SOME NEW HARDY–TYPE INEQUALITIES IN q–ANALYSIS

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Abstract. We derive necessary and sufficient conditions (of Muckenhoupt-Bradley type) for the validity of q-analogs of (r,p)-weighted Hardy-type inequalities for all possible positive values of the parameters r and p. We also point out some possibilities to further develop the theory of Hardy-type inequalities in this new direction.

1. Introduction

G. H. Hardy announced in 1920 [17] and finally proved in 1925 [18] (also see [19, p. 240]) his famous inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)dx, \ p > 1, \tag{1.1}$$

for all non-negative functions f (in the sequel we assume that all functions are non-negative). The constant $\left(\frac{p}{p-1} \right)^p$ in (1.1) is sharp. Since then it has been an enormous activity to develop and apply what is today known as Hardy-type inequalities, see e.g the books [21], [23] and [24] and the references there.

One central problem in this development was to characterize the weights $u(x)$ and $v(x)$ so that the more general Hardy-type inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt \right)^r u(x)dx \right)^{\frac{1}{r}} \leqslant C \left(\int_0^\infty f^p(x)v(x)dx \right)^{\frac{1}{p}} \tag{1.2}$$

holds for some constant C and various parameters p and r.

To make our introduction clear we just concentrate on the case $1 \leqslant p \leqslant r < \infty$. In this case e.g the following result is well-known:

PROPOSITION A. *Let $1 < p \leqslant r < \infty$. Then the inequality (1.2) holds if and only if*

$$A_1 := \sup_{0<x<\infty} (U(x))^{\frac{1}{r}} (V(x))^{\frac{1}{p'}} < \infty$$

or

$$A_2 := \sup_{0<x<\infty} \left(\int_0^x u(t)V^r(t)dt \right)^{\frac{1}{r}} V^{-\frac{1}{p}}(x) < \infty$$

or

$$A_3 := \sup_{0<x<\infty} \left(\int_x^\infty v^{1-p'}(t)U^{p'}(t)dt \right)^{\frac{1}{p'}} U^{-\frac{1}{r}}(x) < \infty,$$

where $U(x) = \int_x^\infty u(t)dt$, $V(x) = \int_0^x v^{1-p'}(t)dt$, $p' = \frac{p}{p-1}$ and $r' = \frac{r}{r-1}$. Moreover, for the sharp constant in (1.2) we have that $C \approx A_1 \approx A_2 \approx A_3$.

REMARK 1.1. A nice proof of the condition $A_1 < \infty$ was given in 1978 by J. S. Bradley [9]. The case $p = r$ was proved by B. Muckenhoupt [28] already in 1972. The condition $A_2 < \infty$ was proved in 2002 by L. E. Persson and V. D. Stepanov [30], but was for the case $p = r$ proved by G. A. Tomaselli [34] already in 1969. The condition $A_3 < \infty$ is just the dual condition of the condition $A_2 < \infty$.

In the beginning G. H. Hardy was most occupied with the discrete version of (1.1). The discrete version of (1.2) reads:

$$\left(\sum_{n=1}^\infty \left(\sum_{k=1}^n f_k \right)^r u_n \right)^{\frac{1}{r}} \leqslant C \left(\sum_{n=1}^\infty f_n^p v_n \right)^{\frac{1}{p}}, \qquad (1.3)$$

where $u = \{u_n\}$ and $v = \{v_n\}$ are non-negative weight sequences and the question is to characterize all such weight sequence so that (1.3) holds for an arbitrary non-negative sequence $f = \{f_n\}$ (in the sequel we assume that the considered sequences are non-negative).

It is interesting that the similar results as that in Proposition A for the discrete case was independently proved by G. Bennett [6] in 1987 (see also [2], [8] and [22, Theorem 7]). It reads:

PROPOSITION B. *Let $1 < p \leqslant r < \infty$. Then the inequality (1.3) holds if and only if*

$$B_1 := \sup_{n \in \mathbb{N}} U_n^{\frac{1}{r}} V_n^{\frac{1}{p'}} < \infty$$

or

$$B_2 := \sup_{n \in \mathbb{N}} \left(\sum_{k=1}^n u_k V_k^r \right)^{\frac{1}{r}} V_n^{-\frac{1}{p}} < \infty$$

or

$$B_3 := \sup_{n \in \mathbb{N}} \left(\sum_{k=n}^{\infty} v_k^{1-p'} U_k^{p'} \right)^{\frac{1}{p'}} U_n^{-\frac{1}{r}} < \infty,$$

where $U_n = \sum\limits_{k=n}^{\infty} u_k$ *and* $V_n = \sum\limits_{k=1}^{n} v_k^{1-p'}$.

Moreover, for the sharp constant C *in* (1.3) *it yields that* $C \approx B_1 \approx B_2 \approx B_3$.

For our purposes we will consider the inequality (1.3) on the following different but equivalent form:

$$\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k f_k \right)^r \right)^{\frac{1}{r}} \leqslant C \left(\sum_{n=1}^{\infty} f_n^p \right)^{\frac{1}{p}}, \tag{1.4}$$

with the obvious changes of the conditions $B_i < \infty$, $i = 1, 2, 3$.

In 1910, F. H. Jackson defined q-derivative and definite q-integral [20] (see also [11]). It was the starting point of q-analysis. Today the interest in the subject has exploded. The q-analysis has numerous applications in various fields of mathematics e.g dynamical systems, number theory, combinatorics, special functions, fractals and also for scientific problems in some applied areas such as computer science, quantum mechanics and quantum physics (see e.g. [3], [5], [12], [13] and [14]). For the further development and recent results in q-analysis we refer to the books [3], [11] and [12] and the references given therein. The first results concerning integral inequalities in q-analysis were proved in 2004 by H. Gauchman [15]. Later on some further q-analogs of the classical inequalities have been proved (see [22], [27], [32] and [33]). We also pronounce the recent book [1] by G.A. Anastassiou, where many important q-inequalities are proved and discussed. Moreover, in 2014 L. Maligranda, R. Oinarov and L.-E. Persson [26] derived a q-analog of the classical Hardy inequality (1.1) and some related inequalities. It seems to be a huge new research area to investigate which of these so called Hardy-type inequalities have their q-analogs.

One main aim in this paper is to prove the q-analog of the results in Propositions A and B (see our Theorem 3.1). We will also prove the corresponding characterization for other possible values of the parameters p and r (see our Theorem 3.3). We also prove the corresponding dual results (see Theorem 3.2 and Theorem 3.4).

