51}$$

$$a_2 y'(x_c) \mathrm{Ai}'(y(x_c)) + a_3 y'(x_c) \mathrm{Bi}'(y(x_c)) = a_4 e^{-i\theta} \tilde{y}'(x_c) \mathrm{Ci}'^+(\tilde{y}(x_c)) \tag{2.52}$$

The second equation (2.52) can be simplified further

$$\begin{aligned} -a_2 2\alpha\varepsilon \mathrm{Ai}'(y(x_c)) - a_3 2\alpha\varepsilon \mathrm{Bi}'(y(x_c)) &= -a_4 2\tilde{\alpha}\varepsilon e^{-i\theta} \mathrm{Ci}'^+(\tilde{y}(x_c)) \\ -a_2 (2\varepsilon)^{\frac{1}{3}} \mathrm{Ai}'(y(x_c)) - a_3 (2\varepsilon)^{\frac{1}{3}} \mathrm{Bi}'(y(x_c)) &= -a_4 \frac{2^{\frac{1}{3}} e^{i3\theta} \varepsilon}{\left(e^{i6\theta} \varepsilon^2\right)^{\frac{1}{3}}} e^{-i\theta} \mathrm{Ci}'^+(\tilde{y}(x_c)) \\ a_2 \mathrm{Ai}'(y(x_c)) + a_3 \mathrm{Bi}'(y(x_c)) &= a_4 \mathrm{Ci}'^+(\tilde{y}(x_c)) \end{aligned} \tag{2.53}$$

From the form of Ci^+ and the fact, that $y(x_c) = \tilde{y}(x_c)$ it is clear, that $a_2 = a_4 i$ and $a_3 = a_4$. The solution can be reduced to the following form

$$\psi(x, \omega_j) = \begin{cases} a_1 \mathrm{Ai}(-\alpha(2\varepsilon x + 2\omega_j)) & x < 0 \\ a_4 \mathrm{Ci}^+(-\alpha(2\varepsilon x + 2\omega_j)) & 0 < x < x_c \\ a_4 \mathrm{Ci}^+\left(-\tilde{\alpha}\left[2\varepsilon x_c e^{i2\theta} + 2\varepsilon e^{i3\theta}(x - x_c) + 2\omega_j e^{i2\theta}\right]\right) & x_c < x \end{cases} \tag{2.54}$$

where the first two regions are exactly the case as in (2.27), (2.28). That means the eigenvalues ω_j stay the same as well. But in order to satisfy all four conditions (2 at $x = 0$ and 2 at $x = x_c$)

we consider for a moment all four coefficients.

$$\psi(x,\omega_j) = \begin{cases} a_1\mathrm{Ai}(-\alpha(2\varepsilon x + 2\omega_j)) & x < 0 \\ a_2\mathrm{Bi}(-\alpha(2\varepsilon x + 2\omega_j)) + a_3\mathrm{Ai}(-\alpha(2\varepsilon x + 2\omega_j)) & 0 < x < x_c \\ a_4\mathrm{Ci}^+\left(-\tilde{\alpha}\left[2\varepsilon x_c e^{i2\theta} + 2\varepsilon e^{i3\theta}(x - x_c) + 2\omega_j e^{i2\theta}\right]\right) & x_c < x \end{cases} \tag{2.55}$$

Now we apply the 4 conditions (2.9), (2.10), (2.51) and (2.52). One should get a system

$$\mathbf{M} = \begin{pmatrix} -\mathrm{Ai}(y(0)) & -\mathrm{Bi}(y(0)) & -\mathrm{Ai}(y(0)) & 0 \\ 2A\mathrm{Ai}(y(0)) - y'(0)\mathrm{Ai}'(y(0)) & y'(0)\mathrm{Bi}'(y(0)) & y'(0)\mathrm{Ai}'(y(0)) & 0 \\ 0 & \mathrm{Bi}(y(x_c)) & \mathrm{Ai}(y(x_c)) & -\mathrm{Ci}^+(\tilde{y}(x_c)) \\ 0 & -y'(x_c)\mathrm{Bi}'(y(x_c)) & -y'(x_c)\mathrm{Ai}'(y(x_c)) & e^{-i\theta}\tilde{y}'(x_c)\mathrm{Ci}^+(\tilde{y}(x_c)) \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.56}$$

After substituting the eigenvalues ω_j into this system, we get a linearly dependent system. Its null-space should be found, but we deal with the same problem as on page 14. ω_j are only numerical values, so in the eigensystem of the matrix (2.56) the eigenvector belonging to the least eigenvalue is chosen. This eigenvector then contains the coefficients.

One question now remains. How should we choose the angle θ? To answer this we solve the Schrödinger equation with Hamiltonian without potential energy to avoid complications, but on the complex line introduced in (2.42).

$$\hat{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{2.57}$$

We use the same solution suggestion as before (2.43) only with a different notation for the function $\psi(x) \to \hat{\psi}(x)$. After substitution into the Schrödinger equation with $\hat{H}$ the equations in two regions are

$$\hat{\psi}_1''(x) = -2\omega\hat{\psi}_1(x), \quad x < x_c \tag{2.58}$$

$$e^{-i2\theta}\hat{\psi}_2''(x) = -2\omega\hat{\psi}_2(x), \quad x_c < x \tag{2.59}$$

where (2.44) was used. Let us now denote $\hat{k}^2 = 2\omega$. The solution to the first equation is

$$\hat{\psi}_1(x) = a_1 e^{i\hat{k}x} + a_2 e^{-i\hat{k}x} \tag{2.60}$$

We look for a wave coming from negative x-direction that is why we set $a_2 = 0$. Using $e^{i\pi} = -1$ in the second equation (2.59), the solution is

$$\hat{\psi}_2(x) = a_3 e^{\hat{k}xe^{i\left(\frac{\pi+2\theta}{2}\right)}} + a_4 e^{-\hat{k}xe^{i\left(\frac{\pi+2\theta}{2}\right)}} \tag{2.61}$$

This solution is for $x \geq x_c$, so it should represent a wave going to the positive x-direction and exponential decay. The imaginary part of the argument in the exponentials should be therefore positive. In this case $\hat{k} > 0$, so the sign depends on $e^{i\left(\frac{\pi+2\theta}{2}\right)} = \cos\left(\frac{\pi+2\theta}{2}\right) + i\sin\left(\frac{\pi+2\theta}{2}\right)$. If we set $a_4 = 0$, then the real part of $e^{i\left(\frac{\pi+2\theta}{2}\right)}$ must be negative and the imaginary part positive. If we set $a_3 = 0$, they must be the opposite. Let us restrict ourself to $-\pi \leq \theta < \pi$ and investigate the first case.

$$\begin{array}{lll} \cos\left(\frac{\pi+2\theta}{2}\right) \leq 0 & \wedge & \sin\left(\frac{\pi+2\theta}{2}\right) \geq 0 \\ \frac{\pi}{2} \leq \frac{\pi}{2} + \theta \leq \frac{3\pi}{2} & \wedge & 0 \leq \frac{\pi}{2} + \theta \leq \pi \\ 0 \leq \theta \leq \pi & \wedge & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \qquad \Rightarrow 0 \leq \theta \leq \frac{\pi}{2} \end{array} \tag{2.62}$$

The second case gives us

$$\begin{array}{lll} \cos\left(\frac{\pi+2\theta}{2}\right) \geq 0 & \wedge & \sin\left(\frac{\pi+2\theta}{2}\right) \leq 0 \\ -\frac{\pi}{2} \leq \frac{\pi}{2} + \theta \leq \frac{\pi}{2} & \wedge & -\pi \leq \frac{\pi}{2} + \theta \leq 0 \\ -\pi \leq \theta \leq 0 & \wedge & -\frac{3\pi}{2} \leq \theta \leq -\frac{\pi}{2} \qquad \Rightarrow -\pi \leq \theta \leq -\frac{\pi}{2} \end{array} \tag{2.63}$$

We choose the first case so $a_4 = 0$ and for the angle we choose $\theta = \frac{\pi}{2}$. Now we apply the 2 continuity conditions at $x = x_c$. In the second condition we must remember that we deal with a derivative on a complex line, so we use the same rule as in (2.52) for the derivative.

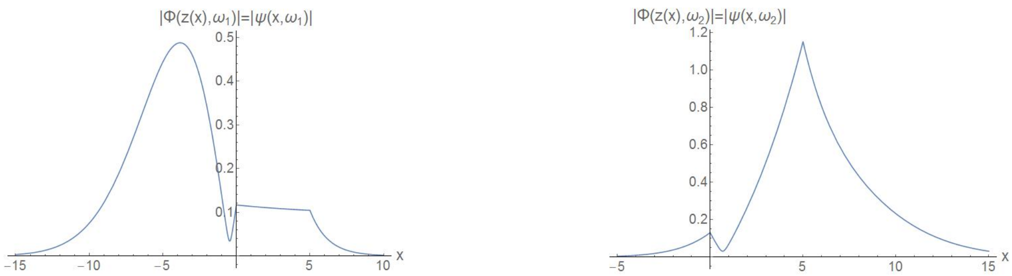
$$a_1 e^{i\hat{k}x_c} = a_3 e^{\hat{k}x_c e^{i\left(\frac{\pi+2\theta}{2}\right)}} \tag{2.64}$$

$$i\hat{k}a_1 e^{i\hat{k}x_c} = \hat{k}e^{i\left(\frac{\pi+2\theta}{2}\right)} a_3 e^{\hat{k}x_c e^{i\left(\frac{\pi+2\theta}{2}\right)}} e^{-i\theta} \Rightarrow a_1 e^{i\hat{k}x_c} = a_3 e^{\hat{k}x_c e^{i\left(\frac{\pi+2\theta}{2}\right)}} \tag{2.65}$$

This system has zero determinant. One of the coefficients can be freely chosen. Let $a_1 = e^{-i\hat{k}x_c}$, then $a_3 = e^{-\hat{k}x_c e^{i\left(\frac{\pi+2\theta}{2}\right)}}$. Hence, the overall solution to the problem is

$$\hat{\psi}(x) = \begin{cases} \hat{\psi}_1(x) = e^{i\hat{k}(x-x_c)} & x < x_c \\ \hat{\psi}_2(x) = e^{i\hat{k}e^{i\theta}(x-x_c)} & x \geq x_c \end{cases} \tag{2.66}$$

We proceed now to the plots of some resonant states namely those in Figure (2.3-3) evaluated on a complex line for $\theta = \frac{\pi}{2}$ belonging to the same eigenvalues as in the very same Figure (2.3-3).



(a) $\omega_1 = 0.192757 - 0.00407091i$ (b) $\omega_2 = -0.109864 - 0.160202i$

Figure 2.3.1-1: Plot of two different resonant states evaluated on a complex line for $\theta = \frac{\pi}{2}$.

For the two results in Figure (2.3.1-1) we used the values $A = 1, x_c = 5, \varepsilon = 0.03$. One can clearly see that the difference is significant comparing to the Figure (2.3-3). After the point x_c the functions decay nicely.

2.3.2 Numerical solution

In this section we will find a numerical solution to the Schrödinger equation for the Hamiltonian (2.22) on the complex line (2.42). We modify the equation into the form

$$\begin{aligned} i\Psi_t(x,t) &= -\frac{1}{2}\partial_{xx}\Psi(x,t) - A\delta(x)\Psi(x,t) - \varepsilon x\Psi(x,t) \\ \Psi_t(x,t) &= i\left[\frac{1}{2}\partial_{xx} + A\delta(x) + \varepsilon x\right]\Psi(x,t) \\ \vec{\psi}'(t) &= i\mathcal{O}(x)\Psi(x,t) = i\mathbf{M}\vec{\psi}(t) \end{aligned} \tag{2.67}$$

The operator $\mathcal{O}(x) = \frac{1}{2}\partial_{xx} + A\delta(x) + \varepsilon x$ will be approximated by a matrix $\mathbf{M}$ and the solution $\Psi(x,t)$ by a discrete vector $\vec{\psi}(t)$, where the discretization is in the x variable as $\psi_i(t) = \Psi(x_i, t)$, where x_i is a point on the spatial grid. To achieve this, we approximate the solution by a polynomial

$$\Psi(x,t) \approx P(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)^2 + \ldots + a_k(x-x_n)^k$$
$$= \sum_{i=0}^{i=k} a_i(x-x_n)^i \tag{2.68}$$

where k is the order of the polynomial and x_n is a discrete point on the axis. First, we choose some boundaries for our space $[-L_1, L_2]$. The space is discretized such as $x_n = x_0 + n\Delta x, n = 0, 1, \ldots, N$, where $x_0 = -L_1, x_N = L_2$. Note, that this discretization takes place on the complex line, so the right boundary L_2 should be chosen such that the breaking point x_c is included in it. The boundary values x_0, x_N are known from the initial condition $\Psi(x,0) = \psi(x,\omega_j)$ in (2.54). Let us denote $P(x_n) = \psi_n$. The matrix $\mathbf{M}$ has then size $(N-1)\times(N-1)$. For a particular point $x_n \neq 0$ the operator $\mathcal{O}(x_n)$ becomes

$$\mathcal{O}(x_n) = a_2 + \varepsilon x_n a_0 \tag{2.69}$$

We will treat the point $x_n = 0$ separately. The coefficients a_i are computed in the following way. Let the order of the polynomial be $k = 2$. In this case we have 3 unknown coefficients in the polynomial, so we need to use 3 points, namely x_{n-1}, x_n, x_{n+1}. The polynomial is now evaluated in these 3 points and we get a system of 3 unknowns.

$$\begin{pmatrix} 1 & x_{n-1}-x_n & (x_{n-1}-x_n)^2 \\ 1 & 0 & 0 \\ 1 & x_{n+1}-x_n & (x_{n+1}-x_n)^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \psi_{n-1}(t) \\ \psi_n(t) \\ \psi_{n+1}(t) \end{pmatrix} \tag{2.70}$$

This gives us a_0 and a_2 as a linear combination of the discrete solutions $\psi_{n-1}(t), \psi_n(t), \psi_{n+1}(t)$. We substitute these into (2.69) and the result will again be a linear combination of the discrete solutions $\psi_{n-1}(t), \psi_n(t), \psi_{n+1}(t)$. The coefficients in this combination then represent the coefficients in the n-th line of the matrix $\mathbf{M}_{n*}$.

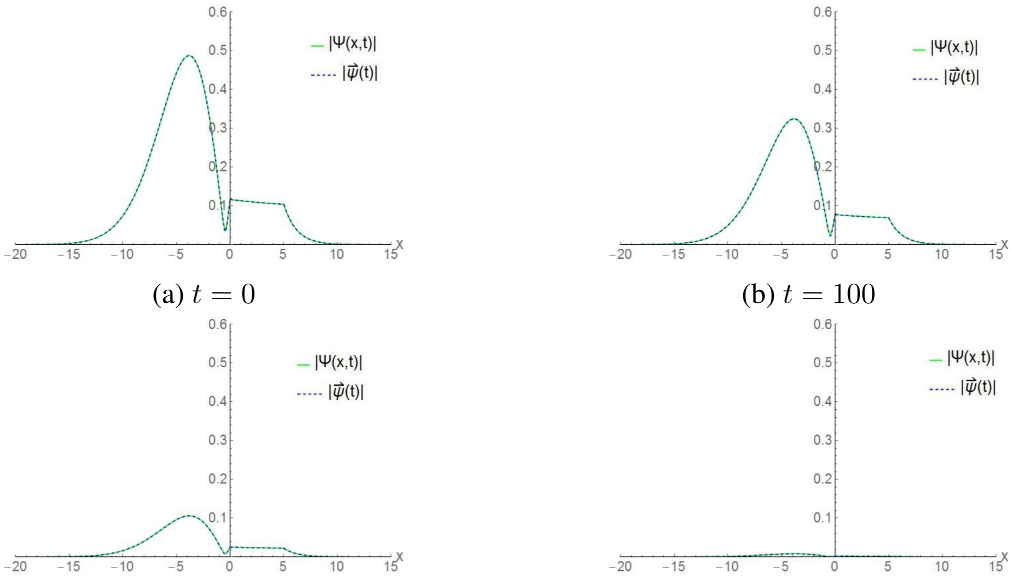
There are two special points that should be mentioned separately. The first is the breaking point x_c where the "complex part" begins. As mentioned before, this point should also be a point on our grid, because this is the transition to the complexity. The second important point is $x_n = 0$. The same applies here too. Also, at this point the Hamiltonian (2.22) is not defined, but the polynomial (2.68) can be evaluated. However, we treat it in a different way. We use the condition (2.10). Using (2.68) in this condition we get

$$a_1^+ - a_1^- = -2A\psi_n(t) \tag{2.71}$$

for $x_n = 0$, where the coefficients a_1^+ and a_1^- are computed as illustrated in (2.70) but using the points x_{n-2}, x_{n-1}, x_n for a_1^- and the points x_n, x_{n+1}, x_{n+2} for a_1^+. Using these coefficients in (2.71) we have

$$\begin{aligned}\psi_n(t) &= F(\psi_{n-2}(t), \psi_{n-1}(t), \psi_{n+1}(t), \psi_{n+2}(t)), \quad /\partial_t \\ \psi_n'(t) &= \psi_{n-2}'\partial_{\psi_{n-2}}F + \psi_{n-1}'\partial_{\psi_{n-1}}F + \psi_{n+1}'\partial_{\psi_{n+1}}F + \psi_{n+2}'\partial_{\psi_{n+2}}F\end{aligned} \tag{2.72}$$

The function $F(\cdot)$ is some linear combination of the arguments, so the derivatives $\partial_{\psi_i}F$ are the coefficients of that linear combination. The derivative $\psi_i'(t)$ is exactly the i-th line in the right hand side of the ODE system (2.67). After substituting all these expressions into (2.72) we get the $\psi_n'(t)$ is equal to a linear combination of $\psi_j(t)$. The coefficients in the right hand-side in this equation make up the line for $x_n = 0$ in matrix $\mathbf{M}$.



(a) $t = 0$ (b) $t = 100$

(c) $t = 375$ (d) $t = 1000$

Figure 2.3.2-1: The exact solution $|\Psi(x,t)|$(blue dashed) and the numeric solution $\left|\vec{\psi}(t)\right|$(green) for Dirac delta potential plotted in one picture at four different times $t = 0, t = 100, t = 375$ and $t = 1000$. The resonant states used as the initial condition belongs to the eigenvalue $\omega_1 = 0.192757 - 0.00407091i$.

On Figure (2.3.2-1) is shown the numerical solution for the case

$\omega_1 = 0.192757 - 0.00407091i$. We used the following parameters: $L_1 = 20, L_2 = 15, N =$

$171, k = 4$. For the resonant state as the initial condition we chose $\varepsilon = 0.03, x_c = 5$ and $A = 1$. One can see that they match almost perfectly at all times.

2.3.3 Resonant states as a linear basis

The resonant states $\Phi_j(z) = \Phi(z(x), \omega_j) = \psi(x, \omega_j)$ can be used as a linear basis to express certain functions $f(x)$ as an infinite sum of a linear combination of these states. In the following computations in this section we assume $\theta = \frac{\pi}{2}$. The expansion has the form

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \Phi_k(z(x)) \tag{2.73}$$

where

$$z(x) = \begin{cases} x & x < x_c \\ x_c + i(x - x_c) & x > x_c \end{cases} \tag{2.74}$$

When we introduced this complex contour and solved the Schrödinger equation on it, the Hamiltonian became no longer self-adjoint. To achieve a resonant mode expansion we need to define some other inner product and introduce a notion of orthogonality with respect to this product. We can start by the fact, that any two resonant states $\Phi(z), \Psi(z)$ should be creating an orthonormal system, that is the inner product should satisfy

$$(\Phi(z), \Psi(z)) = \int_C \overline{\Phi}(z)\Psi(z)\mathrm{d}z = \delta_{kj} \tag{2.75}$$

where C is our complex contour and δ_{kj} is the Kronecker delta symbol which is 1 for $k = j$ and 0 otherwise. From complex analysis we know that the complex conjugate of an analytic complex function becomes non-analytic. Therefore, we need to define a different conjugation of a function, which is

$$\overline{\Phi}(z) = \overline{\Phi\left(\overline{z}\right)} \tag{2.76}$$

We will use this in (2.75) and write the inner product in terms of x variable using (2.74). The integration variable then changes into $\mathrm{d}z = i\mathrm{d}x$ for $x_c < x$ and the whole integration space is divided into two parts $C_1 = (-\infty, x_c]$ and $C_2 = [x_c, \infty)$. Let us use the notation $\Phi(z(x)) =$

$\varphi(x)$ and $\Psi(z(x)) = \psi(x)$. Then we have

$$(\Phi(z), \Psi(z)) = \int_{C_1} \overline{\Phi(z)}\Psi(z)\mathrm{d}z + \int_{C_2} \overline{\Phi(z)}\Psi(z)\mathrm{d}z$$
$$= \int_{-\infty}^{x_c} \overline{\varphi(x)}\psi(x)\mathrm{d}x + i\int_{x_c}^{\infty} \overline{\varphi(x)}\psi(x)\mathrm{d}x \tag{2.77}$$

where the functions $\varphi(x)$ and $\psi(x)$ are defined on the real axis as

$$\varphi(x) = \begin{cases} \Phi(x) & x < x_c \\ \Phi(x_c - i(x - x_c)) & x > x_c \end{cases} \tag{2.78}$$

$$\psi(x) = \begin{cases} \Psi(x) & x < x_c \\ \Psi(x_c + i(x - x_c)) & x > x_c \end{cases} \tag{2.79}$$

and satisfy the following conditions at $x = x_c$

$$\varphi^{x_c^-} = \varphi^{x_c^+} \tag{2.80}$$

$$\varphi'^{x_c^-} = -i\varphi'^{x_c^+} \tag{2.81}$$

and

$$\psi^{x_c^-} = \psi^{x_c^+} \tag{2.82}$$

$$\psi'^{x_c^-} = i\psi'^{x_c^+} \tag{2.83}$$

where for any function $f(x)$ define $f^{a^\pm} = \lim_{\epsilon\to 0} f(a \pm \epsilon)$. One can observe, that these two function satisfy different continuity conditions. As a result of this the two functions belong to different vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ defined as

$$\{\varphi\} \in \mathcal{V}_1 = \left\{\varphi(x)|\varphi'^{x_c^+} = -i\varphi'^{x_c^-}\right\} \tag{2.84}$$

$$\{\psi\} \in \mathcal{V}_2 = \left\{\psi(x)|\psi'^{x_c^+} = i\psi'^{x_c^-}\right\} \tag{2.85}$$

where we also assume, that the functions in both spaces vanish at $\pm\infty$. Let us consider two particular eigenfunctions φ_λ and ψ_μ belonging to the eigenvectors λ and μ. Using the eigenvalue problems $H\varphi_\lambda = \lambda\varphi_\lambda$, $H^\dagger\psi_\mu = \mu\psi_\mu$, where $H^\dagger$ is the adjoint to H, we have

$$(\lambda - \mu)(\varphi_\lambda, \psi_\mu) = (\lambda\varphi_\lambda, \psi_\mu) - (\varphi_\lambda, \mu\psi_\mu) = (H\varphi_\lambda, \psi_\mu) - \left(\varphi_\lambda, H^\dagger\psi_\mu\right) \tag{2.86}$$

The expression (2.86) is zero, if the Hamiltonian H is self-adjoint, which means that if $\lambda \neq \mu$ then $(\varphi_\lambda, \psi_\mu) = 0$ but if $\lambda = \mu$ then $(\varphi_\lambda, \psi_\mu) \neq 0$ and by definition the functions $\varphi_\lambda, \psi_\mu$ are bi-orthogonal. Let us show that the Hamiltonian H is in fact self-adjoint. For two functions $\varphi_\lambda \in \mathcal{V}_1$ and $\psi_\mu \in \mathcal{V}_2$ and the fact that $\overline{H} = H$ we have

$$\begin{aligned}
(H\varphi_\lambda, \psi_\mu) &= \int_{-\infty}^{x_c} \left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - A\delta(x) - \varepsilon x\right)\left(\overline{\varphi_\lambda}\right)\psi_\mu \mathrm{d}x \\
&+ i\int_{x_c}^{\infty} \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - A\delta(x) - \varepsilon x\right)\left(\overline{\varphi_\lambda}\right)\psi_\mu \mathrm{d}x \\
&= -\frac{1}{2}\overline{\varphi_\lambda'}\psi_\mu\Big|_{-\infty}^{x_c} + \int_{-\infty}^{x_c} \frac{1}{2}\overline{\varphi_\lambda'}\psi_\mu' + (-A\delta(x) - \varepsilon x)\left(\overline{\varphi_\lambda}\right)\psi_\mu \mathrm{d}x + \frac{i}{2}\overline{\varphi_\lambda'}\psi_\mu\Big|_{x_c}^{\infty} \\
&+ i\int_{x_c}^{\infty} -\frac{1}{2}\overline{\varphi_\lambda'}\psi_\mu' + (-A\delta(x) - \varepsilon x)\left(\overline{\varphi_\lambda}\right)\psi_\mu \mathrm{d}x \\
&= -\frac{1}{2}\overline{\varphi_\lambda'^{x_c^-}}\psi_\mu(x_c) + \frac{1}{2}\overline{\varphi_\lambda}\psi_\mu'\Big|_{-\infty}^{x_c} \\
&+ \int_{-\infty}^{x_c} \left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - A\delta(x) - \varepsilon x\right)(\psi_\mu)\overline{\varphi_\lambda}\mathrm{d}x - \frac{i}{2}\overline{\varphi_\lambda'^{x_c^+}}\psi_\mu(x_c) - \frac{i}{2}\overline{\varphi_\lambda}\psi_\mu'\Big|_{x_c}^{\infty} \\
&+ i\int_{x_c}^{\infty} \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - A\delta(x) - \varepsilon x\right)(\psi_\mu)\overline{\varphi_\lambda}\mathrm{d}x \\
&= -\frac{1}{2}\overline{\varphi_\lambda'^{x_c^-}}\psi_\mu(x_c) + \frac{1}{2}\overline{\varphi_\lambda}(x_c)\psi_\mu'^{x_c^-} - \frac{i}{2}\overline{\varphi_\lambda'^{x_c^+}}\psi_\mu(x_c) \\
&+ \frac{i}{2}\overline{\varphi_\lambda}(x_c)\psi_\mu'^{x_c^+} + (\varphi_\lambda, H\psi_\mu) = (\varphi_\lambda, H\psi_\mu)
\end{aligned} \tag{2.87}$$

where we used the Hamiltonian from (2.22) and properties of the functions in (2.84), (2.85) such as

$$\begin{aligned}
\tfrac{1}{2}\overline{\varphi_\lambda}(x_c)\psi_\mu'^{x_c^-} &= -\tfrac{i}{2}\overline{\varphi_\lambda}(x_c)\psi_\mu'^{x_c^+} & \qquad \tfrac{1}{2}\overline{\varphi_\lambda'^{x_c^-}}\psi_\mu(x_c) &= -\tfrac{i}{2}\overline{\varphi_\lambda'^{x_c^+}}\psi_\mu(x_c) \\
i\psi_\mu'^{x_c^-} &= \psi_\mu'^{x_c^+} & i\overline{\varphi_\lambda'^{x_c^-}} &= \overline{\varphi_\lambda'^{x_c^+}}
\end{aligned} \tag{2.88}$$

Thus, the expression (2.86) is zero, so the states $\Phi_j(z)$ do create an bi-orthogonal system. We now determine how the functions $\overline{\varphi_\lambda(x)}$ look like and what is the relation between the eigenvalues λ, μ.

$$
\begin{aligned}
&H\psi_\mu = \mu\psi_\mu \\
&\overline{H\varphi_\lambda} = \overline{\lambda\varphi_\lambda} \\
&H\overline{\varphi_\lambda} = \overline{\lambda}\overline{\varphi_\lambda} \Rightarrow \overline{\varphi_\lambda} = \psi_\mu, \overline{\lambda} = \mu
\end{aligned} \tag{2.89}
$$

So the expression under the integral $\int \overline{\varphi_\lambda}\psi_\mu \mathrm{d}x$, where $\overline{\varphi_\lambda}$ belongs to the eigenvalue $\overline{\lambda}$, becomes $\int \psi_\mu\psi_\mu \mathrm{d}x$. This property can be used to expand any function from $\mathcal{V}_2$ using ψ_μ and (2.73) can be rewritten as

$$
f(x) = \sum_\mu c_\mu \psi_\mu \tag{2.90}
$$

where

$$
c_\mu = \frac{(\varphi_\lambda, f)}{(\varphi_\lambda, \psi_\mu)}, \quad \overline{\lambda} = \mu \tag{2.91}
$$

Let us try to approximate some functions using these states as (2.90). We start with a random function. Let the coefficients $\tilde{c}_\mu$ be some real random numbers normally distributed in range from -1 to 1. These create a function $\tilde{f}(x)$ using (2.90). We now compute the coefficients c_k using (2.91) that forms a new function $f(x)$ and compare it with $\tilde{f}(x)$. If the computations are correct, they should match $\tilde{f}(x) = f(x)$.

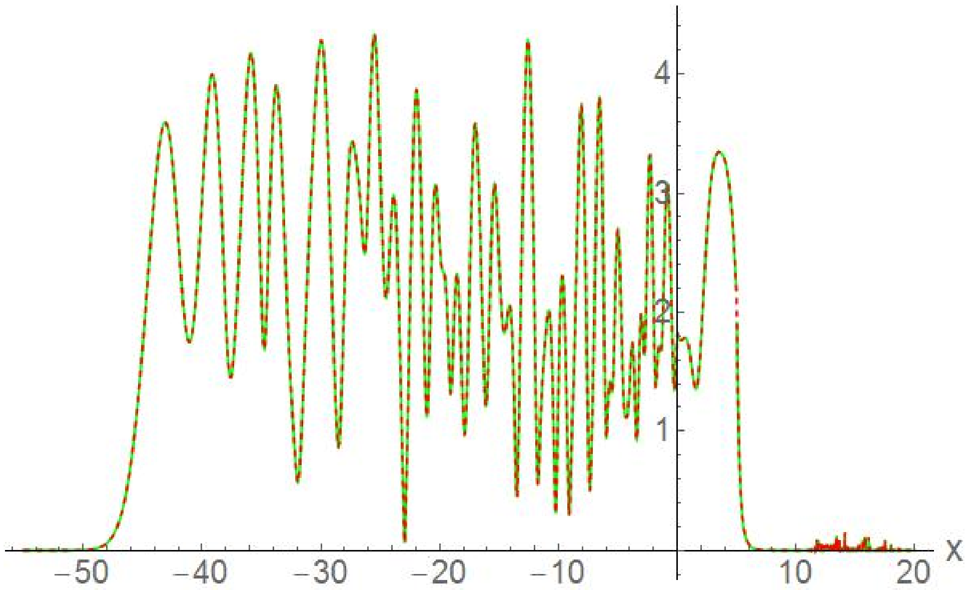
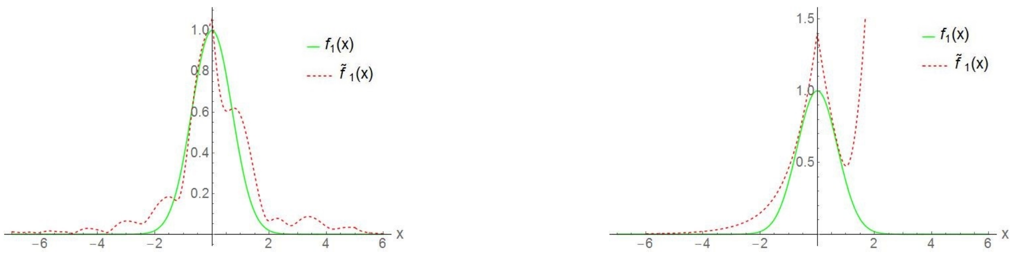


Figure 2.3.3-1: Comparing the function $f(x)$ (red dashed), that was computed using the biorthogonal product of resonant states with the generated function $\tilde{f}(x)$ (green).

The two functions $f(x), \tilde{f}(x)$ are shown on Figure (2.3.3-1). The parameters for the resonant states are $\varepsilon = 0.3, A = 1$ and $x_c = 5$ where we used $N = 50$ resonant states. The same values were used also in Figures (2.3.3-2) and (2.3.3-3). The eigenvalues were all from A-series, which we discussed on page 15 plus the ground state. This plot (2.3.3-1 shows that the two functions are quite the same. In the following we will approximate some particular functions. The first function would be a simple Gaussian $f_1(x) = e^{-x^2}$. We can see the results on Figure (2.3.3-2).

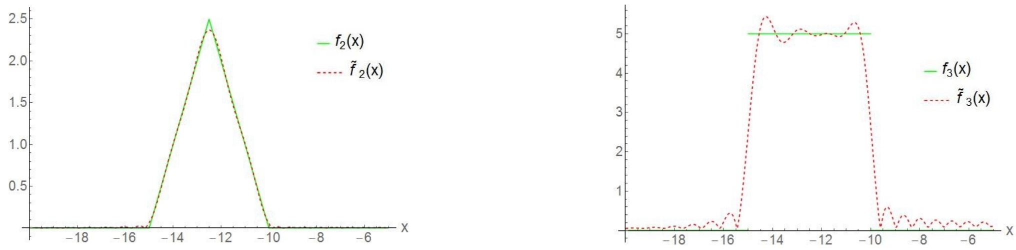


(a) A-series

(b) C-series

Figure 2.3.3-2: A Gaussian function $f_1(x)$ (green) approximated with the resonant states expansion (red dashed) using eigenvalues both from A-series and C-series.

Two different cases are shown. In the first case (2.3.3-2a) only resonant states belonging to eigenvalues from A-series were used and in the other case (2.3.3-2b) from C-series. We can notice, that the one using C-series becomes very inaccurate in the positive x. This is caused by the behaviour of the resonant states. We can see one in (2.3-3b). They grow very quickly in this region. We will later prove that the series approximation works well only for functions that have their support in the negative regions. That is why we choose the next test functions in that way.



(a) (b)

Figure 2.3.3-3: A triangle function $f_2(x)$ (green) and a rectangle function $f_3(x)$ (green) approximated by series using resonant states (red dashed).

The other special functions are the so called triangle $f_2(x)$ and rectangle function $f_3(x)$. They both have support only on the negative axis and only resonant states belonging to eigenvalues from A-series were used because the states from C-series are practically zero in this area.

2.4 The completeness of the resonant states for Dirac delta potential

This section will contain 2 proofs of the completeness of resonant states. We will discuss under what conditions does the expansion (2.90) converge. The first proof will be the weaker proof in which we first look at the scattering form of resonant states and then using complex integration we show the completeness. This can be expressed as

$$\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega = \delta(x - x') \tag{2.92}$$

2.4.1 Weaker proof

We start with the scattering form of the resonant states using the solution (2.27), (2.28).

$$\psi_\omega(x) = \begin{cases} a_1\mathrm{Ai}(y(x)) & x < 0 \\ a_2\mathrm{Ci}^+(y(x)) + a_3\mathrm{Ci}^-(y(x)) & x > 0 \end{cases} \tag{2.93}$$

where $y(x)$ comes from (2.24). From the asymptotic expansion of $\mathrm{Ci}^\pm$ for large x in (2.31) it is clear, that while Ci^+ represents an outgoing wave at positive x then Ci^- is and ingoing wave from positive infinity and both decay algebraically. Applying the conditions (2.9), (2.10) to the expression (2.93) and using $y_0 = y(0)$ we get a linear system

$$\mathbf{M}(y_0) = \begin{pmatrix} \mathrm{Ai}(y_0) & -\mathrm{Ci}^+(y_0) \\ -\frac{A}{\alpha\varepsilon}\mathrm{Ai}(y_0) - \mathrm{Ai}'(y_0) & \mathrm{Ci}'^+(y_0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_3 \begin{pmatrix} \mathrm{Ci}^-(y_0) \\ -\mathrm{Ci}'^-(y_0) \end{pmatrix} \tag{2.94}$$

whose determinant is the same as (2.35) and the solution to the system we have now is

$$a_1 = \frac{a_3}{\det \mathbf{M}(y_0)} \Big(\mathrm{Ci}^-(y_0)\mathrm{Ci}'^+(y_0) - \mathrm{Ci}^+(y_0)\mathrm{Ci}'^-(y_0) \Big) \tag{2.95}$$

$$a_2 = \frac{a_3}{\det \mathbf{M}(y_0)} \left(-\mathrm{Ai}(y_0)\mathrm{Ci}'^-(y_0) + \mathrm{Ci}^-(y_0)\frac{A}{\alpha\varepsilon}\mathrm{Ai}(y_0) + \mathrm{Ci}^-(y_0)\mathrm{Ai}'(y_0) \right) \tag{2.96}$$

With the Wronskian rule of Airy functions in [1] (10.4.10) can this solution be simplified as follows.

$$\begin{aligned} a_1 &= \frac{a_3}{\det \mathbf{M}(y_0)} \left(\mathrm{Bi}(y_0)\mathrm{Bi}'(y_0) + i\mathrm{Bi}(y_0)\mathrm{Ai}'(y_0) - i\mathrm{Ai}(y_0)\mathrm{Bi}'(y_0) + \mathrm{Ai}(y_0)\mathrm{Ai}'(y_0) \right. \\ &\left. -\mathrm{Bi}(y_0)\mathrm{Bi}'(y_0) - i\mathrm{Ai}'(y_0)\mathrm{Bi}(y_0) - i\mathrm{Ai}(y_0)\mathrm{Bi}'(y_0) - \mathrm{Ai}(y_0)\mathrm{Ai}'(y_0) \right) \\ &= -a_3 \frac{2i}{\pi \det \mathbf{M}(y_0)} \end{aligned} \tag{2.97}$$

$$\begin{aligned} a_2 &= \frac{a_3}{\det \mathbf{M}(y_0)} \left(-\mathrm{Ai}(y_0)\mathrm{Bi}'(y_0) + i\mathrm{Ai}(y_0)\mathrm{Ai}'(y_0) + \mathrm{Bi}(y_0)\mathrm{Ai}'(y_0) - i\mathrm{Ai}(y_0)\mathrm{Ai}'(y_0) \right. \\ &\left. +\mathrm{Ci}^-(y_0)\frac{A}{\alpha\varepsilon}\mathrm{Ai}(y_0) \right) = \frac{a_3}{\det \mathbf{M}(y_0)} \left(-\frac{1}{\pi} + \mathrm{Ci}^-(y_0)\frac{A}{\alpha\varepsilon}\mathrm{Ai}(y_0) \right) \\ &= \frac{a_3}{\det \mathbf{M}(y_0)} \left(-\frac{1}{\pi} + \frac{1}{\pi} - \overline{\det \mathbf{M}}(y_0) \right) = -a_3 \frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)} \end{aligned} \tag{2.98}$$

where we used the determinant (2.35) to simplify a_2. Since a_3 is an arbitrary constant, it can be freely chosen. For convenience, let us choose $a_3 = i \left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)} \right)^{\frac{1}{2}}$. Then we can write the scattering solution (2.93) in the form

$$\psi_\omega(x) = \chi \begin{cases} \frac{2}{\pi |\det \mathbf{M}(y_0)|} \mathrm{Ai}(y(x)) & x < 0 \\ -i \left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)} \right)^{\frac{1}{2}} \mathrm{Ci}^+(y(x)) + i \left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)} \right)^{\frac{1}{2}} \mathrm{Ci}^-(y(x)) & x > 0 \end{cases} \tag{2.99}$$

with a normalization constant χ. This constant is computed in Appendix A and it is

$$\chi = 2^{-\frac{2}{3}} \varepsilon^{-\frac{1}{6}} \tag{2.100}$$

We return back to the completeness relation of the resonant states (2.92) and define a function for $R > 0$

$$\mathcal{F}_R(x, x') = \int_{-R}^{R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega \tag{2.101}$$

which converges to $\delta(x - x')$ as $R \to \infty$. Let us now introduce the following closed contour Γ_R in the lower complex half-plane as in Figure (2.4.1-1)

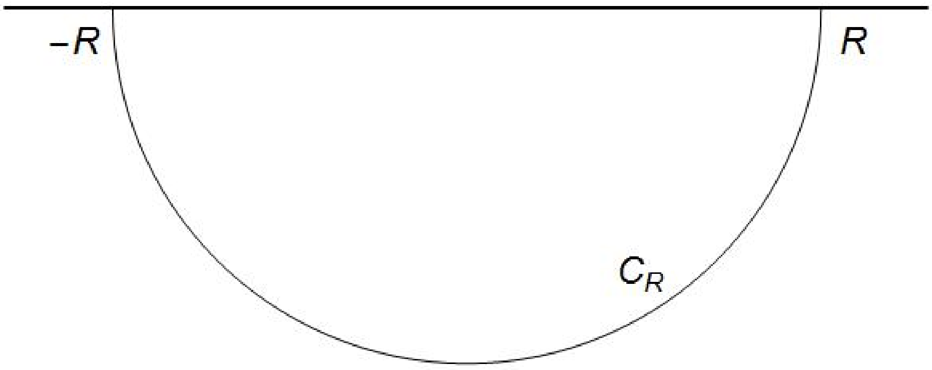


Figure 2.4.1-1: Closed contour in the lower complex half-plane for frequency ω.

Then we have $\Gamma_R = C_R \cup [-R, R]$. Note, that the curve Γ_R we integrate along, must be negatively oriented, that is the interior of the curve is on the right side when travelling along the curve, because we chose $\mathcal{F}_R(x, x')$ to be an integral from $-R$ to R. With the use of the Residue theorem from complex analysis we can write

$$\begin{aligned}
\int_{\Gamma_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega &= \mathcal{F}_R(x, x') + \int_{C_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega \\
-2\pi i \sum_j \mathrm{Res}(\psi_\omega(x)\psi_\omega(x'), \omega_j) &= \mathcal{F}_R(x, x') + \int_{C_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega \\
\mathcal{F}_R(x, x') &= \frac{2\pi}{i} \sum_j \mathrm{Res}(\psi_\omega(x)\psi_\omega(x'), \omega_j) \\
&\quad - \int_{C_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega
\end{aligned} \tag{2.102}$$

where ω_j are the poles of the integrand located on the lower half-plane. These poles are the complex frequencies of the resonant states. We first expand the integrand so we can express the residue. For $x > 0, x' > 0$ we have

$$\psi_\omega(x)\psi_\omega(x') = -\chi^2 \left\{ \frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)} \mathrm{Ci}^-(y(x))\mathrm{Ci}^-(y(x')) \right.$$
$$\left. + \frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)} \mathrm{Ci}^+(y(x))\mathrm{Ci}^+(y(x')) - \mathrm{Ci}^-(y(x))\mathrm{Ci}^+(y(x')) - \mathrm{Ci}^-(y(x'))\mathrm{Ci}^+(y(x)) \right\} \tag{2.103}$$

for $x > 0, x' < 0$

$$\psi_\omega(x)\psi_\omega(x') = \frac{2\chi^2 i}{\pi} \left(\frac{1}{\overline{\det \mathbf{M}}(y_0)} \mathrm{Ci}^-(y(x))\mathrm{Ai}(y(x')) - \frac{1}{\det \mathbf{M}(y_0)} \mathrm{Ci}^+(y(x))\mathrm{Ai}(y(x')) \right) \tag{2.104}$$

for $x < 0, x' > 0$

$$\psi_\omega(x)\psi_\omega(x') = \frac{2\chi^2 i}{\pi} \left(\frac{1}{\overline{\det \mathbf{M}}(y_0)} \mathrm{Ci}^-(y(x'))\mathrm{Ai}(y(x)) - \frac{1}{\det \mathbf{M}(y_0)} \mathrm{Ci}^+(y(x'))\mathrm{Ai}(y(x)) \right) \tag{2.105}$$

and for $x < 0, x' < 0$

$$\psi_\omega(x)\psi_\omega(x') = \frac{4}{\pi^2 \det \mathbf{M}(y_0)\overline{\det \mathbf{M}}(y_0)} \mathrm{Ai}(y(x))\mathrm{Ai}(y(x')) \tag{2.106}$$

Airy functions are analytical and we can see from these expressions, that the poles which are confined in the closed contour Γ_R are determined by the equation

$$\det \mathbf{M}(y_0) = 0 \tag{2.107}$$

Zeros of the complex conjugate of this equation are not in Γ_R so we are not considering the terms containing them. All zeros are simple as we could see from our computations and Figure

(2.3-2), so the residues are

$$
\begin{aligned}
&\frac{2\pi}{i}\mathrm{Res}(\psi_\omega(x)\psi_\omega(x'),\omega_j) = \frac{2\pi}{i}\lim_{\omega\to\omega_j}\left[(\omega-\omega_j)\psi_\omega(x)\psi_\omega(x')\right] \\
&= \frac{2\pi}{i}\chi^2\lim_{\omega\to\omega_j}\left(\frac{\omega-\omega_j}{\det\mathbf{M}(y_0)}\right)\begin{cases} -\overline{\det\mathbf{M}_j}\mathrm{Ci}_j^+(y(x))\mathrm{Ci}_j^+(y(x')) & x>0, x'>0 \\ -\frac{2i}{\pi}\mathrm{Ci}_j^+(y(x))\mathrm{Ai}_j(y(x')) & x>0, x'<0 \\ -\frac{2i}{\pi}\mathrm{Ci}_j^+(y(x'))\mathrm{Ai}_j(y(x)) & x<0, x'>0 \\ \frac{4}{\pi^2\overline{\det\mathbf{M}_j}}\mathrm{Ai}_j(y(x))\mathrm{Ai}_j(y(x')) & x<0, x'<0 \end{cases} \\
&= \psi_j(x)\psi_j(x')
\end{aligned} \tag{2.108}
$$

where

$$
\psi_j(x) = \chi\sqrt{\frac{2\pi}{i}\lim_{\omega\to\omega_j}\left(\frac{\omega-\omega_j}{\det\mathbf{M}(y_0)}\right)}\begin{cases} i\sqrt{\overline{\det\mathbf{M}_j}}\mathrm{Ci}_j^+(y(x)) & x>0 \\ -\frac{2}{\pi\sqrt{\overline{\det\mathbf{M}_j}}}\mathrm{Ai}_j(y(x)) & x<0 \end{cases} \tag{2.109}
$$

and where $\mathrm{Ai}_j(y(x)) = \mathrm{Ai}(y(x))\Big|_{\omega=\omega_j}$, $\mathrm{Ci}_j^\pm(y(x)) = \mathrm{Ci}^\pm(y(x))\Big|_{\omega=\omega_j}$ and $\overline{\det\mathbf{M}_j} = \overline{\det\mathbf{M}}(y_0)\Big|_{\omega=\omega_j}$. It is easy to see that the functions $\psi_j(x)$ are proportional to the resonant states $\psi_\omega(x)$ up to a constant. Hence, for $R\to\infty$ we can rewrite (2.102) in the form

$$
\delta(x-x') = \sum_j \psi_j(x)\psi_j(x') - \lim_{R\to\infty}\int_{C_R}\psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega \tag{2.110}
$$

Whether the resonant states form a complete set or not, depends on the limit of the second term in (2.110). The radius R has huge values, so we can use the asymptotic representations of Airy functions. These computations were done in Appendix B. Using (B.28) for sector $-\frac{2\pi}{3}<\theta<0$ and (B.53), (B.54) for sector $-\pi<\theta<-\frac{2\pi}{3}$, which we concluded that vanishes, and $\mathrm{d}\omega = Re^{i\theta}i\mathrm{d}\theta$ we can write the integral in (2.110) as

$$
\begin{aligned}
&\int_{C_R}\psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega \approx \int_{-\frac{2\pi}{3}}^{0}\frac{\chi(2\alpha\varepsilon)^{\frac12}e^{i\frac{\pi}{4}}}{(\pi A)^{\frac12}}e^{\varpi x R^{\frac12}}\frac{\chi(2\alpha\varepsilon)^{\frac12}e^{i\frac{\pi}{4}}}{(\pi A)^{\frac12}}e^{\varpi x' R^{\frac12}}Re^{i\theta}i\mathrm{d}\theta \\
&= -\frac{\chi^2 2\alpha\varepsilon R}{\pi A}\int_{-\frac{2\pi}{3}}^{0} e^{\varpi R^{\frac12}(x+x')}e^{i\theta}\mathrm{d}\theta \\
&= -\frac{\chi^2 2\alpha\varepsilon R}{\pi A}\int_{-\frac{2\pi}{3}}^{0} e^{-\varepsilon(2\alpha)^{\frac32}\sin\left(\frac12\theta\right)(x+x')R^{\frac12}}e^{i\varepsilon(2\alpha)^{\frac32}\cos\left(\frac12\theta\right)(x+x')R^{\frac12}}e^{i\theta}\mathrm{d}\theta
\end{aligned} \tag{2.111}
$$

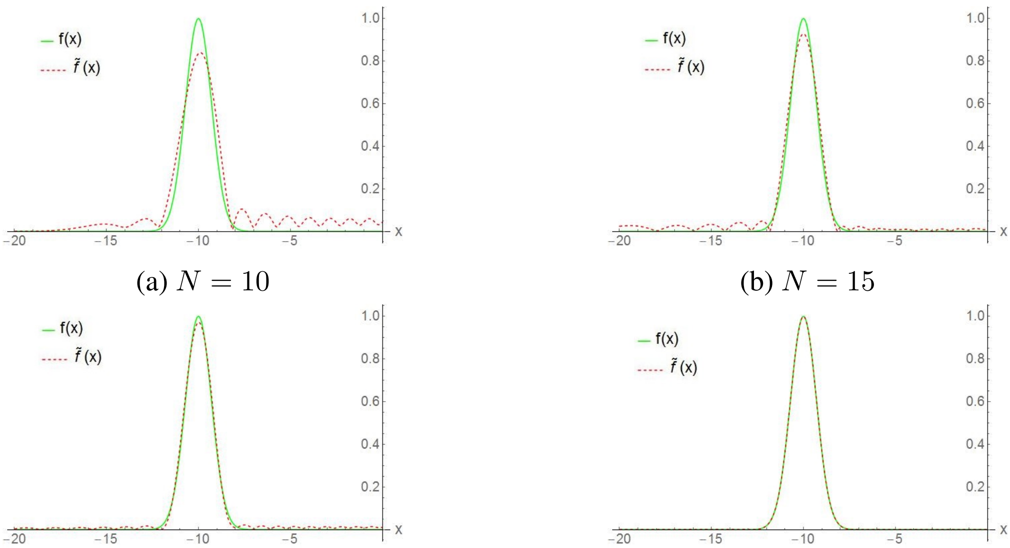
where ϖ is defined in (B.21), (B.22). Let $f(x)$ be any function of a compact support. Multiplying (2.110) by $f(x')$and integrating with respect to x' we get

$$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x-x')\mathrm{d}x' = \sum_j \psi_j(x) \int_{-\infty}^{\infty} \psi_j(x') f(x')\mathrm{d}x' + \lim_{R\to\infty} \frac{\chi^2 2\alpha\varepsilon R}{\pi A}$$
$$\int_{-\infty}^{\infty}\int_{-\frac{2\pi}{3}}^{0} f(x') e^{-\varepsilon(2\alpha)^{\frac{3}{2}}\sin\left(\frac{1}{2}\theta\right)(x+x')R^{\frac{1}{2}}} e^{i\varepsilon(2\alpha)^{\frac{3}{2}}\cos\left(\frac{1}{2}\theta\right)(x+x')R^{\frac{1}{2}}} e^{i\theta}\mathrm{d}\theta\mathrm{d}x'$$
$$f(x) = \sum_j c_j\psi_j(x)$$
$$+ \lim_{R\to\infty} \frac{\chi^2 2\alpha\varepsilon R}{\pi A} \int_{-\frac{2\pi}{3}}^{0} F(R,\theta) e^{-\varepsilon(2\alpha)^{\frac{3}{2}}\sin\left(\frac{1}{2}\theta\right)xR^{\frac{1}{2}}} e^{i\varepsilon(2\alpha)^{\frac{3}{2}}\cos\left(\frac{1}{2}\theta\right)xR^{\frac{1}{2}}} e^{i\theta}\mathrm{d}\theta \qquad (2.112)$$

where

$$F(R,\theta) = \int_{-\infty}^{\infty} f(x') e^{-\varepsilon(2\alpha)^{\frac{3}{2}}\sin\left(\frac{1}{2}\theta\right)x'R^{\frac{1}{2}}} e^{i\varepsilon(2\alpha)^{\frac{3}{2}}\cos\left(\frac{1}{2}\theta\right)x'R^{\frac{1}{2}}}\mathrm{d}x' \qquad (2.113)$$

From (B.21), (B.22) we have that $\varpi_r, \varpi_i < 0$. We can conclude that if the function $f(x)$ has a support that is located on the negative real axis, then the limit in (2.112) vanishes, because (2.113) vanishes and in this case the resonant states form a complete set for such functions. The following Figure (2.4.1-2) shows us examples of this resonant states expansion for a Gaussian function $f(x)$ using various number of terms. We can see that already for 50 terms, the approximation is very good.



(a) $N = 10$ (b) $N = 15$

(c) $N = 20$ (d) $N = 50$

Figure 2.4.1-2: A Gaussian function $f(x) = e^{-(x+10)^2}$ (green) compared with the resonant states expansion $\tilde{f}(x)$ (red dashed) with various number of terms.

