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# Weighted Hardy type inequalities for supremum operators on the cones of monotone functions

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**Abstract**

The complete characterization of the weighted $L^p - L^r$ inequalities of supremum operators on the cones of monotone functions for all $0 < p, r \leq \infty$ is given.

**MSC:** Primary 26D15; secondary 47G10

**Keywords:** Hardy type inequality; weight; supremum operator; Lebesgue space; monotone function

## 1 Introduction

Let $\mathbb{R}_+ := [0,\infty)$. Denote $\mathfrak{M}$ the set of all measurable functions on $\mathbb{R}_+$, $\mathfrak{M}^+ \subset \mathfrak{M}$ the subset of all non-negative functions and $\mathfrak{M}^\downarrow \subset \mathfrak{M}^+$ ($\mathfrak{M}^\uparrow \subset \mathfrak{M}^+$) is the cone of all non-increasing (non-decreasing) functions. Also denote by $\mathscr{C} \subset \mathfrak{M}$ the set of all continuous functions on $\mathbb{R}_+$. If $0 < p \leq \infty$ and $v \in \mathfrak{M}^+$ we define

$$L_v^p := \left\{ f \in \mathfrak{M} : \|f\|_{L_v^p} := \left( \int_0^\infty |f(x)|^p v(x)\,dx \right)^{\frac{1}{p}} < \infty \right\},$$
$$L_v^\infty := \left\{ f \in \mathfrak{M} : \|f\|_{L_v^\infty} := \operatorname*{ess\,sup}_{x \geq 0} v(x)|f(x)| < \infty \right\}.$$

Let $w \in \mathfrak{M}^+$ and $k(x,y) \geq 0$ is a Borel function on $[0,\infty)^2$ satisfying Oinarov's condition: $k(x,y) = 0$ if $x < y$, and there is a constant $D \geq 1$ independent of $x \geq z \geq y \geq 0$ such that

$$\frac{1}{D}\big(k(x,z) + k(z,y)\big) \leq k(x,y) \leq D\big(k(x,z) + k(z,y)\big). \tag{1.1}$$

The mapping properties between weighted $L^p$ spaces of Hardy type operators involved are very well studied. See *e.g.* the books [1, 2] and [3] and the references therein. We also mention the following examples of articles in this area: [4–8] and [9]. Recently, it has been discovered that it is of great interest to study also some corresponding supremum operators instead of the usual such Hardy type (arithmetic mean) operators. The interest comes both from purely mathematical point of view but also from various applications where such kernels many times are the unit impulse answers to the problem at hand and the best constants means the operator norms of the corresponding transfer of the energy of the 'signals' measured in weighted $L^p$ spaces.

We consider supremum operators of the form

$$(Tf)(x) = \operatorname*{ess\,sup}_{y\geq x} k(y,x)w(y)f(y), \quad f \in \mathfrak{M}^{\uparrow},$$

$$(Sf)(x) = \operatorname*{ess\,sup}_{y\geq x} k(y,x)w(y)f(y), \quad f \in \mathfrak{M}^{\downarrow},$$

$$(\mathscr{T}f)(x) = \operatorname*{ess\,sup}_{0\leq y\leq x} k(x,y)w(y)f(y), \quad f \in \mathfrak{M}^{\downarrow},$$

$$(\mathscr{S}f)(x) = \operatorname*{ess\,sup}_{0\leq y\leq x} k(x,y)w(y)f(y), \quad f \in \mathfrak{M}^{\uparrow}.$$

Let $0 < p, r \leq \infty$ and $u, v \in \mathfrak{M}^+$. The paper is devoted to the necessary and sufficient conditions for the inequalities

$$\|Tf\|_{L^r_u} \leq C_T \|f\|_{L^p_v}, \quad f \in \mathfrak{M}^{\uparrow}, \tag{1.2}$$

$$\|Sf\|_{L^r_u} \leq C_S \|f\|_{L^p_v}, \quad f \in \mathfrak{M}^{\downarrow}, \tag{1.3}$$

$$\|\mathscr{T}f\|_{L^r_u} \leq C_{\mathscr{T}} \|f\|_{L^p_v}, \quad f \in \mathfrak{M}^{\downarrow}, \tag{1.4}$$

$$\|\mathscr{S}f\|_{L^r_u} \leq C_{\mathscr{S}} \|f\|_{L^p_v}, \quad f \in \mathfrak{M}^{\uparrow}, \tag{1.5}$$

where the constants $C_T$ and others are taken as the least possible.

This problem was first studied for the inequality (1.3) in [10], Theorem 3.2, in a case when $k(x,y) = 1$, $w \in \mathscr{C}$. This result was extended in [11] for the case $k(x,y)$ satisfying (1.1) with a discrete form of a criterion for $0 < r < p < \infty$. With different supremum operators some similar problems were studied in [12–22]. This area is currently developing intensively and finds many interesting applications.

Section 2 is devoted to preliminaries. The border cases $0 < r < p = \infty$, $0 < p < r = \infty$ and $r = p = \infty$ are solved in Section 3. In Section 4 we characterize the case $k(x,y) = 1$, which is essentially used in Section 5 with the main results of the paper.

We use signs := and =: for determining new quantities and $\mathbb{Z}$ for the set of all integers. For positive functionals $F$ and $G$ we write $F \lesssim G$, if $F \leq cG$ with some positive constant $c$, which depends only on irrelevant parameters. $F \approx G$ means $F \lesssim G \lesssim F$ or $F = cG$. $\chi_E$ denotes the characteristic function (indicator) of a set $E$. Uncertainties of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be zero. □ stands for the end of a proof.

## 2 Preliminaries

We denote

$$V(t) := \int_0^t v, \qquad V_*(t) := \int_t^\infty v.$$

Let $0 < p, r < \infty$. By [11], Lemma 2.1, and the monotone convergence theorem the inequality (1.2) is equivalent to

$$\left(\int_0^\infty \left[\operatorname*{ess\,sup}_{y\geq x}\bigl(k(y,x)w(y)\bigr)^p \int_0^y h\right]^{\frac{r}{p}} u(x)\,dx\right)^{\frac{p}{r}} \leq C_T^p \int_0^\infty h V_*, \quad h \in \mathfrak{M}^+. \tag{2.1}$$

If

$$T_p h(x) := \operatorname*{ess\,sup}_{y\geq x} \big(k(y,x)w(y)\big)^p \int_0^y h,$$

then (2.1) is equivalent to

$$\|T_p h\|_{L_u^{\frac{r}{p}}} \leq C_T^p \|h\|_{L^1_{V_*}}, \quad h \in \mathfrak{M}^+. \tag{2.2}$$

Analogously, if

$$S_p h(x) := \operatorname*{ess\,sup}_{y\geq x} \big(k(y,x)w(y)\big)^p \int_y^\infty h,$$

$$\mathscr{T}_p h(x) := \operatorname*{ess\,sup}_{0\leq y\leq x} \big(k(x,y)w(y)\big)^p \int_y^\infty h,$$

$$\mathscr{S}_p h(x) := \operatorname*{ess\,sup}_{0\leq y\leq x} \big(k(x,y)w(y)\big)^p \int_0^y h,$$

then (1.3), (1.4), and (1.5) are equivalent to

$$\|S_p h\|_{L_u^{\frac{r}{p}}} \leq C_S^p \|h\|_{L^1_V}, \quad h \in \mathfrak{M}^+, \tag{2.3}$$

$$\|\mathscr{T}_p h\|_{L_u^{\frac{r}{p}}} \leq C_{\mathscr{T}}^p \|h\|_{L^1_V}, \quad h \in \mathfrak{M}^+, \tag{2.4}$$

and

$$\|\mathscr{S}_p h\|_{L_u^{\frac{r}{p}}} \leq C_{\mathscr{S}}^p \|h\|_{L^1_{V_*}}, \quad h \in \mathfrak{M}^+, \tag{2.5}$$

respectively.