Our paper is organized as follows: The main results are stated Section 3 and proved in Section 4. In order not to disturb our discussions there some preliminaries are given in Section 2. In particular, we present some basic facts from q-analysis and also state Proposition B on a formally more general form namely where $\sum\limits_{1}^{\infty}$ is replaced by $\sum\limits_{-\infty}^{\infty}$ (see Proposition 2.2). We also state this result for other parameters which is important for our proof of the Theorem 3.3 (see Proposition 2.3). Finally, in Section 5 we present some remarks and in particular point out the possibility to generalize our results even to modern forms of Propositions A and B, where these three conditions even can be replaced by four scales of conditions (For the continuous case, see the review article [25] and for the discrete case see [29]).

2. Preliminaries

2.1. Some basic facts in q-analysis

This subsection gives the definitions and notions of q-analysis [11] (see also [12]). Let the function f defined on $(0,b)$, $0 < b \leqslant \infty$ and $0 < q < 1$. Then

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \in (0,b) \tag{2.1}$$

is called the q-derivative of the function f. This definition was introduced by F. H. Jackson in 1910.

Let $x \in (0,b)$. Then

$$\int_0^x f(t)d_q t := (1-q)x \sum_{k=0}^{\infty} q^k f(xq^k), \tag{2.2}$$

is called q-integral or Jackson integral.

If $b = \infty$ the improper q-integral is defined by

$$\int_0^{\infty} f(t)d_q t := (1-q) \sum_{k=-\infty}^{\infty} q^k f(q^k). \tag{2.3}$$

The integrals (2.2) and (2.3) are meaningful, if the series on the right hand sides converge.

Let $0 < a < b \leqslant \infty$. Then we have that

$$\int_a^b f(t)d_q t := \int_0^b f(t)d_q t - \int_0^a f(t)d_q t. \tag{2.4}$$

We also need the following fact:

PROPOSITION 2.1. *Let* $k \in \mathbb{Z}$. *Then*

$$\int_{q^{k+1}}^{\infty} f(t)d_q t = (1-q) \sum_{j=-\infty}^{k} q^j f(q^j). \tag{2.5}$$

Proof of Proposition 2.1. By using (2.2), (2.3) and (2.4) with $b = \infty$, $a = q^{k+1}$ we have that

$$\int_{q^{k+1}}^{\infty} f(t)d_q t = \int_0^{\infty} f(t)d_q t - \int_0^{q^{k+1}} f(t)d_q t$$

$$= (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j) - (1-q) \sum_{i=0}^{\infty} q^{i+k+1} f(q^{i+k+1})$$

$$= (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j) - (1-q) \sum_{i=k+1}^{\infty} q^i f(q^i)$$

$$= (1-q) \sum_{j=-\infty}^{k} q^j f(q^j),$$

i.e. (2.5) holds. The proof is complete. $\square$

Let Ω be a subset of $(0,\infty)$ and $\mathscr{X}_{\Omega}(t)$ denote the characteristic function of the set Ω. Let $z > 0$. Then from (2.3) we can deduce that

$$\int_0^{\infty} \mathscr{X}_{(0,z]}(t)f(t)d_q t = (1-q) \sum_{i=-\infty}^{\infty} q^i \mathscr{X}_{(0,z]}(q^i)f(q^i) = (1-q) \sum_{q^i \leqslant z} q^i f(q^i), \qquad (2.6)$$

and

$$\int_0^{\infty} \mathscr{X}_{[z,\infty)}(t)f(t)d_q t = (1-q) \sum_{q^i \geqslant z} q^i f(q^i). \qquad (2.7)$$

Moreover,

$$\int_0^{\infty} \mathscr{X}_{(qz,z]}(t)f(t)d_q t = (1-q)q^k f(q^k), \qquad (2.8)$$

for $q^k \leqslant z < q^{k-1}$, $k \in \mathbb{Z}$,

$$\int_0^{\infty} \mathscr{X}_{[z,q^{-1}z)}(t)f(t)d_q t = (1-q)q^m f(q^m), \qquad (2.9)$$

for $q^{m+1} < z \leqslant q^m$, $m \in \mathbb{Z}$.

2.2. An important variant of Proposition B

We consider the inequality:

$$\left(\sum_{n=-\infty}^{\infty} \left(u_n \sum_{k=-\infty}^{n} v_k f_k \right)^r \right)^{\frac{1}{r}} \leqslant C \left(\sum_{n=-\infty}^{\infty} f_n^p \right)^{\frac{1}{p}}, \quad f_n \geqslant 0. \qquad (2.10)$$

We need the following formal extension of Proposition B, of independent interest:

PROPOSITION 2.2. *Let* $1 < p \leqslant r < \infty$. *Then the inequality* (2.10) *holds if and only if*

$$C_1 = \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{1}{r}} \left(\sum_{i=-\infty}^{n} v_i^{p'} \right)^{\frac{1}{p'}} < \infty \qquad (2.11)$$

or

$$C_2 = \sup_{n \in \mathbb{Z}} \left(\sum_{i=-\infty}^{n} v_i^{p'} \right)^{-\frac{1}{p}} \left(\sum_{k=-\infty}^{n} u_k^r \left(\sum_{i=-\infty}^{k} v_i^{p'} \right)^{r} \right)^{\frac{1}{r}} < \infty \qquad (2.12)$$

or

$$C_3 = \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{-\frac{1}{r}} \left(\sum_{i=n}^{\infty} v_i^{p'} \left(\sum_{k=i}^{\infty} u_k^r \right)^{p'} \right)^{\frac{1}{p'}} < \infty. \qquad (2.13)$$

Moreover, for the sharp constant C in (2.10) it yields that $C \approx C_1 \approx C_2 \approx C_3$.

This proposition is even equivalent to Proposition B, which can be seen from the proof below we give for the reader's convenience.

Proof of Proposition 2.2. Let $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\} \cup \{-\infty\}$, $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. The function $\varphi : \mathbb{Z} \to \mathbb{N}$, given by

$$\forall n \in \bar{\mathbb{Z}} : \varphi(n) = \begin{cases} +\infty & n = +\infty, \\ 2n & n > 0, \\ -2n+3 & n \leqslant 0, \\ 1 & n = -\infty, \end{cases}$$

is a bijection.