2.4.2 Stronger proof

This proof is again a proof of the completeness of resonant states (2.99) as in the previous section only in a different way, but many things will be used from the weaker proof too. It will contain a more precise outcome for the condition set to the expanded function $f(x)$. We start again writing down the resonant states in a scattering form.

$$\psi_\omega(x) = \chi \begin{cases} \frac{2}{\pi|\det \mathbf{M}(y_0)|}\mathrm{Ai}(y(x)) & x<0 \\ -i\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y(x)) + i\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y(x)) & x>0 \end{cases} \tag{2.114}$$

with the normalization coefficient (A.29). We use the continuity of these states at $x=0$ to write

$$\begin{aligned} \frac{2}{\pi|\det \mathbf{M}(y_0)|}\mathrm{Ai}(y_0) &= -i\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_0) + i\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_0) \\ 1 &= \frac{\pi|\det \mathbf{M}(y_0)|}{2\mathrm{Ai}(y_0)} \\ &\quad \left(-i\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_0) + i\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_0)\right) \end{aligned} \tag{2.115}$$

so we can rewrite our expression (2.114) as

$$\psi_\omega(x) = \chi \begin{cases} \left(-i\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_0) + i\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_0)\right)\frac{\mathrm{Ai}(y(x))}{\mathrm{Ai}(y_0)} & x<0 \\ -i\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y(x)) + i\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y(x)) & x>0 \end{cases} \tag{2.116}$$

As we discussed in (2.31) the linear combination of Airy functions Ci^+ represents a wave moving in the positive x direction, hence an outgoing wave and Ci^- is an ingoing wave. The resonant states can be then expressed as a sum of outgoing $\psi_\omega^+(x)$ and ingoing waves $\psi_\omega^-(x)$ as

$$\psi_\omega(x) = \psi_\omega^+(x) + \psi_\omega^-(x) \tag{2.117}$$

where we defined

$$\psi_\omega^+(x) = \chi \begin{cases} -i\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_0)\frac{\mathrm{Ai}(y(x))}{\mathrm{Ai}(y_0)} & x < 0 \\ -i\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y(x)) & x > 0 \end{cases} \tag{2.118}$$

$$\psi_\omega^-(x) = \chi \begin{cases} i\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_0)\frac{\mathrm{Ai}(y(x))}{\mathrm{Ai}(y_0)} & x < 0 \\ i\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y(x)) & x > 0 \end{cases} \tag{2.119}$$

Note that these functions $\psi_\omega^\pm(x)$ are continuous at $x = 0$ but do not satisfy the jump condition (2.10). The completeness relation for scattering states as before is

$$\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega = \delta(x - x') \tag{2.120}$$

Let $f(x)$ be any function with a compact support. Then using the completeness (2.120) we have

$$f(x) = \int_{-\infty}^{\infty} \delta(x-s)f(s)\mathrm{d}s = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(s)\mathrm{d}\omega f(s)\mathrm{d}s \tag{2.121}$$

We use second times as we multiply (2.120) with $f(x)$ and integrate over the whole space:

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_{\omega'}(x)\psi_{\omega'}(s)f(x)\mathrm{d}\omega'\mathrm{d}x &= \int_{-\infty}^{\infty} \delta(x-s)f(x)\mathrm{d}x \\ \int_{-\infty}^{\infty} a(\omega')\psi_{\omega'}(s)\mathrm{d}\omega' &= f(s) \end{aligned} \tag{2.122}$$

where $a(\omega') = \int_{-\infty}^{\infty} \psi_{\omega'}(x)f(x)\mathrm{d}x$. We substitute (2.122) back into (2.121) and we get

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(s)\mathrm{d}\omega \int_{-\infty}^{\infty} a(\omega')\psi_{\omega'}(s)\mathrm{d}\omega'\mathrm{d}s \\ &= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x) \int_{-\infty}^{\infty} \psi_\omega(s)\psi_{\omega'}(s)\mathrm{d}s\mathrm{d}\omega\mathrm{d}\omega' \end{aligned} \tag{2.123}$$

Using the separation of the resonant states into outgoing and ingoing waves (2.117) we can rewrite (2.123) as

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x) \int_{-\infty}^{\infty} \psi_\omega(s)(\psi^+_{\omega'}(s) + \psi^-_{\omega'}(s))\mathrm{d}s\mathrm{d}\omega\mathrm{d}\omega' \\
&= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x) \int_{-\infty}^{\infty} \psi_\omega(s)\psi^+_{\omega'}(s)\mathrm{d}s\mathrm{d}\omega\mathrm{d}\omega' \\
&+ \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x) \int_{-\infty}^{\infty} \psi_\omega(s)\psi^-_{\omega'}(s)\mathrm{d}s\mathrm{d}\omega\mathrm{d}\omega' \\
&= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x)\Upsilon^+(\omega,\omega')\mathrm{d}\omega\mathrm{d}\omega' + \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x)\Upsilon^-(\omega,\omega')\mathrm{d}\omega\mathrm{d}\omega' \\
&= f^+(x) + f^-(x) \qquad (2.124)
\end{aligned}
$$

where we have defined the following quantities

$$
\Upsilon^\pm(\omega,\omega') = \int_{-\infty}^{\infty} \psi_\omega(s)\psi^\pm_{\omega'}(s)\mathrm{d}s \qquad (2.125)
$$

$$
f^\pm(x) = \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x)\Upsilon^\pm(\omega,\omega')\mathrm{d}\omega\mathrm{d}\omega' \qquad (2.126)
$$

The integrand in (2.125) does not converge. It comes from the standard asymptotic expression of Airy functions in (2.31), where we see that $\mathrm{Ci}^\pm(x)$ is decaying algebraically, but not fast enough to converge. That is why we introduce the following correction in the variable ω'

$$
\Upsilon^\pm_\xi(\omega,\omega') = \int_{-\infty}^{\infty} \psi_\omega(s)\psi^\pm_{\omega'\pm i\xi}(s)\mathrm{d}s \qquad (2.127)
$$

because the exponential part in the asymptotic behaviour in (2.31) $e^{\pm i\left(\zeta+\frac{\pi}{4}\right)}$ contains ω' through $\zeta = \frac{2}{3}(-y(x))^{\frac{3}{2}}$ and (2.24) so we can write

$$
\begin{aligned}
e^{\pm i\left(\zeta+\frac{\pi}{4}\right)} &= e^{\pm i\left(\frac{2}{3}(2\alpha(\varepsilon x+\omega'))^{\frac{3}{2}}+\frac{\pi}{4}\right)} = e^{\pm i\left(\frac{2}{3}\left(2\alpha\omega'\left(1+\frac{\varepsilon x}{\omega'}\right)\right)^{\frac{3}{2}}+\frac{\pi}{4}\right)} \\
&\approx e^{\pm i\left(\frac{2}{3}(2\alpha\omega')^{\frac{3}{2}}\left(1+\frac{3\varepsilon x}{2\omega'}\right)+\frac{\pi}{4}\right)} = e^{\pm i\frac{2}{3}(2\alpha\omega')^{\frac{3}{2}}} e^{\pm i(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}\varepsilon x} e^{\pm i\frac{\pi}{4}} \\
&\to e^{\pm i\frac{2}{3}(2\alpha)^{\frac{3}{2}}(\omega'\pm i\xi)^{\frac{3}{2}}} e^{\pm i(2\alpha)^{\frac{3}{2}}(\omega'\pm i\xi)^{\frac{1}{2}}\varepsilon x} e^{\pm i\frac{\pi}{4}} \\
&\approx e^{\pm i\frac{2}{3}(2\alpha\omega')^{\frac{3}{2}}\left(1\pm i\frac{3\xi}{2\omega'}\right)} e^{\pm i(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}\left(1\pm i\frac{\xi}{2\omega'}\right)\varepsilon x} e^{\pm i\frac{\pi}{4}} \\
&= e^{\pm i\frac{2}{3}(2\alpha\omega')^{\frac{3}{2}}} e^{-(2\alpha\omega')^{\frac{3}{2}}\frac{\xi}{\omega'}} e^{\pm i(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}} e^{-(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}\frac{\xi}{2\omega'}\varepsilon x} e^{\pm i\frac{\pi}{4}} \qquad (2.128)
\end{aligned}
$$

taking the absolute value of the last line we get

$$\left| e^{-(2\alpha\omega')^{\frac{3}{2}}\frac{\xi}{\omega'}} \right| \left| e^{-(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}\frac{\xi}{2\omega'}\varepsilon x} \right| \tag{2.129}$$

where the exponential decay for large x can be clearly seen. We will perform calculations with a finite ξ and in the end we remove this correction by letting it go to 0 in a limit. Let us focus on the integrals $\Upsilon^{\pm}$. Observe that we can write the resonant states as

$$\psi_\omega(s) = \Psi\left(-2\alpha\varepsilon\left(s + \frac{\omega}{\varepsilon}\right)\right) \tag{2.130}$$

$$\psi^{\pm}_{\omega'\pm i\xi}(s) = \Psi^{\pm}\left(-2\alpha\varepsilon\left(s + \frac{\omega' \pm i\xi}{\varepsilon}\right)\right) \tag{2.131}$$

where the functions Ψ, $\Psi^{\pm}$ are linear combinations of Airy functions in general on intervals $(-\infty, 0)$ and $(0, \infty)$. For the next step we can use the formula (3.53) in [2] which says that if $A(s)$ and $B(s)$ are any linear combinations of Airy functions then

$$\begin{aligned}\int A(\rho(s+\beta_1))B(\rho(s+\beta_2))\mathrm{d}s = \frac{1}{\rho^2(\beta_1-\beta_2)}\left[A'(\rho(s+\beta_1))B(\rho(s+\beta_2))\right)\\ -A(\rho(s+\beta_1))B'(\rho(s+\beta_2))]\end{aligned} \tag{2.132}$$

Using this in (2.127) and the fact that the resonant states $\psi_\omega(s)$ decay at $\pm\infty$, letting $A(s) = \Psi(s), B(s) = \Psi^{\pm}(s)$ and defying $\rho = 2\alpha\varepsilon$ we get

$$\begin{aligned}\Upsilon^{\pm}_{\xi}(\omega,\omega') &= \int_{-\infty}^{0}\Psi\left(-\rho\left(s+\frac{\omega}{\varepsilon}\right)\right)\Psi^{\pm}\left(-\rho\left(s+\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\mathrm{d}s\\ &+\int_{0}^{\infty}\Psi\left(-\rho\left(s+\frac{\omega}{\varepsilon}\right)\right)\Psi^{\pm}\left(-\rho\left(s+\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\mathrm{d}s\\ &=\frac{\varepsilon}{\rho^2(\omega-\omega'\mp i\xi)}\left[\Psi'\left(-\rho\left(s+\frac{\omega}{\varepsilon}\right)\right)\Psi^{\pm}\left(-\rho\left(s+\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\right.\\ &\left.-\Psi\left(-\rho\left(s+\frac{\omega}{\varepsilon}\right)\right)\Psi'^{\pm}\left(-\rho\left(s+\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\right]\Bigg|_{0^+}^{0^-}\end{aligned} \tag{2.133}$$

We go further and use the continuity jump of the derivative at $x = 0$ (2.10) and we get

$$\begin{aligned}
\Upsilon_\xi^\pm(\omega,\omega') = {} & \frac{\varepsilon}{\rho^2(\omega-\omega'\mp i\xi)}\left[\Psi'\left(-\rho\left(0^-+\frac{\omega}{\varepsilon}\right)\right)\Psi^\pm\left(-\rho\left(\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\right.\\
& -\Psi\left(-\rho\left(\frac{\omega}{\varepsilon}\right)\right)\Psi'^\pm\left(-\rho\left(0^-+\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\\
& -\Psi'\left(-\rho\left(0^++\frac{\omega}{\varepsilon}\right)\right)\Psi^\pm\left(-\rho\left(\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\\
& \left.+\Psi\left(-\rho\left(\frac{\omega}{\varepsilon}\right)\right)\Psi'^\pm\left(-\rho\left(0^++\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\right]\\
= {} & \frac{\varepsilon}{\rho^2(\omega-\omega'\mp i\xi)}\left\{\Psi^\pm\left(-\rho\left(\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\left[\Psi'\left(-\rho\left(0^-+\frac{\omega}{\varepsilon}\right)\right)\right.\right.\\
& \left.-\Psi'\left(-\rho\left(0^++\frac{\omega}{\varepsilon}\right)\right)\right]+\Psi\left(-\rho\left(\frac{\omega}{\varepsilon}\right)\right)\left[\Psi'^\pm\left(-\rho\left(0^++\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\right.\\
& \left.\left.-\Psi'^\pm\left(-\rho\left(0^-+\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\right]\right\}\\
= {} & \frac{\varepsilon}{\rho^2(\omega-\omega'\mp i\xi)}\left\{-\Psi^\pm\left(-\rho\left(\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\frac{2A}{\rho}\Psi\left(-\rho\left(\frac{\omega}{\varepsilon}\right)\right)\right.\\
& \left.+\Psi\left(-\rho\left(\frac{\omega}{\varepsilon}\right)\right)\left[\Psi'^\pm\left(-\rho\left(0^++\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)-\Psi'^\pm\left(-\rho\left(0^-+\frac{\omega'\pm i\xi}{\varepsilon}\right)\right)\right]\right\}\\
= {} & \frac{\varepsilon\psi_\omega(y_0)}{\rho^2(\omega-\omega'\mp i\xi)}\left[-\Psi^\pm\left(-\rho\left(\frac{\omega'}{\varepsilon}\right)\right)\frac{2A}{\rho}+\Psi'^\pm\left(-\rho\left(0^++\frac{\omega'}{\varepsilon}\right)\right)\right.\\
& \left.-\Psi'^\pm\left(-\rho\left(0^-+\frac{\omega'}{\varepsilon}\right)\right)\right]
\end{aligned}\tag{2.134}$$

where in the last line we set $\xi = 0$ in all numerators without loss of generality. From (2.118) and (2.119) through (2.131) and using the determinant (2.35) we can simplify this result as

$$\begin{aligned}
& -\Psi^\pm\left(-\rho\left(\frac{\omega'}{\varepsilon}\right)\right)\frac{2A}{\rho}+\Psi'^\pm\left(-\rho\left(0^++\frac{\omega'}{\varepsilon}\right)\right)-\Psi'^\pm\left(-\rho\left(0^-+\frac{\omega'}{\varepsilon}\right)\right)\\
& = \mp i\chi\left(\frac{\det\mathbf{M}^\mp(y_0)}{\det\mathbf{M}^\pm(y_0)}\right)^{\frac{1}{2}}\left[-\frac{2A}{\rho}\mathrm{Ci}^\pm(y_0')+\mathrm{Ci}'^\pm(y_0')-\mathrm{Ci}^\pm(y_0')\frac{\mathrm{Ai}'(y_0')}{\mathrm{Ai}(y_0')}\right]\\
& = \mp i\chi\left(\frac{\det\mathbf{M}^\mp(y_0)}{\det\mathbf{M}^\pm(y_0)}\right)^{\frac{1}{2}}\left[-\frac{2A}{2\alpha\varepsilon}\left(\frac{1}{\pi}-\det\mathbf{M}^\pm(y_0')\right)\frac{\alpha\varepsilon}{A\mathrm{Ai}(y_0')}+\mathrm{Ci}'^\pm(y_0')\right.\\
& \left.-\mathrm{Ci}^\pm(y_0')\frac{\mathrm{Ai}'(y_0')}{\mathrm{Ai}(y_0')}\right]\\
& = \mp i\chi\left(\frac{\det\mathbf{M}^\mp(y_0)}{\det\mathbf{M}^\pm(y_0)}\right)^{\frac{1}{2}}\left[\frac{\det\mathbf{M}^\pm(y_0')}{\mathrm{Ai}(y_0')}+\frac{1}{\mathrm{Ai}(y_0')}\left(-\frac{1}{\pi}+\mathrm{Ci}'^\pm(y_0')\mathrm{Ai}(y_0')\right.\right.\\
& \left.\left.-\mathrm{Ci}^\pm(y_0')\mathrm{Ai}'(y_0')\right)\right]\\
& = \mp i\chi\left(\frac{\det\mathbf{M}^\mp(y_0)}{\det\mathbf{M}^\pm(y_0)}\right)^{\frac{1}{2}}\frac{\det\mathbf{M}^\pm(y_0')}{\mathrm{Ai}(y_0')} = \mp i\chi\frac{|\det\mathbf{M}(y_0')|}{\mathrm{Ai}(y_0')}
\end{aligned}\tag{2.135}$$

where we defined $y_0' = -2\alpha\varepsilon\omega'$ and denoted $\det \mathbf{M}^+(y_0) = \det \mathbf{M}(y_0)$ and $\det \mathbf{M}^-(y_0) = \overline{\det \mathbf{M}}(y_0)$. Hence we can write (2.134) using (2.114) as

$$\begin{aligned}\Upsilon_\xi^\pm(\omega,\omega') &= \mp \frac{\varepsilon\psi_\omega(y_0)}{\rho^2(\omega-\omega'\mp i\xi)} i\chi \frac{|\det \mathbf{M}(y_0')|}{\mathrm{Ai}(y_0')} \\ &= \mp \frac{2i\chi\varepsilon}{\pi\rho^2(\omega-\omega'\mp i\xi)} \frac{|\det \mathbf{M}(y_0')|\,\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|\,\mathrm{Ai}(y_0')}\end{aligned} \tag{2.136}$$

It is easy to verify that the following statement is true.

$$\lim_{\xi\to\infty} \left(\Upsilon_\xi^+(\omega,\omega') + \Upsilon_\xi^-(\omega,\omega')\right) = \delta(\omega-\omega') \tag{2.137}$$

Let us go back to (2.126). Using (2.136) for to express $f^\pm(x)$ we get the following expression

$$\begin{aligned}f^\pm(x) &= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x)\Upsilon^\pm(\omega,\omega')\mathrm{d}\omega\mathrm{d}\omega' \\ &= \mp\frac{2i\chi\varepsilon}{\pi\rho^2} \int_{-\infty}^{\infty} a(\omega')\frac{|\det \mathbf{M}(y_0')|}{\mathrm{Ai}(y_0')} \int_{-\infty}^{\infty} \psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|}\mathrm{d}\omega\mathrm{d}\omega'\end{aligned} \tag{2.138}$$

Our whole focus is now on the integral

$$P_\xi(\omega') = \int_{-\infty}^{\infty} \psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|}\mathrm{d}\omega \tag{2.139}$$

which we can solve using the residue theorem just as we did in (2.102) on a closed contour (2.4.1-1). In order to do this we define

$$P_\xi^R(\omega') = \int_{-R}^{R} \psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|}\mathrm{d}\omega \tag{2.140}$$

which converges to (2.139) as $R\to\infty$. We use also the same notation for this contour Γ_R and we get

$$\begin{aligned}&\int_{\Gamma_R} \psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|}\mathrm{d}\omega \\ &\qquad = P_\xi^R(\omega') + \int_{C_R} \psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|}\mathrm{d}\omega \\ &\frac{2\pi}{i}\sum_j \mathrm{Res}\left(\psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|},\omega_j\right) \\ &\qquad = P_\xi^R(\omega') + \int_{C_R} \psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|}\mathrm{d}\omega\end{aligned} \tag{2.141}$$

from which we can see, that if the the integral on the right hand side vanishes as R approaches infinity, then our function $P_\xi(\omega')$ will equal to the sum of residues. We prove that it in fact does vanish using the knowledge we gained in Appendix B about asymptotic behaviours of Airy functions along rays in the lower half of the complex frequency plane $\omega = Re^{i\theta}$, where we had two sectors $-\frac{2\pi}{3} < \theta < 0$ and $-\pi < \theta < -\frac{2\pi}{3}$. Starting with the first one we have for $\mathrm{Ai}(y_0)$ the formulas (B.8),(B.9) and (B.11), for $|\det \mathbf{M}(y_0)|$ we use (B.18) and for the resonant states $\psi_\omega(x)$ (B.28). Putting it all together we get

$$
\begin{aligned}
&\psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|}\\
&\approx -\frac{\chi(2\alpha\varepsilon)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{(Re^{i\theta}-\omega'\mp i\xi)(\pi A)^{\frac{1}{2}}}e^{\varpi x R^{\frac{1}{2}}}\frac{\pi(2\alpha\varepsilon)^{\frac{1}{2}}}{iA^{\frac{1}{2}}}(\kappa R)^{\frac{1}{4}}e^{-i\vartheta R^{\frac{3}{2}}}\frac{1}{2i\pi^{\frac{1}{2}}}(\kappa R)^{-\frac{1}{4}}\\
&\left(e^{i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)}-e^{-i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\\
&=\frac{\chi\alpha\varepsilon e^{i\frac{\pi}{4}}}{(Re^{i\theta}-\omega'\mp i\xi)A}e^{\varpi x R^{\frac{1}{2}}}e^{-i\vartheta R^{\frac{3}{2}}}e^{-i\frac{\pi}{4}}\left(e^{i\frac{\pi}{2}}e^{i\vartheta R^{\frac{3}{2}}}-e^{-i\vartheta R^{\frac{3}{2}}}\right)\\
&=\frac{\chi\alpha\varepsilon}{(Re^{i\theta}-\omega'\mp i\xi)A}e^{\varpi x R^{\frac{1}{2}}}\left(i-e^{-2i\vartheta R^{\frac{3}{2}}}\right)\\
&\approx\frac{i\chi\alpha\varepsilon}{(Re^{i\theta}-\omega'\mp i\xi)A}e^{\varpi x R^{\frac{1}{2}}}
\end{aligned}
\tag{2.142}
$$

where from (B.13) we know that $\vartheta_i < 0$. For the sector $-\pi < \theta < -\frac{2\pi}{3}$ we use the formula (B.34) for $\mathrm{Ai}(y_0)$, formula (B.42) for $|\det \mathbf{M}(y_0)|$ and (B.53), (B.54) for the resonant states $\psi_\omega(x)$.

$$
\begin{aligned}
&\psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|}\\
&\approx\frac{\chi\pi^{-\frac{1}{2}}}{Re^{i\theta}-\omega'\mp i\xi}(\tilde\kappa R)^{-\frac{1}{4}}e^{-\tilde\vartheta R^{\frac{3}{2}}-\tilde\varpi x R^{\frac{1}{2}}}\frac{1}{2}\pi^{-\frac{1}{2}}(\tilde\kappa R)^{-\frac{1}{4}}e^{-\tilde\vartheta R^{\frac{3}{2}}}\pi\\
&=\frac{\chi}{2(Re^{i\theta}-\omega'\mp i\xi)}(\tilde\kappa R)^{-\frac{1}{2}}e^{-2\tilde\vartheta R^{\frac{3}{2}}}e^{-\tilde\varpi x R^{\frac{1}{2}}}
\end{aligned}
\tag{2.143}
$$

From (B.32), (B.48) and (B.49) we know that $\tilde\vartheta_r, \tilde\vartheta_i, \tilde\varpi_r, \tilde\varpi_i > 0$ so for the sector $-\pi < \theta < -\frac{2\pi}{3}$ the integrand decays independently of x but for the first sector $-\frac{2\pi}{3} < \theta < 0$ the integrand decays only if $x < 0$ in the limit $R \to \infty$. So from (2.141) we get in the limit $R \to \infty$

$$
\frac{2\pi}{i}\sum_j \mathrm{Res}\left(\psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det \mathbf{M}(y_0)|},\omega_j\right) = P_\xi(\omega') \tag{2.144}
$$

In the following we compute the residues through the usual formula, but before that we express the argument using (2.114) for resonant states. For $x > 0$ we have

$$\begin{aligned}
&\psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det\mathbf{M}(y_0)|}\\
&= i\chi\left[-\left(\frac{\overline{\det\mathbf{M}}(y_0)}{\det\mathbf{M}(y_0)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y(x))+\left(\frac{\det\mathbf{M}(y_0)}{\overline{\det\mathbf{M}}(y_0)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y(x))\right]\\
&\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det\mathbf{M}(y_0)|}\\
&= i\chi\left[-\frac{1}{\det\mathbf{M}(y_0)}\mathrm{Ci}^+(y(x))+\frac{1}{\overline{\det\mathbf{M}}(y_0)}\mathrm{Ci}^-(y(x))\right]\frac{\mathrm{Ai}(y_0)}{\omega-\omega'\mp i\xi}
\end{aligned}\tag{2.145}$$

and for $x < 0$ we have

$$\begin{aligned}
\psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det\mathbf{M}(y_0)|} &= \frac{2\chi}{\pi|\det\mathbf{M}(y_0)|}\mathrm{Ai}(y(x))\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det\mathbf{M}(y_0)|}\\
&= \frac{2\chi}{\pi|\det\mathbf{M}(y_0)|^2}\frac{\mathrm{Ai}(y(x))\mathrm{Ai}(y_0)}{\omega-\omega'\mp i\xi}
\end{aligned}\tag{2.146}$$

Since the closed contour Γ_R is on the lower half of the complex frequency plane, we are interested in those poles that are located in that area. One can see, that the poles are determined by the zeros of $\det\mathbf{M}(y_0)$. From previous experience we know that they are on the lower half-plane so the zeros of $\overline{\det\mathbf{M}}(y_0)$ are on the upper half, so we consider only those terms with $\det\mathbf{M}(y_0)$. We can then write the residues in (2.144) as

$$\begin{aligned}
&\mathrm{Res}\left(\psi_\omega(x)\frac{1}{\omega-\omega'\mp i\xi}\frac{\mathrm{Ai}(y_0)}{|\det\mathbf{M}(y_0)|},\omega_j\right)\\
&= \chi\lim_{\omega\to\omega_j}\left(\frac{\omega-\omega_j}{\det\mathbf{M}(y_0)}\right)\begin{cases}\frac{2\mathrm{Ai}_j(y_0)}{\pi\overline{\det\mathbf{M}}_j(y_0)(\omega_j-\omega'\mp i\xi)}\mathrm{Ai}_j(y(x)) & x<0\\ -i\frac{\mathrm{Ai}_j(y_0)}{\omega_j-\omega'\mp i\xi}\mathrm{Ci}_j^+(y(x)) & x>0\end{cases}\\
&= \frac{1}{\omega_j-\omega'\mp i\xi}\psi_j(x)
\end{aligned}\tag{2.147}$$

with

$$\psi_j(x) = \lim_{\omega\to\omega_j}\left(\frac{\omega-\omega_j}{\det\mathbf{M}(y_0)}\right)\chi\begin{cases}\frac{2\mathrm{Ai}_j(y_0)}{\pi\overline{\det\mathbf{M}}_j(y_0)}\mathrm{Ai}_j(y(x)) & x<0\\ -i\mathrm{Ai}_j(y_0)\mathrm{Ci}_j^+(y(x)) & x>0\end{cases}\tag{2.148}$$

where we denoted $\mathrm{Ai}_j(y(x)) = \mathrm{Ai}(y(x))\Big|_{\omega\to\omega_j}$, $\det\mathbf{M}_j(y_0) = \det\mathbf{M}(y_0)\Big|_{\omega\to\omega_j}$ and so on. We can see that the functions $\psi_j(x)$ are proportional to the resonant states (2.114). The expression

(2.144) can be then rewritten as

$$P_\xi(\omega') = \frac{2\pi}{i} \sum_j \frac{\psi_j(x)}{\omega_j - \omega' \mp i\xi} \tag{2.149}$$

Returning back to (2.138) we can use (2.149) taking the limit for ξ and we get

$$\begin{aligned}
\lim_{\xi\to 0} f^{\pm}(x) &= \lim_{\xi\to 0} \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x)\Upsilon^{\pm}(\omega,\omega')\mathrm{d}\omega\mathrm{d}\omega' \\
&= \mp\frac{2i\chi\varepsilon}{\pi\rho^2} \int_{-\infty}^{\infty} a(\omega') \frac{|\det \mathbf{M}(y_0')|}{\mathrm{Ai}(y_0')} \sum_j \frac{2\pi}{i(\omega_j - \omega')} \psi_j(x)\mathrm{d}\omega' \\
&= \sum_j \mp\frac{4\chi\varepsilon}{\rho^2} \int_{-\infty}^{\infty} a(\omega') \frac{|\det \mathbf{M}(y_0')|}{\mathrm{Ai}(y_0')(\omega_j - \omega')} \mathrm{d}\omega' \psi_j(x) \\
&= \sum_j c_j \psi_j(x)
\end{aligned} \tag{2.150}$$

where

$$c_j = \mp\frac{4\chi\varepsilon}{\rho^2} \int_{-\infty}^{\infty} a(\omega') \frac{|\det \mathbf{M}(y_0')|}{\mathrm{Ai}(y_0')(\omega_j - \omega')} \mathrm{d}\omega' \tag{2.151}$$

which ends the proof.

After we have done both of the proofs, we can discuss the qualitative difference between them. In the weaker proof from (2.112) and (2.113) we have that the requirement for the integrand in (2.110) to decay was, that the function $f(x)$ has to have its support in the negative real axis. It basically means, that the support can not enter the positive axis at all. On the other hand the stronger proof gives us a better specification. The integrand in (2.141) vanished if $x < 0$. This gives a possibility for $f(x)$ to have its compact support mixed on the negative and positive real axis. The infinite sum in (2.141) will converge for $x < 0$ and diverge otherwise. It is therefore a stronger condition.

2.5 Time dependent energy field for Dirac delta potential

Until now we have dealt with a constant energy field ε. Let us now consider its time dependency $\varepsilon(t)$. We are not considering space dependency because the size of the atom is very small compared to the space change of any laser fields. The form of this strong energy field takes some assumptions. First of all, the energy field which is a laser pulse should have a smaller frequency than the frequency of the wave function. The proposed form of this pulse is a Gaussian wave packet

$$\varepsilon(t) = e^{-\gamma(t-t_0)^2} \cos\left(\omega_\varepsilon(t-t_0) - \delta\right) \tag{2.152}$$

where ω_ε is the center frequency and γ, δ, t_0 are parameters. On Figure (2.5-1) we can see its plot for $\omega_\varepsilon = 0.02, \gamma = 10^{-5}, \delta = 0, t_0 = 900$ compared to the function $\mathrm{Re}\,[e^{i\omega_0 t}]$(red), where ω_0 is the eigenvalue of the ground eigenstate, which will be used as the initial condition for the numerical scheme $\Psi(0,t) = \psi(0)e^{-i\omega_0 t}$ (2.67), where $\psi(x)$ is (2.16) and $\omega_0 = -\frac{A^2}{2}$.

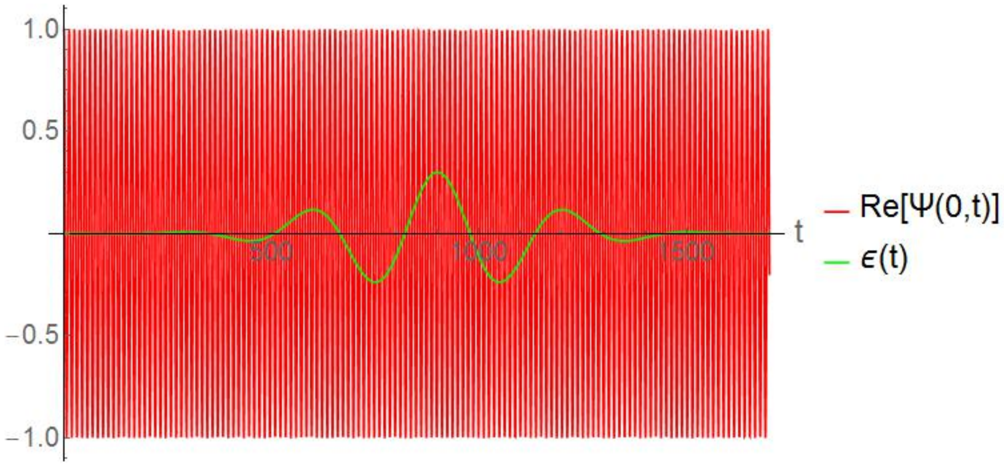


Figure 2.5-1: Plot of $\varepsilon(t)$ and the real part of the time dependency $\mathrm{Re}\left[e^{i\omega_0 t}\right]$ of the solution to the Schrödinger equation for Dirac delta potential. The eigenvalue $\omega_0 = -\frac{1}{2}$ belongs to the ground states computed using the parameters $A = 1$ (2.16).

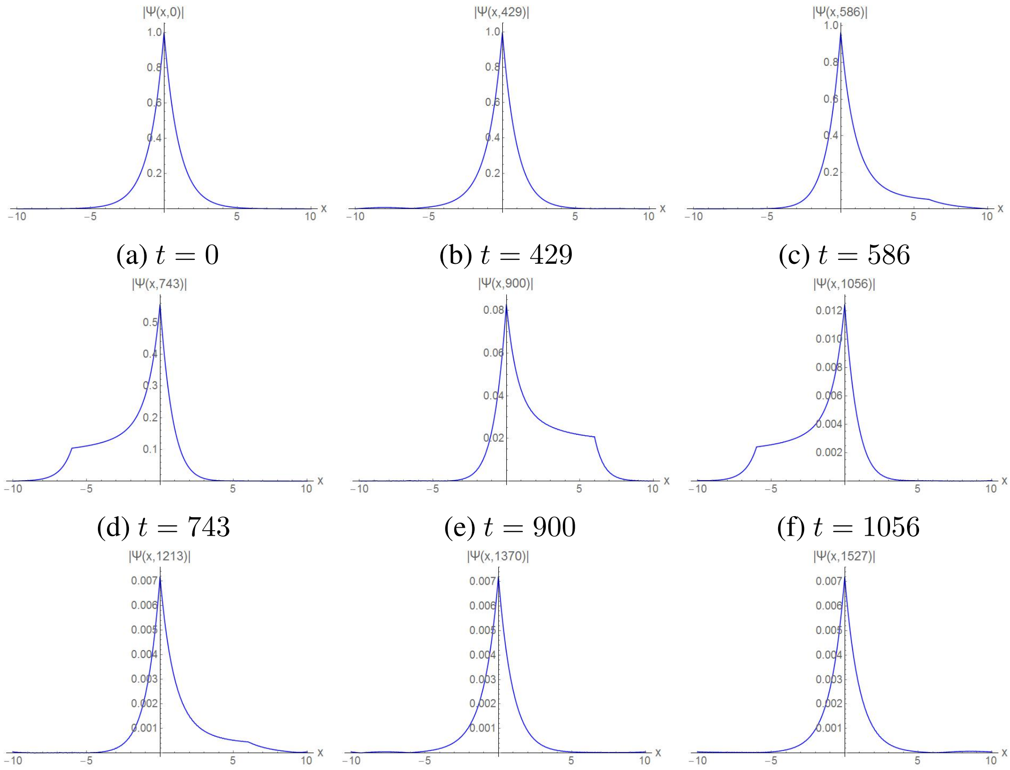
In our previous computation we assumed positive $\varepsilon > 0$. Here we deal with ε that changes signs. We need to figure out what the solution is for $\varepsilon < 0$. Let us define a new variable $\tilde{x} = -x$. Then we have

$$\begin{aligned} i\Psi_t(x,t) &= -\frac{1}{2}\partial_{xx}\Psi(x,t) - A\delta(x)\Psi(x,t) + |\varepsilon|x\Psi(x,t) \\ i\Psi_t(\tilde{x},t) &= -\frac{1}{2}\partial_{\tilde{x}\tilde{x}}\Psi(\tilde{x},t) - A\delta(\tilde{x})\Psi(\tilde{x},t) - |\varepsilon|\tilde{x}\Psi(\tilde{x},t) \end{aligned} \tag{2.153}$$

so for $\varepsilon < 0$ the solution is $\Psi(-x,t;|\varepsilon|)$, where $\Psi(x,t;|\varepsilon|)$ is the solution for $\varepsilon > 0$. We can solve the Schrödinger equation numerically with the same principle as before. The only difference will be that we solve it on the flipped x-axis across the y-axis when $\varepsilon < 0$, so these regions in time should be determined. From (2.152) it is easy to compute the intervals $[t_j, t_{j+1}]$, where t_j are the zeros of (2.152.

$$
\begin{aligned}
e^{-\gamma(t-t_0)^2}\cos\left(\omega_\varepsilon(t-t_0)-\delta\right) &< 0\\
\cos\left(\omega_\varepsilon(t-t_0)-\delta\right) &< 0\\
\frac{\pi}{2}+2j\pi < \omega_\varepsilon(t-t_0)-\delta &< \frac{3\pi}{2}+2j\pi, \quad j\in\mathbb{Z}\\
t_j=\frac{\pi}{2\omega_\varepsilon}+\frac{2j\pi}{\omega_\varepsilon}+\frac{\delta}{\omega_\varepsilon}+t_0 < t &< \frac{3\pi}{2\omega_\varepsilon}+\frac{2j\pi}{\omega_\varepsilon}+\frac{\delta}{\omega_\varepsilon}+t_0=t_{j+1}, \quad j\in\mathbb{Z}
\end{aligned} \tag{2.154}
$$

In those intervals, where $\varepsilon(t)<0$, the Schrödinger equation is solved on the flipped x-axis.



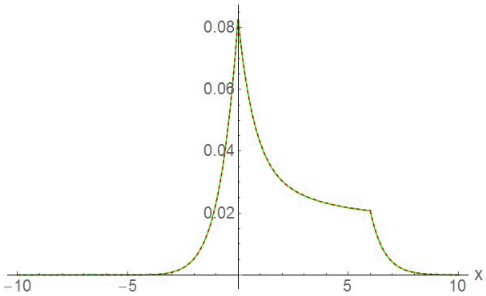
(a) $t=0$ (b) $t=429$ (c) $t=586$

(d) $t=743$ (e) $t=900$ (f) $t=1056$

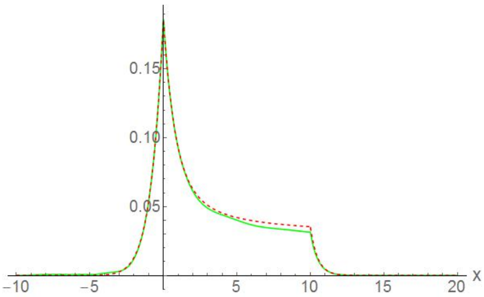
(g) $t=1213$ (h) $t=1370$ (i) $t=1527$

Figure 2.5-2: Absolute value of the numerical solution $\Psi(x,t)$ in particular times. The parameters that were used are $x_c=6, A=1, \theta=\frac{\pi}{2}$ and the numerical parameters are $L_1=L_2=10, N=265, k=4$.

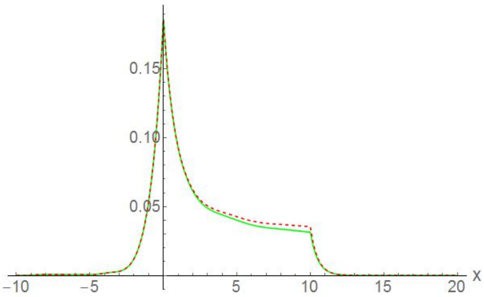
On Figure (2.5-2) were the parameters for the laser pulse the same as in previous figure and the data for the equation were $A=1, x_c=6, \theta=\frac{\pi}{2}$. We used the numerical parameters $L_1=L_2=10, N=265, k=4$ so the spatial grid parameter was $\Delta x=0.0759494$, where L_1, L_2 are the boundaries of the spatial space, N is the number of grid points and k is the order of the numerical scheme. The particular times were picked as the centres of the intervals (2.154). Let us choose the time corresponding to the middle of the 7-th interval $t=900$. The numerical solution to the Schrödinger equation at this time is in Figure (2.5-2e). We will compare this particular solution with an expansion using analytical resonant states as in (2.90) to see if we can find a detectable deviation. In this expansion we will use the perturbed ground state with ε as the starting state.



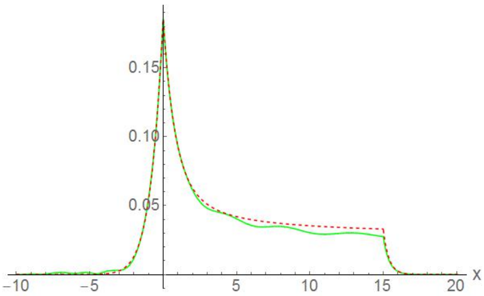
(a) $t = 900, x_c = 6, N = 10$



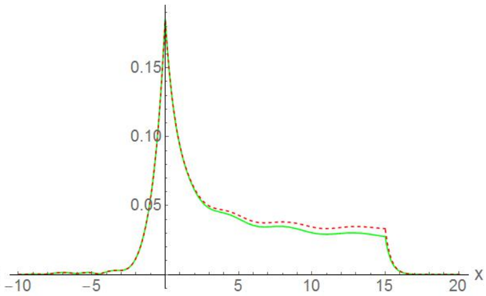
(b) $t = 873, x_c = 10, N = 1$



(c) $t = 873, x_c = 10, N = 20$



(d) $t = 873, x_c = 15, N = 1$



(e) $t = 873, x_c = 15, N = 50$

Figure 2.5-3: Comparing numerical solutions at times $t = 900, 873$ with $x_c = 6, 10, 15$ with the resonant states expansion using the various number of states from A-series.

In Figure (2.5-3a) we used the same parameters as in (2.5-2) but in (2.5-3b), (2.5-3c) we used $x_c = 10$ and picked the time $t = 873$ as in the last two subfigures, where we used $x_c = 15$. We can see that in (2.5-3a) the expansion looks perfect comparing to the numerical version, even if we used only ten modes. In the other cases, since we picked a different time, where the solution is not that smooth, we needed more modes to get the two functions closer. The improvement can be seen even if the matches are not perfect. However, for negative x, after we took more modes, the improvement is very satisfying. If we picked the exact same time in these cases as for (2.5-3a), then the match would be perfect.

We did not include any modes from C-series, because they do not give any significant contribution to the expansion, but on the other hand, modes from A-series showed themselves to be the uppermost of the modes, especially in the negative axis. We talked about these series right after the equation (2.41).

3 Finite well potential

This section contains a process of solving the Schrödinger equation with a different kind of potential. As for the delta case, we continue with ground states and then with resonant states. After that, we look into the completeness of these states through a weaker and a stronger proof.

We consider now the following potential

$$V(x) = \begin{cases} -V_0 & -d < x < d \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for some positive constants d and V_0. Figure (3-1) shows its plot.

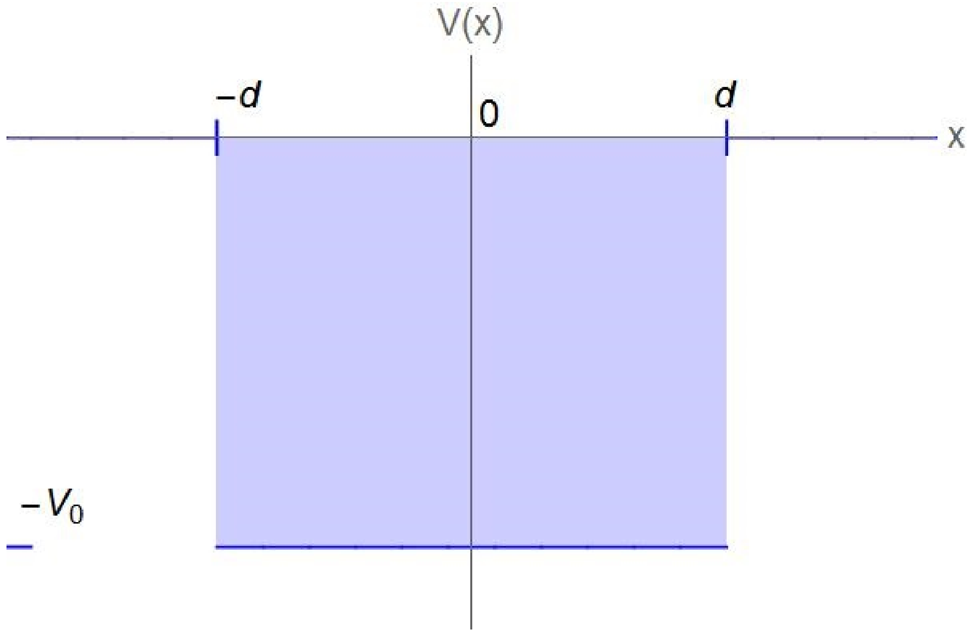


Figure 3-1: Finite well potential $V(x)$.

The equation we are going to solve will again be (3.2) with the overall energy (3.3).

$$i\partial_t \Psi(x,t) = H\Psi(x,t) \tag{3.2}$$

$$H = -\frac{1}{2}\partial_{xx} + V(x) - \varepsilon x \tag{3.3}$$

and the proposed form of the solution $\Psi(x,t)$ will be (2.3). After substituting this form, we get

$$\omega\psi(x) = -\frac{1}{2}\psi''(x) + V(x)\psi(x) - \varepsilon x\psi(x) \tag{3.4}$$

In the following sections, we investigate the ground state, that is $\varepsilon = 0$, and the resonant states $\varepsilon > 0$ as we did with the delta potential.

3.1 Ground state

In this part we consider no laser pulse $\varepsilon = 0$. The form of the potential creates regions where the solutions will be different, so let us denote $\psi_1(x)$ for $x < -d$ (region 1), $\psi_2(x)$ for $-d \leq x \leq d$ (region 2) and $\psi_3(x)$ for $d < x$ (region 3). The equations we are solving are

$$\psi_1''(x) = -2\omega\psi_1(x) \tag{3.5}$$

$$\psi_2''(x) = -2(\omega + V_0)\psi_2(x) \tag{3.6}$$

$$\psi_3''(x) = -2\omega\psi_3(x) \tag{3.7}$$

where ω is in fact a scaled energy of the state and since this is a ground state, we expect $0 < \omega < -V_0$ so we can write $\omega = -\gamma^2$, where $\gamma > 0$. Then the solutions to (3.5) - (3.7 are

$$\psi_1(x) = a_1 e^{k_1 x} + a_2 e^{-k_1 x} \tag{3.8}$$

$$\psi_2(x) = a_3 \cos(k_2 x) + a_4 \sin(k_2 x) \tag{3.9}$$

$$\psi_3(x) = a_5 e^{k_1 x} + a_6 e^{-k_1 x} \tag{3.10}$$

where $k_1^2 = 2\gamma^2$ and $k_2^2 = 2\left(V_0 - \gamma^2\right)$. In this case we prefer writing the solution in region 2 with goniometric functions rather than exponentials for convenience. Assuming that the solution should decay for large negative and positive x, we need to put $a_1 = a_6 = 0$. From the conditions it should obey, we get four equations for 4 left unknowns.

$$\psi_1(-d) = \psi_2(-d) \tag{3.11}$$

$$\psi_1'(-d) = \psi_2'(-d) \tag{3.12}$$

$$\psi_2(d) = \psi_3(d) \tag{3.13}$$

$$\psi_2'(d) = \psi_3'(d) \tag{3.14}$$

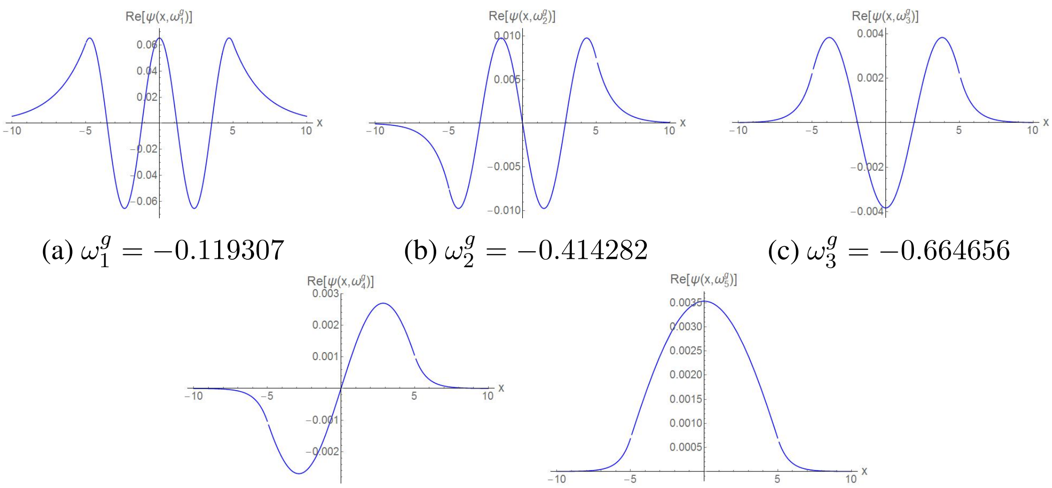
These conditions can be written in a system of equations in a matrix form as follows.

$$\mathbf{M} = \begin{pmatrix} e^{k_1 d} & -\cos(k_2 d) & \sin(k_2 d) & 0 \\ -k_1 e^{k_1 d} & -k_2 \sin(k_2 d) & -k_2 \cos(k_2 d) & 0 \\ 0 & \cos(k_2 d) & \sin(k_2 d) & -e^{k_1 d} \\ 0 & -k_2 \sin(k_2 d) & k_2 \cos(k_2 d) & -k_1 e^{k_1 d} \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.15}$$

To avoid the simple zero solution, we must set the determinant to be zero. This gives us an equation for ground state energy eigenvalues ω_j^g. The equation is

$$\begin{aligned} e^{2dk_1}\left(\left(k_1^2 - k_2^2\right)\sin(2dk_2) - 2k_1 k_2 \cos(2dk_2)\right) &= 0 \\ \frac{\left(k_1^2 - k_2^2\right)}{2k_1 k_2} + \cot\left(2dk_2\right) &= 0 \\ \det \mathbf{M}(\omega) = \frac{2\gamma^2 - 1}{2\sqrt{\gamma^2\left(V_0 - \gamma^2\right)}} + \cot\left(2d\sqrt{2\left(V_0 - \gamma^2\right)}\right) &= 0 \end{aligned} \tag{3.16}$$

The eigenvalues ω_j^g can be computed using some numerical method. The eigenvalue equation (3.16) has γ^2 as a variable which is positive. The starting points can be therefore chosen as grid points on the real line between 0 and V_0. The vector of coefficients are computed as in previous cases. The numerically computed ground state eigenvalues $\tilde{\omega}_j^g$ take non-exact form so $\det \mathbf{M}(\tilde{\omega}_j^g) \approx 0$, but not 0. With computing the eigenvalues of $\mathbf{M}$ we pick the least one and the its corresponding eigenvector will be the vector of coefficients. We have done this when we discussed the Dirac delta case on page 15.



(a) $\omega_1^g = -0.119307$ (b) $\omega_2^g = -0.414282$ (c) $\omega_3^g = -0.664656$

(d) $\omega_4^g = -0.849498$ (e) $\omega_5^g = -0.962183$

Figure 3.1-1: The real part of the ground eigenstates for a well potential for $V_0 = 1, d = 5$.

In our example of ground states in Figure (3.1-1) we used $V_0 = 1$ and $d = 5$. We got totally 5 eigenvalues ω_j^g for which correspond 5 eigenstates.

In order to predict how many of these eigenstates we get or how many solutions does (3.16) have, we separate the solution $\psi_2(x)$ into an odd, in which we set $a_3 = 0, a_4 \neq 0$ and an even part, where $a_3 \neq 0, a_4 = 0$. Applying the conditions (3.11)-(3.14) for the even case, we get

$$a_2 e^{-k_1 d} = a_4 \cos(k_2 d) \qquad a_2 k_1 e^{-k_1 d} = a_4 k_2 \sin(k_2 d), \quad \text{at } x = -d \tag{3.17}$$

$$a_5 e^{-k_1 d} = a_4 \cos(k_2 d) \qquad -a_5 k_1 e^{-k_1 d} = -a_4 k_2 \sin(k_2 d), \quad \text{at } x = d \tag{3.18}$$

and the odd case is

$$a_2 e^{-k_1 d} = -a_4 \sin(k_2 d) \qquad a_2 k_1 e^{-k_1 d} = a_4 k_2 \cos(k_2 d), \quad \text{at } x = -d \tag{3.19}$$

$$a_5 e^{-k_1 d} = a_4 \sin(k_2 d) \qquad -a_5 k_1 e^{-k_1 d} = a_4 k_2 \cos(k_2 d), \quad \text{at } x = d \tag{3.20}$$

Taking the ration in each case for $x = -d$ and $x = d$ we get

$$\frac{a_2 e^{-k_1 d}}{a_2 k_1 e^{-k_1 d}} = \frac{a_4 \cos(k_2 d)}{a_4 k_2 \sin(k_2 d)} \qquad \frac{a_5 e^{-k_1 d}}{-a_5 k_1 e^{-k_1 d}} = \frac{a_4 \cos(k_2 d)}{-a_4 k_2 \sin(k_2 d)} \qquad \text{even} \tag{3.21}$$

$$\frac{a_2 e^{-k_1 d}}{a_2 k_1 e^{-k_1 d}} = \frac{-a_4 \sin(k_2 d)}{a_4 k_2 \cos(k_2 d)} \qquad \frac{a_5 e^{-k_1 d}}{-a_5 k_1 e^{-k_1 d}} = \frac{a_4 \sin(k_2 d)}{a_4 k_2 \cos(k_2 d)} \qquad \text{odd} \tag{3.22}$$

from which we can see, that we the ratios for $x = -d$ and $x = d$ are the same for both cases, so we can write

$$k_2 \tan(k_2 d) = k_1 \qquad \text{even} \tag{3.23}$$

$$k_2 \cot(k_2 d) = -k_1 \qquad \text{odd} \tag{3.24}$$

Let us multiply these two equations with d, so we get

$$\mu \tan(\mu) = \nu \qquad \text{even} \tag{3.25}$$

$$\mu \cot(\mu) = -\nu \qquad \text{odd} \tag{3.26}$$

where $\mu = k_2 d = d\sqrt{2(V_0 - \gamma^2)}$ and $\nu = k_1 d = d\sqrt{2\gamma^2}$. Observe, that for μ and ν we have the following equation

$$\mu^2 + \nu^2 = 2d^2(V_0 - \gamma^2) + 2d^2\gamma^2 = 2d^2V_0 = \lambda^2 \tag{3.27}$$

where $\lambda^2 = 2d^2V_0$. All the points that satisfy (3.25) and (3.26) can be determined as the intersections of the circle (3.27) with the radius $d\sqrt{2V_0}$ and the curve $\mu\tan(\mu) = \nu$ for even states and $\mu\cot(\mu) = -\nu$ for odd states. Graphically it looks like on Figure (3.1-2).

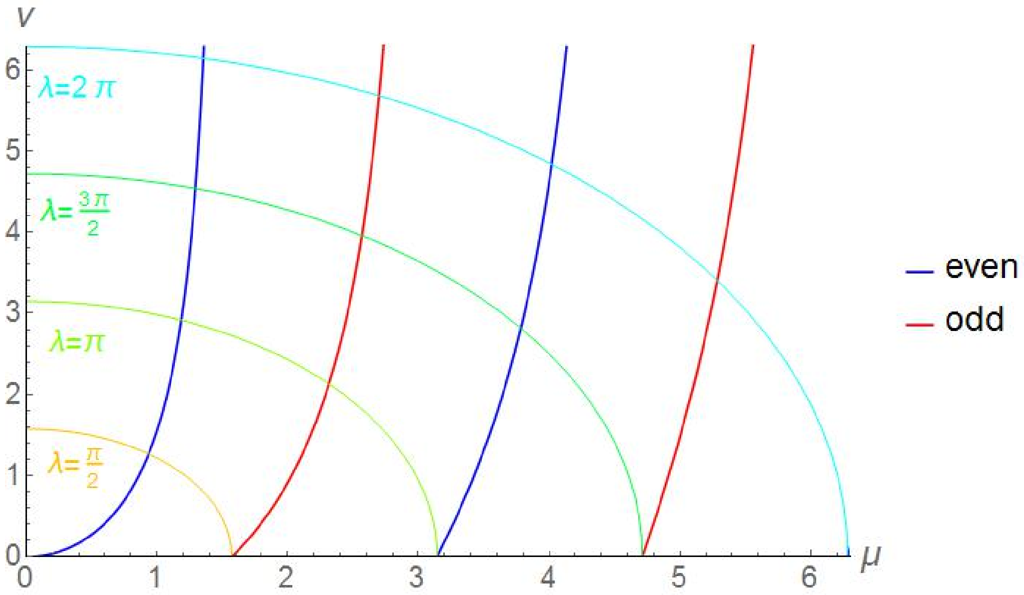


Figure 3.1-2: Graphical determination of the energy eigenvalues ω_j^g. The blue lines represent the curves for even states $\mu\tan(\mu) = \nu$ and the red line the odd states $\mu\cot(\mu) = -\nu$. The four different radii are $\lambda = \frac{\pi}{2}, \lambda = \pi, \lambda = \frac{3\pi}{2}$ and $\lambda = 2\pi$.

Here we can see the curves belonging to the even (3.25) and odd states (3.26) as well as four different quarter circles with different radius $\lambda = \frac{\pi}{2}, \lambda = \pi, \lambda = \frac{3\pi}{2}$ and $\lambda = 2\pi$. It is easy to deduce how the number of intersection depends on the radius. If the radius is $0 < \lambda < \frac{\pi}{2}$, then we have 1 intersection. For a radius $\frac{\pi}{2} < \lambda < \pi$ we have 2 intersections. Generally, we get N_i intersections or eigenstates if the radius λ is

$$(N_i - 1)\frac{\pi}{2} < \lambda < N_i\frac{\pi}{2} \tag{3.28}$$

We can get from this an explicit formula for this number by dividing the range (3.28) by $\frac{\pi}{2}$ and rounding up the number we get in the middle. We then get

$$N_i = \left\lceil \frac{2\lambda}{\pi} \right\rceil = \left\lceil \frac{2\sqrt{2d^2V_0}}{\pi} \right\rceil \tag{3.29}$$

For example in the case $\lambda = 5$ we would get $N_i = \lceil 3.1831 \rceil = 4$, which we can check in Figure (3.1-2) that it is correct.

