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SOME NEW TWO–SIDED INEQUALITIES CONCERNING THE FOURIER TRANSFORM

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Abstract. The classical Hausdorff-Young and Hardy-Littlewood-Stein inequalities do not hold for $p > 2$. In this paper we prove that if we restrict to net spaces we can even derive a two-sided estimate for all $p > 1$. In particular, this result generalizes a recent result by Liflyand E. and Tikhonov S. [7] (MR 2464253)

1. Introduction

Let

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx} dx, \quad x \in \mathbb{R},$$

be the Fourier transform of a function $f \in L_1(\mathbb{R})$.

Let $1 < p < 2$, $p' = \frac{p}{p-1}$ and $0 < q \leqslant \infty$. Then we have the following inequalities

$$\|\widehat{f}\|_{L_{p'}(\mathbb{R})} \leqslant c_1 \|f\|_{L_p(\mathbb{R})}, \tag{1}$$

$$\|\widehat{f}\|_{L_{p',q}(\mathbb{R})} \leqslant c_2 \|f\|_{L_{p,q}(\mathbb{R})}, \tag{2}$$

where $L_{p,q}(\mathbb{R})$ is the classical Lorentz space. These inequalities are called the Hausdorff-Young inequality and the Hardy-Littlewood-Stein inequality, respectively, (see e.g. [15] and [16]).

Note that these inequalities (1) and (2) hold with equality for $p = q = 2$ (Plancherel's theorem) but do not hold in general for $2 < p < \infty$.

Let $0 < p, q \leqslant \infty$, M be the set of the segments $[a, b]$ in $\mathbb{R}$ and $|e| = b - a$.

The net space $N_{p'q}(M)$ is defined as the set of all measurable functions f such that the quasinorm

$$\|f\|_{N_{p'q}(M)} = \left(\int_0^{\infty} \left(t^{\frac{1}{p'}} \overline{f}(t, M) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

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for $q<\infty$, and

$$\|f\|_{N_{p'\infty}(M)} = \sup_{t>0} t^{\frac{1}{p'}} \overline{f}(t,M) < \infty$$

for $q=\infty$, where

$$\overline{f}(t;M) := \sup_{\substack{|e|\geqslant t \\ e\in M}} \frac{1}{|e|} \left| \int_e f(x)dx \right|.$$

These spaces were introduced in [11] (see also [12] and [13]). In particular, the following result was proved:

THEOREM A. *Let* $2<p<\infty,\ 0<q\leqslant\infty$. *Then*

$$\|\widehat{f}\|_{N_{p'q}(M)} \leqslant c_3 \|f\|_{L_{pq}(\mathbb{R})}. \tag{3}$$

The inequality (3) complements the Hardy-Littlewood-Stein inequality. Similar results for the Fourier transform in the periodic case were obtained in [10] and [5].

The main aims of this paper are to derive the sufficient condition so that the Fourier transform $\widehat{f}$ belongs to L_p-space ($1<p<\infty$) and to obtain conditions so that the norm of the Fourier transform $\widehat{f}$ in L_p-space ($1<p<\infty$) has both upper and lower estimates.

The main results are formulated in Section 3. The proofs can be found in Section 4 and in Section 2 we present some necessary preliminaries, including new lemmas of independent interest.

CONVENTIONS. The letter $c(c_1, c_2, etc.)$ means a constant which does not depend on the involved functions and it can be different in different occurrences. Moreover, for $A,B>0$ the notation $A \asymp B$ means that there exists positive constants c_1 and c_2 such that $c_1 A \leqslant B \leqslant c_2 A$. For $1<p<\infty$ we denote $p' = p/(p-1)$.

2. Preliminaries

The total variation of the function f, defined on an interval $[a,b]\subset\mathbb{R}$ is the quantity

$$V_a^b(f) := \sup_{\mathfrak{P}} \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)|,$$

where the supremum is taken over all partitions of [a, b]:

$$\mathfrak{P}:\ \ a = x_0 < x_1 < \ldots < x_n = b,\ \ n\in\mathbb{Z}_+.$$

We say that the measurable function $f(x)\in V([a,b])$ if $V_a^b(f)<\infty$.

The total variations $V_a^\infty(f)$ and $V_{-\infty}^b(f)$ can be defined as follows:

$$V_a^\infty(f) := \lim_{b\to\infty} V_a^b(f)$$

and

$$V_{-\infty}^b(f) := \lim_{a\to-\infty} V_a^b(f).$$

We need the following lemma (see [3]).