Therefore, $\varphi(n) = m$, $m = 1, 2, \cdots$ and $\varphi(k) = j$, $j = 1, 2, \cdots m$, so that

$$\left(\sum_{n=-\infty}^{\infty} \left(u_n \sum_{k=-\infty}^{n} v_k f_k \right)^{r} \right)^{\frac{1}{r}} = \left(\sum_{\varphi(n)=\varphi(-\infty)}^{\varphi(\infty)} \left(u_{\varphi(n)} \sum_{\varphi(k)=\varphi(-\infty)}^{\varphi(n)} v_{\varphi(k)} f_{\varphi(k)} \right)^{r} \right)^{\frac{1}{r}}$$

$$= \left(\sum_{m=1}^{\infty} \left(\tilde{u}_m \sum_{j=1}^{m} \tilde{v}_j \tilde{f}_j \right)^{r} \right)^{\frac{1}{r}}, \qquad (2.14)$$

and

$$\left(\sum_{n=-\infty}^{\infty} f_n^p \right)^{\frac{1}{p}} = \left(\sum_{\varphi(n)=\varphi(-\infty)}^{\varphi(\infty)} f_{\varphi(n)}^p \right)^{\frac{1}{p}} = \left(\sum_{m=1}^{\infty} \tilde{f}_m^p \right)^{\frac{1}{p}}, \qquad (2.15)$$

where $\tilde{f}_m = f_{\varphi(n)}$, $\tilde{u}_m = u_{\varphi(n)}$, $\tilde{v}_j = v_{\varphi(k)}$.

By (2.14) and (2.15), we obtain that (2.10) holds if and only if the inequality

$$\left(\sum_{m=1}^{\infty} \left(\tilde{u}_m \sum_{j=1}^{m} \tilde{v}_j \tilde{f}_j \right)^{r} \right)^{\frac{1}{r}} \leqslant C \left(\sum_{m=1}^{\infty} \tilde{f}_m^p \right)^{\frac{1}{p}} \qquad (2.16)$$

holds.

Let $1 < p \leqslant r < \infty$. By Proposition B we get that the inequality (2.16) holds if and only if

$$B_1 = \sup_{m \in \mathbb{N}} \left(\sum_{j=m}^{\infty} \tilde{u}_j^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^{m} \tilde{v}_i^{p'} \right)^{\frac{1}{p'}} < \infty$$

holds. Moreover, since the function $\varphi^{-1} : \mathbb{N} \to \mathbb{Z}$ is a bijection, we find that

$$B_1 = \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{1}{r}} \left(\sum_{i=-\infty}^{n} v_i^{p'} \right)^{\frac{1}{p'}} = C_1. \tag{2.17}$$

Hence, according to (2.14), (2.15) and (2.17), we obtain that the inequality (2.10) holds if and only if $C_1 < \infty$. Moreover, by Proposition B we find that $C \approx C_1$, where C is the sharp constant in (2.10).

The proofs of the facts that also $C_2 < \infty$ and $C_3 < \infty$ are necessary and sufficient conditions for the characterization of (2.10), and also that $C \approx C_2 \approx C_3$, are similar so we leave out the details. The proof is complete. $\square$

We also need the corresponding result for other cases of possible parameters p and r.

PROPOSITION 2.3. *(i). Let $0 < p \leqslant 1$, $p \leqslant r < \infty$. Then the inequality (2.10) holds if and only if*

$$C_4 = \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{1}{r}} v_n < \infty. \tag{2.18}$$

(ii). Let $1 < p < \infty$, $0 < r < p$. Then the inequality (2.10) holds if and only if

$$C_5 = \left(\sum_{n=-\infty}^{\infty} \left(\sum_{i=-\infty}^{n} v_i^{p'} \right)^{\frac{r(p-1)}{p-r}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{r}{p-r}} u_n^r \right)^{\frac{p-r}{pr}} < \infty. \tag{2.19}$$

(iii). Let $0 < r < p = 1$. Then the inequality (2.10) is satisfied if and only if

$$C_6 = \left(\sum_{n=-\infty}^{\infty} \max_{i \leqslant n} v_i^{\frac{r}{1-r}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{r}{1-r}} u_n^r \right)^{\frac{1-r}{r}} < \infty. \tag{2.20}$$

In all cases (i)–(iii) for the best constant in (2.10) it yields that $C \approx B_i, i = 4, 5, 6$, respectively.

Proof of Proposition 2.3. By using well-known characterizations (see [6], [7], [8], [10], [16] and [21, p. 58]) for the cases (i)–(iii) where $\sum\limits_{-\infty}^{\infty}$ is replaced by $\sum\limits_{1}^{\infty}$, the proof can be performed exactly as the proof of Proposition 2.2. We leave out the details.

2.3. Some q-analogs of weighted Hardy-type inequalities

Let $0 < r, p \leqslant \infty$. Then the q-analog of the discrete Hardy-type inequality of the form (1.4) can be rewritten in the following way:

$$\left(\int_0^\infty \left(u(x) \int_0^x v(t) f(t) d_q t \right)^r d_q x \right)^{\frac{1}{r}} \leqslant C \left(\int_0^\infty f^p(x) d_q x \right)^{\frac{1}{p}}. \qquad (2.21)$$

By Proposition 2.1 we find that the inequality (2.21) can be rewritten on the following dual form:

$$\left(\int_0^\infty \left(v(x) \int_{qx}^\infty u(t) g(t) d_q t \right)^{p'} d_q x \right)^{\frac{1}{p'}} \leqslant C \left(\int_0^\infty g^{r'}(x) d_q x \right)^{\frac{1}{r'}}. \qquad (2.22)$$

We see that the (2.22) lacks some symmetry as in classical analysis.

We consider the operator $(H_q f)(x) = \int_0^\infty \mathscr{X}_{(0,x]}(t) v(t) f(t) d_q t$, which is defined for all $x > 0$. Although it does not coincide with the operator $\int_0^x v(t) f(t) d_q t$ (they coincide at the points $x = q^k$, $k \in \mathbb{Z}$) we have the equality

$$\int_0^\infty \left(u(x) \int_0^x v(t) f(t) d_q t \right)^r d_q x = \int_0^\infty \left(u(x) \int_0^\infty \mathscr{X}_{(0,x]}(t) v(t) f(t) d_q t \right)^r d_q x.$$

Therefore, the inequality (2.21) can be rewritten as

$$\left(\int_0^\infty \left(u(x) \int_0^\infty \mathscr{X}_{(0,x]}(t) v(t) f(t) d_q t \right)^r d_q x \right)^{\frac{1}{r}} \leqslant C \left(\int_0^\infty f^p(x) d_q x \right)^{\frac{1}{p}}, \qquad (2.23)$$

which will be called the q-integral analog of the weighted Hardy-type inequality. The dual inequality of the inequality (2.23) (equivalent of (2.22)) reads:

$$\left(\int_0^\infty \left(v(t) \int_0^\infty \mathscr{X}_{[x,\infty)}(x) u(x) g(x) d_q x \right)^{p'} d_q t \right)^{\frac{1}{p'}} \leqslant C \left(\int_0^\infty g^{r'}(t) d_q t \right)^{\frac{1}{r'}}.$$