3.2 Resonant state on a complex line

We solve now the well potential with a laser pulse for $\varepsilon > 0$ in (3.3). We already know from the Dirac delta case, that for large positive x, the solution grows beyond measure, so we skip this solution and jump right into solving it on a complex line as in (2.42). Let us therefore apply the knowledge from (2.43), (2.44). The Hamiltonian then is

$$H = D + V(x) - \varepsilon x, \quad D = \begin{cases} -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} & x < x_c \\ -\frac{1}{2}e^{-i2\theta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} & x > x_c \end{cases} \tag{3.30}$$

Hence, the x-axis is divided into 4 regions, where the differential equations using H in the equation (3.2) are

$$\psi_1''(x) + 2(\omega + \varepsilon x)\psi_1(x) = 0, \quad x < -d \tag{3.31}$$

$$\psi_2''(x) + 2(\omega + V_0 + \varepsilon x)\psi_2(x) = 0, \quad -d \leq x \leq d \tag{3.32}$$

$$\psi_3''(x) + 2(\omega + \varepsilon x)\psi_3(x) = 0, \quad d < x < x_c \tag{3.33}$$

$$\psi_4''(x) + \left[2\varepsilon x_c e^{i2\theta} + 2\varepsilon e^{i3\theta}(x - x_c) + 2\omega e^{i2\theta}\right]\psi_4(x) = 0, \quad x_c < x \tag{3.34}$$

which construct the overall solution $\psi(x)$. The solutions to these equations are Airy functions, but under some variable transformations we are going to introduce. It is clear that the same transformation $y_1(x)$ is for $\psi_1(x)$ and $\psi_3(x)$, that is $y_1(x) = y(x)$, where we introduced $y(x)$ in (2.24).

$$y_1(x) = y(x) = -2\alpha(\varepsilon x + \omega) \tag{3.35}$$

where $\alpha = (2\varepsilon)^{-\frac{2}{3}}$. The transformation $y_4(x)$ for $\psi_4(x)$ was introduced in (2.47). The only new transformation is for $\psi_2(x)$.

$$y_2(x) = -2\alpha_2(\omega + V_0 + \varepsilon x) \tag{3.36}$$

for some constant α_2. With this change of variable the equation (3.32) becomes

$$\alpha_2^2 4\varepsilon^2 \psi_2''(y_2(x)) - \frac{y_2(x)}{\alpha_2}\psi_2(y_2(x)) = 0$$
$$\psi_2''(y_2(x)) - \frac{y_2(x)}{\alpha_2^3 4\varepsilon^2}\psi_2(y_2(x)) = 0 \Rightarrow \alpha_2 = \left(4\varepsilon^2\right)^{-\frac{1}{3}} \tag{3.37}$$
$$\psi_2''(y_2) - y_2\psi_2(y_2) = 0 \tag{3.38}$$

Observe, that $\alpha_2 = \alpha$. We use all the information to write the overall solution as

$$\psi(x) = \begin{cases} \psi_1(x) = a_1\text{Ai}(y_1(x)) + a_2\text{Bi}(y_1(x)) & x < -d \\ \psi_2(x) = a_3\text{Ai}(y_2(x)) + a_4\text{Bi}(y_2(x)) & -d < x < d \\ \psi_3(x) = a_5\text{Ai}(y_1(x)) + a_6\text{Bi}(y_1(x)) & d < x < x_c \\ \psi_4(x) = a_7\text{Ci}^+(y_4(x)) + a_8\text{Ci}^-(y_4(x)) & x_c < x \end{cases} \tag{3.39}$$

We set $a_8 = 0$ because of the same reasons as explained at the page (12), that Ci^- represents an ingoing wave. Also, we set $a_2 = 0$, because for large negative x the function $\text{Ai}(y_1(x))$ decays. The form of $\psi_3(x)$ is decided based on the same principles as we did in (2.51) - (2.53). These principles were 2 conditions at $x = x_c$: the continuity and the continuity of the derivative. According to this $a_5 = ia_7, a_6 = a_7$. This tells us, that $\psi_3(x) = a_7\text{Ci}^+(y_1(x))$, but we did not apply the continuity conditions at the points $x = -d, d$, which is going to determine the rest of the coefficients. Summing it up, we get

$$\psi(x) = \begin{cases} \psi_1(x) = a_1\text{Ai}(y_1(x)) & x < -d \\ \psi_2(x) = a_3\text{Ai}(y_2(x)) + a_4\text{Bi}(y_2(x)) & -d < x < d \\ \psi_3(x) = a_7\text{Ci}^+(y_1(x)) & d < x < x_c \\ \psi_4(x) = a_7\text{Ci}^+(y_4(x)) & x_c \leq x \end{cases} \tag{3.40}$$

As usual, we set up a system of equations based on continuity conditions at $x = -d, d$. In this system we do not consider $\psi_4(x)$ and the conditions at $x = x_c$ either, because those were already used. Let us denote the following constants to make the computations easier.

$$\begin{aligned} &A_0 = \mathrm{Ai}(y_1(-d)) \quad A_1 = \mathrm{Ai}(y_2(-d)) \quad B_1 = \mathrm{Bi}(y_2(-d)) \\ &A_2 = \mathrm{Ai}(y_2(d)) \quad\;\; B_2 = \mathrm{Bi}(y_2(d)) \quad\;\; C_3 = \mathrm{Ci}^+(y_1(d)) \end{aligned} \tag{3.41}$$

With this notation and the fact that $y_1'(x) = y_2'(x) = y_3'(x)$ we can write the conditions as

$$\begin{aligned} a_1 A_0 &= a_3 A_1 + a_4 B_1 \\ a_1 A_0' &= a_3 A_1' + a_4 B_1' \\ a_3 A_2 + a_4 B_2 &= a_5 C_3 \\ a_3 A_2' + a_4 B_2' &= a_5 C_3' \end{aligned} \tag{3.42}$$

which we can write in a matrix form as

$$\mathbf{M}(\omega) = \begin{pmatrix} A_0 & -A_1 & -B_1 & 0 \\ A_0' & -A_1' & -B_1' & 0 \\ 0 & A_2 & B_2 & -C_3 \\ 0 & A_2' & B_2' & -C_3' \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.43}$$

Using a mathematical engine, the determinant $E(\omega) = \det \mathbf{M}(\omega)$ can be easily found.

$$\begin{aligned} \det \mathbf{M}(\omega) = E(\omega) &= A_1 C_3 A_0' B_2' - A_1 B_2 A_0' C_3' - B_1 C_3 A_0' A_2' + A_2 B_1 A_0' C_3' - A_0 C_3 A_1' B_2' \\ &+ A_0 B_2 A_1' C_3' + A_0 C_3 A_2' B_1' - A_0 A_2 B_1' C_3' \\ &= A_1 A_0' (C_3 B_2' - B_2 C_3') - B_1 A_0' (C_3 A_2' - A_2 C_3') - A_0 A_1' (C_3 B_2' - B_2 C_3') \\ &+ A_0 B_1' (C_3 A_2' - A_2 C_3') \\ &= (A_0 A_1' - A_0' A_1)(B_2 C_3' - B_2' C_3) - (A_0 B_1' - A_0' B_1)(A_2 C_3' - A_2' C_3) \end{aligned} \tag{3.44}$$

We are interested for which ω the expression above $E(\omega)$ is zero. We must not forget that in transformations being in (3.41) is the variable ω present.

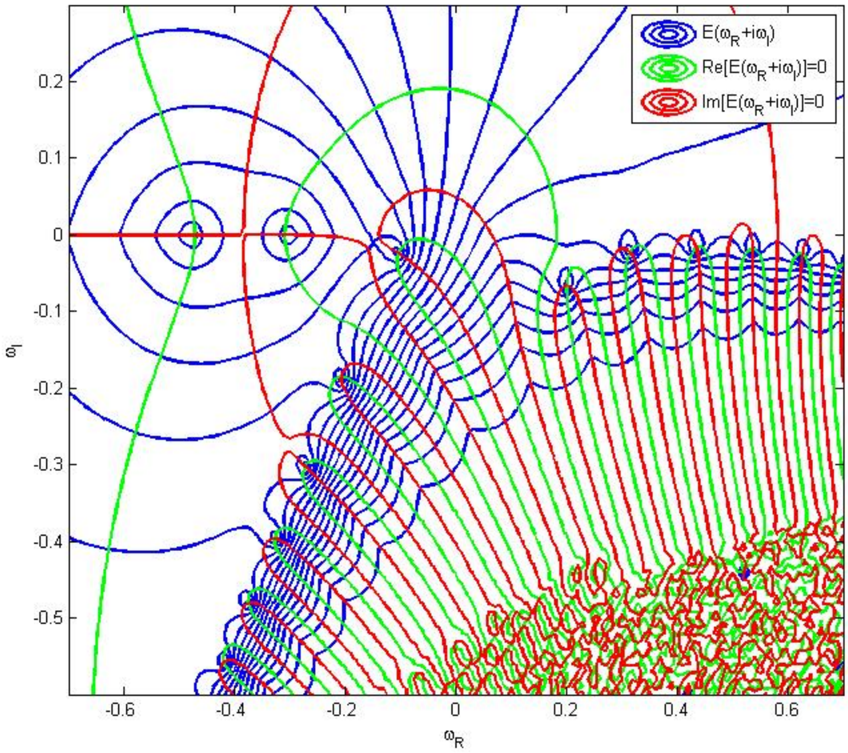
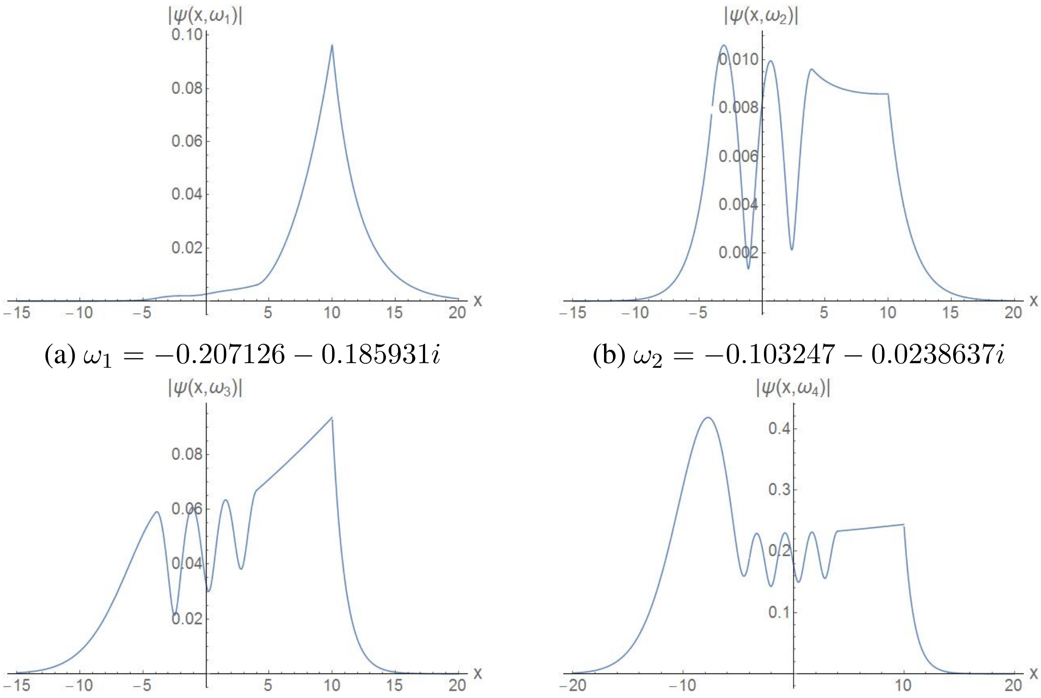


Figure 3.2-1: Contour plot of $E(\omega_R + i\omega_I)$. The green and red lines are the zero contours of the real and imaginary part of $E(\omega_R + i\omega_I)$, where the parameters were $\varepsilon = 0.03, V_0 = 0.5, d = 4$. The blue lines are contours of various values of the modified eigenvalue formula $\tilde{E}(\omega_R + i\omega_I, \rho)$ (3.45) for $\rho = 0.4$.

To see better the zeros of E, we introduce a modified eigenvalue formula via a conversion of the absolute value $|E|$ in the following way

$$\tilde{E} = \left(1 - \left(1 + |E|^{0.3}\right)^{-1} + \rho\right)^{-1} \tag{3.45}$$

On Figure (3.2-1 is the contour plot both of E and the modified $\tilde{E}$. The green and the red lines are the zero contours of E and the blue line are contours of $\tilde{E}$. The zeros can be easily located on the picture.



(a) $\omega_1 = -0.207126 - 0.185931i$

(b) $\omega_2 = -0.103247 - 0.0238637i$

(c) $\omega_3 = 0.201005 - 0.0670688i$

(d) $\omega_4 = 0.31411 - 0.022537i$

Figure 3.2-2: Plots of four resonant states belonging to four different eigenvalues.

To have the full form of resonant states for well potential we use the same principle as we did earlier in the Dirac delta case. The conditions (3.42) are written in a matrix form as

$$\mathbf{M}(\omega) = \begin{pmatrix} A_0 & -A_1 & -B_1 & 0 \\ A_0' & -A_1' & -B_1' & 0 \\ 0 & A_2 & B_2 & -C_3 \\ 0 & A_2' & B_2' & -C_3' \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.46}$$

For a particular ω_j computed numerically from $E(\omega)$ (3.44), the determinant of $\mathbf{M}(\omega_j)$ is zero. The vector of coefficients is therefore the eigenvector belonging to the least eigenvalue of $\mathbf{M}(\omega_j)$. We can recall this procedure from our previous case where we computed the eigenvalues for the Dirac delta potential on page (14). In Figure (3.2-2 we picked 4 eigenvalues and plotted the corresponding resonant states.

3.2.1 Numerical solution

In this section we will find a numerical solution to the Schrödinger equation for the Hamiltonian (3.30). We modify the equation into the form

$$\begin{aligned} i\Psi_t(x,t) &= -\frac{1}{2}\partial_{xx}\Psi(x,t) + V(x)\Psi(x,t) - \varepsilon x\Psi(x,t) \\ \Psi_t(x,t) &= i\left[\frac{1}{2}\partial_{xx} - V(x) + \varepsilon x\right]\Psi(x,t) \\ \vec{\psi}'(t) &= i\mathcal{O}(x)\Psi(x,t) = i\mathbf{M}\vec{\psi}(t) \end{aligned} \tag{3.47}$$

The operator $\mathcal{O}(x) = \frac{1}{2}\partial_{xx} - V(x) + \varepsilon x$ will be approximated by a matrix $\mathbf{M}$ and the solution $\Psi(x,t)$ by a discrete vector $\vec{\psi}(t)$, where the discretization is in the x variable as $\psi_i(t) = \Psi(x_i,t)$, where x_i is explained later. To achieve this, we approximate the solution by a polynomial

$$\begin{aligned} \Psi(x,t) \approx P(x) &= a_0 + a_1(x-x_n) + a_2(x-x_n)^2 + \ldots + a_k(x-x_n)^k \\ &= \sum_{i=0}^{i=k} a_i(x-x_n)^i \end{aligned} \tag{3.48}$$

where k is the order of the polynomial and x_n is a discrete point on the axis. First, we choose the boundaries for our space to be $[-L_1, L_2]$. The axis is discretized such that $x_n = x_0 + n\Delta x, n = 0, 1, \ldots, N$, where $x_0 = -L_1, x_N = L_2$. The matrix $\mathbf{M}$ has then size $(N-1) \times (N-1)$. Note, that this discretization takes place on the complex line, so the right boundary L_2 should be chosen such that the breaking point x_c (2.42) is included in it. The boundary values x_0, x_N are known from the initial condition $\Psi(x,0) = \psi(x,\omega_j)$, where $\psi(x,\omega_j)$ is the ground states belonging to the eigenvalue ω_j. Let us denote $P(x_n) = \psi_n$. For a particular point x_n the operator $\mathcal{O}(x_n)$ becomes

$$\mathcal{O}(x_n) = a_2 + \varepsilon x_n a_0, \quad |d| < x_n \tag{3.49}$$

$$\mathcal{O}(x_n) = a_2 + (\varepsilon x_n + V_0)a_0, \quad |d| > x_n \tag{3.50}$$

The coefficients a_i are computed in the following way. Let the order of the polynomial be $k = 2$. In this case we have 3 unknown coefficients in the polynomial, so we need to use 3 points, namely x_{n-1}, x_n, x_{n+1}. The polynomial is now evaluated in these 3 points and we get a system of 3 unknowns.

$$\begin{pmatrix} 1 & x_{n-1} - x_n & (x_{n-1} - x_n)^2 \\ 1 & 0 & 0 \\ 1 & x_{n+1} - x_n & (x_{n+1} - x_n)^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \psi_{n-1}(t) \\ \psi_n(t) \\ \psi_{n+1}(t) \end{pmatrix} \tag{3.51}$$

This gives us a_0 and a_2 as a linear combination of the discrete solutions. We substitute these into (3.49) or (3.50) and the result is a linear combination of $\psi_{n-1}(t), \psi_n(t), \psi_{n+1}(t)$. The coefficients of this combination represent the n-th line in the matrix $\mathbf{M}_{n*}$.

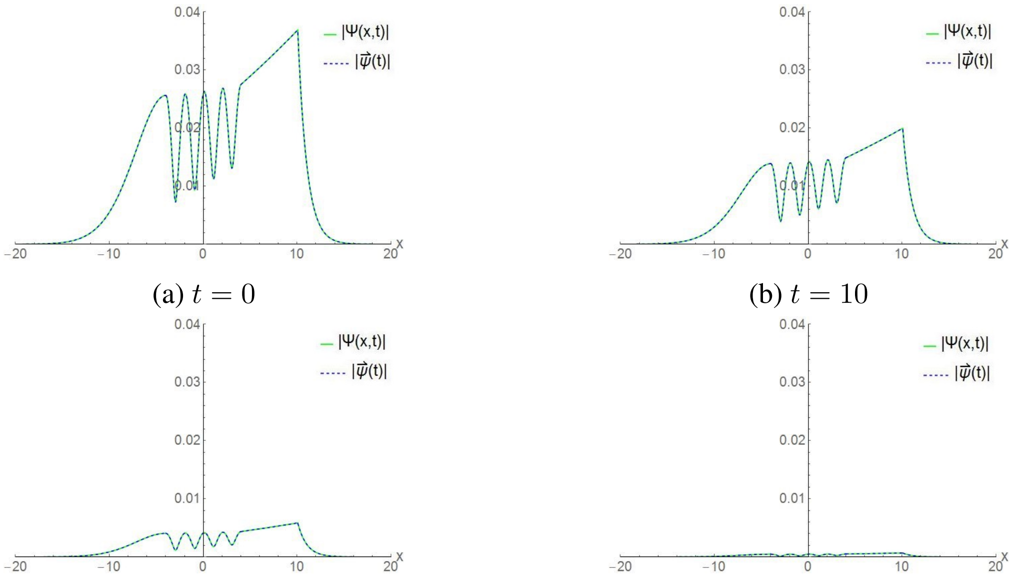
There are three special points that should be mentioned separately. The first is the breaking point where the "complex part" begins x_c. This point should also be a point on our grid, because this is the changing point to the complex part of the contour. The two other important points are $x_n = -d, d$. The same applies here too except the complex part. Since these are transition points, we treat them in a different way. We use the conditions for the derivatives (3.12), (3.14). Using (3.48) we get

$$a_1^+ - a_1^- = 0 \tag{3.52}$$

for $x_n = -d, d$, where the coefficients a_1^+ and a_1^- are computed as illustrated in (3.51) but using the points x_{n-2}, x_{n-1}, x_n for a_1^- and the points x_n, x_{n+1}, x_{n+2} for a_1^+. After doing so, these are then substituted back to (3.52) and we get a linear combination which we can write

$$\begin{aligned} \psi_n(t) &= F(\psi_{n-2}(t), \psi_{n-1}(t), \psi_{n+1}(t), \psi_{n+2}(t)), \quad /\partial_t \\ \psi_n'(t) &= \psi_{n-2}' \partial_{\psi_{n-2}} F + \psi_{n-1}' \partial_{\psi_{n-1}} F + \psi_{n+1}' \partial_{\psi_{n+1}} F + \psi_{n+2}' \partial_{\psi_{n+2}} F \end{aligned} \tag{3.53}$$

where the function $F(\cdot)$ is some linear combination of the arguments, so the derivatives $\partial_{\psi_i} F$ are the coefficients of that linear combination. The derivatives $\psi_i'(t)$ in (3.53) are exactly the i-th lines in the right hand-side of the ODE system (3.47). For each of them we use the corresponding i-th line in the right hand-side. After substituting all these expressions into (3.53) we get the representation of the lines corresponding to the points $x_n = -d, d$. The coefficients in this line are then the coefficients in matrix $\mathbf{M}$.



(a) $t = 0$ (b) $t = 10$ (c) $t = 30$ (d) $t = 65$

Figure 3.2.1-1: The exact solution $|\Psi(x,t)|$(blue dashed) and the numeric solution $\left|\vec{\psi}(t)\right|$(green) for square well potential plotted in one picture at four different times $t = 0, t = 10, t = 30$ and $t = 65$. This resonant state belongs to the eigenvalue $\omega_1 = 0.227766 - 0.0613823i$.

Figure (3.2.1-1) shows the numerical solution for the case $\omega_1 = 0.227766 - 0.0613823i$. We used the following parameters: $L_1 = 20, L_2 = 15, N = 171, k = 4$. For the initial condition we chose $\varepsilon = 0.03, x_c = 5$ and $A = 1$. One can see that they match almost perfectly at all times and that it completely vanishes after some time.

3.2.2 Resonant states as a linear basis

Referring to the section about the Dirac delta resonant states as a linear basis, we can show in a similar way that the resonant states of well potential also share this property of bi-orthogonality. We show that the modes $\psi_k(x) = \psi(x, \omega_k)$ in (3.40) can be used for expansion of any function $f(x)$ as

$$f(x) = \sum_{i=-\infty}^{\infty} c_k \psi_k(x) \tag{3.54}$$

The eigenvalues ω_k are of course entailed in the variable transformations $y_1(x), y_2(x)$ that can be found at (3.35) and (3.36). The same inner product of the modes will be used (2.77) as well as the same conjugation (2.76). Although, we have extra conditions at $x = -d, d$ this time, but the vector spaces defined in (2.84), (2.85) do not change, because it is actually the condition at

$x = x_c$ that defines the two spaces. Let φ_λ and ψ_μ be two eigenfunctions corresponding to the eigenvalues λ and μ. The inner product

$$(\varphi_\lambda, \psi_\mu) = \int_{-\infty}^{x_c} \overline{\varphi_\lambda(x)}\psi_\mu(x)\mathrm{d}x + i\int_{x_c}^{\infty} \overline{\varphi_\lambda(x)}\psi_\mu(x)\mathrm{d}x \tag{3.55}$$

works with functions from the following spaces

$$\{\varphi_\lambda\} \in \mathcal{V}_1 = \left\{\varphi_\lambda(x) | \varphi_\lambda'^{x_c^+} = -i\varphi_\lambda'^{x_c^-}\right\} \tag{3.56}$$

$$\{\psi_\mu\} \in \mathcal{V}_2 = \left\{\psi_\mu(x) | \psi_\mu'^{x_c^+} = i\psi_\mu'^{x_c^-}\right\} \tag{3.57}$$

where $\varphi'^{a^\pm} = \lim_{\epsilon\to 0} \varphi'(a \pm \epsilon)$. Using the eigenvalue problems $H\varphi_\lambda = \lambda\varphi_\lambda$, $H^\dagger\psi_\mu = \mu\psi_\mu$, where $H^\dagger$ is the adjoint to H, we have

$$(\lambda - \mu)(\varphi_\lambda, \psi_\mu) = (\lambda\varphi_\lambda, \psi_\mu) - (\varphi_\lambda, \mu\psi_\mu) = (H\varphi_\lambda, \psi_\mu) - \left(\varphi_\lambda, H^\dagger\psi_\mu\right) \tag{3.58}$$

Showing the bi-orthogonality then reduces to showing that the Hamiltonian H is self-adjoint. If we look at the actual form of H for the well potential (3.30) with $\theta = \frac{\pi}{2}$ and the previous computations (2.87), we can see that all the trouble was at the second derivative. The other terms in H did not get involved whatsoever, which means that H is self-adjoint also in this case and the resonant states for well potential form a bi-orthogonal system of functions.

We can now try to test it out by creating a random linear combination of the modes $\tilde{f}(x)$ as we did in the Dirac case. So first, we have the function $\tilde{f}(x)$ created with random coefficients $\tilde{c}_k$ in (3.54) and then we use this to compute the actual coefficients c_k as in (2.91) to form a new function $f(x)$ (2.90).

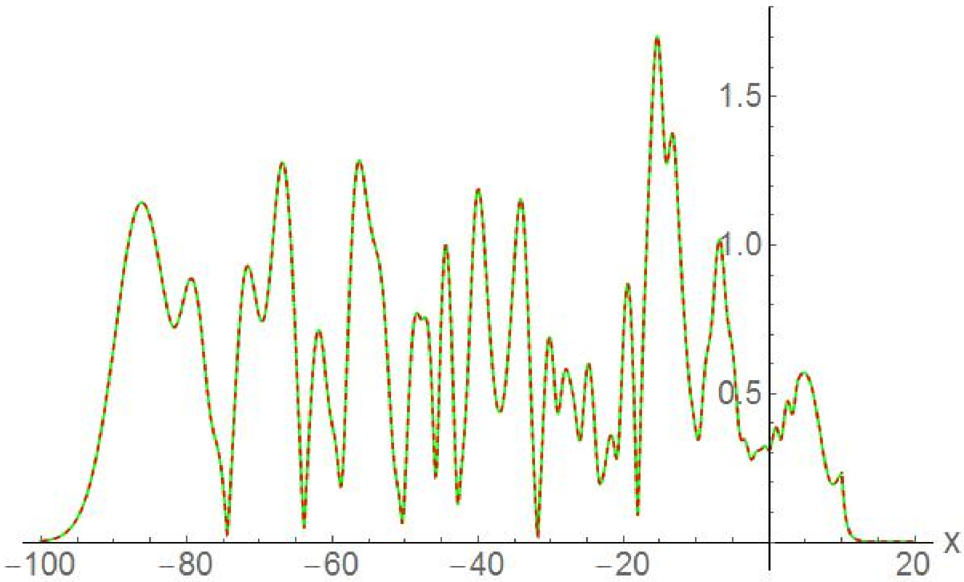
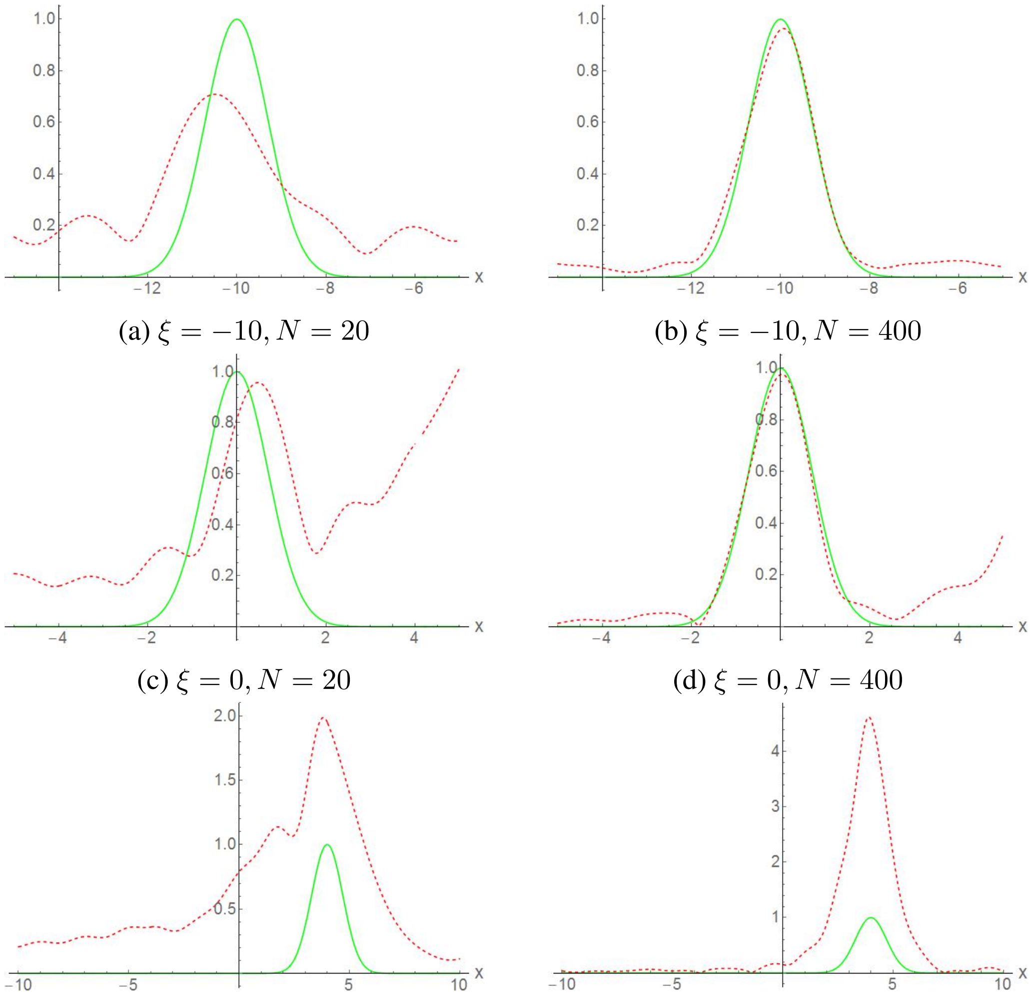


Figure 3.2.2-1: Comparing the function $f(x)$ (red dashed), that was computed using the biorthogonal product of resonant states and the randomly generated function $\tilde{f}(x)$ (green).

We can see the results on Figure (3.2.2-1). The parameters we used for the resonant states were $d = 4, x_c = 10, \varepsilon = 0.03$ and $V_0 = 0.5$. We totally used $N = 50$ states from A-series. On the next figures we tried to approximate a Gaussian $g(x) = e^{-(x-\xi)^2}$ using first $N = 20$ modes and then $N = 400$ modes. Also, the location of the Gaussian was changed three times, in particular $\xi = -10, 0$ and $\xi = 4$. The parameters of the modes were the same as for Figure (3.2.2-1).



(a) $\xi = -10, N = 20$ (b) $\xi = -10, N = 400$

(c) $\xi = 0, N = 20$ (d) $\xi = 0, N = 400$

(e) $\xi = 4, N = 20$ (f) $\xi = 4, N = 400$

Figure 3.2.2-2: Comparing the Gaussian $g(x) = e^{-(x-\xi)^2}$ (green) located at three different places and approximated with $N = 20$ and $N = 400$ resonant modes (red dashed).

From Figure (3.2.2-2) it can be seen, that in the case $\xi = -20$ an improvement happened. We can conclude that in this case the expansion is a very good approximation. The same can be stated in the second case, but no in the third one. Although, if we look on the part for negative x in these cases, some improvement is present. It appears that the situation can be the same as in the Dirac delta case, where we proved that we the expansion converges only for those functions which have their support on the negative real axis. We will see later, that this condition is slightly different for the well potential.

In the following section we repeat the proofs of the completeness, but this time for the well potential modes.

3.3 On the completeness of the resonant states for well potential

This section is about inquiring into the completeness of the resonant states for well potential. It will contain again two kinds of proofs: a weaker and a stronger. Before that, we first derive the scattering form of the resonant states. It is to allow not only outgoing but also ingoing waves in the positive x direction represented by Ci^+ for outgoing and Ci^- for ingoing. We take the form (3.40) and rewrite it including the ingoing wave, rearranging the coefficients and dropping the complex line, because that was just for decaying purposes. Instead, the asymptotic behaviour will be important. So the scattering form of (3.40) is

$$\psi_\omega(x) = \begin{cases} a_1 \mathrm{Ai}(y_1(x)) & x < -d \\ a_2 \mathrm{Ai}(y_2(x)) + a_3 \mathrm{Bi}(y_2(x)) & -d < x < d \\ a_4 \mathrm{Ci}^+(y_1(x)) + a_5 \mathrm{Ci}^-(y_1(x)) & d < x \end{cases} \tag{3.59}$$

where $y_1(x)$ and $y_2(x)$ are from (2.24) and (3.36). For this function $\psi_\omega(x)$ there are 4 continuity condition at the walls of the well $x = -d, d$. Applying these we get a system

$$\mathbf{M}(\omega)\vec{a} = \begin{pmatrix} A_0 & -A_1 & -B_1 & 0 \\ A_0' & -A_1' & -B_1' & 0 \\ 0 & A_2 & B_2 & -C_3 \\ 0 & A_2' & B_2' & -C_3' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = a_5 \begin{pmatrix} 0 \\ 0 \\ D_3 \\ D_3' \end{pmatrix} \tag{3.60}$$

where we used the same notations as in (3.41) and with additional $D_3 = \mathrm{Ci}^-(y_1(d))$. Using a mathematical engine we can obtain a solution to the system (3.60).

$$a_1 = -a_5 \frac{(B_1 A_1' - B_1' A_1)(C_3 D_3' - C_3' D_3)}{\det \mathbf{M}(\omega)} = a_5 \frac{2i(A_2 B_2' - A_2' B_2)}{\pi \det \mathbf{M}(\omega)} = a_5 \frac{2i}{\pi^2 \det \mathbf{M}(\omega)} \tag{3.61}$$

$$a_2 = -a_5 \frac{(B_1 A_0' - B_1' A_0)(C_3 D_3' - C_3' D_3)}{\det \mathbf{M}(\omega)} = -a_5 \frac{2i(B_1 A_0' - B_1' A_0)}{\pi \det \mathbf{M}(\omega)} \tag{3.62}$$

$$a_3 = -a_5 \frac{(A_1 A_0' - A_1' A_0)(D_3 C_3' - D_3' C_3)}{\det \mathbf{M}(\omega)} = a_5 \frac{2i(A_1 A_0' - A_1' A_0)}{\pi \det \mathbf{M}(\omega)} \tag{3.63}$$

$$a_4 = a_5 \left(\frac{A_0'(B_1 D_3 A_2' - A_1 D_3 B_2' - D_3'(A_2 B_1 - A_1 B_2))}{\det \mathbf{M}(\omega)} \right. \\ \left. + \frac{A_0(-B_1' D_3 A_2' + A_2 D_3' B_1' + A_1'(B_2' D_3 - B_2 D_3'))}{\det \mathbf{M}(\omega)} \right) = -a_5 \frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)} \tag{3.64}$$

where we recognized that the numerator in (3.64) is actually $-\overline{\det \mathbf{M}}(\omega)$. We get to choose a_5 as we please. Let it be the same as in (2.98) $a_5 = -i\left(\frac{\det \mathbf{M}(\omega)}{\det \overline{\mathbf{M}}(\omega)}\right)^{\frac{1}{2}}$. Substituting this into (3.61)-(3.64) we can write the resonant states (3.59) as

$$\psi_\omega(x) = \chi \begin{cases} \frac{2}{\pi^2 |\det \mathbf{M}(\omega)|} \mathrm{Ai}(y_1(x)) & x < -d \\ \frac{2}{\pi |\det \mathbf{M}(\omega)|} \left[(B_1' A_0 - B_1 A_0') \mathrm{Ai}(y_2(x)) \right. & \\ \qquad \left. + (A_1 A_0' - A_1' A_0) \mathrm{Bi}(y_2(x)) \right] & -d < x < d \\ i \left(\frac{\det \overline{\mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)} \right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(x)) - i \left(\frac{\det \mathbf{M}(\omega)}{\det \overline{\mathbf{M}}(\omega)} \right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(x)) & d < x \end{cases} \quad (3.65)$$

with the normalization constant χ computed in Appendix (C).

3.3.1 Weaker proof

This subsection will contain the mentioned weaker proof of the completeness of the resonant states (3.65). Our staring point is the completeness relation for the scattering form of resonant states.

$$\int_{-\infty}^{\infty} \psi_\omega(x) \psi_\omega(x') \mathrm{d}\omega = \delta(x - x') \quad (3.66)$$

Define a function for each R

$$\mathcal{F}_R(x, x') = \int_{-R}^{R} \psi_\omega(x) \psi_\omega(x') \mathrm{d}\omega \quad (3.67)$$

that converges to $\delta(x - x')$ as R goes to infinity. We introduce a closed contour Γ_R as on Figure (3.3.1-1) on the lower complex frequency half-plane.

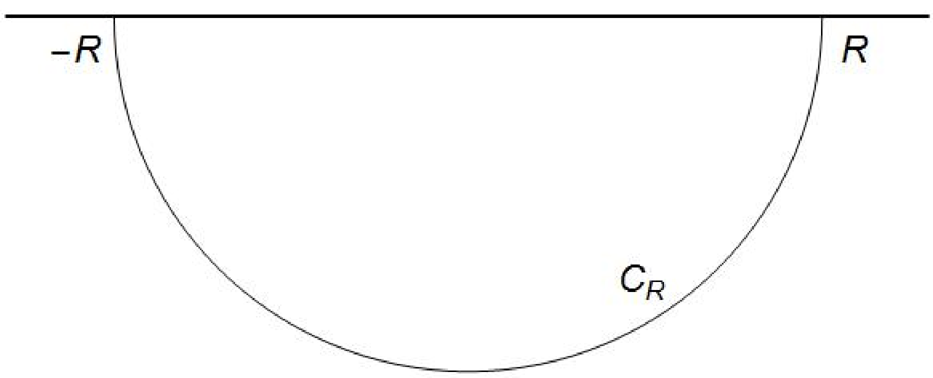


Figure 3.3.1-1: Closed contour Γ_R on the lower frequency half-plane.

Note, that the curve Γ_R we integrate along, must be negatively oriented, that is the interior of the curve is on the right side when travelling along the curve, because we chose $\mathcal{F}_R(x,x')$ to be an integral from $-R$ to R. This contour can be divided into two parts and using the Residue theorem, we get

$$\begin{aligned}
\int_{\Gamma_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega &= \int_{C_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega + \mathcal{F}_R(x,x') \\
-2\pi i \sum_j \mathrm{Res}(\psi_\omega(x)\psi_\omega(x'),\omega_j) &= \int_{C_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega + \mathcal{F}_R(x,x') \\
\mathcal{F}_R(x,x') &= \frac{2\pi}{i}\sum_j \mathrm{Res}(\psi_\omega(x)\psi_\omega(x'),\omega_j) \\
&\quad - \int_{C_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega
\end{aligned} \tag{3.68}$$

where ω_j are the poles of the integrand located on the lower half of the complex frequency plane. Let us now express the residue. From (3.65) we have for $x > d, x' > d$

$$\begin{aligned}
\psi_\omega(x)\psi_\omega(x') = -\chi^2 \Bigg\{ &\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)} \mathrm{Ci}^-(y_1(x))\mathrm{Ci}^-(y_1(x')) - \mathrm{Ci}^-(y_1(x))\mathrm{Ci}^+(y_1(x')) \\
+ &\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)} \mathrm{Ci}^+(y_1(x))\mathrm{Ci}^+(y_1(x')) - \mathrm{Ci}^-(y_1(x'))\mathrm{Ci}^+(y_1(x)) \Bigg\}
\end{aligned} \tag{3.69}$$

for $x > d, x' < -d$

$$\begin{aligned}
&\psi_\omega(x)\psi_\omega(x') \\
&= \frac{2\chi^2 i}{\pi^2}\left(\frac{1}{\det \mathbf{M}(\omega)}\mathrm{Ci}^+(y_1(x))\mathrm{Ai}(y_1(x')) - \frac{1}{\overline{\det \mathbf{M}}(\omega)}\mathrm{Ci}^-(y_1(x))\mathrm{Ai}(y_1(x'))\right)
\end{aligned} \tag{3.70}$$

for $x' > d, x < -d$

$$\begin{aligned}
&\psi_\omega(x)\psi_\omega(x') \\
&= \frac{2\chi^2 i}{\pi^2}\left(\frac{1}{\det \mathbf{M}(\omega)}\mathrm{Ci}^+(y_1(x'))\mathrm{Ai}(y_1(x)) - \frac{1}{\overline{\det \mathbf{M}}(\omega)}\mathrm{Ci}^-(y_1(x'))\mathrm{Ai}(y_1(x))\right)
\end{aligned} \tag{3.71}$$

for $x < -d, x' < -d$

$$\psi_\omega(x)\psi_\omega(x') = \frac{4\chi^2}{\pi^4 \det \mathbf{M}(\omega)\overline{\det \mathbf{M}}(\omega)}\mathrm{Ai}(y_1(x))\mathrm{Ai}(y_1(x')) \tag{3.72}$$

for $-d < x < d, x' < -d$

$$\psi_\omega(x)\psi_\omega(x') = \frac{4\chi^2}{\pi^3 \det \mathbf{M}(\omega)\overline{\det \mathbf{M}}(\omega)} \mathrm{Ai}(y_1(x'))$$
$$[(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x)) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x))] \tag{3.73}$$

for $-d < x' < d, x < -d$

$$\psi_\omega(x)\psi_\omega(x') = \frac{4\chi^2}{\pi^3 \det \mathbf{M}(\omega)\overline{\det \mathbf{M}}(\omega)} \mathrm{Ai}(y_1(x))$$
$$[(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x')) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x'))] \tag{3.74}$$

for $-d < x < d, x' > d$

$$\psi_\omega(x)\psi_\omega(x') = \frac{2i\chi^2}{\pi} [(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x)) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x))]$$
$$\left(\frac{1}{\det \mathbf{M}(\omega)} \mathrm{Ci}^+(y_1(x')) - \frac{1}{\overline{\det \mathbf{M}}(\omega)} \mathrm{Ci}^-(y_1(x'))\right) \tag{3.75}$$

for $-d < x' < d, x > d$

$$\psi_\omega(x)\psi_\omega(x') = \frac{2i\chi^2}{\pi} [(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x')) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x'))]$$
$$\left(\frac{1}{\det \mathbf{M}(\omega)} \mathrm{Ci}^+(y_1(x)) - \frac{1}{\overline{\det \mathbf{M}}(\omega)} \mathrm{Ci}^-(y_1(x))\right) \tag{3.76}$$

and for $-d < x < d, -d < x' < d$ we have

$$\psi_\omega(x)\psi_\omega(x') = \frac{4\chi^2}{\pi^2 \det \mathbf{M}(\omega)\overline{\det \mathbf{M}}(\omega)}$$
$$[(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x)) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x))]$$
$$[(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x')) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x'))] \tag{3.77}$$

We are looking for the poles confined in the lower half-plane ω_j of these expressions and we can see that these poles are only when $\det \mathbf{M}(\omega)$ is zero. The complex conjugate of the determinant has zeros on the upper half-plane and thus they are not inside the contour Γ_R. From Figure

(3.2-1) it is clear that the poles are simple. We can then write

$$\frac{2\pi}{i}\mathrm{Res}(\psi_\omega(x)\psi_\omega(x'),\omega_j) = \frac{2\pi}{i}\lim_{\omega\to\omega_j}\{(\omega-\omega_j)\psi_\omega(x)\psi_\omega(x')\} = \frac{2\pi}{i}\chi^2\lim_{\omega\to\omega_j}\left\{\frac{\omega-\omega_j}{\det\mathbf{M}(\omega)}\right\}$$
$$\begin{cases} -\overline{\det\mathbf{M}}_j\mathrm{Ci}_j^+(y_1(x))\mathrm{Ci'}_j^+(y_1(x')) & x>d, x'>d \\ \frac{2i}{\pi^2}\mathrm{Ci}_j^+(y_1(x))\mathrm{Ai}_j(y_1(x')) & x>d, x'<-d \\ \frac{2i}{\pi^2}\mathrm{Ci}_j^+(y_1(x'))\mathrm{Ai}_j(y_1(x)) & x<-d, x'>d \\ \frac{4}{\pi^4\overline{\det\mathbf{M}_j}}\mathrm{Ai}_j(y_1(x))\mathrm{Ai}_j(y_1(x')) & x<-d, x'<-d \\ \frac{4}{\pi^3\overline{\det\mathbf{M}_j}}\mathrm{Ai}_j(y_1(x'))\left[a_j\mathrm{Ai}_j(y_2(x))+b_j\mathrm{Bi}_j(y_2(x))\right] & -d<x<d, x'<-d \\ \frac{4}{\pi^3\overline{\det\mathbf{M}_j}}\mathrm{Ai}_j(y_1(x))\left[a_j\mathrm{Ai}_j(y_2(x'))+b_j\mathrm{Bi}_j(y_2(x'))\right] & x<-d, -d<x'<d \\ \frac{2i}{\pi}\mathrm{Ci}_j^+(y_1(x'))\left[a_j\mathrm{Ai}_j(y_2(x))+b_j\mathrm{Bi}_j(y_2(x))\right] & -d<x<d, x'>d \\ \frac{2i}{\pi}\mathrm{Ci}_j^+(y_1(x))\left[a_j\mathrm{Ai}_j(y_2(x'))+b_j\mathrm{Bi}_j(y_2(x'))\right] & d<x, -d<x'<d \\ \frac{4}{\pi^2\overline{\det\mathbf{M}_j}}\left[a_j\mathrm{Ai}_j(y_2(x))+b_j\mathrm{Bi}_j(y_2(x))\right] & -d<x<d, -d<x'<d \\ \left[a_j\mathrm{Ai}_j(y_2(x'))+b_j\mathrm{Bi}_j(y_2(x'))\right] & \end{cases}$$
$$= \psi_j(x)\psi_j(x') \tag{3.78}$$

with

$$\psi_j(x) = \chi\left(\frac{2\pi}{i}\lim_{\omega\to\omega_j}\left\{\frac{\omega-\omega_j}{\det\mathbf{M}(\omega)}\right\}\right)^{\frac{1}{2}}\begin{cases} i\left(\overline{\det\mathbf{M}}_j\right)^{\frac{1}{2}}\mathrm{Ci}_j^+(y_1(x)) & x>d \\ \frac{2}{\pi\left(\overline{\det\mathbf{M}}_j\right)^{\frac{1}{2}}} & \\ \left[a_j\mathrm{Ai}_j(y_2(x))+b_j\mathrm{Bi}_j(y_2(x))\right] & -d<x<d \\ \frac{2}{\pi^2\left(\overline{\det\mathbf{M}}_j\right)^{\frac{1}{2}}}\mathrm{Ai}_j(y_1(x)) & -d<x \end{cases} \tag{3.79}$$

where $a_j = (B_1'A_0 - B_1A_0')\Big|_{\omega=\omega_j}$, $b_j = (A_1A_0' - A_1'A_0)\Big|_{\omega=\omega_j}$ and $\mathrm{Ai}_j(y_1(x)) = \mathrm{Ai}(y_1(x))\Big|_{\omega=\omega_j}$ etc. It is easy to verify that the functions $\psi_j(x)$ are proportional to the resonant states (3.65). We can thus rewrite (3.68) using (3.79) in the form

$$\delta(x-x') = \sum_j \psi_j(x)\psi_j(x') - \lim_{R\to\infty}\int_{C_R}\psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega \tag{3.80}$$

The completeness of the resonant states depends on the limit of the integral. Let us perform the variable transformation $\omega = Re^{i\theta}$ so we get $\mathrm{d}\omega = Re^{i\theta}i\mathrm{d}\theta$. We use the asymptotic expressions

(D.51) and (D.93) from Appendix D. For the sector $-\frac{2\pi}{3} < \theta < 0$ the asymptotic expression is

$$\psi_\omega(x) = \chi \begin{cases} \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x R^{\frac{1}{2}}} & x < -d \\ \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i(\beta+\sigma)\varrho R^{\frac{1}{2}}} e^{\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} & -d < x < d \\ -\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{3}{4}}\pi^{\frac{1}{2}}} e^{i\varrho R^{\frac{1}{2}}(\sigma-3\beta)} e^{-\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} & \\ i\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x R^{\frac{1}{2}}} - i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}} e^{i\beta\varrho R^{\frac{1}{2}}} e^{-\varpi x R^{\frac{1}{2}}} & d < x \end{cases} \tag{3.81}$$

For the sector $-\pi < \theta < -\frac{2\pi}{3}$ (D.93) decays independently of x. Now let $f(x)$ be any function with a compact support. We will multiply (3.80) with $f(x')$ and integrate the equation with respect to x'.

$$\int_{-\infty}^{\infty} f(x')\delta(x-x')\mathrm{d}x' = \sum_j \psi_j(x) \int_{-\infty}^{\infty} f(x')\psi_j(x')\mathrm{d}x'$$

$$+ \lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{-\frac{2\pi}{3}}^{0} f(x')\psi_\omega(x)\psi_\omega(x') R e^{i\theta} i\mathrm{d}\theta\mathrm{d}x'$$

$$f(x) = \sum_j c_j\psi_j(x) + \lim_{R\to\infty} \int_{-\frac{2\pi}{3}}^{0} F(R,\theta)\psi_\omega(x) R e^{i\theta} i\mathrm{d}\theta \tag{3.82}$$

where we used $\omega = Re^{i\theta}$ and

$$F(R,\theta) = \int_{-\infty}^{\infty} f(x')\psi_\omega(x')\mathrm{d}x' \tag{3.83}$$

$$c_j = \int_{-\infty}^{\infty} f(x')\psi_j(x')\mathrm{d}x' \tag{3.84}$$

If the function $f(x)$ has its support confined in a region, then both variables x, x' are non-zero only in this region. Therefore, we do not need to consider the cross-terms in the product $\psi_\omega(x)\psi_\omega(x')$.

Let $f(x)$ have its support in the region $x < -d$, so for $x, x' < -d$ we have

$$
\begin{aligned}
f(x) &= \sum_j c_j \psi_j(x) + \lim_{R\to\infty} \int_{-\frac{2\pi}{3}}^{0} F(R,\theta) \frac{\chi 2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x R^{\frac{1}{2}}} R e^{i\theta} i \mathrm{d}\theta \\
&= \sum_j c_j \psi_j(x) \\
&+ \frac{\chi 2}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} \lim_{R\to\infty} R^{\frac{5}{4}} \int_{-\frac{2\pi}{3}}^{0} F(R,\theta) \kappa^{\frac{1}{4}} e^{-i\beta\varrho_r R^{\frac{1}{2}}} e^{R^{\frac{1}{2}}(x|\varpi_r| - \beta|\varrho_i|)} e^{i\varpi_i x R^{\frac{1}{2}}} e^{i\theta} \mathrm{d}\theta \\
&= \sum_j c_j \psi_j(x) + \frac{\chi 2}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} \\
&\lim_{R\to\infty} R^{\frac{5}{4}} \int_{-\frac{2\pi}{3}}^{0} F(R,\theta) \kappa^{\frac{1}{4}} e^{-i\beta\varrho_r R^{\frac{1}{2}}} e^{(2\alpha)^{\frac{1}{2}} \left|\sin\left(\frac{1}{2}\theta\right)\right| R^{\frac{1}{2}}(x\varepsilon 2\alpha - \beta)} e^{i\varpi_i x R^{\frac{1}{2}}} e^{i\theta} \mathrm{d}\theta
\end{aligned}
\tag{3.85}
$$

where $\alpha = (2\varepsilon)^{-\frac{2}{3}}$, $\beta = 2\alpha\varepsilon d$, $\sigma = 2\alpha V_0$, $\kappa = 2\alpha e^{i\theta}$ and

$$
\varrho_r = (2\alpha)^{\frac{1}{2}} \cos\left(\frac{1}{2}\theta\right) \qquad \varpi_r = -\varepsilon(2\alpha)^{\frac{3}{2}} \sin\left(\frac{1}{2}\theta\right) \tag{3.86}
$$

$$
\varrho_i = (2\alpha)^{\frac{1}{2}} \sin\left(\frac{1}{2}\theta\right) \qquad \varpi_i = \varepsilon(2\alpha)^{\frac{3}{2}} \cos\left(\frac{1}{2}\theta\right) \tag{3.87}
$$

Note, that from (D.23) and (D.39), (D.40) we know that $\varrho_i < 0$ and $\varpi_r, \varpi_i > 0$. The function $F(R,\theta)$ in (3.85) becomes

$$
\begin{aligned}
F(R,\theta) &\approx \int_{-\infty}^{\infty} f(x') \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x' R^{\frac{1}{2}}} \mathrm{d}x' \\
&= \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x') e^{-i\beta\varrho_r R^{\frac{1}{2}}} e^{(2\alpha)^{\frac{1}{2}} \left|\sin\left(\frac{1}{2}\theta\right)\right| R^{\frac{1}{2}}(x'\varepsilon 2\alpha - \beta)} e^{i\varpi_i x' R^{\frac{1}{2}}} \mathrm{d}x'
\end{aligned}
\tag{3.88}
$$

This expression decays for $x'\varepsilon 2\alpha - \beta < 0$ which can be also written as

$$
\begin{aligned}
x' &< \frac{\beta}{\varepsilon 2\alpha} \\
x' &< \frac{2\alpha\varepsilon d}{\varepsilon 2\alpha} \Rightarrow x' < d
\end{aligned}
\tag{3.89}
$$

and essentially the same goes for the x variable in (3.85), which means that in the current region $x, x' < -d$ we have a point-wise convergence for $f(x)$.

Consider $f(x)$ now having its support in the region $-d < x < d$, so for $-d < x, x' < d$

(3.82) becomes

$$
\begin{aligned}
f(x) &= \sum_j c_j\psi_j(x) - \lim_{R\to\infty}\chi\int_{-\frac{2\pi}{3}}^{0} F(R,\theta)\left(\frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i(\beta+\sigma)\varrho R^{\frac{1}{2}}}e^{\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}\right.\\
&\left.-\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{3}{4}}\pi^{\frac{1}{2}}}e^{i\varrho R^{\frac{1}{2}}(\sigma-3\beta)}e^{-\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}\right)Re^{i\theta}i\mathrm{d}\theta\\
&= \sum_j c_j\psi_j(x) - \frac{\chi 2}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{5}{4}}\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{\frac{1}{4}}e^{-i(\beta+\sigma)\varrho R^{\frac{1}{2}}}e^{\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}e^{i\theta}\mathrm{d}\theta\\
&+\chi i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{1}{4}}\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{-\frac{3}{4}}e^{i\varrho R^{\frac{1}{2}}(\sigma-3\beta)}e^{-\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}e^{i\theta}\mathrm{d}\theta\\
&= \sum_j c_j\psi_j(x) - \frac{\chi 2}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{5}{4}}\\
&\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{\frac{1}{4}}e^{-i(\beta+\sigma)\varrho_r R^{\frac{1}{2}}}e^{R^{\frac{1}{2}}\left[|\varpi_r|\left(x+\frac{V_0}{\varepsilon}\right)-(\beta+\sigma)|\varrho_i|\right]}e^{i\varpi_i R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}e^{i\theta}\mathrm{d}\theta\\
&+\chi i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{1}{4}}\\
&\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{-\frac{3}{4}}e^{i\varrho_r R^{\frac{1}{2}}(\sigma-3\beta)}e^{R^{\frac{1}{2}}\left[-|\varpi_r|\left(x+\frac{V_0}{\varepsilon}\right)+|\varrho_i|(\sigma-3\beta)\right]}e^{-i\varpi_i R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}e^{i\theta}\mathrm{d}\theta\\
&= \sum_j c_j\psi_j(x) - \frac{\chi 2}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{5}{4}}\\
&\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{\frac{1}{4}}e^{-i(\beta+\sigma)\varrho_r R^{\frac{1}{2}}}e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}\left[\varepsilon 2\alpha\left(x+\frac{V_0}{\varepsilon}\right)-(\beta+\sigma)\right]}e^{i\varpi_i R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}e^{i\theta}\mathrm{d}\theta\\
&+\chi i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{1}{4}}\int_{-\frac{2\pi}{3}}^{0}\Big[F(R,\theta)\\
&\kappa^{-\frac{3}{4}}e^{i\varrho_r R^{\frac{1}{2}}(\sigma-3\beta)}e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}\left[-\varepsilon 2\alpha\left(x+\frac{V_0}{\varepsilon}\right)+(\sigma-3\beta)\right]}e^{-i\varpi_i R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}e^{i\theta}\Big]\mathrm{d}\theta \qquad (3.90)
\end{aligned}
$$

The function $F(R,\theta)$ in (3.83) in this case becomes

$$
\begin{aligned}
&F(R,\theta) \approx \int_{-\infty}^{\infty} f(x') \left(\frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i(\beta+\sigma)\varrho R^{\frac{1}{2}}} e^{\varpi R^{\frac{1}{2}}\left(x'+\frac{V_0}{\varepsilon}\right)} \right. \\
&\left. -\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{3}{4}}\pi^{\frac{1}{2}}} e^{i\varrho R^{\frac{1}{2}}(\sigma-3\beta)} e^{-\varpi R^{\frac{1}{2}}\left(x'+\frac{V_0}{\varepsilon}\right)} \right) \mathrm{d}x' \\
&= \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} \\
&\int_{-\infty}^{\infty} f(x') e^{-i(\beta+\sigma)\varrho_r R^{\frac{1}{2}}} e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right| R^{\frac{1}{2}}\left[\varepsilon 2\alpha\left(x'+\frac{V_0}{\varepsilon}\right)-(\beta+\sigma)\right]} e^{i\varpi_i R^{\frac{1}{2}}\left(x'+\frac{V_0}{\varepsilon}\right)} \mathrm{d}x' \\
&- \frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{3}{4}}\pi^{\frac{1}{2}}} \\
&\int_{-\infty}^{\infty} f(x') e^{i\varrho_r R^{\frac{1}{2}}(\sigma-3\beta)} e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right| R^{\frac{1}{2}}\left[-\varepsilon 2\alpha\left(x'+\frac{V_0}{\varepsilon}\right)+(\sigma-3\beta)\right]} e^{-i\varpi_i R^{\frac{1}{2}}\left(x'+\frac{V_0}{\varepsilon}\right)} \mathrm{d}x' \qquad (3.91)
\end{aligned}
$$

where $\alpha = (2\varepsilon)^{-\frac{2}{3}}$, $\beta = 2\alpha\varepsilon d$, $\sigma = 2\alpha V_0$ and $\kappa = 2\alpha e^{i\theta}$. We can see that we have two integrals both in (3.90) and (3.91). Let us investigate under what conditions each of them decay. In both expression we get the same results for x and x'. The first integral vanishes if

$$
\begin{aligned}
\varepsilon 2\alpha\left(x+\frac{V_0}{\varepsilon}\right)-(\beta+\sigma) &< 0 \\
x &< \frac{\beta+\sigma}{\varepsilon 2\alpha} - \frac{V_0}{\varepsilon} \\
x &< \frac{2\alpha\varepsilon d + 2\alpha V_0 - V_0 2\alpha}{\varepsilon 2\alpha} \\
x &< d \qquad (3.92)
\end{aligned}
$$

and the second one vanishes under the condition

$$
\begin{aligned}
-\varepsilon 2\alpha\left(x+\frac{V_0}{\varepsilon}\right)+(\sigma-3\beta) &< 0 \\
x &> \frac{\sigma-3\beta}{\varepsilon 2\alpha} - \frac{V_0}{\varepsilon} \\
x &> \frac{2\alpha V_0 - 6\alpha\varepsilon d - V_0 2\alpha}{\varepsilon 2\alpha} \\
x &> -3d \qquad (3.93)
\end{aligned}
$$

These are rather nice results, because both of them are obeyed, since we are in the region $-d < x < d$.