For the border cases $0 < p < r = \infty$, $0 < r < p = \infty$, and $r = p = \infty$ we have the following four groups of inequalities:

$$\operatorname*{ess\,sup}_{x\geq 0} \big[u(x)\big]^p T_p h(x) \leq C_{T,p}^p \int_0^\infty h V_*, \quad h \in \mathfrak{M}^+, \tag{2.6}$$

$$\left(\int_0^\infty \Big[\operatorname*{ess\,sup}_{y\geq x} k(y,x)w(y)f(y)\Big]^r u(x)\,dx\right)^{\frac{1}{r}} \leq C_{T,r} \|f\|_{L_v^\infty}, \quad f \in \mathfrak{M}^\uparrow, \tag{2.7}$$

$$\operatorname*{ess\,sup}_{x\geq 0} u(x) \Big[\operatorname*{ess\,sup}_{y\geq x} k(y,x)w(y)f(y)\Big] \leq C_{T,\infty} \|f\|_{L_v^\infty}, \quad f \in \mathfrak{M}^\uparrow, \tag{2.8}$$

for the operator $T$;

$$\operatorname*{ess\,sup}_{x\geq 0} \big[u(x)\big]^p S_p h(x) \leq C_{S,p}^p \int_0^\infty h V, \quad h \in \mathfrak{M}^+, \tag{2.9}$$

$$\left(\int_0^\infty \Big[\operatorname*{ess\,sup}_{y\geq x} k(y,x)w(y)f(y)\Big]^r u(x)\,dx\right)^{\frac{1}{r}} \leq C_{S,r} \|f\|_{L_v^\infty}, \quad f \in \mathfrak{M}^\downarrow, \tag{2.10}$$

$$\operatorname*{ess\,sup}_{x\geq 0} u(x) \Big[\operatorname*{ess\,sup}_{y\geq x} k(y,x)w(y)f(y)\Big] \leq C_{S,\infty} \|f\|_{L_v^\infty}, \quad f \in \mathfrak{M}^\downarrow, \tag{2.11}$$

for the operator $S$;

$$\operatorname*{ess\,sup}_{x\geq 0}\big[u(x)\big]^p\mathscr{T}_p h(x)\leq C^p_{\mathscr{T},p}\int_0^\infty hV,\quad h\in\mathfrak{M}^+,\tag{2.12}$$

$$\left(\int_0^\infty\Big[\operatorname*{ess\,sup}_{0\leq y\leq x}k(x,y)w(y)f(y)\Big]^r u(x)\,dx\right)^{\frac{1}{r}}\leq C_{\mathscr{T},r}\|f\|_{L^\infty_v},\quad f\in\mathfrak{M}^\downarrow,\tag{2.13}$$

$$\operatorname*{ess\,sup}_{x\geq 0}u(x)\Big[\operatorname*{ess\,sup}_{0\leq y\leq x}k(x,y)w(y)f(y)\Big]\leq C_{\mathscr{T},\infty}\|f\|_{L^\infty_v},\quad f\in\mathfrak{M}^\downarrow,\tag{2.14}$$

for the operator $\mathscr{T}$, and

$$\operatorname*{ess\,sup}_{x\geq 0}\big[u(x)\big]^p\mathscr{S}_p h(x)\leq C^p_{\mathscr{S},p}\int_0^\infty hV_*,\quad h\in\mathfrak{M}^+,\tag{2.15}$$

$$\left(\int_0^\infty\Big[\operatorname*{ess\,sup}_{0\leq y\leq x}k(x,y)w(y)f(y)\Big]^r u(x)\,dx\right)^{\frac{1}{r}}\leq C_{\mathscr{S},r}\|f\|_{L^\infty_v},\quad f\in\mathfrak{M}^\uparrow,\tag{2.16}$$

$$\operatorname*{ess\,sup}_{x\geq 0}u(x)\Big[\operatorname*{ess\,sup}_{0\leq y\leq x}k(x,y)w(y)f(y)\Big]\leq C_{\mathscr{S},\infty}\|f\|_{L^\infty_v},\quad f\in\mathfrak{M}^\uparrow,\tag{2.17}$$

for the operator $\mathscr{S}$. We characterize the inequalities (2.6)-(2.17) in the next section.

To deal with the inequalities (2.1)-(2.5) we study first the case $k(x,y)=1$ and then a general case.

## 3 Border cases of summation parameters

For a measurable function $v\in\mathfrak{M}^+$ we define monotone envelopes (see [23], Section 2) as follows:

$$v^\downarrow(x):=\operatorname*{ess\,sup}_{y\geq x}v(y),$$

$$v^\uparrow(x):=\operatorname*{ess\,sup}_{0\leq y\leq x}v(y).$$

**Theorem 3.1** *For the best possible constants of the inequalities* (2.6)-(2.8) *we have*

$$C_{T,p}\approx\sup_{x\geq 0}u^\uparrow(x)\operatorname*{ess\,sup}_{y\geq x}\frac{k(y,x)w(y)}{V_*^{1/p}(y)},\tag{3.1}$$

$$C_{T,r}=\left(\int_0^\infty\left[\operatorname*{ess\,sup}_{y\geq x}\frac{k(y,x)w(y)}{v^\downarrow(y)}\right]^r u(x)\,dx\right)^{\frac{1}{r}},\tag{3.2}$$

$$C_{T,\infty}=\operatorname*{ess\,sup}_{x\geq 0}u(x)\left[\operatorname*{ess\,sup}_{y\geq x}\frac{k(y,x)w(y)}{v^\downarrow(y)}\right].\tag{3.3}$$

*Proof* Observe that if $k(x,y)$ satisfies (1.1), then $[k(x,y)]^p$ satisfies (1.1) too with a constant $D_p\geq 1$. If $x\leq t$, then

$$\begin{aligned}T_p h(t)&=\operatorname*{ess\,sup}_{y\geq t}\big(k(y,t)w(y)\big)^p\int_0^y h\\&\leq D_p\operatorname*{ess\,sup}_{y\geq t}\big(k(y,x)w(y)\big)^p\int_0^y h\\&\leq D_p\operatorname*{ess\,sup}_{y\geq x}\big(k(y,x)w(y)\big)^p\int_0^y h=D_pT_ph(x).\end{aligned}$$

Hence,

$$T_p h(x) \approx \sup_{t\geq x} T_p h(t) := \varphi(x) \in \mathfrak{M}^{\downarrow}.$$

It implies (see [11], Proposition 3.1)

$$\begin{aligned}\operatorname*{ess\,sup}_{x\geq 0}\big[u(x)\big]^p T_p h(x) &\approx \operatorname*{ess\,sup}_{x\geq 0}\big[u(x)\big]^p \varphi(x)\\ &= \operatorname*{ess\,sup}_{x\geq 0}\big[u(x)\big]^p \sup_{t\geq x}\varphi(t) = \sup_{t\geq 0}\varphi(t)\big[u^{\uparrow}(t)\big]^p\\ &\approx \sup_{x\geq 0}\big[u^{\uparrow}(x)\big]^p T_p h(x),\end{aligned}$$

and (2.6) is equivalent to

$$\sup_{x\geq 0}\big[u^{\uparrow}(x)\big]^p \|H_x h\|_{L^{\infty}_{(k(\cdot,x)w(\cdot))^p}} \lesssim C^p_{T,p}\int_0^{\infty} hV_*, \quad h\in\mathfrak{M}^+, \tag{3.4}$$

where

$$H_x h(y) := \chi_{[x,\infty)}(y)\int_0^y h.$$

Thus,

$$C^p_{T,p} \approx \sup_{x\geq 0}\big[u^{\uparrow}(x)\big]^p \|H_x\|_{L^1_{V_*}\to L^{\infty}_{(k(\cdot,x)w(\cdot))^p}}.$$

Since by a well-known theorem ([24], Theorem 1.1)

$$\|H_x\|_{L^1_{V_*}\to L^{\infty}_{(k(\cdot,x)w(\cdot))^p}} = \operatorname*{ess\,sup}_{y\geq x}\frac{(k(y,x)w(y))^p}{V_*(y)},$$

we obtain (3.1).

Now, (2.7) is equivalent to the inequality

$$\left(\int_0^{\infty}\Big[\operatorname*{ess\,sup}_{y\geq x} k(y,x)w(y)f(y)\Big]^r u(x)\,dx\right)^{\frac{1}{r}} \leq C_{T,r}\|f\|_{L^{\infty}_{v^{\downarrow}}}, \quad f\in\mathfrak{M}^{\uparrow}. \tag{3.5}$$

The lower bound of (3.2) follows from (3.5) with $f = \frac{1}{v^{\downarrow}}$ and the upper bound from the estimate $f(y) \leq \frac{\|f\|_{L^{\infty}_{v^{\downarrow}}}}{v^{\downarrow}(y)}$.

The proof of (3.3) is the same. □

Analogously, we can prove the following.

