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Hopf algebras and monoidal categories

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HOPF ALGEBRAS AND MONOIDAL CATEGORIES

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Contents

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Abstract. In this thesis we study the correspondence between categorical notions and bialgebra notions, and make a kind of dictionary and grammar book for translation between these notions. We will show how to obtain an antipode, and how to define braidings and quantizations. The construction is done in two ways. First we use the properties of a bialgebra to define a monoidal structure on (co)modules over this bialgebra. Then we go from a (strict) monoidal category and use a certain monoidal functor from this category to reconstruct bialgebra and (co)module structures. We will show that these constructions in a sense are inverse to each other. In some cases the correspondence is 1-1, and in the final Part we conjecture when this is the case for the category of comodules that are finitely generated and projective over the base ring k. We also briefly discuss how to transfer the results to non-strict categories.

Introduction

The purpose of this thesis is to examine the correspondence between categorical notions and bialgebra notions. There is a close connection between constructions in monoidal categories and constructions on (co)modules over bialgebras, and the categorical language can be a useful tool in studying these. We will examine this correspondence closely, and show that in some special cases there is a 1-1 correspondence between the structures. Most of the results have been known in various versions for some years and used in a variety of mathematical literature. The main idea in this thesis is to bring together these results to make a kind of dictionary and grammar book for translation of notions and methods from the bialgebra language to the categorical language and back. We will examine the following correspondences:

- A monoidal structure on the category of H-(co)modules over a (co)algebra corresponds to a bialgebra structure on H.
- Rigidity of a category corresponds to the existence of an antipode for H.
- Braidings and quantizations in the category are determined by (co)braiders and (co)quantizers as elements in $H \otimes H$ (or in $(H \otimes H)^*$).

The first Part deals with bi- and Hopf algebras. Throughout the thesis the basis for the constructions is the category Mod_k of modules over a base ring k. We define (co)algebra structures, (co)modules over these, and we define bialgebras. We then state some important Lemmas concerning duality of (co)modules. It turns out that most constructions on modules can be achieved by dualizing the corresponding structures on comodules. Vice versa, if we make some restrictions on Mod_k we can go from modules to comodules. We will also see that when Mod_k is the category of finitely generated projective modules, the dual of a bialgebra is still a bialgebra. The Part closes with the definition of an antipode and shows that for modules the dual of a Hopf algebra is also a Hopf algebra.

Remark 0.1. *For the rest of the paper we will use the shorthand notation f.g. projective for "finitely generated and projective"*

In Part II we describe monoidal categories and define various structures in them; braidings, quantizations and rigidity. When H is a bialgebra, the bialgebra structure can be used to define a monoidal structure on the categories Mod^H and Mod_H, the categories of comodules, resp. modules over H. We can then describe braidings and quantizations in these categories. We show that Mod^H is a braided category if and only if the underlying bialgebra is cobraided. The braiding is given by a cobraiding element

$$r \in Hom\,(H \otimes H, k)\,.$$

Likewise, a quantization is determined by a coquantizer

$$q \in Hom\,(H \otimes H, k)\,.$$

If H is a Hopf algebra, we can use the antipode to show that Mod^H is a rigid category. These concepts have mostly been described for categories of H-modules, but we have done a full description of these structures for comodules, as well. This is useful for showing duality between Mod^H and

Mod_H, and is necessary for the reconstructions in Part III. The construction of similar structures for Mod_H follows thereafter. The last section of the Part describes how the constructions in Mod_H and Mod^H in a sense are dual to each other. This duality is then used for the inverse constructions in Part III.

While we used the bialgebra and (co)module structures to establish structures of monoidal categories in Part II, in Part III we will go the opposite way. It turns out that given a monoidal category and a forgetting monoidal functor to an underlying category, it is possible to derive structures of bi- and Hopf algebras, (co)modules, braidings and quantizations. These reconstructions are usually done for a monoidal category $\mathcal{C}$ and a functor

$$G : \mathcal{C} \longrightarrow vec,$$

the category of finite dimensional vector spaces. The reconstructions will be generalized in this thesis to the category of finitely generated projective modules whenever possible. The reconstruction process mainly follows ideas from [LR97], [Ulb90], [Sch92] and [Par96]. The idea is to construct a **coend** for the functor G. We can then construct a coalgebra structure on

$$H = coend\,(G^* \otimes G)\,,$$

and we can give $G\,(X)$ a H-comodule structure. The monoidal structure of $\mathcal{C}$ can then be used to define a bialgebra structure on H, thus defining a monoidal category Mod^H. We then get a functor

$$F : \mathcal{C} \longrightarrow Mod^H$$

such that G factorizes through F. It was our intention to find reasonable restrictions on $\mathcal{C}$, G and k to show that we could get an equivalence between $\mathcal{C}$ and the category Mod^H of H-comodules, but this appeared to be too timeconsuming and too complicated for this thesis. A reasonable conjecture on such an equivalence is formulated in Section 11. However, the proof is only sketched, not completed. That is why the statement is **not** called a Theorem, and is placed in Part "Further perspectives". We will also shortly refer to results from [SR72] and [Sch92] concerning equivalence.

If $\mathcal{C}$ is rigid, we can construct an antipode for H, thereby making it a Hopf algebra. If we take $\mathcal{C}$ to be the category of comodules we constructed in Part II, we can show that the two methods of construction in II and III in a sense are inverse to each other.

We can also dualize this process to reconstruct a category of modules over an algebra. We use a functor $F : \mathcal{C} \longrightarrow Mod_k$ and construct

$$E = end\,(Hom\,(F, F))\,.$$

It can then be showed that

$$E^* \approx H = coend\,(G^* \otimes G)\,,$$

and the duality results from previous Parts are then used to reconstruct the bi- (and Hopf) algebra and module structure. Likewise we show how to construct braidings and quantizations.

It was the aim of this thesis to examine the same processes for non-strict categories, but this appeared to be too large for a cand. sci. thesis. This work is therefore only partially done for some concepts. In Section 9 we have presented the ideas and some partial results. When we have a multiplication that is not associative, it is not possible to get a bialgebra structure on H. But we can still make a "quasi" - associativity, just like braidings give quasisymmetries. To do this we use the structures of coquasibialgebras.

Remark 0.2. *The notion* ***quasibialgebras*** *has been widely used. Our notion* ***coquasibialgebras*** *seems to be relatively new. The difference between the two notions is that quasibialgebras are associative, but not coassociative, while coquasibialgebras are coassociative, but not associative.*

We can then use these structures to define braidings and quantizations in Mod^H. We also sketch how to reconstruct a coquasibialgebra structure and how to reconstruct braidings and quantizations in Mod^H. Finally we make a conjecture on equivalence between $\mathcal{C}$ and Mod^H in the case where Mod_k is the category of f.g. projective k-modules.

Part I. Hopf algebras

1. Bialgebras

In the following let k be a commutative ring with unit. Throughout the paper, the symbol $\otimes$ will denote tensoring over k:

$$\otimes := \otimes_k.$$

Definition 1.1. *A k-**algebra** (H, μ, η) is a k-module H together with k-module homomorphisms*

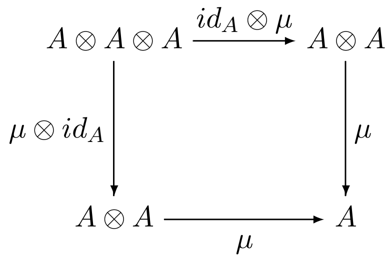
$$\mu : H \otimes H \longrightarrow H,$$

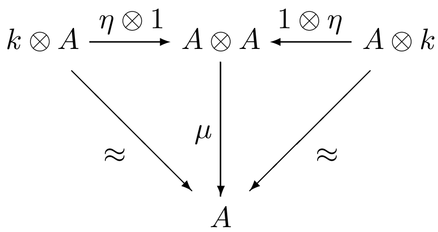
*called **multiplication**, and*

$$\eta : k \longrightarrow H,$$

*called **unit**, such that the two following diagrams commute:*

(1.1)





The first diagram shows accociativity of μ, while the second shows that η is a two-sided unit for μ. The commutativity of the above diagrams is equivalent to the following equations

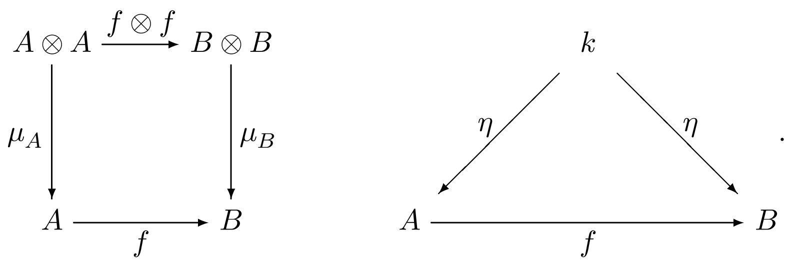
$$\begin{aligned} \mu \circ (\mu \otimes id_A) &= \mu \circ (id_A \otimes \mu) \\ \mu \circ (\eta \otimes id_A) &= \mu \circ (id_A \otimes \eta)\,. \end{aligned} \tag{1.2}$$

A k-module homomorphism

$$f : A \longrightarrow B$$

*where A and B are algebras is an **algebra homomorphism** provided the following diagrams commute*

(1.3)



An algebra is ***commutative*** *if* $\mu \circ \tau = \mu$*, where* τ *is the* ***twist***

$$\tau(a \otimes b) = b \otimes a.$$

Dually,

Definition 1.2. *a* k*-**coalgebra*** C *is a* k*-module together with a* k*-module homomorphism*

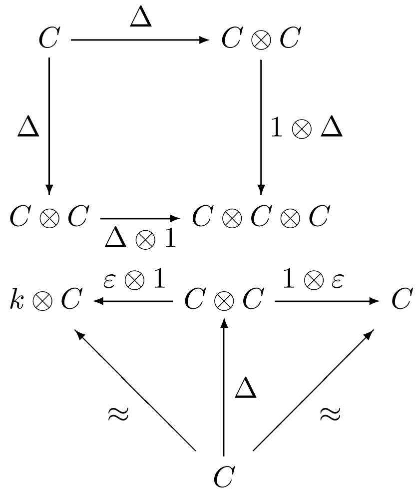
$$\Delta : C \longrightarrow C \otimes C$$

called ***diagonal or comultiplication****, and a* k*-module homomorphism*

$$\varepsilon : C \longrightarrow k$$

called ***counit****, such that the following diagrams commute:*

(1.4)



This can be expressed through the following equations:

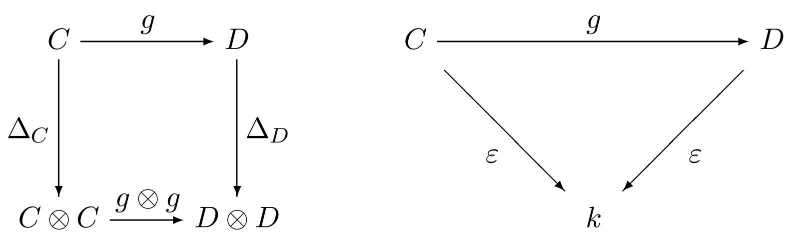
$$(\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta \tag{1.5}$$
$$(\varepsilon \otimes id_C) \circ \Delta = (id_C \otimes \varepsilon) \circ \Delta.$$

The first equation shows that Δ *is* ***coassociative*** *.*

A k*-module homomorphism*

$$g : C \longrightarrow D$$

where C *and* D *are coalgebras is a* ***coalgebra homomorphism*** *provided the following diagram commutes:*



A coalgebra is said to be ***cocommutative*** *if* $\tau \circ \Delta = \Delta$*.*

Definition 1.3. *A* ***bialgebra*** *is an algebra that is also a coalgebra, and where* Δ *and* ε *are algebra morphism. The latter is equivalent to requiring that* μ *and* η *are coalgebra morphisms.*

1.1. **Comodules.**

Definition 1.4. *A (right)* ***comodule*** V *over a* k*-coalgebra* C *is a* k*-module together with a* k*-module homomorphism*

$$\delta_V : V \longrightarrow V \otimes C$$

such that

$$\begin{aligned} (\delta_V \otimes 1) \circ \delta_V &= (1 \otimes \Delta) \circ \delta_V \\ (1 \otimes \varepsilon) \circ \delta_V &= id_V. \end{aligned} \tag{1.6}$$

A C*-comodule morphism is a morphism* $f : V \longrightarrow W$ *such that*

$$\delta_W \circ f = (f \otimes 1) \circ \delta_V.$$

1.2. **Modules.**

Throughout the paper, "A-module" will mean "**left** A-module".

Definition 1.5. *A* ***module*** M *over a* k*-algebra* A *is a* k*-module together with a* k*-module homomorphism*

$$\rho_M : A \otimes M \longrightarrow M$$

such that

$$\begin{aligned} \rho \circ (1 \otimes \rho) &= \rho \circ (\mu \otimes 1) \\ \rho \circ (\eta \otimes 1) &= id_M. \end{aligned} \tag{1.7}$$

An A*-module morphism*
is a map $g : M \longrightarrow N$ *obeying*

$$\rho \circ (1 \otimes g) = g \circ \rho$$

1.3. **Duality.**

We can relate algebras and coalgebras by duality. We define the **dual** module to a k-module M to be the module

$$M^* = Hom_k(M, k).$$

First we state some useful Lemmas.

Given two modules A, B we have a natural homomorphism

$$A^* \otimes B \xrightarrow{\varphi} Hom(A, B),$$
$$(\varphi(f \otimes b))(a) \longmapsto f(a)\, b$$

Lemma 1.6. *The natural homomorphism*

$$\varphi : A^* \otimes B \longrightarrow Hom(A, B)$$

is an isomorphism when A *is a finitely generated projective* k*-module.*

Proof. First suppose that A is free with basis $e_1, \ldots, e_n$. Then $f \in Hom(A, B)$ is uniquely determined by its values on the elements of the basis. This means that any f is uniquely determined by a set of elements $b_1, \ldots, b_n \in B$. Let $e^1, \ldots, e^n$ be the dual basis in A^*. Then any element in $A^* \otimes B$ is uniquely represented by $\sum e^i \otimes b_i$. But $\varphi\left(\sum e^i \otimes b_i\right)$ takes e_i to b_i, so the map is an isomorphism. Now let A be f.g. projective. There is a free module $F \approx A \oplus A'$, where both A and A' are f.g. projective, and $F^* \approx A^* \oplus A'^*$. This gives an isomorphism

$$\begin{aligned} F^* \otimes B &\approx \left(A^* \oplus A'^*\right) \otimes B \approx A^* \otimes B \oplus A'^* \otimes B \\ &\approx Hom(A, B) \oplus Hom\left(A', B\right), \end{aligned}$$

hence the isomorphism

$$A^* \otimes B \approx Hom\,(A, B)$$

□

Lemma 1.7. *Let k be a commutative ring. Then for any k-modules A, B and C we have a natural isomorphism*

$$\pi : Hom_k\,(A \otimes B, C) \longrightarrow Hom_k\,(B, Hom\,(A, C))$$

given by

$$((\pi f)\, b)\, a = f\,(a \otimes b)$$

where $f \in Hom_k\,(A \otimes B, C)$, $a \in A$ and $b \in B$.

Proof. First,

$$(\pi f)\, h : B \longrightarrow C$$

is a k-module homomorphism by the properties of the tensor product. Since f is a k-module homomorphism,

$$\pi f : B \longrightarrow Hom\,(A, C)$$

is also. Now let

$$g \in B \longrightarrow Hom\,(A, C)$$

We define

$$\omega : Hom_k\,(B, Hom\,(A, C)) \longrightarrow Hom_k\,(A \otimes B, C)$$

by the k-module homomorphism

$$\omega\,(g)\,(a \otimes b) = (g\,(b))\,(a)$$

This gives an inverse for π, so we have the desired isomorphism, which is natural in all three arguments. □

Let the map

$$M^* \otimes N^* \longrightarrow (N \otimes M)^* \tag{1.8}$$

be defined by

$$(f \otimes g)\,(m \otimes n) \mapsto g\,(n)\, f\,(m)\,.$$

This is a natural homomorphism: it is the composition

$$M^* \otimes N^* \xrightarrow{\varphi} Hom\,(M, N^*) = Hom\,(M, Hom\,(N, k)) \approx Hom\,(N \otimes M, k)$$

where the last isomorphism is given by Lemma 1.7.

Corollary 1.8. *Let A, B be k-modules. The map $\lambda : M^* \otimes N^* \longrightarrow (N \otimes M)^*$ is an isomorphism if M, N are finitely generated and projective.*

Proof. First note that Lemma 1.6 can be stated as

$$N \otimes M^* \overset{\phi}{\approx} Hom_k\,(M, N)\,,$$
$$\phi\,(n \otimes f)\,(m) = f\,(m)\, n$$

Then λ is the composition

$$M^* \otimes N^* \xrightarrow{\varphi} Hom\,(N, M^*) = Hom\,(N, Hom\,(M, k)) \approx Hom\,(N \otimes M, k) = (N \otimes M)^*\,.$$

By Lemma 1.6 this is an isomorphism when M and N are f.g. projective as k-modules. □

Remark 1.9. *For the rest of this document λ will refer to this isomorphism.*

For the next Proposition we need the following definition:

Definition 1.10. *Let A be an algebra and C a coalgebra. The* ***convolution*** *$f \star g$ of $f, g : C \longrightarrow A$ is defined by the following diagram:*

$$\begin{array}{ccc} C & \xrightarrow{f \star g} & A \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\uparrow{\scriptstyle \mu} \\ C \otimes C & \xrightarrow{f \otimes g} & A \otimes A \end{array}$$

Proposition 1.11. *Let C be a coalgebra. Then C^* is an algebra.*

Proof. Let $f, g \in Hom\,(C, k)$. Using Sweedler notation (see e.g. [Kas95, III, 1.6]) we can write the diagonal as

$$\Delta\,(x) = \sum x_{(0)} \otimes x_{(1)}.$$

We can define a multiplication μ on C^* by

$$\begin{aligned} \mu\,(f \otimes g)\,(x) &= \sum f\left(x_{(0)}\right) g\left(x_{(1)}\right) = (f \star g)\,(x)\,, \\ f, g &\in C^*, x \in C \end{aligned}$$

Associativity follows from the associativity of Δ and in k.

Define η by

$$\eta\,(1) = id_{C^*}.$$

Then

$$\begin{aligned} \mu \circ (\eta \otimes id_{C^*})\,(x) &= \mu\left(\sum \eta\left(x_{(0)}\right) \otimes x_{(1)}\right) \\ &= x = \mu \circ (id_{C^*} \otimes \eta)\,(x) \\ &= \mu\left(\sum x_{(0)} \otimes \eta\left(x_{(1)}\right)\right) \end{aligned}$$

This shows that μ is associative and that $\eta\,(1)$ is a left and right unit for μ, so (C^*, μ, η) is an algebra. □

Lemma 1.12. *Given two k-modules M, V we have an isomorphism*

$$\pi : Hom\,(M, V) \approx Hom\,(V^*, M^*)\,.$$

Proof. By applying Lemma 1.6 and its "twisted" version we get the following:

$$Hom\,(M, V) \approx V \otimes M^* \approx Hom\,(V^*, M^*)\,.$$

□

Now let $f : M \longrightarrow V$ be a k-module homomorphism. We define the **transpose** $f^* : V^* \longrightarrow M^*$ to be the image of f under the above map, that is,

$$f^* = \pi\,(f)\,.$$

Proposition 1.13. *Let A be an algebra that is finitely generated and projective as a k-module. Then A^* is a coalgebra.*

Proof. From corollary 1.8 we see that

$$\lambda : A^* \otimes A^* \longrightarrow (A \otimes A)^* .$$

is an isomorphism. We then define diagonal

$$\Delta' = \tau\lambda^{-1} \circ \mu^*$$

and counit

$$\varepsilon' = \eta^* .$$

The transposition transforms the diagrams 1.1 into the proper diagrams for a coalgebra definition. □

Proposition 1.14. *If H is a bialgebra and a finitely generated projective k-module, then H^* is a bialgebra.*

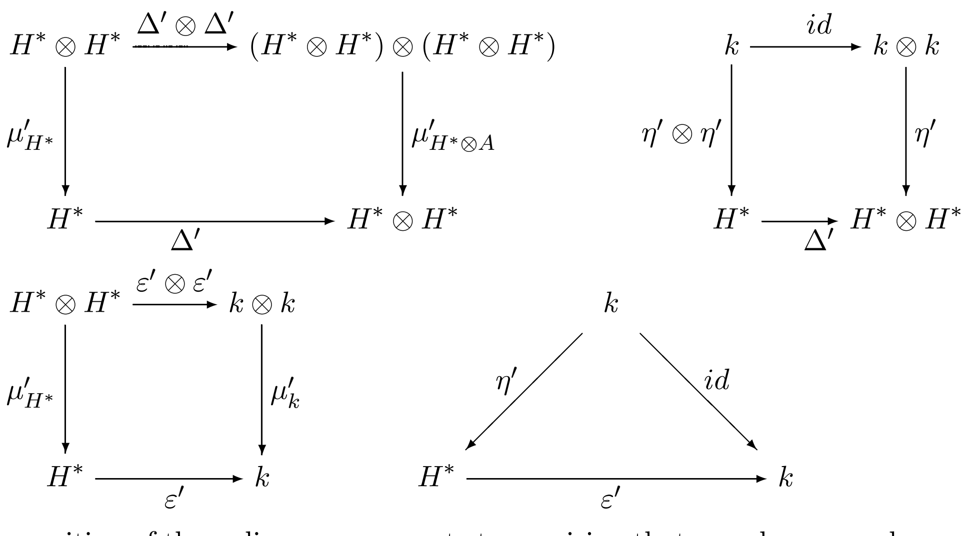
Proof. From the previous Propositions H^* has an algebra and a coalgebra structure. The coalgebra structure was given by transposing the algebra structure of H, with coalgebra structure

$$\begin{aligned} \Delta' &:= \tau\lambda^{-1} \circ \mu^*, \\ \varepsilon' &:= \eta^* \end{aligned}$$

When H is finitely generated and projective the algebra structure from the proof of Proposition 1.11 can be rephrased as

$$\begin{aligned} \mu' &:= \Delta^* \circ \tau\lambda, \\ \eta' &:= \varepsilon^* \end{aligned}$$

We need to show that Δ' and ε' are algebra homomorphisms, so 1.3 we need the following diagrams to commute:



Transposition of these diagrams amounts to requiring that μ and η are coalgebra morphisms. But this we know from the fact that H is a bialgebra, so H^* is a bialgebra. □

Proposition 1.15. *Let $(H, \Delta, \varepsilon, \mu, \eta)$ be a bialgebra which is finitely generated and projective as a k-module. Then for any right H-comodule M, M^* is a left H^*-module. Conversely, if V is a left H-module, V^* is a right H^*-comodule.*

Proof. From the previous Proposition we know that $(H^*, \Delta', \varepsilon', \mu', \eta')$ is a bialgebra when we define $\Delta', \varepsilon', \mu', \eta'$ as in the previous proof. First let

$$V \xrightarrow{\delta_M} V \otimes H$$

be the H-comodule structure on V^*. Define

$$\rho' : H^* \otimes V^* \xrightarrow{\lambda} (V \otimes H)^* \xrightarrow{\delta_M^*} V^*.$$

We want ρ' to satisfy the following equations:

$$\begin{aligned} \rho' \circ \left(1 \otimes \rho'\right) &= \rho' \circ \left(\mu' \otimes 1\right), \\ \rho' \circ (\eta \otimes 1) &= id \end{aligned}$$