Finally, suppose that $f(x)$ has a support confined in $d < x$. For $d < x, x'$ the equation (3.82) reads

$$
\begin{aligned}
f(x) = &\sum_j c_j\psi_j(x) - \lim_{R\to\infty}\int_{-\frac{2\pi}{3}}^{0} F(R,\theta)\left(i\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi x R^{\frac{1}{2}}}\right.\\
&\left.-i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi x R^{\frac{1}{2}}}\right)Re^{i\theta}i\mathrm{d}\theta\\
&= \sum_j c_j\psi_j(x) + \frac{2}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{5}{4}}\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{\frac{1}{4}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi x R^{\frac{1}{2}}}e^{i\theta}\mathrm{d}\theta\\
&-\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{1}{4}}\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{-\frac{3}{4}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi x R^{\frac{1}{2}}}e^{i\theta}\mathrm{d}\theta\\
&= \sum_j c_j\psi_j(x) + \frac{2}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{5}{4}}\\
&\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{\frac{1}{4}}e^{-i\beta\varrho_r R^{\frac{1}{2}}}e^{R^{\frac{1}{2}}(|\varpi_r|x-\beta|\varrho_i|)}e^{i\varpi_i x R^{\frac{1}{2}}}e^{i\theta}\mathrm{d}\theta\\
&-\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{1}{4}}\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{-\frac{3}{4}}e^{i\beta\varrho_r R^{\frac{1}{2}}}e^{R^{\frac{1}{2}}(-|\varpi_r|x+\beta|\varrho_i|)}e^{-i\varpi_i x R^{\frac{1}{2}}}e^{i\theta}\mathrm{d}\theta\\
&= \sum_j c_j\psi_j(x) + \frac{2}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{5}{4}}\\
&\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{\frac{1}{4}}e^{-i\beta\varrho_r R^{\frac{1}{2}}}e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}(\varepsilon 2\alpha x-\beta)}e^{i\varpi_i x R^{\frac{1}{2}}}e^{i\theta}\mathrm{d}\theta\\
&-\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\lim_{R\to\infty}R^{\frac{1}{4}}\int_{-\frac{2\pi}{3}}^{0}F(R,\theta)\kappa^{-\frac{3}{4}}e^{i\beta\varrho_r R^{\frac{1}{2}}}e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}(-\varepsilon 2\alpha x+\beta)}e^{-i\varpi_i x R^{\frac{1}{2}}}e^{i\theta}\mathrm{d}\theta \qquad (3.94)
\end{aligned}
$$

The function $F(R,\theta)$ in (3.83) in this case becomes

$$
\begin{aligned}
F(R,\theta) \approx &\int_{-\infty}^{\infty}f(x')\left(i\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi x' R^{\frac{1}{2}}} - i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi x' R^{\frac{1}{2}}}\right)\mathrm{d}x'\\
&= \frac{2i(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\int_{-\infty}^{\infty}f(x')e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi x' R^{\frac{1}{2}}}\mathrm{d}x' - \frac{\sigma^{\frac{1}{2}}i}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}\int_{-\infty}^{\infty}f(x')e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi x' R^{\frac{1}{2}}}\mathrm{d}x'\\
&= \frac{2i(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\int_{-\infty}^{\infty}f(x')e^{-i\beta\varrho_r R^{\frac{1}{2}}}e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}(\varepsilon 2\alpha x'-\beta)}e^{i\varpi_i x' R^{\frac{1}{2}}}\mathrm{d}x'\\
&-\frac{\sigma^{\frac{1}{2}}i}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}\int_{-\infty}^{\infty}f(x')e^{i\beta\varrho_r R^{\frac{1}{2}}}e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}(-\varepsilon 2\alpha x'+\beta)}e^{-i\varpi_i x' R^{\frac{1}{2}}}\mathrm{d}x' \qquad (3.95)
\end{aligned}
$$

where as before $\alpha = (2\varepsilon)^{-\frac{2}{3}}$, $\beta = 2\alpha\varepsilon d$, $\sigma = 2\alpha V_0$ and $\kappa = 2\alpha e^{i\theta}$. We have again two integrals both in (3.94) and (3.95). This time we will look at the conditions when these integrals grow exponentially in the limit $R \to \infty$. In both expression we get the same results for x and x'. The first integral grows if

$$\begin{aligned}\varepsilon 2\alpha x - \beta &> 0\\ x &> \frac{\beta}{\varepsilon 2\alpha}\\ x &> \frac{2\alpha\varepsilon d}{\varepsilon 2\alpha}\\ x &> d\end{aligned} \tag{3.96}$$

For the second one we have growth if

$$\begin{aligned}-\varepsilon 2\alpha x + \beta &> 0\\ x &< \frac{\beta}{\varepsilon 2\alpha}\\ x &< d\end{aligned} \tag{3.97}$$

At this point we can conclude that in region $d < x$, the first integral in (3.94) and (3.95) grows while the second decays, so overall we get a divergence in this region. Summing it up, we showed, that $f(x)$ can be represented by the resonant states (3.65) only when $f(x)$ has its support confined in $x < d$.

3.3.2 Stronger proof

This proof is again a proof of the completeness of resonant states (3.65) from a theoretical point of view as in the previous section only in a different way with more specific conclusions. We start again writing down the resonant states in a scattering form.

$$\psi_\omega(x) = \chi \begin{cases} \frac{2}{\pi^2|\det\mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x)) & x < -d\\ \frac{2}{\pi|\det\mathbf{M}(\omega)|}\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(x))\right. & \\ \qquad\qquad \left.+(A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(x))\right] & -d < x < d\\ i\left(\frac{\overline{\det\mathbf{M}(\omega)}}{\det\mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(x)) - i\left(\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}(\omega)}}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(x)) & d < x\end{cases} \tag{3.98}$$

with the normalization constant χ computed in Appendix (C) and

$$\chi = 2^{-\frac{2}{3}}\varepsilon^{-\frac{1}{6}} \tag{3.99}$$

$$y_1(x) = -2\alpha(\varepsilon x + \omega) \tag{3.100}$$

$$y_2(x) = -2\alpha(\varepsilon x + V_0 + \omega) \tag{3.101}$$

$$A_0 = \mathrm{Ai}(y_1(-d)), A_1 = \mathrm{Ai}(y_2(-d)), B_1 = \mathrm{Bi}(y_2(-d))$$

with $\alpha = (2\varepsilon)^{-\frac{2}{3}}$ and the determinant

$$\det \mathbf{M}(\omega) = (A_0A_1' - A_0'A_1)(B_2C_3' - B_2'C_3) - (A_0B_1' - A_0'B_1)(A_2C_3' - A_2'C_3) \tag{3.102}$$

We can use the continuity conditions at $x = d$ and $x = -d$ to write (3.98) in a different form. At $x = d$ we have

$$\begin{aligned}
&\frac{2}{\pi|\det \mathbf{M}(\omega)|}\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(d)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(d))\right] \\
&= i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(d)) \\
&1 = \frac{\pi|\det \mathbf{M}(\omega)|}{2\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(d)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(d))\right]} \\
&\left(i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(d))\right)
\end{aligned} \tag{3.103}$$

which we use to rewrite the region $-d < x < d$ as

$$\begin{aligned}
&\frac{2\pi|\det \mathbf{M}(\omega)|\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(x)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(x))\right]}{2\pi|\det \mathbf{M}(\omega)|\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(d)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(d))\right]} \\
&\left(i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(d))\right) \\
&= \frac{\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(x)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(x))\right]}{\left[(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2\right]} \\
&\left(i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(d))\right)
\end{aligned} \tag{3.104}$$

Similarly we can use the continuity condition at $x = -d$ and the newly rewritten region $-d < x < d$ (3.104).

$$
\begin{aligned}
&\frac{2}{\pi^2|\det \mathbf{M}(\omega)|}\mathrm{Ai}(y_1(-d)) = \\
&\frac{[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(-d)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(-d))]}{[(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2]} \\
&\left(i\left(\frac{\overline{\det \mathbf{M}(\omega)}}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d))\right) \\
1 = &\frac{\pi^2|\det \mathbf{M}(\omega)|\,[(B_1'A_0 - B_1A_0')A_1 + (A_1A_0' - A_1'A_0)B_1]}{2A_0\,[(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2]} \\
&\left(i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d))\right) \\
1 = &\frac{\pi|\det \mathbf{M}(\omega)|}{2\,[(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2]} \\
&\left(i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d))\right)
\end{aligned}
\tag{3.105}
$$

where we used [1] (10.4.10). We can use this result (3.105) in (3.98) for the region $x < -d$ and it becomes

$$
\begin{aligned}
&\frac{\pi|\det \mathbf{M}(\omega)|}{2\,[(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2]}\,\frac{2}{\pi^2|\det \mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x)) \\
&\left(i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d))\right) \\
&= \frac{\mathrm{Ai}(y_1(x))}{\pi\,[(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2]} \\
&\left(i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d))\right)
\end{aligned}
\tag{3.106}
$$

Denoting $p(\omega) = (B_1' A_0 - B_1 A_0')A_2 + (A_1 A_0' - A_1' A_0)B_2$ and using (3.104), (3.106) we write (3.98) as

$$\psi_\omega(x) = \chi \begin{cases} i\left(\left(\frac{\det \overline{\mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d)) - \left(\frac{\det \mathbf{M}(\omega)}{\det \overline{\mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d))\right) \frac{\mathrm{Ai}(y_1(x))}{\pi p(\omega)} & x < -d \\ i\left(\left(\frac{\det \overline{\mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d)) - \left(\frac{\det \mathbf{M}(\omega)}{\det \overline{\mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d))\right) & -d < x < d \\ \frac{\left[(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x)) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x))\right]}{p(\omega)} & \\ i\left(\frac{\det \overline{\mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(x)) - i\left(\frac{\det \mathbf{M}(\omega)}{\det \overline{\mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(x)) & d < x \end{cases} \tag{3.107}$$

We showed earlier that the Airy functions $\mathrm{Ci}^{\pm}(x)$ can be interpreted as in- and outgoing waves (2.31), which means, we can define in- and outgoing waves $\psi_\omega^{\pm}(x)$ in terms of the representation (3.107).

$$\psi_\omega^+(x) = \chi \begin{cases} i\left(\frac{\det \overline{\mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d)) \frac{\mathrm{Ai}(y_1(x))}{\pi p(\omega)} & x < -d \\ i\left(\frac{\det \overline{\mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(d))) & -d < x < d \\ \frac{\left[(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x)) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x))\right]}{p(\omega)} & \\ i\left(\frac{\det \overline{\mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y_1(x)) & d < x \end{cases} \tag{3.108}$$

$$\psi_\omega^-(x) = \chi \begin{cases} -i\left(\frac{\det \mathbf{M}(\omega)}{\det \overline{\mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d)) \frac{\mathrm{Ai}(y_1(x))}{\pi p(\omega)} & x < -d \\ -i\left(\frac{\det \mathbf{M}(\omega)}{\det \overline{\mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(d)) & -d < x < d \\ \frac{\left[(B_1' A_0 - B_1 A_0')\mathrm{Ai}(y_2(x)) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(y_2(x))\right]}{p(\omega)} & \\ -i\left(\frac{\det \mathbf{M}(\omega)}{\det \overline{\mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y_1(x)) & d < x \end{cases} \tag{3.109}$$

Observe, that the functions $\psi_\omega^{\pm}(x)$ are continuous at $x = -d, d$, also the derivative at $x = -d$ is continuous, but their derivative at $x = d$ is not continuous. Also observe, that by this construction we have

$$\psi_\omega(x) = \psi_\omega^+(x) + \psi_\omega^-(x) \tag{3.110}$$

The completeness relation for scattering states as before is

$$\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega = \delta(x - x') \tag{3.111}$$

Let $f(x)$ be any wave-function. Then using the completeness (3.111) we get

$$f(x) = \int_{-\infty}^{\infty} \delta(x - s)f(s)\mathrm{d}s = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(s)\mathrm{d}\omega f(s)\mathrm{d}s \tag{3.112}$$

We use second times as we multiply (3.111) with $f(x)$ and integrate over the whole space:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_{\omega'}(x)\psi_{\omega'}(s)f(x)\mathrm{d}\omega'\mathrm{d}x &= \int_{-\infty}^{\infty} \delta(x - s)f(x)\mathrm{d}x \\ \int_{-\infty}^{\infty} a(\omega')\psi_{\omega'}(s)\mathrm{d}\omega' &= f(s)\end{aligned} \tag{3.113}$$

where $a(\omega') = \int_{-\infty}^{\infty} \psi_{\omega'}(x)f(x)\mathrm{d}x$. We substitute (3.113) back into (3.112 and we get

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(s)\mathrm{d}\omega \int_{-\infty}^{\infty} a(\omega')\psi_{\omega'}(s)\mathrm{d}\omega'\mathrm{d}s \\ &= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x) \int_{-\infty}^{\infty} \psi_\omega(s)\psi_{\omega'}(s)\mathrm{d}s\mathrm{d}\omega\mathrm{d}\omega'\end{aligned} \tag{3.114}$$

Using the separation of the resonant states into outgoing and ingoing waves (3.110) we can rewrite (3.114) as

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x) \int_{-\infty}^{\infty} \psi_\omega(s)(\psi_{\omega'}^{+}(s) + \psi_{\omega'}^{-}(s))\mathrm{d}s\mathrm{d}\omega\mathrm{d}\omega' \\ &= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x) \int_{-\infty}^{\infty} \psi_\omega(s)\psi_{\omega'}^{+}(s)\mathrm{d}s\mathrm{d}\omega\mathrm{d}\omega' \\ &+ \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x) \int_{-\infty}^{\infty} \psi_\omega(s)\psi_{\omega'}^{-}(s)\mathrm{d}s\mathrm{d}\omega\mathrm{d}\omega' \\ &= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x)\Upsilon^{+}(\omega, \omega')\mathrm{d}\omega\mathrm{d}\omega' + \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_\omega(x)\Upsilon^{-}(\omega, \omega')\mathrm{d}\omega\mathrm{d}\omega' \\ &= f^{+}(x) + f^{-}(x)\end{aligned} \tag{3.115}$$

where we have defined the following quantities

$$\Upsilon^{\pm}(\omega,\omega') = \int_{-\infty}^{\infty} \psi_{\omega}(s)\psi^{\pm}_{\omega'}(s)\mathrm{d}s \tag{3.116}$$

$$f^{\pm}(x) = \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_{\omega}(x)\Upsilon^{\pm}(\omega,\omega')\mathrm{d}\omega\mathrm{d}\omega' \tag{3.117}$$

The integrand in (3.116) does not converge. It can be seen from the standard asymptotic expression of Airy functions in (2.31), where we see that $\mathrm{Ci}^{\pm}(x)$ is decaying algebraically, but not fast enough to converge. That is why we introduce the following correction in the variable ω'

$$\Upsilon^{\pm}_{\xi}(\omega,\omega') = \int_{-\infty}^{\infty} \psi_{\omega}(s)\psi^{\pm}_{\omega'\pm i\xi}(s)\mathrm{d}s \tag{3.118}$$

because the exponential part in the asymptotic behaviour in (2.31) $e^{\pm i\left(\zeta+\frac{\pi}{4}\right)}$ contains ω' through $\zeta = \frac{2}{3}(-y_1(x))^{\frac{3}{2}}$ and (3.100) so we can write

$$\begin{aligned} e^{\pm i\left(\zeta+\frac{\pi}{4}\right)} &= e^{\pm i\left(\frac{2}{3}(2\alpha(\varepsilon x+\omega'))^{\frac{3}{2}}+\frac{\pi}{4}\right)} = e^{\pm i\left(\frac{2}{3}\left(2\alpha\omega'\left(1+\frac{\varepsilon x}{\omega'}\right)\right)^{\frac{3}{2}}+\frac{\pi}{4}\right)} \\ &\approx e^{\pm i\left(\frac{2}{3}(2\alpha\omega')^{\frac{3}{2}}\left(1+\frac{3\varepsilon x}{2\omega'}\right)+\frac{\pi}{4}\right)} = e^{\pm i\frac{2}{3}(2\alpha\omega')^{\frac{3}{2}}} e^{\pm i(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}\varepsilon x} e^{\pm i\frac{\pi}{4}} \\ &\to e^{\pm i\frac{2}{3}(2\alpha)^{\frac{3}{2}}(\omega'\pm i\xi)^{\frac{3}{2}}} e^{\pm i(2\alpha)^{\frac{3}{2}}(\omega'\pm i\xi)^{\frac{1}{2}}\varepsilon x} e^{\pm i\frac{\pi}{4}} \\ &\approx e^{\pm i\frac{2}{3}(2\alpha\omega')^{\frac{3}{2}}\left(1\pm i\frac{3\xi}{2\omega'}\right)} e^{\pm i(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}\left(1\pm i\frac{\xi}{2\omega'}\right)\varepsilon x} e^{\pm i\frac{\pi}{4}} \\ &= e^{\pm i\frac{2}{3}(2\alpha\omega')^{\frac{3}{2}}} e^{-(2\alpha\omega')^{\frac{3}{2}}\frac{\xi}{\omega'}} e^{\pm i(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}} e^{-(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}\frac{\xi}{2\omega'}\varepsilon x} e^{\pm i\frac{\pi}{4}} \end{aligned} \tag{3.119}$$

taking the absolute value of the last line we get

$$\left| e^{-(2\alpha\omega')^{\frac{3}{2}}\frac{\xi}{\omega'}} \right| \left| e^{-(2\alpha)^{\frac{3}{2}}(\omega')^{\frac{1}{2}}\frac{\xi}{2\omega'}\varepsilon x} \right| \tag{3.120}$$

where it holds for the region $|x| > d$ and the exponential decay for large x can be clearly seen. We will perform calculations with a finite ξ and in the end we remove this correction by letting it go to 0 in a limit. Let us now focus on the integrals $\Upsilon^{\pm}$. Observe that we can write the resonant states as

$$\psi_{\omega}(s) = \Psi\left(y_{1,2}(s,\omega)\right) \tag{3.121}$$

$$\psi^{\pm}_{\omega'\pm i\xi}(s) = \Psi^{\pm}\left(y_{1,2}(s,\omega'\pm i\xi)\right) \tag{3.122}$$

where the functions Ψ, $\Psi^{\pm}$ are linear combinations of Airy functions in general on intervals $(-\infty, -d)$ and (d, ∞) and $y_{1,2}(x, \omega)$ are the usual variable transformations written with ω in the argument which one can write as

$$y_{1,2}(s, \omega) = -2\alpha\varepsilon \left(s_{1,2}(s) + \frac{\omega}{\varepsilon} \right) \tag{3.123}$$

where we defined

$$s_i(s) = \begin{cases} s & i = 1 \\ s + \frac{V_0}{\varepsilon} & i = 2 \end{cases} \tag{3.124}$$

For the next step we can use the formula (3.53) in [2] which says that if $A(s)$ and $B(s)$ are any linear combinations of Airy functions then

$$\int A(\rho(s + \beta_1))B(\rho(s + \beta_2))\mathrm{d}s = \frac{1}{\rho^2(\beta_1 - \beta_2)} \left[A'(\rho(s + \beta_1))B(\rho(s + \beta_2)) \right.$$
$$\left. -A(\rho(s + \beta_1))B'(\rho(s + \beta_2)) \right] \tag{3.125}$$

Using this in (3.118) and the fact that the resonant states $\psi_\omega(s)$ decay at $\pm\infty$, letting $A(s) = \Psi\left(y_{1,2}(s, \omega)\right), B(s) = \Psi^{\pm}\left(y_{1,2}(s, \omega' \pm i\xi)\right)$ and defying $\rho = 2\alpha\varepsilon$ we get

$$\begin{aligned}
\Upsilon_\xi^{\pm}(\omega, \omega') &= \int_{-\infty}^{-d} \Psi\left(-\rho\left(s_1(s) + \frac{\omega}{\varepsilon}\right)\right) \Psi^{\pm}\left(-\rho\left(s_1(s) + \frac{\omega' \pm i\xi}{\varepsilon}\right)\right) \mathrm{d}s \\
&+ \int_{d}^{\infty} \Psi\left(-\rho\left(s_1(s) + \frac{\omega}{\varepsilon}\right)\right) \Psi^{\pm}\left(-\rho\left(s_1(s) + \frac{\omega' \pm i\xi}{\varepsilon}\right)\right) \mathrm{d}s \\
&+ \int_{-d}^{d} \Psi\left(-\rho\left(s_2(s) + \frac{\omega}{\varepsilon}\right)\right) \Psi^{\pm}\left(-\rho\left(s_2(s) + \frac{\omega' \pm i\xi}{\varepsilon}\right)\right) \mathrm{d}s \\
&= \frac{\varepsilon}{\rho^2(\omega - \omega' \mp i\xi)} \left[\Psi'\left(y_1(s, \omega)\right) \Psi^{\pm}\left(y_1(s, \omega' \pm i\xi)\right) \right. \\
&\left. -\Psi\left(y_1(s, \omega)\right) \Psi'^{\pm}\left(y_1(s, \omega' \pm i\xi)\right) \right] \Big|_{d^+}^{-d^-} \\
&+ \frac{\varepsilon}{\rho^2(\omega - \omega' \mp i\xi)} \left[\Psi'\left(y_2(s, \omega)\right) \Psi^{\pm}\left(y_2(s, \omega' \pm i\xi)\right) \right. \\
&\left. -\Psi\left(y_2(s, \omega)\right) \Psi'^{\pm}\left(y_2(s, \omega' \pm i\xi)\right) \right] \Big|_{-d^+}^{d^-}
\end{aligned} \tag{3.126}$$

We go further and use the continuity of the derivative at $x = -d, d$ for $\Psi(s)$, the continuity of the derivative at $x = -d$ for $\Psi^{\pm}(s)$ (3.121), (3.122) and from (3.126) we get

$$
\begin{aligned}
&\Upsilon^{\pm}_{\xi}(\omega,\omega') = \frac{\varepsilon}{\rho^2(\omega-\omega'\mp i\xi)} \\
&\left[\Psi'\left(y_1(s,\omega)\right)\Psi^{\pm}\left(y_1(s,\omega'\pm i\xi)\right) - \Psi\left(y_1(s,\omega)\right)\Psi'^{\pm}\left(y_1(s,\omega'\pm i\xi)\right)\right]\Big|^{-d^-}_{d^+} \\
&+ \frac{\varepsilon}{\rho^2(\omega-\omega'\mp i\xi)} \\
&\left[\Psi'\left(y_2(s,\omega)\right)\Psi^{\pm}\left(y_2(s,\omega'\pm i\xi)\right) - \Psi\left(y_2(s,\omega)\right)\Psi'^{\pm}\left(y_2(s,\omega'\pm i\xi)\right)\right]\Big|^{d^-}_{-d^+} \\
&= \frac{\varepsilon}{\rho^2(\omega-\omega'\mp i\xi)}\left[\Psi'\left(y_1(-d^-,\omega)\right)\Psi^{\pm}\left(y_1(-d^-,\omega'\pm i\xi)\right)\right. \\
&-\Psi\left(y_1(-d^-,\omega)\right)\Psi'^{\pm}\left(y_1(-d^-,\omega'\pm i\xi)\right) - \Psi'\left(y_1(d^+,\omega)\right)\Psi^{\pm}\left(y_1(d^+,\omega'\pm i\xi)\right) \\
&+\Psi\left(y_1(d^+,\omega)\right)\Psi'^{\pm}\left(y_1(d^+,\omega'\pm i\xi)\right) + \Psi'\left(y_2(d^-,\omega)\right)\Psi^{\pm}\left(y_2(d^-,\omega'\pm i\xi)\right) \\
&-\Psi\left(y_2(d^-,\omega)\right)\Psi'^{\pm}\left(y_2(d^-,\omega'\pm i\xi)\right) - \Psi'\left(y_2(-d^+,\omega)\right)\Psi^{\pm}\left(y_2(-d^+,\omega'\pm i\xi)\right) \\
&\left.+\Psi\left(y_2(-d^+,\omega)\right)\Psi'^{\pm}\left(y_2(-d^+,\omega'\pm i\xi)\right)\right] \\
&= \frac{\varepsilon}{\rho^2(\omega-\omega'\mp i\xi)}\left[\Psi\left(y_1(d^+,\omega)\right)\Psi'^{\pm}\left(y_1(d^+,\omega'\pm i\xi)\right)\right. \\
&\left.-\Psi\left(y_2(d^-,\omega)\right)\Psi'^{\pm}\left(y_2(d^-,\omega'\pm i\xi)\right)\right] \\
&= \frac{\varepsilon\psi_\omega(d^+)}{\rho^2(\omega-\omega'\mp i\xi)}\left[\Psi'^{\pm}\left(y_1(d^+,\omega')\right) - \Psi'^{\pm}\left(y_2(d^-,\omega')\right)\right] \qquad (3.127)
\end{aligned}
$$

where in the last line we have put $\xi = 0$ in all numerators without any loss of generality. From our explicit expressions for the states $\psi^{\pm}_{\omega}(x)$ in (3.108), (3.109) and its derivative at $x = d$ we get

$$
\begin{aligned}
&\Psi'^{\pm}\left(y_1(d^+,\omega')\right) - \Psi'^{\pm}\left(y_2(d^-,\omega')\right) \\
&= \pm\chi i\left(\frac{\det\mathbf{M}^{\mp}(\omega')}{\det\mathbf{M}^{\pm}(\omega')}\right)^{\frac{1}{2}}\mathrm{Ci}'^{\pm}(y_1(d^+)) \mp \chi i\left(\frac{\det\mathbf{M}^{\mp}(\omega')}{\det\mathbf{M}^{\pm}(\omega')}\right)^{\frac{1}{2}}\mathrm{Ci}^{\pm}(y_1(d^-,\omega'))) \\
&\frac{\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}'(y_2(d^-,\omega')) + (A_1A_0' - A_1'A_0)\mathrm{Bi}'(y_2(d^-,\omega'))\right]}{p(\omega')} \\
&= \pm\chi i\left(\frac{\det\mathbf{M}^{\mp}(\omega')}{\det\mathbf{M}^{\pm}(\omega')}\right)^{\frac{1}{2}}\mathrm{Ci}'^{\pm}(y_1(d,\omega')) \mp \chi i\left(\frac{\det\mathbf{M}^{\mp}(\omega')}{\det\mathbf{M}^{\pm}(\omega')}\right)^{\frac{1}{2}}\mathrm{Ci}^{\pm}(y_1(d,\omega'))) \\
&\frac{(B_1'A_0 - B_1A_0')A_2' + (A_1A_0' - A_1'A_0)B_2'}{(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2} \qquad (3.128)
\end{aligned}
$$

where we denoted $\det \mathbf{M}^{+}(\omega') = \det \mathbf{M}(\omega')$ and $\det \mathbf{M}^{-}(\omega') = \overline{\det \mathbf{M}}(\omega')$ and every quantity A, B, C contains ω'. Let us look at the determinant $\det \mathbf{M}(\omega)$ (3.102):

$$
\begin{aligned}
&\det \mathbf{M}(\omega) = (A_0A_1' - A_0'A_1)(B_2C_3' - B_2'C_3) - (A_0B_1' - A_0'B_1)(A_2C_3' - A_2'C_3) \\
&= (A_0A_1' - A_0'A_1)B_2C_3' - (A_0A_1' - A_0'A_1)B_2'C_3 - (A_0B_1' - A_0'B_1)A_2C_3' \\
&+ (A_0B_1' - A_0'B_1)A_2'C_3 \\
&= C_3\left[(A_0B_1' - A_0'B_1)A_2' + (A_0'A_1 - A_0A_1')B_2'\right] \\
&- C_3'\left[(A_0B_1' - A_0'B_1)A_2 + (A_0'A_1 - A_0A_1')B_2\right] \qquad (3.129)
\end{aligned}
$$

Using this observation and $\mathrm{Ci}^{\pm}(y_1(d,\omega')) = C_3^{\pm}$, where $C_3^{+} = C_3$ and $C_3^{-} = D_3$, (3.128) becomes

$$
\begin{aligned}
&\Psi'^{\pm}\left(y_1(d^+,\omega')\right) - \Psi'^{\pm}\left(y_2(d^-,\omega')\right) \\
&= \pm\chi i\left(\frac{\det \mathbf{M}^{\mp}(\omega')}{\det \mathbf{M}^{\pm}(\omega')}\right)^{\frac{1}{2}} C_3'^{\pm} \mp \chi i\left(\frac{\det \mathbf{M}^{\mp}(\omega')}{\det \mathbf{M}^{\pm}(\omega')}\right)^{\frac{1}{2}} \\
&\frac{\det \mathbf{M}^{\pm}(\omega') + C_3'^{\pm}\left[(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2\right]}{(B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2} \\
&= \pm\chi i\left(\frac{\det \mathbf{M}^{\mp}(\omega')}{\det \mathbf{M}^{\pm}(\omega')}\right)^{\frac{1}{2}} C_3'^{\pm} \mp \chi i\left(\frac{\det \mathbf{M}^{\mp}(\omega')}{\det \mathbf{M}^{\pm}(\omega')}\right)^{\frac{1}{2}} C_3'^{\pm} \mp \chi i\frac{|\det \mathbf{M}(\omega')|}{p(\omega')} \\
&= \mp\chi i\frac{|\det \mathbf{M}(\omega')|}{p(\omega')} \qquad (3.130)
\end{aligned}
$$

where $p(\omega') = (B_1'A_0 - B_1A_0')A_2 + (A_1A_0' - A_1'A_0)B_2$. Hence we can write (3.127) using (3.130) as

$$
\Upsilon_{\xi}^{\pm}(\omega,\omega') = \frac{\chi\varepsilon}{\rho^2}\frac{\psi_\omega(d^+)}{\omega - \omega' \mp i\xi}\left(\mp i\frac{|\det \mathbf{M}(\omega')|}{p(\omega')}\right) \qquad (3.131)
$$

It is easy to verify that the following statement is true.

$$
\lim_{\xi\to 0}\left(\Upsilon_{\xi}^{+}(\omega,\omega') + \Upsilon_{\xi}^{-}(\omega,\omega')\right) = \delta(\omega - \omega') \qquad (3.132)
$$

Let us go back to (3.117). Using (3.131) for to express $f^{\pm}(x)$ we get the following expression

$$\begin{aligned} f^{\pm}(x) &= \int_{-\infty}^{\infty} a(\omega') \int_{-\infty}^{\infty} \psi_{\omega}(x) \Upsilon^{\pm}(\omega, \omega') \mathrm{d}\omega \mathrm{d}\omega' \\ &= \frac{\chi\varepsilon}{\rho^2} \int_{-\infty}^{\infty} a(\omega') \left[\mp i \frac{|\det \mathbf{M}(\omega')|}{p(\omega')} \right] \int_{-\infty}^{\infty} \psi_{\omega}(x) \frac{\psi_{\omega}(d^+)}{\omega - \omega' \mp i\xi} \mathrm{d}\omega \mathrm{d}\omega' \end{aligned} \tag{3.133}$$

The goal is now to solve the integral

$$P_{\xi}(\omega') = \int_{-\infty}^{\infty} \psi_{\omega}(x) \frac{\psi_{\omega}(d^+)}{\omega - \omega' \mp i\xi} \mathrm{d}\omega \tag{3.134}$$

using the residue theorem just as we did in (3.68) on a closed contour Γ_R that we can see on Figure (2.4.1-1). In order to do this we define

$$P_{\xi}^{R}(\omega') = \int_{-R}^{R} \psi_{\omega}(x) \frac{\psi_{\omega}(d^+)}{\omega - \omega' \mp i\xi} \mathrm{d}\omega \tag{3.135}$$

which converges to (3.134) as $R \to \infty$. We use also the same notation for this contour Γ_R and we get

$$\begin{aligned} \int_{\Gamma_R} \psi_{\omega}(x) \frac{\psi_{\omega}(d^+)}{\omega - \omega' \mp i\xi} \mathrm{d}\omega &= P_{\xi}^{R}(\omega') + \int_{C_R} \psi_{\omega}(x) \frac{\psi_{\omega}(d^+)}{\omega - \omega' \mp i\xi} \mathrm{d}\omega \\ \frac{2\pi}{i} \sum_j \mathrm{Res} \left(\psi_{\omega}(x) \frac{\psi_{\omega}(d^+)}{\omega - \omega' \mp i\xi}, \omega_j \right) &= P_{\xi}^{R}(\omega') + \int_{C_R} \psi_{\omega}(x) \frac{\psi_{\omega}(d^+)}{\omega - \omega' \mp i\xi} \mathrm{d}\omega \end{aligned} \tag{3.136}$$

from which we can see, that if the the integral on the right hand side vanishes as R approaches infinity, then our function $P_{\xi}(\omega')$ will equal to the sum of residues. We will use the knowledge we gained in Appendix D about asymptotic behaviours of Airy functions along rays in the lower half of the complex frequency plane $\omega = Re^{i\theta}$, where we had two sectors $-\frac{2\pi}{3} < \theta < 0$ and $-\pi < \theta < -\frac{2\pi}{3}$. The asymptotic expression for the second sector decays in the limit $R \to \infty$ independently of x. In the first one first sector we will use (D.51) for the resonant states.

$$\psi_{\omega}(x) = \chi \begin{cases} \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x R^{\frac{1}{2}}} & x < -d \\ \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i(\beta+\sigma)\varrho R^{\frac{1}{2}}} e^{\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} & -d < x < d \\ -\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{3}{4}}\pi^{\frac{1}{2}}} e^{i\varrho R^{\frac{1}{2}}(\sigma-3\beta)} e^{-\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} & \\ i\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x R^{\frac{1}{2}}} - i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}} e^{i\beta\varrho R^{\frac{1}{2}}} e^{-\varpi x R^{\frac{1}{2}}} & d < x \end{cases} \tag{3.137}$$

where $\alpha = (2\varepsilon)^{-\frac{2}{3}}$, $\beta = 2\alpha\varepsilon d$, $\sigma = 2\alpha V_0$ and $\kappa = 2\alpha e^{i\theta}$. From (D.23), (D.39) and (D.40) we know that $\varrho_i < 0$ and $\varpi_r, \varpi_i > 0$ with

$$\varrho_r = (2\alpha)^{\frac{1}{2}} \cos\left(\frac{1}{2}\theta\right) \qquad \varpi_r = -\varepsilon(2\alpha)^{\frac{3}{2}} \sin\left(\frac{1}{2}\theta\right) \tag{3.138}$$

$$\varrho_i = (2\alpha)^{\frac{1}{2}} \sin\left(\frac{1}{2}\theta\right) \qquad \varpi_i = \varepsilon(2\alpha)^{\frac{3}{2}} \cos\left(\frac{1}{2}\theta\right) \tag{3.139}$$

Let us start analysing the various regions. For $x < -d$ The integrand on the right hand side in (3.136) becomes

$$\begin{aligned}
&\psi_\omega(x)\frac{\psi_\omega(d^+)}{\omega-\omega'\mp i\xi} \\
&\approx \chi^2 \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x R^{\frac{1}{2}}} \frac{\left(i\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi d R^{\frac{1}{2}}} - i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}} e^{i\beta\varrho R^{\frac{1}{2}}} e^{-\varpi d R^{\frac{1}{2}}}\right)}{\omega-\omega'\mp i\xi} \\
&= \chi^2 \frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}(\omega-\omega'\mp i\xi)} \left(-\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}} e^{\varpi R^{\frac{1}{2}}(x-d)} + \frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i2\beta\varrho R^{\frac{1}{2}}} e^{\varpi R^{\frac{1}{2}}(x+d)}\right) \\
&= -\chi^2 \frac{1}{\pi(\kappa R)^{\frac{1}{2}}(\omega-\omega'\mp i\xi)} e^{|\varpi_r| R^{\frac{1}{2}}(x-d)} e^{i\varpi_i R^{\frac{1}{2}}(x-d)} \\
&+ \chi^2 \frac{4(\kappa R)^{\frac{1}{2}}}{\pi\sigma(\omega-\omega'\mp i\xi)} e^{-i2\beta\varrho_r R^{\frac{1}{2}}} e^{R^{\frac{1}{2}}(|\varpi_r|(x+d)-2\beta|\varrho_i|)} e^{i\varpi_i R^{\frac{1}{2}}(x+d)} \\
&= -\chi^2 \frac{1}{\pi(\kappa R)^{\frac{1}{2}}(\omega-\omega'\mp i\xi)} e^{\left|\sin\left(\frac{1}{2}\theta\right)\right| R^{\frac{1}{2}}\varepsilon(2\alpha)^{\frac{3}{2}}(x-d)} e^{i\varpi_i R^{\frac{1}{2}}(x-d)} \\
&+ \chi^2 \frac{4(\kappa R)^{\frac{1}{2}}}{\pi\sigma(\omega-\omega'\mp i\xi)} e^{-i2\beta\varrho_r R^{\frac{1}{2}}} e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right| R^{\frac{1}{2}}(\varepsilon 2\alpha(x+d)-2\beta)} e^{i\varpi_i R^{\frac{1}{2}}(x+d)}
\end{aligned} \tag{3.140}$$

The first part of (3.140) decays in the limit $R \to \infty$ if

$$\begin{aligned}
\varepsilon(2\alpha)^{\frac{3}{2}}(x-d) &< 0 \\
x &< d
\end{aligned} \tag{3.141}$$

and for the second part if

$$\begin{aligned}
\varepsilon 2\alpha(x+d) - 2\beta &< 0 \\
x &< \frac{2\beta}{\varepsilon 2\alpha} - d \\
x &< \frac{4\alpha\varepsilon d}{\varepsilon 2\alpha} - d \\
x &< d
\end{aligned} \tag{3.142}$$

which is satisfied in the region $x < -d$, so the expression (3.140) decays.

Let us proceed to the region $-d < x < d$. Here we have

$$
\begin{aligned}
&\psi_\omega(x)\frac{\psi_\omega(d^+)}{\omega-\omega'\mp i\xi} \\
&\approx \chi^2\left(\frac{2(\kappa R)^{\frac14}}{i\pi^{\frac12}\sigma^{\frac12}}e^{-i(\beta+\sigma)\varrho R^{\frac12}}e^{\varpi R^{\frac12}\left(x+\frac{V_0}{\varepsilon}\right)}-\frac{\sigma^{\frac12}}{2(\kappa R)^{\frac34}\pi^{\frac12}}e^{i\varrho R^{\frac12}(\sigma-3\beta)}e^{-\varpi R^{\frac12}\left(x+\frac{V_0}{\varepsilon}\right)}\right) \\
&\frac{\left(i\frac{2(\kappa R)^{\frac14}}{\pi^{\frac12}\sigma^{\frac12}}e^{-i\beta\varrho R^{\frac12}}e^{\varpi dR^{\frac12}}-i\frac{\sigma^{\frac12}}{2\pi^{\frac12}(\kappa R)^{\frac34}}e^{i\beta\varrho R^{\frac12}}e^{-\varpi dR^{\frac12}}\right)}{\omega-\omega'\mp i\xi} \\
&=-\frac{\chi^2}{\omega-\omega'\mp i\xi}\left(\frac{2(\kappa R)^{\frac14}}{\pi^{\frac12}\sigma^{\frac12}}e^{-i(\beta+\sigma)\varrho R^{\frac12}}e^{\varpi R^{\frac12}\left(x+\frac{V_0}{\varepsilon}\right)}\frac{\sigma^{\frac12}}{2\pi^{\frac12}(\kappa R)^{\frac34}}e^{i\beta\varrho R^{\frac12}}e^{-\varpi dR^{\frac12}}\right. \\
&-\frac{\sigma^{\frac12}}{2(\kappa R)^{\frac34}\pi^{\frac12}}e^{i\varrho R^{\frac12}(\sigma-3\beta)}e^{-\varpi R^{\frac12}\left(x+\frac{V_0}{\varepsilon}\right)}\frac{\sigma^{\frac12}}{2\pi^{\frac12}(\kappa R)^{\frac34}}e^{i\beta\varrho R^{\frac12}}e^{-\varpi dR^{\frac12}} \\
&-\frac{2(\kappa R)^{\frac14}}{\pi^{\frac12}\sigma^{\frac12}}e^{-i(\beta+\sigma)\varrho R^{\frac12}}e^{\varpi R^{\frac12}\left(x+\frac{V_0}{\varepsilon}\right)}\frac{2(\kappa R)^{\frac14}}{\pi^{\frac12}\sigma^{\frac12}}e^{-i\beta\varrho R^{\frac12}}e^{\varpi dR^{\frac12}} \\
&\left.+\frac{\sigma^{\frac12}}{2(\kappa R)^{\frac34}\pi^{\frac12}}e^{i\varrho R^{\frac12}(\sigma-3\beta)}e^{-\varpi R^{\frac12}\left(x+\frac{V_0}{\varepsilon}\right)}\frac{2(\kappa R)^{\frac14}}{\pi^{\frac12}\sigma^{\frac12}}e^{-i\beta\varrho R^{\frac12}}e^{\varpi dR^{\frac12}}\right) \\
&=-\frac{\chi^2}{\omega-\omega'\mp i\xi}\left(\frac{1}{\pi(\kappa R)^{\frac12}}e^{-i\sigma\varrho_r R^{\frac12}}e^{R^{\frac12}\left[|\varpi_r|\left(x+\frac{V_0}{\varepsilon}-d\right)-\sigma|\varrho_i|\right]}e^{i\varpi_i R^{\frac12}\left(x+\frac{V_0}{\varepsilon}-d\right)}\right. \\
&-\frac{\sigma}{8\pi(\kappa R)^{\frac32}}e^{i\varrho_r R^{\frac12}(\sigma-2\beta)}e^{R^{\frac12}\left[-|\varpi_r|\left(x+\frac{V_0}{\varepsilon}+d\right)+|\varrho_i|(\sigma-2\beta)\right]}e^{-i\varpi_i R^{\frac12}\left(x+\frac{V_0}{\varepsilon}+d\right)} \\
&-\frac{4(\kappa R)^{\frac12}}{\pi\sigma}e^{-i(2\beta+\sigma)\varrho_r R^{\frac12}}e^{R^{\frac12}\left[|\varpi_r|\left(x+\frac{V_0}{\varepsilon}+d\right)-(2\beta+\sigma)|\varrho_i|\right]}e^{i\varpi_i R^{\frac12}\left(x+\frac{V_0}{\varepsilon}+d\right)} \\
&\left.+\frac{1}{\pi(\kappa R)^{\frac12}}e^{i\varrho_r R^{\frac12}(\sigma-4\beta)}e^{R^{\frac12}\left[-|\varpi_r|\left(x+\frac{V_0}{\varepsilon}-d\right)+|\varrho_i|(\sigma-4\beta)\right]}e^{-i\varpi_i R^{\frac12}\left(x+\frac{V_0}{\varepsilon}-d\right)}\right) \\
&=-\frac{\chi^2}{\omega-\omega'\mp i\xi}\left(\frac{1}{\pi(\kappa R)^{\frac12}}e^{-i\sigma\varrho_r R^{\frac12}}e^{(2\alpha)^{\frac12}\left|\sin\left(\frac12\theta\right)\right|R^{\frac12}\left[\varepsilon 2\alpha\left(x+\frac{V_0}{\varepsilon}-d\right)-\sigma\right]}e^{i\varpi_i R^{\frac12}\left(x+\frac{V_0}{\varepsilon}-d\right)}\right. \\
&-\frac{\sigma}{8\pi(\kappa R)^{\frac32}}e^{i\varrho_r R^{\frac12}(\sigma-2\beta)}e^{(2\alpha)^{\frac12}\left|\sin\left(\frac12\theta\right)\right|R^{\frac12}\left[-\varepsilon 2\alpha\left(x+\frac{V_0}{\varepsilon}+d\right)+(\sigma-2\beta)\right]}e^{-i\varpi_i R^{\frac12}\left(x+\frac{V_0}{\varepsilon}+d\right)} \\
&-\frac{4(\kappa R)^{\frac12}}{\pi\sigma}e^{-i(2\beta+\sigma)\varrho_r R^{\frac12}}e^{(2\alpha)^{\frac12}\left|\sin\left(\frac12\theta\right)\right|R^{\frac12}\left[\varepsilon 2\alpha\left(x+\frac{V_0}{\varepsilon}+d\right)-(2\beta+\sigma)\right]}e^{i\varpi_i R^{\frac12}\left(x+\frac{V_0}{\varepsilon}+d\right)} \\
&\left.+\frac{1}{\pi(\kappa R)^{\frac12}}e^{i\varrho_r R^{\frac12}(\sigma-4\beta)}e^{(2\alpha)^{\frac12}\left|\sin\left(\frac12\theta\right)\right|R^{\frac12}\left[-\varepsilon 2\alpha\left(x+\frac{V_0}{\varepsilon}-d\right)+(\sigma-4\beta)\right]}e^{-i\varpi_i R^{\frac12}\left(x+\frac{V_0}{\varepsilon}-d\right)}\right)
\end{aligned}
\tag{3.143}
$$

We ended up with four different parts that needs to be checked separately. Let us take the first two of them at once. In the following computations we analyse when do they decay. The first column represents the first part and the second column the second part.

$$\begin{aligned} \varepsilon 2\alpha \left(x + \frac{V_0}{\varepsilon} - d \right) - \sigma < 0 & \qquad -\varepsilon 2\alpha \left(x + \frac{V_0}{\varepsilon} + d \right) + (\sigma - 2\beta) < 0 \\ \frac{\sigma}{\varepsilon 2\alpha} + d - \frac{V_0}{\varepsilon} > x & \qquad \frac{\sigma - 2\beta}{\varepsilon 2\alpha} - \frac{V_0}{\varepsilon} - d < x \\ \frac{2\alpha V_0}{\varepsilon 2\alpha} + d - \frac{V_0}{\varepsilon} > x & \qquad \frac{2\alpha V_0 - 4\alpha\varepsilon d}{\varepsilon 2\alpha} - \frac{V_0}{\varepsilon} - d < x \\ d > x & \qquad -3d < x \end{aligned} \tag{3.144}$$

These conditions are satisfied in the current region $-d < x < d$. The second two parts of (3.143) converge to zero if

$$\begin{aligned} \varepsilon 2\alpha \left(x + \frac{V_0}{\varepsilon} + d \right) - (2\beta + \sigma) < 0 & \qquad -\varepsilon 2\alpha \left(x + \frac{V_0}{\varepsilon} - d \right) + (\sigma - 4\beta) < 0 \\ \frac{2\beta + \sigma}{\varepsilon 2\alpha} - \frac{V_0}{\varepsilon} - d > x & \qquad \frac{\sigma - 4\beta}{\varepsilon 2\alpha} - \frac{V_0}{\varepsilon} + d < x \\ \frac{4\alpha\varepsilon d + 2\alpha V_0}{\varepsilon 2\alpha} - \frac{V_0}{\varepsilon} - d > x & \qquad \frac{2\alpha V_0 - 8\alpha\varepsilon d}{\varepsilon 2\alpha} - \frac{V_0}{\varepsilon} + d < x \\ d > x & \qquad -3d < x \end{aligned} \tag{3.145}$$

which are obeyed as well for the same reasons. This leads to the fact that the integral in (3.136) vanishes for this region.

Let us move to the last region $d < x$. Here we get using (3.137) the following.

$$
\begin{aligned}
&\psi_\omega(x)\frac{\psi_\omega(d^+)}{\omega-\omega'\mp i\xi} \approx \chi^2\left(i\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi xR^{\frac{1}{2}}} - i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi xR^{\frac{1}{2}}}\right) \\
&\frac{\left(i\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi dR^{\frac{1}{2}}} - i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi dR^{\frac{1}{2}}}\right)}{\omega-\omega'\mp i\xi} \\
&= \frac{\chi^2}{\omega-\omega'\mp i\xi}\left(\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi xR^{\frac{1}{2}}}\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi dR^{\frac{1}{2}}}\right. \\
&- \frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi xR^{\frac{1}{2}}}\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi dR^{\frac{1}{2}}} \\
&- \frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi xR^{\frac{1}{2}}}\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi dR^{\frac{1}{2}}} \\
&\left.+\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi xR^{\frac{1}{2}}}\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi dR^{\frac{1}{2}}}\right) \\
&= \frac{\chi^2}{\omega-\omega'\mp i\xi}\left(\frac{1}{\pi(\kappa R)^{\frac{1}{2}}}e^{|\varpi_r|R^{\frac{1}{2}}(x-d)}e^{i\varpi_i R^{\frac{1}{2}}(x-d)}\right. \\
&- \frac{\sigma}{4(\kappa R)^{\frac{3}{2}}}e^{i2\beta\varrho_r R^{\frac{1}{2}}}e^{R^{\frac{1}{2}}[-|\varpi_r|(x+d)+2\beta|\varrho_i|]}e^{-i\varpi_i R^{\frac{1}{2}}(x+d)} \\
&- \frac{4(\kappa R)^{\frac{1}{2}}}{\pi\sigma}e^{-i2\beta\varrho_r R^{\frac{1}{2}}}e^{R^{\frac{1}{2}}[|\varpi_r|(x+d)-2\beta|\varrho_i|]}e^{i\varpi_i R^{\frac{1}{2}}(x+d)} \\
&\left.+\frac{1}{\pi(\kappa R)^{\frac{1}{2}}}e^{-|\varpi_r|R^{\frac{1}{2}}(x-d)}e^{-i\varpi_i R^{\frac{1}{2}}(x-d)}\right) \\
&= \frac{\chi^2}{\omega-\omega'\mp i\xi}\left(\frac{1}{\pi(\kappa R)^{\frac{1}{2}}}e^{\varepsilon(2\alpha)^{\frac{3}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}(x-d)}e^{i\varpi_i R^{\frac{1}{2}}(x-d)}\right. \\
&- \frac{\sigma}{4(\kappa R)^{\frac{3}{2}}}e^{i2\beta\varrho_r R^{\frac{1}{2}}}e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}[-\varepsilon 2\alpha(x+d)+2\beta]}e^{-i\varpi_i R^{\frac{1}{2}}(x+d)} \\
&- \frac{4(\kappa R)^{\frac{1}{2}}}{\pi\sigma}e^{-i2\beta\varrho_r R^{\frac{1}{2}}}e^{(2\alpha)^{\frac{1}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}[\varepsilon 2\alpha(x+d)-2\beta]}e^{i\varpi_i R^{\frac{1}{2}}(x+d)} \\
&\left.+\frac{1}{\pi(\kappa R)^{\frac{1}{2}}}e^{\varepsilon(2\alpha)^{\frac{3}{2}}\left|\sin\left(\frac{1}{2}\theta\right)\right|R^{\frac{1}{2}}(d-x)}e^{-i\varpi_i R^{\frac{1}{2}}(x-d)}\right) \qquad (3.146)
\end{aligned}
$$

This is a similar situation as in the previous case. We proceed therefore in an almost identical way. The first two terms in this expression decay exponentially if

$$
\begin{aligned}
x - d &< 0 & -\varepsilon 2\alpha(x+d) + 2\beta &< 0 \\
x &< d & x &> \frac{2\beta}{\varepsilon 2\alpha} - d \\
& & x &> \frac{4\alpha\varepsilon d}{\varepsilon 2\alpha} - d \\
& & x &> d
\end{aligned} \tag{3.147}
$$

and for the second two terms we have

$$
\begin{aligned}
\varepsilon 2\alpha(x+d) - 2\beta &< 0 & d - x &< 0 \\
x &< \frac{2\beta}{\varepsilon 2\alpha} - d & d &< x \\
x &< \frac{4\alpha\varepsilon d}{\varepsilon 2\alpha} - d & & \\
x &< d & &
\end{aligned} \tag{3.148}
$$

We got a variety of answers from which we can deduce that according to the current region $d < x$ the first and the third term in (3.146) grow while the second and fourth term decay for $R \to \infty$. The sum of these four terms gives us exponential growth, so the integrand in (3.136) diverges for $d < x$. This is exactly what the numerical evidence on page 62 indicated.

After analysing the integrand on the right hand side in (3.136) we can conclude, that it decays exponentially only when $x < d$ in the limit $R \to \infty$. If we go back to (3.136), then in this region and the limit $R \to \infty$ we have

$$
\frac{2\pi}{i} \sum_j \operatorname{Res}\left(\psi_\omega(x) \frac{\psi_\omega(d^+)}{\omega - \omega' \mp i\xi}, \omega_j \right) = P_\xi(\omega') \tag{3.149}
$$

and $f^{\pm}(x)$ in (3.133) becomes

$$
f^{\pm}(x) = \frac{\chi\varepsilon}{\rho^2} \int_{-\infty}^{\infty} a(\omega') \left[\mp i \frac{|\det \mathbf{M}(\omega')|}{p(\omega')} \right] \frac{2\pi}{i} \sum_j \operatorname{Res}\left(\psi_\omega(x) \frac{\psi_\omega(d^+)}{\omega - \omega' \mp i\xi}, \omega_j \right) \mathrm{d}\omega' \tag{3.150}
$$

In the following we compute the residues in (3.150). The expression in the residue function should first be expanded. Using (3.98) we get

$$\psi_\omega(x)\frac{\psi_\omega(d^+)}{\omega-\omega'\mp i\xi}=\frac{\chi^2}{\omega-\omega'\mp i\xi}$$

$$\begin{cases} \frac{2}{\pi^2|\det\mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x)) & x<-d \\ i\left[\left(\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(d))-\left(\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(d))\right] & \\ \frac{2}{\pi|\det\mathbf{M}(\omega)|}\left[(B_1'A_0-B_1A_0')\mathrm{Ai}(y_2(x)+(A_1A_0'-A_1'A_0)\mathrm{Bi}(y_2(x))\right] & \\ i\left[\left(\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(d))-\left(\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(d))\right] & -d<x<d \\ i\left[\left(\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(x))-\left(\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(x))\right] & \\ i\left[\left(\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(d))-\left(\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(d))\right] & d<x \end{cases} \tag{3.151}$$

which can be simplified further as

$$\psi_\omega(x)\frac{\psi_\omega(d^+)}{\omega-\omega'\mp i\xi}=\frac{\chi^2}{\omega-\omega'\mp i\xi}$$

$$\begin{cases} \frac{2}{\pi^2}\left(\frac{1}{\overline{\det\mathbf{M}}(\omega)}\mathrm{Ci}^+(y_1(d))-\frac{1}{\det\mathbf{M}(\omega)}\mathrm{Ci}^-(y_1(d))\right)\mathrm{Ai}(y_1(x)) & x<-d \\ \frac{2}{\pi}\left[(B_1'A_0-B_1A_0')\mathrm{Ai}(y_2(x)+(A_1A_0'-A_1'A_0)\mathrm{Bi}(y_2(x))\right] & \\ \left(\frac{1}{\overline{\det\mathbf{M}}(\omega)}\mathrm{Ci}^+(y_1(d))-\frac{1}{\det\mathbf{M}(\omega)}\mathrm{Ci}^-(y_1(d))\right) & -d<x<d \\ \left[\mathrm{Ci}^+(y_1(x))\mathrm{Ci}^-(y_1(d))+\mathrm{Ci}^-(y_1(x))\mathrm{Ci}^+(y_1(d))\right] & \\ -\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\mathrm{Ci}^+(y_1(x))\mathrm{Ci}^+(y_1(d))-\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}}(\omega)}\mathrm{Ci}^-(y_1(x))\mathrm{Ci}^-(y_1(d)) & d<x \end{cases} \tag{3.152}$$

The resonant eigenstates ω_j in the residue are the zero points of $\det\mathbf{M}(\omega)$ located in the lower complex frequency half-plane and similarly the zeros of $\overline{\det\mathbf{M}}(\omega)$ are in the upper half. It is clear, that the poles of (3.152) are determined by zeros of $\det\mathbf{M}(\omega)$, since C_R is in the lower half-plane, see (3.136). Remembering this we can ignore those terms that do not have ω_j as poles in the lower complex half-plane. Then we get

$$\psi_\omega(x)\frac{\psi_\omega(d^+)}{\omega-\omega'\mp i\xi}=\frac{\chi^2}{\omega-\omega'\mp i\xi}$$

$$\begin{cases} -\frac{2}{\pi^2}\frac{1}{\det\mathbf{M}(\omega)}\mathrm{Ci}^-(y_1(d))\mathrm{Ai}(y_1(x)) & x<-d \\ -\frac{2}{\pi}\left[(B_1'A_0-B_1A_0')\mathrm{Ai}(y_2(x)+(A_1A_0'-A_1'A_0)\mathrm{Bi}(y_2(x))\right] & -d<x<d \\ \frac{1}{\det\mathbf{M}(\omega)}\mathrm{Ci}^-(y_1(d)) & \\ -\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\mathrm{Ci}^+(y_1(x))\mathrm{Ci}^+(y_1(d)) & d<x \end{cases} \tag{3.153}$$

According to the residue theorem and using (3.153), we have

$$\operatorname{Res}\left(\psi_\omega(x)\frac{\psi_\omega(d^+)}{\omega-\omega'\mp i\xi},\omega_j\right)=\lim_{\omega\to\omega_j}\left\{\frac{\omega-\omega_j}{\omega-\omega'\mp i\xi}\psi_\omega(x)\psi_\omega(d^+)\right\}$$
$$=\lim_{\omega\to\omega_j}\left\{\frac{\omega-\omega_j}{\det\mathbf{M}(\omega)}\right\}$$
$$\frac{\chi^2}{\omega_j-\omega'\mp i\xi}\begin{cases}-\frac{2}{\pi^2}\mathrm{Ci}_j^-(y_1(d))\mathrm{Ai}_j(y_1(x)) & x<-d\\ -\frac{2}{\pi}\left[(B_1'A_0-B_1A_0')_j\mathrm{Ai}_j(y_2(x))\right. & -d<x<d\\ \left.+(A_1A_0'-A_1'A_0)_j\mathrm{Bi}_j(y_2(x))\right]\mathrm{Ci}_j^-(y_1(d)) & \\ -\overline{\det\mathbf{M}}(\omega_j)\mathrm{Ci}_j^+(y_1(d))\mathrm{Ci}_j^+(y_1(x)) & d<x\end{cases}$$
$$=\frac{\chi}{\omega_j-\omega'\mp i\xi}\psi_j(x) \tag{3.154}$$

with

$$\psi_j(x)=\chi\lim_{\omega\to\omega_j}\left\{\frac{\omega-\omega_j}{\det\mathbf{M}(\omega)}\right\}\begin{cases}-\frac{2}{\pi^2}\mathrm{Ci}_j^-(y_1(d))\mathrm{Ai}_j(y_1(x)) & x<-d\\ -\frac{2}{\pi}\left[(B_1'A_0-B_1A_0')_j\mathrm{Ai}_j(y_2(x))\right. & -d<x<d\\ \left.+(A_1A_0'-A_1'A_0)_j\mathrm{Bi}_j(y_2(x))\right] & \\ \mathrm{Ci}_j^-(y_1(d)) & \\ -\overline{\det\mathbf{M}}(\omega_j)\mathrm{Ci}_j^+(y_1(d))\mathrm{Ci}_j^+(y_1(x)) & d<x\end{cases} \tag{3.155}$$

where we denoted $\mathrm{Ai}_j(y_1(x))=\mathrm{Ai}(y_1(x))\Big|_{\omega=\omega_j}$, $(B_1'A_0-B_1A_0')_j=(B_1'A_0-B_1A_0')\Big|_{\omega=\omega_j}$ etc. One can see that the functions $\psi_j(x)$ are proportional to the resonant states (3.65). Substituting (3.154) back into (3.150) and taking the limit $\xi\to 0$ we get

$$\lim_{\xi\to 0}f^\pm(x)=\frac{\chi\varepsilon}{\rho^2}\int_{-\infty}^{\infty}a(\omega')\left[\mp i\frac{|\det\mathbf{M}(\omega')|}{p(\omega')}\right]\frac{2\pi}{i}\sum_j\frac{\chi}{\omega_j-\omega'\mp i\xi}\psi_j(x)\mathrm{d}\omega'$$
$$=\sum_j\frac{\chi 2\pi\varepsilon}{\rho^2}\int_{-\infty}^{\infty}a(\omega')\left[\mp\frac{|\det\mathbf{M}(\omega')|}{p(\omega')}\right]\frac{\chi}{\omega_j-\omega'}\mathrm{d}\omega'\psi_j(x)$$
$$=\sum_j c_j\psi_j(x) \tag{3.156}$$

where

$$c_j = \frac{\chi 2\pi\varepsilon}{\rho^2} \int_{-\infty}^{\infty} a(\omega') \left[\mp \frac{|\det \mathbf{M}(\omega')|}{p(\omega')} \right] \frac{\chi}{\omega_j - \omega'} \mathrm{d}\omega' \tag{3.157}$$

with $\rho = 2\alpha\varepsilon$.