3. The main results

Our main result reads:

THEOREM 3.1. *Let* $1 < p \leqslant r < \infty$. *Then the inequality (2.23) holds if and only if*

$$D_1 = \sup_{z>0} \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(x)u^r(x)d_qx \right)^{\frac{1}{r}} \left(\int_0^\infty \mathscr{X}_{(0,z]}(t)v^{p'}(t)d_qt \right)^{\frac{1}{p'}} < \infty$$

or

$$D_2 = \sup_{z>0} \left(\int_0^\infty \mathscr{X}_{(0,z]}(t)v^{p'}(t)d_qt \right)^{-\frac{1}{p}}$$

$$\left(\int_0^\infty \mathscr{X}_{(0,z]}(x)u^r(x) \left(\int_0^\infty \mathscr{X}_{(0,z]}(t)v^{p'}(t)d_qt \right)^r d_qx \right)^{\frac{1}{r}} < \infty$$

or

$$D_3 = \sup_{z>0} \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(x)u^r(x)d_qx \right)^{-\frac{1}{r}}$$

$$\left(\int_0^\infty \mathscr{X}_{[z,\infty)}(t)v^{p'}(t) \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(x)u^r(x)d_qx \right)^{p'} d_qt \right)^{\frac{1}{p'}} < \infty.$$

Moreover, for the sharp constant in (2.23) we have that $C \approx D_1 \approx D_2 \approx D_3$.

Next, we will consider the corresponding inequality

$$\left(\int_0^\infty \left(u(x) \int_0^\infty \mathscr{X}_{[x,\infty)}(t)v(t)f(t)d_qt \right)^r d_qx \right)^{\frac{1}{r}} \leqslant C \left(\int_0^\infty f^p(x)d_qx \right)^{\frac{1}{p}}, \qquad (3.1)$$

for the dual operator of H_q.

THEOREM 3.2. *Let* $1 < p \leqslant r < \infty$. *Then the inequality (3.1) holds if and only if*

$$D_1^* = \sup_{z>0} \left(\int_0^\infty \mathscr{X}_{(0,z]}(x)u^r(x)d_qx \right)^{\frac{1}{r}} \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(t)v^{p'}(t)d_qt \right)^{\frac{1}{p'}} < \infty$$

or

$$D_2^* = \sup_{z>0} \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(t) v^{p'}(t) d_q t \right)^{-\frac{1}{p}}$$

$$\left(\int_0^\infty \mathscr{X}_{[z,\infty)}(x) u^r(x) \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(t) v^{p'}(t) d_q t \right)^r d_q x \right)^{\frac{1}{r}} < \infty$$

or

$$D_3^* = \sup_{z>0} \left(\int_0^\infty \mathscr{X}_{(0,z]}(x) u^r(x) d_q x \right)^{-\frac{1}{r'}}$$

$$\left(\int_0^\infty \mathscr{X}_{(0,z]}(t) v^{p'}(t) \left(\int_0^\infty \mathscr{X}_{(0,z]}(x) u^r(x) d_q x \right)^{p'} d_q t \right)^{\frac{1}{p'}} < \infty.$$

Moreover, for the sharp constant in (3.1) *we have that* $C \approx D_1^* \approx D_2^* \approx D_3^*$.

Concerning other possible parameters of p and r we have the following complement of Theorem 3.1:

THEOREM 3.3. *(i). Let* $0 < p \leqslant 1$, $p \leqslant r < \infty$. *Then the inequality* (2.23) *holds if and only if*

$$D_4 = \sup_{z>0} \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(x) u^r(x) d_q x \right)^{\frac{1}{r}} \left(\int_0^\infty \mathscr{X}_{(qz,z]}(t) v^{p'}(t) d_q t \right)^{\frac{1}{p'}} < \infty.$$

(ii). Let $1 < p < \infty$, $0 < r < p$. *Then the inequality* (2.23) *holds if and only if*

$$D_5 = \left(\int_0^\infty \left(\int_0^\infty \mathscr{X}_{(0,z]}(t) v^{p'}(t) d_q t \right)^{\frac{r(p-1)}{p-r}} \right.$$

$$\left. \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(x) u^r(x) d_q x \right)^{\frac{r}{p-r}} u^r(z) d_q z \right)^{\frac{p-r}{pr}} < \infty.$$

(iii). Let $0 < r < p = 1$. Then the inequality (2.23) is satisfied if and only if

$$D_6 = \left(\int_0^\infty \sup_{y < z} \left(\int_0^\infty \mathscr{X}_{(qy,y]}(t) \frac{v(t)}{(1-q)t} d_q t \right)^{\frac{r}{1-r}} \right.$$

$$\left. \left(\left(\int_0^\infty \mathscr{X}_{[z,\infty)}(x) u^r(x) d_q x \right)^{\frac{r}{1-r}} u^r(z) d_q z \right)^{\frac{1-r}{r}} < \infty.$$

In all cases (i)–(iii), for the best constant in (2.23) it yields that $C \approx D_i$, $i = 4, 5, 6$, respectively.

Finally, the corresponding complement of Theorem 3.2 reads:

THEOREM 3.4. *(i). Let $0 < p \leqslant 1$, $p \leqslant r < \infty$. Then the inequality (3.1) holds if and only if*

$$D_4^* = \sup_{z > 0} \left(\int_0^\infty \mathscr{X}_{(0,z]}(x) u^r(x) d_q x \right)^{\frac{1}{r}} \left(\int_0^\infty \mathscr{X}_{[z,q^{-1}z)}(t) v^{p'}(t) d_q t \right)^{\frac{1}{p'}} < \infty.$$

(ii). Let $1 < p < \infty$, $0 < r < p$. Then the inequality (3.1) holds if and only if

$$D_5^* = \left(\int_0^\infty \left(\int_0^\infty \mathscr{X}_{(0,z]}(x) u^r(x) d_q x \right)^{\frac{r}{p-r}} \right.$$

$$\left. \left(\int_0^\infty \mathscr{X}_{[z,\infty)}(t) v^{p'}(t) d_q t \right)^{\frac{r(p-1)}{p-r}} u^r(z) d_q z \right)^{\frac{p-r}{pr}} < \infty.$$

(iii). Let $0 < r < p = 1$. Then the inequality (3.1) holds if and only if

$$D_6^* = \left(\int_0^\infty \sup_{y \geqslant z} \left(\int_0^\infty \mathscr{X}_{[y,q^{-1}y)}(t) \frac{v(t)}{(1-q)t} d_q t \right)^{\frac{r}{1-r}} \right.$$

$$\left. \left(\int_0^\infty \mathscr{X}_{(0,z]}(x) u^r(x) d_q x \right)^{\frac{r}{1-r}} u^r(z) d_q z \right)^{\frac{1-r}{r}} < \infty.$$

In all cases (i)–(iii), for the best constant in (3.1) it yields that $C \approx D_i^*$, $i = 4, 5, 6$, respectively.