Transposing the equations 1.6 will give the desired result. We show the first equation:

$$\begin{aligned} \rho' \circ \left(1 \otimes \rho'\right) &= (\delta^* \circ \lambda) \circ (1 \otimes (\delta^* \circ \lambda)) \\ &= (\delta^* \circ \lambda) \circ ((\Delta^* \circ \tau \circ \lambda) \otimes 1) \\ &= (\delta^* \circ \lambda) \circ \left(\mu' \otimes 1\right) \\ &= \rho' \circ \left(\mu' \otimes 1\right). \end{aligned}$$

The second equation follows:

$$\begin{aligned} \rho' \circ \left(\eta' \otimes 1\right) &= (\delta^* \circ \lambda) \circ (\varepsilon^* \otimes 1) \\ &= \delta^* \circ \lambda \circ (\varepsilon^* \otimes 1) \\ &= id. \end{aligned}$$

To go the other way, let

$$H \otimes V \xrightarrow{\rho_V} V$$

be the H-module structure on V. Define

$$\delta' : V^* \xrightarrow{\rho_V^*} (H \otimes V)^* \xrightarrow{\lambda^{-1}} V^* \otimes H^*$$

Then δ' gives a H^*-comodule on V^*. The proof is similar to the opposite case. □

2. Antipode

Let H be a bialgebra. We define an **antipode** as an endomorphism

$$s : H \longrightarrow H$$

satisfying

$$\mu \circ (s \otimes id_H) \circ \Delta = \eta \circ \varepsilon = \mu \circ (id_H \otimes s) \circ \Delta,$$

or in other words,

$$s \star id_H = id_H \star s = \eta \circ \varepsilon.$$

Definition 2.1. *A* ***Hopf algebra*** *is a bialgebra H with an antipode s, that is, an endomorphism*

$$s : H \longrightarrow H$$

satisfying

$$s \star id_H = id_H \star s = \eta \circ \varepsilon. \tag{2.1}$$

Proposition 2.2. *If H is a Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, s)$, then H^* is a Hopf algebra with antipode s^*, the transpose of s.*

Proof. From 1.14 we know that H^* is a bialgebra, so we only need to find an antipode for H^*. The equations 2.1 can be described by requiring commutativity of

$$
\begin{array}{ccc}
H & \xrightarrow{\Delta} & H\otimes H \\
\Big\downarrow & & \Big\downarrow {\scriptstyle id_H\otimes s} \\
{\scriptstyle\varepsilon} & & H\otimes H \\
\Big\downarrow & & \Big\downarrow {\scriptstyle\mu} \\
k & \xrightarrow[\eta]{} & H
\end{array}
\qquad\qquad
\begin{array}{ccc}
H & \xrightarrow{\Delta} & H\otimes H \\
\Big\downarrow & & \Big\downarrow {\scriptstyle s\otimes id_H} \\
{\scriptstyle\varepsilon} & & H\otimes H \\
\Big\downarrow & & \Big\downarrow {\scriptstyle\mu} \\
k & \xrightarrow[\eta]{} & H
\end{array}
\tag{2.2}
$$

Transposition of these diagrams shows that s^* is an antipode for H^*. We show this explicitly for the first diagram. Transposing gives

$$
\begin{array}{ccccc}
H^* & \xrightarrow{\eta^*} & k & \xrightarrow{\varepsilon^*} & H^* \\
{\scriptstyle\mu^*}\Big\downarrow & & & & \Big\uparrow{\scriptstyle\Delta^*} \\
(H\otimes H)^* & & & & (H\otimes H)^* \\
{\scriptstyle\lambda^{-1}}\Big\downarrow & & & & \Big\uparrow{\scriptstyle\lambda} \\
H^*\otimes H^* & & \xrightarrow{id\otimes s^*} & & H^*\otimes H^*
\end{array}
$$

using

$$(id_H\otimes s)^* = \lambda\circ(id_{H^*}\otimes s^*)\circ\lambda^{-1}.$$

By the definitions of the bialgebra structure on H^* from the proof of Proposition 1.14 the diagram transforms to

$$
\begin{array}{ccccc}
H^* & \xrightarrow{\varepsilon'} & k & \xrightarrow{\eta'} & H^* \\
{\scriptstyle\Delta'}\Big\downarrow & & & & \Big\uparrow{\scriptstyle\mu'} \\
H^*\otimes H^* & & \xrightarrow{id\otimes s^*} & & H^*\otimes H^*
\end{array}
$$

The commutativity of the diagram gives

$$id\otimes s^* = \eta'\circ\varepsilon'.$$

Switching $id\otimes s$ with $s\otimes id$ and applying the same procedure gives

$$s^*\otimes id = \eta'\circ\varepsilon',$$

so s^* is an antipode for H^*. □

Part II. Monoidal categories

3. General monoidal categories

Definition 3.1. *A **monoidal category** is a category $\mathcal{C}$ with a bifunctor $\Box : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and a **unit object** e together with natural isomorphisms*

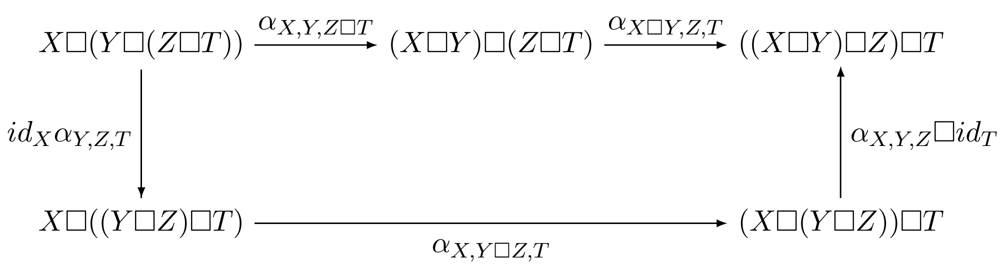
$$\alpha = \alpha_{X,Y,Z} : X\Box\left(Y\Box Z\right) \longrightarrow \left(X\Box Y\right)\Box Z,$$

*called the **associativity constraint**, and*

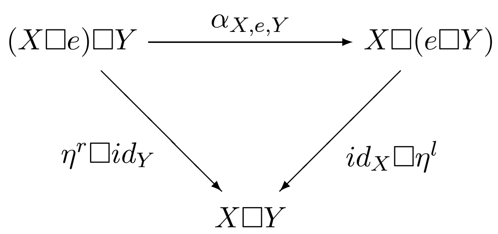
$$\eta^r : X\Box e \longrightarrow X \text{ and } \eta^l : e\Box X \longrightarrow X,$$

*called **unity constraints,** such that the following coherence conditions (see* [ML98, ch. VII]*) holds:*

- ***pentagon axiom***



- ***unity axiom***



*A monoidal category is **strict** when the associativity and unity constraints are identity morphisms.*

Definition 3.2. *A **monoidal functor** (F,ξ_2,ξ_0) consists of*

- *a functor*

$$F : \mathcal{C} \longrightarrow \mathcal{C}'$$

between monoidal categories

- *a natural morphism*

$$\xi_2\left(X,Y\right) : F\left(X\right)\Box F\left(Y\right) \longrightarrow F\left(X\Box Y\right)$$

for $X,Y \in \mathcal{C}'$

- *a natural morphism*

$$\xi_0 : e' \longrightarrow F\left(e\right)$$

for e, e' the units in $\mathcal{C}$ and $\mathcal{C}'$ respectively. Together these must make all the following diagrams commute

$$
\begin{array}{ccc}
F(X)\square(F(Y)\square F(Z)) & \xrightarrow{\alpha'} & (F(X)\square F(Y))\square F(Z) \\
\downarrow{\scriptstyle 1\otimes\xi_2} & & \downarrow{\scriptstyle \xi_2\otimes 1} \\
F(X)\square(F(Y\tilde{\square}Z) & & F(X\tilde{\square}Y)\square F(Z) \\
\downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi_2} \\
F(X\tilde{\square}(Y\tilde{\square}Z) & \xrightarrow[F(\alpha)]{} & F((X\tilde{\square}Y)\tilde{\square}Z)
\end{array}
\tag{3.1}
$$

$$
\begin{array}{ccc}
F(X)\square e' & \xrightarrow{(\eta^r)'} & F(X) \\
\downarrow{\scriptstyle 1\square\xi_0} & & \uparrow{\scriptstyle F(\eta^r)} \\
F(X)\square F(E) & \xrightarrow[F_2]{} & F(X\tilde{\square}e)
\end{array}
\qquad
\begin{array}{ccc}
e'\square F(X) & \xrightarrow{(\eta^l)'} & F(X) \\
\downarrow{\scriptstyle \xi_0\square 1} & & \uparrow{\scriptstyle F(\eta^l)} \\
F(e)\square F(X) & \xrightarrow[F_2]{} & F(e\tilde{\square}X)
\end{array}
\tag{3.2}
$$

*where $\tilde{\square}$ and $\square$ are in $\mathcal{C}$ and $\mathcal{C}'$ respectively. The functor is said to be **strong** when ξ_0 and ξ_2 are isomorphisms, and **strict** when they are the identity.*

Remark 3.3. *For the rest of the text we will write $\otimes$ for the functor $\square$ when there is no risk of confusion. We will also occasionally call it the **product**. We will also assume that categories and functors are strict when nothing else is said.*

Definition 3.4. *An object X^* in a monoidal category K is called a **left dual** if there are K-morphisms*

$$X^* \otimes X \xrightarrow{ev} I,$$
$$I \xrightarrow{db} X \otimes X^*$$

such that

$$X \approx I \otimes X \xrightarrow{db\otimes 1} X \otimes X^* \otimes X \xrightarrow{1\otimes ev} X \otimes I \approx X \tag{3.3}$$

and

$$X^* \approx X^* \otimes I \xrightarrow{1\otimes db} X^* \otimes X \otimes X^* \xrightarrow{ev\otimes 1} I \otimes X^* \approx X^* \tag{3.4}$$

*are the identity maps. Likewise we can define a **right dual** to be an object X^* with K-morphisms*

$$X \otimes X^* \xrightarrow{ve} I,$$
$$I \xrightarrow{bd} X^* \otimes X$$

such that

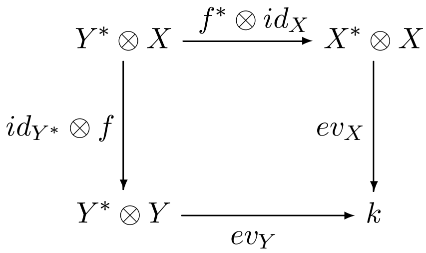
$$X^* \approx I \otimes X^* \xrightarrow{bd\otimes 1} X^* \otimes X \otimes X^* \xrightarrow{1\otimes ve} X^* \otimes I \approx X^*$$

and

$$X \approx X \otimes I \stackrel{1\otimes bd}{\longrightarrow} X \otimes X^* \otimes X \stackrel{ve\otimes 1}{\longrightarrow} I \otimes X \approx X.$$

If every $X \in K$ has a left (right) dual, the category is ***left (right) rigid****. A category where all elements have both left and right duals, is called* ***rigid****.*

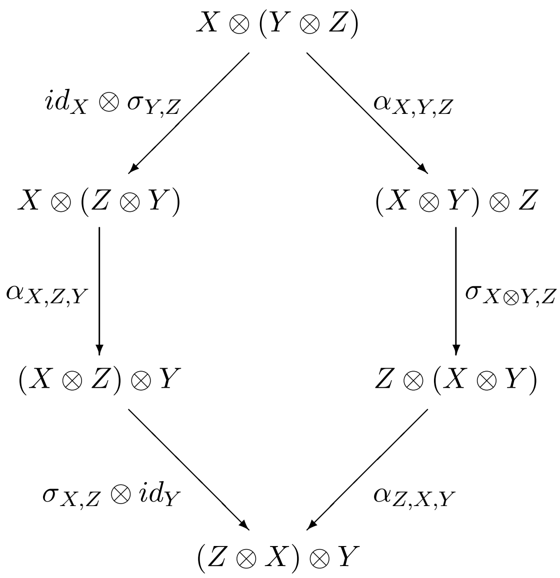
When a category is (left) rigid, we can give an alternative description of the transpose of a morphism $f : X \longrightarrow Y$: it is the unique morphism f^* making the following diagram commutative:

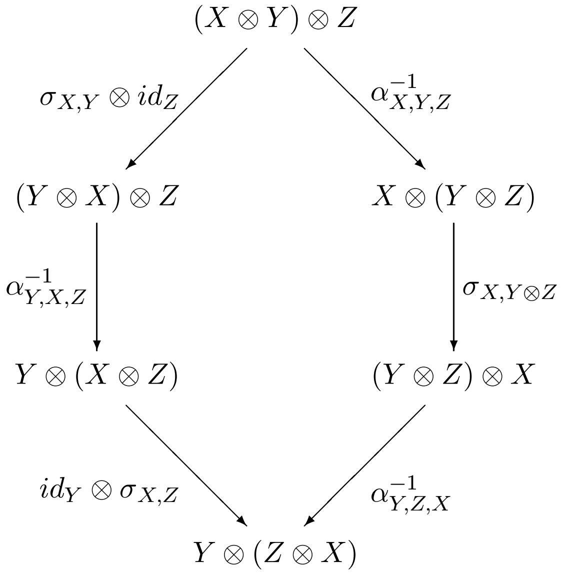


We can also equally define f^* by the following:

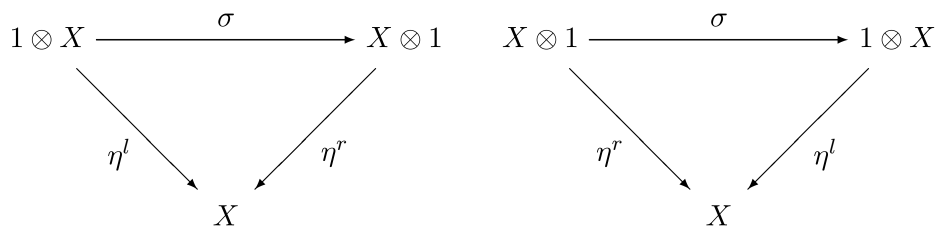
$$f^* : Y^* \stackrel{1\otimes db_X}{\longrightarrow} Y^* \otimes X \otimes X^* \stackrel{1\otimes f\otimes 1}{\longrightarrow} Y^* \otimes Y \otimes X^* \stackrel{ev_Y\otimes 1}{\longrightarrow} X^*.$$

Definition 3.5. *A* ***braiding*** *in a monoidal k-linear category is a natural k-bilinear isomorphism $\sigma_{X,Y} : X\otimes Y \longrightarrow Y\otimes X$ that satisfies commutativity of the hexagon diagrams:*



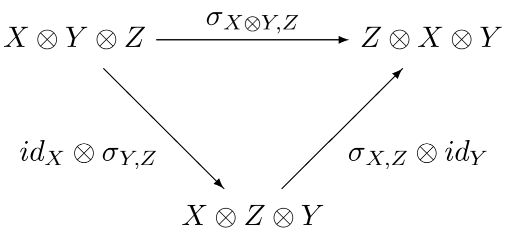


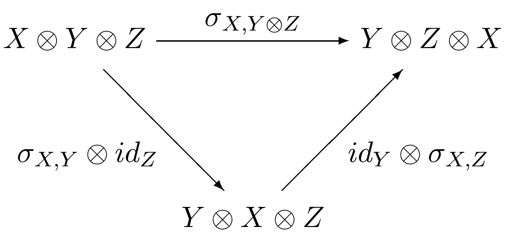
and of the diagrams



In the case of strict monoidal categories the hexagon diagrams take the following form:

(3.5)





or, equivalently,

$$(\sigma_{X,Z} \otimes id_Y)\,(id_X \otimes \sigma_{Y,Z}) = (\sigma_{X \otimes Y,Z}) \tag{3.6}$$

and

$$(id_Y \otimes \sigma_{X,Z})\,(\sigma_{X,Y} \otimes id_Z) = (\sigma_{X,Y \otimes Z})\,. \tag{3.7}$$

A monoidal functor (F, ξ_2, ξ_0) *is said to be braided if the following diagram commutes naturally*

$$\begin{array}{ccc} F(X)\otimes F(Y) & \xrightarrow{\xi_2} & F(X\otimes Y) \\ \Big\downarrow{\scriptstyle \sigma_{F(X),F(Y)}} & & \Big\downarrow{\scriptstyle F(\sigma_{X,Y})} \\ F(Y)\otimes F(X) & \xrightarrow[\xi_2]{} & F(Y\otimes X) \end{array}$$

Definition 3.6. *A* ***quantization*** *(due to V. Lychagin, see e.g.* [LP99]*) in a monoidal category* $\mathcal{C}$ *is a natural isomorphism*

$$Q = Q_{X,Y} : X \otimes Y \longrightarrow X \otimes Y,$$

such that the coherence conditions

(3.8)

$$\begin{array}{ccccc} X\otimes(Y\otimes X) & \xrightarrow{\alpha_{X,Y,Z}} & (X\otimes Y)\otimes Z & \xrightarrow{Q_{X,Y}\otimes id_Z} & (X\otimes Y)\otimes Z \\ \Big\downarrow{\scriptstyle id_X\otimes Q_{Y,Z}} & & & & \Big\downarrow{\scriptstyle Q_{X\otimes Y,Z}} \\ X\otimes(Y\otimes Z) & \xrightarrow[Q_{X,Y\otimes Z}]{} & X\otimes(Y\otimes Z) & \xrightarrow{\alpha_{X,Y,Z}} & (X\otimes Y)\otimes Z \end{array}$$

and

(3.9)

$$\begin{array}{ccc} X\otimes k & \xrightarrow{Q_{X,k}} & X\otimes k \\ & {\scriptstyle \iota^r_X}\searrow & \Big\downarrow{\scriptstyle \iota^r_X} \\ & & X \end{array}$$

$$\begin{array}{ccc} k\otimes X & \xrightarrow{Q_{k,X}} & k\otimes X \\ & {\scriptstyle \iota^l_X}\searrow & \Big\downarrow{\scriptstyle \iota^l_X} \\ & & X \end{array}$$

holds for all $X, Y \in \mathcal{C}$. *For strict monoidal categories the diagram 3.8 reduces to*

(3.10)

$$\begin{array}{ccc} X\otimes Y\otimes Z & \xrightarrow{Q_{X,Y}\otimes id_Z} & X\otimes Y\otimes Z \\ \Big\downarrow{\scriptstyle id_X\otimes Q_{Y,Z}} & & \Big\downarrow{\scriptstyle Q_{X\otimes Y,Z}} \\ X\otimes Y\otimes Z & \xrightarrow[Q_{X,Y\otimes Z}]{} & X\otimes Y\otimes Z \end{array}$$

*A **quantization** of a functor $G : A \longrightarrow B$ is a natural isomorphism*

$$Q : G(X)\widehat{\otimes}G(Y) \longrightarrow G(X \otimes Y),$$

where $\otimes$ and $\widehat{\otimes}$ are the products in A and B, respectively, together with the coherence conditions

$$\begin{array}{ccc}
G(X)\widehat{\otimes}(G(Y)\widehat{\otimes}G(Z)) & \xrightarrow{\beta_{G(X),G(Y),G(Z)}} & (G(X)\widehat{\otimes}G(Y))\widehat{\otimes}G(Z) \\
\downarrow{\scriptstyle id_X\widehat{\otimes}Q_{Y,Z}} & & \downarrow{\scriptstyle Q_{X,Y}\widehat{\otimes}id_Z} \\
G(X)\widehat{\otimes}G(Y\otimes Z) & & G(X\otimes Y)\widehat{\otimes}G(Z) \\
\downarrow{\scriptstyle Q_{X,Y\otimes Z}} & & \downarrow{\scriptstyle Q_{X\otimes Y,Z}} \\
G(X\otimes(Y\otimes Z)) & \xrightarrow{G(\alpha_{X,Y,Z})} & G((X\otimes Y)\otimes Z)
\end{array}$$

and

(3.11)
$$\begin{array}{ccc}
G(X)\widehat{\otimes}G(k) & \xrightarrow{Q_{X,k}} & G(X\otimes k) \\
& {\scriptstyle \iota^r_B}\searrow & \downarrow{\scriptstyle G(\iota^r_A)} \\
& & G(X)
\end{array}$$

$$\begin{array}{ccc}
G(k)\widehat{\otimes}G(X) & \xrightarrow{Q_{k,X}} & G(k\otimes X) \\
& {\scriptstyle \iota^l_B}\searrow & \downarrow{\scriptstyle G(\iota^l_A)} \\
& & G(X)
\end{array}$$

4. Monoidal structure on the category of H-comodules

First, note that the category Mod_k of modules over k is a monoidal category with the usual tensor product.

4.1. **Comodules over a bialgebra.** Let H be a bialgebra. The category Mod^H of H-comodules can be given a monoidal structure if we define the product

$$\otimes : Mod^H \times Mod^H \longrightarrow Mod^H$$

to be the ordinary tensor product $\otimes_k$. The pentagon and unity axioms are satisfied through the properties of the tensor product. We give a H-comodule structure of the tensor product by

$$\delta_{V\otimes W} : V\otimes W \overset{\delta_V\otimes\delta_W}{\longrightarrow} V\otimes H\otimes W\otimes H \overset{1\otimes\tau\otimes 1}{\longrightarrow} V\otimes W\otimes H\otimes H \overset{1\otimes 1\otimes\mu}{\longrightarrow} V\otimes W\otimes H,$$

where

$$\tau : V \otimes W \longrightarrow V \otimes W$$

is the twist.

Remark 4.1. *By abuse of notation we will write* $\delta_V \otimes \delta_W$ *for the composition*

$$(1 \otimes \tau \otimes 1) \circ (\delta_V \otimes \delta_W)$$

whenever there are no possibility for confusion. We will also write $\sum v_{(1)} \otimes v_{(2)} = \delta_V(v)$ *whenever the context make the notation clear.*

We must check that the conditions 1.6 holds:

$$\begin{aligned} & ((\delta_{V\otimes W} \otimes 1) \circ \delta_{V\otimes W})(v \otimes w) \\ = \; & (\delta_{V\otimes W} \otimes 1) \sum v_{(0)} \otimes w_{(0)} \otimes \mu\left(v_{(1)} \otimes w_{(1)}\right) \\ = \; & \sum v_{(0)} \otimes w_{(0)} \otimes \mu\left(v_{(1)} \otimes w_{(1)}\right) \otimes \mu\left(v_{(2)} \otimes w_{(2)}\right) \\ = \; & (1 \otimes \Delta) \sum v_{(0)} \otimes w_{(0)} \otimes \mu\left(v_{(1)} \otimes w_{(1)}\right) \\ = \; & (1 \otimes \Delta) \circ \delta_{V\otimes W} \end{aligned}$$

and

$$\begin{aligned} & ((1 \otimes \varepsilon) \circ \delta_{V\otimes W})(v \otimes w) \\ = \; & (1 \otimes \varepsilon) \sum \left(v_{(0)} \otimes w_{(0)}\right) \otimes \mu\left(v_{(1)} \otimes w_{(1)}\right) \\ = \; & \sum \left(v_{(0)} \otimes w_{(0)}\right) \\ = \; & v \otimes w. \end{aligned}$$

4.2. **Comodules over a Hopf algebra. Rigidity.** Now let H be a Hopf algebra with the antipode s and let Mod^H be the category of H-comodules.

