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A NOTE ON THE MAXIMAL OPERATORS OF VILENKIN–NÖRLUND MEANS WITH NON-INCREASING COEFFICIENTS*

N. MEMIĆ[1], L. E. PERSSON[2,3] and G. TEPHNADZE[2,4]

[1] Department of Mathematics, University of Sarajevo, Zmaja od Bosne 33-35, Sarajevo, Bosnia and Herzegovina
e-mail: nacima.o@gmail.com

[2] Department of Engineering Sciences and Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden
e-mail: larserik@ltu.se

[3] Narvik University College, P.O. Box 385, N-8505, Narvik, Norway

[4] Department of Mathematics, Faculty of Exact and Natural Sciences, Iv. Javakhishvili Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia
e-mail: giorgitephnadze@gmail.com

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Abstract

In [14] we investigated some Vilenkin–Nörlund means with non-increasing coefficients. In particular, it was proved that under some special conditions the maximal operators of such summabily methods are bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space *weak*-$L_{1/(1+\alpha)}$, $(0<\alpha\leqq 1)$. In this paper we construct a martingale in the space $H_{1/(1+\alpha)}$, which satisfies the conditions considered in [14], and so that the maximal operators of these Vilenkin–Nörlund means with non-increasing coefficients are not bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha)}$. In particular, this shows that the conditions under which the result in [14] is proved are in a sense sharp. Moreover, as further applications, some well-known and new results are pointed out.

1. Introduction and statement of the main result

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$. Let $m:=(m_0, m_1,\dots)$ be a sequence of the positive integers not less than 2. Denote by $Z_{m_k}:=\{0,1,\dots,m_k-1\}$ the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i}, with the product of the discrete topologies of Z_{m_j}.

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The direct product μ of the measures $\mu_k\big(\{j\}\big) := 1/m_k$ $(j \in Z_{m_k})$ is the Haar measure on G_m with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$.

The elements of G_m are represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$ $(x_j \in Z_{m_j})$.

It is easy to give a base for the neighborhoods of G_m:

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\},$$

where $x \in G_m$, $n \in \mathbb{N}$. Denote $I_n := I_n(0)$ for $n \in \mathbb{N}_+$, and $\overline{I_n} := G_m \backslash I_n$.

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of n_j's differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp\left(2\pi i x_k / m_k\right), \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh–Paley system when $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [17]).

The norm (or quasi-norm) of the space $L_p(G_m)$ and *weak*-$L_p(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \quad \|f\|_{\text{weak-}L_p}^p := \sup_{\lambda > 0} \lambda^p \mu\left(f > \lambda\right) < +\infty.$$

If $f \in L_1(G_m)$ we can respectively define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n \, d\mu, \quad (n \in \mathbb{N}),$$

$$S_nf := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+)$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } \ x \in I_n, \\ 0, & \text{if } \ x \notin I_n. \end{cases} \tag{1}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\digamma_n$ $(n \in \mathbb{N})$. Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ a martingale with respect to $\digamma_n$ $(n \in \mathbb{N})$. (for details see e.g. [18]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f = (f^{(n)}, n \in \mathbb{N})$ is a martingale, then the Vilenkin–Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)} \overline{\psi}_i \, d\mu.$$

A bounded measurable function a is a p-atom $(p > 0)$, if there exists an interval I, such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \leqq \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

We also need the following auxiliary result (see [19]):

Lemma 1. *A martingale* $f = \left(f^{(n)}, \ n \in \mathbb{N}\right)$ *is in* H_p $(0 < p \leqq 1)$ *if and only if there exists a sequence* $(a_k, \ k \in \mathbb{N})$ *of p-atoms and a sequence* $(\mu_k, \ k \in \mathbb{N})$ *of real numbers, such that, for every* $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{2}$$

Moreover, $\|f\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}$*, where the infimum is taken over all decompositions of* f *of the form* (2)*.*

Let $\{q_k : k \geqq 0\}$ be a sequence of nonnegative numbers. The n-th Nörlund mean for a Fourier series of f is defined by

$$t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \tag{3}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$.

We always assume that $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$. In this case it is well-known that the summability method generated by $\{q_k : k \geqq 0\}$ is regular if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

Concerning this fact and related basic results, we refer to [6].