To prove these theorems, we need some Lemmas of independent interest:

LEMMA 3.5. *Let f and g be nonnegative functions and*

$$I(z) := \left(\int_0^\infty \mathcal{X}_{(0,z]}(t)f(t)d_qt \right)^\alpha \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x)g(x)d_qx \right)^\beta,$$

for $\alpha, \beta \in \mathbb{R}$, *and where at least one of the numbers* α, β *is positive. Then*

$$\sup_{z>0} I(z) = (1-q)^{\alpha+\beta} \sup_{k\in\mathbb{Z}} \left(\sum_{j=k}^\infty q^j f(q^j) \right)^\alpha \left(\sum_{i=-\infty}^k q^i g(q^i) \right)^\beta. \qquad (3.2)$$

LEMMA 3.6. *Let* $\alpha, \beta \in \mathbb{R}^+$,

$$I^+(z) := \left(\int_0^\infty \mathcal{X}_{(0,z]}(x)f(x)d_qx \right)^\alpha \left(\int_0^\infty \mathcal{X}_{[z,q^{-1}z)}(t)g(t)d_qt \right)^\beta,$$

and

$$I^-(z) := \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x)f(x)d_qx \right)^\alpha \left(\int_0^\infty \mathcal{X}_{(qz,z]}(t)g(t)d_qt \right)^\beta.$$

Then

$$\sup_{z>0} I^+(z) = (1-q)^{\alpha+\beta} \sup_{k\in\mathbb{Z}} \left(\sum_{i=k}^\infty q^i f(q^i) \right)^\alpha \left(q^k g(q^k) \right)^\beta, \qquad (3.3)$$

and

$$\sup_{z>0} I^-(z) = (1-q)^{\alpha+\beta} \sup_{k\in\mathbb{Z}} \left(\sum_{i=-\infty}^k q^i f(q^i) \right)^\alpha \left(q^k g(q^k) \right)^\beta. \qquad (3.4)$$

LEMMA 3.7. *Let f, φ and g be nonnegative functions. Then*

$$D \equiv \int_0^\infty \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(t)f(t)d_qt \right)^\alpha \left(\int_0^\infty \mathcal{X}_{(0,z]}(x)g(x)d_qx \right)^\beta \varphi(z)d_qz$$

$$= (1-q)^{\alpha+\beta} \sum_{k=-\infty}^\infty \left[\left(\sum_{i=-\infty}^k q^i f(q^i) \right)^\alpha \left(\sum_{j=k}^\infty q^j g(q^j) \right)^\beta q^k \varphi(q^k) \right],$$

for $\alpha, \beta \in \mathbb{R}$.

LEMMA 3.8. *Let* $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}$ *and*

$$F(y) := \left(\int_0^\infty \mathscr{X}_{[y,q^{-1}y)}(t)f(t)d_q t \right)^\alpha.$$

Then

$$\sup_{y \geqslant q^k} F(y) = (1-q)^\alpha \sup_{i \leqslant k} \left(q^i f(q^i) \right)^\alpha. \tag{3.5}$$

4. Proofs

Proof of Lemma 3.5. From (2.6) and (2.7) it follows that

$$I(z) = (1-q)^{\alpha+\beta} \left(\sum_{q^j \leqslant z} q^j f(q^j) \right)^\alpha \left(\sum_{q^i \geqslant z} q^i g(q^i) \right)^\beta.$$

If $z = q^k$, then, for $k \in \mathbb{Z}$,

$$I(z) = I(q^k) = (1-q)^{\alpha+\beta} \left(\sum_{j=k}^\infty q^j f(q^j) \right)^\alpha \left(\sum_{i=-\infty}^k q^i g(q^i) \right)^\beta.$$

If $q^k < z < q^{k-1}$, then, for $k \in \mathbb{Z}$,

$$I(z) = (1-q)^{\alpha+\beta} \left(\sum_{j=k}^\infty q^j f(q^j) \right)^\alpha \left(\sum_{i=-\infty}^{k-1} q^i g(q^i) \right)^\beta.$$

Hence, for $k \in \mathbb{Z}$ and $\beta > 0$ we find that

$$\sup_{q^k \leqslant z < q^{k-1}} I(z) = I(q^k) = (1-q)^{\alpha+\beta} \left(\sum_{j=k}^\infty q^j f(q^j) \right)^\alpha \left(\sum_{i=-\infty}^k q^i g(q^i) \right)^\beta.$$

Therefore

$$\sup_{z>0} I(z) = \sup_{k \in \mathbb{Z}} \sup_{q^k \leqslant z < q^{k-1}} I(z)$$

$$= (1-q)^{\alpha+\beta} \sup_{k \in \mathbb{Z}} \left(\sum_{j=k}^\infty q^j f(q^j) \right)^\alpha \left(\sum_{i=-\infty}^k q^i g(q^i) \right)^\beta.$$

We have proved that (3.1) holds wherever $\beta > 0$.
Next we assume that $\alpha > 0$. Let $q^{k+1} < z < q^k$, $k \in \mathbb{Z}$. Then we get that

$$I(z) = (1-q)^{\alpha+\beta} \sup_{k \in \mathbb{Z}} \left(\sum_{j=k+1}^\infty q^j f(q^j) \right)^\alpha \left(\sum_{i=-\infty}^k q^i g(q^i) \right)^\beta$$

and analogously as above we find that

$$\sup_{q^{k+1}<z\leqslant q^k} I(z) = I(q^k) = (1-q)^{\alpha+\beta}\left(\sum_{j=k}^{\infty}q^j f(q^j)\right)^{\alpha}\left(\sum_{i=-\infty}^{k}q^i g(q^i)\right)^{\beta}$$

and (3.1) holds also for the case $\alpha > 0$. The proof is complete.   □

Proof of Lemma 3.6. According to (2.6) and (2.9) we have that

$$I^+(q^k) = (1-q)^{\alpha+\beta}\left(\sum_{i=k}^{\infty}q^i f(q^i)\right)^{\alpha}\left(q^k g(q^k)\right)^{\beta},$$

for $z = q^k$, $k \in \mathbb{Z}$, and

$$I^+(z) = (1-q)^{\alpha+\beta}\left(\sum_{i=k+1}^{\infty}q^i f(q^i)\right)^{\alpha}\left(q^k g(q^k)\right)^{\beta},$$

for $q^{k+1} < z < q^k$, $k \in \mathbb{Z}$.
Therefore,

$$\sup_{q^{k+1}<z\leqslant q^k} I^+(z) = (1-q)^{\alpha+\beta}\left(\sum_{i=k}^{\infty}q^i f(q^i)\right)^{\alpha}\left(q^k g(q^k)\right)^{\beta}.$$