**Theorem 3.2** *For the best possible constants of the inequalities* (2.9)-(2.17) *we have*

$$C_{S,p} \approx \sup_{x\geq 0} u^{\uparrow}(x)\operatorname*{ess\,sup}_{y\geq x}\frac{k(y,x)w(y)}{V^{1/p}(y)}, \tag{3.6}$$

$$C_{S,r} = \left(\int_0^\infty \left[\operatorname*{ess\,sup}_{y\ge x} \frac{k(y,x)w(y)}{v^{\uparrow}(y)}\right]^r u(x)\,dx\right)^{\frac{1}{r}}, \tag{3.7}$$

$$C_{S,\infty} = \operatorname*{ess\,sup}_{x\ge 0} u(x)\left[\operatorname*{ess\,sup}_{y\ge x} \frac{k(y,x)w(y)}{v^{\uparrow}(y)}\right]. \tag{3.8}$$

$$C_{\mathscr{T},p} \approx \sup_{x\ge 0} u^{\downarrow}(x) \operatorname*{ess\,sup}_{0\le y\le x} \frac{k(x,y)w(y)}{V^{1/p}(y)}, \tag{3.9}$$

$$C_{\mathscr{T},r} = \left(\int_0^\infty \left[\operatorname*{ess\,sup}_{0\le y\le x} \frac{k(x,y)w(y)}{v^{\uparrow}(y)}\right]^r u(x)\,dx\right)^{\frac{1}{r}}, \tag{3.10}$$

$$C_{\mathscr{T},\infty} = \operatorname*{ess\,sup}_{x\ge 0} u(x)\left[\operatorname*{ess\,sup}_{0\le y\le x} \frac{k(x,y)w(y)}{v^{\uparrow}(y)}\right], \tag{3.11}$$

$$C_{\mathscr{S},p} \approx \sup_{x\ge 0} u^{\downarrow}(x) \operatorname*{ess\,sup}_{0\le y\le x} \frac{k(y,x)w(y)}{V_*^{1/p}(y)}, \tag{3.12}$$

$$C_{\mathscr{S},r} = \left(\int_0^\infty \left[\operatorname*{ess\,sup}_{0\le y\le x} \frac{k(y,x)w(y)}{v^{\downarrow}(y)}\right]^r u(x)\,dx\right)^{\frac{1}{r}}, \tag{3.13}$$

$$C_{\mathscr{S},\infty} = \operatorname*{ess\,sup}_{x\ge 0} u(x)\left[\operatorname*{ess\,sup}_{0\le y\le x} \frac{k(y,x)w(y)}{v^{\downarrow}(y)}\right]. \tag{3.14}$$

## 4 The case $k(x,y) = 1$

Let $u, v_0, w_0 \in \mathfrak{M}^+$ be weights. We suppose for simplicity that $0 < \int_0^t u < \infty$, for all $t > 0$, $\int_0^\infty u = \infty$ and define the functions $\sigma : [0;\infty) \to [0;\infty)$, $\sigma^{-1} : [0;\infty) \to [0;\infty)$, by

$$\sigma(x) := \inf\left\{y > 0 : \int_0^y u \ge 2\int_0^x u\right\},$$
$$\sigma^{-1}(x) := \inf\left\{y > 0 : \int_0^y u \ge \frac{1}{2}\int_0^x u\right\}.$$

Let $\sigma^2 := \sigma(\sigma)$. For $0 \le c < d < \infty$ and $h \in \mathfrak{M}^+$ we put

$$H_c h(x) := \chi_{[c,\infty)}(x) \int_0^x h,$$
$$H_{c,d} h(x) := \chi_{[c,d)}(x) \int_{\sigma^{-1}(c)}^x h,$$
$$H_c^* h(x) := \chi_{[c,\infty)}(x) \int_x^\infty h,$$
$$H_{c,d}^* h(x) := \chi_{[c,d)}(x) \int_x^{\sigma(d)} h.$$

We need the following partial cases of [21], Theorems 2.1 and 2.3 (see also [19, 20]).

**Theorem 4.1** *Let* $0 < r < \infty$. *Then*:

(a) *For validity of the inequality*

$$\left(\int_0^\infty \left[\operatorname*{ess\,sup}_{y\ge x} w_0(y)\int_0^y h\right]^r u(x)\,dx\right)^{\frac{1}{r}} \le C_0 \|h\|_{L^1_{v_0}}, \quad h \in \mathfrak{M}^+, \tag{4.1}$$

*it is necessary and sufficient that the inequality*

$$\left(\int_0^\infty u(x)\left[w_0^{\downarrow}(x)\right]^r\left(\int_0^x h\right)^r dx\right)^{\frac{1}{r}} \le A_0\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*holds and the constant*

$$A_1 := \begin{cases} \sup_{t>0}(\int_0^t u)^{\frac{1}{r}}\|H_t\|_{L^1_{v_0}\to L^\infty_{w_0}}, & r\ge 1,\\ (\int_0^\infty u(x)(\int_0^x u)^{\frac{r}{1-r}}\|H_{[\sigma^{-1}(x),\sigma(x)]}\|^{\frac{r}{1-r}}_{L^1_{v_0}\to L^\infty_{w_0}}\,dx)^{\frac{1-r}{r}}, & 0<r<1,\end{cases}$$

*is finite. Moreover,* $C_0 \approx A_0 + A_1$.

(b) *For validity of the inequality*

$$\left(\int_0^\infty\left[\operatorname*{ess\,sup}_{y\ge x} w_0(y)\int_y^\infty h\right]^r u(x)\,dx\right)^{\frac{1}{r}} \le C_1\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+, \tag{4.2}$$

*it is necessary and sufficient that the inequality*

$$\left(\int_0^\infty u(x)\left[\operatorname*{ess\,sup}_{x\le y\le\sigma^2(x)} w_0(y)\right]^r\left(\int_{\sigma^2(x)}^\infty h\right)^r dx\right)^{\frac{1}{r}} \le B_0\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*holds and the constant*

$$B_1 := \begin{cases} \sup_{t>0}(\int_0^t u)^{\frac{1}{r}}\|H^*_t\|_{L^1_{v_0}\to L^\infty_{w_0}}, & r\ge 1,\\ (\int_0^\infty u(x)(\int_0^x u)^{\frac{r}{1-r}}\|H^*_{[\sigma^{-1}(x),\sigma(x)]}\|^{\frac{r}{1-r}}_{L^1_{v_0}\to L^\infty_{w_0}}\,dx)^{\frac{1-r}{r}}, & 0<r<1,\end{cases}$$

*is finite. Moreover,* $C_1 \approx B_0 + B_1$.

Using Theorem 4.1 we characterize (1.2) and (1.3) with $k(x,y) = 1$.

**Theorem 4.2** *Let* $0 < p, r < \infty$ *and* $k(x,y) = 1$. *Then, for the best possible constants of the inequalities* (1.2) *and* (1.3) *the following equivalences hold*:

$$C_T \approx \mathscr{A}_0 + \mathscr{A}_1, \qquad C_S \approx \mathscr{B}_0 + \mathscr{B}_1, \tag{4.3}$$

*where*

$$\mathscr{A}_0 = \sup_{t>0}\left[V_*(t)\right]^{-\frac{1}{p}}\left(\int_t^\infty u\left[w^{\downarrow}\right]^r\right)^{\frac{1}{r}}, \quad r\ge p,$$

$$\mathscr{A}_0 = \left(\int_0^\infty\left(\left[V_*(x)\right]^{-1}\int_x^\infty u\left[w^{\downarrow}\right]^r\right)^{\frac{r}{p-r}} u(x)\left[w^{\downarrow}(x)\right]^r dx\right)^{\frac{p-r}{pr}}, \quad 0<r<p,$$

$$\mathscr{A}_1 = \sup_{t>0}\left(\int_0^t u\right)^{\frac{1}{r}}\sup_{y\ge t}\frac{w^{\downarrow}(y)}{[V_*(y)]^{\frac{1}{p}}}, \quad r\ge p,$$

$$\mathscr{A}_1 = \left(\int_0^\infty u(x)\left(\int_0^x u\right)^{\frac{r}{p-r}}\left(\operatorname*{ess\,sup}_{\sigma^{-1}(x)\le y\le\sigma(x)}\frac{[w(y)]^p}{V_*(y)}\right)^{\frac{r}{p-r}} dx\right)^{\frac{p-r}{pr}}, \quad 0<r<p,$$

$$\mathscr{B}_0 = \sup_{t>0}\big[V\big(\sigma^2(t)\big)\big]^{-\frac{1}{p}}\bigg(\int_0^t u(x)\Big[\operatorname*{ess\,sup}_{x\le y\le\sigma^2(x)} w(y)\Big]^r dx\bigg)^{\frac{1}{r}}, \quad r\ge p,$$

$$\mathscr{B}_0 = \bigg(\int_0^\infty\bigg(\big[V\big(\sigma^2(z)\big)\big]^{-1}\int_0^z u(x)\Big[\operatorname*{ess\,sup}_{x\le y\le\sigma^2(x)} w(y)\Big]^r dx\bigg)^{\frac{r}{p-r}}$$
$$\times u(z)\Big[\operatorname*{ess\,sup}_{z\le y\le\sigma^2(z)} w(y)\Big]^r dz\bigg)^{\frac{p-r}{pr}}, \quad 0<r<p,$$

$$\mathscr{B}_1 = \sup_{t>0}\bigg(\int_0^t u\bigg)^{\frac{1}{r}} \operatorname*{ess\,sup}_{y\ge t}\frac{w(y)}{[V(y)]^{\frac{1}{p}}}, \quad r\ge p,$$

$$\mathscr{B}_1 = \bigg(\int_0^\infty u(x)\bigg(\int_0^x u\bigg)^{\frac{r}{p-r}}\bigg(\operatorname*{ess\,sup}_{\sigma^{-1}(x)\le y\le\sigma(x)}\frac{[w(y)]^p}{V(y)}\bigg)^{\frac{r}{p-r}} dx\bigg)^{\frac{p-r}{pr}}, \quad 0<r<p.$$