The (C, α)-means (Cesáro means) of the Vilenkin–Fourier series are defined by

$$\sigma_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha = 0, \quad A_n^\alpha = \frac{(\alpha+1)\dots(\alpha+n)}{n!}, \quad \alpha \neq -1, -2, \dots$$

When $\alpha = 1$ the Cesáro means coincide with the Fejér means

$$\sigma_n f = \frac{1}{n} \sum_{k=1}^{n} S_k f.$$

For the martingale f we consider the following maximal operators:

$$t^* f := \sup_{n\in\mathbb{N}} |t_n f|, \quad \sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|, \quad \sigma^{\alpha,*} f := \sup_{n\in\mathbb{N}} \left| \sigma_n^\alpha f \right|.$$

In the one-dimensional case the result with respect to the a.e. convergence of Fejér is due to Pál and Simon [11] (c.f. also [2]) for bounded Vilenkin series. Weisz [20] proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. Simon [12] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given in [16].

In [4] Goginava investigated the behaviour of Cesáro means of Walsh–Fourier series in detail. The a.e. convergence of Cesáro means of $f \in L_1$ was proved in [5]. Furthermore, Simon and Weisz [13] showed that the maximal operator $\sigma^{\alpha,*}$ $(0<\alpha<1)$ of the (C,α) means is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space *weak*-$L_{1/(1+\alpha)}$. Moreover, Goginava [3] gave a counterexample, which shows that boundedness does not hold for $0<p\leqq 1/(1+\alpha)$.

Móricz and Siddiqi [7] investigated the approximation properties of some special Nörlund means of Walsh–Fourier series of L_p functions in norm. In the two-dimensional case approximation properties of Nörlund was considered by Nagy (see [8]–[10]). In [1] and [15] it was proved strong convergence theorems for Nörlund means of Vilenkin–Fourier series with monotone coefficients. Moreover, there was also shown boundedness of weighted maximal operators of such Nörlund means on martingale Hardy spaces. Recently, in [14] it was proved that the following is true:

THEOREM A. a) *Let* $0<\alpha\leqq 1$. *Then the maximal operator* t^* *of summability method* (3) *with non-increasing sequence* $\{q_k : k\geqq 0\}$, *satisfying the conditions*

$$\frac{n^{\alpha}q_0}{Q_n}=O(1),\qquad \frac{|q_n-q_{n+1}|}{n^{\alpha-2}}=O(1),\qquad \text{as } n\to\infty, \tag{4}$$

is bounded from the Hardy space $H_{1/(1+\alpha)}$ *to the space weak*-$L_{1/(1+\alpha)}$.

b) *Let* $0<\alpha\leqq 1$, $0\leqq p<1/(1+\alpha)$ *and* $\{q_k : k\geqq 0\}$ *be a non-increasing sequence, satisfying the condition*

$$\frac{q_0}{Q_n}\geqq\frac{c}{n^{\alpha}},\qquad (c>0). \tag{5}$$

Then there exists a martingale $f\in H_p$, *such that*

$$\sup_{n\in\mathbb{N}}\|t_nf\|_{weak\text{-}L_p}=\infty.$$

c) *Let* $\{q_k : k\geqq 0\}$ *be a non-increasing sequence, satisfying the condition*

$$\overline{\lim_{n\to\infty}}\,\frac{q_0n^{\alpha}}{Q_n}=\infty,\qquad (0<\alpha\leqq 1). \tag{6}$$

Then there exists an martingale $f\in H_{1/(1+\alpha)}$, *such that*

$$\sup_{n\in\mathbb{N}}\|t_nf\|_{weak\text{-}L_{1/(1+\alpha)}}=\infty.$$

In this paper we complement this result by proving sharpness of both conditions of (4). Our main result reads:

Theorem 1. *Let* $0 < \alpha \leqq 1$ *and* $\{q_k : k \geqq 0\}$ *be a non-increasing sequence, satisfying the conditions*

$$\overline{\lim_{n\to\infty}} \frac{|q_n - q_{n+1}|}{n^{\alpha-2}} \geqq c, \quad (c > 0), \tag{7}$$

and

$$\frac{n^\alpha q_0}{Q_n} \geqq c, \quad (c > 0,\ n \in \mathbb{N}). \tag{8}$$

Then there exists a martingale $f \in H_{1/(1+\alpha)}$*, such that*

$$\sup_{n\in\mathbb{N}} \|t_n f\|_{1/(1+\alpha)} = \infty.$$

The proof can be found in the Section 2 and some applications and final remark in the Section 3.