Since $\sup_{z>0}I^+(z) = \sup_{k\in\mathbb{Z}}\sup_{q^{k+1}<z\leqslant q^k}I^+(z)$, we conclude that (3.3) holds.
Next, by using (2.7) and (2.8) we find that

$$I^-(q^k) = (1-q)^{\alpha+\beta}\left(\sum_{i=-\infty}^{k}q^i f(q^i)\right)^{\alpha}\left(q^k g(q^k)\right)^{\beta}, \qquad (4.1)$$

for $z = q^k$, $k \in \mathbb{Z}$, and

$$I^-(z) = (1-q)^{\alpha+\beta}\left(\sum_{i=-\infty}^{k-1}q^i f(q^i)\right)^{\alpha}\left(q^k g(q^k)\right)^{\beta},$$

for $q^k < z < q^{k-1}$, $k \in \mathbb{Z}$.
Thus,

$$\sup_{q^k\leqslant z<q^{k-1}} I^-(z) = (1-q)^{\alpha+\beta}\left(\sum_{i=-\infty}^{k}q^i f(q^i)\right)^{\alpha}\left(q^k g(q^k)\right)^{\beta}.$$

Since $\sup_{z>0}I^-(z) = \sup_{k\in\mathbb{Z}}\sup_{q^k\leqslant z<q^{k-1}}I^-(z)$, we have that (3.4) holds. The proof is complete.   □

Proof of Lemma 3.7. By using (2.3), (2.6) and (2.7), we have that

$$D = (1-q) \sum_{k=-\infty}^{\infty} q^k \left(\int_0^\infty \mathscr{X}_{[q^k,\infty)}(t)f(t)d_qt \right)^\alpha \left(\int_0^\infty \mathscr{X}_{(0,q^k]}(x)g(x)d_qx \right)^\beta \varphi(q^k)$$

$$= (1-q)^{\alpha+\beta} \sum_{k=-\infty}^{\infty} q^k \left(\sum_{i=-\infty}^{k} q^i f(q^i) \right)^\alpha \left(\sum_{j=k}^\infty q^j g(q^j) \right)^\beta \varphi(q^k).$$

The proof is complete. □

Proof of Lemma 3.8. By using (2.9), we get that

$$F(q^k) = \left(\int_0^\infty \mathscr{X}_{[q^k,q^{k-1})}(t)f(t)d_qt \right)^\alpha = (1-q)^\alpha \left(q^k f(q^k) \right)^\alpha, \tag{4.2}$$

for $y = q^k$, $k \in Z$, and

$$\sup_{y>q^k} F(y) = \sup_{i\leqslant k} \sup_{q^i<y\leqslant q^{i-1}} F(y)$$

$$= (1-q)^\alpha \sup_{i\leqslant k} \left(q^{i-1} f(q^{i-1}) \right)^\alpha$$

$$= (1-q)^\alpha \sup_{i\leqslant k-1} \left(q^i f(q^i) \right)^\alpha, \tag{4.3}$$

for $i \leqslant k$ and $q^i < y \leqslant q^{i-1}$.

From (4.2) and (4.3) it follows that

$$\sup_{y\geqslant q^k} F(y) = \max\{ \sup_{y>q^k} F(y), F(q^k) \} = (1-q)^\alpha \sup_{i\leqslant k} \left(q^i f(q^i) \right)^\alpha.$$

Thus, (3.5) holds so the proof is complete. □

Proof of Theorem 3.2. By using (2.3) and (2.7), we have that

$$\left(\int_0^\infty f^p(x)d_qx \right)^{\frac{1}{p}} = (1-q)^{\frac{1}{p}} \left(\sum_{j=-\infty}^{\infty} q^j f^p(q^j) \right)^{\frac{1}{p}}, \tag{4.4}$$

and

$$\left(\int_0^\infty \left(u(x) \int_0^\infty \mathscr{X}_{[x,\infty)}(t)v(t)f(t)d_qt \right)^r d_qx \right)^{\frac{1}{r}}$$

$$= (1-q)^{\frac{1}{r}} \left(\sum_{j=-\infty}^{\infty} q^j u^r(q^j) \left(\int_0^\infty \mathscr{X}_{[q^j,\infty)}(t)v(t)f(t)d_qt \right)^r \right)^{\frac{1}{r}}$$

$$= (1-q)^{1+\frac{1}{r}} \left(\sum_{j=-\infty}^{\infty} q^j u^r(q^j) \left(\sum_{q^i \geqslant q^j} q^i v(q^i) f(q^i) \right)^r \right)^{\frac{1}{r}}$$

$$= (1-q)^{1+\frac{1}{r}} \left(\sum_{j=-\infty}^{\infty} q^j u^r(q^j) \left(\sum_{i=-\infty}^{j} q^i v(q^i) f(q^i) \right)^r \right)^{\frac{1}{r}}. \tag{4.5}$$

By now using (3.1), (4.4) and (4.5) we find that

$$(1-q)^{\frac{1}{p'}+\frac{1}{r}} \left(\sum_{j=-\infty}^{\infty} q^j \left(u(q^j) \sum_{i=-\infty}^{j} q^i v(q^i) f(q^i) \right)^r \right)^{\frac{1}{r}} \leqslant C \left(\sum_{j=-\infty}^{\infty} q^j f^p(q^j) \right)^{\frac{1}{p}}.$$

Let

$$q^j f^p(q^j) = f_j^p, \ v_j = q^{\frac{j}{p'}} v(q^j)(1-q)^{\frac{1}{p'}}, \ u_j = (1-q)^{\frac{1}{r}} q^{\frac{j}{r}} u(q^j), \ j \in \mathbb{Z}. \tag{4.6}$$

Then we see that the inequality (3.1) is equivalent to the inequality (2.10). The best constants in inequalities (3.1) and (2.10) are the same.

Since the inequality (3.1) is equivalent to the inequality (2.10) we can use Proposition 2.2 to conclude that the inequality (3.1) holds if and only if at least one of the conditions $C_1 < \infty$, $C_2 < \infty$ and $C_3 < \infty$ holds. Moreover, for the best constant C in (3.1) it yields that $C \approx C_1 \approx C_2 \approx C_3$.