*Proof* Since (1.2) $\Leftrightarrow$ (2.2) and (1.3) $\Leftrightarrow$ (2.3), the proof follows by applying Theorem 4.1 with $r$ replaced by $\frac{r}{p}$, $w_0 = w^p$, $v_0 = V_*$ in (4.1) and $v_0 = V$ in (4.2). Thus, $C_T \approx \mathscr{A}_0' + \mathscr{A}_1'$, where $\mathscr{A}_0'$ is the best constant in the inequality

$$\bigg(\int_0^\infty u(x)\big[w^{\downarrow}(x)\big]^r\bigg(\int_0^x h\bigg)^{\frac{r}{p}} dx\bigg)^{\frac{p}{r}} \le \big[\mathscr{A}_0'\big]^p\|h\|_{L^1_{V_*}}, \quad h\in\mathfrak{M}^+, \tag{4.4}$$

and

$$\big[\mathscr{A}_1'\big]^p = \begin{cases} \sup_{t>0}(\int_0^t u)^{\frac{p}{r}}\|H_t\|_{L^1_{V_*}\to L^\infty_{w^p}}, & r\ge p,\\ (\int_0^\infty u(x)(\int_0^x u)^{\frac{r}{p-r}}\|H_{[\sigma^{-1}(x),\sigma(x)]}\|^{\frac{r}{p-r}}_{L^1_{V_*}\to L^\infty_{w^p}} dx)^{\frac{p-r}{r}}, & 0<r<p.\end{cases}$$

If $k(x,y)\ge 0$ is a measurable kernel on $\mathbb{R}_+\times\mathbb{R}_+$ and

$$Kf(x) := \int_0^\infty k(x,y)f(y)\,dy,$$

then by well-known results ([25], Chapter XI, Section 1.5, Theorem 4, see also [24], Theorem 1.1)

$$\|K\|_{L^1\to L^q} = \operatorname*{ess\,sup}_{s\ge 0}\big\|k(\cdot,s)\big\|_{L^q}, \quad 1\le q\le\infty. \tag{4.5}$$

If $k(x,y) = w(x)\chi_{[0,x]}(y)u(y)$ and $0<q<1$, then ([26], Theorem 3.3)

$$\|K\|_{L^1\to L^q} \approx \bigg(\int_0^\infty\Big[\operatorname*{ess\,sup}_{0\le y\le x} u(y)\Big]^{\frac{q}{1-q}}\bigg(\int_x^\infty w^q\bigg)^{\frac{q}{1-q}}\big[w(x)\big]^q dx\bigg)^{\frac{1-q}{q}}. \tag{4.6}$$

Applying (4.5) and (4.6) to (4.4) we find that $\mathscr{A}_0 \approx \mathscr{A}_0'$. Again, applying (4.5), when

$$k(x,y) = w^p(x)\chi_{[t,\infty)}(x)\frac{\chi_{[0,x]}(y)}{V_*(y)}$$

we obtain

$$\begin{aligned}\|H_t\|_{L^1_{V_*}\to L^\infty_{w^p}} &= \operatorname*{ess\,sup}_{s\ge 0}\bigl\|k(\cdot,s)\bigr\|_{L^\infty} = \operatorname*{ess\,sup}_{s\ge 0}\frac{1}{V_*(s)}\operatorname*{ess\,sup}_{\{x\ge t\}\cap\{x\ge s\}} w^p(x)\\ &= \sup_{s\ge 0}\frac{1}{V_*(s)}\bigl[w^\downarrow\bigl(\max(t,s)\bigr)\bigr]^p\\ &= \max\biggl(\sup_{0\le s\le t}\frac{[w^\downarrow(t)]^p}{V_*(s)},\sup_{s\ge t}\frac{[w^\downarrow(s)]^p}{V_*(s)}\biggr) = \sup_{s\ge t}\frac{[w^\downarrow(s)]^p}{V_*(s)}.\end{aligned}$$

Similarly, using the monotonicity of $V_*$, we find

$$\begin{aligned}\|H_{[\sigma^{-1}(x),\sigma(x)]}\|^{\frac{r}{p-r}}_{L^1_{V_*}\to L^\infty_{w^p}} &= \operatorname*{ess\,sup}_{s\ge\sigma^{-2}(x)}\frac{1}{V_*(s)}\operatorname*{ess\,sup}_{\{\sigma^{-1}(x)\le y\le\sigma(x)\}\cap\{y\ge s\}} w^p(y)\\ &= \sup_{\sigma^{-1}(x)\le s\le\sigma(x)}\frac{1}{V_*(s)}\sup_{\sigma^{-1}(x)\le y\le\sigma(x)}\operatorname*{ess\,sup}_{s\le y\le\sigma(x)} w^p(y)\\ &= \operatorname*{ess\,sup}_{\sigma^{-1}(x)\le y\le\sigma(x)}\frac{w^p(y)}{V_*(y)}\end{aligned}$$

and the estimate $\mathscr{A}_1 \approx \mathscr{A}_1'$ follows.

For the second part we observe that $C_S \approx \mathscr{B}_0' + \mathscr{B}_1'$, where $\mathscr{B}_0'$ is the least constant in the inequality

$$\biggl(\int_0^\infty u(x)\Bigl[\operatorname*{ess\,sup}_{x\le y\le\sigma^2(x)} w(y)\Bigr]^r\biggl(\int_{\sigma^2(x)}^\infty h\biggr)^{\frac{r}{p}}dx\biggr)^{\frac{p}{r}} \le \bigl[\mathscr{B}_0'\bigr]^p\|h\|_{L^1_V},\quad h\in\mathfrak{M}^+, \tag{4.7}$$

and

$$\bigl[\mathscr{B}_1'\bigr]^p = \begin{cases}\sup_{t>0}(\int_0^t u)^{\frac{p}{r}}\|H_t^*\|_{L^1_V\to L^\infty_{w^p}}, & r\ge p,\\ (\int_0^\infty u(x)(\int_0^x u)^{\frac{r}{p-r}}\|H^*_{[\sigma^{-1}(x),\sigma(x)]}\|^{\frac{r}{p-r}}_{L^1_V\to L^\infty_{w^p}}dx)^{\frac{p-r}{r}}, & 0<r<p.\end{cases}$$

By a change of variables we see that (4.7) is equivalent to

$$\biggl(\int_0^\infty u(x)\Bigl[\operatorname*{ess\,sup}_{x\le y\le\sigma^2(x)} w(y)\Bigr]^r\biggl(\int_x^\infty h\biggr)^{\frac{r}{p}}dx\biggr)^{\frac{p}{r}} \le \bigl[\mathscr{B}_0'\bigr]^p\|h\|_{L^1_{V_{\sigma^2}}},\quad h\in\mathfrak{M}^+, \tag{4.8}$$

where $V_{\sigma^2}(y) := V(\sigma^2(t))$. By the same argument as above it follows that $\mathscr{B}_0' \approx \mathscr{B}_0$ and $\mathscr{B}_1' \approx \mathscr{B}_1$. □

Analogously, we obtain the sharp estimates for the best constants in (1.4) and (1.5).

Suppose for simplicity that $0 < \int_t^\infty u < \infty$ for all $t > 0$, $\int_0^\infty u = \infty$ and define the functions $\zeta : [0;\infty) \to [0;\infty)$, $\zeta^{-1} : [0;\infty) \to [0;\infty)$, by

$$\begin{aligned}\zeta(x) &:= \sup\biggl\{y>0 : \int_y^\infty u \ge \frac{1}{2}\int_x^\infty u\biggr\},\\ \zeta^{-1}(x) &:= \sup\biggl\{y>0 : \int_y^\infty u \ge 2\int_x^\infty u\biggr\}.\end{aligned}$$

Let $\zeta^2 := \zeta(\zeta)$. For $0 \le c < d < \infty$ and $h \in \mathfrak{M}^+$ we put

$$\mathscr{H}_d h(x) := \chi_{(0,d]}(x) \int_x^\infty h,$$

$$\mathscr{H}_{c,d} h(x) := \chi_{(c,d]}(x) \int_x^{\zeta(d)} h,$$

$$\mathscr{H}_c^* h(x) := \chi_{(0,d]}(x) \int_0^x h,$$

$$\mathscr{H}_{c,d}^* h(x) := \chi_{(c,d]}(x) \int_{\zeta^{-1}(c)}^x h.$$

We need the following partial cases of [21], Theorems 3.1 and 3.2.