It is worth mentioning that this result is rather surprising, because the convergence of the sum in (3.82) does not depend on the depth of the well V_0. As a consequence of this, one might ask what if we set $V_0 = 0$? Then the width of the well d would lost its meaning, so would we then get convergence everywhere? To answer this we must remember that for this case we need to consider $V_0 = 0$ all the way from the beginning. Let us see what do the scattering states look like in this special case. For convenience we choose $d > 0$ although we know that it has no significant meaning. The variable transformation $y_2(x)$ in the region $-d < x < d$ now becomes $y_1(x)$ as the rest. So we have from (3.59)

$$\psi_\omega(x) = \begin{cases} a_1 \mathrm{Ai}(y_1(x)) & x < -d \\ a_2 \mathrm{Ai}(y_1(x)) + a_3 \mathrm{Bi}(y_1(x)) & -d < x < d \\ a_4 \mathrm{Ci}^+(y_1(x)) + a_5 \mathrm{Ci}^-(y_1(x)) & d < x \end{cases} \tag{3.158}$$

For this function $\psi_\omega(x)$ there are 4 continuity condition at the walls of the well $x = -d, d$. Applying these we get a system

$$\mathbf{M}_0(\omega)\vec{a} = \begin{pmatrix} A_0 & -A_0 & -B_0 & 0 \\ A_0' & -A_0' & -B_0' & 0 \\ 0 & A_3 & B_3 & -C_3 \\ 0 & A_3' & B_3' & -C_3' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = a_5 \begin{pmatrix} 0 \\ 0 \\ D_3 \\ D_3' \end{pmatrix} \tag{3.159}$$

where we used the same notations as in (3.41) and with additional $A_3 = \mathrm{Ai}(y_1(d))$, $B_3 = \mathrm{Bi}(y_1(d))$ and $D_3 = \mathrm{Ci}^-(y_1(d))$. The determinant of this matrix is

$$\det \mathbf{M}_0(\omega) = C_3 A_3' - C_3' A_3 = (B_3 + iA_3)A_3' - (B_3' + iA_3')A_3 = B_3 A_3' - B_3' A_3 = -\frac{1}{\pi} \tag{3.160}$$

where we used the formula (10.4.10) from [1]. The solution to this system is then

$$a_1 = a_5 \frac{C_3 D_3' - C_3' D_3}{\det \mathbf{M}_0(\omega)} = a_5 \frac{2i}{\pi \det \mathbf{M}_0(\omega)} = -a_5 2i \tag{3.161}$$

$$a_2 = a_5 \frac{C_3 D_3' - C_3' D_3}{\det \mathbf{M}_0(\omega)} = a_5 \frac{2i}{\pi \det \mathbf{M}_0(\omega)} = -a_5 2i \tag{3.162}$$

$$a_3 = 0 \tag{3.163}$$

$$a_4 = -a_5 \frac{D_3 A_3' - D_3' A_3}{\det \mathbf{M}_0(\omega)} = a_5 \frac{1}{\pi \det \mathbf{M}_0(\omega)} = -a_5 \tag{3.164}$$

We choose a_5 to be -1. Substituting this back into (3.161)-(3.164) the resonant states (3.158) become

$$\begin{aligned}\psi_\omega(x) &= \chi \begin{cases} 2i\mathrm{Ai}(y_1(x)) & x < d \\ \mathrm{Ci}^+(y_1(x)) - \mathrm{Ci}^-(y_1(x)) & d < x \end{cases} \\ &= \chi 2i\mathrm{Ai}(y_1(x)) \quad \forall x \end{aligned} \tag{3.165}$$

with some normalization constant χ which can be obtained in a similar way as we did in Appendix C (C.80) - (C.84) for the regions $x, x' < 0$. We find out that this constant in this case is $\chi = 2^{-\frac{2}{3}}\varepsilon^{-\frac{1}{6}}$. In both of the proofs we used the complex contour Γ_R to prove that the function $f(x)$ can be represented as a linear combination of the resonant states and we got the equations (3.68) and (3.136). We needed to compute the integral along this contour using Cauchy's residue theorem which gives us the sum over the poles of the function that we are interested in. In both cases, the poles were determined by the zero points ω_j of the determinant $\det \mathbf{M}(\omega)$. In our case $V_0 = 0$, the determinant is $\det \mathbf{M}_0(\omega) = -\frac{1}{\pi}$ which has no zeros, hence the residues become zero. The equation we are left with is now from (3.80)

$$\delta(x - x') = -\lim_{R\to\infty} \int_{C_R} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega \tag{3.166}$$

This equation is true and it follows from the Cauchy's residue theorem.

As another consequence of the fact that the determinant has no zeros is, that there are no energy eigenvalues, so resonant states can not exist. However, scattering states can exist. Scattering states are states where an ingoing wave with amplitude a_5 generates a transmitted and reflected wave. A resonant state is a situation where there is no ingoing wave $a_5 = 0$. Such a solutions are non-trivial only if the determinant is equal to zero which in the case when $V_0 = 0$

is not possible, because $\det \mathbf{M}_0(\omega) = -\frac{1}{\pi}$.

As we did after the weaker and stronger proof in the Dirac delta case, we can discuss the difference between these two proofs for the square well as well. The weaker proof showed us that the integrand in (3.82) vanishes if the function $f(x)$ has its compact support confined in the region $x < d$. However, the stronger proof provides a more exact condition. The integrand in (3.82) decays pointwise for all $x < d$. This means, that if the center of the compact support of the function $f(x)$ is located on $x = d$, then the part where $x < d$ will converge, whereas the part $x > d$ will diverge.

3.4 Time dependent energy field for the square well potential

In this section we will numerically solve the Schrödinger equation for square well potential but considering a time dependent energy field $\varepsilon(t)$. As before in the Dirac delta case, we are not considering space dependency because the size of the atom is very small compared to the space change to any laser fields. The proposed form of the pulse is a Gaussian wave packet

$$\varepsilon(t) = e^{-\gamma(t-t_0)^2} \cos\left(\omega_\varepsilon(t-t_0) - \delta\right) \tag{3.167}$$

where ω_ε is the center frequency and γ, δ, t_0 are parameters. On Figure (3.4-1) we can see this function(green) for $\omega_\varepsilon = 0.02, \gamma = 10^{-5}, \delta = 0, t_0 = 900$. The red function is $\mathrm{Re}\left[e^{i\omega_0 t}\right]$, where ω_0 is the eigenvalue corresponding to one of the ground states, in particular $\omega_0 = -0.779988$. This ground state $\psi_{\omega_0}(x)$ of the square well was computed using $V_0 = 1, d = 4$. This state will be used as the initial condition for our scheme in (3.47) as $\Psi(x,0) = \vec{\psi}(0) = \psi_{\omega_0}(x)$.

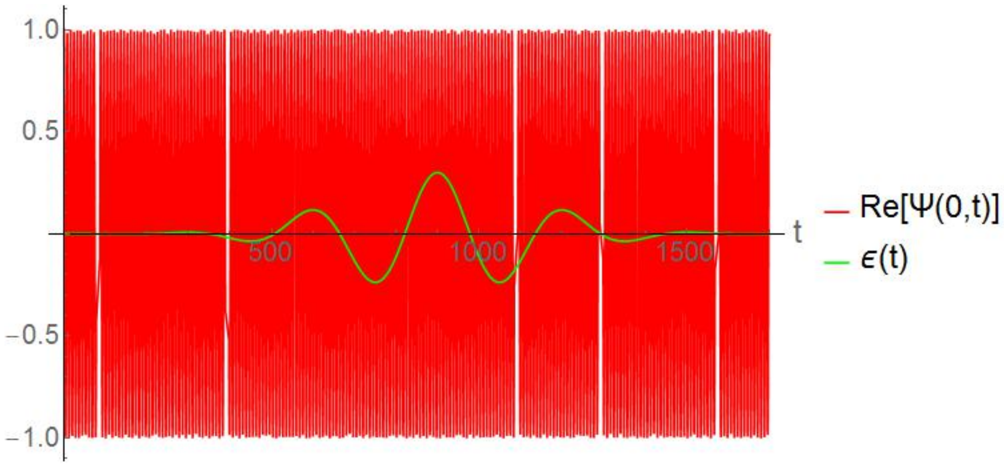
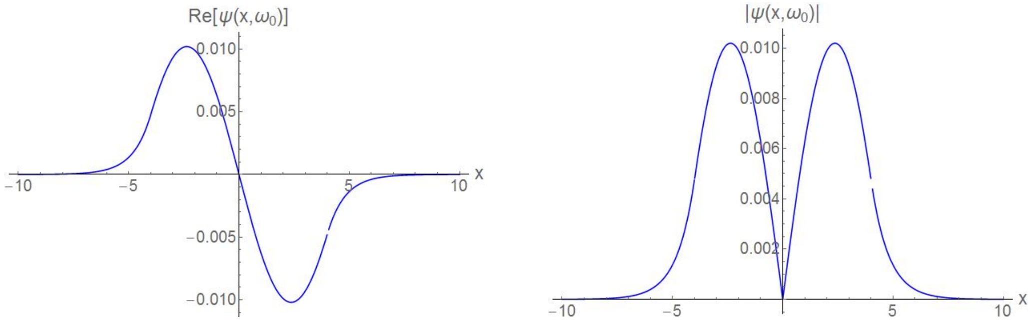


Figure 3.4-1: Plot of $\varepsilon(t)$ and the real part of the time dependency $\text{Re}\left[e^{i\omega_0 t}\right]$ of the solution to the Schrödinger equation for square well. The eigenvalue $\omega_0 = -0.779988$ belongs to one of the ground states computed using the parameters $V_0 = 1, d = 4$.

The ground state $\psi_{\omega_0}(x)$ can be seen on Figure (3.4-2).



(a) (b)

Figure 3.4-2: Plot of the real part and absolute value of the ground state $\psi_{\omega_0}(x)$ for square well potential, where $\omega_0 = -0.779988$.

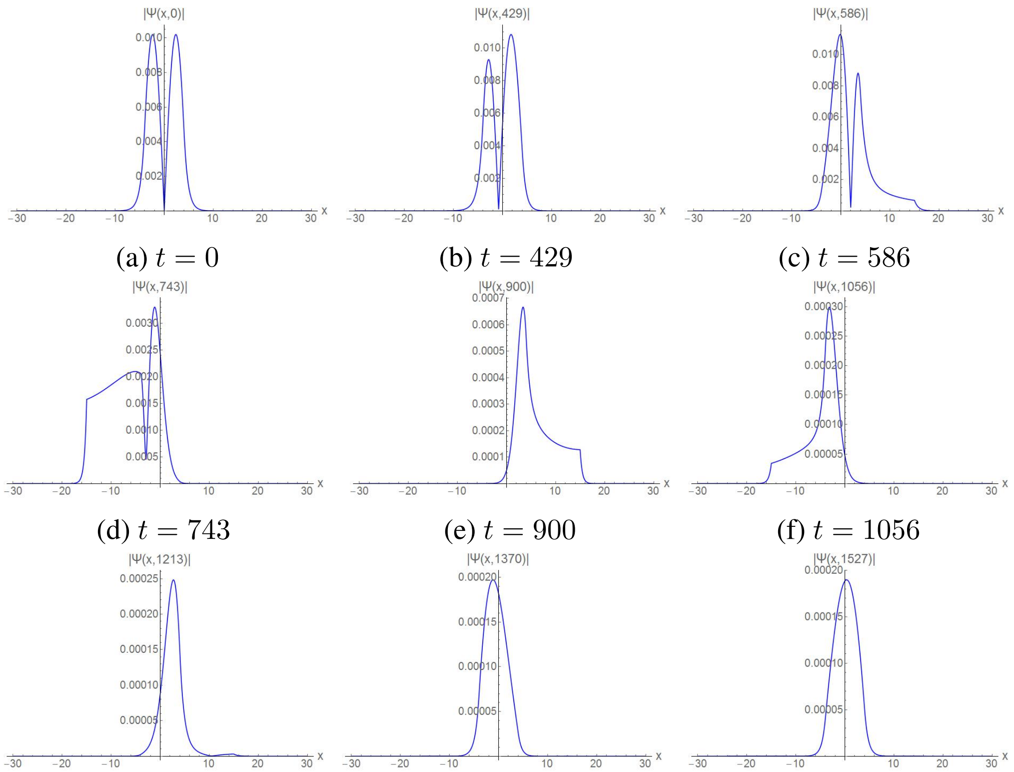
In our previous computation we assumed positive $\varepsilon > 0$. Here we deal with ε that changes signs. We need to figure out what the solution is for $\varepsilon < 0$. Let us define a new variable $\tilde{x} = -x$. Then we have

$$\begin{aligned} i\Psi_t(x,t) &= -\frac{1}{2}\partial_{xx}\Psi(x,t) - A\delta(x)\Psi(x,t) + |\varepsilon|x\Psi(x,t) \\ i\Psi_t(\tilde{x},t) &= -\frac{1}{2}\partial_{\tilde{x}\tilde{x}}\Psi(\tilde{x},t) - A\delta(\tilde{x})\Psi(\tilde{x},t) - |\varepsilon|\tilde{x}\Psi(\tilde{x},t) \end{aligned} \tag{3.168}$$

so for $\varepsilon < 0$ the solution is $\Psi(-x, t; |\varepsilon|)$, where $\Psi(x, t; |\varepsilon|)$ is the solution for $\varepsilon > 0$. We can solve the Schrödinger equation numerically in the same way as before as (3.47). The only difference will be that we solve it on the flipped x-axis across the y-axis when $\varepsilon < 0$, so these regions in time should be determined. We determine them from (3.167) and we get the intervals $[t_j, t_{j+1}]$, where t_j are the zeros of (3.167.

$$
\begin{aligned}
e^{-\gamma(t-t_0)^2}\cos\left(\omega_\varepsilon(t-t_0)-\delta\right) &< 0 \\
\cos\left(\omega_\varepsilon(t-t_0)-\delta\right) &< 0 \\
\frac{\pi}{2}+2j\pi < \omega_\varepsilon(t-t_0)-\delta &< \frac{3\pi}{2}+2j\pi, \quad j\in\mathbb{Z} \\
t_j=\frac{\pi}{2\omega_\varepsilon}+\frac{2j\pi}{\omega_\varepsilon}+\frac{\delta}{\omega_\varepsilon}+t_0 < t &< \frac{3\pi}{2\omega_\varepsilon}+\frac{2j\pi}{\omega_\varepsilon}+\frac{\delta}{\omega_\varepsilon}+t_0=t_{j+1}, \quad j\in\mathbb{Z} \qquad (3.169)
\end{aligned}
$$

In those intervals, where $\varepsilon(t) < 0$, the Schrödinger equation is solved on the flipped x-axis.



(a) $t = 0$ (b) $t = 429$ (c) $t = 586$

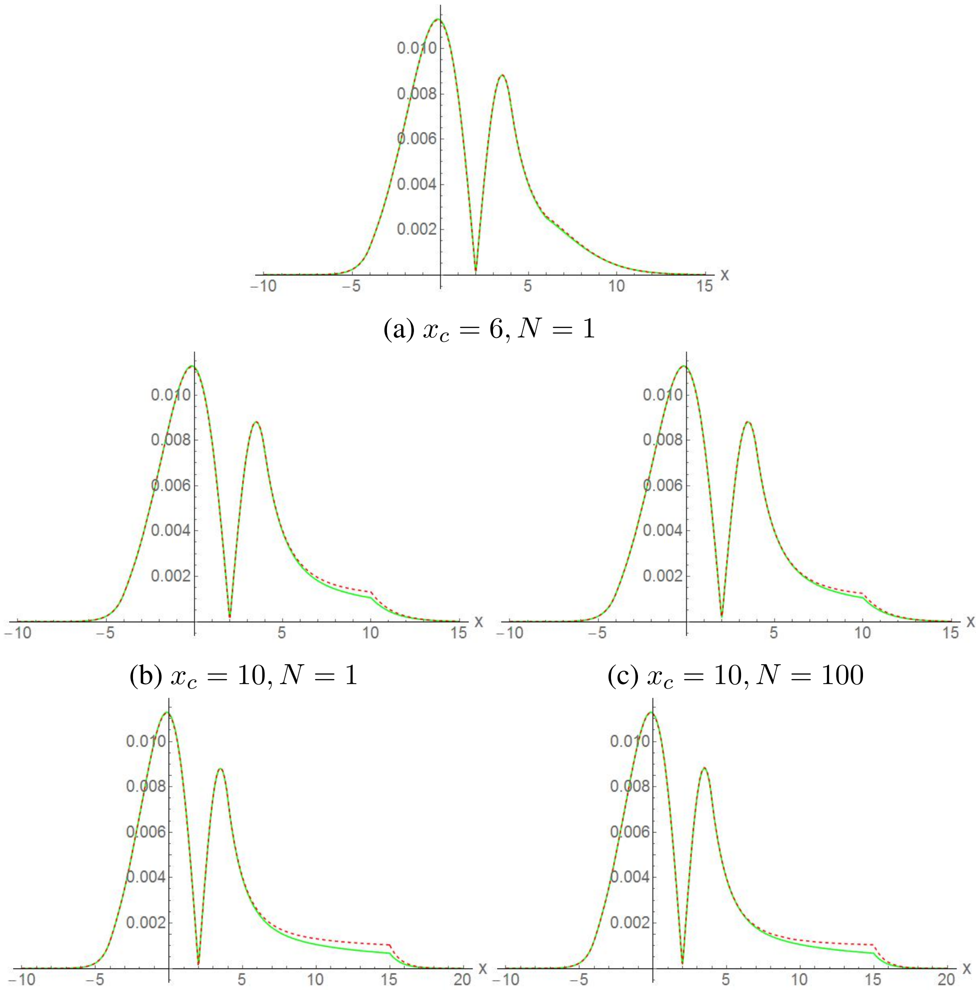
(d) $t = 743$ (e) $t = 900$ (f) $t = 1056$

(g) $t = 1213$ (h) $t = 1370$ (i) $t = 1527$

Figure 3.4-3: Absolute value of the numerical solution $\Psi(x, t)$ in particular times. The parameters that were used are $x_c = 15, V_0 = 1.d = 4, \theta = \frac{\pi}{2}$ and the numerical parameters are $L_1 = L_2 = 30, N = 801, k = 4$.

On Figure (3.4-3) were the parameters for the laser pulse the same as in previous figure and the data for the equation were $x_c = 15, V_0 = 1.d = 4, \theta = \frac{\pi}{2}$. We used the numerical parameters $L_1 = L_2 = 30, N = 801, k = 4$ so the spatial grid parameter was $\Delta x = 0.1$, where L_1, L_2 are the boundaries of the spatial space, N is the number of grid points and k is the order of the numerical scheme. The particular times were picked as the centres of the intervals (3.169).

Let us choose the time corresponding to the middle of the 5-th interval $t = 586$. The numerical solution to the Schrödinger equation at this time is in figure (3.4-3c). We will compare this particular solution with an expansion using analytical resonant states as in (3.54) to see if we can find a detectable deviation. In this expansion we will use the perturbed ground state corresponding to $\omega_0 = -0.779988$ as the starting state.



(a) $x_c = 6, N = 1$

(b) $x_c = 10, N = 1$

(c) $x_c = 10, N = 100$

(d) $x_c = 15, N = 1$

(e) $x_c = 15, N = 100$

Figure 3.4-4: Comparing numerical solutions (green) at time $t = 586$ with $x_c = 6, 10, 15$ with the resonant states expansion (red dashed) using the various number of states from A-series.

The parameters for the numerical solution in Figure (3.4-4) were the same as in (3.4-3) except for x_c for which we used three different values. In (3.4-4a) it was $x_c = 6$, while in the other subfigures it was $x_c = 10$ and $x_c = 15$. We can see that in (3.4-4a) the expansion looks perfect comparing to the numerical version, even if we used only one mode. In the other cases, since we picked a different value of x_c, we needed more modes to bring the two functions closer. A smaller improvement can be seen from (3.4-4b) to (3.4-4c) even if the matches are not perfect.

In the last case where $x_c = 15$ we can not see any clear improvement, however it is present, but it would take much more modes to manifest itself. These expansion were done using only resonant states from A-series, because the modes from C-series do not give any significant contribution to the expansion. On the other hand, the A-series are the dominant ones, especially in the negative axis, where the C-series modes are practically zero.

4 Conclusion

At the beginning of this paper we introduced the Schrödinger equation which we solved successfully for two kinds of potentials. We analytically found the resonant states for both cases and derived the numerical solution to the same tasks to compare. Due to the exponential growth of the resonant states, we replaced the real axis with a complex line upon which these states were evaluated. This prevented the growth and as a consequence of this, the bi-orthogonal product was introduced. Later on, we assumed the energy field or the laser pulse to be time dependent and managed to solve the equation numerically.

The main effort in this work was, however, the analysing of the completeness of the resonant states. In other words, we investigated whether the resonant states can form a complete set of functions to be used to represent any function with a compact support as a linear basis. The theory behind the completeness was provided in chapters 2.4 and 3.3 for Dirac delta potential and square well potential respectively. At the end of both proofs we found some interesting results. In the Dirac delta case the conclusion was that any function $f(x)$ with a compact support confined in the negative real axis can be expanded using the resonant states. In the stronger version of this proof we found out that this compact support does not have to be all the way in the negative part. The desired convergence happened for $x < 0$. This result was a bit different for the square well. Instead of the negative real axis, we talked about the region $x < d$, where d was the positive boundary of the well. The convergence was present for $x < d$ and it did not depend on the depth of the well V_0. We also discussed the option when $V_0 = 0$, where we concluded that it is not possible to get any resonant states in this case, so we must have $V_0 > 0$. This conclusion gives rise to an interesting thought. Let us have two such square wells next to each other with different depths. Using the same methods, similar results could be shown. If we would proceed further, we could consider a continuous potential well $V(x)$ over the whole space which decays at $\pm\infty$ approximated with narrow square wells with suitable depths V_i and widths Δx. The outermost square wells would be then more and more shallow such that $0 < V_i \ll 1$. Without our result one would think, that we would loose the convergence in those places. Our result however tells us, it is not necessarily the case. On the other side we do not know if this result can be generalized to this case when one has many square wells one after each other, therefore it is a conjecture as we stated in the introduction. This statement could be supported only by explicit computations of more examples. If our result turns out to be true for these more complicated examples, one could think of constructing a formal proof of this statement. It would be interesting to see the properties of those resonant states obtained by eventually taking the limit in that potential discretization $\lim_{i\to\infty,\Delta x\to 0} V_i = V(x)$. We can see an example of this approximation on Figure 4-1.

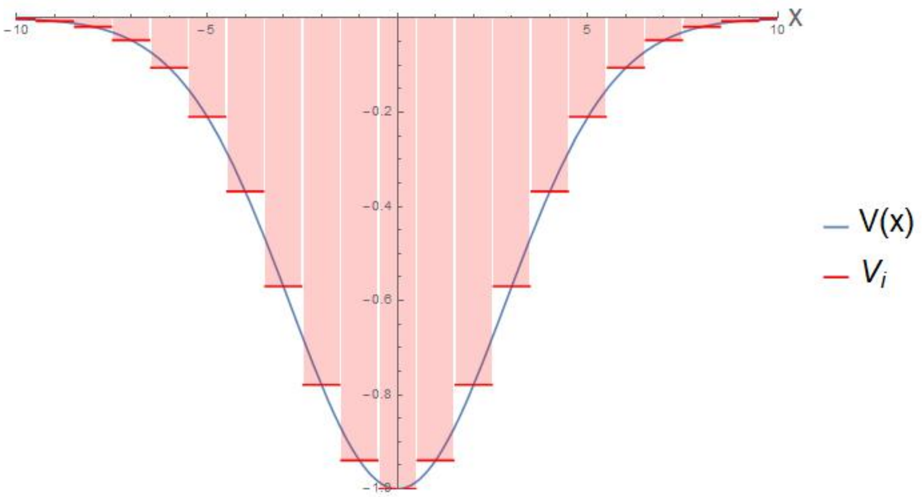


Figure 4-1: A continuous potential $V(x)$ with a suitable disretization V_i.

As we mentioned in the introduction, this idea of a continuous potential without a compact support is considerably an important conjecture. It can be formulated for example as a statement which says that with a potential we just described, we would get a convergence of the series (1.9) for all x. It is a very important result of this work since the solution to that discovery would reveal other interesting facts as well. That is, however, a topic for a possibly near future.

Appendix A

This Appendix provides a computation of the normalisation constant χ for the scattering form of the resonant states in (2.99). We know, that the Hamiltonian in (2.22) is Hermitian, so the scattering states (2.99) form a set with a continuous spectrum. The completeness of these states can be therefore expressed through the relation

$$\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega = \delta(x - x') \tag{A.1}$$

where the scattering states are

$$\psi_\omega(x) = \chi \begin{cases} \frac{2}{\pi|\det \mathbf{M}(y_0)|}\mathrm{Ai}(y(x)) & x \leq 0 \\ -i\left(\frac{\det \overline{\mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y(x)) + i\left(\frac{\det \mathbf{M}(y_0)}{\det \overline{\mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y(x)) & x > 0 \end{cases} \tag{A.2}$$

with

$$y(x) = -2\alpha(\varepsilon x + \omega) \tag{A.3}$$

$$\det \mathbf{M}(y_0) = \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\mathrm{Ai}(y_0)\mathrm{Ci}^+(y_0) \tag{A.4}$$

where $\alpha = (2\varepsilon)^{-\frac{2}{3}}$ and $y_0 = y(0)$. Let us integrate (A.1) over the interval $I_\epsilon = [x - \epsilon, x + \epsilon]$ and take the limit as ϵ approaches zero. We get the formula that determines χ.

$$\lim_{\epsilon\to 0}\int_{I_\epsilon}\left(\int_{-\infty}^{\infty}\chi\tilde{\psi}_\omega(x)\chi\tilde{\psi}_\omega(x')\mathrm{d}\omega\right)\mathrm{d}x' = 1$$

$$\lim_{\epsilon\to 0}\int_{I_\epsilon}\left(\int_{-\infty}^{\infty}\tilde{\psi}_\omega(x)\tilde{\psi}^*_\omega(x')\mathrm{d}\omega\right)\mathrm{d}x' = \chi^{-2} \tag{A.5}$$

where we denoted $\tilde{\psi}_\omega$ as ψ_ω with $\chi = 1$. Since we deal with an infinitely small interval, any finite part of the integral

$$Q(x, x') = \int_{-\infty}^{\infty}\tilde{\psi}_\omega(x)\tilde{\psi}_\omega(x')\mathrm{d}\omega \tag{A.6}$$

will give no contribution to the value of χ. This has a very useful consequence. In this integral we integrate over real ω. We integrate with respect to ω. We can see from (A.3) that the positions of x and ω are basically the same. Let us see what happens when $\omega \to -\infty$. The argument $y(x)$ turns into a big positive value, thus we can use the expression (10.4.59) and (10.4.63) from [1] for Airy functions to conclude that the function $\psi_\omega(x)$ decays exponentially for all x. This fact allows us to write

$$Q(x,x') = \int_q^\infty \tilde{\psi}_\omega(x)\tilde{\psi}_\omega(x')\mathrm{d}\omega \tag{A.7}$$

for some positive value q, because the part from $-\infty$ to q is finite. Also, q can be chosen as large as we like, so we can use the asymptotic expressions of the Airy functions $\mathrm{Ci}^\pm$ (2.31) and (10.4.59) from [1]. Define

$$z(x) = -y(x) = 2\alpha(\varepsilon x + \omega) = \beta x + \gamma\omega \tag{A.8}$$

where $\beta = 2\alpha\varepsilon$ and $\gamma = 2\alpha$. Then the Airy functions are

$$\mathrm{Ai}(-z(x)) \approx \frac{1}{2i}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\frac{2}{3}(z(x))^{\frac{3}{2}}+\frac{\pi}{4}\right)} - e^{-i\left(\frac{2}{3}(z(x))^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right) \tag{A.9}$$

$$\mathrm{Ci}^+(-z(x)) \approx \pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{i\left(\frac{2}{3}(z(x))^{\frac{3}{2}}+\frac{\pi}{4}\right)} \tag{A.10}$$

$$\mathrm{Ci}^-(-z(x)) \approx \pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{-i\left(\frac{2}{3}(z(x))^{\frac{3}{2}}+\frac{\pi}{4}\right)} \tag{A.11}$$

As the first case, we take $x, x' > 0$. We now expand the integrand in $Q(x,x')$ and we get

$$\begin{aligned} Q(x,x') = &\int_q^\infty \left\{\mathrm{Ci}^+(-z(x))\mathrm{Ci}^-(-z(x')) + \mathrm{Ci}^-(-z(x))\mathrm{Ci}^+(-z(x'))\right\}\mathrm{d}\omega \\ &- \int_q^\infty \left\{\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\mathrm{Ci}^-(-z(x))\mathrm{Ci}^-(-z(x')) \right. \\ &\left. + \frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\mathrm{Ci}^+(-z(x))\mathrm{Ci}^+(-z(x'))\right\}\mathrm{d}\omega \end{aligned} \tag{A.12}$$

Let us consider the various terms and start with the term

$$\begin{aligned} &\int_q^\infty \mathrm{Ci}^-(-z(x))\mathrm{Ci}^-(-z(x'))\mathrm{d}\omega \\ &\approx \frac{1}{\pi}\int_q^\infty \left[(\beta x+\gamma\omega)(\beta x'+\gamma\omega)\right]^{-\frac{1}{4}} e^{-i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}+(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]-i\frac{\pi}{2}}\mathrm{d}\omega \\ &\approx \frac{1}{i\pi}\int_q^\infty \left[(\gamma\omega)^2\left(\frac{\beta x}{\gamma\omega}+1\right)\left(\frac{\beta x'}{\gamma\omega}+1\right)\right]^{-\frac{1}{4}} e^{-i\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}\left[\left(\frac{\beta x}{\gamma\omega}+1\right)^{\frac{3}{2}}+\left(\frac{\beta x'}{\gamma\omega}+1\right)^{\frac{3}{2}}\right]}\mathrm{d}\omega \\ &\approx \frac{1}{i\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}e^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\mathrm{d}\omega \end{aligned} \tag{A.13}$$

Due to large ω, we dropped the terms in the second line, where ω was in the denominator. It is easy to see that this integral converges, hence it is finite and gives no contribution to χ.

Similarly, we get for the next term in (A.12)

$$\int_q^\infty \mathrm{Ci}^+(-z(x))\mathrm{Ci}^+(-z(x'))\mathrm{d}\omega \approx \frac{i}{\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}e^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\mathrm{d}\omega \tag{A.14}$$

which does not give any contribution either. We proceed to the cross terms.

$$\begin{aligned}
&\int_q^\infty \mathrm{Ci}^+(-z(x))\mathrm{Ci}^-(-z(x'))\mathrm{d}\omega \\
&\approx \frac{1}{\pi}\int_q^\infty [(\beta x+\gamma\omega)(\beta x'+\gamma\omega)]^{-\frac{1}{4}}\, e^{i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}-(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]}\mathrm{d}\omega \\
&\approx \frac{1}{\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}e^{i\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}\left[\left(\frac{\beta x}{\gamma\omega}+1\right)^{\frac{3}{2}}-\left(\frac{\beta x'}{\gamma\omega}+1\right)^{\frac{3}{2}}\right]}\mathrm{d}\omega \\
&\approx \frac{1}{\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}e^{i\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}\left[\frac{3\beta x}{2\gamma\omega}+1-\frac{3\beta x'}{2\gamma\omega}-1\right]}\mathrm{d}\omega \approx \frac{1}{\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}e^{i(\gamma\omega)^{\frac{1}{2}}\beta[x-x']}\mathrm{d}\omega \\
&= \left|\begin{array}{c} \beta\gamma^{\frac{1}{2}}\omega^{\frac{1}{2}}=u \\ \frac{1}{2}\beta\gamma^{\frac{1}{2}}\omega^{-\frac{1}{2}}\mathrm{d}\omega=\mathrm{d}u \\ \omega^{-\frac{1}{2}}\mathrm{d}\omega=\frac{2}{\beta\gamma^{\frac{1}{2}}}\mathrm{d}u \end{array}\right| = \frac{2}{\pi\beta\gamma}\int_r^\infty e^{iu[x-x']}\mathrm{d}u
\end{aligned} \tag{A.15}$$

where $r=\beta\sqrt{\gamma q}$. In a almost exactly similar way we compute the second cross term too.

$$\begin{aligned}
&\int_q^\infty \mathrm{Ci}^-(-z(x))\mathrm{Ci}^+(-z(x'))\mathrm{d}\omega \\
&\approx \frac{1}{\pi}\int_q^\infty [(\beta x+\gamma\omega)(\beta x+\gamma\omega)]^{-\frac{1}{4}}\, e^{-i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}-(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]}\mathrm{d}\omega \\
&\approx \frac{1}{\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}e^{-i(\gamma\omega)^{\frac{1}{2}}\beta[x-x']}\mathrm{d}\omega = \left|\begin{array}{c} -\beta\gamma^{\frac{1}{2}}\omega^{\frac{1}{2}}=u \\ -\frac{1}{2}\beta\gamma^{\frac{1}{2}}\omega^{-\frac{1}{2}}\mathrm{d}\omega=\mathrm{d}u \\ \omega^{-\frac{1}{2}}\mathrm{d}\omega=-\frac{2}{\beta\gamma^{\frac{1}{2}}}\mathrm{d}u \end{array}\right| \\
&= -\frac{2}{\pi\beta\gamma}\int_{-r}^{-\infty} e^{iu[x-x']}\mathrm{d}u = \frac{2}{\pi\beta\gamma}\int_{-\infty}^{-r} e^{iu[x-x']}\mathrm{d}u
\end{aligned} \tag{A.16}$$

Summing the terms up in (A.12) which give contribution, we have

$$\begin{aligned}
Q(x,x') &= \frac{2}{\pi\beta\gamma}\int_r^\infty e^{iu[x-x']}\mathrm{d}u + \frac{2}{\pi\beta\gamma}\int_{-\infty}^{-r} e^{iu[x-x']}\mathrm{d}u = \frac{1}{\pi 2\alpha^2\varepsilon}\int_{-\infty}^{\infty} e^{iu[x-x']}\mathrm{d}u \\
&= \frac{1}{\alpha^2\varepsilon}\delta(x-x')
\end{aligned} \tag{A.17}$$

We proceed with the next region which is $x<0, 0<x'$. In this case, the integral (A.7)

becomes

$$
\begin{aligned}
&Q(x,x') = \int_q^\infty \frac{2}{\pi\,|\det \mathbf{M}(y_0)|}\mathrm{Ai}(y(x)) \\
&\left(i\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y(x')) - i\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y(x'))\right) \mathrm{d}\omega \\
&= \frac{2i}{\pi}\int_q^\infty \left(\frac{1}{\overline{\det \mathbf{M}}(y_0)}\mathrm{Ai}(y(x))\mathrm{Ci}^-(y(x')) - \frac{1}{\det \mathbf{M}(y_0)}\mathrm{Ai}(y(x))\mathrm{Ci}^+(y(x'))\right)\mathrm{d}\omega
\end{aligned} \tag{A.18}
$$

To get the asymptotic expression for this integrand we are going to need an expression for $\det \mathbf{M}(y_0)$. We define $z_0 = z(0)$ and from (A.4) we have

$$
\begin{aligned}
&\det \mathbf{M}(y_0) = \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\mathrm{Ai}(-z_0)\mathrm{Ci}^+(-z_0) \\
&\approx \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\frac{1}{2i}\pi^{-\frac{1}{2}}(z_0)^{-\frac{1}{4}}\left(e^{i\left(\frac{2}{3}(z_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)} - e^{-i\left(\frac{2}{3}(z_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\pi^{-\frac{1}{2}}(z_0)^{-\frac{1}{4}}e^{i\left(\frac{2}{3}(z_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)} \\
&= \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\frac{1}{2\pi}(\gamma\omega)^{-\frac{1}{2}}e^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} \approx \frac{1}{\pi}
\end{aligned} \tag{A.19}
$$

Similarly for the conjugate $\overline{\det \mathbf{M}}(y_0)$ we get

$$
\begin{aligned}
&\overline{\det \mathbf{M}}(y_0) = \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\mathrm{Ai}(-z_0)\mathrm{Ci}^-(-z_0) \\
&\approx \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\frac{1}{2i}\pi^{-\frac{1}{2}}(z_0)^{-\frac{1}{4}}\left(e^{i\left(\frac{2}{3}(z_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)} - e^{-i\left(\frac{2}{3}(z_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\pi^{-\frac{1}{2}}(z_0)^{-\frac{1}{4}}e^{-i\left(\frac{2}{3}(z_0)^{\frac{3}{2}}+\frac{\pi}{4}\right)} \\
&= \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\frac{1}{2\pi}(\gamma\omega)^{-\frac{1}{2}}e^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} \approx \frac{1}{\pi}
\end{aligned} \tag{A.20}
$$

We can also separately examine the term $\mathrm{Ai}(y(x))\mathrm{Ci}^-(y(x'))$.

$$
\begin{aligned}
&\mathrm{Ai}(-z(x))\mathrm{Ci}^-(-z(x')) \approx \frac{1}{2i}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\frac{2}{3}(z(x))^{\frac{3}{2}}+\frac{\pi}{4}\right)} - e^{-i\left(\frac{2}{3}(z(x))^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right) \\
&\pi^{-\frac{1}{2}}(z(x'))^{-\frac{1}{4}}e^{-i\left(\frac{2}{3}(z(x'))^{\frac{3}{2}}+\frac{\pi}{4}\right)} \\
&= \frac{1}{2i\pi}(\beta x+\gamma\omega)^{-\frac{1}{4}}(\beta x'+\gamma\omega)^{-\frac{1}{4}}\left(e^{i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}-(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]}\right. \\
&\left.+ie^{-i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}+(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]}\right) \\
&\approx \frac{1}{2i\pi}(\gamma\omega)^{-\frac{1}{2}}\left(e^{i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')} + ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)
\end{aligned} \tag{A.21}
$$

where we used the same arguments as in (A.13) and (A.15). By conjugating it we get the next term.

$$\mathrm{Ai}(-z(x))\mathrm{Ci}^{+}(-z(x')) \approx \frac{1}{2i\pi}(\gamma\omega)^{-\frac{1}{2}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} - e^{-i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}\right) \tag{A.22}$$

It can be seen the the terms including $e^{\pm i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}$ after integrating will be finite numbers and hence will give no contribution to the constant χ. We can therefore exclude them and we can write (A.18) as

$$\begin{aligned}
Q(x,x') &= \frac{2i}{\pi}\int_q^\infty \frac{1}{\overline{\det \mathbf{M}(y_0)}}\mathrm{Ai}(y(x))\mathrm{Ci}^{-}(y(x')) - \frac{1}{\det \mathbf{M}(y_0)}\mathrm{Ai}(y(x))\mathrm{Ci}^{+}(y(x'))\mathrm{d}\omega \\
&\approx \frac{2i}{\pi}\int_q^\infty \left(\frac{1}{\frac{1}{\pi}}\frac{1}{2i\pi}(\gamma\omega)^{-\frac{1}{2}}e^{i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')} + \frac{1}{\frac{1}{\pi}}\frac{1}{2i\pi}(\gamma\omega)^{-\frac{1}{2}}e^{-i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}\right)\mathrm{d}\omega \\
&= \frac{1}{\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}\left(e^{i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')} + e^{-i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}\right)\mathrm{d}\omega
\end{aligned} \tag{A.23}$$

This can be further transformed as we did in (A.15) and (A.16) into

$$\begin{aligned}
Q(x,x') &= \frac{1}{\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}\left(e^{i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')} + e^{-i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}\right)\mathrm{d}\omega \\
&= \frac{2}{\pi\beta\gamma}\left(\int_q^\infty e^{iu(x-x')}\mathrm{d}u + \int_{-\infty}^{-q} e^{iu(x-x')}\mathrm{d}u\right) = \frac{2}{\pi\beta\gamma}\int_{-\infty}^\infty e^{iu(x-x')}\mathrm{d}u \\
&= \frac{4}{\beta\gamma}\delta(x-x') = \frac{1}{\alpha^2\varepsilon}\delta(x-x')
\end{aligned} \tag{A.24}$$

The last region we need to cover is $x, x' < 0$. The function $Q(x,x')$ now reads

$$\begin{aligned}
Q(x,x') &= \int_q^\infty \frac{2}{\pi\left|\det \mathbf{M}(y_0)\right|}\mathrm{Ai}(y(x))\frac{2}{\pi\left|\det \mathbf{M}(y_0)\right|}\mathrm{Ai}(y(x'))\mathrm{d}\omega \\
&= \int_q^\infty \frac{4}{\pi^2\left|\det \mathbf{M}(y_0)\right|^2}\mathrm{Ai}(y(x))\mathrm{Ai}(y(x'))\mathrm{d}\omega
\end{aligned} \tag{A.25}$$

Before we write the asymptotic expression for the whole integrand, let us first take the term

$\mathrm{Ai}(y(x))\mathrm{Ai}(y(x'))$.

$$
\begin{aligned}
&\mathrm{Ai}(-z(x))\mathrm{Ai}(-z(x')) \approx \frac{1}{2i}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\frac{2}{3}(z(x))^{\frac{3}{2}}+\frac{\pi}{4}\right)}-e^{-i\left(\frac{2}{3}(z(x))^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\\
&\frac{1}{2i}\pi^{-\frac{1}{2}}(z(x'))^{-\frac{1}{4}}\left(e^{i\left(\frac{2}{3}(z(x'))^{\frac{3}{2}}+\frac{\pi}{4}\right)}-e^{-i\left(\frac{2}{3}(z(x'))^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\\
&=-\frac{1}{4\pi}(\beta x+\gamma\omega)^{-\frac{1}{4}}(\beta x'+\gamma\omega)^{-\frac{1}{4}}\left(e^{i\left(\frac{2}{3}(\beta x+\gamma\omega)^{\frac{3}{2}}+\frac{\pi}{4}\right)}-e^{-i\left(\frac{2}{3}(\beta x+\gamma\omega)^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\\
&\left(e^{i\left(\frac{2}{3}(\beta x'+\gamma\omega)^{\frac{3}{2}}+\frac{\pi}{4}\right)}-e^{-i\left(\frac{2}{3}(\beta x'+\gamma\omega)^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\\
&\approx-\frac{1}{4\pi}(\gamma\omega)^{-\frac{1}{2}}\left(ie^{i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}+(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]}-e^{i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}-(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]}\right.\\
&\left.-e^{-i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}-(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]}-ie^{-i\frac{2}{3}\left[(\beta x+\gamma\omega)^{\frac{3}{2}}+(\beta x'+\gamma\omega)^{\frac{3}{2}}\right]}\right)\\
&\approx-\frac{1}{4\pi}(\gamma\omega)^{-\frac{1}{2}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}-e^{i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}-e^{-i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}-ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)
\end{aligned}
\tag{A.26}
$$

where we used the same simplifying approach as in (A.13) and (A.15). Using this expression and the fact that we drop the terms including $e^{\pm i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}$ because they give no contribution to χ, the integral (A.25) becomes

$$
\begin{aligned}
Q(x,x') &= \int_q^\infty \frac{4}{\pi^2\left|\det\mathbf{M}(y_0)\right|^2}\mathrm{Ai}(y(x))\mathrm{Ai}(y(x'))\mathrm{d}\omega\\
&\approx \int_q^\infty \frac{4}{\pi^2\left(\frac{1}{\pi}\right)^2}\frac{1}{4\pi}(\gamma\omega)^{-\frac{1}{2}}\left(e^{i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}+e^{-i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}\right)\mathrm{d}\omega\\
&= \frac{1}{\pi\gamma^{\frac{1}{2}}}\int_q^\infty \omega^{-\frac{1}{2}}\left(e^{i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}+e^{-i(\gamma\omega)^{\frac{1}{2}}\beta(x-x')}\right)\mathrm{d}\omega
\end{aligned}
\tag{A.27}
$$

and we end up with exactly identical situation as in (A.23) so the result is

$$
Q(x,x') = \frac{1}{\alpha^2\varepsilon}\delta(x-x') \tag{A.28}
$$

It is helpful to stress out that for the switched regions (for example instead of $x<0, 0<x'$ the switched is $x'<0, 0<x$) the results are the same, because the Dirac delta function is even ($\delta(x)=\delta(-x)$). The conclusion from (A.17), (A.24) and (A.28) is that the equation (A.5) turns

into

$$
\begin{aligned}
\chi^{-2} &= \frac{1}{\alpha^2\varepsilon} \lim_{\epsilon\to 0} \int_{I_\epsilon} \delta(x-x')\mathrm{d}x' = \frac{1}{\alpha^2\varepsilon} = \left(\alpha^2\varepsilon\right)^{-1} \\
\chi &= \left(\alpha^2\varepsilon\right)^{\frac{1}{2}} = \left(2^{-\frac{4}{3}}\varepsilon^{-\frac{4}{3}}\varepsilon\right)^{\frac{1}{2}} = 2^{-\frac{2}{3}}\varepsilon^{-\frac{1}{6}}
\end{aligned}
\tag{A.29}
$$

which is the desired normalization constant.

Appendix B

In this Appendix, we investigate the integral term in (2.110) by using asymptotic expressions of resonant states along complex rays in the lower half of the frequency plane for large R. The resonant state have the form

$$\psi_\omega(x) = \chi \begin{cases} \frac{2}{\pi|\det \mathbf{M}(y_0)|}\mathrm{Ai}(y(x)) & x \leq 0 \\ -i\left(\frac{\det \overline{\mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y(x)) + i\left(\frac{\det \mathbf{M}(y_0)}{\det \overline{\mathbf{M}}(y_0)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y(x)) & x > 0 \end{cases} \tag{B.1}$$

with

$$\chi = 2^{-\frac{2}{3}}\varepsilon^{-\frac{1}{6}} \tag{B.2}$$

$$y(x) = -2\alpha(\varepsilon x + \omega) \tag{B.3}$$

$$\det \mathbf{M}(y_0) = \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\mathrm{Ai}(y_0)\mathrm{Ci}^+(y_0) \tag{B.4}$$

where $\alpha = (2\varepsilon)^{-\frac{2}{3}}$ and $y_0 = y(0)$. Define the following quantity

$$z(x) = -y(x) = 2\alpha(\varepsilon x + \omega) \tag{B.5}$$

Let us express the rays on the lower half of the frequency plane as

$$\omega = Re^{i\theta}, -\pi < \theta < 0 \tag{B.6}$$

where $R \to \infty$. Using this, the transformation then takes form

$$z(x) = 2\alpha(\varepsilon x + Re^{i\theta}) \tag{B.7}$$

$$z_0 = z(0) = 2\alpha e^{i\theta}R = \kappa R \tag{B.8}$$

where $\kappa = 2\alpha e^{i\theta}$. There are 2 different asymptotic forms of Airy functions for this complex region $-\pi < \theta < -\frac{2\pi}{3}$ and $-\frac{2\pi}{3} < \theta < 0$. We will investigate them separately. Let us first consider the sector $-\frac{2\pi}{3} < \theta < 0$. The variable has now changed from x to radius R and for large positive radius the asymptotic expansions are in y_0 which is large negative. From [1]

(10.4.60), (10.4.64) we have

$$\mathrm{Ai}(-z_0) \approx \pi^{-\frac{1}{2}}(z_0)^{-\frac{1}{4}} \sin\left(\zeta_0 + \frac{\pi}{4}\right) = \frac{1}{2i\sqrt{\pi}}(z_0)^{-\frac{1}{4}}\left(e^{i\left(\zeta_0+\frac{\pi}{4}\right)} - e^{-i\left(\zeta_0+\frac{\pi}{4}\right)}\right) \tag{B.9}$$

$$\mathrm{Ci}^{\pm}(-z_0) \approx \pi^{-\frac{1}{2}}(z_0)^{-\frac{1}{4}} e^{\pm i\left(\zeta_0+\frac{\pi}{4}\right)} \tag{B.10}$$

where we defined $\zeta_0 = \frac{2}{3}(z_0)^{\frac{3}{2}}$ that we can write using (B.8) as

$$\zeta_0 = \underbrace{\frac{2}{3}\kappa^{\frac{3}{2}}}_{\vartheta} R^{\frac{3}{2}} = \vartheta R^{\frac{3}{2}} = (\vartheta_r + i\vartheta_i)R^{\frac{3}{2}} \tag{B.11}$$

with

$$\vartheta_r = \frac{2}{3}(2\alpha)^{\frac{3}{2}} \cos\left(\frac{3}{2}\theta\right) \tag{B.12}$$

$$\vartheta_i = \frac{2}{3}(2\alpha)^{\frac{3}{2}} \sin\left(\frac{3}{2}\theta\right) \tag{B.13}$$

Since we are in the sector $-\frac{2}{3} < \theta < 0$, we have $-\pi < \frac{3}{2}\theta < 0$ and therefore for these rays we have $\vartheta_i < 0$.

The reason we expressed the formulas (B.9) and (B.10) is to help us with the determinants in the resonant states (2.99). This determinant can be found in (2.35).

$$\begin{aligned}
&\det \mathbf{M}(y_0) = \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\mathrm{Ai}(-z_0)\mathrm{Ci}^{+}(-z_0) \\
&\approx \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\frac{1}{2i\sqrt{\pi}}(\kappa R)^{-\frac{1}{4}}\left(e^{i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)} - e^{-i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\pi^{-\frac{1}{2}}(\kappa R)^{-\frac{1}{4}} e^{i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)} \\
&= \frac{1}{\pi} - \frac{A}{2i\pi\alpha\varepsilon}(\kappa R)^{-\frac{1}{2}}\left(e^{2i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)} - 1\right) \approx -\frac{A}{2\pi\alpha\varepsilon}(\kappa R)^{-\frac{1}{2}} e^{2i\vartheta R^{\frac{3}{2}}}
\end{aligned} \tag{B.14}$$

Its complex conjugate will be

$$\begin{aligned}
&\overline{\det \mathbf{M}}(y_0) = \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\mathrm{Ai}(-z_0)\mathrm{Ci}^{-}(-z_0) \\
&\approx \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\frac{1}{2i\sqrt{\pi}}(\kappa R)^{-\frac{1}{4}}\left(e^{i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)} - e^{-i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right)\pi^{-\frac{1}{2}}(\kappa R)^{-\frac{1}{4}} e^{-i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)} \\
&= \frac{1}{\pi} - \frac{A}{2i\pi\alpha\varepsilon}(\kappa R)^{-\frac{1}{2}}\left(1 - e^{-2i\left(\vartheta R^{\frac{3}{2}}+\frac{\pi}{4}\right)}\right) \approx \frac{1}{\pi}
\end{aligned} \tag{B.15}$$

Observe that the complex conjugate determinant was reduced to a constant, because $\vartheta_i < 0$

which made the exponential decay as $R \to \infty$. Looking at the terms in (2.99) we have

$$\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \approx \frac{iA^{\frac{1}{2}}}{(2\alpha\varepsilon)^{\frac{1}{2}}}(\kappa R)^{-\frac{1}{4}} e^{i\vartheta R^{\frac{3}{2}}} \tag{B.16}$$

$$\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \approx \frac{(2\alpha\varepsilon)^{\frac{1}{2}}}{iA^{\frac{1}{2}}}(\kappa R)^{\frac{1}{4}} e^{-i\vartheta R^{\frac{3}{2}}} \tag{B.17}$$

$$|\det \mathbf{M}(y_0)| = \left(\det \mathbf{M}(y_0)\overline{\det \mathbf{M}}(y_0)\right)^{\frac{1}{2}} \approx \frac{iA^{\frac{1}{2}}}{\pi(2\alpha\varepsilon)^{\frac{1}{2}}}(\kappa R)^{-\frac{1}{4}} e^{i\vartheta R^{\frac{3}{2}}} \tag{B.18}$$

In the following we find the asymptotic behaviour in $y(x)$ too using the transformation (B.7). We simplify the expression $\zeta = \frac{2}{3}(z(x))^{\frac{3}{2}}$ used in asymptotic formulas.

$$\begin{aligned}
\zeta = \frac{2}{3}(z(x))^{\frac{3}{2}} &= \frac{2}{3}\left[2\alpha\left(\varepsilon x + Re^{i\theta}\right)\right]^{\frac{3}{2}} = \frac{2}{3}\left[2\alpha e^{i\theta}R\left(\varepsilon x e^{-i\theta}R^{-1} + 1\right)\right]^{\frac{3}{2}} \\
&= \frac{2}{3}\left[\kappa R\left(1 + \varepsilon x e^{-i\theta}R^{-1}\right)\right]^{\frac{3}{2}} = \vartheta R^{\frac{3}{2}}\left(1 + \varepsilon x e^{-i\theta}R^{-1}\right)^{\frac{3}{2}} \\
&\approx \vartheta R^{\frac{3}{2}}\left(1 + \frac{3}{2}\varepsilon x e^{-i\theta}R^{-1}\right) = \vartheta R^{\frac{3}{2}} + \underbrace{\frac{3}{2}\vartheta\varepsilon i e^{-i\theta}}_{\varpi}(-i)xR^{\frac{1}{2}} \\
&= \vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}}
\end{aligned} \tag{B.19}$$

where we defined a new quantity ϖ which can be simplified further.

$$\begin{aligned}
\varpi &= \frac{3}{2}\vartheta\varepsilon i e^{-i\theta} = \frac{3}{2}\varepsilon i\frac{2}{3}(2\alpha)^{\frac{3}{2}}\left[\cos\left(\frac{3}{2}\theta\right) + i\sin\left(\frac{3}{2}\theta\right)\right]\left[\cos(\theta) - i\sin(\theta)\right] \\
&= \varepsilon i(2\alpha)^{\frac{3}{2}}\left[\cos\left(\frac{3}{2}\theta\right)\cos(\theta) + \sin\left(\frac{3}{2}\theta\right)\sin(\theta)\right. \\
&\left. + i\left(\sin\left(\frac{3}{2}\theta\right)\cos(\theta) - \cos\left(\frac{3}{2}\theta\right)\sin(\theta)\right)\right] \\
&= \varepsilon i(2\alpha)^{\frac{3}{2}}\left[\cos\left(\frac{1}{2}\theta\right) + i\sin\left(\frac{1}{2}\theta\right)\right] \\
&= \varepsilon(2\alpha)^{\frac{3}{2}}\left[-\sin\left(\frac{1}{2}\theta\right) + i\cos\left(\frac{1}{2}\theta\right)\right] = \varpi_r + i\varpi_r
\end{aligned} \tag{B.20}$$

where

$$\varpi_r = -\varepsilon(2\alpha)^{\frac{3}{2}}\sin\left(\frac{1}{2}\theta\right) \tag{B.21}$$

$$\varpi_i = \varepsilon(2\alpha)^{\frac{3}{2}}\cos\left(\frac{1}{2}\theta\right) \tag{B.22}$$

Note, that in the current sector which is $-\frac{2\pi}{3} < \theta < 0$ we have $-\frac{\pi}{3} < \frac{1}{2}\theta < 0$ and therefore

$\varpi_r, \varpi_i > 0$.