As we have seen, Mod^H has the structure of a monoidal category. Let M^* be the dual module $Hom_k(M, k)$. To define a H-comodule morphism

$$ev : M^* \otimes M \longrightarrow k$$

we need to have a H-comodule structure on M^*. First, to do calculations about rigidity, we use the following Lemma:

Lemma 4.2. *A k-module M is f.g. projective if and only if there are elements $m_1, \ldots, m_n \in M$ and $m^1, \ldots, m^n \in M^*$ such that*

$$\forall x \in M, \; x = \sum m^i(x)\, m_i.$$

We then call $\left\{m^i, m_i\right\}$ a ***dual basis*** *for M.*

Proof. The following proof is adopted from [DI71, Lemma 1.3]. We assume that M is finitely generated and projective. Therefore there exists a f.g. free module F and homomorphisms

$$\begin{aligned} \pi &: F \longrightarrow M, \\ \rho &: M \longrightarrow F, \end{aligned}$$

such that

$$\pi \circ \rho = Id_M.$$

As F is free, $F \approx k^I$ for some finite set I. Thinking of k^I as a set of functions from I to k, define

$$\varphi_i : k^I \longrightarrow k$$

by

$$\begin{aligned} \forall f &\in k^I, \\ \varphi_i(f) &= f(e_i) \end{aligned}$$

Then we have

$$\sum \varphi_i(f)\, e^j = f$$

Define

$$\begin{aligned} m^i &= \varphi_i \circ \rho, \\ m_i &= \pi\left(e^i\right). \end{aligned}$$

We get the following.

$$\begin{aligned} \sum m^i(x)\, m_i &= \sum (\varphi_i \circ \rho)(x)\, \pi\left(e^i\right) \\ &= \sum \varphi_i(\rho(x))\, \pi\left(e^i\right) \\ &= \pi \sum \varphi_i(\rho(x))\, e^i \\ &= \pi(\rho(x)) \\ &= x \end{aligned}$$

Conversely, assume $\left\{m_i, m^i\right\}$ forms a dual basis for M in the sense defined above. Define

$$\begin{aligned} \pi &: F \longrightarrow M, \\ \pi_i(f) &= \sum f(e_i)\, m_i \end{aligned}$$

and

$$\begin{aligned} \rho &: M \longrightarrow F, \\ \rho(x)\left(e^i\right) &= m^i(x). \end{aligned}$$

Then

$$\begin{aligned} \pi(\rho(x)) &= \sum m^i(x)\, m_i, \\ &= x \end{aligned}$$

Thus

$$\pi \circ \rho = id_M$$

and therefore M is isomorphic to a direct summand of F and thus projective. □

Remark 4.3. *In the rest of this paper we will use the term dual basis just defined whenever there is no risk for confusion.*

Lemma 4.4. *Let H be a Hopf algebra with antipode s. Then $Hom_k(M, k)$ becomes an H-comodule by*

$$\delta(f)(m) = \sum f\left(m_{(0)}\right) \otimes s\left(m_{(1)}\right)$$

Proof. We must show that

$$(\delta \otimes 1) \circ \delta = (1 \otimes \Delta) \circ \delta.$$

Using the definition we get

$$\begin{aligned}
& ((\delta \otimes 1) \circ \delta)(m) \\
= \;& (\delta \otimes 1) \sum f\left(m_{(0)}\right) \otimes s\left(m_{(1)}\right) \\
= \;& \sum f\left(m_{(0)}\right) \otimes s\left(m_{(1)}\right) \otimes s\left(m_{(2)}\right) \\
= \;& (1 \otimes \Delta) \sum f\left(m_{(0)}\right) \otimes s\left(m_{(1)}\right) \\
= \;& (1 \otimes \Delta) \circ \delta
\end{aligned}$$

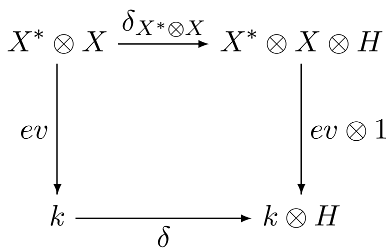
□

Theorem 4.5. *Let H be a Hopf algebra with antipode s. Then Mod^H is left rigid.*

Proof. Define ev to be the evaluation

$$\begin{aligned}
ev &: M^* \otimes M \longrightarrow k, \\
ev(f \otimes m) &: = f(m)
\end{aligned}$$

where $f \in Hom_k(M, k)$ and $m \in M$. We want ev to be a H-comodule homomorphism, that is, the following diagram has to commute:



Going right, down gives the following:

$$\begin{aligned}
& (ev \otimes 1) \circ (\delta_{M^* \otimes M})(f \otimes m) \\
= \;& (ev \otimes 1) \sum f \otimes m_{(0)} \otimes s\left(m_{(1)}\right) m_{(2)} \\
= \;& \sum f\left(m_{(0)}\right) \otimes s\left(m_{(1)}\right) m_{(2)} \\
= \;& f(m) \otimes 1
\end{aligned}$$

while going down, right gives

$$\begin{aligned}
\delta_k \circ ev(f \otimes m) &= \delta_k \sum f(m) \\
&= \sum f\left(m_{(0)}\right) \otimes 1 \\
&= f(m) \otimes 1
\end{aligned}$$

Since we assume that M is finitely generated and projective, we have a dual basis $\{m_i, m^i\}$, $m_i \in M$ and $m^i \in M^*$ such that $x = \sum m^i(x) m_i$. Define

$$\begin{aligned}
db &: k \longrightarrow M \otimes M^*, \\
db(1) &= \sum m_i \otimes m^i.
\end{aligned}$$

We then get the following equations:

$$\begin{aligned}\delta_{M\otimes M^*}\circ db\,(1_k) &= \delta_{M\otimes M^*}\sum m_i\otimes m^i\\ &= \sum m_{i(0)}\otimes m^i\otimes m_{i(1)}s\left(m_{i(2)}\right)\\ &= \sum m_i\otimes m^i\otimes 1_H\\ &= (db\otimes 1)\,\delta\,(1_k)\,,\end{aligned}$$

so db is also a H-comodule morphism.

The equations 3.3 and 3.4 follows from the definition of ev and db: First, 3.3 gives

$$\begin{aligned}& (1\otimes ev)\circ(db\otimes 1)\,(m)\\ =\;& (1\otimes ev)\left(\sum m_i\otimes m^i\otimes m\right)\\ =\;& \sum m_i m^i\,(m) = m\end{aligned}$$

3.4 follows:

$$\begin{aligned}& (ev\otimes 1)\circ(1\otimes db)\,(f\,(m))\\ =\;& (ev\otimes 1)\left(\sum f\otimes m_i\otimes m^i\right)(m)\\ =\;& \sum f\,(m_i)\,m^i\,(m)\\ =\;& f\,(m)\end{aligned}$$

□

4.3. **Braidings and quantizations.**

Definition 4.6. *A* ***cobraided bialgebra*** *is a bialgebra* $(H,\mu,\eta,\Delta,\varepsilon,r)$ *where* $r\in Hom_k\,(H\otimes H,k)$*, called the* ***cobraiding element*** *or* ***cobraider,*** *satisfies the following properties:*

(4.1)
(1) r *is* $\star$*-invertible (with inverse* $\bar{r}$*)*
(2) $\mu\circ\tau = r\star\mu\star\bar{r}$
(3) $r\circ(\mu\otimes 1) = r^{13}\star r^{12}$
(4) $r\circ(1\otimes\mu) = r^{13}\star r^{23}$

where

$$r^{12} = (r\otimes\varepsilon)\,,\; r^{23} = (\varepsilon\otimes r)\,,\; r^{13} = (\varepsilon\otimes r)\,(\tau_{H,H}\otimes id_H)$$

A Hopf algebra is cobraided if the underlying bialgebra is.

A braiding in Mod^H is uniquely determined by H being a cobraided bialgebra.

Theorem 4.7. *The category* Mod^H *is braided if* H *is a cobraided bialgebra. The braiding is given by*

$$\sigma_{X,Y}\,(x\otimes y) = \sum\left(y_{(0)}\otimes x_{(0)}\right)r\left(x_{(1)}\otimes y_{(1)}\right).$$

Proof. The definition comes from the H-comodule structure via the following composition:

$$X \otimes Y \overset{\delta_X \otimes \delta_Y}{\longrightarrow} X \otimes Y \otimes H \otimes H \overset{\tau \otimes 1 \otimes 1}{\longrightarrow} Y \otimes X \otimes H \otimes H \overset{1 \otimes 1 \otimes r}{\longrightarrow} Y \otimes X$$

(and thus can be seen as a generalization of the ordinary twist). First we must check that $\sigma_{X,Y}$ is a H-comodule homomorphism. This means that the following equation must hold:

$$\delta_{Y \otimes X} \circ \sigma_{X,Y} = (\sigma_{X,Y} \otimes 1) \circ \delta_{X \otimes Y}.$$

The left hand side is

$$\begin{aligned} & \delta_{Y \otimes X} \circ \sigma_{X,Y} (x \otimes y) \\ = \; & \delta_{Y \otimes X} \left(\sum \left(y_{(0)} \otimes x_{(0)} \right) \cdot r \left(x_{(1)} \otimes y_{(1)} \right) \right) \\ = \; & \sum \left(y_{(0)} \otimes x_{(0)} \right) \otimes \mu \left(y_{(1)} \otimes x_{(1)} \right) \cdot r \left(x_{(2)} \otimes y_{(2)} \right) \end{aligned}$$

which is the same as $(\tau \otimes (\mu\tau \star r))\, \delta\, (x \otimes y)$

$$\begin{aligned} & (\sigma_{X,Y} \otimes 1) \circ \delta_{X \otimes Y} (x \otimes y) \\ = \; & (\sigma_{X,Y} \otimes 1) \left(\sum x_{(0)} \otimes y_{(0)} \otimes \mu \left(x_{(1)} \otimes y_{(1)} \right) \right) \\ = \; & \sum \left(y_{(0)} \otimes x_{(0)} \right) \cdot r \left(x_{(1)} \otimes y_{(1)} \right) \otimes \mu \left(x_{(2)} \otimes y_{(2)} \right) \end{aligned}$$

and this is the same as $(\tau \otimes (r \star \mu))\, \delta\, (x \otimes y)$. By 4.1, eq. (2) these two actions are the same, so $\sigma_{X,Y}$ is a H-comodule homomorphism. To see that $\sigma_{X,Y}$ actually gives a braiding, we check that the triangles 3.5 commutes. We check the first: The top arrow gives

$$\begin{aligned} & (\sigma_{X \otimes Y, Z}) (x \otimes y \otimes z) \\ = \; & \sum z_{(0)} \otimes \left(x_{(0)} \otimes y_{(0)} \right) \cdot r \left(z_{(1)} \otimes (x \otimes y)_{(1)} \right) \\ = \; & \sum z_{(0)} \otimes x_{(0)} \otimes y_{(0)} \cdot r \left(z_{(1)} \otimes \mu \left(x_{(1)} \otimes y_{(1)} \right) \right) \\ & \text{by the } H\text{-comodule structure on } X \otimes Y, \end{aligned}$$

while the bottom arrows gives

$$\begin{aligned} & (\sigma_{X,Z} \otimes id_Y) (id_X \otimes \sigma_{Y,Z}) (x \otimes y \otimes z) \\ = \; & (\sigma_{X,Z} \otimes id_Y) \left(\sum x \otimes z_{(0)} \otimes y_{(0)} \cdot r \left(z_{(1)} \otimes y_{(1)} \right) \right) \\ = \; & \sum z_{(0)} \otimes x_{(0)} \otimes y_{(0)} \cdot r \left(z_{(1)} \otimes x_{(1)} \right) \cdot r \left(z_{(1)} \otimes y_{(2)} \right) \end{aligned}$$

But

$$r \left(z_{(1)} \otimes x_{(1)} \right) r \left(z_{(1)} \otimes y_{(2)} \right) = r^{13} \star r^{23} (x \otimes y \otimes z) = r \left(z_{(1)} \otimes \mu \left(x_{(1)} \otimes y_{(1)} \right) \right),$$

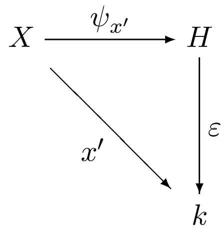
so we have the desired equality. The commutativity of the second triangle follows similarly. □

We can also show the converse (see Theorem 4.9 below). We need first the following Lemma:

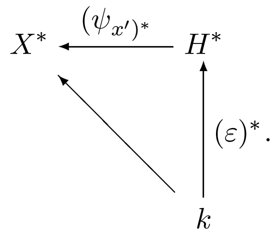
Lemma 4.8. *For any $x' \in X^*$ there exists a unique H-comodule homomorphism*

$$\psi_{x'} : X \longrightarrow H$$

such that the following diagram is commutative:



Proof. Take the dual of the above diagram:



One gets an H^*-module homomorphism

$$(\psi_{x'})^* : H^* \longrightarrow X^*$$

such that

$$(\psi_{x'})^* (\varepsilon^*) = x' \Longleftrightarrow \varepsilon \circ \psi_{x'} = x'.$$

Such a homomorphism $(\psi_{x'})^*$ exists and is unique because ε^* is the unit of an algebra H^*. Actually,

$$(\psi_{x'})^* (h') = h' \cdot x'$$

where $h' \in H^*$ and $\cdot$ is the multiplication. Finally,

$$\psi_{x'} = ((\psi_{x'})^*)^*,$$

and we are done. □

Theorem 4.9. *Let*

$$\sigma : Mod^H \times Mod^H \longrightarrow Mod^H$$

be a braiding, and let

$$r = (\varepsilon \otimes \varepsilon) \circ \sigma_{H \otimes H}.$$

Then for any $X, Y \in Ob\left(Mod^H\right)$ the homomorphism $\sigma_{X,Y}$ is equal to the composition

$$X \otimes Y \overset{\delta_X \otimes \delta_Y}{\longrightarrow} X \otimes Y \otimes H \otimes H \overset{\tau \otimes 1 \otimes 1}{\longrightarrow} Y \otimes X \otimes H \otimes H \overset{id \otimes r}{\longrightarrow} Y \otimes X.$$

Proof. Let $x' \in X^*, y' \in Y^*$. The following diagram is commutative:

$$\begin{array}{ccc} X^* \otimes Y^* & \xrightarrow{(\sigma_{X,Y})^*} & Y^* \otimes X^* \\ (\psi_{y'} \otimes \psi_{x'})^* \uparrow & & \uparrow (\psi_{x'} \otimes \psi_{y'})^* \\ H^* \otimes H^* & \xrightarrow[(\sigma_{H,H})^*]{} & H^* \otimes H^* \end{array}$$

It follows that

$$\begin{aligned} (\sigma_{X,Y})^* \left(x' \otimes y'\right) &= \\ (\sigma_{X,Y})^* \left((\varepsilon \circ \psi_{x'}) \otimes \left(\varepsilon \circ \psi_{y'}\right)\right) &= \\ (\sigma_{X,Y})^* \circ \left(\psi_{y'} \otimes \psi_{x'}\right)^* (\varepsilon^* \otimes \varepsilon^*) &= \\ \left(\psi_{x'} \otimes \psi_{y'}\right)^* \circ (\sigma_{H,H})^* (\varepsilon^* \otimes \varepsilon^*) &= \\ \left(\psi_{x'} \otimes \psi_{y'}\right)^* (r) &= \\ \left(\left(\psi_{y'}\right)^* \otimes (\psi_{x'})^*\right) (r \cdot (\varepsilon^* \otimes \varepsilon^*)) &= r \cdot \left(y' \otimes x'\right). \end{aligned}$$

Let

$$\begin{aligned} x &= \left(x'\right)^*, \\ y &= \left(y'\right)^*, \end{aligned}$$

and take now the dual of the above equality:

$$\begin{aligned} \sigma_{X,Y} (x \otimes y) &= \sigma_{X,Y} \left(\left(x'\right)^* \otimes \left(y'\right)^*\right) \\ &= (Id \otimes r) \circ (\tau \otimes 1 \otimes 1) \circ (\delta_X \otimes \delta_Y) (x \otimes y), \end{aligned}$$

and we are done. □

A quantization in Mod^H is a H-comodule morphism such that the condition 3.8 and 3.9 hold.

Theorem 4.10. *A quantization*

$$Q = Q_{X,Y} : X \otimes Y \longrightarrow X \otimes Y$$

can be defined by an element $q \in Hom_k (H \otimes H, k)$, called a ***coquantizer,*** *satisfying the following properties:*

(4.2) (1) $\mu \star q = q \star \mu$

(4.3) (2) $(q \circ (1 \otimes m)) \star (\varepsilon \otimes q) = (q \circ (m \otimes 1)) \star (q \otimes \varepsilon)$

(3) $q \circ (\eta \otimes 1) = \varepsilon \otimes \varepsilon = q \circ (1 \otimes \eta)$.

The quantization is then given by

$$Q_{X,Y} (x \otimes y) = \sum \left(x_{(0)} \otimes y_{(0)}\right) \cdot q \left(x_{(1)} \otimes y_{(1)}\right)$$

Proof. We define a morphism Q by the composition

$$X \otimes Y \overset{\delta_{x \otimes Y}}{\longrightarrow} X \otimes Y \otimes H \otimes H \overset{1 \otimes 1 \otimes q}{\longrightarrow} X \otimes Y,$$

and we want q to be a quantization. We must prove that Q is a H-comodule morphism and that it satisfies the conditions for a quantization. The proof that $Q_{X,Y}$ is a H-comodule morphism follows in the same way as in the proof of Theorem 4.7. We need the following diagram to commute:

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\delta_{X\otimes Y}} & X \otimes Y \otimes H \\ \Big\downarrow Q_{X,Y} & & \Big\downarrow Q_{X,Y} \otimes 1 \\ X \otimes Y & \xrightarrow{\delta_{X\otimes Y}} & X \otimes Y \otimes H \end{array}$$

Going right-down gives

$$\begin{aligned} & (Q_{X,Y} \otimes 1) \circ \delta_{X\otimes Y} (x \otimes y) \\ = \; & (Q_{X,Y} \otimes 1) \left(\sum x_{(0)} \otimes y_{(0)} \otimes \mu \left(x_{(1)} \otimes y_{(1)} \right) \right) \\ = \; & \sum \left(x_{(0)} \otimes y_{(0)} \right) \cdot q \left(x_{(2)} \otimes y_{(2)} \right) \otimes \mu \left(x_{(1)} \otimes y_{(1)} \right) \end{aligned}$$

which is $(1 \otimes (q \star \mu)) \circ \delta (x \otimes y)$. Going down-right gives

$$\begin{aligned} & \delta_{Y\otimes X} \circ Q_{X,Y} (x \otimes y) \\ = \; & \delta_{Y\otimes X} \left(\sum \left(x_{(0)} \otimes y_{(0)} \right) \cdot q \left(x_{(1)} \otimes y_{(1)} \right) \right) \\ = \; & \sum \left(x_{(0)} \otimes y_{(0)} \right) \otimes \mu \left(x_{(2)} \otimes y_{(2)} \right) \cdot q \left(x_{(1)} \otimes y_{(1)} \right) \end{aligned}$$

which is $(1 \otimes (\mu \star q)) \circ \delta (x \otimes y)$. We can now show that the definition of $Q_{X,Y}$ actually gives a quantization. First we see that equation (2) gives commutativity of the coherence diagram. The down-bottom part of the coherence diagram is the morphism

$$\begin{aligned} & Q_{X,Y\otimes Z} \circ (1 \otimes Q_{Y,Z}) \\ = \; & Q_{X,Y\otimes Z} \left(x \otimes \sum \left(y_{(0)} \otimes z_{(0)} \right) \cdot q \left(y_{(1)} \otimes z_{(1)} \right) \right) \\ = \; & \sum x_{(0)} \otimes y_{(0)} \otimes z_{(0)} \cdot q \left(x_{(1)} \otimes (y \otimes z)_1 \right) q \left(y_{(1)} \otimes z_{(1)} \right) \\ = \; & \sum x_{(0)} \otimes y_{(0)} \otimes z_{(0)} \cdot q \left(x_{(1)} \otimes \mu \left(y_{(1)} \otimes z_{(1)} \right) \right) q \left(y_{(1)} \otimes z_{(1)} \right) \\ = \; & \sum x_{(0)} \otimes y_{(0)} \otimes z_{(0)} \cdot \left((q \circ (1 \otimes m)) \star (\varepsilon \otimes q) \right) \left(x_{(1)} \otimes y_{(1)} \otimes z_{(1)} \right) \end{aligned}$$

while the top-down is described on elements by

$$\begin{aligned} & Q_{X\otimes Y,Z} \circ (Q_{X,Y} \otimes id_Z) (x \otimes y \otimes z) \\ = \; & Q_{X\otimes Y,Z} \left(\sum x_{(0)} \otimes y_{(0)} \otimes z_{(0)} \cdot q \left(x_{(1)} \otimes y_{(1)} \right) \right) \\ = \; & \sum x_{(0)} \otimes y_{(0)} \otimes z_{(0)} \cdot q \left(\mu \left(x_{(1)} \otimes y_{(1)} \right) \otimes z_{(1)} \right) q \left(x_{(1)} \otimes y_{(1)} \right) \\ = \; & \sum x_{(0)} \otimes y_{(0)} \otimes z_{(0)} \cdot \left((q \circ (m \otimes 1)) \star (q \otimes \varepsilon) \right) \end{aligned}$$

From this we see that the condition

$$(q \circ (1 \otimes m)) \star (\varepsilon \otimes q) = (q \circ (m \otimes 1)) \star (q \otimes \varepsilon)$$

is the same as requiring diagram 3.10 to commute. To show that the third condition of 4.2 is satisfied, we first note the following: As

$$\varepsilon \otimes \varepsilon = q \circ (1 \otimes \eta),$$

the morphism

$$(1 \otimes 1 \otimes (q \circ (1 \otimes \eta))) \circ (\delta_X \otimes \delta_k)$$

is the identity. Then the following diagram commutes:

$$\begin{array}{ccccc} X \otimes k & \xrightarrow{\delta} & X \otimes k \otimes H \otimes k & \xrightarrow{1 \otimes 1 \otimes 1 \otimes \eta} & X \otimes k \otimes H \otimes H \\ \downarrow{\iota^r} & & & & \downarrow{1 \otimes 1 \otimes q} \\ X & & \xleftarrow[\iota^r]{} & & X \otimes k \end{array}$$

But

$$\left[(1 \otimes 1 \otimes (q \circ (1 \otimes \eta))) \circ (\delta_X \otimes \delta_k)\right](x \otimes k) = Q_{X,k}$$

so we see that

$$i^r \circ Q_{X,k} = i^r.$$

In a similar manner the equality

$$q \circ (\eta \otimes 1) = \varepsilon \otimes \varepsilon$$

gives the equality

$$\iota^l \circ Q_{X,k} = i^l.$$

The converse implication follows the same procedure as in the proof of Theorem 4.9. □

5. Monoidal structure on the category of H'-modules

5.1. **Monoidality and rigidity.** Let H' be a bialgebra. The category $Mod_{H'}$ of H'-modules can be given a structure of a monoidal category by defining

$$\otimes : Mod_{H'} \times Mod_{H'} \longrightarrow Mod_{H'}$$

to be $\otimes_k$, the ordinary tensor product over k. As in the case of Mod^H, the pentagon and unity axioms are fulfilled through the properties of $\otimes_k$.We can define the H'-module structure on the tensor product by

$$H' \otimes M \otimes N \overset{\Delta\otimes1\otimes1}{\longrightarrow} H' \otimes H' \otimes M \otimes N \overset{1\otimes\tau\otimes1}{\longrightarrow} H' \otimes M \otimes H' \otimes N \overset{\rho'_M\otimes\rho'_N}{\longrightarrow} M \otimes N$$

Lemma 5.1. *If H' has an antipode s' then M^* has a H'-module structure by*

$$h \cdot f(v) = f\left(s'(h) \cdot v\right)$$

Proof. See, e.g., [Kas95, III, (5.6)]. □

Theorem 5.2. *If H' is a Hopf algebra with antipode s', then $Mod_{H'}$ is left rigid.*

Proof. Define

$$\begin{aligned} ev &: M^* \otimes M \longrightarrow k, \\ ev(f \otimes m) &= f(m). \end{aligned}$$

Using the H'-module structure on M^*we just defined, we can show that ev is a H'-module morphism.

$$\begin{aligned}
\left(ev \circ \rho'_{M^* \otimes M}\right)(h \otimes f \otimes m) &= ev\left(\sum h_1 \cdot f \otimes h_2 \cdot m\right) \\
&= ev\left(\sum f\left(s'\left(h_1\right)\right) \otimes h_2 \cdot m\right) \\
&= \sum h_2\left(f\left(s'\left(h_1\right) m\right)\right) \\
&= h \cdot f(m) \\
&= \rho'_k(h \otimes f(m)) \\
&= \rho'_k(1 \otimes ev)(h \otimes f \otimes m)
\end{aligned}$$

Now define

$$\begin{aligned}
db &: k \longrightarrow M \otimes M^*, \\
db(1) &= \sum m_i \otimes m^i
\end{aligned}$$

Then

$$\begin{aligned}
\left(db \circ \rho_k\right)(h \otimes 1) &= db(h) \\
&= \sum h_1 m_i \otimes h_2 m^i \\
&= h \sum m_i \otimes m^i \\
&= \rho_{M \otimes M^*}\left(h \otimes \sum m_i \otimes m^i\right) \\
&= \left(\rho_{M \otimes M^*} \circ (1 \otimes db)\right)(h \otimes 1),
\end{aligned}$$

so db and ev are H'-module morphisms. The validity of 3.3 and 3.4 follows as in the proof of 4.5. □

Remark 5.3. *We will also show the opposite implication in a more general setting in Part III.*

5.2. **Braidings and quantizations.** The Definitions and constructions of braidings in $Mod_{H'}$ follow similar to the comodule case.