**Theorem 4.3** *Let* $0 < r < \infty$. *Then*:

(a) *For validity of the inequality*

$$\left( \int_0^\infty \left[ \operatorname*{ess\,sup}_{0 \le y \le x} w_0(y) \int_y^\infty h \right]^r u(x)\,dx \right)^{\frac{1}{r}} \le C_2 \|h\|_{L^1_{v_0}}, \quad h \in \mathfrak{M}^+,$$

*it is necessary and sufficient that the inequality*

$$\left( \int_0^\infty u(x) \left[ w_0^\uparrow(x) \right]^r \left( \int_x^\infty h \right)^r dx \right)^{\frac{1}{r}} \le D_0 \|h\|_{L^1_{v_0}}, \quad h \in \mathfrak{M}^+,$$

*holds and the constant*

$$D_1 := \begin{cases} \sup_{t>0} (\int_t^\infty u)^{\frac{1}{r}} \|\mathscr{H}_t\|_{L^1_{v_0} \to L^\infty_{w_0}}, & r \ge 1, \\ (\int_0^\infty u(x) (\int_x^\infty u)^{\frac{r}{1-r}} \|\mathscr{H}_{[\zeta^{-1}(x), \zeta(x)]}\|^{\frac{r}{1-r}}_{L^1_{v_0} \to L^\infty_{w_0}}\,dx)^{\frac{1-r}{r}}, & 0 < r < 1, \end{cases}$$

*is finite*. *Moreover*, $C_2 \approx D_0 + D_1$.

(b) *For validity of the inequality*

$$\left( \int_0^\infty \left[ \operatorname*{ess\,sup}_{0 \le y \le x} w_0(y) \int_0^y h \right]^r u(x)\,dx \right)^{\frac{1}{r}} \le C_3 \|h\|_{L^1_{v_0}}, \quad h \in \mathfrak{M}^+,$$

*it is necessary and sufficient that the inequality*

$$\left( \int_0^\infty u(x) \left[ \operatorname*{ess\,sup}_{\zeta^{-2}(x) \le y \le x} w_0(y) \right]^r \left( \int_0^{\zeta^{-2}(x)} h \right)^r dx \right)^{\frac{1}{r}} \le E_0 \|h\|_{L^1_{v_0}}, \quad h \in \mathfrak{M}^+,$$

*holds and the constant*

$$E_1 := \begin{cases} \sup_{t>0} (\int_t^\infty u)^{\frac{1}{r}} \|\mathscr{H}_t^*\|_{L^1_{v_0} \to L^\infty_{w_0}}, & r \ge 1, \\ (\int_0^\infty u(x) (\int_x^\infty u)^{\frac{r}{1-r}} \|\mathscr{H}^*_{[\zeta^{-1}(x), \zeta(x)]}\|^{\frac{r}{1-r}}_{L^1_{v_0} \to L^\infty_{w_0}}\,dx)^{\frac{1-r}{r}}, & 0 < r < 1, \end{cases}$$

*is finite*. *Moreover*, $C_3 \approx E_0 + E_1$.

Using Theorem 4.3 we characterize (1.4) and (1.5) with $k(x,y) = 1$.

**Theorem 4.4** *Let* $0 < p, r < \infty$ *and* $k(x,y) = 1$. *Then for the best possible constants of the inequalities* (1.4) *and* (1.5) *the following equivalences hold*:

$$C_{\mathscr{T}} \approx \mathscr{D}_0 + \mathscr{D}_1, \qquad C_{\mathscr{S}} \approx \mathscr{E}_0 + \mathscr{E}_1,$$

*where*

$$\mathscr{D}_0 = \sup_{t>0} \bigl[V(t)\bigr]^{-\frac{1}{p}} \left(\int_0^t u\bigl[w^{\uparrow}\bigr]^r\right)^{\frac{1}{r}}, \quad r \geq p,$$

$$\mathscr{D}_0 = \left(\int_0^\infty \left(\bigl[V(x)\bigr]^{-1} \int_0^x u\bigl[w^{\uparrow}\bigr]^r\right)^{\frac{r}{p-r}} u(x)\bigl[w^{\uparrow}(x)\bigr]^r\,dx\right)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathscr{D}_1 = \sup_{t>0} \left(\int_t^\infty u\right)^{\frac{1}{r}} \sup_{0<y<t} \frac{w^{\uparrow}(y)}{[V(y)]^{\frac{1}{p}}}, \quad r \geq p,$$

$$\mathscr{D}_1 = \left(\int_0^\infty u(x)\left(\int_x^\infty u\right)^{\frac{r}{p-r}} \left(\operatorname*{ess\,sup}_{\zeta^{-1}(x)\leq y\leq \zeta(x)} \frac{[w(y)]^p}{V(y)}\right)^{\frac{r}{p-r}} dx\right)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathscr{E}_0 = \sup_{t>0} \bigl[V_*\bigl(\zeta^{-2}(t)\bigr)\bigr]^{-\frac{1}{p}} \left(\int_t^\infty u(x)\Bigl[\operatorname*{ess\,sup}_{\zeta^{-2}(x)\leq y\leq x} w(y)\Bigr]^r dx\right)^{\frac{1}{r}}, \quad r \geq p,$$

$$\begin{aligned}\mathscr{E}_0 = {}&\left(\int_0^\infty \left(\bigl[V_*\bigl(\zeta^{-2}(z)\bigr)\bigr]^{-1} \int_z^\infty u(x)\Bigl[\operatorname*{ess\,sup}_{\zeta^{-2}(x)\leq y\leq x} w(y)\Bigr]^r dx\right)^{\frac{r}{p-r}}\right.\\ &\left.\times\, u(z)\Bigl[\operatorname*{ess\,sup}_{\zeta^{-2}(z)\leq y\leq z} w(y)\Bigr]^r dz\right)^{\frac{p-r}{pr}}, \quad 0 < r < p,\end{aligned}$$

$$\mathscr{E}_1 = \sup_{t>0} \left(\int_t^\infty u\right)^{\frac{1}{r}} \operatorname*{ess\,sup}_{0\leq y\leq t} \frac{w(y)}{[V_*(y)]^{\frac{1}{p}}}, \quad r \geq p,$$

$$\mathscr{E}_1 = \left(\int_0^\infty u(x)\left(\int_x^\infty u\right)^{\frac{r}{p-r}} \left(\operatorname*{ess\,sup}_{\zeta^{-1}(x)\leq y\leq \zeta(x)} \frac{[w(y)]^p}{V_*(y)}\right)^{\frac{r}{p-r}} dx\right)^{\frac{p-r}{pr}}, \quad 0 < r < p.$$

## 5 Main results

To deal with the kernel transformation we need the following extension of Theorem 4.1 following from [21], Theorems 4.1 and 4.3.

**Theorem 5.1** *Let* $0 < r < \infty$, $u, v_0, w_0 \in \mathfrak{M}^+$ *and* $k_0(x,y)$ *satisfies Oinarov's condition* (1.1). *Then*:

(a) *For validity of the inequality*

$$\left(\int_0^\infty \left[\operatorname*{ess\,sup}_{y\geq x} k_0(y,x) w_0(y) \int_0^y h\right]^r u(x)\,dx\right)^{\frac{1}{r}} \leq \mathbf{C}_0 \|h\|_{L^1_{v_0}}, \quad h \in \mathfrak{M}^+, \tag{5.1}$$

*it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x)\Bigl[\operatorname*{ess\,sup}_{y\geq x} k_0(y,x) w_0(y)\Bigr]^r \left(\int_0^x h\right)^r dx\right)^{\frac{1}{r}} \leq \mathbf{A}_0 \|h\|_{L^1_{v_0}}, \quad h \in \mathfrak{M}^+,$$

*and*

$$\left(\int_0^\infty u(x)\big[k_0\big(\sigma^2(x),x\big)\big]^r\left(\operatorname*{ess\,sup}_{y\ge\sigma^2(x)} w_0(y)\int_0^y h\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{A}_1\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*hold and the constant*

$$\mathbf{A}_2 := \begin{cases} \sup_{t>0}(\int_0^t u)^{\frac{1}{r}}\|H_t\|_{L^1_{v_0}\to L^\infty_{w_0(\cdot)k_0(\cdot,t)}}, & r\ge 1,\\ (\int_0^\infty u(x)(\int_0^x u)^{\frac{r}{1-r}}\|H_{[\sigma^{-1}(x),\sigma^2(x)]}\|^{\frac{r}{1-r}}_{L^1_{v_0}\to L^\infty_{w_0(\cdot)k_0(\cdot,\sigma^{-1}(x))}}\,dx)^{\frac{1-r}{r}}, & r<1,\end{cases}$$

*is finite. Moreover,* $\mathbf{C}_0\approx\mathbf{A}_0+\mathbf{A}_1+\mathbf{A}_2$.