Hence, according to Lemma 3.5 we have that

$$C_1 = \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{1}{r}} \left(\sum_{i=-\infty}^{n} v_i^{p'} \right)^{\frac{1}{p'}}$$

$$= (1-q)^{\frac{1}{r}+\frac{1}{p'}} \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} q^k u^r(q^k) \right)^{\frac{1}{r}} \left(\sum_{i=-\infty}^{n} q^i v^p(q^i) \right)^{\frac{1}{p'}}$$

$$= \sup_{z>0} \left(\int_0^{\infty} \mathcal{X}_{(0,z]}(x) u^r(x) d_q x \right)^{\frac{1}{r}} \left(\int_0^{\infty} \mathcal{X}_{[z,\infty]}(t) v^{p'}(t) d_q t \right)^{\frac{1}{p'}} = D_1^*.$$

In particular, $C \approx D_1^*$. Moreover, by arguing as above and using Lemma 3.6 we obtain that $C_2 \approx D_2^*$ and $C_3 \approx D_3^*$. Hence, for the best constant C in (3.1) it yields that $C \approx D_1^* \approx D_2^* \approx D_3^*$. The proof is complete. $\square$

Proof of Theorem 3.4. In a similarly way as in the proof of Theorem 3.2, by using (2.3), (2.7) and (4.6), we find that the inequality (2.10) is equivalent to the inequality (3.1).

Since the inequality (3.1) is equivalent to the inequality (2.10) we can use Proposition 2.3 to conclude that the inequality (3.1) holds if and only if the conditions (2.18), (2.19) and (2.20) hold, for considered cases $0 < p < 1$, $p \leqslant r$; $1 < p < \infty$, $0 < r < p$ and $0 < r < p = 1$, respectively.

Next, we prove that the conditions (2.18), (2.19) and (2.20) are equivalent to the conditions $D_4^* < \infty$, $D_5^* < \infty$ and $D_6^* < \infty$, respectively.

By using Lemma 3.6 from (2.18) and (4.6) we obtain that

$$C_4 = \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{1}{r}} v_n = (1-q)^{\frac{1}{r}+\frac{1}{p'}} \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} q^k u^r(q^k) \right)^{\frac{1}{r}} \left(q^n v^{p'}(q^n) \right)^{\frac{1}{p'}}$$

$$= \sup_{z>0} \left(\int_0^{\infty} \mathscr{X}_{(0,z]}(x) u^r(x) d_q x \right)^{\frac{1}{r}} \left(\int_0^{\infty} \mathscr{X}_{[z,-q^{-1}z)}(t) v^{p'}(t) d_q t \right)^{\frac{1}{p'}} = D_4^*.$$

Moreover, by Lemma 3.7 we have that

$$C_5 = \sum_{n=-\infty}^{\infty} \left(\sum_{i=-\infty}^{n} v_i^{p'} \right)^{\frac{r(p-1)}{p-r}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{r}{p-r}} u_n^r$$

$$= (1-q)^{\frac{rp}{p-r}+1} \int_{n=-\infty}^{\infty} \left(\sum_{i=-\infty}^{n} q^i v^{p'}(q^i) \right)^{\frac{r(p-1)}{p-r}} \left(\sum_{k=n}^{\infty} q^k u^r(q^k) \right)^{\frac{r}{p-r}} q^n u^r(q^n)$$

$$= \int_0^{\infty} \left(\int_0^{\infty} \mathscr{X}_{[z,\infty)}(t) v^{p'}(t) d_q t \right)^{\frac{r(p-1)}{p-r}} \left(\int_0^{\infty} \mathscr{X}_{(0,z]}(x) u^r(x) d_q x \right)^{\frac{r}{p-r}} u^r(z) d_q z = D_5^*$$

Now let $p = 1$ so that $p' = \infty$. Then $v_i = v(q^i)$ in (4.6). By Lemma 3.8 we find that

$$\max_{i\leqslant n} v_i^{\frac{r}{1-r}} = (\max_{i\leqslant n} v(q^i))^{\frac{r}{1-r}} = \left((1-q) \max_{i\leqslant n} \frac{q^i v(q^i)}{(1-q)q^i} \right)^{\frac{r}{1-r}}$$

$$= \left(\sup_{y\geqslant q^n} \int_0^{\infty} \mathscr{X}_{[y,q^{-1}y)}(t) \frac{v(t)}{(1-q)t} d_q t \right)^{\frac{r}{1-r}} = \sup_{y\geqslant q^n} \left(\int_0^{\infty} \mathscr{X}_{[y,q^{-1}y)}(t) \frac{v(t)}{(1-q)t} d_q t \right)^{\frac{r}{1-r}}.$$

Therefore,

$$C_6 = \sum_{n=-\infty}^{\infty} \max_{i\leqslant n} v_i^{\frac{r}{1-r}} \left(\sum_{k=n}^{\infty} u_k^r \right)^{\frac{r}{1-r}} u_n^r$$

$$= \sum_{n=-\infty}^{\infty} q^n \max_{i\leqslant n} v_i^{\frac{r}{1-r}} \left((1-q) \sum_{k=n}^{\infty} q^k u_k^r(q^k) \right)^{\frac{r}{1-r}} u^r(q^n)$$

$$= (1-q) \sum_{n=-\infty}^{\infty} q^n \sup_{y\geqslant q^n} \left(\int_0^{\infty} \mathscr{X}_{[y,q^{-1}y)}(t) \frac{v(t)}{(1-q)t} d_q t \right)^{\frac{r}{1-r}}$$

$$\times \left(\int_0^{\infty} \mathscr{X}_{(0,q^n]}(x) u^r(x) d_q x \right)^{\frac{r}{1-r}} u^r(q^n)$$

$$= \int_0^\infty \sup_{y \geqslant z} \left(\int_0^\infty \mathcal{X}_{[y,q^{-1}y)}(t) \frac{v(t)}{(1-q)t} d_q t \right)^{\frac{r}{1-r}} \left(\int_0^\infty \mathcal{X}_{(0,z]}(x) u^r(x) d_q x \right)^{\frac{r}{1-r}} u^r(z) d_q z$$

$$= D_6^*.$$

Thus, in all cases (i)–(iii), for the best constant in (3.1) it yields that $C \approx D_i^*$, $i = 4, 5, 6$, respectively. The proof is complete. $\square$