Definition 5.4. *A **braided bialgebra** is a bialgebra $(H', \mu, \eta, \Delta, \varepsilon, R)$ where $R \in H' \otimes H'$, called the **braiding element** or **braider**, satisfies the following properties:*

(1) *R is invertible (with inverse $\bar{R}$)*
(2) $\tau\Delta = \bar{R} \cdot \Delta \cdot R$
(3) $(1_{'} \otimes \Delta) r = R_{12} \cdot R_{13}$
(4) $(\Delta \otimes 1) r = R_{23} \cdot R_{13}$

where

$$R_{12} = (R \otimes 1), \; R_{23} = (1 \otimes R), \; R_{13} = (id_{H'} \otimes \tau)(R \otimes 1)$$

A Hopf algebra is braided if the underlying bialgebra is.

Theorem 5.5. *The category $Mod_{H'}$ is braided if and only if H' is a braided bialgebra. The braiding is given by*

$$\begin{aligned}\sigma_{X,Y}(x\otimes y) &= R\cdot(y\otimes x)\\ &= \sum R^1 y\otimes R^2 x\end{aligned}$$

where

$$R=\sum R^1\otimes R^2$$

Proof. The definition comes from the H'-module structure via the following composition:

$$X\otimes Y\overset{R\otimes 1\otimes 1}{\longrightarrow}H'\otimes H'\otimes X\otimes Y\overset{1\otimes 1\otimes\tau}{\longrightarrow}H'\otimes H'\otimes Y\otimes X\overset{\rho'_{Y\otimes X}}{\longrightarrow}Y\otimes X$$

(and thus can be seen as a generalization of the ordinary twist). Assume H' is a braided bialgebra. First we must check that $\sigma_{X,Y}$ is a H-comodule homomorphism. This means that the following equation must hold:

$$\rho_{Y\otimes X}\circ\sigma_{X,Y}=(\sigma_{X,Y}\otimes 1)\circ\rho_{X\otimes Y}.$$

From the definition of the H-module structure of the tensor product we get the following:

$$\begin{aligned}&\left(\sigma_{X,Y}\circ\rho_{X\otimes Y}\right)(h\otimes x\otimes y)\\ =\ &\sigma_{X,Y}\left(\Delta(h)\cdot(x\otimes y)\right)\\ =\ &\sigma_{X,Y}\sum h_{(1)}x\otimes h_{(2)}y\\ =\ &\sum R^1h_{(2)}y\otimes R^2h_{(1)}x\\ =\ &(R\cdot\tau\Delta(h))\cdot(x\otimes y)\end{aligned}$$

The left hand side gives

$$\begin{aligned}&\left(\rho_{Y\otimes X}\circ(1\otimes\sigma_{X,Y})\right)(h\otimes x\otimes y)\\ =\ &\rho_{Y\otimes X}\left(h\otimes\left(\sum R^1y\otimes R^2x\right)\right)\\ =\ &\Delta(h)\left(\sum R^1y\otimes R^2x\right)\\ =\ &\Delta(h)\cdot R\cdot(x\otimes y)\end{aligned}$$

Now $R\cdot\tau\Delta(h)=\Delta(h)\cdot R$ by assumption, so $\sigma_{X,Y}$ is a H-module morphism. To see that $\sigma_{X,Y}$ actually gives a braiding, we check that the triangles 3.5 commutes. We check the second: The top arrow gives

$$\begin{aligned}&\left(\sigma_{X\otimes Y,Z}\right)(x\otimes y\otimes z)\\ =\ &\sum R^1z\otimes R^2(x\otimes y)\\ =\ &\sum R^1z\otimes\sum R^{2'}y\otimes R^{2''}x\\ =\ &\sum R^1z\otimes\Delta\left(R^2\right)(y\otimes x)\\ =\ &(1\otimes\Delta)\cdot R\cdot(z\otimes y\otimes x)\end{aligned}$$

while the bottom arrows gives

$$\begin{aligned} & \left(\left(\sigma_{X,Z} \otimes id_Y\right) \circ \left(id_X \otimes \sigma_{Y,Z}\right)\right)(x \otimes y \otimes z) \\ = & \left(\sigma_{X,Z} \otimes id_Y\right)\left(x \otimes \sum R^1 z \otimes R^2 y\right) \\ = & \sum R^1 R^1 z \otimes R^2 x \otimes R^2 y \\ = & R_{12} \cdot R_{13}\,(z \otimes y \otimes x) \end{aligned}$$

But

$$(1 \otimes \Delta) \cdot R = R_{12} \cdot R_{13}$$

by assumption, so the braiding triangle commutes. Commutativity of the other triangle follows by the same procedure.

For the other way round, suppose that we have a braiding σ. We can identify elements $x \in X$ with morphisms

$$\begin{aligned} \phi_x & : & H \longrightarrow X, \\ \phi_x(h) & = & x \cdot h \end{aligned}$$

The following diagram commutes by the naturality of a braiding:

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\sigma_{X,Y}} & Y \otimes X \\ {\scriptstyle \varphi_x \otimes \varphi_y} \uparrow & & \uparrow {\scriptstyle \varphi_y \otimes \varphi_x} \\ H \otimes H & \xrightarrow{\sigma_{H,H}} & H \otimes H \end{array}$$

If we define

$$R := \sigma_{H' \otimes H'}(1 \otimes 1)$$

we see that

$$\sigma_{X,Y}(x \otimes y) = R \cdot (y \otimes x)$$

As we have seen above, commutativity of the diagrams 3.5 shows conditions (3) and (4). Likewise, condition (2) is satisfied by σ being a H'-module homomorphism.

Defining

$$\bar{R} := \left(\sigma_{H' \otimes H'}\right)^{-1}(1 \otimes 1)$$

gives an inverse. □

Theorem 5.6. *A quantization in* $Mod_{H'}$

$$Q = Q_{X,Y} : X \otimes Y \longrightarrow X \otimes Y$$

is determined by an element $q \in H' \otimes H'$ *called* ***quantizer****, that satisfies the following properties:*

(5.1) (1) $q \cdot \Delta = \Delta \cdot q$

(5.2) (2) $(\Delta \otimes id_{H'})(q) \cdot (q \otimes 1) = (id_{H'} \otimes \Delta)(q) \cdot (1 \otimes q)$

(3) $(\varepsilon \otimes id_{H'})(q) = (id_{H'} \otimes \varepsilon)(q) = 1.$

The quantization is given by

$$Q_{X,Y}(x \otimes y) = q \cdot (x \otimes y) = \sum q_{(1)} x \otimes q_{(2)} y$$

where

$$q = \sum q_{(1)} \otimes q_{(2)}$$

Proof. Observe that $H' \otimes H' \approx Hom(k, H' \otimes H')$, so we can define a morphism Q by the composition

$$X \otimes Y \overset{q\otimes 1\otimes 1}{\longrightarrow} H' \otimes H' \otimes X \otimes Y \overset{\rho_X \otimes \rho_Y}{\longrightarrow} X \otimes Y$$

For Q to be a quantization we must show that Q is a H-module morphism and that it satisfies the conditions for a quantization. For Q to be a H-module morphism we must show that

$$Q_{X,Y} \circ \rho_{X\otimes Y} = \rho_{Y\otimes X} \circ (1 \otimes Q_{X,Y}).$$

$$\begin{aligned} & \left(Q_{X,Y} \circ \rho_{X\otimes Y}\right)(h \otimes x \otimes y) \\ = \; & Q_{X,Y}\left(\Delta(h) \cdot (x \otimes y)\right) \\ = \; & Q_{X,Y} \sum h_{(1)}x \otimes h_{(2)}y \\ = \; & \sum q_{(1)}h_{(1)}x \otimes q_{(1)}h_{(2)}y \\ = \; & (q \cdot \Delta(h)) \cdot (x \otimes y) \end{aligned}$$

The left hand side gives

$$\begin{aligned} & \left(\rho_{Y\otimes X} \circ (1 \otimes Q_{X,Y})\right)(h \otimes x \otimes y) \\ = \; & \rho_{Y\otimes X}\left(h \otimes \left(\sum q_{(1)}x \otimes q_{(2)}y\right)\right) \\ = \; & \Delta(h)\left(\sum q_{(1)}x \otimes q_{(1)}y\right) \\ = \; & \Delta(h) \cdot q \cdot (x \otimes y) \end{aligned}$$

We see that the condition $q \cdot \Delta(h) = \Delta(h) \cdot q$ makes $Q_{X,Y}$ a H-module morphism. We can now show when $Q_{X,Y}$ actually gives a quantization. First we see that equation (2) gives commutativity of the coherence diagram. The down-bottom part of the coherence diagram is the morphism

$$\begin{aligned} & Q_{X,Y\otimes Z} \circ (1 \otimes Q_{Y,Z}) \\ = \; & Q_{X,Y\otimes Z}\left(x \otimes \sum q_{(1)}y \otimes q_{(2)}z\right) \\ = \; & \sum q_{(1)}x \otimes q_{(2)} \sum q_{(1)}y \otimes q_{(2)}z \\ = \; & \sum q_{(1)}x \otimes \sum \left(q'_{(2)}\right) q_{(1)}y \otimes \left(q'_{(2)}\right) q_{(2)}z \\ = \; & (id_{H'} \otimes \Delta)(q) \cdot (1 \otimes q) \cdot (x \otimes y \otimes z) \end{aligned}$$

while the top-down is described on elements by

$$\begin{aligned} & \left(Q_{X\otimes Y,Z} \circ (Q_{X,Y} \otimes 1)\right)(x \otimes y \otimes z) \\ = \; & Q_{X\otimes Y,Z}\left(\left(\sum q_{(1)}x \otimes q_{(2)}y\right) \otimes z\right) \\ = \; & \sum \left(q'_{(1)}\right) q_{(1)}x \otimes \left(q''_{(1)}\right) q_{(2)}y \otimes q_{(2)}z \\ = \; & (\Delta \otimes id_H)(q) \cdot (q \otimes 1) \cdot (x \otimes y \otimes z), \end{aligned}$$

so for the diagram to commute we need

$$(id_{H'} \otimes \Delta)(q) \cdot (1 \otimes q) = (\Delta \otimes id_H)(q) \cdot (q \otimes 1).$$

Property (3) in the Theorem are the same same as requiring the diagrams 3.9 to commute, so if all three conditions are fulfilled, Q is a quantization.

It is left to show that any quantization is on the form

$$Q_{X,Y}(x \otimes y) = \sum q_{(1)}x \otimes q_{(2)}y.$$

Let us identify elements $x \in X$ with morphisms

$$\begin{array}{rcl} \varphi_x & : & H \longrightarrow X, \\ \varphi_x(h) & = & hx \end{array}$$

The following diagram commutes by the naturality of a quantization:

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{Q_{X,Y}} & X \otimes Y \\ \varphi_x \otimes \varphi_y \uparrow & & \uparrow \varphi_x \otimes \varphi_y \\ H \otimes H & \xrightarrow{Q_{H,H}} & H \otimes H \end{array}$$

If we define

$$q := Q_{H' \otimes H'}(1 \otimes 1)$$

we see that

$$Q_{X,Y}(x \otimes y) = q \cdot (x \otimes y) = \sum q_{(1)}x \otimes q_{(2)}y$$

□

6. Duality

Let H be a bialgebra that is finitely generated and projective as a k-module. In this case it is possible to obtain all the above structures in $Mod_{H'}$ by dualizing the constructions for Mod^H. Recall the following results from earlier Sections:

- the dual module $M^* = Hom(M, k)$ is a left dual in the category Mod^H and $Mod_{H'}$. (see Section 4.2).
- If H is a Hopf algebra then H^* is a Hopf algebra (see Proposition 2.2)
- If V is a right H-comodule, then V^* is a left H^*-module. Vice versa, if M is a left H-module, then M^* is a right H^*-comodule (see Proposition 1.15)

6.1. **Rigidity.**

Theorem 6.1. *Let H be left rigid. Then Mod_{H^*} is right rigid.*

Proof. Let V be a H-comodule. Then V^* is a H^*-module. The transpose of the map

$$ev : V^* \otimes V \longrightarrow k$$

is the map

$$k \longrightarrow (V^* \otimes V)^*$$

defined by

$$ev^*(1) = \left(\sum m^i \otimes m_i\right)^*$$

and for V f.g. projective we have isomorphisms

$$(V^* \otimes V)^* \overset{\lambda^{-1}}{\approx} V^* \otimes V^{**} \approx V^* \otimes V.$$

Define

$$bd' = \lambda^{-1} \circ ev^*.$$

Then db' is a H^*-module morphism since ev is a H-comodule morphism. We can similarly define $ve' = \lambda^{-1} \circ db^*$and show that ev' is a H^*-module morphism. Then the following holds by transposing 3.3

$$X^* \approx I \otimes X^* \overset{bd\otimes 1}{\longrightarrow} X^* \otimes X \otimes X^* \overset{1\otimes ve}{\longrightarrow} X^* \otimes I \approx X^*.$$

Similarly for the other equation defining right rigidity. □

Remark 6.2. *By defining $db' = \tau\lambda^{-1}ev^*$ and similarly for ev' we can formulate an alternative Theorem stating that Mod_{H^*} is left rigid.*

6.2. **Braidings.**

Theorem 6.3. *If H is a cobraided bialgebra with cobraiding element r, then H^* is a braided bialgebra with braiding element $R = \tau\lambda^{-1} \circ r^*$.*

Proof. Recall that a cobraided bialgebra is determined by an element

$$r \in Hom\,(H \otimes H, k) = (H \otimes H)^*$$

satisfying the set of equations 4.1. Define

$$R = \tau\lambda^{-1} \circ r^*.$$

We will show that R satisfies the equations determining a braided bialgebra.

The second equation in 4.1 gives

$$\begin{aligned} R \cdot (\tau \circ \Delta') &= (\tau\lambda^{-1} \circ r^*) \cdot (\tau \circ (\tau\lambda^{-1} \circ \mu^*)) \\ &= \tau\lambda^{-1}(r^* \cdot \tau\mu^*) \\ &= ((\tau\mu \star r)\,\lambda\tau)^* \\ &= ((r \star \mu)\,\lambda\tau)^* \\ &= \tau\lambda^{-1}(\mu^* \cdot r^*) \\ &= (\tau\lambda^{-1} \circ \mu^*) \cdot (\tau\lambda^{-1} \circ r^*) \\ &= \Delta' \cdot R \end{aligned}$$

The third equation gives

$$\begin{aligned} (\Delta' \otimes 1) \circ R &= ((\tau\lambda^{-1} \circ \mu^*) \otimes 1) \circ (\tau\lambda^{-1} \circ r^*) \\ &= (\tau\lambda^{-1}\mu^* \otimes 1) \circ \tau\lambda^{-1}r^* \\ &= (r\lambda\tau \circ (1 \otimes \mu\lambda\tau))^* \\ &= \left((r\lambda\tau)^{13} \star (r\lambda\tau)^{12}\right)^* \\ &= \left((r\lambda\tau)^{12}\right)^* \cdot \left((r\lambda\tau)^{13}\right)^* \\ &= (r\lambda\tau \otimes \varepsilon)^* \cdot ((\varepsilon \otimes r\lambda\tau) \circ (\tau \otimes 1))^* \\ &\quad (1 \otimes \tau\lambda^{-1}r^*) \cdot ((1 \otimes \tau) \circ (\tau\lambda^{-1}r^* \otimes 1)) \\ &= R_{23} \cdot R_{13} \end{aligned}$$

The rest follows similarly. Together this shows that R makes H^* a braided bialgebra and thus determines a braiding in Mod_{H^*}. □

6.3. **Quantizations.** Let $q' = \tau\lambda^{-1} \circ q^*$. We will show that q' determines a quantization in $Mod_{H'}$. First

$$\begin{aligned} q' \cdot \Delta' &= \tau\lambda^{-1} \circ q^* \cdot \tau\lambda^{-1} \circ \mu^* \\ &= \tau\lambda^{-1} (q^* \cdot \mu^*) \\ &= ((\mu \star q) \lambda\tau)^* \\ &= ((q \star \mu) \lambda\tau)^* \\ &= \tau\lambda^{-1} (\mu^* \cdot q^*) \\ &= \Delta' \cdot q' \end{aligned}$$

The other equations determining a quantizer follows similarly. This proves the following

Theorem 6.4. *Let H be a bialgebra that is f.g. projective as a k-module. Let q be a coquantizer in Mod^H. Then Mod_{H^*} is quantized with quantizer $q' = \tau\lambda^{-1} \circ q^*$.*

Part III. The inverse problem

7. Monoidal categories are comodule categories

We have seen how we can give a structure of monoidal category to comodules and modules over a Hopf algebra H. It is also possible to go the other way round. Given a suitable monoidal category and a forgetting functor to the category Mod_k, we can show that this category is equivalent to a category of (co-)modules over a bialgebra. The construction of braidings, quantizations and antipode can also be derived from the structure of the monoidal category.

In the following let k be a commutative ring and Mod_k be the category of f.g. projective k-modules.

Let C be a small monoidal category and let

$$G : \mathcal{C} \longrightarrow Mod_k$$

be a monoidal functor preserving sums. Let

$$G^* \otimes G : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow Mod_k$$

be the functor

$$(G^* \otimes G)(X) := G(X)^* \otimes G(X)$$

and let

$$H = Coend(G^* \otimes G)$$

It means that we have morphisms

$$f_X : G(X)^* \otimes G(X) \longrightarrow H$$

such that the diagram

$$\begin{array}{ccc} G(Y)^* \otimes G(X) & \xrightarrow{Id \ \otimes G(a)} & G(Y)^* \otimes G(Y) \\ {\scriptstyle G(a)^* \otimes Id}\Big\downarrow & & \Big\downarrow{\scriptstyle f_Y} \\ G(X)^* \otimes G(X) & \xrightarrow[f_X]{} & H \end{array} \tag{7.1}$$

commutes for each

$$a : X \longrightarrow Y$$

in $\mathcal{C}$, and such that H is universal object for this property. The diagram is a component of a dinatural transformation, called a **wedge,** and we use the notation $G^* \otimes G \overset{\cdot\cdot}{\longrightarrow} H$. We want to show that a wedge $G^* \otimes G \overset{\cdot\cdot}{\longrightarrow} V$ is equivalent to a natural transformation $G \longrightarrow G \otimes V$.

Lemma 7.1. *Given U, V, and $W \in Mod_k$, there is a natural isomorphism*

$$Hom_k(U^* \otimes V, W) \approx Hom_k(V, U \otimes W)$$

Proof. By Lemma 1.6 we have the isomorphism

$$Hom_k(V, U^* \otimes W) \approx Hom_k(V, Hom_k(U, W))$$

But we also have a natural isomorphism $\xi : U \longrightarrow U^{**}$given by $(\xi u)(h) = h(u)$. Substituting U with U^* in the above isomorphism, we get the natural isomorphism

$$f : U \otimes W \to Hom_k(U^*, W),$$

given by

$$f(u \otimes w)\, h := h(u)\, w.$$

This gives a natural isomorphism

$$Hom_k(V, U \otimes W) \approx Hom_k(V, Hom_k(U^*, W)).$$

By Lemma 1.7 we have the isomorphism

$$Hom_k(U^* \otimes V, W) \approx Hom_k(V, Hom_k(U^*, W))$$

Combining these two isomorphisms we get the desired isomorphism □

Proposition 7.2. *A natural transformation*

$$G \longrightarrow G \otimes V$$

is equivalent to a wedge $G^* \otimes G \overset{\cdot\cdot}{\longrightarrow} V$

Proof. Set

$$U = V = G(X), W = H$$

in the above Lemma. To the homomorphisms

$$Hom(G(X)^* \otimes G(X), H) \ni f_X : G(X)^* \otimes G(X) \longrightarrow H$$

it then correspond homomorphisms

$$Hom(G(X), G(X) \otimes H) \ni g_X : G(X) \longrightarrow G(X) \otimes H$$

We will show that these homomorphisms form a natural transformation of functors $G \longrightarrow G \otimes H$. A wedge can be described as follows: for $\alpha : X \longrightarrow Y$ in C we have a diagram

$$\begin{array}{ccc} Hom(G(Y)^* \otimes G(Y), H) & & \\ & \searrow^{t_2} & \\ & & Hom(G(Y)^* \otimes G(X), H) \\ & \nearrow_{t_1} & \\ Hom(G(X)^* \otimes G(X), H) & & \end{array} \tag{7.2}$$

and morphisms

$$\begin{aligned} f_X &\in Hom(G(X)^* \otimes G(X), H), \\ f_Y &\in Hom(G(Y)^* \otimes G(Y), H) \end{aligned}$$

such that $t_2(f_Y) = t_1(f_X)$, where

$$\begin{aligned} t_2(f_Y) &= f_Y \circ (1 \otimes G(\alpha)), \\ t_1(f_X) &= f_X \circ (G(\alpha)^* \otimes 1). \end{aligned}$$

By the above Lemma this transforms to the diagram

$$\begin{array}{ccc} Hom(G(Y), G(Y) \otimes H) & & \\ & \searrow^{s_2} & \\ & & Hom(G(X), G(Y) \otimes H) \\ & \nearrow_{s_1} & \\ Hom(G(X), G(X) \otimes H) & & \end{array}$$

where

$$\begin{aligned} s_1 (g_X) &= (G (\alpha) \otimes 1) \circ g_X, \\ s_2 (g_Y) &= g_Y \circ G (\alpha) \end{aligned}$$

The g_X, g_Y corresponding to the f_X, f_Y in the first diagram are exactly those that fulfils $s_1 (g_X) = s_2 (g_Y)$. This means that the following diagram has to commute for all α:

$$\begin{array}{ccc} G(X) & \xrightarrow{g_X} & G(X) \otimes H \\ {\scriptstyle G(\alpha)} \downarrow & & \downarrow {\scriptstyle G(\alpha) \otimes 1} \\ G(Y) & \xrightarrow{g_Y} & G(Y) \otimes H \end{array},$$

which is exactly the condition that the g_- form a natural transformation $G \longrightarrow G \otimes V$. Thus we have established a 1-1 correspondence between the f_- in the coend diagram and the g_- in $Nat (G, G \otimes V)$. □

Remark 7.3. *The family of $f\_$'s above form an end for the functor*

$$Hom (G (-)^* \otimes G (-) , H) .$$

By the above isomorphism this transforms to an end of

$$Hom (G (-) , G (-) \otimes H) ,$$

which is exactly $Nat (G, G \otimes H)$.