(b) *For validity of the inequality*

$$\left(\int_0^\infty\left[\operatorname*{ess\,sup}_{y\ge x} k_0(y,x)w_0(y)\int_y^\infty h\right]^r u(x)\,dx\right)^{\frac{1}{r}} \le \mathbf{C}_1\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+, \tag{5.2}$$

*it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x)\Big[\operatorname*{ess\,sup}_{x\le y\le\sigma^3(x)} k_0(y,x)w_0(y)\Big]^r\left(\int_{\sigma^3(x)}^\infty h\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{B}_0\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*and*

$$\left(\int_0^\infty u(x)\big[k_0\big(\sigma^2(x),x\big)\big]^r\left(\operatorname*{ess\,sup}_{y\ge\sigma^2(x)} w_0(y)\int_y^\infty h\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{B}_1\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*hold and the constant*

$$\mathbf{B}_2 := \begin{cases} \sup_{t>0}(\int_0^t u)^{\frac{1}{r}}\|H_t^*\|_{L^1_{v_0}\to L^\infty_{w_0(\cdot)k_0(\cdot,t)}}, & r\ge 1,\\ (\int_0^\infty u(x)(\int_0^x u)^{\frac{r}{1-r}}\|H^*_{[\sigma^{-1}(x),\sigma^2(x)]}\|^{\frac{r}{1-r}}_{L^1_{v_0}\to L^\infty_{w_0(\cdot)k_0(\cdot,\sigma^{-1}(x))}}\,dx)^{\frac{1-r}{r}}, & r<1,\end{cases}$$

*is finite. Moreover,* $\mathbf{C}_0\approx\mathbf{B}_0+\mathbf{B}_1+\mathbf{B}_2$.

Using Theorem 5.1 we obtain the characterization of (1.2) and (1.3) for $0<p,r<\infty$. Denote

$$W_k(x) := \operatorname*{ess\,sup}_{y\ge x} k(y,x)w(y), \qquad \mathscr{W}_k(x) := \operatorname*{ess\,sup}_{x\le y\le\sigma^3(x)} k(y,x)w(y),$$

$$\mathbf{w}_\sigma(w)(x) := \operatorname*{ess\,sup}_{x\le y\le\sigma^2(x)} w(y),$$

$$k_\sigma(x) := k\big(\sigma^2(x),x\big), \qquad g_{\sigma^k}(y) := g\big(\sigma^k(y)\big),$$

$$\sigma_0(x) := \inf\left\{y>0 : \int_0^y u[k_\sigma]^r \ge 2\int_0^x u[k_\sigma]^r\right\},$$

$$\sigma_0^{-1}(x) := \inf\left\{y>0 : \int_0^y u[k_\sigma]^r \ge \frac{1}{2}\int_0^x u[k_\sigma]^r\right\}.$$

**Theorem 5.2** *Let* $0 < p, r < \infty$. *Then, for the best possible constants of the inequalities* (1.2) *and* (1.3) *the following equivalences hold*:

$$C_T \approx \mathbb{A}_0 + \mathbb{A}_{1,0} + \mathbb{A}_{1,1} + \mathbb{A}_2, \qquad C_S \approx \mathbb{B}_0 + \mathbb{B}_{1,0} + \mathbb{B}_{1,1} + \mathbb{B}_2, \tag{5.3}$$

*where*

$$\mathbb{A}_0 = \sup_{t>0}\bigl[V_*(t)\bigr]^{-\frac{1}{p}}\biggl(\int_t^\infty u[W_k]^r\biggr)^{\frac{1}{r}}, \quad r \ge p,$$

$$\mathbb{A}_0 = \biggl(\int_0^\infty\biggl(\bigl[V_*(x)\bigr]^{-1}\int_x^\infty u[W_k]^r\biggr)^{\frac{r}{p-r}} u(x)\bigl[W_k(x)\bigr]^r\,dx\biggr)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathbb{A}_{1,0} = \sup_{t>0}\bigl[(V_*)_{\sigma^2}(t)\bigr]^{-\frac{1}{p}}\biggl(\int_t^\infty u\bigl[k_\sigma w^{\downarrow}_{\sigma^2}\bigr]^r\biggr)^{\frac{1}{r}}, \quad r \ge p,$$

$$\mathbb{A}_{1,0} = \biggl(\int_0^\infty\biggl(\bigl[(V_*)_{\sigma^2}(x)\bigr]^{-1}\int_x^\infty u\bigl[k_\sigma w^{\downarrow}_{\sigma^2}\bigr]^r\biggr)^{\frac{r}{p-r}} \times u(x)\bigl[k_\sigma(x)w^{\downarrow}_{\sigma^2}(x)\bigr]^r\,dx\biggr)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathbb{A}_{1,1} = \sup_{t>0}\biggl(\int_0^t u[k_\sigma]^r\biggr)^{\frac{1}{r}} \operatorname*{ess\,sup}_{y\ge t}\frac{w_{\sigma^2}(y)}{[(V_*)_{\sigma^2}(y)]^{\frac{1}{p}}}, \quad r \ge p,$$

$$\mathbb{A}_{1,1} = \biggl(\int_0^\infty u(x)\bigl[k_\sigma(x)\bigr]^r\biggl(\int_0^x u[k_\sigma]^r\biggr)^{\frac{r}{p-r}} \times\biggl(\operatorname*{ess\,sup}_{\sigma_0^{-1}(x)\le y\le\sigma_0(x)}\frac{[w_{\sigma^2}(y)]^p}{(V_*)_{\sigma^2}(y)}\biggr)^{\frac{r}{p-r}}dx\biggr)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathbb{A}_2 = \sup_{t>0}\biggl(\int_0^t u\biggr)^{\frac{1}{r}} \operatorname*{ess\,sup}_{y\ge t}\frac{w(y)k(y,t)}{[V_*(y)]^{\frac{1}{p}}}, \quad r \ge p,$$

$$\mathbb{A}_2 = \biggl(\int_0^\infty u(x)\biggl(\int_0^x u\biggr)^{\frac{r}{p-r}} \times\biggl(\operatorname*{ess\,sup}_{\sigma^{-1}(x)\le y\le\sigma^2(x)}\frac{[w(y)k(y,\sigma^{-1}(x))]^p}{V_*(y)}\biggr)^{\frac{r}{p-r}}dx\biggr)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathbb{B}_0 = \sup_{t>0}\bigl[V_{\sigma^3}(t)\bigr]^{-\frac{1}{p}}\biggl(\int_0^t u[\mathscr{W}_k]^r\biggr)^{\frac{1}{r}}, \quad r \ge p,$$

$$\mathbb{B}_0 = \biggl(\int_0^\infty\biggl(\bigl[V_{\sigma^3}(z)\bigr]^{-1}\int_0^z u[\mathscr{W}_k]^r\biggr)^{\frac{r}{p-r}} u(z)\bigl[\mathscr{W}_k(z)\bigr]^r\,dz\biggr)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathbb{B}_{1,0} = \sup_{t>0}\bigl[V_{\sigma^3}\bigl(\sigma_0^2(t)\bigr)\bigr]^{-\frac{1}{p}}\biggl(\int_0^t u\bigl[k_\sigma\mathbf{w}_{\sigma_0}(w_{\sigma^3})\bigr]^r\biggr)^{\frac{1}{r}}, \quad r \ge p,$$

$$\mathbb{B}_{1,0} = \biggl(\int_0^\infty\biggl(\bigl[V_{\sigma^2}\bigl(\sigma_0^2(t)\bigr)(z)\bigr]^{-1}\int_0^z u[k_\sigma]^r\bigl[\mathbf{w}_{\sigma_0}(w_{\sigma^3})\bigr]^r\biggr)^{\frac{r}{p-r}} \times u(z)\bigl[k_\sigma(z)\mathbf{w}_{\sigma_0}(w_{\sigma^3})(z)\bigr]^r\,dz\biggr)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathbb{B}_{1,1} = \sup_{t>0}\left(\int_0^t u[k_\sigma]^r\right)^{\frac{1}{r}} \operatorname*{ess\,sup}_{y\geq t} \frac{w_{\sigma^3}(y)}{[V_{\sigma^3}(y)]^{\frac{1}{p}}}, \quad r \geq p,$$

$$\mathbb{B}_{1,1} = \left(\int_0^\infty u(x)\big[k_\sigma(x)\big]^r \left(\int_0^x u[k_\sigma]^r\right)^{\frac{r}{p-r}} \times \left(\operatorname*{ess\,sup}_{\sigma_0^{-1}(x)\leq y\leq \sigma_0(x)} \frac{[w_{\sigma^3}(y)]^p}{V_{\sigma^3}(y)}\right)^{\frac{r}{p-r}} dx\right)^{\frac{p-r}{pr}}, \quad 0 < r < p,$$

$$\mathbb{B}_2 = \sup_{t>0}\left(\int_0^t u\right)^{\frac{1}{r}} \operatorname*{ess\,sup}_{y\geq t} \frac{w(y)k(y,t)}{[V(y)]^{\frac{1}{p}}}, \quad r \geq p,$$

$$\mathbb{B}_2 = \left(\int_0^\infty u(x)\left(\int_0^x u\right)^{\frac{r}{p-r}} \times \left(\operatorname*{ess\,sup}_{\sigma^{-1}(x)\leq y\leq \sigma^2(x)} \frac{[w(y)k(y,\sigma^{-1}(x))]^p}{V(y)}\right)^{\frac{r}{p-r}} dx\right)^{\frac{p-r}{pr}}, \quad 0 < r < p.$$