Using (B.19), we have in the current sector $-\frac{2\pi}{3} < \theta < 0$ the following asymptotic relations

$$\begin{aligned}\mathrm{Ai}(-z(x)) &\approx \frac{1}{2i}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} - e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right)\\ &= \frac{1}{2i}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}}+\frac{\pi}{4}\right)} - e^{-i\left(\vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}}+\frac{\pi}{4}\right)}\right)\\ &= \frac{1}{2i}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}\left(e^{i\frac{\pi}{4}}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi x R^{\frac{1}{2}}} - e^{-i\frac{\pi}{4}}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi x R^{\frac{1}{2}}}\right) \qquad \text{(B.23)}\end{aligned}$$

$$\mathrm{Ci}^{+}(-z(x)) \approx \pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{i\left(\zeta+\frac{\pi}{4}\right)} = \pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{i\frac{\pi}{4}}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi x R^{\frac{1}{2}}} \qquad \text{(B.24)}$$

$$\mathrm{Ci}^{-}(-z(x)) \approx \pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{-i\left(\zeta+\frac{\pi}{4}\right)} = \pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{-i\frac{\pi}{4}}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi x R^{\frac{1}{2}}} \qquad \text{(B.25)}$$

At this point we are ready to find the asymptotic behaviour of the resonant states (2.99) along rays in the lower half of the complex frequency plane. For $x > 0$ we get

$$\begin{aligned}\psi_\omega(x) &= i\chi\left[\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}}\mathrm{Ci}^{-}(-z(x)) - \left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}}\mathrm{Ci}^{+}(-z(x))\right]\\ &\approx i\chi\left[\frac{iA^{\frac{1}{2}}}{(2\alpha\varepsilon)^{\frac{1}{2}}}(\kappa R)^{-\frac{1}{4}}e^{i\vartheta r^{\frac{3}{2}}}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{-i\frac{\pi}{4}}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi x R^{\frac{1}{2}}}\right.\\ &\left.-\frac{(2\alpha\varepsilon)^{\frac{1}{2}}}{iA^{\frac{1}{2}}}(\kappa R)^{\frac{1}{4}}e^{-i\vartheta r^{\frac{3}{2}}}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{i\frac{\pi}{4}}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi x R^{\frac{1}{2}}}\right]\\ &= i\chi\left[\frac{iA^{\frac{1}{2}}}{(2\alpha\varepsilon)^{\frac{1}{2}}}(\kappa R)^{-\frac{1}{4}}\pi^{-\frac{1}{2}}(\kappa R)^{-\frac{1}{4}}(1+\varepsilon x R^{-1}e^{-i\theta})^{-\frac{1}{4}}e^{-i\frac{\pi}{4}}e^{-\varpi x R^{\frac{1}{2}}}\right.\\ &\left.-\frac{(2\alpha\varepsilon)^{\frac{1}{2}}}{iA^{\frac{1}{2}}}(\kappa R)^{\frac{1}{4}}\pi^{-\frac{1}{2}}(\kappa R)^{-\frac{1}{4}}(1+\varepsilon x R^{-1}e^{-i\theta})^{-\frac{1}{4}}e^{i\frac{\pi}{4}}e^{\varpi x R^{\frac{1}{2}}}\right]\\ &\approx i\chi\left[\frac{iA^{\frac{1}{2}}}{(2\pi\alpha\varepsilon\kappa R)^{\frac{1}{2}}}\left(1-\frac{1}{4}\varepsilon x R^{-1}e^{-i\theta}\right)e^{-i\frac{\pi}{4}}e^{-\varpi x R^{\frac{1}{2}}}\right.\\ &\left.-\frac{(2\alpha\varepsilon)^{\frac{1}{2}}}{i(\pi A)^{\frac{1}{2}}}\left(1-\frac{1}{4}\varepsilon x R^{-1}e^{-i\theta}\right)e^{i\frac{\pi}{4}}e^{\varpi x R^{\frac{1}{2}}}\right]\\ &\approx -\frac{\chi(2\alpha\varepsilon)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{(\pi A)^{\frac{1}{2}}}e^{\varpi x R^{\frac{1}{2}}}\left(1-\frac{1}{4}\varepsilon x R^{-1}e^{-i\theta}\right) \approx -\frac{\chi(2\alpha\varepsilon)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{(\pi A)^{\frac{1}{2}}}e^{\varpi x R^{\frac{1}{2}}} \qquad \text{(B.26)}\end{aligned}$$

The first term in the last line disappears for exponential decay because of $\varpi_r, \varpi_i > 0$. For the region $x < 0$ we get

$$\psi_\omega(x) = \frac{2\chi}{\pi\left|\det \mathbf{M}(y_0)\right|}\mathrm{Ai}(-z(x)) \approx \frac{2\chi}{\pi}\frac{\pi(2\alpha\varepsilon)^{\frac{1}{2}}}{iA^{\frac{1}{2}}}(\kappa R)^{\frac{1}{4}}e^{-i\vartheta R^{\frac{3}{2}}}\frac{1}{2i}\pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}$$
$$\left(e^{i\frac{\pi}{4}}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi x R^{\frac{1}{2}}} - e^{-i\frac{\pi}{4}}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi x R^{\frac{1}{2}}}\right)$$
$$\approx -\frac{\chi(2\alpha\varepsilon)^{\frac{1}{2}}}{(\pi A)^{\frac{1}{2}}}(\kappa R)^{\frac{1}{4}}e^{-i\vartheta R^{\frac{3}{2}}}(\kappa R)^{-\frac{1}{4}}(1+\varepsilon x R^{-1}e^{-i\theta})^{-\frac{1}{4}}e^{i\frac{\pi}{4}}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi x R^{\frac{1}{2}}}$$
$$\approx -\frac{\chi(2\alpha\varepsilon)^{\frac{1}{2}}}{(\pi A)^{\frac{1}{2}}}\left(1-\frac{1}{4}\varepsilon x R^{-1}e^{-i\theta}\right)e^{i\frac{\pi}{4}}e^{\varpi x R^{\frac{1}{2}}} \approx -\frac{\chi(2\alpha\varepsilon)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{(\pi A)^{\frac{1}{2}}}e^{\varpi x R^{\frac{1}{2}}} \tag{B.27}$$

Summing it up, the approximation of the resonant states along rays for $-\frac{2\pi}{3} < \theta < 0$ in the lower complex frequency plane is

$$\psi_\omega(x) \approx -\frac{\chi(2\alpha\varepsilon)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{(\pi A)^{\frac{1}{2}}}e^{\varpi x R^{\frac{1}{2}}}, \quad \forall x \tag{B.28}$$

We proceed to the second sector in the lower half-plane $-\pi < \theta < -\frac{2\pi}{3}$. According to (10.4.59) and (10.4.63) in [1] we can see that comparing to the previous cases, the sign of the argument stays the same in the expression. That means we would use a square root of a negative quantity y_0, so we adjust it as

$$y_0 = -2\alpha e^{i\theta}R = \underbrace{2\alpha e^{i(\theta+\pi)}}_{\tilde{\kappa}}R = \tilde{\kappa}R \tag{B.29}$$

from which we can observe that $0 < \theta + \pi < \frac{\pi}{3}$. The asymptotic formulas for Airy functions for this sector are

$$\mathrm{Ai}(y_0) \approx \frac{1}{2}\pi^{-\frac{1}{2}}y_0^{-\frac{1}{4}}e^{-\zeta_0} \tag{B.30}$$
$$\mathrm{Bi}(y_0) \approx \pi^{-\frac{1}{2}}y_0^{-\frac{1}{4}}e^{\zeta_0} \tag{B.31}$$

where

$$\zeta_0 = \frac{2}{3} y_0^{\frac{3}{2}} = \underbrace{\frac{2}{3}\tilde{\kappa}^{\frac{3}{2}}}_{\tilde{\vartheta}} R^{\frac{3}{2}} = \tilde{\vartheta} R^{\frac{3}{2}} = \frac{2}{3}(2\alpha)^{\frac{3}{2}} e^{i\frac{3}{2}(\theta+\pi)} R^{\frac{3}{2}}$$
$$= \left[\underbrace{\frac{2}{3}(2\alpha)^{\frac{3}{2}} \cos\left(\frac{3}{2}(\theta+\pi)\right)}_{\tilde{\vartheta}_r} + i \underbrace{\frac{2}{3}(2\alpha)^{\frac{3}{2}} \sin\left(\frac{3}{2}(\theta+\pi)\right)}_{\tilde{\vartheta}_i} \right] R^{\frac{3}{2}} = \left(\tilde{\vartheta}_r + i\tilde{\vartheta}_i\right) R^{\frac{3}{2}} \quad \text{(B.32)}$$

Note, that in the current sector $-\pi < \theta < -\frac{2\pi}{3}$ we have for the argument of $\tilde{\vartheta}$ the range $0 < \frac{3}{2}(\theta+\pi) < \frac{\pi}{2}$ where

$$\tilde{\vartheta}_r, \tilde{\vartheta}_i > 0 \quad \text{(B.33)}$$

Using (B.32) we rewrite the asymptotic formulas into the form

$$\mathrm{Ai}(y_0) \approx \frac{1}{2}\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}}} \quad \text{(B.34)}$$
$$\mathrm{Bi}(y_0) \approx \pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{\tilde{\vartheta}R^{\frac{3}{2}}} \quad \text{(B.35)}$$

then for the Airy functions $\mathrm{Ci}^{\pm}$ we have

$$\mathrm{Ci}^{+}(y_0) \approx \pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{\tilde{\vartheta}R^{\frac{3}{2}}} + i\frac{1}{2}\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}}} \approx \pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{\tilde{\vartheta}R^{\frac{3}{2}}} \quad \text{(B.36)}$$
$$\mathrm{Ci}^{-}(y_0) \approx \pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{\tilde{\vartheta}R^{\frac{3}{2}}} - i\frac{1}{2}\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}}} \approx \pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{\tilde{\vartheta}R^{\frac{3}{2}}} \quad \text{(B.37)}$$

where we used the fact that $\tilde{\vartheta}_r, \tilde{\vartheta}_i > 0$. The determinant terms (2.99) can be expressed then as

$$\det \mathbf{M}(y_0) = \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\mathrm{Ai}(y_0)\mathrm{Ci}^{+}(y_0)$$
$$\approx \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\frac{1}{2}\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}}} \pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{\tilde{\vartheta}R^{\frac{3}{2}}} = \frac{1}{\pi} - \frac{A}{\alpha\varepsilon}\frac{1}{2}\pi^{-1}(\tilde{\kappa}R)^{-\frac{1}{2}}$$
$$\approx \frac{1}{\pi} \quad \text{(B.38)}$$

Since in this case we have $\mathrm{Ci}^{+}(y_0) = \mathrm{Ci}^{-}(y_0)$ then we also have

$$\overline{\det \mathbf{M}}(y_0) \approx \frac{1}{\pi} \quad \text{(B.39)}$$

and hence we get

$$\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}}(y_0)}\right)^{\frac{1}{2}} \approx 1 \tag{B.40}$$

$$\left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \approx 1 \tag{B.41}$$

$$|\det \mathbf{M}(y_0)| = \left(\det \mathbf{M}(y_0)\overline{\det \mathbf{M}}(y_0)\right)^{\frac{1}{2}} \approx \frac{1}{\pi} \tag{B.42}$$

As a next step we write the standard asymptotic relations for Airy functions with the argument $y(x) = -2\alpha(\varepsilon x + Re^{i\theta})$.

$$\mathrm{Ai}(y(x)) = \frac{1}{2}\pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}}e^{-\zeta} \tag{B.43}$$

$$\mathrm{Bi}(y(x)) = \pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}}e^{\zeta} \tag{B.44}$$

where in this case we rewrite $y(x)$ as

$$y(x) = -2\alpha\left(\varepsilon x + Re^{i(\theta)}\right) = 2\alpha\left(\varepsilon x e^{i\pi} + Re^{i(\theta+\pi)}\right) \tag{B.45}$$

and we have

$$\begin{aligned}\zeta &= \frac{2}{3}y(x)^{\frac{3}{2}} = \frac{2}{3}\left[2\alpha\left(\varepsilon x e^{i\pi} + e^{i(\theta+\pi)}R\right)\right]^{\frac{3}{2}} = \frac{2}{3}\left[2\alpha e^{i(\theta+\pi)}R\left(\varepsilon x e^{-i\theta}R^{-1} + 1\right)\right]^{\frac{3}{2}} \\ &= \frac{2}{3}\left[\tilde{\kappa}R\left(\varepsilon x e^{-i\theta}R^{-1} + 1\right)\right]^{\frac{3}{2}} = \tilde{\vartheta}R^{\frac{3}{2}}\left(1 + \varepsilon x e^{-i\theta}R^{-1}\right)^{\frac{3}{2}} \\ &\approx \tilde{\vartheta}R^{\frac{3}{2}}\left(1 + \frac{3}{2}\varepsilon x e^{-i\theta}R^{-1}\right) = \tilde{\vartheta}R^{\frac{3}{2}} + \underbrace{\frac{3}{2}\varepsilon e^{-i\theta}\tilde{\vartheta}}_{\tilde{\varpi}}\, xR^{\frac{1}{2}} = \tilde{\vartheta}R^{\frac{3}{2}} + \tilde{\varpi}xR^{\frac{1}{2}}\end{aligned} \tag{B.46}$$

where we defined the quantity $\tilde{\varpi}$ which is

$$\begin{aligned}\tilde{\varpi} &= \frac{3}{2}\varepsilon e^{-i\theta}\tilde{\vartheta} = \frac{3}{2}\varepsilon e^{-i\theta}\frac{2}{3}\left(2\alpha\right)^{\frac{3}{2}} e^{i\frac{3}{2}(\theta+\pi)} \\ &= \varepsilon\left(2\alpha\right)^{\frac{3}{2}} e^{i\frac{1}{2}(\theta+3\pi)} = \varepsilon\left(2\alpha\right)^{\frac{3}{2}}\left[\cos\left(\frac{1}{2}(\theta+3\pi)\right) + i\sin\left(\frac{1}{2}(\theta+3\pi)\right)\right] \\ &= \tilde{\varpi}_r + i\tilde{\varpi}_i\end{aligned} \tag{B.47}$$

with

$$\tilde{\varpi}_r = \varepsilon\,(2\alpha)^{\frac{3}{2}} \cos\left(\frac{1}{2}(\theta + 3\pi)\right) \tag{B.48}$$

$$\tilde{\varpi}_i = \varepsilon\,(2\alpha)^{\frac{3}{2}} \sin\left(\frac{1}{2}(\theta + 3\pi)\right) \tag{B.49}$$

We can say that in the sector $-\pi < \theta < -\frac{2\pi}{3}$ we have $\pi < \frac{1}{2}(\theta + 3\pi) < \frac{7\pi}{6}$ and therefore

$$\tilde{\varpi}_r, \tilde{\varpi}_i < 0 \tag{B.50}$$

Substituting (B.46) into (B.43) and (B.44) we get

$$\mathrm{Ai}(y(x)) = \frac{1}{2}\pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}} - \tilde{\varpi}xR^{\frac{1}{2}}} \tag{B.51}$$

$$\mathrm{Bi}(y(x)) = \pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}} e^{\tilde{\vartheta}R^{\frac{3}{2}} + \tilde{\varpi}xR^{\frac{1}{2}}} \tag{B.52}$$

Finally we got to the point where we can find the asymptotic expressions for the resonant states (2.99) along rays in the sector $-\pi < \theta < -\frac{2\pi}{3}$, which is a part of a the lower half of the complex frequency plane. Using (B.40), (B.41) we have for $x > 0$

$$\begin{aligned}
\psi_\omega(x) &= i\chi\left[\left(\frac{\det \mathbf{M}(y_0)}{\overline{\det \mathbf{M}(y_0)}}\right)^{\frac{1}{2}} \mathrm{Ci}^-(y(x)) - \left(\frac{\overline{\det \mathbf{M}}(y_0)}{\det \mathbf{M}(y_0)}\right)^{\frac{1}{2}} \mathrm{Ci}^+(y(x))\right] \\
&\approx i\chi\left(\mathrm{Ci}^-(y(x)) - \mathrm{Ci}^+(y(x))\right) = i\chi(-2i)\mathrm{Ai}(y(x)) \\
&\approx 2\chi\frac{1}{2}\pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}} - \tilde{\varpi}xR^{\frac{1}{2}}} \\
&= \chi\pi^{-\frac{1}{2}}\left[2\alpha\left(\varepsilon x + e^{i(\theta+\pi)}\right)R\right]^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}} - \tilde{\varpi}xR^{\frac{1}{2}}} \\
&= \chi\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}}\left(1 + \varepsilon x e^{-i(\theta+\pi)}R^{-1}\right)^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}} - \tilde{\varpi}xR^{\frac{1}{2}}} \\
&\approx \chi\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}}\left(1 - \frac{1}{4}\varepsilon x e^{-i(\theta+\pi)}R^{-1}\right) e^{-\tilde{\vartheta}R^{\frac{3}{2}} - \tilde{\varpi}xR^{\frac{1}{2}}} \\
&\approx \chi\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}} e^{-\tilde{\vartheta}R^{\frac{3}{2}} - \tilde{\varpi}xR^{\frac{1}{2}}}
\end{aligned} \tag{B.53}$$

and using (B.42) for $x < 0$

$$
\begin{aligned}
\psi_\omega(x) &= \frac{2\chi}{\pi \left|\det \mathbf{M}(y_0)\right|}\mathrm{Ai}(y(x)) \approx 2\chi \mathrm{Ai}(y(x)) \\
&\approx 2\frac{\chi}{2}\pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \\
&= \chi\pi^{-\frac{1}{2}}\left[2\alpha\left(\varepsilon x + e^{i(\theta+\pi)}\right)R\right]^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \\
&\approx \chi\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}}\left(1-\frac{1}{4}\varepsilon x e^{-i(\theta+\pi)}R^{-1}\right)e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \\
&\approx \chi\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}}
\end{aligned}
\tag{B.54}
$$

Since in the current sector we have $\tilde{\vartheta}_r, \tilde{\vartheta}_i > 0$ and $\tilde{\varpi}_r, \tilde{\varpi}_i < 0$, both (B.53) and (B.54) decay exponentially even for $x > 0$, because the term $e^{-\tilde{\vartheta}R^{\frac{3}{2}}}$ decays much faster in the limit $R \to \infty$. Summing it up, the asymptotic expression for the resonant states (2.99) along rays $-\pi < \theta < -\frac{2\pi}{3}$ is

$$
\psi_\omega(x) \approx \chi\pi^{-\frac{1}{2}}(\tilde{\kappa}R)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}}, \quad \forall x \tag{B.55}
$$

Appendix C

This Appendix provides a computation of the normalisation constant χ for the scattering form of the resonant states of the square well potential in (3.65). As for the Dirac delta potential, the Hamiltonian (3.3) is Hermitian, the scattering states (3.65) form a set with a continuous spectrum. Therefore there must exist a normalization constant. This constant can be computed from the completeness relation. The completeness relation is

$$\int_{-\infty}^{\infty} \psi_\omega(x)\psi_\omega(x')\mathrm{d}\omega = \delta(x-x') \tag{C.1}$$

where the scattering state have the form

$$\psi_\omega(x) = \chi \begin{cases} \frac{2}{\pi^2|\det \mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x)) & x < -d \\ \frac{2}{\pi|\det \mathbf{M}(\omega)|}\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(x)) \right. & \\ \qquad\qquad \left. +(A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(x))\right] & -d < x < d \\ i\left(\frac{\overline{\det \mathbf{M}(\omega)}}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(x)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}(\omega)}}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(x)) & d < x \end{cases} \tag{C.2}$$

In this formula we have defined

$$y_1(x) = -2\alpha(\varepsilon x + \omega) \tag{C.3}$$

$$y_2(x) = -2\alpha(\varepsilon x + V_0 + \omega) \tag{C.4}$$

$$A_0 = \mathrm{Ai}(y_1(-d)), A_1 = \mathrm{Ai}(y_2(-d)), B_1 = \mathrm{Bi}(y_2(-d))$$

with $\alpha = (2\varepsilon)^{-\frac{2}{3}}$. Integrating the relation (C.1) through the interval $I_\epsilon = [x-\epsilon, x+\epsilon]$ and taking the limit we obtain a formula for χ.

$$\lim_{\epsilon\to 0}\int_{I_\epsilon}\left(\int_{-\infty}^{\infty}\chi\tilde{\psi}_\omega(x)\chi\tilde{\psi}_\omega(x')\mathrm{d}\omega\right)\mathrm{d}x' = 1$$
$$\lim_{\epsilon\to 0}\int_{I_\epsilon}\left(\int_{-\infty}^{\infty}\tilde{\psi}_\omega(x)\tilde{\psi}_\omega(x')\mathrm{d}\omega\right)\mathrm{d}x' = \chi^{-2} \tag{C.5}$$

where $\tilde{\psi}_\omega(x)$ is the resonant state setting $\chi = 1$. The interval I_ϵ is infinitely small which means that any finite part of the integral

$$Q(x,x') = \int_{-\infty}^{\infty}\tilde{\psi}_\omega(x)\tilde{\psi}_\omega(x')\mathrm{d}\omega \tag{C.6}$$

gives no contribution to χ. The goal then is to recognize those parts and exclude them. We integrate with respect to ω. Essentially, we can see from (C.3) and (C.4) that the positions of x and ω are the same. Let us see what happens when $\omega \to -\infty$. The arguments $y_1(x)$ and $y_2(x)$ turn into big positive values, hence we can use the expression (10.4.59), (10.4.61), (10.4.63) and (10.4.66) from [1] for Airy functions to conclude that the function $\psi_\omega(x)$ decays exponentially for all x. Therefore it gives a finite contribution so we can write

$$Q(x,x') = \int_q^\infty \tilde{\psi}_\omega(x)\tilde{\psi}_\omega(x')\mathrm{d}\omega \tag{C.7}$$

for some positive q, which can be chosen as big as we like. Therefore we can treat the integrands in their asymptotic forms. We are going to investigate each regions separately. Let us start first with $-d < x < d$ and $d < x'$. In this regions we have

$$\begin{aligned}
Q(x,x') = &\int_q^\infty \frac{2}{\pi|\det\mathbf{M}(\omega)|}\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(x)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(x))\right] \\
&\left[-i\left(\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(x')) + i\left(\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(x'))\right]\mathrm{d}\omega \\
= &-\int_q^\infty \left[\frac{2i(B_1'A_0 - B_1A_0')}{\pi\overline{\det\mathbf{M}}(\omega)}\mathrm{Ai}(y_2(x))\mathrm{Ci}^-(y_1(x'))\right. \\
&- \frac{2i(B_1'A_0 - B_1A_0')}{\pi\det\mathbf{M}(\omega)}\mathrm{Ai}(y_2(x))\mathrm{Ci}^+(y_1(x')) \\
&+ \frac{2i(A_1A_0' - A_1'A_0)}{\pi\overline{\det\mathbf{M}}(\omega)}\mathrm{Bi}(y_2(x))\mathrm{Ci}^-(y_1(x')) \\
&\left.- \frac{2i(A_1A_0' - A_1'A_0)}{\pi\det\mathbf{M}(\omega)}\mathrm{Bi}(y_2(x))\mathrm{Ci}^+(y_1(x'))\right]\mathrm{d}\omega
\end{aligned} \tag{C.8}$$

Let us first begin with $\overline{\det\mathbf{M}}(\omega)$. Its form can be seen in (3.64) as the numerator. Instead of working with that form, we can use (3.44).

$$\det\mathbf{M}(\omega) = (A_0A_1' - A_0'A_1)(B_2C_3' - B_2'C_3) - (A_0B_1' - A_0'B_1)(A_2C_3' - A_2'C_3) \tag{C.9}$$

$$\overline{\det\mathbf{M}}(\omega) = (A_0A_1' - A_0'A_1)(B_2D_3' - B_2'D_3) - (A_0B_1' - A_0'B_1)(A_2D_3' - A_2'D_3) \tag{C.10}$$

Define

$$z_{1,2}(x) = -y_{1,2}(x) \tag{C.11}$$

We will use $z(x)$ without an index for the general asymptotic expressions of Airy functions. The asymptotic behaviours of Airy functions according to [1] (10.4.60) and (10.4.64) are

$$\mathrm{Ai}(-z(x)) \approx \frac{1}{2i\sqrt{\pi}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} - e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right) \tag{C.12}$$

$$\mathrm{Bi}(-z(x)) \approx \frac{1}{2\sqrt{\pi}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} + e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right) \tag{C.13}$$

$$\mathrm{Ci}^{\pm}(-z(x)) \approx \pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{\pm i\left(\zeta+\frac{\pi}{4}\right)} \tag{C.14}$$

with $\zeta = \frac{2}{3}(z(x))^{\frac{3}{2}}$. We get the asymptotic behaviour for their derivatives from [1] (10.4.62) and (10.4.67).

$$\mathrm{Ai}'(-z(x)) \approx -\frac{1}{2\sqrt{\pi}}(z(x))^{\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} + e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right) \tag{C.15}$$

$$\mathrm{Bi}'(-z(x)) \approx \frac{1}{2i\sqrt{\pi}}(z(x))^{\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} - e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right) \tag{C.16}$$

$$\begin{aligned}\mathrm{Ci}'^{-}(-z(x)) &= \mathrm{Bi}'(-z(x)) - i\mathrm{Ai}'(-z(x))\\ &\approx \pi^{-\frac{1}{2}}(z(x))^{\frac{1}{4}}\sin\left(\zeta+\frac{\pi}{4}\right) + i\pi^{-\frac{1}{2}}(z(x))^{\frac{1}{4}}\cos\left(\zeta+\frac{\pi}{4}\right)\\ &= i\pi^{-\frac{1}{2}}(z(x))^{\frac{1}{4}}e^{-i\left(\zeta+\frac{\pi}{4}\right)}\end{aligned} \tag{C.17}$$

$$\mathrm{Ci}'^{+}(-z(x)) \approx -i\pi^{-\frac{1}{2}}(z(x))^{\frac{1}{4}}e^{i\left(\zeta+\frac{\pi}{4}\right)} \tag{C.18}$$

Define the following quantities

$$z_1(-d) = 2\alpha\varepsilon d - 2\alpha\omega = -\beta + \gamma\omega \qquad z_1(d) = 2\alpha\varepsilon d - 2\alpha\omega = \beta + \gamma\omega \tag{C.19}$$

$$\begin{aligned}z_2(-d) = 2\alpha(\varepsilon d - V_0) + 2\alpha\omega = -\beta + \sigma + \gamma\omega \qquad z_2(d) &= 2\alpha(\varepsilon d + V_0) + 2\alpha\omega\\ &= \beta + \sigma + \gamma\omega\end{aligned} \tag{C.20}$$

where $\beta = 2\alpha\varepsilon d, \sigma = 2\alpha V_0$ and $\gamma = 2\alpha$ and consequently define

$$\begin{aligned}\zeta_1^{\pm} &= \frac{2}{3}(z_1(\pm d))^{\frac{3}{2}} = \frac{2}{3}(\pm\beta + \gamma\omega)^{\frac{3}{2}} = \frac{2}{3}(\gamma\omega)^{\frac{3}{2}}\left(\pm\frac{\beta}{\gamma\omega} + 1\right)^{\frac{3}{2}}\\ &\approx \frac{2}{3}(\gamma\omega)^{\frac{3}{2}}\left(1 \pm \frac{3\beta}{2\gamma\omega}\right) = \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \pm \beta(\gamma\omega)^{\frac{1}{2}}\end{aligned} \tag{C.21}$$

$$\begin{aligned}\zeta_2^{\pm} &= \frac{2}{3}(z_2(\pm d))^{\frac{3}{2}} = \frac{2}{3}(\pm\beta + \sigma + \gamma\omega)^{\frac{3}{2}} = \frac{2}{3}(\gamma\omega)^{\frac{3}{2}}\left(\frac{\sigma \pm \beta}{\gamma\omega} + 1\right)^{\frac{3}{2}}\\ &\approx \frac{2}{3}(\gamma\omega)^{\frac{3}{2}}\left(1 + \frac{3(\sigma \pm \beta)}{2\gamma\omega}\right) = \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} + (\sigma \pm \beta)(\gamma\omega)^{\frac{1}{2}}\end{aligned} \tag{C.22}$$

In our proceeding calculations we will use the following computations

$$\begin{aligned}
z_1(\pm d)^{-\frac{1}{4}} z_2(\pm d)^{\frac{1}{4}} &= (\pm\beta + \gamma\omega)^{-\frac{1}{4}} (\sigma \pm \beta + \gamma\omega)^{\frac{1}{4}} \\
&= (\gamma\omega)^{-\frac{1}{4}} \left(1 \pm \frac{\beta}{\gamma\omega}\right)^{-\frac{1}{4}} (\gamma\omega)^{\frac{1}{4}} \left(1 + \frac{\sigma \pm \beta}{\gamma\omega}\right)^{\frac{1}{4}} \\
&= \left(1 \mp \frac{\beta}{4\gamma\omega}\right)\left(1 + \frac{\sigma \pm \beta}{4\gamma\omega}\right) = 1 + \frac{\sigma \pm \beta}{4\gamma\omega} \mp \frac{\beta}{4\gamma\omega} \mp \frac{\beta(\sigma \pm \beta)}{16(\gamma\omega)^2} \\
&\approx 1 + \frac{\sigma}{4\gamma\omega}
\end{aligned} \tag{C.23}$$

and we also have

$$\begin{aligned}
z_1(\pm d)^{\frac{1}{4}} z_2(\pm d)^{-\frac{1}{4}} &= (\pm\beta + \gamma\omega)^{\frac{1}{4}} (\sigma \pm \beta + \gamma\omega)^{-\frac{1}{4}} \\
&= (\gamma\omega)^{\frac{1}{4}} \left(1 \pm \frac{\beta}{\gamma\omega}\right)^{\frac{1}{4}} (\gamma\omega)^{-\frac{1}{4}} \left(1 + \frac{\sigma \pm \beta}{\gamma\omega}\right)^{-\frac{1}{4}} \\
&= \left(1 \pm \frac{\beta}{4\gamma\omega}\right)\left(1 - \frac{\sigma \pm \beta}{4\gamma\omega}\right) = 1 - \frac{\sigma \pm \beta}{4\gamma\omega} \pm \frac{\beta}{4\gamma\omega} \mp \frac{\beta(\sigma \pm \beta)}{16(\gamma\omega)^2} \\
&\approx 1 - \frac{\sigma}{4\gamma\omega}
\end{aligned} \tag{C.24}$$

Let us take the various terms in (C.9). First, we calculate the term

$$\begin{aligned}
&A_0 A_1' - A_0' A_1 = \mathrm{Ai}(-z_1(-d))\mathrm{Ai}'(-z_2(-d)) - \mathrm{Ai}'(-z_1(-d))\mathrm{Ai}(-z_2(-d)) \\
&\approx -\frac{1}{4i\pi}(z_1(-d))^{-\frac{1}{4}} \left(e^{i\left(\zeta_1^- + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_1^- + \frac{\pi}{4}\right)}\right)(z_2(-d))^{\frac{1}{4}} \left(e^{i\left(\zeta_2^- + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2^- + \frac{\pi}{4}\right)}\right) \\
&+ \frac{1}{4i\pi}(z_1(-d))^{\frac{1}{4}} \left(e^{i\left(\zeta_1^- + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_1^- + \frac{\pi}{4}\right)}\right)(z_2(-d))^{-\frac{1}{4}} \left(e^{i\left(\zeta_2^- + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2^- + \frac{\pi}{4}\right)}\right) \\
&\approx -\frac{1}{4i\pi}\left(1 + \frac{\sigma}{4\gamma\omega}\right)\left(e^{i\left(\zeta_1^- + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_1^- + \frac{\pi}{4}\right)}\right)\left(e^{i\left(\zeta_2^- + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2^- + \frac{\pi}{4}\right)}\right) \\
&+ \frac{1}{4i\pi}\left(1 - \frac{\sigma}{4\gamma\omega}\right)\left(e^{i\left(\zeta_1^- + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_1^- + \frac{\pi}{4}\right)}\right)\left(e^{i\left(\zeta_2^- + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2^- + \frac{\pi}{4}\right)}\right) \\
&= -\frac{1}{4i\pi}\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} + e^{i\left(\zeta_1^- - \zeta_2^-\right)} - e^{-i\left(\zeta_1^- - \zeta_2^-\right)} + ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&- \frac{1}{4i\pi}\frac{\sigma}{4\gamma\omega}\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} + e^{i\left(\zeta_1^- - \zeta_2^-\right)} - e^{-i\left(\zeta_1^- - \zeta_2^-\right)} + ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&+ \frac{1}{4i\pi}\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} - e^{i\left(\zeta_1^- - \zeta_2^-\right)} + e^{-i\left(\zeta_1^- - \zeta_2^-\right)} + ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&- \frac{1}{4i\pi}\frac{\sigma}{4\gamma\omega}\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} - e^{i\left(\zeta_1^- - \zeta_2^-\right)} + e^{-i\left(\zeta_1^- - \zeta_2^-\right)} + ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&= -\frac{1}{2i\pi}\left(e^{i\left(\zeta_1^- - \zeta_2^-\right)} - e^{-i\left(\zeta_1^- - \zeta_2^-\right)}\right) - \frac{\sigma}{8\pi\gamma\omega}\left(e^{i\left(\zeta_1^- + \zeta_2^-\right)} + e^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right)
\end{aligned} \tag{C.25}$$

where we used (C.23) and we proceed to the next one.

$$
\begin{aligned}
&B_2C_3' - B_2'C_3 = \mathrm{Bi}(-z_2(d))\mathrm{Ci}'^{+}(-z_1(d)) - \mathrm{Bi}'(-z_2(d))\mathrm{Ci}^{+}(-z_1(d)) \\
&\approx \frac{1}{2\sqrt{\pi}}(z_2(d))^{-\frac{1}{4}}\left(e^{i\left(\zeta_2^+ + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2^+ + \frac{\pi}{4}\right)}\right)\left(-i\pi^{-\frac{1}{2}}(z_1(d))^{\frac{1}{4}}e^{i\left(\zeta_1^+ + \frac{\pi}{4}\right)}\right) \\
&- \frac{1}{2i\sqrt{\pi}}(z_2(d))^{\frac{1}{4}}\left(e^{i\left(\zeta_2^+ + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2^+ + \frac{\pi}{4}\right)}\right)\pi^{-\frac{1}{2}}(z_1(d))^{-\frac{1}{4}}e^{i\left(\zeta_1^+ + \frac{\pi}{4}\right)} \\
&\approx \frac{1}{2i\pi}\left(1 - \frac{\sigma}{4\gamma\omega}\right)\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} + e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) \\
&- \frac{1}{2i\pi}\left(1 + \frac{\sigma}{4\gamma\omega}\right)\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} - e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) \\
&= \frac{1}{2i\pi}\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} + e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) - \frac{1}{2i\pi}\frac{\sigma}{4\gamma\omega}\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} + e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) \\
&- \frac{1}{2i\pi}\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} - e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) - \frac{1}{2i\pi}\frac{\sigma}{4\gamma\omega}\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} - e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) \\
&= \frac{1}{i\pi}e^{i\left(\zeta_1^+ - \zeta_2^+\right)} - \frac{\sigma}{4\pi\gamma\omega}e^{i\left(\zeta_1^+ + \zeta_2^+\right)}
\end{aligned} \tag{C.26}
$$

and similarly

$$
\begin{aligned}
&B_2D_3' - B_2'D_3 = \mathrm{Bi}(-z_2(d))\mathrm{Ci}'^{-}(-z_1(d)) - \mathrm{Bi}'(-z_2(d))\mathrm{Ci}^{-}(-z_1(d)) \\
&\approx -\frac{1}{i\pi}e^{-i\left(\zeta_1^+ - \zeta_2^+\right)} - \frac{\sigma}{4\pi\gamma\omega}e^{-i\left(\zeta_1^+ + \zeta_2^+\right)}
\end{aligned} \tag{C.27}
$$

We proceed to the next term in (C.9) which is

$$
\begin{aligned}
&A_0B_1' - A_0'B_1 = \mathrm{Ai}(-z_1(-d))\mathrm{Bi}'(-z_2(-d)) - \mathrm{Ai}'(-z_1(-d))\mathrm{Bi}(-z_2(-d)) \\
&\approx \frac{1}{2i\sqrt{\pi}}(z_1(-d))^{-\frac{1}{4}}\left(e^{i\left(\zeta_1^- + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_1^- + \frac{\pi}{4}\right)}\right) \\
&\frac{1}{2i\sqrt{\pi}}(z_2(-d))^{\frac{1}{4}}\left(e^{i\left(\zeta_2^- + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2^- + \frac{\pi}{4}\right)}\right) \\
&+ \frac{1}{2\sqrt{\pi}}(z_1(-d))^{\frac{1}{4}}\left(e^{i\left(\zeta_1^- + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_1^- + \frac{\pi}{4}\right)}\right) \\
&\frac{1}{2\sqrt{\pi}}(z_2(-d))^{-\frac{1}{4}}\left(e^{i\left(\zeta_2^- + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2^- + \frac{\pi}{4}\right)}\right) \\
&\approx -\frac{1}{4\pi}\left(1 + \frac{\sigma}{4\gamma\omega}\right)\left(e^{i\left(\zeta_1^- + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_1^- + \frac{\pi}{4}\right)}\right)\left(e^{i\left(\zeta_2^- + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2^- + \frac{\pi}{4}\right)}\right) \\
&+ \frac{1}{4\pi}\left(1 - \frac{\sigma}{4\gamma\omega}\right)\left(e^{i\left(\zeta_1^- + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_1^- + \frac{\pi}{4}\right)}\right)\left(e^{i\left(\zeta_2^- + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2^- + \frac{\pi}{4}\right)}\right) \\
&= -\frac{1}{4\pi}\left(1 + \frac{\sigma}{4\gamma\omega}\right)\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} - e^{i\left(\zeta_1^- - \zeta_2^-\right)} - e^{-i\left(\zeta_1^- - \zeta_2^-\right)} - ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&+ \frac{1}{4\pi}\left(1 - \frac{\sigma}{4\gamma\omega}\right)\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} + e^{i\left(\zeta_1^- - \zeta_2^-\right)} + e^{-i\left(\zeta_1^- - \zeta_2^-\right)} - ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&= -\frac{1}{4\pi}\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} - e^{i\left(\zeta_1^- - \zeta_2^-\right)} - e^{-i\left(\zeta_1^- - \zeta_2^-\right)} - ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&- \frac{1}{4\pi}\frac{\sigma}{4\gamma\omega}\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} - e^{i\left(\zeta_1^- - \zeta_2^-\right)} - e^{-i\left(\zeta_1^- - \zeta_2^-\right)} - ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&+ \frac{1}{4\pi}\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} + e^{i\left(\zeta_1^- - \zeta_2^-\right)} + e^{-i\left(\zeta_1^- - \zeta_2^-\right)} - ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&- \frac{1}{4\pi}\frac{\sigma}{4\gamma\omega}\left(ie^{i\left(\zeta_1^- + \zeta_2^-\right)} + e^{i\left(\zeta_1^- - \zeta_2^-\right)} + e^{-i\left(\zeta_1^- - \zeta_2^-\right)} - ie^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&= \frac{1}{2\pi}\left(e^{i\left(\zeta_1^- - \zeta_2^-\right)} + e^{-i\left(\zeta_1^- - \zeta_2^-\right)}\right) - \frac{\sigma i}{8\pi\gamma\omega}\left(e^{i\left(\zeta_1^- + \zeta_2^-\right)} - e^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \qquad \text{(C.28)}
\end{aligned}
$$

With the same approach we take the next term.

$$
\begin{aligned}
&A_2C_3' - A_2'C_3 = \mathrm{Ai}(-z_2(d))\mathrm{Ci}'^{+}(-z_1(d)) - \mathrm{Ai}'(-z_2(d))\mathrm{Ci}^{+}(-z_1(d)) \\
&\approx \frac{1}{2i\sqrt{\pi}}(z_2(d))^{-\frac{1}{4}}\left(e^{i\left(\zeta_2^+ + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2^+ + \frac{\pi}{4}\right)}\right)\left(-i\pi^{-\frac{1}{2}}(z_1(d))^{\frac{1}{4}}e^{i\left(\zeta_1^+ + \frac{\pi}{4}\right)}\right) \\
&- \left(-\frac{1}{2\sqrt{\pi}}(z_2(d))^{\frac{1}{4}}\left(e^{i\left(\zeta_2^+ + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2^+ + \frac{\pi}{4}\right)}\right)\right)\pi^{-\frac{1}{2}}(z_1(d))^{-\frac{1}{4}}e^{i\left(\zeta_1^+ + \frac{\pi}{4}\right)} \\
&\approx -\frac{1}{2\pi}\left(1 - \frac{\sigma}{4\gamma\omega}\right)\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} - e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) \\
&+ \frac{1}{2\pi}\left(1 + \frac{\sigma}{4\gamma\omega}\right)\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} + e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) \\
&= -\frac{1}{2\pi}\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} - e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) + \frac{1}{2\pi}\frac{\sigma}{4\gamma\omega}\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} - e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) \\
&+ \frac{1}{2\pi}\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} + e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) + \frac{1}{2\pi}\frac{\sigma}{4\gamma\omega}\left(ie^{i\left(\zeta_1^+ + \zeta_2^+\right)} + e^{i\left(\zeta_1^+ - \zeta_2^+\right)}\right) \\
&= \frac{1}{\pi}e^{i\left(\zeta_1^+ - \zeta_2^+\right)} + \frac{\sigma i}{4\pi\gamma\omega}e^{i\left(\zeta_1^+ + \zeta_2^+\right)}
\end{aligned}
\tag{C.29}
$$

and similarly its complex conjugate

$$
\begin{aligned}
&A_2D_3' - A_2'D_3 = \mathrm{Ai}(-z_2(d))\mathrm{Ci}'^{-}(-z_1(d)) - \mathrm{Ai}'(-z_2(d))\mathrm{Ci}^{-}(-z_1(d)) \\
&\approx \frac{1}{\pi}e^{-i\left(\zeta_1^+ - \zeta_2^+\right)} - \frac{\sigma i}{4\pi\gamma\omega}e^{-i\left(\zeta_1^+ + \zeta_2^+\right)}
\end{aligned}
\tag{C.30}
$$

We can now conclude that the asymptotic expression for $\det \mathbf{M}(\omega)$ using (C.25), (C.26), (C.28) and (C.29) is

$$
\begin{aligned}
&\det \mathbf{M}(\omega) = (A_0 A_1' - A_0' A_1)(B_2 C_3' - B_2' C_3) - (A_0 B_1' - A_0' B_1)(A_2 C_3' - A_2' C_3) \\
&\approx \left[-\frac{1}{2i\pi} \left(e^{i\left(\zeta_1^- - \zeta_2^-\right)} - e^{-i\left(\zeta_1^- - \zeta_2^-\right)} \right) - \frac{\sigma}{8\pi\gamma\omega} \left(e^{i\left(\zeta_1^- + \zeta_2^-\right)} + e^{-i\left(\zeta_1^- + \zeta_2^-\right)} \right) \right] \\
&\left(\frac{1}{i\pi} e^{i\left(\zeta_1^+ - \zeta_2^+\right)} - \frac{\sigma}{4\pi\gamma\omega} e^{i\left(\zeta_1^+ + \zeta_2^+\right)} \right) \\
&- \left[\frac{1}{2\pi} \left(e^{i\left(\zeta_1^- - \zeta_2^-\right)} + e^{-i\left(\zeta_1^- - \zeta_2^-\right)} \right) - \frac{\sigma i}{8\pi\gamma\omega} \left(e^{i\left(\zeta_1^- + \zeta_2^-\right)} - e^{-i\left(\zeta_1^- + \zeta_2^-\right)} \right) \right] \\
&\left(\frac{1}{\pi} e^{i\left(\zeta_1^+ - \zeta_2^+\right)} + \frac{\sigma i}{4\pi\gamma\omega} e^{i\left(\zeta_1^+ + \zeta_2^+\right)} \right) \\
&= \frac{1}{2\pi^2} \left(e^{i\left(\zeta_1^- + \zeta_1^+ - \zeta_2^- - \zeta_2^+\right)} - e^{-i\left(\zeta_1^- - \zeta_1^+ - \zeta_2^- + \zeta_2^+\right)} \right) \\
&+ \frac{\sigma}{8i\pi^2\gamma\omega} \left(e^{i\left(\zeta_1^- + \zeta_1^+ - \zeta_2^- + \zeta_2^+\right)} - e^{-i\left(\zeta_1^- - \zeta_1^+ - \zeta_2^- - \zeta_2^+\right)} \right) \\
&- \frac{\sigma}{8i\pi^2\gamma\omega} \left(e^{i\left(\zeta_1^- + \zeta_1^+ + \zeta_2^- - \zeta_2^+\right)} + e^{-i\left(\zeta_1^- - \zeta_1^+ + \zeta_2^- + \zeta_2^+\right)} \right) \\
&+ \frac{\sigma^2}{32\pi^2(\gamma\omega)^2} \left(e^{i\left(\zeta_1^- + \zeta_1^+ + \zeta_2^- + \zeta_2^+\right)} + e^{-i\left(\zeta_1^- - \zeta_1^+ + \zeta_2^- - \zeta_2^+\right)} \right) \\
&- \frac{1}{2\pi^2} \left(e^{i\left(\zeta_1^- + \zeta_1^+ - \zeta_2^- - \zeta_2^+\right)} + e^{-i\left(\zeta_1^- - \zeta_1^+ - \zeta_2^- + \zeta_2^+\right)} \right) \\
&+ \frac{\sigma}{8i\pi^2\gamma\omega} \left(e^{i\left(\zeta_1^- + \zeta_1^+ - \zeta_2^- + \zeta_2^+\right)} + e^{-i\left(\zeta_1^- - \zeta_1^+ - \zeta_2^- - \zeta_2^+\right)} \right) \\
&- \frac{\sigma}{8i\pi^2\gamma\omega} \left(e^{i\left(\zeta_1^- + \zeta_1^+ + \zeta_2^- - \zeta_2^+\right)} - e^{-i\left(\zeta_1^- - \zeta_1^+ + \zeta_2^- + \zeta_2^+\right)} \right) \\
&- \frac{\sigma^2}{32\pi^2(\gamma\omega)^2} \left(e^{i\left(\zeta_1^- + \zeta_1^+ + \zeta_2^- + \zeta_2^+\right)} - e^{-i\left(\zeta_1^- - \zeta_1^+ + \zeta_2^- - \zeta_2^+\right)} \right) \\
&= -\frac{1}{\pi^2} e^{-i\left(\zeta_1^- - \zeta_1^+ - \zeta_2^- + \zeta_2^+\right)} + \frac{\sigma}{4i\pi^2\gamma\omega} e^{i\left(\zeta_1^- + \zeta_1^+ - \zeta_2^- + \zeta_2^+\right)} \\
&- \frac{\sigma}{4i\pi^2\gamma\omega} e^{i\left(\zeta_1^- + \zeta_1^+ + \zeta_2^- - \zeta_2^+\right)} + \frac{\sigma^2}{16\pi^2(\gamma\omega)^2} e^{-i\left(\zeta_1^- - \zeta_1^+ + \zeta_2^- - \zeta_2^+\right)}
\end{aligned}
\tag{C.31}
$$

We can use (C.21) and (C.22) to simplify this expression.

$$\begin{aligned}
&\det \mathbf{M}(\omega) \approx -\frac{1}{\pi^2} \\
&e^{-i\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}-\beta(\gamma\omega)^{\frac{1}{2}}-\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+\beta(\gamma\omega)^{\frac{1}{2}}\right)-\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma-\beta)(\gamma\omega)^{\frac{1}{2}}\right)+\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma+\beta)(\gamma\omega)^{\frac{1}{2}}\right)} \\
&+\frac{\sigma}{4i\pi^2\gamma\omega} \\
&e^{i\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}-\beta(\gamma\omega)^{\frac{1}{2}}+\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+\beta(\gamma\omega)^{\frac{1}{2}}-\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma-\beta)(\gamma\omega)^{\frac{1}{2}}\right)+\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma+\beta)(\gamma\omega)^{\frac{1}{2}}\right)} \\
&-\frac{\sigma}{4i\pi^2\gamma\omega} \\
&e^{i\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}-\beta(\gamma\omega)^{\frac{1}{2}}+\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+\beta(\gamma\omega)^{\frac{1}{2}}+\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma-\beta)(\gamma\omega)^{\frac{1}{2}}-\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma+\beta)(\gamma\omega)^{\frac{1}{2}}\right)\right)} \\
&+\frac{\sigma^2}{16\pi^2(\gamma\omega)^2} \\
&e^{-i\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}-\beta(\gamma\omega)^{\frac{1}{2}}-\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+\beta(\gamma\omega)^{\frac{1}{2}}\right)+\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma-\beta)(\gamma\omega)^{\frac{1}{2}}-\left(\frac{2}{3}(\gamma\omega)^{\frac{3}{2}}+(\sigma+\beta)(\gamma\omega)^{\frac{1}{2}}\right)\right)} \\
&= -\frac{1}{\pi^2}+\frac{\sigma}{4i\pi^2\gamma\omega}e^{i\left(\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}+2\beta(\gamma\omega)^{\frac{1}{2}}\right)}-\frac{\sigma}{4i\pi^2\gamma\omega}e^{i\left(\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}-2\beta(\gamma\omega)^{\frac{1}{2}}\right)} \\
&+\frac{\sigma^2}{16\pi^2(\gamma\omega)^2}e^{i4\beta(\gamma\omega)^{\frac{1}{2}}} \\
&\approx -\frac{1}{\pi^2}+\frac{\sigma}{4i\pi^2\gamma\omega}e^{i\left(\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}+2\beta(\gamma\omega)^{\frac{1}{2}}\right)}-\frac{\sigma}{4i\pi^2\gamma\omega}e^{i\left(\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}-2\beta(\gamma\omega)^{\frac{1}{2}}\right)} \approx -\frac{1}{\pi^2}
\end{aligned} \tag{C.32}$$

where in the last line we used the fact that ω is real. In the same way using (C.25), (C.27), (C.28) and (C.30) we have the complex conjugate of the determinant

$$\begin{aligned}
\overline{\det \mathbf{M}}(\omega) &\approx -\frac{1}{\pi^2}-\frac{\sigma}{4i\pi^2\gamma\omega}e^{-i\left(\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}+2\beta(\gamma\omega)^{\frac{1}{2}}\right)}+\frac{\sigma}{4i\pi^2\gamma\omega}e^{-i\left(\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}-2\beta(\gamma\omega)^{\frac{1}{2}}\right)} \\
&\approx -\frac{1}{\pi^2}
\end{aligned} \tag{C.33}$$

We continue with the expression in (C.8). Let us introduce the following quantities for transformations $z_1(x)$ and $z_2(x)$.

$$z_1(x) = 2\alpha(\varepsilon x+\omega) = \mu x+\gamma\omega \tag{C.34}$$

$$z_2(x) = 2\alpha(\varepsilon x+V_0+\omega) = \mu x+\sigma+\gamma\omega \tag{C.35}$$

where $\gamma = 2\alpha, \mu = 2\alpha\varepsilon$ and $\sigma = 2\alpha V_0$. Also, the following computations will be helpful.

$$
\begin{aligned}
&(z_1(x'))^{-\frac{1}{4}}(z_2(x))^{-\frac{1}{4}} = (\mu x' + \gamma\omega)^{-\frac{1}{4}} (\mu x + \sigma + \gamma\omega)^{-\frac{1}{4}} \\
&= (\gamma\omega)^{-\frac{1}{4}} \left(1 + \frac{\mu x'}{\gamma\omega}\right)^{-\frac{1}{4}} (\gamma\omega)^{-\frac{1}{4}} \left(1 + \frac{\mu x + \sigma}{\gamma\omega}\right)^{-\frac{1}{4}} \\
&\approx (\gamma\omega)^{-\frac{1}{2}} \left(1 - \frac{\mu x'}{4\gamma\omega}\right) \left(1 - \frac{\mu x + \sigma}{4\gamma\omega}\right) = (\gamma\omega)^{-\frac{1}{2}} \qquad \text{(C.36)} \\
&\zeta_2 - \zeta_1' = \frac{2}{3}(z_2(x))^{\frac{3}{2}} - \frac{2}{3}(z_1(x'))^{\frac{3}{2}} = \frac{2}{3}(\mu x + \sigma + \gamma\omega)^{\frac{3}{2}} - \frac{2}{3}(\mu x' + \gamma\omega)^{\frac{3}{2}} \\
&= \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \left[\left(\frac{\mu x + \sigma}{\gamma\omega} + 1\right)^{\frac{3}{2}} - \left(\frac{\mu x'}{\gamma\omega} + 1\right)^{\frac{3}{2}}\right] \\
&\approx \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \left(1 + \frac{3(\mu x + \sigma)}{2\gamma\omega} - 1 - \frac{3\mu x'}{2\gamma\omega}\right) = \mu(\gamma\omega)^{\frac{1}{2}} (x - x') + \sigma(\gamma\omega)^{\frac{1}{2}} \qquad \text{(C.37)} \\
&\zeta_2 + \zeta_1' = \frac{2}{3}(z_2(x))^{\frac{3}{2}} + \frac{2}{3}(z_1(x'))^{\frac{3}{2}} = \frac{2}{3}(\mu x + \sigma + \gamma\omega)^{\frac{3}{2}} + \frac{2}{3}(\mu x' + \gamma\omega)^{\frac{3}{2}} \\
&= \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \left[\left(\frac{\mu x + \sigma}{\gamma\omega} + 1\right)^{\frac{3}{2}} + \left(\frac{\mu x'}{\gamma\omega} + 1\right)^{\frac{3}{2}}\right] \\
&\approx \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \left(1 + \frac{3(\mu x + \sigma)}{2\gamma\omega} + 1 + \frac{3\mu x'}{2\gamma\omega}\right) \approx \frac{4}{3}(\gamma\omega)^{\frac{3}{2}} \qquad \text{(C.38)}
\end{aligned}
$$

where $\zeta_{1,2} = \frac{2}{3}(z_{1,2}(x))^{\frac{3}{2}}$. With this notation, the first term in the integral (C.8) becomes

$$
\begin{aligned}
&\mathrm{Ai}(-z_2(x))\mathrm{Ci}^-(-z_1(x')) \\
&\approx \frac{1}{2i\sqrt{\pi}}(z_2(x))^{-\frac{1}{4}} \left(e^{i\left(\zeta_2 + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2 + \frac{\pi}{4}\right)}\right) \pi^{-\frac{1}{2}}(z_1(x'))^{-\frac{1}{4}} e^{-i\left(\zeta_1' + \frac{\pi}{4}\right)} \\
&= \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\left(\zeta_2 - \zeta_1'\right)} - e^{-i\left(\zeta_2 + \zeta_1' + \frac{\pi}{2}\right)}\right) \\
&\approx \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x - x')} e^{i\sigma(\gamma\omega)^{\frac{1}{2}}} + ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right) \\
&= \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x - x' + \frac{\sigma}{\mu}\right)} + ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right) \qquad \text{(C.39)}
\end{aligned}
$$

where we used the help of (C.36)-(C.38). Similarly we have

$$
\mathrm{Ai}(-z_2(x))\mathrm{Ci}^+(-z_1(x')) \approx \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}} \left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} - e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x - x' + \frac{\sigma}{\mu}\right)}\right) \qquad \text{(C.40)}
$$

We are ready to return back to our integral (C.8) and collect the gained knowledge to compute the first two terms. We have

$$\frac{2i(B_1'A_0 - B_1A_0')}{\pi\det\mathbf{M}(\omega)} \approx \frac{2i}{\pi}\frac{\frac{1}{2\pi}\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}} + e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)}{-\frac{1}{\pi^2}} = -i\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}} + e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right) \qquad \text{(C.41)}$$

then the first two terms in (C.8) are

$$\begin{aligned}
&\frac{2i(B_1'A_0 - B_1A_0')}{\pi\det\mathbf{M}(\omega)}\mathrm{Ai}(-z_2(x))\mathrm{Ci}^-(-z_1(x')) \\
&-\frac{2i(B_1'A_0 - B_1A_0')}{\pi\det\mathbf{M}(\omega)}\mathrm{Ai}(-z_2(x))\mathrm{Ci}^+(-z_1(x')) \\
&\approx i\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}} + e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)\left[\mathrm{Ai}(-z_2(x))\mathrm{Ci}^+(-z_1(x')) - \mathrm{Ai}(-z_2(x))\mathrm{Ci}^-(-z_1(x'))\right] \\
&\approx \frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}} + e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)\left[\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} - e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)}\right)\right. \\
&\left.-\left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)} + ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)\right] \qquad \text{(C.42)}
\end{aligned}$$

It is easy to verify, that the integrals that contain the term $\frac{i}{(\gamma\omega)^{\frac{1}{2}}}e^{\pm i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}$ are convergent for $\omega \to \infty$ and therefore they give no contribution to χ. Dropping those terms, the expression (C.42) becomes

$$\begin{aligned}
&-\frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}} + e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)} \\
&-\frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}} + e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)} \\
&= -\frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+2\frac{\sigma}{\mu}\right)} + e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x'-x)}\right. \\
&\left.+e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} + e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+2\frac{\sigma}{\mu}\right)}\right) \qquad \text{(C.43)}
\end{aligned}$$