Remark 7.4. *In general, for a functor $G : \mathcal{C} \longrightarrow \mathcal{A}$ where $\mathcal{A}$ is rigid (see definition 3.4), the correspondence above can be given by the following: A natural transformation $g : G \longrightarrow G \otimes M$ defines a wedge with components*

$$G (X)^* \otimes G (X) \stackrel{1 \otimes g_X}{\longrightarrow} G (X)^* \otimes G (X) \otimes M \stackrel{ev \otimes 1}{\longrightarrow} M,$$

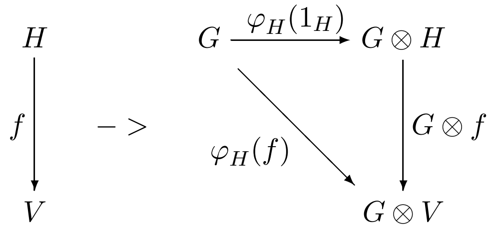
while a wedge defines a natural transformation with components

$$G (X) \stackrel{db \otimes 1}{\longrightarrow} G (X) \otimes G (X)^* \otimes G (X) \stackrel{1 \otimes f_X}{\longrightarrow} G (X) \otimes M$$

Corollary 7.5. *H represents the functor $V \longrightarrow Nat (G, G \otimes V)$, in other words, there is a natural isomorphism*

$$Hom_k (H, V) \stackrel{\varphi_V}{\longrightarrow} Nat (G, G \otimes V) .$$

Proof. By the universality of H there is a unique $f : H \longrightarrow V$ for any wedge $G^* \otimes G \overset{..}{\longrightarrow} V$. As the wedges are in $1-1$ correspondence wit natural transformations $G \longrightarrow G \otimes V$, we have the desired isomorphism. By Yoneda's Lemma the isomorphism is determined by $\varphi_H(1_H)$. The isomorphism is then given by $f \mapsto (1 \otimes f) \circ \varphi_H(1_H)$ as in the following diagram:



The components of a natural transformation

$$\phi : G \longrightarrow G \otimes M$$

can then be written as follows: let $\alpha \in \mathcal{C}$,

$$\alpha : X \longrightarrow Y.$$

Then ϕ can be expressed by the following composition:

$$G(X) \overset{\varphi_H(1_H)}{\longrightarrow} G(X) \otimes H \overset{G(\alpha)\otimes f}{\longrightarrow} G(Y) \otimes M,$$

where

$$f = \varphi_M^{-1}(\phi).$$

□

7.1. **Coalgebra and H-comodule structure.** From corollary 7.5 we have an isomorphism $Hom_k(H, H) \overset{\varphi_H}{\longrightarrow} Mor(G, G \otimes H)$ This gives a morphism

$$G \overset{\varphi_H(1)}{\longrightarrow} G \otimes H \overset{\varphi_H(1)\otimes 1}{\longrightarrow} G \otimes H \otimes H.$$

Define

$$\Delta = \varphi_{H\otimes H}^{-1}((\varphi_H(1_H) \otimes 1) \circ \varphi_H(1_H)) : H \longrightarrow H \otimes H$$

We also have an isomorphism

$$Hom_k(H, k) \overset{\varphi_k}{\longrightarrow} Mor(G, G \otimes k),$$

and the isomorphism $G(X) \approx G(X) \otimes k$ gives an $e \in Mor(G, G \otimes k)$ .Define

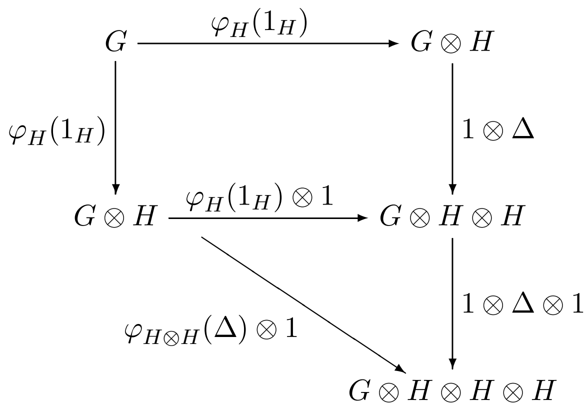
$$\varepsilon = \varphi_k^{-1}(e) : H \longrightarrow k$$

We will show that Δ and ε gives a coalgebra structure for H over k.

1: Δ is coassociative, that is $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$: First, the definition of Δ can be written as

$$\varphi_{H\otimes H}(\Delta) = (\varphi_H(1_H) \otimes 1) \circ \varphi_H(1_H)$$

The diagram



commutes: The upper rectangle commutes by the above definition of Δ , while the triangle commutes by the same definition, tensored with H on the right. From this diagram, and by repeated using the definition of Δ and the definition of the isomorphism φ_H given in the above proof, we get the following:

$$
\begin{aligned}
& \varphi_{H\otimes H\otimes H}\left((\Delta \otimes 1) \circ \Delta\right) \\
= \; & \left(\varphi_{H\otimes H}\left(\Delta\right) \otimes 1\right) \circ \varphi_H\left(1_H\right) \\
& \text{(from the diagram)} \\
= \; & \left(\left(\left(\varphi_H\left(1_H\right) \otimes 1\right) \circ \varphi_H\left(1_H\right)\right) \otimes 1\right) \circ \varphi_H\left(1_H\right) \\
& \text{(by the definition of } \Delta\text{)} \\
= \; & \left(\varphi_H\left(1_H\right) \otimes 1 \otimes 1\right) \circ \left(\varphi_H\left(1_H\right) \otimes 1\right) \circ \varphi_H\left(1_H\right) \\
& \text{by rearranging} \\
= \; & \left(\varphi_H\left(1_H\right) \otimes 1 \otimes 1\right) \circ \left(1 \otimes \Delta\right) \circ \varphi_H\left(1_H\right) \\
& \text{(by the definition of } \Delta\text{)} \\
= \; & \left(1 \otimes 1 \otimes \Delta\right) \circ \left(\varphi_H\left(1_H\right) \otimes 1\right) \circ \varphi_H\left(1_H\right) \\
& \text{by rearranging} \\
= \; & \left(1 \otimes 1 \otimes \Delta\right) \circ \left(1 \otimes \Delta\right) \circ \varphi_H\left(1_H\right) \\
& \text{(by the definition of } \Delta\text{)} \\
= \; & \left(1 \otimes \left((1 \otimes \Delta) \circ \Delta\right)\right) \circ \varphi_H\left(1_H\right) \\
= \; & \varphi_{H\otimes H\otimes H}\left((1 \otimes \Delta) \circ \Delta\right) \\
& \text{by corollary 7.5,}
\end{aligned}
$$

so comultiplication is coassociative.

2. ε is a unit: We must show that $(\varepsilon \otimes 1) \circ \Delta = 1_H = (1 \otimes \varepsilon) \circ \Delta$. First we show the equality $(\varepsilon \otimes 1) \circ \Delta = 1_H$. The following diagram commutes

by the isomorphism described in Lemma 7.5 :

$$\begin{array}{ccccc} G & \xrightarrow{\varphi_H(1_H)} & G\otimes H & & \\ \Big\downarrow{\scriptstyle \varphi_H(1_H)} & & \Big\downarrow{\scriptstyle \varphi_H(1_H)\otimes 1} & \searrow^{\varphi_k(\varepsilon)\otimes 1} & \\ G\otimes H & \xrightarrow{1\otimes\Delta} & G\otimes H\otimes H & \xrightarrow[1\otimes\varepsilon\otimes 1]{} & G\otimes k\otimes H \end{array}$$

The "bottom" part is the morphism $\varphi_{k\otimes H}\left((\varepsilon\otimes 1)\circ\Delta\right)$, so we have

$$\begin{aligned} \varphi_{k\otimes H}\left((\varepsilon\otimes 1)\circ\Delta\right) &= \left(\varphi_k\left(\varepsilon\right)\otimes 1\right)\circ\varphi_H\left(1_H\right) \\ &= \varphi_{k\otimes H}\left(1_H\right) \end{aligned}$$

By some small changes we get the following diagram

$$\begin{array}{ccccc} G & \xrightarrow{\varphi_H(1_H)} & G\otimes H & & \\ \Big\downarrow{\scriptstyle \varphi_H(1_H)} & & \Big\downarrow{\scriptstyle \varphi_H(1_H)\otimes 1} & \searrow^{\varphi_H(1)\otimes\varepsilon} & \\ G\otimes H & \xrightarrow{1\otimes\Delta} & G\otimes H\otimes H & \xrightarrow[1\otimes 1\otimes\varepsilon]{} & G\otimes H\otimes k \end{array}.$$

The right triangle is still commutative, so we have

$$\begin{aligned} \varphi_{H\otimes k}\left((1\otimes\varepsilon)\,\Delta\right) &= \left(\varphi_H\left(1_H\right)\otimes\varepsilon\right)\varphi_H\left(1_H\right) \\ &= \varphi_{H\otimes k}\left(1_H\right) \end{aligned}$$

1. and 2. together makes (H,Δ,ε) a coalgebra over k .

We can also define a H-comodule structure on $G(X)$ by the map

$$\delta_X = \varphi_H\left(1_H\right) : G\left(X\right) \longrightarrow G\left(X\right)\otimes H$$

To see that this actually defines a comodule structure we must show that the diagram

$$\begin{array}{ccc} G(X) & \xrightarrow{\delta} & G(X)\otimes H \\ \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \delta\otimes 1} \\ G(X)\otimes H & \xrightarrow[1\otimes\Delta]{} & G(X)\otimes H\otimes H \end{array}$$

commutes. But $(1\otimes\Delta)\circ\delta = \varphi_{H\otimes H}\left(\Delta\right)$, so for the above diagram to commute we must require that $\varphi_{H\otimes H}\left(\Delta\right) = \left(\varphi_H\left(1_H\right)\otimes 1\right)\circ\varphi_H\left(1_H\right)$. This is the definition of Δ, so we have a H-comodule structure on $G(X)$ by δ.

We have then proved the following:

Proposition 7.6. *Let*

$$\varphi_H : Hom_k\left(H,H\right) \longrightarrow Nat\left(G, G\otimes H\right)$$

and

$$\varphi_k : Hom\left(H,k\right) \longrightarrow Nat\left(G, G\otimes k\right).$$

Define

$$\Delta = \varphi_{H\otimes H}^{-1}\left(\left(\varphi_H\left(1_H\right)\otimes 1\right)\circ\varphi_H\left(1_H\right)\right) : H \longrightarrow H\otimes H$$

and

$$\varepsilon = \varphi_k^{-1}\left(e\right) : H \longrightarrow k.$$

Then (H,Δ,ε) *is a coalgebra over* k.

Furthermore, let

$$\delta = \varphi_H\left(1_H\right) : G\left(X\right) \longrightarrow G\left(X\right)\otimes H.$$

Then δ *defines a* H*-comodule structure on all* $G\left(X\right)$, $X\in Ob\left(C\right)$.

7.2. **Relations between $\mathcal{C}$ and Mod^H.** Let $U_H : Mod^H \longrightarrow Mod_k$ be the forgetting functor. The comodule structure in 7.1 gives a functor $F : \mathcal{C} \longrightarrow Mod^H$ such that $G = U_H F$. It would be interesting to know when this functor is actually an equivalence. We need some definitions:

Definition 7.7. *A functor* $S : \mathcal{A} \longrightarrow \mathcal{B}$ *between two categories* $\mathcal{A}$ *and* $\mathcal{B}$ *is an* ***equivalence*** *if there exist a functor* $T : \mathcal{B} \longrightarrow \mathcal{A}$ *such that*

$$\begin{aligned} ST &\approx I_A : \mathcal{A} \longrightarrow \mathcal{A} \text{ and} \\ TS &\approx I_B : \mathcal{B} \longrightarrow \mathcal{B}. \end{aligned}$$

$\mathcal{A}$ *and* $\mathcal{B}$ *are then called* ***equivalent****.*

A functor $S : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be **full** when to every pair a, a' in $\mathcal{A}$ and to every arrow $g : S\left(a\right) \longrightarrow S\left(a'\right)$ in B there is an arrow $f : a \longrightarrow a'$ in $\mathcal{A}$ such that $g = S\left(f\right)$. A functor $S : \mathcal{A} \longrightarrow \mathcal{B}$ is **faithful** if when to every pair a, a' in A and to every pair of parallel arrows $f, f' : a \longrightarrow a'$ the equality $S\left(f\right) = S\left(f'\right)$ implies that $f = f'$. Finally, a functor $S : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be **essentially surjective** if every $b \in \mathcal{B}$ is isomorphic to $S\left(a\right)$ for some $a \in \mathcal{A}$.

Theorem 7.8. *For a functor* $S : \mathcal{A} \longrightarrow \mathcal{B}$ *to be an equivalence it is necessary and sufficient that* S *is full, faithful and essentially surjective.*

Proof. See [ML98, Thm 1, p.93] □

This problem has been thoroughly studied by Saavedra Rivano in [SR72], where he gives a complete characterization of monoidal categories which are equivalent to categories of comodules over a bialgebra. The reasonings and proofs are too complicated to include in this thesis, so we only refer to some of the results that are close to our case. If Mod_k is the category of f.g. k-modules, the equivalence is proven under the assumptions that k is Noetherian, $\mathcal{C}$ is abelian and that G is faithful and exact. If in addition the ring k is a local ring of dimension ≤ 1 we get the following result:

Theorem 7.9. *Let* $\mathcal{C}$ *be a* k*-linear abelian category and* $G : \mathcal{C} \longrightarrow Mod_k$ *a faithful and exact functor. Then there exist a flat* k*-coalgebra* H *and an equivalence*

$$F : \mathcal{C} \longrightarrow Mod^H$$

such that $G = UF$, *where* U *is the forgetting functor, if and only if the following is satisfied:*

Let $\mathcal{C}_0$ *be the subcategory of* $\mathcal{C}$ *consisting of all* X *such that* $G\left(X\right)$ *is a f.g. projective* k*-module. Then every object in* $\mathcal{C}$ *is a quotient of an object in* $\mathcal{C}_0$.

Proof. See [SR72], thm.2.6.1. □

In the case where G is a functor to *vec*, the category of finite dimensional vector spaces, Peter Schauenburg has proved the following in [Sch92]

Theorem 7.10. *Assume that k is a field. Let $\mathcal{C}$ be a k-linear abelian category and let*

$$G : \mathcal{C} \longrightarrow vec$$

be a k-linear, exact and faithful functor. Let

$$H = coend\,(G^* \otimes G)$$

and let

$$U : Mod^H \longrightarrow vec$$

be the forgetting functor. Then there exist a monoidal equivalence

$$F : \mathcal{C} \longrightarrow Mod^H,$$

such that $G = UF$.

Proof. See [Sch92, Theorem 2.2.8]. □

It was our intention to find under which conditions we could get this equivalence when G is a monoidal functor from $\mathcal{C}$ into the category of f.g. projective modules over a ring k, but due to lack of time this has not been accomplished. However, a reasonable Conjecture has been formulated (Conjecture 11.9), and a plan how to prove it is proposed. Some steps are fulfilled completely, others are made only partially, and it is reasonable to expect that the Conjecture will have been finally proved.

7.3. H is a bialgebra.

Lemma 7.11. *The map*

$$Hom_k\,(H \otimes H, V) \xrightarrow{\Phi_V} Nat\,(G \otimes G, G \otimes G \otimes V)$$

given by

$$(\Phi\,(\alpha)\,(x \otimes y) = \sum x_{(0)} \otimes y_{(0)} \otimes \alpha\left(x_{(1)} \otimes y_{(1)}\right)$$

for $\alpha : H \otimes H \longrightarrow V, x \in G\,(X)$ and $y \in G\,(Y)$ is an isomorphism.

Proof. Let a morphism $G\,(X) \otimes G\,(Y) \longrightarrow G\,(X) \otimes G\,(Y) \otimes V$ be defined by the following composition:

$$\begin{aligned} & G\,(X) \otimes G\,(Y) \xrightarrow{\delta_X \otimes \delta_Y} G\,(X) \otimes H \otimes G\,(Y) \otimes H \xrightarrow{1 \otimes \tau \otimes 1} \\ \longrightarrow\quad & G\,(X) \otimes G\,(Y) \otimes H \otimes H \xrightarrow{1 \otimes 1 \otimes \alpha} G\,(X) \otimes G\,(Y) \otimes V \end{aligned}$$

where $\alpha \in Hom_k\,(H \otimes H, V)$. The naturality of Φ_V makes this a natural transformation $G \otimes G \longrightarrow G \otimes G \otimes V$ that is uniquely defined by α. On the other hand, $H \otimes H$ is $coend\,(G \otimes G) = coend\,(G) \otimes coend\,(G)$, so we have a 1-1 correspondence between $Hom_k\,(H \otimes H, V)$ and $Nat\,(G \otimes G, G \otimes G \otimes V)$, where the universal morphism can be written as

$$\varrho = (1 \otimes \tau \otimes 1) \circ (\delta \otimes \delta)$$

The isomorphism is given by a mapping

$$Hom_k\,(H \otimes H, V) \xrightarrow{\Phi_V} Nat\,(G \otimes G, G \otimes G \otimes V)$$

making the following diagram commutative

$$\begin{array}{ccc} G(X)\otimes G(Y) & \xrightarrow{\varrho} & G(X)\otimes G(Y)\otimes H\otimes H \\ & \searrow^{\Phi_V(\alpha)} & \Big\downarrow{\scriptstyle 1\otimes 1\otimes\alpha} \\ & & G(X)\otimes G(Y)\otimes V \end{array}$$

Let ϱ be described on elements by

$$\varrho\,(x\otimes y)=\sum\left(x_{(0)}\otimes y_{(0)}\right)\otimes\left(x_{(1)}\otimes y_{(1)}\right).$$

We see that

$$\left(\Phi_V\,(\alpha)\right)(x\otimes y)=\sum\left(x_{(0)}\otimes y_{(0)}\right)\otimes\alpha\sum\left(x_{(1)}\otimes y_{(1)}\right)$$

gives the desired isomorphism. □

We now want to give a bialgebra structure on H.

Proposition 7.12. *Let*

$$\delta'_{X\otimes Y}:G\,(X)\otimes G\,(Y)\approx G\,(X\otimes Y)\overset{\delta_{X\otimes Y}}{\longrightarrow}G\,(X\otimes Y)\otimes H\approx G\,(X)\otimes G\,(Y)\otimes H$$

and

$$\mu=\Phi_H^{-1}\left(\delta'\right):H\otimes H\longrightarrow H.$$

Let

$$\eta:k\approx G\,(e)\overset{\delta}{\longrightarrow}G\,(e)\otimes H\approx H.$$

Then $(H,\mu,\eta,\Delta,\varepsilon)$ is a bialgebra over k.

Proof. If (H,μ,η) is an algebra, it is enough to show that μ and η are k-coalgebra morphisms. Let $\alpha\in Hom_k\,(H\otimes H,H\otimes H)$ be the homomorphism

$$H\otimes H\overset{\Delta}{\longrightarrow}H\otimes H\otimes H\otimes H\overset{\mu\otimes\mu}{\longrightarrow}H\otimes H$$

and $\beta\in Hom_k\,(H\otimes H,H\otimes H)$ be

$$H\otimes H\overset{\mu}{\longrightarrow}H\overset{\Delta}{\longrightarrow}H\otimes H.$$

α is mapped to the commutative diagram

$$\begin{array}{ccc} G\otimes G & \xrightarrow{\delta\otimes\delta} & G\otimes G\otimes H\otimes H \\ \Big\downarrow{\scriptstyle \Phi(\alpha)} & & \Big\downarrow{\scriptstyle 1\otimes 1\otimes\Delta} \\ G\otimes G\otimes H\otimes H & \xleftarrow[1\otimes 1\otimes\mu\otimes\mu]{} & G\otimes G\otimes H\otimes H\otimes H\otimes H \end{array}$$

If we denote $\mu\,(a \otimes b)$ by ab the previous diagram can be described on elements by

$$\begin{array}{ccc}
x \otimes y & \xrightarrow{\delta \otimes \delta} & x_1 \otimes y_1 \otimes x_2 \otimes y_2 \\
\downarrow & & \downarrow {\scriptstyle 1 \otimes 1 \otimes \Delta} \\
x_1 \otimes y_1 \otimes (x_2 y_2)_1 \otimes (x_2 y_2)_2 & \xleftarrow[1 \otimes 1 \otimes \mu \otimes \mu]{} & x_1 \otimes y_1 \otimes (x_2 \otimes y_2)_1 \otimes (x_2 \otimes y_2)_2
\end{array}$$

(omitting the summation signs). In the same manner we can describe β on elements by the following diagram:

$$\begin{array}{ccc}
x \otimes y & \xrightarrow{\delta \otimes \delta} & x_1 \otimes y_1 \otimes x_2 \otimes y_2 \\
\downarrow & & \downarrow {\scriptstyle 1 \otimes 1 \otimes \mu} \\
x_1 \otimes y_1 \otimes (x_2 y_2)_1 \otimes (x_2 y_2)_2 & \xleftarrow[1 \otimes 1 \otimes \Delta]{} & x_1 \otimes y_1 \otimes x_2 y_2
\end{array}$$

We then see that

$$\Phi_{H \otimes H}\left((\mu \otimes \mu) \circ \Delta\right) = \Phi_{H \otimes H}\left(\Delta \mu\right).$$

Since Φ is an isomorphism, we have

$$(\mu \otimes \mu) \circ \Delta = \Delta \circ \mu.$$

We also must show that $\varepsilon \mu = \varepsilon \otimes \varepsilon$. The following diagram commutes:

$$\begin{array}{ccc}
G \otimes G & \xrightarrow{\delta \otimes \delta} & G \otimes G \otimes H \otimes H \\
{\scriptstyle \xi} \downarrow & & \downarrow {\scriptstyle \xi \otimes 1 \otimes \mu} \\
(G \otimes G) & \xrightarrow[\delta]{} & (G \otimes G) \otimes H \\
& {\scriptstyle \Phi(\epsilon)} \searrow & \downarrow {\scriptstyle 1 \otimes \epsilon} \\
& & (G \otimes G) \otimes k
\end{array}$$

The bottom triangle commutes by the definition of ε, while the upper quadrate commutes by the definition of μ. The left hand side is the morphism $\Phi_{H \otimes H}\left(\varepsilon \otimes \varepsilon\right)$, while the right side is $\Phi_{H \otimes H}\left(\varepsilon \mu\right)$.

To show that η is a k-coalgebra morphism we must show that

$$(\eta \otimes \eta) \circ \Delta = \Delta \circ \eta$$

We can associate η with 1_H, and by the definition of Δ we have that $\Delta\left(1_H\right) = (1 \otimes 1)$, which was to be proved.