*Proof* We start with the inequality (1.2). Since (1.2) ⇔ (2.1), then applying Theorem 5.1 we see that

$$C_T \approx \mathbb{A}_0' + \mathbb{A}_1' + \mathbb{A}_2',$$

where $\mathbb{A}_0'$ and $\mathbb{A}_1'$ are the best constants in the inequalities

$$\begin{aligned} &\left(\int_0^\infty u(x)\Big[\operatorname*{ess\,sup}_{y\geq x} k(y,x)w(y)\Big]^r \left(\int_0^x h\right)^{\frac{r}{p}} dx\right)^{\frac{p}{r}} \leq \big[\mathbb{A}_0'\big]^p \|h\|_{L^1_{V_*}}, \quad h \in \mathfrak{M}^+, \\ &\left(\int_0^\infty u(x)\big[k\big(\sigma^2(x),x\big)\big]^r \left(\operatorname*{ess\,sup}_{y\geq\sigma^2(x)}\big[w(y)\big]^p \int_0^y h\right)^{\frac{r}{p}} dx\right)^{\frac{p}{r}} \leq \big[\mathbb{A}_1'\big]^p \|h\|_{L^1_{V_*}}, \quad h \in \mathfrak{M}^+, \end{aligned} \tag{5.4}$$

and

$$\big[\mathbb{A}_2'\big]^p := \begin{cases} \sup_{t>0}(\int_0^t u)^{\frac{p}{r}} \|H_t\|_{L^1_{V_*}\to L^\infty_{[w(\cdot)k(\cdot,t)]^p}}, & r\geq p, \\ (\int_0^\infty u(x)(\int_0^x u)^{\frac{r}{p-r}} \|H_{[\sigma^{-1}(x),\sigma^2(x)]}\|^{\frac{r}{p-r}}_{L^1_{V_*}\to L^\infty_{[w(\cdot)k(\cdot,\sigma^{-1}(x))]^p}} dx)^{\frac{p-r}{r}}, & 0<r<p. \end{cases}$$

Applying (4.5) and (4.6) we see that $\mathbb{A}_0' \approx \mathbb{A}_0$ and $\mathbb{A}_2' \approx \mathbb{A}_2$. By a change of variable we find that (5.4) is equivalent to the inequality

$$\left(\int_0^\infty u(x)\big[k_\sigma(x)\big]^r \left(\operatorname*{ess\,sup}_{y\geq x}\big[w_{\sigma^2}(y)\big]^p \int_0^y h\right)^{\frac{r}{p}} dx\right)^{\frac{p}{r}} \leq \big[\mathbb{A}_1'\big]^p \|h\|_{L^1_{[V_*]_{\sigma^2}}}, \quad h \in \mathfrak{M}^+, \tag{5.5}$$

which is governed by Theorem 4.1. Arguing analogously to the proof of Theorem 4.2 we see that

$$\mathbb{A}_1' \approx \mathbb{A}_{1,0}' + \mathbb{A}_{1,1}',$$

where $\mathbb{A}'_{1,0}$ is the best constant of the inequality

$$\left(\int_0^\infty u(x)\big[k_\sigma(x)\big]^r\big[w^{\downarrow}_{\sigma^2}(x)\big]^r\left(\int_0^x h\right)^{\frac{r}{p}}dx\right)^{\frac{p}{r}} \le \big[\mathbb{A}'_{1,0}\big]^p\|h\|_{L^1_{[V*]_{\sigma^2}}}, \quad h\in\mathfrak{M}^+,$$

and

$$\big[\mathbb{A}'_{1,1}\big]^p := \sup_{t>0}\left(\int_0^t u[k_\sigma]^r\right)^{\frac{p}{r}}\|H_t\|_{L^1_{[V*]_{\sigma^2}}\to L^\infty_{w^p_{\sigma^2}}}, \quad r\ge p,$$

$$\big[\mathbb{A}'_{1,1}\big]^p := \left(\int_0^\infty u(x)\big[k_\sigma(x)\big]^r\left(\int_0^x u[k_\sigma]^r\right)^{\frac{r}{p-r}}\|H_{[\sigma_0^{-1}(x),\sigma_0(x)]}\|^{\frac{r}{p-r}}_{L^1_{[V*]_{\sigma^2}}\to L^\infty_{w^p_{\sigma^2}}}dx\right)^{\frac{p-r}{r}},$$

for $0<r<p$. Again applying (4.5) and (4.6) we see that $\mathbb{A}'_{1,0}\approx\mathbb{A}_{1,0}$ and $\mathbb{A}'_{1,1}\approx\mathbb{A}_{1,1}$.

The proof for the inequality (1.3) is similar. □

Analogously, we obtain the sharp estimates for the best constants in (1.4) and (1.5). To this end we need the following extension of Theorem 4.3 from [21], Theorems 5.1 and 5.2.

**Theorem 5.3** *Let* $0<r<\infty$, $u,v_0,w_0\in\mathfrak{M}^+$ *and* $k_0(x,y)$ *satisfy Oinarov's condition* (1.1). *Then*:

(a) *For validity of the inequality*

$$\left(\int_0^\infty\left[\operatorname*{ess\,sup}_{0\le y\le x}k_0(x,y)w_0(y)\int_y^\infty h\right]^r u(x)\,dx\right)^{\frac{1}{r}}\le \mathbf{C}_0\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x)\Big[\operatorname*{ess\,sup}_{0\le y\le x}k_0(x,y)w_0(y)\Big]^r\left(\int_x^\infty h\right)^r dx\right)^{\frac{1}{r}}\le \mathbf{A}_0\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*and*

$$\left(\int_0^\infty u(x)\big[k_0\big(x,\zeta^{-2}(x)\big)\big]^r\left(\operatorname*{ess\,sup}_{0\le y\le\zeta^{-2}(x)}w_0(y)\int_y^\infty h\right)^r dx\right)^{\frac{1}{r}}$$
$$\le \mathbf{A}_1\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*hold and the constant*

$$\mathbf{A}_2 := \begin{cases}\sup_{t>0}(\int_t^\infty u)^{\frac{1}{r}}\|\mathscr{H}_t\|_{L^1_{v_0}\to L^\infty_{w_0(\cdot)k_0(t,\cdot)}}, & r\ge 1,\\ (\int_0^\infty u(x)(\int_x^\infty u)^{\frac{r}{1-r}}\|\mathscr{H}_{[\zeta^{-1}(x),\zeta^2(x)]}\|^{\frac{r}{1-r}}_{L^1_{v_0}\to L^\infty_{w_0(\cdot)k_0(\zeta^2(x),\cdot)}}dx)^{\frac{1-r}{r}}, & r<1,\end{cases}$$

*is finite. Moreover,* $\mathbf{C}_0\approx\mathbf{A}_0+\mathbf{A}_1+\mathbf{A}_2$.

(b) *For validity of the inequality*

$$\left(\int_0^\infty\left[\operatorname*{ess\,sup}_{0\le y\le x}k_0(y,x)w_0(y)\int_0^y h\right]^r u(x)\,dx\right)^{\frac{1}{r}}\le \mathbf{C}_1\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x)\Big[\operatorname*{ess\,sup}_{\zeta^{-3}(x)\le y\le x} k_0(x,y)w_0(y)\Big]^r\left(\int_0^{\zeta^{-3}(x)} h\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{B}_0\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*and*

$$\left(\int_0^\infty u(x)\big[k_0\big(x,\zeta^{-2}(x)\big)\big]^r\left(\operatorname*{ess\,sup}_{0\le y\le \zeta^{-2}(x)} w_0(y)\int_0^y h\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{B}_1\|h\|_{L^1_{v_0}}, \quad h\in\mathfrak{M}^+,$$

*hold and the constant*

$$\mathbf{B}_2 := \begin{cases} \sup_{t>0}(\int_t^\infty u)^{\frac{1}{r}}\|\mathscr{H}_t^*\|_{L^1_{v_0}\to L^\infty_{w_0(\cdot)k_0(t,\cdot)}}, & r\ge 1, \\ (\int_0^\infty u(x)(\int_x^\infty u)^{\frac{r}{1-r}}\|\mathscr{H}^*_{[\zeta^{-1}(x),\zeta^2(x)]}\|^{\frac{r}{1-r}}_{L^1_{v_0}\to L^\infty_{w_0(\cdot)k_0(\zeta^2(x),\cdot)}}\,dx)^{\frac{1-r}{r}}, & r<1, \end{cases}$$

*is finite. Moreover,* $\mathbf{C}_0 \approx \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2$.