The integral of this can be transformed just as we did in Appendix A (A.15) and (A.16). Using this, the integral of the first two terms in (C.8) becomes

$$\int_q^\infty -\frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}+e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+2\frac{\sigma}{\mu}\right)}\right.$$
$$\left.+e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x'-x)}+e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+2\frac{\sigma}{\mu}\right)}\right)\mathrm{d}\omega$$
$$=-\frac{1}{\pi\mu\gamma}\left[\int_r^\infty\left(e^{iu(x-x')}+e^{iu\left(x-x'+2\frac{\sigma}{\mu}\right)}\right)\mathrm{d}u\right.$$
$$\left.+\int_{-\infty}^{-r}\left(e^{iu(x-x')}+e^{iu\left(x-x'+2\frac{\sigma}{\mu}\right)}\right)\mathrm{d}u\right]$$
$$=-\frac{1}{\pi\mu\gamma}\int_{-\infty}^{\infty}\left(e^{iu(x-x')}+e^{iu\left(x-x'+2\frac{\sigma}{\mu}\right)}\right)\mathrm{d}u$$
$$=-\frac{2}{\mu\gamma}\left[\delta(x-x')+\delta\left(x-x'+2\frac{\sigma}{\mu}\right)\right] \tag{C.44}$$

with $r=\mu\sqrt{\gamma q}$ and the whole integral (C.8) can be written as

$$Q(x,x')=\frac{2}{\mu\gamma}\left[\delta(x-x')+\delta\left(x-x'+2\frac{\sigma}{\mu}\right)\right]$$
$$-\int_q^\infty\left[\frac{2i(A_1A_0'-A_1'A_0)}{\pi\overline{\det\mathbf{M}}(\omega)}\mathrm{Bi}(-z_2(x))\mathrm{Ci}^-(-z_1(x'))\right.$$
$$\left.-\frac{2i(A_1A_0'-A_1'A_0)}{\pi\det\mathbf{M}(\omega)}\mathrm{Bi}(-z_2(x))\mathrm{Ci}^+(-z_1(x'))\right]\mathrm{d}\omega \tag{C.45}$$

We take now the other part which is going to be very similar to the previous one. With the use of (C.25) and (C.33) we have

$$\frac{2i(A_1A_0'-A_1'A_0)}{\pi\overline{\det\mathbf{M}}(\omega)}\approx\frac{-2i\frac{1}{2i\pi}\left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)}{-\pi\frac{1}{\pi^2}}=e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}} \tag{C.46}$$

and similarly to (C.40) $\mathrm{Bi}(-z_2(x))\mathrm{Ci}^+(-z_1(x'))$ can be expressed as

$$\mathrm{Bi}(-z_2(x))\mathrm{Ci}^+(-z_1(x'))$$
$$\approx\frac{1}{2\sqrt{\pi}}(z_2(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta_2+\frac{\pi}{4}\right)}+e^{-i\left(\zeta_2+\frac{\pi}{4}\right)}\right)\pi^{-\frac{1}{2}}(z_1(x'))^{-\frac{1}{4}}e^{i\left(\zeta_1'+\frac{\pi}{4}\right)}$$
$$\approx\frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\left(\zeta_2+\zeta_1'+\frac{\pi}{2}\right)}+e^{-i\left(\zeta_2-\zeta_1'\right)}\right)$$
$$\approx\frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)$$
$$=\frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)}\right) \tag{C.47}$$

and its complex conjugate is

$$\mathrm{Bi}(-z_2(x))\mathrm{Ci}^-(-z_1(x')) \approx \frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)} - ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right) \tag{C.48}$$

then the integral in (C.45) becomes

$$\begin{aligned}
&\int_q^\infty \left[\frac{2i(A_1A_0' - A_1'A_0)}{\pi\overline{\det \mathbf{M}}(\omega)}\mathrm{Bi}(-z_2(x))\mathrm{Ci}^-(-z_1(x'))\right.\\
&\left.-\frac{2i(A_1A_0' - A_1'A_0)}{\pi \det \mathbf{M}(\omega)}\mathrm{Bi}(-z_2(x))\mathrm{Ci}^+(-z_1(x'))\right] \mathrm{d}\omega\\
&\approx \int_q^\infty \left\{\frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}} - e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)\left[\left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)} - ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)\right.\right.\\
&\left.\left.-\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} + e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)}\right)\right]\right\} \mathrm{d}\omega
\end{aligned} \tag{C.49}$$

Those terms containing $\frac{i}{(\gamma\omega)^{\frac{1}{2}}}e^{\pm i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}$ can be dropped considering the fact that the integral of them is finite and give no contribution to χ. After this we are left with

$$\begin{aligned}
&\int_q^\infty \frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}} - e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)\left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)} - e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+\frac{\sigma}{\mu}\right)}\right)\mathrm{d}\omega\\
&= \int_q^\infty \frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+2\frac{\sigma}{\mu}\right)} - e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}\right.\\
&\qquad\qquad\left.-e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x'-x)} + e^{-i\mu(\gamma\omega)^{\frac{1}{2}}\left(x-x'+2\frac{\sigma}{\mu}\right)}\right)\mathrm{d}\omega\\
&= \frac{1}{\pi\mu\gamma}\int_r^\infty \left(e^{iu\left(x-x'+2\frac{\sigma}{\mu}\right)} - e^{iu(x-x')}\right)\mathrm{d}u\\
&\qquad\qquad + \int_{-\infty}^{-r}\left(-e^{iu(x'-x)} + e^{iu\left(x-x'+2\frac{\sigma}{\mu}\right)}\right)\mathrm{d}u\\
&= \frac{1}{\pi\mu\gamma}\left(\int_{-\infty}^\infty e^{iu\left(x-x'+2\frac{\sigma}{\mu}\right)}\mathrm{d}u - \int_{-\infty}^\infty e^{iu(x-x')}\mathrm{d}u\right)\\
&= \frac{2}{\mu\gamma}\left[\delta\left(x-x'+2\frac{\sigma}{\mu}\right) - \delta(x-x')\right]
\end{aligned} \tag{C.50}$$

where we used the same method as in Appendix A (A.15) and (A.16). Substituting this into

(C.45) we get

$$Q(x,x') = \frac{2}{\mu\gamma}\left[\delta(x-x') + \delta\left(x - x' + 2\frac{\sigma}{\mu}\right)\right] - \frac{2}{\mu\gamma}\left[\delta\left(x - x' + 2\frac{\sigma}{\mu}\right) - \delta(x-x')\right] = \frac{4}{\mu\gamma}\delta(x-x') = \frac{1}{\alpha^2\varepsilon}\delta(x-x') \tag{C.51}$$

Note, that this is for the region $-d < x < d$ and $d < x'$. In an entirely similar way we could do the region $-d < x' < d$ and $d < x$. Here, the function $Q(x,x')$ reads

$$Q(x,x') = \frac{1}{\alpha^2\varepsilon}\delta(x'-x) = \frac{1}{\alpha^2\varepsilon}\delta(x-x') \tag{C.52}$$

since the Dirac delta function is an even distribution. The result for $x > d$ and $x' > d$ is also known. We have done this in Appendix A (A.12). The solution in this region is (A.17)

$$Q(x,x') = \frac{1}{\alpha^2\varepsilon}\delta(x-x') \tag{C.53}$$

The next region we will investigate is $-d < x < d$ and $-d < x' < d$. The integral (C.7) in this region reads

$$\begin{aligned} Q(x,x') = \int_q^\infty & \frac{2}{\pi|\det \mathbf{M}(\omega)|} \\ & [(B_1'A_0 - B_1A_0')\mathrm{Ai}(-z_2(x)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(-z_2(x))] \\ & \frac{2}{\pi|\det \mathbf{M}(\omega)|}\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(-z_2(x')) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(-z_2(x'))\right]\mathrm{d}\omega \\ & = \int_q^\infty \frac{4}{\pi^2|\det \mathbf{M}(\omega)|^2}\big[(B_1'A_0 - B_1A_0')^2\mathrm{Ai}(-z_2(x))\mathrm{Ai}(-z_2(x')) \\ & +(A_1A_0' - A_1'A_0)^2\mathrm{Bi}(-z_2(x))\mathrm{Bi}(-z_2(x'))\big]\,\mathrm{d}\omega \\ & + \int_q^\infty \frac{4}{\pi^2|\det \mathbf{M}(\omega)|^2}(B_1'A_0 - B_1A_0')(A_1A_0' - A_1'A_0) \\ & [\mathrm{Ai}(-z_2(x))\mathrm{Bi}(-z_2(x')) + \mathrm{Ai}(-z_2(x'))\mathrm{Bi}(-z_2(x))]\,\mathrm{d}\omega \end{aligned} \tag{C.54}$$

There will be needed to additional smaller computations.

$$
\begin{aligned}
&(z_2(x'))^{-\frac{1}{4}}(z_2(x))^{-\frac{1}{4}} = (\mu x' + \sigma + \gamma\omega)^{-\frac{1}{4}} (\mu x + \sigma + \gamma\omega)^{-\frac{1}{4}} \\
&= (\gamma\omega)^{-\frac{1}{4}} \left(1 + \frac{\mu x' + \sigma}{\gamma\omega}\right)^{-\frac{1}{4}} (\gamma\omega)^{-\frac{1}{4}} \left(1 + \frac{\mu x + \sigma}{\gamma\omega}\right)^{-\frac{1}{4}} \\
&\approx (\gamma\omega)^{-\frac{1}{2}} \left(1 - \frac{\mu x' + \sigma}{4\gamma\omega}\right) \left(1 - \frac{\mu x + \sigma}{4\gamma\omega}\right) \approx (\gamma\omega)^{-\frac{1}{2}} \qquad \text{(C.55)} \\
&\zeta_2 - \zeta_2' = \frac{2}{3}(z_2(x))^{\frac{3}{2}} - \frac{2}{3}(z_2(x'))^{\frac{3}{2}} = \frac{2}{3}(\mu x + \sigma + \gamma\omega)^{\frac{3}{2}} - \frac{2}{3}(\mu x' + \sigma + \gamma\omega)^{\frac{3}{2}} \\
&= \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \left[\left(\frac{\mu x + \sigma}{\gamma\omega} + 1\right)^{\frac{3}{2}} - \left(\frac{\mu x' + \sigma}{\gamma\omega} + 1\right)^{\frac{3}{2}}\right] \\
&\approx \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \left(1 + \frac{3(\mu x + \sigma)}{2\gamma\omega} - 1 - \frac{3(\mu x' + \sigma)}{2\gamma\omega}\right) = \mu(\gamma\omega)^{\frac{1}{2}} (x - x') \qquad \text{(C.56)} \\
&\zeta_2 + \zeta_2' = \frac{2}{3}(z_2(x))^{\frac{3}{2}} + \frac{2}{3}(z_2(x'))^{\frac{3}{2}} = \frac{2}{3}(\mu x + \sigma + \gamma\omega)^{\frac{3}{2}} + \frac{2}{3}(\mu x' + \sigma + \gamma\omega)^{\frac{3}{2}} \\
&= \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \left[\left(\frac{\mu x + \sigma}{\gamma\omega} + 1\right)^{\frac{3}{2}} + \left(\frac{\mu x' + \sigma}{\gamma\omega} + 1\right)^{\frac{3}{2}}\right] \\
&\approx \frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \left(1 + \frac{3(\mu x + \sigma)}{2\gamma\omega} + 1 + \frac{3(\mu x' + \sigma)}{2\gamma\omega}\right) \approx \frac{4}{3}(\gamma\omega)^{\frac{3}{2}} \qquad \text{(C.57)}
\end{aligned}
$$

Let us start with the first integral. For each term we found an asymptotic expression except for

$$
\begin{aligned}
&\mathrm{Ai}(-z_2(x))\mathrm{Ai}(-z_2(x')) \\
&\approx \frac{1}{2i\sqrt{\pi}}(z_2(x))^{-\frac{1}{4}} \left(e^{i\left(\zeta_2 + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2 + \frac{\pi}{4}\right)}\right) \frac{1}{2i\sqrt{\pi}}(z_2(x'))^{-\frac{1}{4}} \left(e^{i\left(\zeta_2' + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2' + \frac{\pi}{4}\right)}\right) \\
&\approx -\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\left(\zeta_2 + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2 + \frac{\pi}{4}\right)}\right) \left(e^{i\left(\zeta_2' + \frac{\pi}{4}\right)} - e^{-i\left(\zeta_2' + \frac{\pi}{4}\right)}\right) \\
&= -\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}} \left(ie^{i\left(\zeta_2 + \zeta_2'\right)} - e^{i\left(\zeta_2 - \zeta_2'\right)} - e^{-i\left(\zeta_2 - \zeta_2'\right)} - ie^{-i\left(\zeta_2 + \zeta_2'\right)}\right) \qquad \text{(C.58)}
\end{aligned}
$$

where we used (C.55). For the next term we have

$$
\begin{aligned}
&\mathrm{Bi}(-z_2(x))\mathrm{Bi}(-z_2(x')) \approx \frac{1}{2\sqrt{\pi}}(z_2(x))^{-\frac{1}{4}} \left(e^{i\left(\zeta_2 + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2 + \frac{\pi}{4}\right)}\right) \\
&\frac{1}{2\sqrt{\pi}}(z_2(x'))^{-\frac{1}{4}} \left(e^{i\left(\zeta_2' + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2' + \frac{\pi}{4}\right)}\right) \\
&\approx \frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\left(\zeta_2 + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2 + \frac{\pi}{4}\right)}\right) \left(e^{i\left(\zeta_2' + \frac{\pi}{4}\right)} + e^{-i\left(\zeta_2' + \frac{\pi}{4}\right)}\right) \\
&= \frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}} \left(ie^{i\left(\zeta_2 + \zeta_2'\right)} + e^{i\left(\zeta_2 - \zeta_2'\right)} + e^{-i\left(\zeta_2 - \zeta_2'\right)} - ie^{-i\left(\zeta_2 + \zeta_2'\right)}\right) \qquad \text{(C.59)}
\end{aligned}
$$

Using and (C.25), (C.28), (C.30) in the first integral of (C.54) we get

$$
\begin{aligned}
&\int_q^\infty \frac{4}{\pi^2|\det \mathbf{M}(\omega)|^2}\left[(B_1'A_0-B_1A_0')^2\mathrm{Ai}(-z_2(x))\mathrm{Ai}(-z_2(x'))\right.\\
&\left.+(A_1A_0'-A_1'A_0)^2\mathrm{Bi}(-z_2(x))\mathrm{Bi}(-z_2(x'))\right]\mathrm{d}\omega\\
&\approx \int_q^\infty \frac{4}{\pi^2\frac{1}{\pi^4}}\left[\left(\frac{1}{2\pi}\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}+e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)\right)^2\mathrm{Ai}(-z_2(x))\mathrm{Ai}(-z_2(x'))\right.\\
&\left.+\left(-\frac{1}{2i\pi}\left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)\right)^2\mathrm{Bi}(-z_2(x))\mathrm{Bi}(-z_2(x'))\right]\mathrm{d}\omega\\
&=\int_q^\infty\left[\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}+e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)^2\mathrm{Ai}(-z_2(x))\mathrm{Ai}(-z_2(x'))\right.\\
&\left.-\left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)^2\mathrm{Bi}(-z_2(x))\mathrm{Bi}(-z_2(x'))\right]\mathrm{d}\omega
\end{aligned}
\tag{C.60}
$$

We examine the first part of the integrand in (C.60) using (C.58.

$$
\begin{aligned}
&\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}+e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)^2\mathrm{Ai}(-z_2(x))\mathrm{Ai}(-z_2(x'))\\
&\approx -\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}+e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)^2\\
&\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\left(\zeta_2+\zeta_2'\right)}-e^{i\left(\zeta_2-\zeta_2'\right)}-e^{-i\left(\zeta_2-\zeta_2'\right)}-ie^{-i\left(\zeta_2+\zeta_2'\right)}\right)\\
&=-\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{-i2\sigma(\gamma\omega)^{\frac{1}{2}}}+e^{i2\sigma(\gamma\omega)^{\frac{1}{2}}}+2\right)\\
&\left(ie^{i\left(\zeta_2+\zeta_2'\right)}-e^{i\left(\zeta_2-\zeta_2'\right)}-e^{-i\left(\zeta_2-\zeta_2'\right)}-ie^{-i\left(\zeta_2+\zeta_2'\right)}\right)
\end{aligned}
\tag{C.61}
$$

and the other part is

$$
\begin{aligned}
&\left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)^2\mathrm{Bi}(-z_2(x))\mathrm{Bi}(-z_2(x'))\\
&\approx \left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)^2\\
&\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\left(\zeta_2+\zeta_2'\right)}+e^{i\left(\zeta_2-\zeta_2'\right)}+e^{-i\left(\zeta_2-\zeta_2'\right)}-ie^{-i\left(\zeta_2+\zeta_2'\right)}\right)\\
&=\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i2\sigma(\gamma\omega)^{\frac{1}{2}}}+e^{-i2\sigma(\gamma\omega)^{\frac{1}{2}}}-2\right)\\
&\left(ie^{i\left(\zeta_2+\zeta_2'\right)}+e^{i\left(\zeta_2-\zeta_2'\right)}+e^{-i\left(\zeta_2-\zeta_2'\right)}-ie^{-i\left(\zeta_2+\zeta_2'\right)}\right)
\end{aligned}
\tag{C.62}
$$

Both (C.61) and (C.62) are integrands. Looking at the form of $\zeta_2+\zeta_2'$ in (C.57) we can see that the integrals containing terms with this exponent can be ignored since they give a finite contribution to χ. Doing so and subtracting (C.62) from (C.61) according to (C.60 we get

$$\int_q^\infty \left[\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}} + e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)^2 \mathrm{Ai}(-z_2(x))\mathrm{Ai}(-z_2(x'))\right.$$
$$\left. - \left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}} - e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)^2 \mathrm{Bi}(-z_(x))\mathrm{Bi}(-z_2(x'))\right] \mathrm{d}\omega$$
$$\approx \int_q^\infty \frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\left(\zeta_2-\zeta_2'-2\sigma(\gamma\omega)^{\frac{1}{2}}\right)} + e^{-i\left(\zeta_2-\zeta_2'+2\sigma(\gamma\omega)^{\frac{1}{2}}\right)} + e^{i\left(\zeta_2-\zeta_2'+2\sigma(\gamma\omega)^{\frac{1}{2}}\right)}\right.$$
$$+e^{-i\left(\zeta_2-\zeta_2'-2\sigma(\gamma\omega)^{\frac{1}{2}}\right)} + 2e^{i\left(\zeta_2-\zeta_2'\right)} + 2e^{-i\left(\zeta_2-\zeta_2'\right)} - e^{i\left(\zeta_2-\zeta_2'+2\sigma(\gamma\omega)^{\frac{1}{2}}\right)}$$
$$-e^{-i\left(\zeta_2-\zeta_2'-2\sigma(\gamma\omega)^{\frac{1}{2}}\right)} - e^{i\left(\zeta_2-\zeta_2'-2\sigma(\gamma\omega)^{\frac{1}{2}}\right)} - e^{-i\left(\zeta_2-\zeta_2'+2\sigma(\gamma\omega)^{\frac{1}{2}}\right)}$$
$$\left.+2e^{i\left(\zeta_2-\zeta_2'\right)} + 2e^{-i\left(\zeta_2-\zeta_2'\right)}\right) \mathrm{d}\omega$$
$$\approx \int_q^\infty \frac{1}{\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\left(\zeta_2-\zeta_2'\right)} + e^{-i\left(\zeta_2-\zeta_2'\right)}\right) \mathrm{d}\omega$$
$$\approx \int_q^\infty \frac{1}{\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} + e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}\right) \mathrm{d}\omega \tag{C.63}$$

where we used (C.56), (C.57). This expression can be simplified further as

$$\int_q^\infty \frac{1}{\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} + e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}\right) \mathrm{d}\omega$$
$$= \frac{2}{\pi\mu\gamma}\left(\int_r^\infty e^{iu(x-x')}\mathrm{d}u + \int_{-\infty}^{-r} e^{iu(x-x')}\mathrm{d}u\right) = \frac{2}{\pi\mu\gamma}\int_{-\infty}^{\infty} e^{iu(x-x')}\mathrm{d}u$$
$$= \frac{4}{\mu\gamma}\delta\left(x-x'\right) = \frac{1}{\alpha^2\varepsilon}\delta\left(x-x'\right) \tag{C.64}$$

Returning back to (C.54) we have the following

$$Q(x,x') = \int_q^\infty \frac{4}{\pi^2|\det\mathbf{M}(\omega)|^2}\left[(B_1'A_0 - B_1A_0')^2\mathrm{Ai}(-z_2(x))\mathrm{Ai}(-z_2(x'))\right.$$
$$\left.+(A_1A_0' - A_1'A_0)^2\mathrm{Bi}(-z_2(x))\mathrm{Bi}(-z_2(x'))\right]\mathrm{d}\omega$$
$$+\int_q^\infty \frac{4}{\pi^2|\det\mathbf{M}(\omega)|^2}(B_1'A_0 - B_1A_0')(A_1A_0' - A_1'A_0)$$
$$[\mathrm{Ai}(-z_2(x))\mathrm{Bi}(-z_2(x')) + \mathrm{Ai}(-z_2(x'))\mathrm{Bi}(-z_2(x))]\,\mathrm{d}\omega$$
$$= \frac{4}{\mu\gamma}\delta\left(x-x'\right) + \int_q^\infty \frac{4}{\pi^2|\det\mathbf{M}(\omega)|^2}(B_1'A_0 - B_1A_0')(A_1A_0' - A_1'A_0)$$
$$[\mathrm{Ai}(-z_2(x))\mathrm{Bi}(-z_2(x')) + \mathrm{Ai}(-z_2(x'))\mathrm{Bi}(-z_2(x))]\,\mathrm{d}\omega \tag{C.65}$$

The next term we need to express asymptotically is

$$
\begin{aligned}
&\mathrm{Ai}(-z_2(x))\mathrm{Bi}(-z_2(x')) \\
&\approx \frac{1}{2i\sqrt{\pi}}(z_2(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta_2+\frac{\pi}{4}\right)}-e^{-i\left(\zeta_2+\frac{\pi}{4}\right)}\right)\frac{1}{2\sqrt{\pi}}(z_2(x'))^{-\frac{1}{4}}\left(e^{i\left(\zeta_2'+\frac{\pi}{4}\right)}+e^{-i\left(\zeta_2'+\frac{\pi}{4}\right)}\right) \\
&\approx \frac{1}{4i\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\left(\zeta_2+\frac{\pi}{4}\right)}-e^{-i\left(\zeta_2+\frac{\pi}{4}\right)}\right)\left(e^{i\left(\zeta_2'+\frac{\pi}{4}\right)}+e^{-i\left(\zeta_2'+\frac{\pi}{4}\right)}\right) \\
&\approx \frac{1}{4i\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\left(\zeta_2+\zeta_2'\right)}+e^{i\left(\zeta_2-\zeta_2'\right)}-e^{-i\left(\zeta_2-\zeta_2'\right)}+ie^{-i\left(\zeta_2+\zeta_2'\right)}\right) \\
&\approx \frac{1}{4i\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}+ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)
\end{aligned} \tag{C.66}
$$

where we used (C.56), (C.57) and similarly we have

$$
\begin{aligned}
&\mathrm{Ai}(-z_2(x'))\mathrm{Bi}(-z_2(x)) \\
&\approx \frac{1}{4i\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}+ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)
\end{aligned} \tag{C.67}
$$

Using (C.66) and (C.67) together with (C.25), (C.28), (C.30) in the integral in (C.65) we get

$$
\begin{aligned}
&\int_q^\infty \frac{4}{\pi^2|\det\mathbf{M}(\omega)|^2}(B_1'A_0-B_1A_0')(A_1A_0'-A_1'A_0) \\
&[\mathrm{Ai}(-z_2(x))\mathrm{Bi}(-z_2(x'))+\mathrm{Ai}(-z_2(x'))\mathrm{Bi}(-z_2(x))]\,\mathrm{d}\omega \\
&\approx -\int_q^\infty \frac{4}{\pi^2\frac{1}{\pi^4}}\frac{1}{2\pi}\left(e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}+e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}\right)\frac{1}{2i\pi}\left(e^{i\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i\sigma(\gamma\omega)^{\frac{1}{2}}}\right) \\
&\left[\frac{1}{4i\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}+ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)\right. \\
&\left.+\frac{1}{4i\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}+ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)\right]\mathrm{d}\omega \\
&=\int_q^\infty \frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i2\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i2\sigma(\gamma\omega)^{\frac{1}{2}}}\right) \\
&\left[ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}+ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right. \\
&\left.+ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}+ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right]\mathrm{d}\omega \\
&=\int_q^\infty \frac{1}{2\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i2\sigma(\gamma\omega)^{\frac{1}{2}}}-e^{-i2\sigma(\gamma\omega)^{\frac{1}{2}}}\right)\left[ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}+ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right]\mathrm{d}\omega
\end{aligned} \tag{C.68}
$$

It is easy to verify that this integral does not give any contribution either. So basically from

(C.65) we have

$$Q(x,x') = \frac{1}{\alpha^2\varepsilon}\delta\left(x-x'\right) \tag{C.69}$$

Continuing with this process, as the next we will investigate the region $x < -d, d < x'$. The integral (C.6) here becomes

$$\begin{aligned}
&Q(x,x') = \int_q^\infty \frac{2}{\pi^2|\det \mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x)) \\
&\left[-i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(x')) + i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(x'))\right]\mathrm{d}\omega \\
&= \int_q^\infty \frac{2i}{\pi^2\det \mathbf{M}(\omega)}\mathrm{Ai}(y_1(x))\mathrm{Ci}^+(y_1(x')) - \frac{2i}{\pi^2\overline{\det \mathbf{M}}(\omega)}\mathrm{Ai}(y_1(x))\mathrm{Ci}^-(y_1(x'))\mathrm{d}\omega
\end{aligned} \tag{C.70}$$

For this, we need the following asymptotic expressions

$$\begin{aligned}
&\mathrm{Ai}(-z_1(x))\mathrm{Ci}^-(-z_1(x')) \\
&\approx \frac{1}{2i\sqrt{\pi}}(z_1(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta_1+\frac{\pi}{4}\right)} - e^{-i\left(\zeta_1+\frac{\pi}{4}\right)}\right)\pi^{-\frac{1}{2}}(z_1(x'))^{-\frac{1}{4}}e^{-i\left(\zeta_1'+\frac{\pi}{4}\right)} \\
&= \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\left(\zeta_1-\zeta_1'\right)} - e^{-i\left(\zeta_1+\zeta_1'+\frac{\pi}{2}\right)}\right) \approx \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} + ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)
\end{aligned} \tag{C.71}$$

$$\begin{aligned}
&\mathrm{Ai}(-z_1(x))\mathrm{Ci}^+(-z_1(x')) \\
&\approx \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} - e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}\right)
\end{aligned} \tag{C.72}$$

The using (C.30), (C.33), (C.71) and (C.72) the integral becomes

$$
\begin{aligned}
&Q(x,x') \\
&= \int_q^\infty \frac{2i}{\pi^2 \det \mathbf{M}(\omega)} \mathrm{Ai}(y_1(x)) \mathrm{Ci}^+(y_1(x')) - \frac{2i}{\pi^2 \overline{\det \mathbf{M}}(\omega)} \mathrm{Ai}(y_1(x)) \mathrm{Ci}^-(y_1(x')) \mathrm{d}\omega \\
&\approx \int_q^\infty \frac{2i}{-\pi^2 \frac{1}{\pi^2}} \mathrm{Ai}(y_1(x)) \mathrm{Ci}^+(y_1(x')) - \frac{2i}{-\pi^2 \frac{1}{\pi^2}} \mathrm{Ai}(y_1(x)) \mathrm{Ci}^-(y_1(x')) \mathrm{d}\omega \\
&\approx 2i \int_q^\infty \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} + ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} \right) \\
&- \frac{1}{2i\pi(\gamma\omega)^{\frac{1}{2}}} \left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} - e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} \right) \mathrm{d}\omega \\
&= \int_q^\infty \frac{1}{\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} + ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} - ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} + e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} \right) \mathrm{d}\omega \\
&= \int_q^\infty \frac{1}{\pi(\gamma\omega)^{\frac{1}{2}}} \left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} + e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} \right) \mathrm{d}\omega
\end{aligned}
\tag{C.73}
$$

which can be further transformed into

$$
\begin{aligned}
&Q(x,x') = \frac{1}{\pi\gamma^{\frac{1}{2}}} \int_q^\infty \omega^{-\frac{1}{2}} \left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} + e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')} \right) \mathrm{d}\omega \\
&= \frac{2}{\pi\mu\gamma} \left(\int_r^\infty e^{iu(x-x')} \mathrm{d}u + \int_{-\infty}^{-r} e^{iu(x-x')} \mathrm{d}u \right) = \frac{4}{\mu\gamma} \delta(x-x') = \frac{1}{\alpha^2\varepsilon} \delta(x-x')
\end{aligned}
\tag{C.74}
$$

where $r = \mu\sqrt{\gamma q}$ and at the same time the region $x' < -d, d < x$ is also determined and it is the same as (C.74) because of the delta function is even.

The following region we discuss is $x < -d, -d < x' < d$. The function $Q(x,x')$ reads

$$
\begin{aligned}
&Q(x,x') = \int_q^\infty \frac{2}{\pi^2 |\det \mathbf{M}(\omega)|} \mathrm{Ai}(y_1(x)) \frac{2}{\pi |\det \mathbf{M}(\omega)|} \left[(B_1' A_0 - B_1 A_0') \mathrm{Ai}(y_2(x')) \right. \\
&\left. + (A_1 A_0' - A_1' A_0) \mathrm{Bi}(y_2(x')) \right] \mathrm{d}\omega \\
&\approx \int_q^\infty \frac{4}{\pi^3 |\det \mathbf{M}(\omega)|^2} \left[(B_1' A_0 - B_1 A_0') \mathrm{Ai}(y_1(x)) \mathrm{Ai}(y_2(x')) \right. \\
&\left. + (A_1 A_0' - A_1' A_0) \mathrm{Ai}(y_1(x)) \mathrm{Bi}(y_2(x')) \right] \mathrm{d}\omega
\end{aligned}
\tag{C.75}
$$

The only expressions we do not have asymptotic versions for are $\mathrm{Ai}(y_1(x))\mathrm{Ai}(y_2(x'))$ and

$\mathrm{Ai}(y_1(x))\mathrm{Bi}(y_2(x'))$. From (C.58) and (C.66) they are easily obtainable.

$$\begin{aligned}
&\mathrm{Ai}(-z_1(x))\mathrm{Ai}(-z_2(x')) \\
&\approx -\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\left(\zeta_1+\zeta_2'\right)} - e^{i\left(\zeta_1-\zeta_2'\right)} - e^{-i\left(\zeta_1-\zeta_2'\right)} - ie^{-i\left(\zeta_1+\zeta_2'\right)}\right) \\
&\approx -\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}} \\
&\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} - e^{i\left(\mu(\gamma\omega)^{\frac{1}{2}}(x-x')-\sigma(\gamma\omega)^{\frac{1}{2}}\right)} - e^{-i\left(\mu(\gamma\omega)^{\frac{1}{2}}(x-x')-\sigma(\gamma\omega)^{\frac{1}{2}}\right)} - ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right) \qquad \text{(C.76)}
\end{aligned}$$

and the second is

$$\begin{aligned}
&\mathrm{Ai}(-z_1(x))\mathrm{Bi}(-z_2(x')) \\
&\approx \frac{1}{4i\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\left(\zeta_1+\zeta_2'\right)} + e^{i\left(\zeta_1-\zeta_2'\right)} - e^{-i\left(\zeta_1-\zeta_2'\right)} + ie^{-i\left(\zeta_1+\zeta_2'\right)}\right) \\
&\approx \frac{1}{4i\pi(\gamma\omega)^{\frac{1}{2}}} \\
&\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}} + e^{i\left(\mu(\gamma\omega)^{\frac{1}{2}}(x-x')-\sigma(\gamma\omega)^{\frac{1}{2}}\right)} - e^{-i\left(\mu(\gamma\omega)^{\frac{1}{2}}(x-x')-\sigma(\gamma\omega)^{\frac{1}{2}}\right)} + ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right) \qquad \text{(C.77)}
\end{aligned}$$

In both cases, the terms including $\frac{1}{(\gamma\omega)^{\frac{1}{2}}}e^{\pm i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}$ can be dropped in the following computations, because they do not give any contribution to χ. Carrying that in mind and also (C.25), (C.28) we have from (C.75)

$$
\begin{aligned}
Q(x,x') &\approx \int_q^\infty \frac{4}{\pi^3|\det \mathbf{M}(\omega)|^2}\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_1(x))\mathrm{Ai}(y_2(x'))\right.\\
&\left.+(A_1A_0' - A_1'A_0)\mathrm{Ai}(y_1(x))\mathrm{Bi}(y_2(x'))\right]\mathrm{d}\omega\\
&\approx \int_q^\infty \frac{4}{\pi^3\left(\frac{1}{\pi^2}\right)^2}\left[-\frac{1}{2\pi}\left(e^{-i\sigma(\gamma\omega)^{\frac12}} + e^{i\sigma(\gamma\omega)^{\frac12}}\right)\frac{1}{4\pi(\gamma\omega)^{\frac12}}\right.\\
&\left(-e^{i\left(\mu(\gamma\omega)^{\frac12}(x-x') - \sigma(\gamma\omega)^{\frac12}\right)} - e^{-i\left(\mu(\gamma\omega)^{\frac12}(x-x') - \sigma(\gamma\omega)^{\frac12}\right)}\right)\\
&-\frac{1}{2i\pi}\left(e^{i\sigma(\gamma\omega)^{\frac12}} - e^{-i\sigma(\gamma\omega)^{\frac12}}\right)\frac{1}{4i\pi(\gamma\omega)^{\frac12}}\\
&\left.\left(e^{i\left(\mu(\gamma\omega)^{\frac12}(x-x') - \sigma(\gamma\omega)^{\frac12}\right)} - e^{-i\left(\mu(\gamma\omega)^{\frac12}(x-x') - \sigma(\gamma\omega)^{\frac12}\right)}\right)\right]\mathrm{d}\omega\\
&= \frac{1}{2\pi}\int_q^\infty \frac{1}{(\gamma\omega)^{\frac12}}\left[e^{i\left(\mu(\gamma\omega)^{\frac12}(x-x') - 2\sigma(\gamma\omega)^{\frac12}\right)} + e^{-i\left(\mu(\gamma\omega)^{\frac12}(x-x')\right)} + e^{i\left(\mu(\gamma\omega)^{\frac12}(x-x')\right)}\right.\\
&+ e^{-i\left(\mu(\gamma\omega)^{\frac12}(x-x') - 2\sigma(\gamma\omega)^{\frac12}\right)} + e^{i\left(\mu(\gamma\omega)^{\frac12}(x-x')\right)} - e^{-i\left(\mu(\gamma\omega)^{\frac12}(x-x') - 2\sigma(\gamma\omega)^{\frac12}\right)}\\
&\left.-e^{i\left(\mu(\gamma\omega)^{\frac12}(x-x') - 2\sigma(\gamma\omega)^{\frac12}\right)} + e^{-i\left(\mu(\gamma\omega)^{\frac12}(x-x')\right)}\right]\mathrm{d}\omega\\
&= \frac{1}{\pi}\int_q^\infty \frac{1}{(\gamma\omega)^{\frac12}}\left[e^{i\mu(\gamma\omega)^{\frac12}(x-x')} + e^{-i\mu(\gamma\omega)^{\frac12}(x-x')}\right]\mathrm{d}\omega \qquad \text{(C.78)}
\end{aligned}
$$

which can be transformed as we did several times into

$$
\begin{aligned}
Q(x,x') &\approx \frac{1}{\pi}\int_q^\infty \frac{1}{(\gamma\omega)^{\frac12}}\left[e^{i\mu(\gamma\omega)^{\frac12}(x-x')} + e^{-i\mu(\gamma\omega)^{\frac12}(x-x')}\right]\mathrm{d}\omega\\
&= \frac{2}{\pi\mu\gamma}\left(\int_r^\infty e^{iu(x-x')}\mathrm{d}u + \int_{-\infty}^{-r} e^{iu(x-x')}\mathrm{d}u\right) = \frac{2}{\pi\mu\gamma}\int_{-\infty}^\infty e^{iu(x-x')}\mathrm{d}u\\
&= \frac{4}{\mu\gamma}\delta\left(x-x'\right) = \frac{1}{\alpha^2\varepsilon}\delta\left(x-x'\right) \qquad \text{(C.79)}
\end{aligned}
$$

with $r = \mu\sqrt{\gamma q}$.

Finally we have come to the last region $x, x' < -d$. Here we deal with the integral

$$
\begin{aligned}
Q(x,x') &= \int_q^\infty \frac{2}{\pi^2|\det \mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x))\frac{2}{\pi^2|\det \mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x'))\mathrm{d}\omega\\
&= \int_q^\infty \frac{4}{\pi^4|\det \mathbf{M}(\omega)|^2}\mathrm{Ai}(y_1(x))\mathrm{Ai}(y_1(x'))\mathrm{d}\omega \qquad \text{(C.80)}
\end{aligned}
$$

The asymptotic expression for the term $\mathrm{Ai}(y_1(x))\mathrm{Ai}(y_1(x'))$ is

$$\begin{aligned}
&\mathrm{Ai}(-z_1(x))\mathrm{Ai}(-z_1(x')) \\
&\approx \frac{1}{2i\sqrt{\pi}}(z_1(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta_1+\frac{\pi}{4}\right)}-e^{-i\left(\zeta_1+\frac{\pi}{4}\right)}\right)\frac{1}{2i\sqrt{\pi}}(z_1(x'))^{-\frac{1}{4}}\left(e^{i\left(\zeta_1'+\frac{\pi}{4}\right)}-e^{-i\left(\zeta_1'+\frac{\pi}{4}\right)}\right) \\
&= -\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\left(\zeta_1+\zeta_1'\right)}-e^{i\left(\zeta_1-\zeta_1'\right)}-e^{-i\left(\zeta_1-\zeta_1'\right)}-ie^{-i\left(\zeta_1+\zeta_1'\right)}\right) \\
&\approx -\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(ie^{i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}-e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-ie^{-i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}\right)
\end{aligned} \tag{C.81}$$

The terms that include $\frac{1}{(\gamma\omega)^{\frac{1}{2}}}e^{\pm i\frac{4}{3}(\gamma\omega)^{\frac{3}{2}}}$ can be removed because the give a finite contribution to χ, so (C.80) becomes

$$\begin{aligned}
Q(x,x') &= \int_q^\infty \frac{4}{\pi^4|\det\mathbf{M}(\omega)|^2}\mathrm{Ai}(y_1(x))\mathrm{Ai}(y_1(x'))\mathrm{d}\omega \\
&\approx \int_q^\infty -\frac{4}{\pi^4\left(\frac{1}{\pi^2}\right)^2}\frac{1}{4\pi(\gamma\omega)^{\frac{1}{2}}}\left(-e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}-e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}\right)\mathrm{d}\omega \\
&= \int_q^\infty \frac{1}{\pi(\gamma\omega)^{\frac{1}{2}}}\left(e^{i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}+e^{-i\mu(\gamma\omega)^{\frac{1}{2}}(x-x')}\right)\mathrm{d}\omega
\end{aligned} \tag{C.82}$$

and we end up with the same situation as in (C.74). The answer for this region therefore is

$$Q(x,x') = \frac{1}{\alpha^2\varepsilon}\delta(x-x') \tag{C.83}$$

Summing it all the regions up, we can see from (C.51), (C.53), (C.69), (C.74), (C.79) and (C.83) that the function $Q(x,x')$ is exactly the same in every region. When it comes to the switched regions, by which I mean for example in stead of $x < -d, d < x'$, the switched regions is $x' < -d, d < x$, then they give us also the same and equal term, namely $\frac{1}{\alpha^2\varepsilon}\delta(x-x')$. The cause of that is that the Dirac delta function is even. So overall we have

$$\begin{aligned}
\chi^{-2} &= \lim_{\epsilon\to 0}\int_{I_\epsilon} Q(x,x')\mathrm{d}x' = \lim_{\epsilon\to 0}\int_{I_\epsilon}\frac{1}{\alpha^2\varepsilon}\delta(x-x')\mathrm{d}x' = \frac{1}{\alpha^2\varepsilon} \\
\chi &= (\alpha^2\varepsilon)^{\frac{1}{2}} = 2^{-\frac{2}{3}}\varepsilon^{-\frac{1}{6}}
\end{aligned} \tag{C.84}$$

which we recognize it to be exactly the same as the normalization constant for the Dirac delta case (A.29).

Appendix D

This Appendix is about acquiring the asymptotic formulas for the scattering form of the resonant states $\psi_\omega(x)$ for the square well potential (3.1) along rays in the lower part of the complex frequency plane. The state have the form

$$\psi_\omega(x) = \chi \begin{cases} \frac{2}{\pi^2|\det \mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x)) & x < -d \\ \frac{2}{\pi|\det \mathbf{M}(\omega)|}\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(x))\right. & \\ \qquad\qquad \left.+(A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(x))\right] & -d < x < d \\ i\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(x)) - i\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(x)) & d < x \end{cases} \tag{D.1}$$

In this formula we have defined

$$\chi = 2^{-\frac{2}{3}}\varepsilon^{-\frac{1}{6}} \tag{D.2}$$

$$y_1(x) = -2\alpha(\varepsilon x + \omega) \tag{D.3}$$

$$y_2(x) = -2\alpha(\varepsilon x + V_0 + \omega) \tag{D.4}$$

$$A_0 = \mathrm{Ai}(y_1(-d)), A_1 = \mathrm{Ai}(y_2(-d)), B_1 = \mathrm{Bi}(y_2(-d))$$

with $\alpha = (2\varepsilon)^{-\frac{2}{3}}$ and the determinant is

$$\det \mathbf{M}(\omega) = (A_0A_1' - A_0'A_1)(B_2C_3' - B_2'C_3) - (A_0B_1' - A_0'B_1)(A_2C_3' - A_2'C_3) \tag{D.5}$$

$$\overline{\det \mathbf{M}}(\omega) = (A_0A_1' - A_0'A_1)(B_2D_3' - B_2'D_3) - (A_0B_1' - A_0'B_1)(A_2D_3' - A_2'D_3) \tag{D.6}$$

We represent the rays in the lower frequency half-plane as

$$\omega = Re^{i\theta}, \quad -\pi < \theta < 0 \tag{D.7}$$

where $R \to \infty$. There are two different asymptotic behaviours of Airy functions depending on the angle of the ray. The two sectors are $-\frac{2\pi}{3} < \theta < 0$ and $-\pi < \theta < -\frac{2\pi}{3}$, which are going to be treated independently.

Let us first take the sector

$$-\frac{2\pi}{3} < \theta < 0 \tag{D.8}$$

and define

$$z_{1,2}(x) = -y_{1,2}(x) \tag{D.9}$$

We will use $z(x)$ without an index for the general asymptotic expressions of Airy functions. In this sector the asymptotic behaviours of Airy functions according to [1] (10.4.60) and (10.4.64) are

$$\mathrm{Ai}(-z(x)) \approx \frac{1}{2i\sqrt{\pi}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} - e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right) \tag{D.10}$$

$$\mathrm{Bi}(-z(x)) \approx \frac{1}{2\sqrt{\pi}}(z(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} + e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right) \tag{D.11}$$

$$\mathrm{Ci}^{\pm}(-z(x)) \approx \pi^{-\frac{1}{2}}(z(x))^{-\frac{1}{4}}e^{\pm i\left(\zeta+\frac{\pi}{4}\right)} \tag{D.12}$$

with $\zeta = \frac{2}{3}(z(x))^{\frac{3}{2}}$. We get the asymptotic behaviour for their derivatives from [1] (10.4.62) and (10.4.67).

$$\mathrm{Ai}'(-z(x)) \approx -\frac{1}{2\sqrt{\pi}}(z(x))^{\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} + e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right) \tag{D.13}$$

$$\mathrm{Bi}'(-z(x)) \approx \frac{1}{2i\sqrt{\pi}}(z(x))^{\frac{1}{4}}\left(e^{i\left(\zeta+\frac{\pi}{4}\right)} - e^{-i\left(\zeta+\frac{\pi}{4}\right)}\right) \tag{D.14}$$

$$\begin{aligned}\mathrm{Ci}'^{-}(-z(x)) &= \mathrm{Bi}'(-z(x)) - i\mathrm{Ai}'(-z(x)) \\ &\approx \pi^{-\frac{1}{2}}(z(x))^{\frac{1}{4}}\sin\left(\zeta+\frac{\pi}{4}\right) + i\pi^{-\frac{1}{2}}(z(x))^{\frac{1}{4}}\cos\left(\zeta+\frac{\pi}{4}\right) \\ &= i\pi^{-\frac{1}{2}}(z(x))^{\frac{1}{4}}e^{-i\left(\zeta+\frac{\pi}{4}\right)}\end{aligned} \tag{D.15}$$

$$\mathrm{Ci}'^{+}(-z(x)) \approx -i\pi^{-\frac{1}{2}}(z(x))^{\frac{1}{4}}e^{i\left(\zeta+\frac{\pi}{4}\right)} \tag{D.16}$$

Define the following quantities

$$z_1(-d) = -2\alpha\varepsilon d + 2\alpha\omega = -\beta + \kappa R \qquad z_1(d) = 2\alpha\varepsilon d + 2\alpha\omega = \beta + \kappa R \tag{D.17}$$

$$\begin{aligned}z_2(-d) = 2\alpha(-\varepsilon d + V_0) + 2\alpha\omega = -\beta + \sigma + \kappa R \qquad z_2(d) &= 2\alpha(\varepsilon d + V_0) + 2\alpha\omega \\ &= \beta + \sigma + \kappa R\end{aligned} \tag{D.18}$$

where $\beta = 2\alpha\varepsilon d, \sigma = 2\alpha V_0$ and $\kappa = 2\alpha e^{i\theta}$ and consequently define

$$\begin{aligned}
\zeta_1^{\pm} &= \frac{2}{3}(z_1(\pm d))^{\frac{3}{2}} = \frac{2}{3}(\pm\beta + \kappa R)^{\frac{3}{2}} = \frac{2}{3}(\kappa R)^{\frac{3}{2}}\left(\pm\frac{\beta}{\kappa R} + 1\right)^{\frac{3}{2}} \\
&\approx \frac{2}{3}(\kappa R)^{\frac{3}{2}}\left(1 \pm \frac{3\beta}{2\kappa R}\right) = \underbrace{\frac{2}{3}\kappa^{\frac{3}{2}}}_{\vartheta} R^{\frac{3}{2}} \pm \beta \underbrace{\kappa^{\frac{1}{2}}}_{\varrho} R^{\frac{1}{2}} = \vartheta R^{\frac{3}{2}} \pm \beta\varrho R^{\frac{1}{2}}
\end{aligned} \tag{D.19}$$

$$\begin{aligned}
\zeta_2^{\pm} &= \frac{2}{3}(z_2(\pm d))^{\frac{3}{2}} = \frac{2}{3}(\pm\beta + \sigma + \kappa R)^{\frac{3}{2}} = \frac{2}{3}(\kappa R)^{\frac{3}{2}}\left(\frac{\pm\beta + \sigma}{\kappa R} + 1\right)^{\frac{3}{2}} \\
&\approx \frac{2}{3}(\kappa R)^{\frac{3}{2}}\left(1 + \frac{3(\pm\beta + \sigma)}{2\kappa R}\right) = \frac{2}{3}\kappa^{\frac{3}{2}} R^{\frac{3}{2}} \pm \beta\kappa^{\frac{1}{2}} R^{\frac{1}{2}} + \sigma\kappa^{\frac{1}{2}} R^{\frac{1}{2}} \\
&= \vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma \pm \beta)
\end{aligned} \tag{D.20}$$

where we defined $\vartheta = \frac{2}{3}\kappa^{\frac{3}{2}}$ and $\varrho = \kappa^{\frac{1}{2}}$ with

$$\vartheta_r = \frac{2}{3}(2\alpha)^{\frac{3}{2}}\cos\left(\frac{3}{2}\theta\right) \qquad \varrho_r = (2\alpha)^{\frac{1}{2}}\cos\left(\frac{1}{2}\theta\right) \tag{D.21}$$

$$\vartheta_i = \frac{2}{3}(2\alpha)^{\frac{3}{2}}\sin\left(\frac{3}{2}\theta\right) \qquad \varrho_i = (2\alpha)^{\frac{1}{2}}\sin\left(\frac{1}{2}\theta\right) \tag{D.22}$$

Observe, that in the current sector $-\frac{2\pi}{3} < \theta < 0$ we have $-\pi < \frac{3}{2}\theta < 0$ and $-\frac{\pi}{3} < \frac{1}{2}\theta < 0$ and therefore

$$\vartheta_i < 0, \quad \varrho_r > 0, \varrho_i < 0 \tag{D.23}$$

Let us start with the determinants. With a help from Appendix C we can see from (C.32) that we can straight use this expression

$$\det \mathbf{M}(\omega) \approx -\frac{1}{\pi^2} + \frac{\sigma}{4i\pi^2\gamma\omega} e^{i\left(\frac{4}{3}(\gamma\omega)^{\frac{3}{2}} + 2\beta(\gamma\omega)^{\frac{1}{2}}\right)} - \frac{\sigma}{4i\pi^2\gamma\omega} e^{i\left(\frac{4}{3}(\gamma\omega)^{\frac{3}{2}} - 2\beta(\gamma\omega)^{\frac{1}{2}}\right)} \tag{D.24}$$

One should remember that in Appendix C the ω was real, whereas now it is complex. We can however see the connection between the involved quantities from (C.21) and (C.22).

$$\gamma\omega \to \kappa R \tag{D.25}$$

$$\frac{2}{3}(\gamma\omega)^{\frac{3}{2}} \to \vartheta R^{\frac{3}{2}} \tag{D.26}$$

$$(\gamma\omega)^{\frac{1}{2}} \to \varrho R^{\frac{1}{2}} \tag{D.27}$$

Using these analogies, we can rewrite (D.24) as

$$\begin{aligned}\det \mathbf{M}(\omega) &\approx -\frac{1}{\pi^2} + \frac{\sigma}{4i\pi^2\kappa R} e^{i\left(2\vartheta R^{\frac{3}{2}} + 2\beta\varrho R^{\frac{1}{2}}\right)} - \frac{\sigma}{4i\pi^2\kappa R} e^{i\left(2\vartheta R^{\frac{3}{2}} - 2\beta\varrho R^{\frac{1}{2}}\right)} \\ &= -\frac{1}{\pi^2} + \frac{\sigma}{4i\pi^2\kappa R} e^{i2\vartheta R^{\frac{3}{2}}} \left(e^{i2\beta\varrho R^{\frac{1}{2}}} - e^{-i2\beta\varrho R^{\frac{1}{2}}}\right) \\ &\approx \frac{\sigma}{4i\pi^2\kappa R} e^{i2\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}\end{aligned} \tag{D.28}$$

where we used the fact that $\varrho_i < 0$ and $\vartheta_i < 0$.

The asymptotic expression for $\overline{\det \mathbf{M}}(\omega)$ according to (C.33) is

$$\begin{aligned}\overline{\det \mathbf{M}}(\omega) &\approx -\frac{1}{\pi^2} - \frac{\sigma}{4i\pi^2\gamma\omega} e^{-i\left(2\vartheta R^{\frac{3}{2}} + 2\beta\varrho R^{\frac{1}{2}}\right)} + \frac{\sigma}{4i\pi^2\gamma\omega} e^{-i\left(2\vartheta R^{\frac{3}{2}} - 2\beta\varrho R^{\frac{1}{2}}\right)} \\ &= -\frac{1}{\pi^2} - \frac{\sigma}{4i\pi^2\kappa R} e^{-i2\vartheta R^{\frac{3}{2}}} \left(e^{-i2\beta\varrho R^{\frac{1}{2}}} - e^{i2\beta\varrho R^{\frac{1}{2}}}\right) \\ &\approx -\frac{1}{\pi^2}\end{aligned} \tag{D.29}$$

where we again used $\vartheta_i < 0$.