2. Proof of Theorem 1

Proof. Under the condition (7), there exists an increasing sequence $\{n_k : k \in \mathbb{N}\}$ of positive integers such that

$$\frac{M^\alpha_{2n_k+1}}{Q_{M_{2n_k}+1}} > c_\alpha > 0, \quad k \in \mathbb{N}. \tag{9}$$

Let $\{\alpha_k : k \in \mathbb{N}\} \in \{n_k : k \in \mathbb{N}\}$ be an increasing sequence of positive integers such that:

$$\sum_{k=0}^{\infty} 1/\alpha_k^{1/(1+\alpha)} < \infty, \tag{10}$$

$$\lambda \sum_{\eta=0}^{k-1} \frac{M^{1+\alpha}_{\alpha_\eta}}{\alpha_\eta} < \frac{M^{1+\alpha}_{\alpha_k}}{\alpha_k} \tag{11}$$

and

$$\frac{32\lambda M^{1+\alpha}_{\alpha_{k-1}}}{\alpha_{k-1}} < \frac{M^{\alpha+1}_{[\alpha_k/2]}}{\alpha_k}, \tag{12}$$

where $\lambda = \sup_n m_n$ and $[\alpha_k/2]$ denotes the integer part of $\alpha_k/2$.

We note that such increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$ which satisfies conditions (10)–(12) can be constructed.

Let the martingale $f := \left(f^{(n)} : n \in \mathbb{N}\right)$ be defined by

$$f^{(n)} = \sum_{\{k:\, \alpha_k < n\}} \lambda_k \theta_{\alpha_k}, \tag{13}$$

where

$$\lambda_k = \frac{\lambda}{\alpha_k} \quad \text{and} \quad \theta_{\alpha_k} = \frac{M_{\alpha_k}^{\alpha}}{\lambda}\left(D_{M_{\alpha_k+1}} - D_{M_{\alpha_k}}\right). \tag{14}$$

Since

$$S_{M_A}\theta_k = \begin{cases} \theta_k, & \text{if } \alpha_k < A, \\ 0, & \text{if } \alpha_k \geqq A, \end{cases}$$

$$\operatorname{supp}(\theta_k) = I_{\alpha_k}, \quad \int_{I_{\alpha_k}} \theta_k \, d\mu = 0, \quad \|\theta_k\|_\infty \leqq M_{\alpha_k}^{1+\alpha} = (\operatorname{supp}\theta_k)^{1+\alpha},$$

if we apply Lemma 1 and (10) we can conclude that $f \in H_{1/(1+\alpha)}$.

Moreover, it is easy to see that

$$\widehat{f}(j) = \begin{cases} \frac{M_{\alpha_k}^{\alpha}}{\alpha_k}, & \text{if } j \in \left\{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1\right\}, \ k = 0, 1, 2 \ldots, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \left\{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1\right\}. \end{cases} \tag{15}$$

Let $s = 0, \ldots, k-1$. We can write that

$$\begin{aligned} & t_{M_{\alpha_k}+M_s} f \\ = & \frac{1}{Q_{M_{\alpha_k}+M_s}} \sum_{j=0}^{M_{\alpha_k}} q_j S_j f + \frac{1}{Q_{M_{\alpha_k}+M_s}} \sum_{j=M_{\alpha_k}+1}^{M_{\alpha_k}+M_s} q_j S_j f \\ := & I + II. \end{aligned}$$

Let $M_{\alpha_s} \leqq j \leqq M_{\alpha_s+1}$, where $s = 0, \ldots, k-1$. Moreover,

$$\left| D_j - D_{M_{\alpha_s}} \right| \leqq 2j \leqq \lambda M_{\alpha_s}, \quad (s \in \mathbb{N})$$

so that, according to (1) and (15), we have that

$$
\begin{aligned}
(16)\qquad & |S_jf| \\
&= \left| \sum_{v=0}^{M_{\alpha_{s-1}+1}-1} \widehat{f}(v)\psi_v + \sum_{v=M_{\alpha_s}}^{j-1} \widehat{f}(v)\psi_v \right| \\
&\leqq \left| \sum_{\eta=0}^{s-1} \sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1} \frac{M^{\alpha}_{\alpha_\eta}}{\alpha_\eta} \psi_v \right| \\
&\quad + \frac{M^{\alpha}_{\alpha_s}}{\alpha_s} \left| \left(D_j - D_{M_{\alpha_s}} \right) \right| \\
&= \left| \sum_{\eta=0}^{s-1} \frac{M^{\alpha}_{\alpha_\eta}}{\alpha_\eta} \left(D_{M_{\alpha_\eta+1}} - D_{M_{\alpha_\eta}} \right) \right| \\
&\quad + \frac{M^{\alpha}_{\alpha_s}}{\alpha_s} \left| \left(D_j - D_{M_{\alpha_s}} \right) \right| \\
&\leqq \lambda \sum_{\eta=0}^{s-1} \frac{M^{\alpha+1}_{\alpha_\eta}}{\alpha_\eta} + \frac{\lambda M^{\alpha+1}_{\alpha_s}}{\alpha_s} \\
&\leqq \frac{\lambda M^{\alpha+1}_{\alpha_s}}{\alpha_s} + \frac{\lambda M^{\alpha+1}_{\alpha_s}}{\alpha_s} \leqq \frac{2\lambda M^{\alpha+1}_{\alpha_{k-1}}}{\alpha_{k-1}}.
\end{aligned}
$$