Proof of Theorem 3.1. As in the proof of Theorem 3.2 we get that the inequality (2.23) is equivalent to the inequality

$$\left(\sum_{j=-\infty}^\infty \left(u_j \sum_{i=j}^\infty v_i f_i \right)^r \right)^{\frac{1}{r}} \leqslant C \left(\sum_{j=-\infty}^\infty f_j^p \right)^{\frac{1}{p}}. \tag{4.7}$$

By using standard dual arguments the characterizations similar to those in Proposition 2.2 hold also in this situation (see e.g. [16, p. 59]). Here it is even simpler to just put $\widetilde{u}_i = u_{-i}$, $\widetilde{v}_i = v_{-i}$, $\widetilde{f}_i = f_{-i}$, $i \in \mathbb{Z}$, and note that then (4.7) reads

$$\left(\sum_{j=-\infty}^\infty \left(\widetilde{u}_j \sum_{i=-\infty}^j \widetilde{v}_i \widetilde{f}_i \right)^r \right)^{\frac{1}{r}} \leqslant C \left(\sum_{j=-\infty}^\infty \widetilde{f}_j^p \right)^{\frac{1}{p}}. \tag{4.8}$$

Now use Proposition 2.2, and find that the inequality (4.8) holds if and only if one of the conditions $\widetilde{C}_i < \infty$, $1 \leqslant i \leqslant 3$ holds. Note that here $\widetilde{C}_i$, $1 \leqslant i \leqslant 3$, are defined by just in the expressions for C_i inserting $\widetilde{u}_j$, $\widetilde{v}_j$, $j \in \mathbb{Z}$. Moreover, for the best constant C in (4.8) it yields that $C \approx \widetilde{C}_1 \approx \widetilde{C}_2 \approx \widetilde{C}_3$.

Next, by replacing $\widetilde{u}_j$ and $\widetilde{v}_j$ by u_j and v_j, $j \in \mathbb{Z}$, in the expressions $\widetilde{C}_i$, $1 \leqslant i \leqslant 3$, respectively, we obtain the corresponding characterizations for the validity of the inequality (4.7). In a similar way as in the proof of Theorem 3.2, from the equivalence of inequalities (2.23) and (4.7) and using Lemma 3.6 we find that the inequality (2.23) holds if and only if $D_1 < \infty$ or $D_2 < \infty$ or $D_3 < \infty$ holds. Moreover, for the best constant C in (2.23) it yields that $C \approx D_1 \approx D_2 \approx D_3$. The proof is complete. $\square$

Proof of Theorem 3.3. The equivalence between (4.7) and (4.8) holds in the case too. Hence, by arguing exactly as in proof of Theorem 3.1 but using Proposition 2.3 instead of Proposition 2.2 the proof can be done analogously, so we leave out the details.

5. Final remarks

REMARK 5.1. Assume that $v(t) = 0$, $u(t) = 0$, $f(t) = 0$, $t > 1$ and the integrals in the expressions D_i, D_i^*, $1 \leqslant i \leqslant 6$ are replaced by the integrals from zero to one and the sets $[z, \infty)$, $[z, q^{-1}z)$ are replaced by the sets $[z, 1]$, $[z, \min\{q^{-1}z, 1\}]$, respectively. Then, by using Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, we obtain

that the corresponding characterizations for the validity of the inequalities

$$\left(\int_0^1 \left(u(x) \int_0^1 \mathscr{X}_{(0,x]}(t)v(t)f(t)d_qt \right)^r d_qx \right)^{\frac{1}{r}} \leqslant C \left(\int_0^1 f^p(t)d_qt \right)^{\frac{1}{p}},$$

and

$$\left(\int_0^1 \left(u(x) \int_0^1 \mathscr{X}_{[x,1]}(t)v(t)f(t)d_qt \right)^r d_qx \right)^{\frac{1}{r}} \leqslant C \left(\int_0^1 f^p(t)d_qt \right)^{\frac{1}{p}},$$

for all parameters r and p in these theorems.

REMARK 5.2. Note that nowadays it is known that the conditions $B_i < \infty$, $i = 1,2,3$, in Proposition B are special cases of more general conditions. More exactly these conditions can be replaced by infinite many conditions, namely the following four scales of conditions (see [29] and also [17, p. 60]):

$$B_1(s) := \sup_{n \in \mathbb{N}} \left(\sum_{k=1}^n v_k^{1-p'} \right)^{\frac{(s-1)}{p}} \left(\sum_{k=n}^\infty u_k \left(\sum_{m=1}^k v_m^{1-p'} \right)^{\frac{r(p-s)}{p}} \right)^{\frac{1}{r}} < \infty,$$

for s satisfying $1 < s \leqslant p$;

$$B_1^*(s) := \sup_{n \in \mathbb{N}} \left(\sum_{k=n}^\infty u_k \right)^{\frac{(s-1)}{r'}} \left(\sum_{k=1}^n v_k^{1-p'} \left(\sum_{m=k}^\infty u_m \right)^{\frac{p'(r'-s)}{r'}} \right)^{\frac{1}{p'}} < \infty,$$

for s satisfying $1 < s \leqslant r'$;

$$B_2(s) := \sup_{n \in \mathbb{N}} \left(\sum_{k=1}^n v_k^{1-p'} \right)^{-s} \left(\sum_{k=1}^n u_k \left(\sum_{m=1}^k v_m^{1-p'} \right)^{r(\frac{1}{p'}+s)} \right)^{\frac{1}{r}} < \infty,$$

for s satisfying $0 < s \leqslant \frac{1}{p}$;

$$B_2^*(s) := \sup_{n \in \mathbb{N}} \left(\sum_{k=n}^\infty u_k \right)^{-s} \left(\sum_{k=n}^\infty v_k^{1-p'} \left(\sum_{m=k}^\infty u_m \right)^{p'(\frac{1}{r}+s)} \right)^{\frac{1}{p'}} < \infty,$$

for s satisfying $0 < s \leqslant \frac{1}{r'}$. Note that $B_1(p) = B_1^*(r') = B_1$, $B_2(\frac{1}{p}) = B_2$ and $B_2^*(\frac{1}{r'}) = B_3$.

Our results in Theorems 3.1 and 3.2 can be generalized in a corresponding way namely that the three alternative conditions in these theorems can be replaced by infinite many equivalent conditions.

REMARK 5.3. The corresponding alternative conditions for the parameters in Proposition 2.3 are not known except for the continuous case $r < p$, $p > 1$ where even four scales of such alternative equivalent conditions are known (see [31]). Hence, at the moment only in this case it seems to be possible to generalize Theorems 3.3 and 3.4 in this direction.

REMARK 5.4. Some similar results as those in this paper can found in [4] (in Russian). However, the results in this paper are more complete and putted to a more general frame. The proofs are also different and more precise and clear.

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