It is left to show that (H, μ, η) actually **is** an algebra.

For the associativity, note that the following diagram commutes:

$$\begin{array}{ccc}
(G(X)\otimes G(Y))\otimes G(Z) & \xrightarrow{\delta^3} & G(X)\otimes G(Y)\otimes G(Z)\otimes (H\otimes H)\otimes H \\
\Big\downarrow {\scriptstyle \xi\otimes 1} & & \Big\downarrow {\scriptstyle \xi\otimes 1\otimes \mu\otimes 1} \\
G(X\otimes Y)\otimes G(Z) & \xrightarrow{\delta\otimes\delta} & G(X\otimes Y)\otimes G(Z)\otimes H\otimes H \\
\Big\downarrow {\scriptstyle \xi} & & \Big\downarrow {\scriptstyle \xi\otimes\mu} \\
G((X\otimes Y)\otimes Z) & \xrightarrow[\delta]{} & G((X\otimes Y)\otimes Z)\otimes H
\end{array}$$

by the definition of μ and the monoidality of G. This diagram describes the morphism

$$G\,(X)\otimes G\,(Y)\otimes G\,(Z)\longrightarrow G\,((X\otimes Y)\otimes Z)\otimes H$$

which by the definition of μ corresponds to the morphism $\mu\circ(\mu\otimes 1)$. In the same way $\mu\circ(1\otimes\mu)$ corresponds to

$$G\,(X)\otimes G\,(Y)\otimes G\,(Z)\longrightarrow G\,(X\otimes(Y\otimes Z))\otimes H.$$

But by the monoidality of G,

$$\begin{array}{rcl}
G\,(X\otimes(Y\otimes Z)) & \approx & G\,(X)\otimes(G\,(Y)\otimes G\,(Z))\approx \\
(G\,(X)\otimes G\,(Y))\otimes G\,(Z) & \approx & G\,((X\otimes Y)\otimes Z)\,,
\end{array}$$

so $\mu\circ(\mu\otimes 1)\approx\mu\circ(1\otimes\mu)$. □

7.4. **Correspondence of the direct and inverse constructions of Mod^H.** While we in Part II used the bialgebra structure on a coalgebra H to give a monoidal structure on Mod^H, we have in this Part used monoidality of the category $\mathcal{C}$ and the forgetting functor G to construct a comodule category $Mod^{H'}$, the right comodules over the coalgebra

$$H' = coend\,(G^*\otimes G)\,.$$

We will show that the two constructions in a sense are inverse to each other. First we see that F preserves the tensor structure, that is, F is a monoidal functor. To show this we must show that ξ_0 and ξ_2 are H-comodule isomorphisms, and that the diagrams 3.1 and 3.2 are commutative diagrams of H-comodule morphisms. Now by construction Mod^H consists of k-modules $G\,(X)$ endowed with an H-comodule structure, so it is enough to validate the diagrams and morphisms for elements $G\,(X)$. That ξ_0 and ξ_2 are H-comodule isomorphisms follows immediately from the monoidality of G. We show this for ξ_2: We know that

$$G\,(X)\otimes G\,(Y)\approx G\,(X\otimes Y)$$

as k-modules. But H is a bialgebra, so we have a H-comodule structure on $G\,(X)\otimes G\,(Y)$. This gives commutativity of the following diagram, which shows that ξ_2 is a H-comodule isomorphism:

$$\begin{array}{ccc} G(X)\otimes G(Y) & \xrightarrow{\delta_{G(X)}\otimes\delta_{G(Y)}} & G(X)\otimes G(Y)\otimes H \\ \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi_2\otimes 1} \\ G(X\otimes Y) & \xrightarrow[\delta_{G(X\otimes Y)}]{} & G(X\otimes Y)\otimes H \end{array}$$

The discussion of the associativity of μ in the proof of Proposition 7.12 shows that the commutativity of 3.1 is taken care of by the associativity of μ. The commutativity of 3.2 again follows from the monoidality of G. It then follows that F is a monoidal functor, so Mod^H is constructed by carrying the monoidal structure of $\mathcal{C}$ over to Mod^H. We have showed:

Proposition 7.13. *Let $\mathcal{C}$ be a monoidal category and $G : \mathcal{C} \longrightarrow Mod_k$ a monoidal functor. Let*

$$H = coend\,(G^* \otimes G)$$

Let

$$F : \mathcal{C} \longrightarrow Mod^H$$

be a functor such that $G = UF$ where

$$U : Mod^H \longrightarrow Mod_k$$

is the forgetting functor. Then F is monoidal.

The monoidal structure on Mod^H is described in the same way as in the direct case: the multiplication in Part III is defined as the inverse under Φ of the homomorphism

$$\begin{aligned} \delta'_{X\otimes Y} \quad &: \quad G(X)\otimes G(Y) \approx G(X\otimes Y) \overset{\delta_{X\otimes Y}}{\longrightarrow} G(X\otimes Y)\otimes H \approx \\ &\approx \quad G(X)\otimes G(Y)\otimes H, \\ \delta' \quad &\in \quad Nat\,(G\otimes G, G\otimes G\otimes H) \end{aligned}$$

But from the proof of Lemma 7.11 we see that then μ is defined by the composition

$$G(X)\otimes G(Y) \overset{\delta_X\otimes\delta_Y}{\longrightarrow} G(X)\otimes G(Y)\otimes H\otimes H \overset{1\otimes 1\otimes \mu}{\longrightarrow} G(X)\otimes G(Y)\otimes H,$$

and this is exactly the same way we defined the monoidal structure of Mod^H in Part II.

Now let us go the opposite way: Let H be a bialgebra and let $\mathcal{C} = Mod^H$. Let $G : Mod^H \longrightarrow Mod_k$ be the forgetting functor and let $H' = coend\,(G^* \otimes G)$.

Lemma 7.14. *Let H be a coalgebra that is finitely generated and projective as a k-module. Let V be a H-comodule. Then we have an isomorphism*

$$\psi : Hom_k\,(H, V) \approx Nat\,(G, G\otimes V)$$

given by

$$\psi\,(f) = (1\otimes f)\circ\delta$$

Proof. We define an inverse mapping as follows: Let

$$\phi \in Nat\,(G, G \otimes V)$$

and let $\bar{\psi}(\phi)(h) = (\varepsilon \otimes 1)\,\phi_H(h)$. Then

$$\begin{aligned}
\bar{\psi} \circ (\psi(f))(h) &= (\varepsilon \otimes 1) \circ \psi(f)(h) \\
&= [(\varepsilon \otimes 1) \circ (1 \otimes f) \circ \delta](h) \\
&= f(h)
\end{aligned}$$

For the other way

$$\begin{aligned}
\left(\psi \circ \bar{\psi}(\phi)\right)_M &= \left(1 \otimes \bar{\psi}(\phi)\right) \circ \delta_M \\
&= (1 \otimes \varepsilon \otimes 1) \circ \left(\phi_{M \otimes H}\right) \circ \delta_M \\
&= (1 \otimes \varepsilon \otimes 1) \circ (1 \otimes \phi_H) \circ \delta_M \\
&= (1 \otimes \varepsilon \otimes 1) \circ (\delta_M \otimes 1) \circ \phi_M \\
&= \phi_M
\end{aligned}$$

The third equality follows from the fact that H itself is a f.g. projective module and also a H-comodule, so we can write $\phi_{M \otimes H} = 1 \otimes \phi_H$ by letting H carry the comodule structure of $M \otimes H$. □

Proposition 7.15. *Let $G : Mod^H \longrightarrow Mod_k$ be the forgetting functor and let $H' = coend\,(G)$. Then $H' \approx H$.*

Proof. The Lemma shows that

$$Hom_k\,(H, V) \approx Nat\,(G, G \otimes V)\,.$$

But corollary 7.5 shows that

$$Hom_k\left(H', V\right) \approx Nat\,(G, G \otimes V)$$

We then have a morphism

$$f : H' \longrightarrow H$$

that give the same module structure: δ'can be uniquely written as

$$X \xrightarrow{\delta'} X \otimes H' \xrightarrow{Id \otimes f} X \otimes H$$

and vice versa. Then it is enough to show that f is a morphism of bialgebras. This means that we have to show that f is both an algebra and a coalgebra morphism. That f is an algebra morphism follows from the fact that the definition of multiplication is essentially the same in $Mod^{H'}$ and Mod^H . So we only have to show that f is a coalgebra morphism, that is, we must show that

$$\Delta \circ f = (f \otimes f) \circ \Delta'$$

and

$$\varepsilon \circ f = \varepsilon$$

Before we go on, note that the correspondence of the comodule structure gives

$$\varphi_H(f) = (1 \otimes id_H) \circ \delta$$

From the definition of Δ' in $Mod^{H'}$ we get the following:

$$\begin{aligned}\varphi_{H\otimes H}\left((f\otimes f)\circ\Delta'\right) &= \left(\varphi_{H'}(f)\otimes id_H\right)\circ\varphi_{H'}(f)\\ &= \left((1\otimes id_H)\circ\delta\otimes id_H\right)\circ(1\otimes id_H)\circ\delta\\ &= (1\otimes id_H\otimes id_H)\circ(\delta\otimes id_H)\circ\delta\\ &= (1\otimes id_H\otimes id_H)\circ(1\otimes\Delta)\circ\delta\\ &= (1\otimes\Delta)\circ\delta\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi_{H\otimes H}(\Delta\circ f) &= (1\otimes\Delta)\circ\varphi_H(f)\\ &= (1\otimes\Delta)\circ(1\otimes id_H)\circ\delta\\ &= (1\otimes\Delta)\circ\delta\end{aligned}$$

so

$$\Delta\circ f=(f\otimes f)\circ\Delta'.$$

We also have

$$\begin{aligned}\varphi_H(\varepsilon\circ f) &= (1\otimes 1)\circ\varphi_H(f)\\ &= (1\otimes 1)\circ\delta\\ &= \varphi_H(\varepsilon).\end{aligned}$$

□

This gives the following Theorem:

Theorem 7.16. *Let $G : Mod^H \longrightarrow Mod_k$ be the forgetting functor and let $H' = coend\,(G^*\otimes G)$. Let $Mod^{H'}$ be the category of H'-comodules we have constructed in this Part. Then the functor*

$$I:\left(Mod^{H'},\otimes'\right)\longrightarrow\left(Mod^H,\otimes\right)$$

gives an isomorphism of monoidal categories.

To sum up: If we take a coalgebra with a bialgebra structure and give it a structure of monoidal category as in Part II, applying the reconstruction of this Part gives us the same coalgebra (up to isomorphism). Conversely, given a monoidal category and a forgetting functor, we can construct a coalgebra $H = coend\,(G^*\otimes G)$ and give it a bialgebra structure such that Mod^H has the structure of a monoidal category.

7.5. **Rigidity and antipode.** Assume that $\mathcal{C}$ is rigid. Recall that Mod_k is left rigid: we can use $V^* = Hom_k(V,k)$. ev is the evaluation

$$ev\,(f,v)=f\,(v)\,.$$

and db is defined by

$$db\,(1_k)=\sum_i v_i\otimes v^i.$$

where $\{v_i, v^i\}$ is the dual basis of V, in the sense of Lemma 4.2. This shows that any $G(X)$ has a dual in Mod_k in the sense defined above.

In the category of f.g. projective modules over a commutative ring k, we have natural isomorphisms (see Lemma 1.6)

$$G\,(X)^*\otimes V\to Hom_k\,(G\,(X)\,,V)\,.$$

Then there is a natural isomorphism

$$Hom_k\left(G\left(X\right)^*, G\left(X\right)^* \otimes V\right) \approx Hom_k\left(G\left(X\right)^*, Hom_k\left(G\left(X\right), V\right)\right)$$

We also have an isomorphism

$$\varphi : Hom_k\left(G\left(X\right)^*, Hom_k\left(G\left(X\right), V\right)\right) \approx Hom_k\left(G\left(X\right)^* \otimes G\left(X\right), V\right)$$

given by $\left(\left(\varphi f\right)a\right)\left(b\right) = f\left(a \otimes b\right)$. From Lemma 7.1 we know that the latter is isomorphic to $Hom_k\left(G\left(X\right), G\left(X\right) \otimes V\right)$. This gives an isomorphism

$$Hom_k\left(G\left(X\right), G\left(X\right) \otimes V\right) \simeq Hom_k\left(G\left(X\right)^*, G\left(X\right)^* \otimes V\right). \tag{7.3}$$

From the monoidality of G it follows that $G(X^*)$ also is a dual for $G\left(X\right)$. Since two duals are isomorphic, we have $G\left(X\right)^* \approx G\left(X^*\right)$. Then the isomorphism 7.3 induces the H-comodule structure on $G\left(X\right)^*$, making Mod^H a left rigid category.

Let $v \in Hom\left(G, G \otimes V\right)$. Then we have a morphism

$$G\left(X\right)^* \approx G\left(X^*\right) \xrightarrow{v} G\left(X^*\right) \otimes V \approx G\left(X\right)^* \otimes V.$$

This morphism has a preimage $\widetilde{v}$ under (7.3). We then have a map

$$Nat\left(G, G \otimes V\right) \longrightarrow Nat\left(G, G \otimes V\right), v \mapsto \widetilde{v}.$$

This corresponds to a map $s : H \longrightarrow H$ making the following diagram commute:

$$\begin{array}{ccc} G(X^*) & \xrightarrow{v} & G(X^*) \otimes H \\ {\scriptstyle iso}\downarrow & & \downarrow{\scriptstyle iso \otimes s} \\ G(X)^* & \xrightarrow[\widetilde{v}]{} & G(X)^* \otimes H \end{array} \tag{7.4}$$

We want to show that s is an antipode. This means that s has to obey the equation

$$\mu \circ (s \otimes 1) \circ \Delta = \eta \circ \varepsilon = \mu \circ (1 \otimes s) \circ \Delta.$$

It is enough to show that

$$\varphi_H\left(\mu \circ (s \otimes 1) \circ \Delta\right) = \varphi_H\left(\eta \circ \varepsilon\right) = \varphi_H\left(\mu \circ (1 \otimes s) \circ \Delta\right)$$

I show the left equality first. We want the following diagram to commute:

$$\begin{array}{ccc} G(X) \otimes H & \xrightarrow{1 \otimes \Delta} & G(X) \otimes H \otimes H \\ & & \downarrow{\scriptstyle 1 \otimes s \otimes 1} \\ {\scriptstyle 1 \otimes \varepsilon}\downarrow & & G(X) \otimes H \otimes H \\ & & \downarrow{\scriptstyle 1 \otimes \mu} \\ G(X) \otimes k & \xrightarrow[1 \otimes \eta]{} & G(X) \otimes H \end{array}$$

or, by elements,

$$x_{(0)} \otimes s\left(x_{(1)}\right) x_{(2)} = x \otimes 1.$$

From the property of dual elements we have a morphism

$$G\left(X^* \otimes X\right) \overset{G(\widetilde{ev})}{\longrightarrow} G\left(I\right)$$

where $\widetilde{ev}$ is the map ev from Definition 3.4 applied to $\mathcal{C}$. Then we get the following commutative diagram:

$$\begin{array}{ccc} G(X^* \otimes X) & \xrightarrow{\ \delta\ } & G(X^* \otimes X) \otimes H \\ {\scriptstyle G(\widetilde{ev})}\Big\downarrow & & \Big\downarrow{\scriptstyle G(\widetilde{ev}) \otimes 1} \\ G(I) = k & \xrightarrow[\ \eta\ ]{} & (G(I) \otimes H) = (k \otimes H) = H \end{array}$$

From the algebra structure of H we then have a commutative diagram

$$\begin{array}{ccccc} G(X^*) \otimes G(X) & & \xrightarrow{\ \delta_{X^*} \otimes \delta_X\ } & & G(X^*) \otimes G(X) \otimes H \otimes H \\ \Big\downarrow{\scriptstyle ev} & & & & {\scriptstyle 1 \otimes \mu}\Big\downarrow \\ k & \xrightarrow{\ \eta\ } & H & \xleftarrow{\ ev \otimes 1\ } & G(X^*) \otimes G(X) \otimes H \end{array} \tag{7.5}$$

By the definition of s the following diagram commutes:

$$\begin{array}{ccc} G(X)^* \otimes G(X) & \xrightarrow{\ \delta^* \otimes \delta\ } & G(X)^* \otimes H \otimes G(X) \otimes H \\ {\scriptstyle 1 \otimes \delta}\Big\downarrow & & \\ G(X)^* \otimes G \otimes H & & \Big\downarrow{\scriptstyle 1 \otimes \tau \otimes 1} \\ {\scriptstyle 1 \otimes 1 \otimes \Delta}\Big\downarrow & & \\ G(X)^* \otimes G(X) \otimes H \otimes H & \xrightarrow{\ 1 \otimes 1 \otimes s \otimes 1\ } & G(X)^* \otimes G(X) \otimes H \otimes H \end{array} \tag{7.6}$$

We are now ready to show the equation

$$\mu \circ (s \otimes 1) \circ \Delta = \eta \circ \varepsilon.$$

The following diagram commutes (writing G as shorthand for $G(X)$):

$$\begin{array}{ccccc}
G & \xrightarrow{db\otimes 1} & G\otimes G^{*}\otimes G & \xrightarrow{1\otimes\delta^{*}\otimes\delta} & G\otimes G^{*}\otimes H\otimes G\otimes H \\
\downarrow{\scriptstyle db\otimes 1} & & & & \\
G\otimes G^{*}\otimes G & & & & \downarrow{\scriptstyle 1\otimes 1\otimes\tau\otimes 1} \\
\downarrow{\scriptstyle 1\otimes 1\otimes\delta} & & & & \\
G\otimes G^{*}\otimes G\otimes H & \xrightarrow{1\otimes 1\otimes 1\otimes\Delta} & G\otimes G^{*}\otimes G\otimes H\otimes H & \xrightarrow{1\otimes 1\otimes 1\otimes s\otimes 1} & G\otimes G^{*}\otimes G\otimes H\otimes H \\
\downarrow{\scriptstyle 1\otimes ev\otimes\varepsilon} & & & & \downarrow{\scriptstyle 1\otimes 1\otimes 1\otimes\mu} \\
G\otimes k & \xrightarrow{1\otimes\eta} & G\otimes H & \xleftarrow{1\otimes ev\otimes 1} & G\otimes G^{*}\otimes G\otimes H
\end{array}$$

The upper rectangle is 7.6 tensored with $G(X)$ on the left and with the arrow

$$G = k\otimes G \stackrel{db\otimes 1}{\longrightarrow} G\otimes G^{*}\otimes G$$

inserted in the upper left corner, while the outer rectangle commutes by 7.5 treated the same way. The diagram describes morphisms

$$G \longrightarrow G\otimes H,$$

so there are corresponding morphims in $Hom(H,H)$. Before we explore the diagram, we note the following equality:

$$(1\otimes 1\otimes\delta)\circ(db\otimes 1) = (db\otimes 1\otimes 1)\circ\delta$$

Now we get the following morphism going "down, down and right":

$$(1\otimes\eta)\circ(1\otimes ev\otimes\varepsilon)\circ(db\otimes 1\otimes 1)\circ\delta.$$

Since

$$(1\otimes ev)\circ(db\otimes 1) = 1$$

we have the morphism

$$(1\otimes(\eta\circ\varepsilon))\circ\delta = \varphi_H(\eta\circ\varepsilon).$$

Going "down, right, right, down" gives the morphism

$$\begin{aligned}
& (1\otimes ev\otimes 1)\circ(1\otimes 1\otimes 1\otimes\mu)\circ\left(1^{\otimes 3}\otimes s\otimes 1\right)\circ\left(\left(1^{\otimes 3}\right)\otimes\Delta\right)\circ(db\otimes 1\otimes 1)\circ\delta \\
= \ & [(1\otimes ev)\circ(db\otimes 1)\otimes(\mu\circ(s\otimes 1)\circ\Delta)]\circ\delta \text{ (by rearranging)} \\
= \ & (1\otimes(\mu\circ(s\otimes 1)\circ\Delta))\circ\delta = \varphi_H(\mu\circ(s\otimes 1)\circ\Delta)
\end{aligned}$$

so we see that

$$\mu\circ(s\otimes 1)\circ\Delta = \eta\circ\varepsilon.$$

The right equality can be established by replacing ev and η in diagram with db and ε. We conclude:

Theorem 7.17. *Let $\mathcal{C}$ be a monoidal category and $G : \mathcal{C} \longrightarrow Mod_k$ the forgetting functor. Let $H = coend(G^{*}\otimes G)$. If $\mathcal{C}$ is rigid, then H has an antipode.*

Corollary 7.18. *Let H be a bialgebra and let Mod^H be the category of H-comodules that are f.g. projective as k-modules. Then H is a Hopf algebra if and only if Mod^H is rigid.*

Proof. This follows from the above Theorem and Proposition 5.2. □

7.6. **Braidings in Mod^H.** Suppose $\mathcal{C}$ is braided with braiding

$$\sigma' : X \otimes Y \xrightarrow{\sigma'} Y \otimes X.$$

Define

$$\sigma : G(X) \otimes G(Y) \approx G(X \otimes Y) \xrightarrow{G(\sigma')} G(Y \otimes X) \approx G(Y) \otimes G(X).$$

Then σ defines a natural isomorphism, and by Lemma 7.11 corresponds to an element $r \in Hom_k(H \otimes H, k)$. We get

$$\sigma(x \otimes y) = \sum y_{(0)} \otimes x_{(0)} \cdot r\left(x_{(1)} \otimes y_{(1)}\right).$$

As G is a monoidal functor, it preserves the commutativity of diagrams defining a braiding. It then follows from the proof of Theorem 4.7 that r has to satisfy the conditions 4.1.

Theorem 7.19. *Suppose C is a braided category with braiding $\tilde{\sigma}$. Define $\sigma \in Nat(G \otimes G, G \otimes G)$ by*

$$\sigma_{X,Y} : G(X) \otimes G(Y) \approx G(X \otimes Y) \xrightarrow{G\left(\tilde{\sigma}_{X,Y}\right)} G(Y \otimes X) \otimes k \approx G(Y) \otimes G(X) \otimes k.$$

Then σ is a braiding in Mod^H given by

$$\sigma_{X,Y}(x \otimes y) = \sum y_{(0)} \otimes x_{(0)} \cdot r\left(x_{(1)} \otimes y_{(1)}\right).$$

where

$$r \in Hom(H \otimes H, k)$$

is the cobraider

Proof. We have seen that the braiding in $\mathcal{C}$ defines an element

$$r \in Hom(H \otimes H, k)$$

that makes H into a cobraided bialgebra. By Theorem 4.7 Mod^H then is braided. □

7.7. **Quantizations in Mod^H.** Suppose $\mathcal{C}$ is quantized. By the monoidality of G then Mod^H is also quantized. Recall from 4.10 that a quantization in Mod^H is uniquely determined by a coquantizer $q \in Hom_k(H \otimes H, k)$. We can give a different proof here by using the universality of H. As a quantization is natural, Q can be viewed as a natural transformation

$$Q : G \otimes G \longrightarrow G \otimes G.$$

Then Q corresponds to an element $q \in Hom_k(H \otimes H, k)$ described by the following composition:

$$G(X) \otimes G(Y) \longrightarrow G(X) \otimes G(Y) \otimes H \otimes H \xrightarrow{1 \otimes 1 \otimes q} G(X) \otimes G(Y)$$

We can describe the isomorphism $Q : G(X) \otimes G(Y)$ by the following diagram:

$$\begin{array}{ccccc} G(X)\otimes G(Y) & \xrightarrow{\varrho} & G(X)\otimes G(Y)\otimes H\otimes H & \xrightarrow{1\otimes 1\otimes q} & G(X)\otimes G(Y) \\ & \searrow^{Q} & & & \downarrow{\scriptstyle \xi_2} \\ & & & & G(X\otimes Y) \end{array}$$

If we assume that C is strict, the coherence diagram reduces to:

$$\begin{CD} G(X)\otimes G(Y)\otimes G(Z) @>{Q_{X,Y}\otimes id_Z}>> G(X\otimes Y)\otimes G(Z) \\ @V{id_X\otimes Q_{Y,Z}}VV @VV{Q_{X\otimes Y,Z}}V \\ G(X)\otimes G(Y\otimes Z) @>>{Q_{X,Y\otimes Z}}> G(X\otimes Y\otimes Z) \end{CD} \tag{7.7}$$

By following the same procedure as in Theorem 4.10 we see that q satisfies the conditions for being a coquantizer.