Let $M_{\alpha_{s-1}+1} + 1 \leqq j \leqq M_{\alpha_s}$, where $s = 1, \dots, k$. Analogously to (16) we find that

$$
|S_jf| = \left| \sum_{v=0}^{M_{\alpha_{s-1}+1}-1} \widehat{f}(v)\psi_v \right| = \left| \sum_{\eta=0}^{s-1} \sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1} \frac{M^{\alpha}_{\alpha_\eta}}{\alpha_\eta} \psi_v \right|
$$

$$
= \left| \sum_{\eta=0}^{s-1} \frac{M^{\alpha}_{\alpha_\eta}}{\alpha_\eta} \left(D_{M_{\alpha_\eta+1}} - D_{M_{\alpha_\eta}} \right) \right| \leqq \frac{2\lambda M^{\alpha+1}_{\alpha_{k-1}}}{\alpha_{k-1}}.
$$

Hence,

$$
(17)\qquad |I| \leqq \frac{1}{Q_{M_{\alpha_k}+M_s}} \sum_{j=0}^{M_{\alpha_k}} q_j |S_jf|
$$

$$\leqq \frac{2\lambda M_{\alpha_{k-1}}^{\alpha+1}}{\alpha_{k-1}} \frac{1}{Q_{M_{\alpha_k}+M_s}} \sum_{j=0}^{M_{\alpha_k}} q_j$$

$$\leqq \frac{2\lambda M_{\alpha_{k-1}}^{\alpha+1}}{\alpha_{k-1}}.$$

Let $x \in I_s/I_{s+1}$. Since

$$D_{j+M_n} = D_{M_n} + \psi_{M_n} D_j = D_{M_n} + r_n D_j, \quad \text{when} \quad j < M_n, \tag{18}$$

if we now apply Abel transformation, (15) and inequalities of (8) and (9) we get that

$$|II| = \frac{1}{Q_{M_{\alpha_k}+M_s}} \left| \frac{M_{\alpha_k}^{\alpha}}{\alpha_k} \sum_{j=M_{\alpha_k}+1}^{M_{\alpha_k}+M_s} q_{M_{\alpha_k}+M_s-j} \left(D_j - D_{M_{\alpha_k}} \right) \right|$$

$$= \frac{1}{Q_{M_{\alpha_k}+M_s}} \left| \frac{M_{\alpha_k}^{\alpha}}{\alpha_k} \sum_{j=1}^{M_s} q_{M_s-j} \left(D_{j+M_{\alpha_k}} - D_{M_{\alpha_k}} \right) \right|$$

$$= \frac{1}{Q_{M_{\alpha_k}+M_s}} \left| \frac{\psi_{M_{\alpha_k}} M_{\alpha_k}^{\alpha}}{\alpha_k} \sum_{j=1}^{M_s} q_{M_s-j} D_j \right|$$

$$= \frac{M_{\alpha_k}^{\alpha}}{\alpha_k Q_{M_{\alpha_k}+M_s}} \left| \sum_{j=1}^{M_s} q_{M_s-j} j \right|$$

$$\geqq \frac{c}{\alpha_k} \left| \sum_{j=1}^{M_s} \left(q_{M_s-j} - q_{M_s-j-1} \right) j^2 \right|$$

$$\geqq \frac{c M_s^2}{\alpha_k} \sum_{j=[M_s/2]}^{M_s} \left| q_{M_s-j} - q_{M_s-j-1} \right|$$

$$\geqq \frac{c M_s^2}{\alpha_k} \sum_{j=0}^{[M_s/2]} |q_j - q_{j+1}|$$

$$\geqq \frac{c M_s^2}{\alpha_k} \sum_{j=0}^{[M_s/2]} j^{\alpha-2}$$

$$\geqq \frac{cM_s^{\alpha-1}M_s^2}{\alpha_k} \geqq \frac{cM_s^{\alpha+1}}{\alpha_k}.$$