The expression $|\det \mathbf{M}(\omega)|$ can be then written as

$$\begin{aligned}|\det \mathbf{M}(\omega)| &= \left(\det \mathbf{M}(\omega)\overline{\det \mathbf{M}}(\omega)\right)^{\frac{1}{2}} \approx \left(\frac{\sigma}{4i\pi^2\kappa R} e^{i2\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)} \left(-\frac{1}{\pi^2}\right)\right)^{\frac{1}{2}} \\ &= \frac{\sigma^{\frac{1}{2}} e^{i\frac{\pi}{4}}}{2\pi^2(\kappa R)^{\frac{1}{2}}} e^{i\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}\end{aligned} \tag{D.30}$$

We also have

$$\begin{aligned}\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} &\approx \left(\frac{-\frac{1}{\pi^2}}{\frac{\sigma}{4i\pi^2\kappa R} e^{i2\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}}\right)^{\frac{1}{2}} = \left(-\frac{1}{\pi^2}\frac{4i\pi^2\kappa R}{\sigma} e^{-i2\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}\right)^{\frac{1}{2}} \\ &= \frac{2(\kappa R)^{\frac{1}{2}}}{e^{i\frac{\pi}{4}}\sigma^{\frac{1}{2}}} e^{-i\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}\end{aligned} \tag{D.31}$$

$$\begin{aligned}\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}} &\approx \left(\frac{\frac{\sigma}{4i\pi^2\kappa R} e^{i2\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}}{-\frac{1}{\pi^2}}\right)^{\frac{1}{2}} = \left(-\pi^2\frac{\sigma}{4i\pi^2\kappa R} e^{i2\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}\right)^{\frac{1}{2}} \\ &= \frac{\sigma^{\frac{1}{2}} e^{i\frac{\pi}{4}}}{2(\kappa R)^{\frac{1}{2}}} e^{i\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}\end{aligned} \tag{D.32}$$

In the region $-d < x < d$ we are going to need the following terms. We can therefore use

(C.25) and (C.28).

$$
\begin{aligned}
&A_0B_1' - A_0'B_1 = \mathrm{Ai}(-z_1(-d))\mathrm{Bi}'(-z_2(-d)) - \mathrm{Ai}'(-z_1(-d))\mathrm{Bi}(-z_2(-d)) \\
&\approx \frac{1}{2\pi}\left(e^{i\left(\zeta_1^- - \zeta_2^-\right)} + e^{-i\left(\zeta_1^- - \zeta_2^-\right)}\right) - \frac{\sigma i}{8\pi\kappa R}\left(e^{i\left(\zeta_1^- + \zeta_2^-\right)} - e^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&\approx \frac{1}{2\pi}\left(e^{i\left(\vartheta R^{\frac{3}{2}} - \beta\varrho R^{\frac{1}{2}} - \left(\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)\right)} + e^{-i\left(\vartheta R^{\frac{3}{2}} - \beta\varrho R^{\frac{1}{2}} - \left(\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)\right)}\right) \\
&- \frac{\sigma i}{8\pi\kappa R}\left(e^{i\left(\vartheta R^{\frac{3}{2}} - \beta\varrho R^{\frac{1}{2}} + \vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)} - e^{-i\left(\vartheta R^{\frac{3}{2}} - \beta\varrho R^{\frac{1}{2}} + \vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)}\right) \\
&= \frac{1}{2\pi}\left(e^{-i\sigma\varrho R^{\frac{1}{2}}} + e^{i\sigma\varrho R^{\frac{1}{2}}}\right) - \frac{\sigma i}{8\pi\kappa R}\left(e^{i\left(2\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)} - e^{-i\left(2\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)}\right) \qquad \text{(D.33)}
\end{aligned}
$$

The other term in this region is

$$
\begin{aligned}
&A_0'A_1 - A_0A_1' = \mathrm{Ai}'(-z_1(-d))\mathrm{Ai}(-z_2(-d)) - \mathrm{Ai}(-z_1(-d))\mathrm{Ai}'(-z_2(-d)) \\
&\approx \frac{1}{2i\pi}\left(e^{i\left(\zeta_1^- - \zeta_2^-\right)} - e^{-i\left(\zeta_1^- - \zeta_2^-\right)}\right) + \frac{\sigma}{8\pi\kappa R}\left(e^{i\left(\zeta_1^- + \zeta_2^-\right)} + e^{-i\left(\zeta_1^- + \zeta_2^-\right)}\right) \\
&= \frac{1}{2i\pi}\left(e^{i\left(\vartheta R^{\frac{3}{2}} - \beta\varrho R^{\frac{1}{2}} - \left(\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)\right)} - e^{-i\left(\vartheta R^{\frac{3}{2}} - \beta\varrho R^{\frac{1}{2}} - \left(\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)\right)}\right) \\
&+ \frac{\sigma}{8\pi\kappa R}\left(e^{i\left(\vartheta R^{\frac{3}{2}} - \beta\varrho R^{\frac{1}{2}} + \vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)} + e^{-i\left(\vartheta R^{\frac{3}{2}} - \beta\varrho R^{\frac{1}{2}} + \vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)}\right) \\
&= \frac{1}{2i\pi}\left(e^{-i\sigma\varrho R^{\frac{1}{2}}} - e^{i\sigma\varrho R^{\frac{1}{2}}}\right) + \frac{\sigma}{8\pi\kappa R}\left(e^{i\left(2\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)} + e^{-i\left(2\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)}\right) \qquad \text{(D.34)}
\end{aligned}
$$

We take (D.30), (D.33) and (D.34) and do separately the following terms in the region $-d < x < d$.

$$
\begin{aligned}
&\frac{2}{\pi|\det\mathbf{M}(\omega)|}(B_1'A_0 - B_1A_0') \approx \frac{2}{\pi\frac{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{2\pi^2(\kappa R)^{\frac{1}{2}}}e^{i\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)}} \\
&\left[\frac{1}{2\pi}\left(e^{-i\sigma\varrho R^{\frac{1}{2}}} + e^{i\sigma\varrho R^{\frac{1}{2}}}\right) - \frac{\sigma i}{8\pi\kappa R}\left(e^{i\left(2\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)} - e^{-i\left(2\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)}\right)\right] \\
&= \frac{4\pi(\kappa R)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}e^{-i\left(\vartheta R^{\frac{3}{2}} + \beta\varrho R^{\frac{1}{2}}\right)} \\
&\left[\frac{1}{2\pi}\left(e^{-i\sigma\varrho R^{\frac{1}{2}}} + e^{i\sigma\varrho R^{\frac{1}{2}}}\right) - \frac{\sigma i}{8\pi\kappa R}\left(e^{i\left(2\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)} - e^{-i\left(2\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)}\right)\right] \\
&= \frac{2(\kappa R)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\left(e^{-i\left(\vartheta R^{\frac{3}{2}} + (\beta+\sigma)\varrho R^{\frac{1}{2}}\right)} + e^{-i\left(\vartheta R^{\frac{3}{2}} + (\beta-\sigma)\varrho R^{\frac{1}{2}}\right)}\right) \\
&- \frac{\sigma^{\frac{1}{2}}i}{2(\kappa R)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\left(e^{i\left(\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-3\beta)\right)} - e^{-i\left(3\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)}\right) \qquad \text{(D.35)}
\end{aligned}
$$

In a similar way using we get

$$
\frac{2}{\pi|\det\mathbf{M}(\omega)|}(A_1A_0'-A_1'A_0)\approx\frac{2}{\pi\frac{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{2\pi^2(\kappa R)^{\frac{1}{2}}}e^{i\left(\vartheta R^{\frac{3}{2}}+\beta\varrho R^{\frac{1}{2}}\right)}}
$$

$$
\frac{1}{2i\pi}\left(e^{-i\sigma\varrho R^{\frac{1}{2}}}-e^{i\sigma\varrho R^{\frac{1}{2}}}\right)+\frac{\sigma}{8\pi\kappa R}\left(e^{i\left(2\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)}+e^{-i\left(2\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)}\right)
$$

$$
=\frac{4\pi(\kappa R)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}e^{-i\left(\vartheta R^{\frac{3}{2}}+\beta\varrho R^{\frac{1}{2}}\right)}
$$

$$
\frac{1}{2i\pi}\left(e^{-i\sigma\varrho R^{\frac{1}{2}}}-e^{i\sigma\varrho R^{\frac{1}{2}}}\right)+\frac{\sigma}{8\pi\kappa R}\left(e^{i\left(2\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)}+e^{-i\left(2\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-2\beta)\right)}\right)
$$

$$
=\frac{2(\kappa R)^{\frac{1}{2}}}{i\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\left(e^{-i\left(\vartheta R^{\frac{3}{2}}+(\beta+\sigma)\varrho R^{\frac{1}{2}}\right)}-e^{-i\left(\vartheta R^{\frac{3}{2}}+(\beta-\sigma)\varrho R^{\frac{1}{2}}\right)}\right)
$$

$$
+\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\left(e^{i\left(\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-3\beta)\right)}+e^{-i\left(3\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-\beta)\right)}\right) \tag{D.36}
$$

For the variable transformations $y_1(x), y_2(x)$ from (2.24) and (3.36) using (D.7) we introduce the following notations for the quantities $\zeta_{1,2}=\frac{2}{3}(-y_{1,2}(x))^{\frac{3}{2}}=\frac{2}{3}(z_{1,2}(x))^{\frac{3}{2}}$ used in asymptotic formulas.

$$
\begin{aligned}
\zeta_1=\frac{2}{3}(z_1(x))^{\frac{3}{2}}&=\frac{2}{3}\left[2\alpha\left(\varepsilon x+Re^{i\theta}\right)\right]^{\frac{3}{2}}=\frac{2}{3}\left[2\alpha e^{i\theta}R\left(\varepsilon xe^{-i\theta}R^{-1}+1\right)\right]^{\frac{3}{2}}\\
&=\frac{2}{3}\left[\kappa R\left(1+\varepsilon xe^{-i\theta}R^{-1}\right)\right]^{\frac{3}{2}}=\vartheta R^{\frac{3}{2}}\left(1+\varepsilon xe^{-i\theta}R^{-1}\right)^{\frac{3}{2}}\\
&\approx\vartheta R^{\frac{3}{2}}\left(1+\frac{3}{2}\varepsilon xe^{-i\theta}R^{-1}\right)=\vartheta R^{\frac{3}{2}}+\underbrace{\frac{3}{2}\vartheta\varepsilon ie^{-i\theta}}_{\varpi}(-i)xR^{\frac{1}{2}}\\
&=\vartheta R^{\frac{3}{2}}-i\varpi xR^{\frac{1}{2}}
\end{aligned} \tag{D.37}
$$

where we defined a new quantity ϖ which can be simplified further.

$$
\begin{aligned}
\varpi&=\frac{3}{2}\vartheta\varepsilon ie^{-i\theta}=\frac{3}{2}\varepsilon i\frac{2}{3}(2\alpha)^{\frac{3}{2}}\left[\cos\left(\frac{3}{2}\theta\right)+i\sin\left(\frac{3}{2}\theta\right)\right]\left[\cos(\theta)-i\sin(\theta)\right]\\
&=\varepsilon i(2\alpha)^{\frac{3}{2}}\left[\cos\left(\frac{3}{2}\theta\right)\cos(\theta)+\sin\left(\frac{3}{2}\theta\right)\sin(\theta)\right.\\
&\left.+i\left(\sin\left(\frac{3}{2}\theta\right)\cos(\theta)-\cos\left(\frac{3}{2}\theta\right)\sin(\theta)\right)\right]\\
&=\varepsilon i(2\alpha)^{\frac{3}{2}}\left[\cos\left(\frac{1}{2}\theta\right)+i\sin\left(\frac{1}{2}\theta\right)\right]\\
&=\varepsilon(2\alpha)^{\frac{3}{2}}\left[-\sin\left(\frac{1}{2}\theta\right)+i\cos\left(\frac{1}{2}\theta\right)\right]=\varpi_r+i\varpi_r
\end{aligned} \tag{D.38}
$$

where

$$\varpi_r = -\varepsilon(2\alpha)^{\frac{3}{2}} \sin\left(\frac{1}{2}\theta\right) \tag{D.39}$$

$$\varpi_i = \varepsilon(2\alpha)^{\frac{3}{2}} \cos\left(\frac{1}{2}\theta\right) \tag{D.40}$$

and in a almost identical way we get for ζ_2

$$\begin{aligned}
\zeta_2 = \frac{2}{3}(z_2(x))^{\frac{3}{2}} &= \frac{2}{3}\left[2\alpha\left(\varepsilon x + V_0 + Re^{i\theta}\right)\right]^{\frac{3}{2}} = \frac{2}{3}\left[2\alpha e^{i\theta}R\left(\frac{\varepsilon x + V_0}{e^{i\theta}R} + 1\right)\right]^{\frac{3}{2}} \\
&= \frac{2}{3}\left[\kappa R\left(1 + \frac{\varepsilon x + V_0}{e^{i\theta}R}\right)\right]^{\frac{3}{2}} = \vartheta R^{\frac{3}{2}}\left(1 + \frac{\varepsilon x + V_0}{e^{i\theta}R}\right)^{\frac{3}{2}} \\
&\approx \vartheta R^{\frac{3}{2}}\left(1 + \frac{3(\varepsilon x + V_0)}{2e^{i\theta}R}\right) = \vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}} + \frac{3}{2}\vartheta\varepsilon i e^{-i\theta}(-i)\frac{V_0}{\varepsilon}R^{\frac{1}{2}} \\
&= \vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}} - i\varpi\frac{V_0}{\varepsilon}R^{\frac{1}{2}} = \vartheta R^{\frac{3}{2}} - i\varpi R^{\frac{1}{2}}\left(x + \frac{V_0}{\varepsilon}\right)
\end{aligned} \tag{D.41}$$

Note, that in the current sector which is $-\frac{2\pi}{3} < \theta < 0$ we have $-\frac{\pi}{3} < \frac{1}{2}\theta < 0$ and therefore $\varpi_r, \varpi_i > 0$.

Using (D.37) and (D.41) we have for Airy function in the current sector the asymptotic relations

$$\begin{aligned}
&\mathrm{Ai}(-z_1(x)) \approx \frac{1}{2i\sqrt{\pi}}(z_1(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta_1+\frac{\pi}{4}\right)} - e^{-i\left(\zeta_1+\frac{\pi}{4}\right)}\right) \\
&\approx \frac{1}{2i\sqrt{\pi}}\left[2\alpha\left(\varepsilon x + Re^{i\theta}\right)\right]^{-\frac{1}{4}}\left(e^{i\left(\vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}} + \frac{\pi}{4}\right)} - e^{-i\left(\vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}} + \frac{\pi}{4}\right)}\right) \\
&\approx \frac{1}{2i\sqrt{\pi}}\left(2\alpha Re^{i\theta}\right)^{-\frac{1}{4}}\left(\frac{\varepsilon x}{Re^{i\theta}} + 1\right)^{-\frac{1}{4}}\left(e^{i\left(\vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}} + \frac{\pi}{4}\right)} - e^{-i\left(\vartheta R^{\frac{3}{2}} - i\varpi x R^{\frac{1}{2}} + \frac{\pi}{4}\right)}\right) \\
&\approx \frac{1}{2i\sqrt{\pi}}(\kappa R)^{-\frac{1}{4}}\left(e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi x R^{\frac{1}{2}}}e^{i\frac{\pi}{4}} - e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi x R^{\frac{1}{2}}}e^{-i\frac{\pi}{4}}\right)
\end{aligned} \tag{D.42}$$

and

$$\mathrm{Ci}^+(-z_1(x)) \approx \pi^{-\frac{1}{2}}(z_1(x))^{-\frac{1}{4}}e^{i\left(\zeta_1+\frac{\pi}{4}\right)} \approx \pi^{-\frac{1}{2}}(\kappa R)^{-\frac{1}{4}}e^{i\frac{\pi}{4}}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi x R^{\frac{1}{2}}} \tag{D.43}$$

$$\mathrm{Ci}^-(-z_1(x)) \approx \pi^{-\frac{1}{2}}(z_1(x))^{-\frac{1}{4}}e^{-i\left(\zeta_1+\frac{\pi}{4}\right)} \approx \pi^{-\frac{1}{2}}(\kappa R)^{-\frac{1}{4}}e^{-i\frac{\pi}{4}}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi x R^{\frac{1}{2}}} \tag{D.44}$$

We also have

$$
\begin{aligned}
&\mathrm{Ai}(-z_2(x)) \approx \frac{1}{2i\sqrt{\pi}}(z_2(x))^{-\frac{1}{4}}\left(e^{i\left(\zeta_2+\frac{\pi}{4}\right)}-e^{-i\left(\zeta_2+\frac{\pi}{4}\right)}\right)\\
&\approx \frac{1}{2i\sqrt{\pi}}\left[2\alpha\left(\varepsilon x+V_0+Re^{i\theta}\right)\right]^{-\frac{1}{4}}\\
&\left(e^{i\left(\vartheta R^{\frac{3}{2}}-i\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)+\frac{\pi}{4}\right)}-e^{-i\left(\vartheta R^{\frac{3}{2}}-i\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)+\frac{\pi}{4}\right)}\right)\\
&\approx \frac{1}{2i\sqrt{\pi}}\left(2\alpha Re^{i\theta}\right)^{-\frac{1}{4}}\left(\frac{\varepsilon x+V_0}{Re^{i\theta}}+1\right)^{-\frac{1}{4}}\\
&\left(e^{i\left(\vartheta R^{\frac{3}{2}}-i\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)+\frac{\pi}{4}\right)}-e^{-i\left(\vartheta R^{\frac{3}{2}}-i\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)+\frac{\pi}{4}\right)}\right)\\
&\approx \frac{1}{2i\sqrt{\pi}}(\kappa R)^{-\frac{1}{4}}\left(e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}-e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi R^{\frac{1}{2}}x}e^{-\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{-i\frac{\pi}{4}}\right)
\end{aligned}
\tag{D.45}
$$

and similarly from (D.11)

$$
\begin{aligned}
&\mathrm{Bi}(-z_2(x))\\
&\approx \frac{1}{2\sqrt{\pi}}(\kappa R)^{-\frac{1}{4}}\left(e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}+e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi R^{\frac{1}{2}}x}e^{-\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{-i\frac{\pi}{4}}\right)
\end{aligned}
\tag{D.46}
$$

At this point we have everything to write the asymptotic expression of the resonant states (D.1). For $x<-d$ we get using (D.30) and (D.42)

$$
\begin{aligned}
\psi_\omega(x) &= \chi\frac{2}{\pi^2|\det\mathbf{M}(\omega)|}\mathrm{Ai}(-z_1(x))\\
&\approx \chi\frac{2}{\pi^2\frac{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{2\pi^2(\kappa R)^{\frac{1}{2}}}e^{i\left(\vartheta R^{\frac{3}{2}}+\beta\varrho R^{\frac{1}{2}}\right)}}\frac{1}{2i\sqrt{\pi}}(\kappa R)^{-\frac{1}{4}}\left(e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi xR^{\frac{1}{2}}}e^{i\frac{\pi}{4}}-e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi xR^{\frac{1}{2}}}e^{-i\frac{\pi}{4}}\right)\\
&= \chi\frac{4(\kappa R)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}e^{-i\left(\vartheta R^{\frac{3}{2}}+\beta\varrho R^{\frac{1}{2}}\right)}\frac{1}{2i\sqrt{\pi}}(\kappa R)^{-\frac{1}{4}}\left(e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi xR^{\frac{1}{2}}}e^{i\frac{\pi}{4}}-e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi xR^{\frac{1}{2}}}e^{-i\frac{\pi}{4}}\right)\\
&= \chi\frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\left(e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi xR^{\frac{1}{2}}}+ie^{-i2\vartheta R^{\frac{3}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi xR^{\frac{1}{2}}}\right)\\
&\approx \chi\frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi xR^{\frac{1}{2}}}
\end{aligned}
\tag{D.47}
$$

where in the last line we used the fact that $\vartheta_i<0$ from (D.23). For $-d<x<d$ using (D.35),

(D.36) and (D.45), (D.46) we have

$$
\begin{aligned}
&\psi_\omega(x) = \chi \frac{2}{\pi|\det \mathbf{M}(\omega)|}\left[(B_1' A_0 - B_1 A_0')\mathrm{Ai}(-z_2(x)) + (A_1 A_0' - A_1' A_0)\mathrm{Bi}(-z_2(x))\right] \\
&\approx \chi \left[\frac{2(\kappa R)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}} e^{i\frac{\pi}{4}}} \left(e^{-i\left(\vartheta R^{\frac{3}{2}} + (\beta+\sigma)\varrho R^{\frac{1}{2}}\right)} + e^{-i\left(\vartheta R^{\frac{3}{2}} + (\beta-\sigma)\varrho R^{\frac{1}{2}}\right)} \right) \right. \\
&\left. - \frac{\sigma^{\frac{1}{2}} i}{2(\kappa R)^{\frac{1}{2}} e^{i\frac{\pi}{4}}} \left(e^{i\left(\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma - 3\beta)\right)} - e^{-i\left(3\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)} \right) \right] \\
&\frac{1}{2i\sqrt{\pi}} (\kappa R)^{-\frac{1}{4}} \left(e^{i\vartheta R^{\frac{3}{2}}} e^{\varpi R^{\frac{1}{2}} x} e^{\varpi R^{\frac{1}{2}} \frac{V_0}{\varepsilon}} e^{i\frac{\pi}{4}} - e^{-i\vartheta R^{\frac{3}{2}}} e^{-\varpi R^{\frac{1}{2}} x} e^{-\varpi R^{\frac{1}{2}} \frac{V_0}{\varepsilon}} e^{-i\frac{\pi}{4}} \right) \\
&+ \chi \left[\frac{2(\kappa R)^{\frac{1}{2}}}{i\sigma^{\frac{1}{2}} e^{i\frac{\pi}{4}}} \left(e^{-i\left(\vartheta R^{\frac{3}{2}} + (\beta+\sigma)\varrho R^{\frac{1}{2}}\right)} - e^{-i\left(\vartheta R^{\frac{3}{2}} + (\beta-\sigma)\varrho R^{\frac{1}{2}}\right)} \right) \right. \\
&\left. + \frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{1}{2}} e^{i\frac{\pi}{4}}} \left(e^{i\left(\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma - 3\beta)\right)} + e^{-i\left(3\vartheta R^{\frac{3}{2}} + \varrho R^{\frac{1}{2}}(\sigma-\beta)\right)} \right) \right] \\
&\frac{1}{2\sqrt{\pi}} (\kappa R)^{-\frac{1}{4}} \left(e^{i\vartheta R^{\frac{3}{2}}} e^{\varpi R^{\frac{1}{2}} x} e^{\varpi R^{\frac{1}{2}} \frac{V_0}{\varepsilon}} e^{i\frac{\pi}{4}} + e^{-i\vartheta R^{\frac{3}{2}}} e^{-\varpi R^{\frac{1}{2}} x} e^{-\varpi R^{\frac{1}{2}} \frac{V_0}{\varepsilon}} e^{-i\frac{\pi}{4}} \right) \qquad \text{(D.48)}
\end{aligned}
$$

We can simplify this expression by dropping those exponential terms, that contain $-i\vartheta R^{\frac{3}{2}}$ in their exponents after multiplying all out. Since $\vartheta_i < 0$, these exponentials will decay very fast.

Bearing this in mind we have from (D.48)

$$
\begin{aligned}
\psi_\omega(x) &\approx \chi\frac{2(\kappa R)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2i\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{-i\left(\vartheta R^{\frac{3}{2}}+(\beta+\sigma)\varrho R^{\frac{1}{2}}\right)}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}\\
&+\chi\frac{2(\kappa R)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2i\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{-i\left(\vartheta R^{\frac{3}{2}}+(\beta-\sigma)\varrho R^{\frac{1}{2}}\right)}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}\\
&-\chi\frac{\sigma^{\frac{1}{2}}i}{2(\kappa R)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2i\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{i\left(\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-3\beta)\right)}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}\\
&+\chi\frac{\sigma^{\frac{1}{2}}i}{2(\kappa R)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2i\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{i\left(\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-3\beta)\right)}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi R^{\frac{1}{2}}x}e^{-\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{-i\frac{\pi}{4}}\\
&+\chi\frac{2(\kappa R)^{\frac{1}{2}}}{i\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{-i\left(\vartheta R^{\frac{3}{2}}+(\beta+\sigma)\varrho R^{\frac{1}{2}}\right)}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}\\
&-\chi\frac{2(\kappa R)^{\frac{1}{2}}}{i\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{-i\left(\vartheta R^{\frac{3}{2}}+(\beta-\sigma)\varrho R^{\frac{1}{2}}\right)}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}\\
&+\chi\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{i\left(\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-3\beta)\right)}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}\\
&+\chi\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{i\left(\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-3\beta)\right)}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi R^{\frac{1}{2}}x}e^{-\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{-i\frac{\pi}{4}}\\
&=2\chi\frac{2(\kappa R)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2i\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{-i\left(\vartheta R^{\frac{3}{2}}+(\beta+\sigma)\varrho R^{\frac{1}{2}}\right)}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi R^{\frac{1}{2}}x}e^{\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{i\frac{\pi}{4}}\\
&+2\chi\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{1}{2}}e^{i\frac{\pi}{4}}}\frac{1}{2i\sqrt{\pi}}\,(\kappa R)^{-\frac{1}{4}}\,e^{i\left(\vartheta R^{\frac{3}{2}}+\varrho R^{\frac{1}{2}}(\sigma-3\beta)\right)}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi R^{\frac{1}{2}}x}e^{-\varpi R^{\frac{1}{2}}\frac{V_0}{\varepsilon}}e^{-i\frac{\pi}{4}}\\
&=\chi\frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i(\beta+\sigma)\varrho R^{\frac{1}{2}}}e^{\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}-\chi\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{3}{4}}\pi^{\frac{1}{2}}}e^{i\varrho R^{\frac{1}{2}}(\sigma-3\beta)}e^{-\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}
\end{aligned}
\tag{D.49}
$$

For the case $x > d$ we get the asymptotic expression using (D.31), (D.32) and (D.43), (D.44).

$$
\begin{aligned}
\psi_\omega(x) &= i\chi\left(\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^{+}(-z_1(x))-i\chi\left(\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^{-}(-z_1(x))\\
&\approx i\chi\frac{2(\kappa R)^{\frac{1}{2}}}{e^{i\frac{\pi}{4}}\sigma^{\frac{1}{2}}}e^{-i\left(\vartheta R^{\frac{3}{2}}+\beta\varrho R^{\frac{1}{2}}\right)}\pi^{-\frac{1}{2}}\,(\kappa R)^{-\frac{1}{4}}\,e^{i\frac{\pi}{4}}e^{i\vartheta R^{\frac{3}{2}}}e^{\varpi xR^{\frac{1}{2}}}\\
&-i\chi\frac{\sigma^{\frac{1}{2}}e^{i\frac{\pi}{4}}}{2(\kappa R)^{\frac{1}{2}}}e^{i\left(\vartheta R^{\frac{3}{2}}+\beta\varrho R^{\frac{1}{2}}\right)}\pi^{-\frac{1}{2}}\,(\kappa R)^{-\frac{1}{4}}\,e^{-i\frac{\pi}{4}}e^{-i\vartheta R^{\frac{3}{2}}}e^{-\varpi xR^{\frac{1}{2}}}\\
&=i\chi\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}}e^{-i\beta\varrho R^{\frac{1}{2}}}e^{\varpi xR^{\frac{1}{2}}}-i\chi\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}}e^{i\beta\varrho R^{\frac{1}{2}}}e^{-\varpi xR^{\frac{1}{2}}}
\end{aligned}
\tag{D.50}
$$

Summing it up, the asymptotic behaviour of the resonant states (D.1) along rays $\omega = Re^{i\theta}$

in the sector $-\frac{2\pi}{3} < \theta < 0$ is

$$\psi_\omega(x) = \chi \begin{cases} \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x R^{\frac{1}{2}}} & x < -d \\ \frac{2(\kappa R)^{\frac{1}{4}}}{i\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i(\beta+\sigma)\varrho R^{\frac{1}{2}}} e^{\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} & -d < x < d \\ -\frac{\sigma^{\frac{1}{2}}}{2(\kappa R)^{\frac{3}{4}}\pi^{\frac{1}{2}}} e^{i\varrho R^{\frac{1}{2}}(\sigma-3\beta)} e^{-\varpi R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} & \\ i\frac{2(\kappa R)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}}} e^{-i\beta\varrho R^{\frac{1}{2}}} e^{\varpi x R^{\frac{1}{2}}} - i\frac{\sigma^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{4}}} e^{i\beta\varrho R^{\frac{1}{2}}} e^{-\varpi x R^{\frac{1}{2}}} & d < x \end{cases} \tag{D.51}$$

Let us continue with the second sector $-\pi < \theta < -\frac{2\pi}{3}$. Here we are not going to use $z_{1,2}(x)$ as the argument, but $y_{1,2}(x)$. For the general expressions of Airy functions we use $y(x)$ without index. In this region the asymptotic expression for Airy functions according to [1] (10.4.59) and (10.4.63) are

$$\mathrm{Ai}(y(x)) \approx \frac{1}{2}\pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}}e^{-\zeta} \tag{D.52}$$

$$\mathrm{Bi}(y(x)) \approx \pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}}e^{\zeta} \tag{D.53}$$

Then the Airy functions $\mathrm{Ci}^{\pm}(y(x))$ are

$$\mathrm{Ci}^{\pm}(y(x)) \approx \pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}}e^{\zeta} \pm i\frac{1}{2}\pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}}e^{-\zeta} = \pi^{-\frac{1}{2}}(y(x))^{-\frac{1}{4}}\left(e^{\zeta} \pm \frac{i}{2}e^{-\zeta}\right) \tag{D.54}$$

where $\zeta = \frac{2}{3}(y(x))^{\frac{3}{2}}$. The asymptotic expression of the derivatives of Airy functions for this sector we can find in [1] (10.4.61) and (10.4.66)

$$\mathrm{Ai}'(y(x)) \approx -\frac{1}{2}\pi^{-\frac{1}{2}}(y(x))^{\frac{1}{4}}e^{-\zeta} \tag{D.55}$$

$$\mathrm{Bi}'(y(x)) \approx \pi^{-\frac{1}{2}}(y(x))^{\frac{1}{4}}e^{\zeta} \tag{D.56}$$

and hence

$$\mathrm{Ci}'^{\pm}(y(x)) \approx \pi^{-\frac{1}{2}}(y(x))^{\frac{1}{4}}e^{\zeta} \mp i\frac{1}{2}\pi^{-\frac{1}{2}}(y(x))^{\frac{1}{4}}e^{-\zeta} = \pi^{-\frac{1}{2}}(y(x))^{\frac{1}{4}}\left(e^{\zeta} \mp \frac{i}{2}e^{-\zeta}\right) \tag{D.57}$$

Our argument to these expressions is $y_{1,2}(\pm d)$, so it needs to be written differently as in the previous sector, where the argument was $-y_{1,2}(\pm d)$.

$$
\begin{aligned}
y_1(-d) &= 2\alpha\varepsilon d - 2\alpha Re^{i\theta} & y_1(d) &= -2\alpha\varepsilon d - 2\alpha Re^{i\theta} \\
&= \beta + \tilde{\kappa}R & &= -\beta + \tilde{\kappa}R \qquad \text{(D.58)} \\
y_2(-d) &= 2\alpha(\varepsilon d - V_0) - 2\alpha Re^{i\theta} & y_2(d) &= -2\alpha(\varepsilon d + V_0) - 2\alpha Re^{i\theta} \\
&= \beta - \sigma + \tilde{\kappa}R & &= -\beta - \sigma + \tilde{\kappa}R \qquad \text{(D.59)}
\end{aligned}
$$

where $\beta = 2\alpha\varepsilon d, \sigma = 2\alpha V_0$ and $\tilde{\kappa} = 2\alpha e^{i(\theta+\pi)}$ and consequently define

$$
\begin{aligned}
\tilde{\zeta}_1^{\pm} &= \frac{2}{3}(y_1(\pm d))^{\frac{3}{2}} = \frac{2}{3}(\mp\beta + \tilde{\kappa}R)^{\frac{3}{2}} = \frac{2}{3}(\tilde{\kappa}R)^{\frac{3}{2}}\left(\mp\frac{\beta}{\tilde{\kappa}R} + 1\right)^{\frac{3}{2}} \\
&\approx \frac{2}{3}(\tilde{\kappa}R)^{\frac{3}{2}}\left(1 \mp \frac{3\beta}{2\tilde{\kappa}R}\right) = \underbrace{\frac{2}{3}\tilde{\kappa}^{\frac{3}{2}}}_{\tilde{\vartheta}} R^{\frac{3}{2}} \mp \beta \underbrace{\tilde{\kappa}^{\frac{1}{2}}}_{\tilde{\varrho}} R^{\frac{1}{2}} = \tilde{\vartheta}R^{\frac{3}{2}} \mp \beta\tilde{\varrho}R^{\frac{1}{2}} \qquad \text{(D.60)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\zeta}_2^{\pm} &= \frac{2}{3}(y_2(\pm d))^{\frac{3}{2}} = \frac{2}{3}(\mp\beta - \sigma + \tilde{\kappa}R)^{\frac{3}{2}} = \frac{2}{3}(\tilde{\kappa}R)^{\frac{3}{2}}\left(\frac{\mp\beta - \sigma}{\tilde{\kappa}R} + 1\right)^{\frac{3}{2}} \\
&\approx \frac{2}{3}(\tilde{\kappa}R)^{\frac{3}{2}}\left(1 + \frac{3(\mp\beta - \sigma)}{2\tilde{\kappa}R}\right) = \frac{2}{3}\tilde{\kappa}^{\frac{3}{2}}R^{\frac{3}{2}} \mp \beta\tilde{\kappa}^{\frac{1}{2}}R^{\frac{1}{2}} - \sigma\tilde{\kappa}^{\frac{1}{2}}R^{\frac{1}{2}} \\
&= \tilde{\vartheta}R^{\frac{3}{2}} - \tilde{\varrho}R^{\frac{1}{2}}(\sigma \pm \beta) \qquad \text{(D.61)}
\end{aligned}
$$

where we defined $\tilde{\vartheta} = \frac{2}{3}\tilde{\kappa}^{\frac{3}{2}}$ and $\tilde{\varrho} = \tilde{\kappa}^{\frac{1}{2}}$ with

$$
\tilde{\vartheta}_r = \frac{2}{3}(2\alpha)^{\frac{3}{2}}\cos\left(\frac{3}{2}(\theta+\pi)\right) \qquad \tilde{\varrho}_r = (2\alpha)^{\frac{1}{2}}\cos\left(\frac{1}{2}(\theta+\pi)\right) \qquad \text{(D.62)}
$$

$$
\tilde{\vartheta}_i = \frac{2}{3}(2\alpha)^{\frac{3}{2}}\sin\left(\frac{3}{2}(\theta+\pi)\right) \qquad \tilde{\varrho}_i = (2\alpha)^{\frac{1}{2}}\sin\left(\frac{1}{2}(\theta+\pi)\right) \qquad \text{(D.63)}
$$

In the current sector $-\pi < \theta < -\frac{2\pi}{3}$ we have $0 < \frac{3}{2}(\theta+\pi) < \frac{\pi}{2}$ and $0 < \frac{1}{2}(\theta+\pi) < \frac{\pi}{6}$ therefore

$$
\tilde{\vartheta}_r, \tilde{\vartheta}_i > 0, \quad \tilde{\varrho}_r, \tilde{\varrho}_i > 0 \qquad \text{(D.64)}
$$

In the next computations we use the same approach for simplification of the following.

$$\begin{aligned} y_1(\pm d)^{\frac{1}{4}} y_2(\pm d)^{-\frac{1}{4}} &= (\mp\beta + \tilde{\kappa}R)^{\frac{1}{4}} (\mp\beta - \sigma + \tilde{\kappa}R)^{-\frac{1}{4}} \\ &= (\tilde{\kappa}R)^{\frac{1}{4}} \left(1 \mp \frac{\beta}{\tilde{\kappa}R}\right)^{\frac{1}{4}} (\tilde{\kappa}R)^{-\frac{1}{4}} \left(1 + \frac{\mp\beta - \sigma}{\tilde{\kappa}R}\right)^{-\frac{1}{4}} \\ &= \left(1 \mp \frac{\beta}{4\tilde{\kappa}R}\right)\left(1 - \frac{\mp\beta - \sigma}{4\tilde{\kappa}R}\right) \\ &= 1 - \frac{\mp\beta - \sigma}{4\tilde{\kappa}R} \mp \frac{\beta}{4\tilde{\kappa}R} \pm \frac{\beta(\mp\beta - \sigma)}{16(\tilde{\kappa}R)^2} \approx 1 + \frac{\sigma}{4\tilde{\kappa}R} \end{aligned} \tag{D.65}$$

and we also have

$$\begin{aligned} y_1(\pm d)^{-\frac{1}{4}} y_2(\pm d)^{\frac{1}{4}} &= (\mp\beta + \tilde{\kappa}R)^{-\frac{1}{4}} (\mp\beta - \sigma + \tilde{\kappa}R)^{\frac{1}{4}} \\ &= (\tilde{\kappa}R)^{-\frac{1}{4}} \left(1 \mp \frac{\beta}{\tilde{\kappa}R}\right)^{-\frac{1}{4}} (\tilde{\kappa}R)^{\frac{1}{4}} \left(1 + \frac{\mp\beta - \sigma}{\tilde{\kappa}R}\right)^{\frac{1}{4}} \\ &= \left(1 \pm \frac{\beta}{4\tilde{\kappa}R}\right)\left(1 + \frac{\mp\beta - \sigma}{4\tilde{\kappa}R}\right) \\ &= 1 + \frac{\mp\beta - \sigma}{4\tilde{\kappa}R} \pm \frac{\beta}{4\tilde{\kappa}R} \pm \frac{\beta(\mp\beta - \sigma)}{16(\tilde{\kappa}R)^2} \approx 1 - \frac{\sigma}{4\tilde{\kappa}R} \end{aligned} \tag{D.66}$$

We are now ready to start writing the asymptotic expressions for the resonant states (D.1) in the current sector. Let us start with the determinant (D.5). The first term is

$$\begin{aligned} & A_0 A_1' - A_0' A_1 = \mathrm{Ai}(y_1(-d))\mathrm{Ai}'(y_2(-d)) - \mathrm{Ai}'(y_1(-d))\mathrm{Ai}(y_2(-d)) \\ & \approx \frac{1}{2}\pi^{-\frac{1}{2}}(y_1(-d))^{-\frac{1}{4}} e^{-\tilde{\zeta}_1^-} \left(-\frac{1}{2}\pi^{-\frac{1}{2}}(y_2(-d))^{\frac{1}{4}} e^{-\tilde{\zeta}_2^-}\right) \\ & - \left(-\frac{1}{2}\pi^{-\frac{1}{2}}(y_1(-d))^{\frac{1}{4}} e^{-\tilde{\zeta}_1^-}\right) \frac{1}{2}\pi^{-\frac{1}{2}}(y_2(-d))^{-\frac{1}{4}} e^{-\tilde{\zeta}_2^-} \\ & = -\frac{1}{4\pi}\left(1 - \frac{\sigma}{4\tilde{\kappa}R}\right) e^{-\tilde{\zeta}_1^- - \tilde{\zeta}_2^-} + \frac{1}{4\pi}\left(1 + \frac{\sigma}{4\tilde{\kappa}R}\right) e^{-\tilde{\zeta}_1^- - \tilde{\zeta}_2^-} \\ & = \frac{\sigma}{8\pi\tilde{\kappa}R} e^{-\left(\tilde{\zeta}_1^- + \tilde{\zeta}_2^-\right)} \end{aligned} \tag{D.67}$$

The next one is

$$
\begin{aligned}
&B_2 C_3' - B_2' C_3 = \mathrm{Bi}(y_2(d))\mathrm{Ci}'^{+}(y_1(d)) - \mathrm{Bi}'(y_2(d))\mathrm{Ci}^{+}(y_1(d)) \\
&\approx \pi^{-\frac{1}{2}}(y_2(d))^{-\frac{1}{4}} e^{\tilde{\zeta}_2^+} \pi^{-\frac{1}{2}}(y_1(d))^{\frac{1}{4}} \left(e^{\tilde{\zeta}_1^+} - \frac{i}{2} e^{-\tilde{\zeta}_1^+} \right) \\
&- \pi^{-\frac{1}{2}}(y_2(d))^{\frac{1}{4}} e^{\tilde{\zeta}_2^+} \pi^{-\frac{1}{2}}(y_1(d))^{-\frac{1}{4}} \left(e^{\tilde{\zeta}_1^+} + \frac{i}{2} e^{-\tilde{\zeta}_1^+} \right) \\
&= \frac{1}{\pi}\left(1 + \frac{\sigma}{4\tilde{\kappa}R}\right)\left(e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+} - \frac{i}{2} e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} \right) - \frac{1}{\pi}\left(1 - \frac{\sigma}{4\tilde{\kappa}R}\right)\left(e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+} + \frac{i}{2} e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} \right) \\
&= \frac{1}{\pi}\left(e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+} - \frac{i}{2} e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} \right) + \frac{1}{\pi}\frac{\sigma}{4\tilde{\kappa}R}\left(e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+} - \frac{i}{2} e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} \right) \\
&- \frac{1}{\pi}\left(e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+} + \frac{i}{2} e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} \right) + \frac{1}{\pi}\frac{\sigma}{4\tilde{\kappa}R}\left(e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+} + \frac{i}{2} e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} \right) \\
&= -\frac{i}{\pi} e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} + \frac{\sigma}{2\pi\tilde{\kappa}R} e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+} \qquad \text{(D.68)}
\end{aligned}
$$

Its complex conjugate is

$$
\begin{aligned}
&B_2 D_3' - B_2' D_3 = \mathrm{Bi}(y_2(d))\mathrm{Ci}'^{-}(y_1(d)) - \mathrm{Bi}'(y_2(d))\mathrm{Ci}^{-}(y_1(d)) \\
&\approx \frac{i}{\pi} e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} + \frac{\sigma}{2\pi\tilde{\kappa}R} e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+} \qquad \text{(D.69)}
\end{aligned}
$$

We take the next term in (D.5).

$$
\begin{aligned}
&A_0 B_1' - A_0' B_1 = \mathrm{Ai}(y_1(-d))\mathrm{Bi}'(y_2(-d)) - \mathrm{Ai}'(y_1(-d))\mathrm{Bi}(y_2(-d)) \\
&\approx \frac{1}{2}\pi^{-\frac{1}{2}}(y_1(-d))^{-\frac{1}{4}} e^{-\tilde{\zeta}_1^-} \pi^{-\frac{1}{2}}(y_2(-d))^{\frac{1}{4}} e^{\tilde{\zeta}_2^-} \\
&- \left(-\frac{1}{2}\pi^{-\frac{1}{2}}(y_1(-d))^{\frac{1}{4}} e^{-\tilde{\zeta}_1^-} \right) \pi^{-\frac{1}{2}}(y_2(-d))^{-\frac{1}{4}} e^{\tilde{\zeta}_2^-} \\
&= \frac{1}{2\pi}\left(1 - \frac{\sigma}{4\tilde{\kappa}R}\right) e^{-\left(\tilde{\zeta}_1^- - \tilde{\zeta}_2^-\right)} + \frac{1}{2\pi}\left(1 + \frac{\sigma}{4\tilde{\kappa}R}\right) e^{-\left(\tilde{\zeta}_1^- - \tilde{\zeta}_2^-\right)} \\
&= \frac{1}{\pi} e^{-\left(\tilde{\zeta}_1^- - \tilde{\zeta}_2^-\right)} \qquad \text{(D.70)}
\end{aligned}
$$

The last term in the determinant (D.5) is

$$
\begin{aligned}
&A_2C_3' - A_2'C_3 = \mathrm{Ai}(y_2(d))\mathrm{Ci}'^{+}(y_1(d)) - \mathrm{Ai}'(y_2(d))\mathrm{Ci}^{+}(y_1(d)) \\
&\approx \frac{1}{2}\pi^{-\frac{1}{2}}(y_2(d))^{-\frac{1}{4}}e^{-\tilde{\zeta}_2^+}\pi^{-\frac{1}{2}}(y_1(d))^{\frac{1}{4}}\left(e^{\tilde{\zeta}_1^+} - \frac{i}{2}e^{-\tilde{\zeta}_1^+}\right) \\
&- \left(-\frac{1}{2}\pi^{-\frac{1}{2}}(y_2(d))^{\frac{1}{4}}e^{-\tilde{\zeta}_2^+}\right)\pi^{-\frac{1}{2}}(y_1(d))^{-\frac{1}{4}}\left(e^{\tilde{\zeta}_1^+} + \frac{i}{2}e^{-\tilde{\zeta}_1^+}\right) \\
&= \frac{1}{2\pi}\left(1 + \frac{\sigma}{4\tilde{\kappa}R}\right)\left(e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} - \frac{i}{2}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}\right) \\
&+ \frac{1}{2\pi}\left(1 - \frac{\sigma}{4\tilde{\kappa}R}\right)\left(e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} + \frac{i}{2}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}\right) \\
&= \frac{1}{2\pi}\left(e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} - \frac{i}{2}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}\right) + \frac{1}{2\pi}\frac{\sigma}{4\tilde{\kappa}R}\left(e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} - \frac{i}{2}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}\right) \\
&+ \frac{1}{2\pi}\left(e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} + \frac{i}{2}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}\right) - \frac{1}{2\pi}\frac{\sigma}{4\tilde{\kappa}R}\left(e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} + \frac{i}{2}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}\right) \\
&= \frac{1}{\pi}e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} - \frac{\sigma i}{8\pi\tilde{\kappa}R}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}
\end{aligned} \tag{D.71}
$$

whose complex conjugate is

$$
\begin{aligned}
&A_2D_3' - A_2'D_3 = \mathrm{Ai}(y_2(d))\mathrm{Ci}'^{-}(y_1(d)) - \mathrm{Ai}'(y_2(d))\mathrm{Ci}^{-}(y_1(d)) \\
&\approx \frac{1}{\pi}e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} + \frac{\sigma i}{8\pi\tilde{\kappa}R}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}
\end{aligned} \tag{D.72}
$$

Using (D.67), (D.68) and (D.70), (D.71) we can write the asymptotic expression for the determinant.

$$
\begin{aligned}
&\det \mathbf{M}(\omega) \approx \frac{\sigma}{8\pi\tilde{\kappa}R}e^{-\left(\tilde{\zeta}_1^- + \tilde{\zeta}_2^-\right)}\left(-\frac{i}{\pi}e^{-\left(\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+\right)} + \frac{\sigma}{2\pi\tilde{\kappa}R}e^{\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+}\right) \\
&- \frac{1}{\pi}e^{-\left(\tilde{\zeta}_1^- - \tilde{\zeta}_2^-\right)}\left(\frac{1}{\pi}e^{\tilde{\zeta}_1^+ - \tilde{\zeta}_2^+} - \frac{\sigma i}{8\pi\tilde{\kappa}R}e^{-\left(\tilde{\zeta}_1^+ + \tilde{\zeta}_2^+\right)}\right) \\
&= -\frac{\sigma i}{8\pi^2\tilde{\kappa}R}e^{-2\tilde{\zeta}_1^-} + \frac{\sigma^2}{16\pi^2(\tilde{\kappa}R)^2} - \frac{1}{\pi^2} + \frac{\sigma i}{8\pi^2\tilde{\kappa}R}e^{-2\tilde{\zeta}_1^-} \approx -\frac{1}{\pi^2}
\end{aligned} \tag{D.73}
$$

and for its complex conjugate we use (D.67), (D.69) and (D.70), (D.72).

$$
\begin{aligned}
\overline{\det \mathbf{M}}(\omega) &\approx \frac{\sigma}{8\pi\tilde{\kappa}R}e^{-\left(\tilde{\zeta}_1^-+\tilde{\zeta}_2^-\right)}\left(\frac{i}{\pi}e^{-\left(\tilde{\zeta}_1^+-\tilde{\zeta}_2^+\right)}+\frac{\sigma}{2\pi\tilde{\kappa}R}e^{\tilde{\zeta}_1^++\tilde{\zeta}_2^+}\right)\\
&-\frac{1}{\pi}e^{-\left(\tilde{\zeta}_1^--\tilde{\zeta}_2^-\right)}\left(\frac{1}{\pi}e^{\tilde{\zeta}_1^+-\tilde{\zeta}_2^+}+\frac{\sigma i}{8\pi\tilde{\kappa}R}e^{-\left(\tilde{\zeta}_1^++\tilde{\zeta}_2^+\right)}\right)\\
&=\frac{\sigma i}{8\pi^2\tilde{\kappa}R}e^{-2\tilde{\zeta}_1^-}+\frac{\sigma^2}{16\pi^2(\tilde{\kappa}R)^2}-\frac{1}{\pi^2}-\frac{\sigma i}{8\pi^2\tilde{\kappa}R}e^{-2\tilde{\zeta}_1^+}\approx-\frac{1}{\pi^2}
\end{aligned}
\tag{D.74}
$$

The absolute value of the determinant can be expressed as

$$
|\det \mathbf{M}(\omega)| = \left(\det \mathbf{M}(\omega)\overline{\det \mathbf{M}}(\omega)\right)^{\frac{1}{2}} \approx \left(\frac{1}{\pi^4}\right)^{\frac{1}{2}} = \frac{1}{\pi^2} \tag{D.75}
$$

and also the expression we need in our resonant states (D.1)

$$
\left(\frac{\overline{\det \mathbf{M}}(\omega)}{\det \mathbf{M}(\omega)}\right)^{\frac{1}{2}} \approx \left(\frac{-\frac{1}{\pi^2}}{-\frac{1}{\pi^2}}\right)^{\frac{1}{2}} = 1 \tag{D.76}
$$

$$
\left(\frac{\det \mathbf{M}(\omega)}{\overline{\det \mathbf{M}}(\omega)}\right)^{\frac{1}{2}} \approx \left(\frac{-\frac{1}{\pi^2}}{-\frac{1}{\pi^2}}\right)^{\frac{1}{2}} = 1 \tag{D.77}
$$

Let us now find the asymptotic relations of the actual functions in (D.1). There is going to be a difference in the quantities $\zeta_{1,2}$ used in asymptotic formulas compared to the previous sector. Denote them $\tilde{\zeta}_{1,2}$ in this case. We use the same trick as in (D.58) or (D.59) and we write

$$
y_1(x) = -2\alpha\left(\varepsilon x + Re^{i\theta}\right) = 2\alpha\left(\varepsilon x e^{i\pi} + Re^{i(\theta+\pi)}\right) \tag{D.78}
$$

$$
y_2(x) = -2\alpha\left(\varepsilon x + V_0 + Re^{i\theta}\right) = 2\alpha\left((\varepsilon x + V_0)e^{i\pi} + Re^{i(\theta+\pi)}\right) \tag{D.79}
$$

so the expression $\tilde{\zeta}_1$ becomes

$$
\begin{aligned}
\tilde{\zeta}_1 = \frac{2}{3}(y_1(x))^{\frac{3}{2}} &= \frac{2}{3}\left[2\alpha\left(\varepsilon x e^{i\pi}+Re^{i(\theta+\pi)}\right)\right]^{\frac{3}{2}} = \frac{2}{3}\left[2\alpha Re^{i(\theta+\pi)}\left(\varepsilon x e^{-i\theta}R^{-1}+1\right)\right]^{\frac{3}{2}}\\
&= \frac{2}{3}\left[\tilde{\kappa}R\left(1+\varepsilon x e^{-i\theta}R^{-1}\right)\right]^{\frac{3}{2}} = \tilde{\vartheta}R^{\frac{3}{2}}\left(1+\varepsilon x e^{-i\theta}R^{-1}\right)^{\frac{3}{2}}\\
&\approx \tilde{\vartheta}R^{\frac{3}{2}}\left(1+\frac{3}{2}\varepsilon x e^{-i\theta}R^{-1}\right) = \tilde{\vartheta}R^{\frac{3}{2}} + \underbrace{\frac{3}{2}\tilde{\vartheta}\varepsilon e^{-i\theta}}_{\tilde{\varpi}}xR^{\frac{1}{2}}\\
&= \tilde{\vartheta}R^{\frac{3}{2}} + \tilde{\varpi}xR^{\frac{1}{2}}
\end{aligned}
\tag{D.80}
$$

where we defined a new quantity $\tilde{\varpi}$ which can be simplified further.

$$\tilde{\varpi} = \frac{3}{2}\tilde{\vartheta}\varepsilon e^{-i\theta} = \frac{3}{2}\varepsilon\frac{2}{3}(2\alpha)^{\frac{3}{2}}e^{i\frac{3}{2}(\theta+\pi)}e^{-i\theta} = \varepsilon(2\alpha)^{\frac{3}{2}}e^{i\frac{3}{2}\left(\frac{1}{3}\theta+\pi\right)} = \tilde{\varpi}_r + i\tilde{\varpi}_r \tag{D.81}$$

where

$$\tilde{\varpi}_r = \varepsilon(2\alpha)^{\frac{3}{2}}\sin\left(\frac{3}{2}\left(\frac{1}{3}\theta+\pi\right)\right) \tag{D.82}$$

$$\tilde{\varpi}_i = \varepsilon(2\alpha)^{\frac{3}{2}}\cos\left(\frac{3}{2}\left(\frac{1}{3}\theta+\pi\right)\right) \tag{D.83}$$

Note, that in the current sector which is $-\pi < \theta < -\frac{2\pi}{3}$ we have $\pi < \frac{3}{2}\left(\frac{1}{3}\theta+\pi\right) < \frac{7\pi}{6}$ or $-\pi < \frac{3}{2}\left(\frac{1}{3}\theta+\pi\right) < -\frac{5\pi}{6}$ and therefore

$$\tilde{\varpi}_r, \tilde{\varpi}_i < 0 \tag{D.84}$$

In a similar way we get for ζ_2

$$\begin{aligned}
\zeta_2 = \frac{2}{3}(y_2(x))^{\frac{3}{2}} &= \frac{2}{3}\left[2\alpha\left((\varepsilon x+V_0)e^{i\pi} + Re^{i(\theta+\pi)}\right)\right]^{\frac{3}{2}} \\
&= \frac{2}{3}\left[2\alpha e^{i(\theta+\pi)}R\left(\frac{(\varepsilon x+V_0)e^{i\pi}}{e^{i(\theta+\pi)}R}+1\right)\right]^{\frac{3}{2}} \\
&= \frac{2}{3}\left[\tilde{\kappa}R\left(1+\frac{\varepsilon x+V_0}{e^{i\theta}R}\right)\right]^{\frac{3}{2}} = \tilde{\vartheta}R^{\frac{3}{2}}\left(1+\frac{\varepsilon x+V_0}{e^{i\theta}R}\right)^{\frac{3}{2}} \\
&\approx \tilde{\vartheta}R^{\frac{3}{2}}\left(1+\frac{3(\varepsilon x+V_0)}{2e^{i\theta}R}\right) = \tilde{\vartheta}R^{\frac{3}{2}} + \tilde{\varpi}xR^{\frac{1}{2}} + \frac{3}{2}\tilde{\vartheta}\varepsilon e^{-i\theta}\frac{V_0}{\varepsilon}R^{\frac{1}{2}} \\
&= \tilde{\vartheta}R^{\frac{3}{2}} + \tilde{\varpi}xR^{\frac{1}{2}} + \tilde{\varpi}\frac{V_0}{\varepsilon}R^{\frac{1}{2}} = \tilde{\vartheta}R^{\frac{3}{2}} + \tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)
\end{aligned} \tag{D.85}$$

Using (D.80), (D.85) and (D.52), (D.54) we have for Airy function in the current sector the asymptotic relations

$$\begin{aligned}
\mathrm{Ai}(y_1(x)) &\approx \frac{1}{2}\pi^{-\frac{1}{2}}\left(y_1(x)\right)^{-\frac{1}{4}}e^{-\tilde{\zeta}_1} \approx \frac{1}{2}\pi^{-\frac{1}{2}}\left(2\alpha\left(\varepsilon x e^{i\pi}+Re^{i(\theta+\pi)}\right)\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \\
&= \frac{1}{2}\pi^{-\frac{1}{2}}\left(2\alpha Re^{i(\theta+\pi)}\left(\varepsilon xR^{-1}e^{-i\theta}+1\right)\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \\
&\approx \frac{1}{2}\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}\left(1-\frac{1}{4}\varepsilon xR^{-1}e^{-i\theta}\right)e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \\
&\approx \frac{1}{2}\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}}
\end{aligned} \tag{D.86}$$

and similarly

$$
\begin{aligned}
\mathrm{Ci}^{\pm}(y_1(x)) &\approx \pi^{-\frac{1}{2}}(y_1(x))^{-\frac{1}{4}}\left(e^{\tilde{\zeta}_1} \pm \frac{i}{2}e^{-\tilde{\zeta}_1}\right) \\
&\approx \pi^{-\frac{1}{2}}\left(2\alpha\left(\varepsilon x e^{i\pi} + Re^{i(\theta+\pi)}\right)\right)^{-\frac{1}{4}}\left(e^{\tilde{\zeta}_1} \pm \frac{i}{2}e^{-\tilde{\zeta}_1}\right) \\
&\approx \pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}\left(e^{\tilde{\vartheta}R^{\frac{3}{2}}+\tilde{\varpi}xR^{\frac{1}{2}}} \pm \frac{i}{2}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}}\right) \qquad \text{(D.87)}
\end{aligned}
$$

For $-d < x < d$ we get

$$
\begin{aligned}
\mathrm{Ai}(y_2(x)) &\approx \frac{1}{2}\pi^{-\frac{1}{2}}\left(y_2(x)\right)^{-\frac{1}{4}}e^{-\tilde{\zeta}_2} \\
&\approx \frac{1}{2}\pi^{-\frac{1}{2}}\left(2\alpha\left((\varepsilon x + V_0)e^{i\pi} + Re^{i(\theta+\pi)}\right)\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} \\
&= \frac{1}{2}\pi^{-\frac{1}{2}}\left(2\alpha Re^{i(\theta+\pi)}\left((\varepsilon x + V_0)R^{-1}e^{-i\theta} + 1\right)\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} \\
&\approx \frac{1}{2}\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}\left(1 - \frac{1}{4}(\varepsilon x + V_0)R^{-1}e^{-i\theta}\right)e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} \\
&\approx \frac{1}{2}\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} \qquad \text{(D.88)}
\end{aligned}
$$

and

$$
\mathrm{Bi}(y_2(x)) \approx \pi^{-\frac{1}{2}}\left(y_2(x)\right)^{-\frac{1}{4}}e^{\tilde{\zeta}_2} \approx \pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{\tilde{\vartheta}R^{\frac{3}{2}}+\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)} \qquad \text{(D.89)}
$$

We are prepared to get the final form of (D.1) for the sector $-\pi < \theta < -\frac{2\pi}{3}$. Using (D.75) and (D.86) we have for $x < -d$

$$
\begin{aligned}
\psi_\omega(x) &= \frac{2\chi}{\pi^2|\det \mathbf{M}(\omega)|}\mathrm{Ai}(y_1(x)) \approx \frac{2\chi}{\pi^2\frac{1}{\pi^2}}\frac{1}{2}\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \\
&= \chi\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \qquad \text{(D.90)}
\end{aligned}
$$

which decays for large R since $\tilde{\vartheta}_r > 0$. For $-d < x < d$ we use (D.75), (D.70), (D.67) and

(D.88), (D.89.

$$\begin{aligned}
\psi_\omega(x) &= \frac{2\chi}{\pi|\det \mathbf{M}(\omega)|}\left[(B_1'A_0 - B_1A_0')\mathrm{Ai}(y_2(x)) + (A_1A_0' - A_1'A_0)\mathrm{Bi}(y_2(x))\right] \\
&\approx \frac{2\chi}{\pi\frac{1}{\pi^2}}\left[\frac{1}{\pi}e^{-\tilde{\varrho}R^{\frac{1}{2}}\sigma}\frac{1}{2}\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}\right. \\
&\left.+0\cdot\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{\tilde{\vartheta}R^{\frac{3}{2}}+\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}\right] \\
&= \chi\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}}e^{-\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}e^{-\tilde{\varrho}R^{\frac{1}{2}}\sigma} \qquad \text{(D.91)}
\end{aligned}$$

which also decays because of (D.64).

For the last region $x > d$ we have using (D.76), (D.77) and (D.87).

$$\begin{aligned}
\psi_\omega(x) &= \chi i\left(\frac{\overline{\det\mathbf{M}}(\omega)}{\det\mathbf{M}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^+(y_1(x)) - \chi i\left(\frac{\det\mathbf{M}(\omega)}{\overline{\det\mathbf{M}}(\omega)}\right)^{\frac{1}{2}}\mathrm{Ci}^-(y_1(x)) \\
&\approx \chi i\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}\left(e^{\tilde{\vartheta}R^{\frac{3}{2}}+\tilde{\varpi}xR^{\frac{1}{2}}} + \frac{i}{2}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}}\right) \\
&- \chi i\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}\left(e^{\tilde{\vartheta}R^{\frac{3}{2}}+\tilde{\varpi}xR^{\frac{1}{2}}} - \frac{i}{2}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}}\right) \\
&= -\chi\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}\frac{1}{2}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} - \chi\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}\frac{1}{2}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \\
&= -\chi\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} \qquad \text{(D.92)}
\end{aligned}$$

that goes to exponentially small values as well because of the term $e^{-\tilde{\vartheta}R^{\frac{3}{2}}}$ which decays faster than $e^{-\tilde{\varpi}xR^{\frac{1}{2}}}$ grows because of (D.64).

Summing it up, the asymptotic expression for the resonant states (D.1) for the sector $-\pi < \theta < -\frac{2\pi}{3}$ is

$$\psi_\omega(x) = \chi\begin{cases}\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} & x < -d \\ \pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}}e^{-\tilde{\varpi}R^{\frac{1}{2}}\left(x+\frac{V_0}{\varepsilon}\right)}e^{-\tilde{\varrho}R^{\frac{1}{2}}\sigma} & -d < x < d \\ -\pi^{-\frac{1}{2}}\left(\tilde{\kappa}R\right)^{-\frac{1}{4}}e^{-\tilde{\vartheta}R^{\frac{3}{2}}-\tilde{\varpi}xR^{\frac{1}{2}}} & d < x\end{cases} \qquad \text{(D.93)}$$

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