Using Theorem 5.3 we obtain the characterization of (1.4) and (1.5) for $0 < p, r < \infty$. Denote

$$W_k^*(x) := \operatorname*{ess\,sup}_{0\le y\le x} k(x,y)w(y), \qquad \mathscr{W}_k^*(x) := \operatorname*{ess\,sup}_{\zeta^{-3}(x)\le y\le x} k(x,y)w(y),$$

$$\Omega_\zeta(w)(x) := \operatorname*{ess\,sup}_{\zeta^{-2}(x)\le y\le x} w(y),$$

$$k_\zeta(x) := k\big(x,\zeta^{-2}(x)\big), \qquad g_{\zeta^{-k}}(y) := g\big(\zeta^{-k}(y)\big),$$

$$\zeta_0(x) := \sup\left\{y>0 : \int_y^\infty u[k_\zeta]^r \ge \frac{1}{2}\int_x^\infty u[k_\zeta]^r\right\},$$

$$\zeta_0^{-1}(x) := \sup\left\{y>0 : \int_y^\infty u[k_\zeta]^r \ge 2\int_x^\infty u[k_\zeta]^r\right\}.$$

**Theorem 5.4** *Let* $0 < p, r < \infty$. *Then for the best possible constants of the inequalities* (1.4) *and* (1.5) *the following equivalences hold*:

$$C_{\mathscr{T}} \approx \mathbb{D}_0 + \mathbb{D}_{1,0} + \mathbb{D}_{1,1} + \mathbb{D}_2, \qquad C_{\mathscr{S}} \approx \mathbb{E}_0 + \mathbb{E}_{1,0} + \mathbb{E}_{1,1} + \mathbb{E}_2, \tag{5.6}$$

*where*

$$\mathbb{D}_0 = \sup_{t>0}\big[V(t)\big]^{-\frac{1}{p}}\left(\int_0^t u\big[W_k^*\big]^r\right)^{\frac{1}{r}}, \quad r\ge p,$$

$$\mathbb{D}_0 = \left(\int_0^\infty\left(\big[V(x)\big]^{-1}\int_0^x u\big[W_k^*\big]^r\right)^{\frac{r}{p-r}} u(x)\big[W_k^*(x)\big]^r\,dx\right)^{\frac{p-r}{pr}}, \quad 0<r<p,$$

$$\mathbb{D}_{1,0} = \sup_{t>0}\big[V_{\zeta^{-2}}(t)\big]^{-\frac{1}{p}}\left(\int_0^t u\big[k_\zeta w^\uparrow_{\zeta^{-2}}\big]^r\right)^{\frac{1}{r}}, \quad r\ge p,$$

$$\mathbb{D}_{1,0} = \left(\int_0^\infty\left(\big[V_{\zeta^{-2}}(x)\big]^{-1}\int_0^x u\big[k_\zeta w^\uparrow_{\zeta^{-2}}\big]^r\right)^{\frac{r}{p-r}}\right.$$

$$\times u(x)\big[k_\zeta(x)w^{\uparrow}_{\zeta^{-2}}(x)\big]^r\,dx\bigg)^{\frac{p-r}{pr}},\quad 0<r<p,$$

$$\mathbb{D}_{1,1}=\sup_{t>0}\bigg(\int_t^\infty u[k_\zeta]^r\bigg)^{\frac{1}{r}}\operatorname*{ess\,sup}_{0\le y\le t}\frac{w_{\zeta^{-2}}(y)}{[V_{\zeta^{-2}}(y)]^{\frac{1}{p}}},\quad r\ge p,$$

$$\mathbb{D}_{1,1}=\bigg(\int_0^\infty u(x)\big[k_\zeta(x)\big]^r\bigg(\int_x^\infty u[k_\zeta]^r\bigg)^{\frac{r}{p-r}}$$

$$\times\bigg(\operatorname*{ess\,sup}_{\zeta_0^{-1}(x)\le y\le\zeta_0(x)}\frac{[w_{\zeta^{-2}}(y)]^p}{V_{\zeta^{-2}}(y)}\bigg)^{\frac{r}{p-r}}dx\bigg)^{\frac{p-r}{pr}},\quad 0<r<p,$$

$$\mathbb{D}_2=\sup_{t>0}\bigg(\int_t^\infty u\bigg)^{\frac{1}{r}}\operatorname*{ess\,sup}_{0\le y\le t}\frac{w(y)k(t,y)}{[V(y)]^{\frac{1}{p}}},\quad r\ge p,$$

$$\mathbb{D}_2=\bigg(\int_0^\infty u(x)\bigg(\int_x^\infty u\bigg)^{\frac{r}{p-r}}$$

$$\times\bigg(\operatorname*{ess\,sup}_{\zeta^{-1}(x)\le y\le\zeta^2(x)}\frac{[w(y)k(\zeta^2(x),y)]^p}{V(y)}\bigg)^{\frac{r}{p-r}}dx\bigg)^{\frac{p-r}{pr}},\quad 0<r<p,$$

$$\mathbb{E}_0=\sup_{t>0}\big[(V_*)_{\zeta^{-3}}(t)\big]^{-\frac{1}{p}}\bigg(\int_t^\infty u\big[\mathscr{W}_k^*\big]^r\bigg)^{\frac{1}{r}},\quad r\ge p,$$

$$\mathbb{E}_0=\bigg(\int_0^\infty\bigg(\big[(V_*)_{\zeta^{-3}}(z)\big]^{-1}\int_z^\infty u\big[\mathscr{W}_k^*\big]^r\bigg)^{\frac{r}{p-r}}u(z)\big[\mathscr{W}_k^*(z)\big]^r\,dz\bigg)^{\frac{p-r}{pr}},\quad 0<r<p,$$

$$\mathbb{E}_{1,0}=\sup_{t>0}\big[(V_*)_{\zeta^{-2}}\big(\zeta_0^{-2}(t)\big)\big]^{-\frac{1}{p}}\bigg(\int_t^\infty u[k_\zeta]^r\big[\Omega_{\zeta_0}(w_{\zeta^{-2}})\big]^r\bigg)^{\frac{1}{r}},\quad r\ge p,$$

$$\mathbb{E}_{1,0}=\bigg(\int_0^\infty\bigg(\big[(V_*)_{\zeta^{-2}}\big(\zeta_0^{-2}(t)\big)(z)\big]^{-1}\int_z^\infty u[k_\zeta]^r\big[\Omega_{\zeta_0}(w_{\zeta^{-2}})\big]^r\bigg)^{\frac{r}{p-r}}$$

$$\times u(z)\big[k_\zeta(z)\big]^r\big[\Omega_{\zeta_0}(w_{\zeta^{-2}})\big]^r\,dz\bigg)^{\frac{p-r}{pr}},\quad 0<r<p,$$

$$\mathbb{E}_{1,1}=\sup_{t>0}\bigg(\int_t^\infty u[k_\zeta]^r\bigg)^{\frac{1}{r}}\operatorname*{ess\,sup}_{0\le y\le t}\frac{w_{\zeta^{-2}}(y)}{[(V_*)_{\zeta^{-2}}(y)]^{\frac{1}{p}}},\quad r\ge p,$$

$$\mathbb{E}_{1,1}=\bigg(\int_0^\infty u(x)\big[k_\zeta(x)\big]^r\bigg(\int_x^\infty u[k_\zeta]^r\bigg)^{\frac{r}{p-r}}$$

$$\times\bigg(\operatorname*{ess\,sup}_{\zeta_0^{-1}(x)\le y\le\zeta_0(x)}\frac{[w_{\zeta^{-2}}(y)]^p}{(V_*)_{\zeta^{-2}}(y)}\bigg)^{\frac{r}{p-r}}dx\bigg)^{\frac{p-r}{pr}},\quad 0<r<p,$$

$$\mathbb{E}_2=\sup_{t>0}\bigg(\int_t^\infty u\bigg)^{\frac{1}{r}}\operatorname*{ess\,sup}_{0\le y\le t}\frac{w(y)k(t,y)}{[V_*(y)]^{\frac{1}{p}}},\quad r\ge p,$$

$$\mathbb{E}_2=\bigg(\int_0^\infty u(x)\bigg(\int_x^\infty u\bigg)^{\frac{r}{p-r}}$$

$$\times\bigg(\operatorname*{ess\,sup}_{\zeta^{-1}(x)\le y\le\zeta^2(x)}\frac{[w(y)k(\zeta^2(x),y)]^p}{V_*(y)}\bigg)^{\frac{r}{p-r}}dx\bigg)^{\frac{p-r}{pr}},\quad 0<r<p.$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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