Let $[\alpha_k/2] < s \leqq \alpha_k$. Therefore, it yields that

$$\int_{G_m} \left| t_{M_{\alpha_k}+M_s} f(x) \right|^{1/(1+\alpha)} d\mu(x) \tag{19}$$

$$\geqq |II| - |I| \geqq \frac{cM_s^{1+\alpha}}{\alpha_k} - \frac{4\lambda M_{\alpha_{k-1}}^{\alpha+1}}{\alpha_{k-1}} \geqq \frac{cM_s^{1+\alpha}}{\alpha_k}.$$

By combining (17) and (19) we get that

$$\int_{G_m} |t^* f|^{1/(1+\alpha)} \, d\mu$$

$$\geqq \sum_{s=[\alpha_k/2]+1}^{\alpha_k} \int_{I_s/I_{s+1}} \left| t_{M_{\alpha_k}+M_s} f \right|^{1/(1+\alpha)} d\mu$$

$$\geqq c \sum_{s=[\alpha_k/2]}^{\alpha_k} \frac{M_s}{M_s \alpha_k^{1/(1+\alpha)}} \geqq c \sum_{s=[\alpha_k/2]}^{\alpha_k-3} \frac{1}{\alpha_k^{1/(1+\alpha)}}$$

$$\geqq \frac{c}{\alpha_k^{1/(1+\alpha)}} \sum_{s=[\alpha_k/2]}^{\alpha_k} 1 \geqq \frac{c\alpha_k}{\alpha_k^{1/(1+\alpha)}}$$

$$\geqq c\alpha_k^{\alpha/(1+\alpha)} \to \infty, \quad \text{as} \quad k \to \infty.$$

The proof is complete. □

3. Applications and final remark

REMARK 1. We note that under the both conditions of (7) in Theorem 1 the conditions (4) in Theorem A can still be fulfilled. So our main result shows that under the both conditions of (7) in part a) of Theorem A are in a sence sharp and the point $p = 1/(1+\alpha)$ is the smallest number for which we have boundedness from the Hardy space $H_{1/(1+\alpha)}$ to the space *weak*-$L_{1/(1+\alpha)}$.

Our main result Theorem 1 immediately implies the following results of Goginava [3] and Tephnadze [16]:

COROLLARY 1 (Goginava). *The maximal operator of the* (C,α)*-means* $\sigma^{\alpha,*}$ *is not bounded from the Hardy space* $H_{1/(1+\alpha)}$ *to the space* $L_{1/(1+\alpha)}$, *where* $0<\alpha\leqq 1$.

COROLLARY 2 (Tephnadze). *The maximal operator of the Fejér means* σ^* *is not bounded from the Hardy space* $H_{1/2}$ *to the space* $L_{1/2}$.

Let θ_n^α denote the Nörlund mean, where $\{q_0=0,\, q_k=k^{\alpha-1}: k\geqq 1\}$, that is

$$\theta_n^\alpha f=\frac{1}{Q_n}\sum_{k=1}^{n}(n-k)^{\alpha-1}S_k f.$$

It is easy to see that

$$\frac{|q_n-q_{n+1}|}{n^{\alpha-2}}=\frac{1}{n^{\alpha-2}}\left(\frac{n^\alpha}{n}-\frac{(n+1)^\alpha}{n+1}\right) \tag{20}$$

$$\leqq\frac{1}{n^{\alpha-2}}\left(\frac{n^\alpha}{n}-\frac{n^\alpha}{n+1}\right)=\frac{1}{n^{\alpha-2}}\frac{n^\alpha}{n(n+1)}$$

$$\leqq\frac{1}{n^{\alpha-2}}\frac{2}{n^{2-\alpha}}=O(1),\quad \text{as}\quad n\to\infty.$$

Since

$$Q_n:=\sum_{k=0}^{n-1}k^{\alpha-1}\geqq\int_1^{n-1}x^{\alpha-1}\,dx\geqq cn^\alpha$$

we obtain that

$$\frac{n^\alpha q_0}{Q_n}=O(1),\quad \text{as}\quad n\to\infty. \tag{21}$$

By combining inequalities (20) and (21) we get the following new result:

COROLLARY 3. *The maximal operator of the* θ_n^α*-means*

$$\theta^{\alpha,*}:=\sup_{n\in\mathbb{N}}|\theta_n f|$$

is not bounded from the martingale Hardy space $H_{1/(1+\alpha)}$ *to the Lebesgue space* $L_{1/(1+\alpha)}$, *where* $0<\alpha\leqq 1$.

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