8. Monoidal categories are module categories

It is also possible to dualize the construction of Section 7. As above, let $\mathcal{C}$ be a small monoidal category and let

$$G : \mathcal{C} \longrightarrow Mod_k$$

be a monoidal functor preserving sums where k is a commutative ring. Let

$$F : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow Mod_k$$

be the functor

$$F(X) := Hom(G(X), G(X))$$

and let

$$E = End(Hom(G, G))$$

(see [ML98, IX.5]). We need additional assumptions in order to prove the existence of the *End*, but for now we assume that *End* exists. It means that we have morphisms

$$f_X : E \longrightarrow Hom(G(X), G(X))$$

such that the diagram

$$\begin{CD} H @>{f_Y}>> Hom(G(Y), G(Y)) \\ @V{f_X}VV @VV{G(\alpha)^*\otimes id}V \\ Hom(G(X), G(X) @>>{id\otimes G(\alpha)}> Hom(G(X), G(Y) \end{CD} \tag{8.1}$$

commutes for each

$$a : X \longrightarrow Y$$

in C, and such that H is universal object for this property.

Proposition 8.1. *If F is a bifunctor from $\mathcal{C}$ to Mod_k then*

$$End\left(F^*\right) \approx coend\left(F\right)^*$$

Proof. First we construct the following wedge:

$$\begin{array}{ccc} F(X,Y) & \xrightarrow{F(\alpha,1)} & F(Y,Y) \\ {\scriptstyle F(1,\alpha^*)}\downarrow & & \downarrow{\scriptstyle f_Y} \\ F(X,X) & \xrightarrow[f_X]{} & k \end{array}$$

Commutativity of this wedge can be reformulated as follows: Let $X, Y \in \mathcal{C}$ and let $\alpha : X \longrightarrow Y$. We want to find the family of all

$$f_Z \in Hom_k\left(F\left(Z,Z\right),k\right)$$

such that the following diagram commutes:
(8.2)

$$Hom_k\left(F\left(X,X\right),k\right) \xrightarrow{h_1} Hom_k\left(F\left(Y,X\right),k\right) \xleftarrow{h_2} Hom_k\left(F\left(Y,Y\right),k\right)$$

where

$$\begin{aligned} h_1\left(f_X\right) &= f_X \circ F\left(\alpha,1\right) \\ h_2\left(f_Y\right) &= f_Y \circ F\left(1,\alpha^{op}\right) \end{aligned}$$

Now let H be *coend*(F). Then the *coend*-diagram shows that these $f\_$ factorizes uniquely through a morphism $e : H \longrightarrow k$. This then makes $Hom_k\left(H,k\right)$ an end for 8.2. Since a *coend* is a colimit and an *End* is a limit, the general isomorphism

$$Hom\left(\amalg_j a_j, x\right) \approx \Pi_j\left(Hom\left(a_j,x\right)\right)$$

gives the isomorphisms

$$Hom\left(coend\left(F\right),k\right) \approx End\left(Hom\left(F,k\right)\right),$$

or

$$coend\left(F\right)^* \approx End\left(F^*\right)$$

in our case. □

Now by Lemma 7.1

$$Hom_k\left(F\left(X\right),F\left(X\right)\right) = Hom_k\left(F\left(X\right)^* \otimes F\left(X\right),k\right) = \left(F\left(X\right)^* \otimes F\left(X\right)\right)^*.$$

Setting $coend\left(G^* \otimes G\right) = H$ as in Section 7 , we see that $E \approx H^*$. H is a bialgebra, so E is a bialgebra by 1.14 . As we have seen in Section 6, the constructions of module structures, antipode, braiders and quantizers can be done by dualizing the constructions in the comodule situation. We will show some of this explicitly:

8.1. **Module structure.**

Proposition 8.2. *E represents the functor*

$$V \longrightarrow Nat\left(V \otimes G, G\right),$$

in other words there is a natural isomorphism

$$\phi : Hom_k\left(V, E\right) \longrightarrow Nat\left(V \otimes G, G\right).$$

Proof. We know that in Mod_k every k-module M has a left dual M^*. There is also an isomorphism $M^{**} \approx M$. From Proposition 7.5 we have a $1-1$ correspondence

$$\varphi : Hom_k\left(H, V^*\right) \approx Nat\left(G, G \otimes V^*\right).$$

Looking at the component φ_{X^*}, we have natural isomorphisms

$$\begin{aligned} Hom_k\left(G\left(X^*\right), G\left(X^*\right) \otimes V^*\right) &\approx Hom_k\left(G\left(X\right)^*, G\left(X\right)^* \otimes V^*\right) \\ &\approx Hom_k\left(G\left(X\right)^* \otimes G\left(X\right), V^*\right) \\ &\approx Hom\left(V, Hom\left(G\left(X\right), G\left(X\right)\right)\right) \\ &\approx Hom\left(V \otimes G\left(X\right), G\left(X\right)\right) \end{aligned}$$

As all these isomorphisms are natural we get

$$Hom_k\left(H, V^*\right) \approx Nat\left(V \otimes G, G\right).$$

But by Lemma 1.12 there are isomorphisms

$$Hom_k\left(H, V^*\right) \approx Hom_k\left(V, H^*\right) \approx Hom\left(V, E\right)$$

and thereby the isomorphism

$$\phi : Hom_k\left(V, E\right) \longrightarrow Nat\left(V \otimes G, G\right).$$

As in Proposition 7.5 the isomorphism is determined by $\phi_E\left(1_E\right)$, and given by

$$\phi\left(f\right) = \phi_E\left(1_E\right) \circ \left(f \otimes 1\right).$$

The components of $Nat\left(G \otimes V, G\right)$ gives the following : for any $\alpha : X \longrightarrow Y$ in $\mathcal{C}$, a morphism

$$\omega : V \otimes G\left(X\right) \longrightarrow G\left(Y\right)$$

is the composition

$$V \otimes G\left(X\right) \stackrel{f \otimes G(\alpha)}{\longrightarrow} E \otimes G\left(Y\right) \stackrel{\phi_E(1_E)}{\longrightarrow} G\left(Y\right)$$

□

Proposition 8.3. *Define*

$$\rho := \phi_E\left(1_E\right)$$

where ϕ is the isomorphism from the above Proposition. Then ρ gives a E-module structure on $G\left(X\right)$

Proof. Let

$$\rho \in Nat\left(V \otimes G, G\right)$$

be defined by

$$\rho := \phi\left(1_E\right).$$

Recall the following isomorphism from Lemma 1.12.

$$Nat\left(G^*, G^* \otimes V^*\right) \approx Nat\left(V \otimes G, G\right).$$

The H^*-comodule structure in Mod^{H^*} was given by

$$\delta := \varphi_{H^*}\left(1_{H^*}\right).$$

But then $\rho = \delta^* \circ \lambda$, and we know from Proposition 1.15 that this gives a E-module structure on $G\left(X\right)$. □

8.2. **Correspondence of the direct and inverse constructions of Mod_E.** This follows dually to Section 7.4. As in the comodule case,

$$F : \mathcal{C} \longrightarrow Mod_E$$

is monoidal by G being so. Dualizing the results from 7.4 gives the following:

Lemma 8.4. *Let E be an algebra that is finitely generated and projective as a k-module. Then we have an isomorphism*

$$\psi : Hom_k\left(V, E\right) \approx Nat\left(V \otimes E, E\right)$$

given by

$$\psi\left(g\right) = \rho \circ \left(g \otimes 1\right)$$

Proof. We know that $E \approx H^*$, so $E^* \approx H^{**} \approx H$. We also know that V^* is a E^*-comodule. Lemma 7.14 combined with the isomorphism 7.3 then gives the following isomorphism:

$$Hom_k\left(H, V^*\right) \approx Nat\left(G^*, G^* \otimes V^*\right).$$

By Lemma 1.12 we then get the isomorphism

$$Hom_k\left(V, H^*\right) \approx Nat\left(\left(G^* \otimes V^*\right)^*, G\right) \approx Nat\left(V \otimes G, G\right).$$

Finally we note that

$$\left(\left(1 \otimes f\right) \circ \delta\right)^* = \left(\lambda^{-1} \circ \delta^*\right) \circ \left(f^* \otimes 1\right) = \rho \circ \left(f^* \otimes 1\right),$$

so we see that the isomorphism can be given by

$$\psi\left(g\right) = \rho \circ \left(g \otimes 1\right).$$

□

Proposition 8.5. *Let E be a bialgebra and let $G : Mod_E \longrightarrow Mod_k$ be the forgetting functor. Let $E' = End\left(Hom_k\left(G, G\right)\right)$. Then $E' \approx E$.*

Proof. This also follows dually to Proposition 7.15. Corollary 8.2 and the previous Lemma give correspondence between the module structures: ρ can be written as

$$\rho : E \otimes V \overset{\phi \otimes 1}{\longrightarrow} E' \otimes V \overset{\rho'}{\longrightarrow} V$$

where

$$\phi \in Hom_k\left(E, E'\right).$$

In the same manner ρ'can be factorized through ρ. That ϕ is a bialgebra morphism follows directly from dualizing the proof of Proposition 7.15. □

We get the following Theorem:

Theorem 8.6. *Let E be a bialgebra that is f.g. projective as a k-module. Let $G : Mod_E \longrightarrow Mod_k$ be the forgetting functor. Let*

$$E' = end\left(Hom_k\left(G, G\right)\right)$$

and let $Mod_{E'}$ be the category of left E'-modules constructed in this Section. Then the functor

$$\begin{array}{rcl} I & : & \left(Mod_{E.}, \otimes'\right) \longrightarrow \left(Mod_E, \otimes\right), \\ I\left(V\right) & = & V \end{array}$$

is an isomorphism of monoidal categories.

8.3. **Braidings and quantizations.** Braidings and quantizations in Mod_E can be reconstructed from the corresponding structures in $\mathcal{C}$, just like in the dual case. The monoidality of F takes the diagrams defining braidings and quantizations over to the appropriate diagrams in Mod_E. A braiding σ' in $\mathcal{C}$ carries over to a natural transformation

$$G\left(X \otimes Y\right) \longrightarrow G\left(Y \otimes X\right),$$

and by the bialgebra structure of E this corresponds to a morphism

$$R \in Hom_k\left(k, H \otimes H\right)$$

which essentially is the same as an element of $H \otimes H$. It can then be shown that R has to satisfy the conditions to make E a braided bialgebra.

The same reasoning follows for quantizations.

8.4. **Rigidity and antipode.** We know that if $\mathcal{C}$ is left rigid, we can construct an antipode for the bialgebra $H = coend\left(G^* \otimes G\right)$. As $E = H^*$, the constructions in Section 7.5 can be dualized to show that right rigidity of $\mathcal{C}$ makes it possible to construct an antipode for E. First we note that we have an isomorphism

$$Hom\left(V \otimes G\left(X\right), G\left(X\right)\right) \approx Hom\left(V \otimes G\left(X\right)^*, G\left(X\right)^*\right)$$

by dualizing 7.3. This gives rise to a map

$$Nat\left(V \otimes G, G\right) \longrightarrow Nat\left(V \otimes G^*, G^*\right)$$

corresponding to a k-morphism

$$s \in Hom_k\left(E, E\right),$$

just as in the dual situation. Now suppose $\mathcal{C}$ is right rigid. The following diagram commutes by the naturality of ρ:

$$\begin{array}{ccc} E \otimes G(X^* \otimes X) & \xrightarrow{\rho} & G(X^* \otimes X) \\ \Big\downarrow{\scriptstyle 1 \otimes G(ve)} & & \Big\downarrow{\scriptstyle G(ve)} \\ E \otimes G(e) & \xrightarrow[\rho]{} & E \end{array}$$

By using the coalgebra structure of E we get the following commutative diagram:

$$\begin{array}{ccccc} E\otimes G(X)\otimes G(X)^* & & \xrightarrow{\Delta\otimes 1\otimes 1} & & E\otimes E\otimes G(X)\otimes G(X)^* \\ \Big\downarrow{\scriptstyle 1\otimes ve} & & & & \Big\downarrow{\scriptstyle \rho\otimes\rho^*} \\ E & \xrightarrow[\varepsilon]{} & k & \xleftarrow[ve]{} & G(X)\otimes G(X)^* \end{array} \tag{8.3}$$

The definition of s gives commutativity of the following diagram:

$$\begin{array}{ccc} E\otimes E\otimes G(X)\otimes G(X)^* & \xrightarrow{\rho\otimes\rho^*} & G(X)\otimes G(X)^* \\ \Big\downarrow{\scriptstyle 1\otimes s\otimes 1\otimes 1} & & \Big\uparrow{\scriptstyle \rho\otimes 1} \\ E\otimes E\otimes G(X)\otimes G(X)^* & \xrightarrow[m\otimes 1\otimes 1]{} & E\otimes G(X)\otimes G(X)^* \end{array} \tag{8.4}$$

Proceeding in the same manner as in 7.5 we glue together the two previous diagrams tensoring on the right with G and adding an upper left corner

$$H\otimes G \stackrel{1\otimes 1\otimes bd}{\longrightarrow} H\otimes G\otimes G^*\otimes G$$

to get the following commutative diagram:

$$\begin{array}{ccccc} E\otimes G & \xrightarrow{1\otimes 1\otimes bd} & E\otimes G\otimes G^*\otimes G & \xrightarrow{1\otimes ve\otimes 1} & E\otimes G \\ \Big\downarrow{\scriptstyle 1\otimes 1\otimes bd} & & & & \Big\downarrow{\scriptstyle \varepsilon\otimes 1} \\ E\otimes G\otimes G^*\otimes G & & & & G \\ \Big\downarrow{\scriptstyle \Delta\otimes 1\otimes 1\otimes 1} & & & & \Big\uparrow{\scriptstyle ve\otimes 1} \\ E\otimes E\otimes G\otimes G^*\otimes G & & \xrightarrow{\rho\otimes\rho^*\otimes 1} & & G\otimes G^*\otimes G \\ \Big\downarrow{\scriptstyle 1\otimes s\otimes 1\otimes 1\otimes 1} & & & & \Big\uparrow{\scriptstyle \rho\otimes 1\otimes 1\otimes 1} \\ E\otimes E\otimes G\otimes G^*\otimes G & & \xrightarrow[\mu\otimes 1\otimes 1\otimes 1]{} & & E\otimes G\otimes G^*\otimes G \end{array}$$

The upper rectangle commutes by 8.3, the lower by 8.4. Following the diagram round along the outer edges, we get the morphisms

$$\begin{aligned} & (ve\otimes 1)\circ\rho\circ(\mu\otimes 1\otimes 1\otimes 1)\circ(1\otimes s\otimes 1\otimes 1\otimes 1)\circ \\ & (\Delta\otimes 1\otimes 1\otimes 1)\circ(1\otimes 1\otimes bd) \\ = \;& \rho\circ((\mu\circ(1\otimes s)\circ\Delta)\otimes(ve\otimes 1)\circ(1\otimes bd)) \\ = \;& \rho\circ((\mu\circ(1\otimes s)\circ\Delta)\otimes 1) \end{aligned}$$

and

$$\begin{aligned} & (\varepsilon \otimes 1) \circ (1 \otimes ve \otimes 1) \circ (1 \otimes 1 \otimes bd) \\ = \quad & (\varepsilon \otimes 1) \circ (1 \otimes 1) \\ = \quad & \rho \circ (\eta \otimes 1) \circ (\varepsilon \otimes 1) \\ = \quad & \rho \circ (\eta \circ \varepsilon \otimes 1)\,. \end{aligned}$$

From this we see that

$$(\mu \circ (1 \otimes s) \circ \Delta) = \eta \circ \varepsilon.$$

Using the same reasoning based on bd instead of ve gives the second equality defining an antipode.

Remark 8.7. *The remark after Proposition 6.1 indicates that left rigidity of $\mathcal{C}$ also makes it possible to construct an antipode.*

Part IV. Further perspectives

The program (to make a complete "Dictionary and Grammar Book" that translates monoidal notions to bialgebra notions and back) that was planned in the beginning of my work on this Thesis, appeared to be too large for a cand. sci. thesis. There are many steps in that program that have not been completed, or have been done only partially. In this Part we are briefly discussing these "missing pages of the Dictionary".

9. Non-strict monoidal categories: towards coquasibialgebras

Until now, the paper has dealt with strict monoidal categories, avoiding discussions about associativity. We have seen that defining a bialgebra structure on Mod^H makes it a strict monoidal category, where we used the multiplication to give a H-comodule structure on the tensor product. To have a non-strict monoidal category it is not necessary to have strict associativity of the multiplication, so we will try to find an "almost" - bialgebra structure that makes Mod^H a non-strict monoidal category. To do this try to find ways of "controlling" the non-associativity of the multiplication, such that we still can give Mod^H a structure of a non-strict monoidal category. We make some definitions:

Definition 9.1. *A* ***quasialgebra*** (A, μ, η, a) *is a k-module A together with morphisms*

$$\mu : A \otimes A \longrightarrow A,$$

called ***quasimultiplication***

$$\eta : k \longrightarrow A,$$

called ***unit****, and an* ***associator***

$$a \in Hom\,(A \otimes A \otimes A, k)$$

These morphisms has to obey the following relations:

$$\begin{aligned} a \star (\mu \circ (\mu \otimes id_A)) &= (\mu \circ (id_A \otimes \mu)) \star a \\ \mu \circ (\eta \otimes id_A) &= \mu \circ (id_A \otimes \eta)\,. \end{aligned}$$

Remark 9.2. *I have not found any general definition of the term quasialgebra, so this is an adaption to our case. Shan Majid has an almost similar definition in* [AM99].

When we put this structure on a coalgebra, we get a **coquasibialgebra.**

Definition 9.3. *A* ***coquasibialgebra*** (H, μ, η, a) *consists of a coalgebra* H*, coalgebra morphisms*

$$\begin{aligned} \mu &: H \otimes H \longrightarrow H, \\ \eta &: k \longrightarrow H \end{aligned}$$

and a $\star$*-invertible*

$$a : H \otimes H \otimes H \longrightarrow k$$

such that the following equations are fulfilled:

$$\mu \circ (\eta \otimes id) = id = \mu \circ (id \otimes \eta) \tag{9.1}$$

$$a \star \mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id) \star a \tag{9.2}$$

$$(9.3)\quad a_{123} \star a \circ (id \otimes \mu \otimes id) \star a_{234} = a \circ (\mu \otimes id \otimes id) \star a \circ (id \otimes \mu \otimes id)$$

where

$$a_{123} = (a \otimes \varepsilon)$$

and

$$a_{234} = (\varepsilon \otimes a) .$$

The motivation for this definition is that it can be used to give a (non-strict) monoidal structure on Mod^H. The definition of a monoidal category 3.1 state that we need to find an associativity and a unity constraint satisfying the pentagon and unity axioms. For unity we can use η just as in the strict case, as the unity constraint does not depend on μ being associative. So we need to find an associativity constraint. Define

$$\alpha (x \otimes (y \otimes z)) = \sum \left(x_{(0)} \otimes y_{(0)}\right) \otimes z_{(0)} \cdot a \left(x_{(1)} \otimes y_{(1)} \otimes z_{(1)}\right) .$$

Before we go on, note that the morphism $a \star \mu \circ (id \otimes \mu)$ can be viewed as applying first $\mu \circ (id \otimes \mu)$, then a to $\delta (x \otimes (y \otimes z))$. We get the following sequence:

$$(X \otimes Y) \otimes Z \xrightarrow{\delta} (X \otimes Y) \otimes Z \otimes H \otimes H \otimes H \overset{1^{\otimes 3} \otimes \mu \circ (id \otimes \mu)}{\longrightarrow} (X \otimes Y) \otimes Z \otimes H$$
$$\xrightarrow{\delta \otimes 1} (X \otimes Y) \otimes Z \otimes H \otimes H \otimes H \otimes H \xrightarrow{\alpha \otimes 1} X \otimes (Y \otimes Z) \otimes H.$$

Recall how we used the multiplication in to define a H-comodule structure on the tensor product: It was defined by the composition

$$\delta_{V \otimes W} : V \otimes W \overset{\delta_V \otimes \delta_W}{\longrightarrow} V \otimes H \otimes W \otimes H \overset{1 \otimes \tau \otimes 1}{\longrightarrow} V \otimes W \otimes H \otimes H \overset{1 \otimes 1 \otimes \mu}{\longrightarrow} V \otimes W \otimes H.$$

Then the previous morphism is essentially the morphism

$$(X \otimes Y) \otimes Z \overset{\delta_{(X \otimes Y) \otimes Z} \circ (\alpha \otimes 1)}{\longrightarrow} X \otimes (Y \otimes Z) \otimes H.$$

In the same manner $\mu \circ (\mu \otimes id) \star a$ gives the morphism

$$(X \otimes Y) \otimes Z \overset{\alpha \circ \delta_{X \otimes (Y \otimes Z)}}{\longrightarrow} X \otimes (Y \otimes Z) \otimes H.$$

The condition 9.2 therefore shows that α is a H-module morphism. Now for the pentagon axiom: For readability we restate the pentagon diagram with the product $\otimes$:

$$\begin{array}{ccccc}
X \otimes (Y \otimes (Z \otimes T)) & \xrightarrow{\alpha_{X,Y,Z \otimes T}} & (X \otimes Y) \otimes (Z \otimes T) & \xrightarrow{\alpha_{X \otimes Y,Z,T}} & ((X \otimes Y) \otimes Z) \otimes T \\
\downarrow {\scriptstyle id_X \otimes \alpha_{Y,Z,T}} & & & & \uparrow {\scriptstyle \alpha_{X,Y,Z} \otimes id_T} \\
X \otimes ((Y \otimes Z) \otimes T) & & \xrightarrow[\alpha_{X,Y \otimes Z,T}]{} & & (X \otimes (Y \otimes Z)) \otimes T
\end{array}$$

By the definition of the associativity the first arrow down can be written as

$$x \otimes (y \otimes (z \otimes t)) \longrightarrow \sum x_{(0)} \otimes \left(\left(y_{(0)} \otimes z_{(0)}\right) \otimes t_{(0)}\right) \cdot \varepsilon \left(x_{(1)}\right) a \left(y_{(0)} \otimes z_{(0)} \otimes t_{(0)}\right) ,$$

or in other words, by the action of a_{234} on $x \otimes (y \otimes (z \otimes t))$. The bottom arrow is the morphism

$$X \otimes ((Y \otimes Z) \otimes T) \longrightarrow (X \otimes (Y \otimes Z)) \otimes T,$$

which induces the action of $a\,(id \otimes \mu \otimes id)$. Following the same procedure for the whole diagram, we see that to have commutativity we need that

$$a_{123} \star a \circ (id \otimes \mu \otimes id) \star a_{234} = a \circ (\mu \otimes id \otimes id) \star a \circ (id \otimes id \otimes \mu) .$$

But this is the condition 9.3 for a coquasibialgebra. We have then proved:

Theorem 9.4. *Let H be a coquasibialgebra. Then Mod^H is a non-strict monoidal category.*

We can then apply the same procedures to describe braidings and quantizations.

9.1. **Braidings in Mod^H.** As for the strict case, we define a natural morphism

$$\begin{aligned} \sigma_{X,Y} &: X \otimes Y \longrightarrow Y \otimes X, \\ \sigma_{X,Y}(x \otimes y) &= \sum y_{(0)} \otimes x_{(0)} \cdot r\left(x_{(1)} \otimes y_{(1)}\right) \end{aligned}$$

where $r \in Hom_H(H \otimes H, k)$. We need to put some conditions on r to make this a braiding: First, to be an isomorphism we need r to be $\star$-invertible. We want σ to be a H-module morphism, so as in the strict case, we need r to fulfill the condition

$$\mu\tau = r \star \mu \star \bar{r}.$$

If we go through the hexagon diagrams the same way as we did for the pentagon diagram in the previous Section, we find that the following conditions have to be satisfied:

$$\begin{aligned} r\,(id \otimes \mu) &= a_{231} \star r_{13} \star \bar{a}_{213} \star r_{12} \star a \\ r\,(\mu \otimes id) &= \bar{a}_{321} \star r_{13} \star a_{132} \star r_{23} \star \bar{a} \end{aligned}$$

This leads to the following definition:

Definition 9.5. *A **cobraided** coquasibialgebra is a coquasibialgebra (H, μ, η, a, r) that satisfies the following properties:*

$$\begin{aligned} & r \text{ is } \star\text{-invertible} \\ \mu\tau &= r \star \mu \star \bar{r} \\ r\,(id \otimes \mu) &= a_{231} \star r_{13} \star \bar{a}_{213} \star r_{12} \star a \\ r\,(\mu \otimes id) &= \bar{a}_{321} \star r_{13} \star a_{132} \star r_{23} \star \bar{a} \end{aligned}$$

The above discussion shows the following Theorem

Theorem 9.6. *Let (H, μ, η, a, r) be a cobraided coquasibialgebra. Then*

$$\begin{aligned} \sigma_{X,Y} &: X \otimes Y \longrightarrow Y \otimes X, \\ \sigma_{X,Y}(x \otimes y) &= \sum y_{(0)} \otimes x_{(0)} \cdot r\left(x_{(1)} \otimes y_{(1)}\right) \end{aligned}$$

is a braiding in Mod^H.

9.2. **Quantizations in Mod^H.** Let us define a morphism

$$\begin{aligned} Q(x\otimes y) &= \sum x_{(0)}\otimes y_{(0)}\cdot q\left(x_{(1)}\otimes y_{(1)}\right), \\ q &\in Hom_k(H\otimes H,k) \end{aligned}$$

We want to see under which conditions on q this can define a quantization on Mod^H. First we need Q to be an H-comodule morphism. But this is similar to the strict case, so at least we need q to satisfy

$$q\star\mu=\mu\star q.$$

We then examine the coherence diagram 3.8 using the above definition of the associator. We then get the following equations:

$$\begin{aligned} & \left[\alpha_{X,Y,Z}\circ(Q_{X,Y\otimes Z})\circ(id_X\otimes Q_{Y,Z})\right](x\otimes(y\otimes z)) \\ = \;& \alpha_{X,Y,Z}\circ(Q_{X,Y\otimes Z})\left(x\otimes\sum y_{(0)}\otimes z_{(0)}\cdot q\left(y_{(1)}\otimes z_{(1)}\right)\right) \\ = \;& \alpha_{X,Y,Z}\left(\sum x_{(0)}\otimes\left(y_{(0)}\otimes z_{(0)}\right)\cdot q\left(x_{(1)}\otimes(y\otimes z)_{(1)}\right)\cdot q\left(y_{(1)}\otimes z_{(1)}\right)\right) \end{aligned}$$

$$=\sum\left(x_{(0)}\otimes y_{(0)}\right)\otimes z_{(0)}\cdot a\left(x_{(1)}\otimes y_{(1)}\otimes z_{(1)}\right)\cdot q\left(x_{(1)}\otimes(y\otimes z)_{(1)}\right)\cdot q\left(y_{(1)}\otimes z_{(1)}\right)$$

$$=\sum\left(x_{(0)}\otimes y_{(0)}\right)\otimes z_{(0)}\cdot a\left(x_{(1)}\otimes y_{(1)}\otimes z_{(1)}\right)\cdot q\left(x_{(1)}\otimes\mu\left(y_{(1)}\otimes z_{(1)}\right)\right)\cdot q\left(y_{(1)}\otimes z_{(1)}\right)$$

$$=\sum\left(x_{(0)}\otimes y_{(0)}\right)\otimes z_{(0)}\cdot a\star\left(q\circ(1\otimes\mu)\star(\varepsilon\otimes q)\left(x_{(1)}\otimes y_{(1)}\otimes z_{(1)}\right)\right).$$

Following the other direction in the same manner gives the equality

$$\begin{aligned} & Q_{X\otimes Y,Z}\circ(Q_{X,Y}\otimes id_Z)\circ\alpha_{X,Y,Z}(x\otimes(y\otimes z)) \\ = \;& \sum\left(x_{(0)}\otimes y_{(0)}\right)\otimes z_{(0)}\cdot\left(q\circ(\mu\otimes 1)\star(q\otimes\varepsilon)\star a\left(x_{(1)}\otimes y_{(1)}\otimes z_{(1)}\right)\right), \end{aligned}$$

so we see that we must require that

$$a\star q\circ(1\otimes\mu)\star(\varepsilon\otimes q)=q\circ(\mu\otimes 1)\star(q\otimes\varepsilon)\star a$$

10. The inverse construction

The following is an informal outline of what can be done, and thus lacks some mathematical formalities.

Now assume that we have a not necessarily strict monoidal category $\mathcal{C}$ and a functor $G:\mathcal{C}\longrightarrow Mod_k$ as earlier. If G is a monoidal functor we can construct $H=coend(G^*\otimes G)$ and get a coalgebra structure on H, just as in the strict case. We also have an associator by taking $G(\alpha)$ to be the associator in Mod^H. We then get a coquasibialgebra structure on Mod^H as described above, and Mod^H is a monoidal category. If $\mathcal{C}$ is braided, we can construct a braiding on Mod^H by the same procedure as in Part III, now by taking the associativity constraint into consideration. We then find a cobraider making H into a cobraided coquasibialgebra, thus defining a braiding on Mod^H. All these constructions rely on an associativity constraint α which essentially arised from G fulfilling diagram 3.1. This gives rise to the following question: what happens if the functor does not satisfy this diagram, but all the other conditions for a monoidal functor? It turns out that we can still construct a coquasibialgebra structure on H based on the associativity constraint in $\mathcal{C}$.

Remark 10.1. *We will call this functor a* ***neutral*** *tensor functor to distinguish it from an ordinary monoidal functor.*

10.1. **Associativity.** Let us define μ and η the same way as in the monoidal case in Part II. The equation 9.1 follows immediately. Let β be the associativity constraint in $\mathcal{C}$. We want to find an associativity α on G such that the diagram

$$\begin{array}{ccc} G(X)\otimes(G(Y)\otimes G(Z)) & \xrightarrow{\alpha} & (G(X)\otimes G(Y))\otimes G(Z) \\ \Big\downarrow{\scriptstyle \xi_2(1\otimes\xi_2)} & & \Big\downarrow{\scriptstyle \xi_2(\xi_2\otimes 1)} \\ G(X\otimes(Y\otimes Z) & \xrightarrow[G(\beta)]{} & G((X\otimes Y)\otimes Z) \end{array}$$

commutes naturally. Let us therefore define a natural morphism α by

$$\begin{aligned} \alpha_{X,Y,Z} \quad : \quad & G(X)\otimes(G(Y)\otimes G(Z)) \overset{\xi_2\circ(1\otimes\xi_2)}{\longrightarrow} G(X\otimes(Y\otimes Z)) \\ & \overset{G(\beta)}{\longrightarrow} G((X\otimes Y)\otimes Z) \overset{\left(1\otimes\xi_2^{-1}\right)\circ\xi_2^{-1}}{\longrightarrow} (G(X)\otimes G(Y))\otimes G(Z). \end{aligned}$$

This is an endomorphism on G. Then $\alpha \in Nat((G\otimes G)\otimes G, G\otimes(G\otimes G)\otimes k)$ and thus corresponds to a morphism

$$a \in Hom_k(H\otimes H\otimes H, k)$$

We can therefore write

$$\alpha(x\otimes(y\otimes x)) = \sum((x_0\otimes y_0)\otimes z_0)\cdot a(x_1\otimes y_1\otimes z_1).$$

By the definition we gave of μ in Part III we have the following diagram for $G(X\otimes(Y\otimes Z))$

$$\begin{array}{ccc} G(X)\otimes(G(Y)\otimes G(Z)) & \longrightarrow & G(X)\otimes(G(Y)\otimes G(Z))\otimes H\otimes H\otimes H \\ \Big\downarrow{\scriptstyle \xi_2(1\otimes\xi_2)} & & \Big\downarrow{\scriptstyle \xi_2(1\otimes\xi_2)\otimes\mu(1\otimes\mu)} \\ G(X\otimes(Y\otimes Z)) & \xrightarrow[\delta]{} & G(X\otimes(Y\otimes Z))\otimes H \end{array}$$

We also have a similar diagram for $G((X\otimes Y)\otimes Z)$. In the strict case these diagrams were linked by the associativity in Mod^H to make multiplication associative. In this Part we can use the morphism a to link these diagrams together. We have seen that for this diagram to commute, a has to satisfy

$$a\star\mu(1\otimes\mu) = \mu(\mu\otimes 1)\star a.$$

The pentagon diagram for the associativity in C maps to the following diagram:

$$
\begin{array}{ccc}
G(A\otimes(B\otimes(C\otimes D))) & \xrightarrow{G(\beta)} & G((A\otimes B)\otimes(C\otimes D)) \\
\downarrow{\scriptstyle G(1\otimes\beta)} & & \\
G(A\otimes((B\otimes C)\otimes D)) & & \downarrow{\scriptstyle G(\beta)} \\
\downarrow{\scriptstyle G(\beta)} & & \\
G((A\otimes(B\otimes C))\otimes D) & \xrightarrow[G(\beta\otimes 1)]{} & G(((A\otimes B)\otimes C)\otimes D)
\end{array}
$$

By the definition of α this transforms to a similar pentagon diagram for α. Chasing the diagram for α gives

$$a_{123}\star a\circ(id\otimes\mu\otimes id)\star a_{234}=a\circ(\mu\otimes id\otimes id)\star a\circ(id\otimes\mu\otimes id)$$

just as we have seen earlier in this Part.

We have then proved

Theorem 10.2. *If C is a monoidal category and $G: C\longrightarrow A$ is a neutral tensor functor, then $H := coend\,(G)$ is a coquasibialgebra .*

From this we get the following corollary:

Corollary 10.3. *Mod^H is a monoidal category*

Proof. μ gives a H-comodule structure on the tensor product just as in Part III, and we have associativity by

$$\alpha\,(x\otimes(y\otimes z))=(x_0\otimes y_0)\otimes z_0\cdot a\,(x_1\otimes y_1\otimes z_1)$$

□

10.2. **Braidings and quantizations.** By using the associator we defined in the previous Section, I assume that we can reconstruct braidings and quantizations in Mod^H in a similar manner as in the strict case. An outline of the process is as follows: The functor G takes the appropriate diagrams defining braidings and quantizations in $\mathcal{C}$ to diagrams in Mod^H. To make this diagrams commutative in Mod^H we define cobraiders and coquantizers as above, and chasing the diagrams will define the requirements of r and q as described earlier in this Part.

11. When is F an equivalence?

Under an assumption that there exists the coend

$$Coend(G^*\otimes G)=:H$$

we have established a bialgebra structure on H. We have also proved that the forgetting functor

$$G:\mathcal{C}\longrightarrow Mod_k$$

factors through the category Mod^H:

$$G : \mathcal{C} \xrightarrow{F} Mod^H \xrightarrow{U} Mod_k$$

where U is the forgetting functor. We remind that Mod_k is the category of f.g. projective k-modules, Mod^H is the category of those H-comodules that are f.g. projective as k-modules. Below we formulate a reasonable conjecture stating necessary and sufficient conditions (on $\mathcal{C}$ and G) for F to be an equivalence. We do not call that statement a Theorem because it is proved only partially.

Let us first examine the forgetting functor

$$U : Mod^H \longrightarrow Mod_k.$$

This functor is evidently faithful. Given $X \in Ob(Mod_k)$, let

$$W(X) := X \otimes H$$

with the following H-comodule structure:

$$\delta : X \otimes H \xrightarrow{1\otimes\Delta} X \otimes H \otimes H.$$

Proposition 11.1. *The functor W is right adjoint to the forgetting functor U.*

Proof. Given

$$X \in Ob(Mod^H)$$

and

$$Y \in Ob(Mod_k).$$

Let

$$f : X \longrightarrow W(Y) = Y \otimes H.$$

Denote by $\varphi(f)$ the following composition

$$\varphi(f) : U(X) = X \xrightarrow{f} Y \otimes H \xrightarrow{1\otimes\varepsilon} Y.$$

Let further

$$g : U(X) = X \longrightarrow Y.$$

Define

$$\psi(g) : X \xrightarrow{\delta} X \otimes H \xrightarrow{g\otimes 1} Y \otimes H = W(Y).$$

The pair (φ, ψ) defines the desired adjointness isomorphisms

$$Hom^H (X, W(Y)) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Hom_k (U(X), Y).$$

□

Corollary 11.2. *The functor*

$$(U(\_))^* : Mod^H \longrightarrow Mod_k$$

is representable.

Proof.

$$(U(X))^* = Hom_k (U(X), k) \approx Hom^H (X, V(k)) = Hom^H (X, H),$$

so $(U(\_))^*$ is representable by $H \in Mod^H$. □

We are going to prove a kind of left exactness of the functor U. However, one should not expect that U is left exact in the **usual** sense. The thing is that neither Mod^H, nor Mod_k is abelian. One cannot guarantee that the kernel

$$\ker(f) \longrightarrow X \longrightarrow Y$$

is projective. We can neither claim that $\ker(f)$ is finitely generated, since we do not require k to be Noetherian. However, a kind of exactness can be stated.

Definition 11.3. *A functor*

$$G : \mathcal{C} \longrightarrow Mod_k$$

is called ***weak left exact*** *if the following is satisfied:*
given

$$f : X \longrightarrow Y$$

in $\mathcal{C}$*, let* $\ker(G(f))$ *be f.g. projective. Then* $\ker(f)$ *exists, and the natural homomorphism*

$$G(\ker(f)) \longrightarrow \ker(G(f))$$

is an isomorphism.

Remark 11.4. *Below we sometimes will consider difference kernels* $\ker(f, g)$*. It will not imply any difficulties because*

$$\ker(f, g) \approx \ker(f - g).$$

Proposition 11.5. *The forgetting functor*

$$U : Mod^H \longrightarrow Mod_k$$

is weak left exact.

Proof. Consider

$$f : X \longrightarrow Y$$

in Mod^H, and let

$$K = \ker(U(f))$$

be f.g. projective. H is f.g. projective, and therefore flat, over k. It follows that the sequence

$$0 \longrightarrow K \otimes H \longrightarrow X \otimes H \longrightarrow Y \otimes H$$

is exact, and therefore the H-comodule structure

$$\delta_X : X \longrightarrow X \otimes H$$

uniquely extends to a H-comodule structure

$$\delta_K : K \longrightarrow K \otimes H$$

on K. Let us denote the resulting comodule by K'. It is easy to prove that

$$K' = \ker(f : X \longrightarrow Y)$$

and that

$$U(K') = K \approx K = \ker(G(f)).$$

□

Finally, both the category Mod^H and Mod_k are **closed under idempotents**:

Definition 11.6. *A category $\mathcal{A}$ is said to have splitting idempotents, or to be **closed under idempotents** if the following is satisfied: Let*

$$f : X \longrightarrow X$$

be an idempotent in $\mathcal{A}$, i.e. $f \circ f = f$. Then there exist a P and morphisms

$$\begin{aligned} g &: X \longrightarrow P \\ h &: P \longrightarrow X \end{aligned}$$

such that

$$\begin{aligned} g \circ h &= f \\ h \circ g &= 1_P \end{aligned}$$

We will also need the following standard Lemma on adjoint functors:

Lemma 11.7. *Let*

$$(U : \mathcal{C} \longrightarrow \mathcal{D}, W : \mathcal{D} \longrightarrow \mathcal{C})$$

be a pair of adjoint k-linear functors between k-linear categories $\mathcal{C}$ and $\mathcal{D}$, and let

$$\begin{aligned} s_X &: X \longrightarrow WU(X), X \in Ob(\mathcal{C}), \\ t_Y &: UW(Y) \longrightarrow Y, Y \in Ob(\mathcal{D}), \end{aligned}$$

be the adjunctions. Then the sequence

$$0 \longrightarrow X \xrightarrow{s_X} WU(X) \underset{sWU}{\overset{WUs}{\rightrightarrows}} WUWU(X)$$

becomes split exact after applying the functor U.

Corollary 11.8. *Let*

$$X \in Ob\left(Mod^H\right).$$

Then the following sequence

$$0 \longrightarrow X \xrightarrow{\delta} X \otimes H \underset{1\otimes\Delta}{\overset{\delta\otimes 1}{\rightrightarrows}} X \otimes H \otimes H$$

is exact in Mod^H, and split exact in Mod_k.

We have now established enough properties of the forgetting functor U in order to formulate our Conjecture.

Conjecture 11.9. *Let*

$$\begin{aligned} G &: \mathcal{C} \longrightarrow Mod_k, \\ G &= UF, \\ F &: \mathcal{C} \longrightarrow Mod^H, \end{aligned}$$

as above. Then F is an equivalence if and only if the following are satisfied:

- *G is faithful;*
- *G is weak left exact;*
- *$(G(\_))^*$ is representable;*

- *$\mathcal{C}$ is closed under idempotents.*

Remark 11.10. *The two last conditions can be replaced by one condition: G admits a right adjoint.*

Below a sketch of the proof is given:

Proof. **Step 1.** The conditions are necessary. Assume F is an equivalence. Since the forgetting functor U satisfies the four conditions above, the same does the functor G.

Step 2. Assume that G satisfies all the four conditions. Since both $G = UF$ and U are faithful, the functor F is faithful as well.

Step 3. Let $\mathbf{H}$ represent the functor $(G(\_))^*$, i.e. for all $X \in Ob(\mathcal{C})$,

$$\begin{aligned} Hom_{\mathcal{C}}(X, \mathbf{H}) &\approx (G(X))^* = Hom_k(G(X), k) \approx Hom_k(UF(X), k) \\ &\approx Hom^H(F(X), W(k)) \approx Hom^H(F(X), H). \end{aligned}$$

Set $X = \mathbf{H}$ in the above sequence of isomorphisms. Then there is a H-comodule morphism

$$i \in Hom^H(F(\mathbf{H}), H)$$

which is the image of

$$id_{\mathbf{H}} \in Hom_{\mathcal{C}}(\mathbf{H}, \mathbf{H}).$$

One can prove that

$$i : F(\mathbf{H}) \approx H$$

as H-comodules.

Step 4. Let us construct a right adjoint R to the functor G. Put

$$\begin{aligned} R(k) &:= \mathbf{H}, \\ R(k^n) &:= \mathbf{H}^n. \end{aligned}$$

Since $\mathbf{H}$ represents $(G(\_))^*$, one has

$$Hom_{\mathcal{C}}(X, \mathbf{H}^n) \approx Hom_k(G(X), k^n).$$

Therefore, both R and the adjunctions are constructed for **free** modules. Since both Mod_k and $\mathcal{C}$ are closed under idempotents, the functor R and the adjunctions can be easily extended to the whole category Mod_k.

Step 5. There exists a natural isomorphism

$$FR(Y) \approx W(Y) = Y \otimes H, Y \in Ob(Mod_k),$$

which commutes with the adjunctions. The latter means the following. If we denote the adjunctions for (G, R) by $(\mathbf{s}, \mathbf{t})$ and the adjunctions for (U, W) by (s, t), then, for $X \in Ob(\mathcal{C})$, the composition

$$s_{F(X)} : F(X) \longrightarrow WUF(X) = WG(X) \approx FRG(X)$$

equals $F(\mathbf{s}_X)$, and, for $Y \in Ob(Mod_k)$, the composition

$$GR(Y) = UFR(Y) \approx UW(Y) \xrightarrow{t_Y} Y$$

equals $\mathbf{t}_Y$.

Step 6. Let

$$X \in Ob\left(Mod^H\right).$$

The sequence

$$0 \longrightarrow X \xrightarrow{\delta} X \otimes H \underset{1\otimes\Delta}{\overset{\delta\otimes 1}{\rightrightarrows}} X \otimes H \otimes H$$

is exact both in Mod^H and in Mod_k. Under the adjunction between U and W, the morphisms δ, $\delta \otimes 1$ and $1 \otimes \Delta$ correspond respectively to

$$\begin{aligned} Id &: X \longrightarrow X, \\ \alpha &: X \otimes H \longrightarrow X \otimes H, \\ Id &: X \otimes H \longrightarrow X \otimes H, \end{aligned}$$

where

$$\alpha (x \otimes h) = \delta(x) \cdot \varepsilon(h).$$

Let us construct a pair of morphisms

$$R(X) \underset{g}{\overset{f}{\rightrightarrows}} R(X \otimes H)$$

in $\mathcal{C}$, such that f and g correspond respectively to

$$\alpha, Id : GR(X) = X \otimes H \longrightarrow X \otimes H.$$

Since G is weak left exact, there exists

$$K = \ker(f, g)$$

in $\mathcal{C}$, and

$$F(K) \approx X$$

in Mod^H. Therefore the functor F is **essentially surjective**.

Step 7. Let

$$X, Y \in Ob(\mathcal{C}).$$

Lemma 11.7 and weak left exactness of G describes Y as

$$Y = \ker (RG(Y) \rightrightarrows RGRG(Y)).$$

It follows that

$$F(Y) = \ker (WUF(Y) \rightrightarrows WUWUF(Y)).$$

Now

$$\begin{aligned} Hom_{\mathcal{C}} (X, Y) &= \ker (Hom_{\mathcal{C}} (X, RG(Y)) \rightrightarrows Hom_{\mathcal{C}} (X, RGRG(Y))) \\ &\approx \ker (Hom_k (G(X), G(Y)) \rightrightarrows Hom_k (G(X), GRG(Y))) \\ &= \ker (Hom_k (UF(X), UF(Y)) \rightrightarrows Hom_k (UF(X), UFRUF(Y))) \\ &\approx \ker \left(Hom^H (F(X), WUF(Y)) \rightrightarrows Hom^H (F(X), WUFRUF(Y))\right) \\ &\approx \ker \left(Hom^H (F(X), WUF(Y)) \rightrightarrows Hom^H (F(X), WUWUF(Y))\right) \\ &\approx Hom^H (F(X), F(Y)), \end{aligned}$$

therefore F is **full**.

Step 8. We have proved that F is full, faithful and essentially surjective. It follows from Theorem 7.8 that F is an equivalence. □

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