LEMMA 1. *Let $f(x)$ be a continuous function on a closed interval $[a,b]$ and $g = g(x) \in V([a,b])$. Then the integral*

$$I = \int_a^b f(x)dg(x),$$

exists and

$$|I| \leqslant \max_{x\in[a,b]} |f(x)| V_a^b(g).$$

We also need the following lemmas of independent interest:

LEMMA 2. *Let $1 < p < \infty$. Then*

$$\|\widehat{f}\|_{N_{p'p}(M)} \leqslant c\|f\|_{L_p(\mathbb{R})}.$$

Proof. The statement of this Lemma 2 follows from Theorem A for the case $2 < p < \infty$. Moreover, taking into account that

$$\|\widehat{f}\|_{N_{p'p}(M)} \leqslant c_1 \|\widehat{f}\|_{L_{p'p}(\mathbb{R})},$$

(see [13]) the statement of this Lemma 2 follows from (2) for the case $1 < p \leqslant 2$. The proof is complete. □

Let $V_{2^k}(f) := V_{[2^k,2^{k+1}]\cup[-2^{k+1},-2^k]}(f)$ be the total variation of the function $f(x)$, defined on the set $[2^k,2^{k+1}]\cup[-2^{k+1},-2^k]$, $k\in\mathbb{Z}$.

LEMMA 3. *Let $\alpha > 0$ and $1 < p < \infty$. If*

$$\left(\sum_{k\in\mathbb{Z}} \left(2^{k\alpha} V_{2^k}(f)\right)^p\right)^{\frac{1}{p}} < \infty,$$

then

$$V_1^\infty(f) < \infty \quad \text{and} \quad V_{-\infty}^{-1}(f) < \infty.$$

Proof. It is obvious that

$$V_1^\infty(f) = \sum_{k=0}^\infty V_{2^k}^{2^{k+1}}(f).$$

By using Hölder's inequality, we obtain that

$$\sum_{k=0}^\infty V_{2^k}^{2^{k+1}}(f) \leqslant \left(\sum_{k=0}^\infty \left(2^{k\alpha} V_{2^k}^{2^{k+1}}(f)\right)^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=0}^\infty 2^{-k\alpha\cdot p'}\right)^{\frac{1}{p'}}$$

$$= c_{\alpha,p}\left(\sum_{k=0}^\infty \left(2^{k\alpha} V_{2^k}^{2^{k+1}}(f)\right)^p\right)^{\frac{1}{p}} \leqslant c_{\alpha,p}\left(\sum_{k=0}^\infty \left(2^{k\alpha} V_{2^k}(f)\right)^p\right)^{\frac{1}{p}} < \infty.$$

The second inequality for $V_{-\infty}^{-1}(f)$ can be proved in a similar way. The proof is complete. □

LEMMA 4. *Let* $1 < p < \infty$ *and* $0 < q \leqslant \infty$. *If* $f \in N_{pq}(M)$, *then the following equivalence*

$$\|f\|_{N_{pq}} \asymp \left(\sum_{k \in \mathbb{Z}} \left(2^{\frac{k}{p}} \overline{f}(2^k, M) \right)^q \right)^{\frac{1}{q}}$$

holds, where $\overline{f}(2^k, M) = \sup\limits_{\substack{|e| \geqslant 2^k \\ e \in M}} \frac{1}{|e|} \left| \int_e f(x) dx \right|$, M *is the set of all segments* $[a,b]$ *in* $\mathbb{R}$.

Proof. Since $f \in N_{pq}(M)$ we have that

$$I = \int_0^\infty \left(t^{\frac{1}{p}} \overline{f}(t, M) \right)^q \frac{dt}{t} < \infty.$$

This integral can be represented as follows

$$I = \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(t^{\frac{1}{p}} \overline{f}(t, M) \right)^q \frac{dt}{t}.$$

Taking into account that the function $\overline{f}$ is monotone, we find that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(t^{\frac{1}{p}} \overline{f}(t, M) \right)^q \frac{dt}{t} &\leqslant \sum_{k \in \mathbb{Z}} \left(2^{\frac{k+1}{p}} \overline{f}(2^k, M) \right)^q \int_{2^k}^{2^{k+1}} \frac{dt}{t} \\ &= c_1 \sum_{k \in \mathbb{Z}} \left(2^{\frac{k}{p}} \overline{f}(2^k, M) \right)^q, \end{aligned}$$

and

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(t^{\frac{1}{p}} \overline{f}(t, M) \right)^q \frac{dt}{t} &\geqslant 2^{-\frac{1}{p}} \sum_{k \in \mathbb{Z}} \left(2^{\frac{k+1}{p}} \overline{f}(2^{k+1}, M) \right)^q \int_{2^k}^{2^{k+1}} \frac{dt}{t} \\ &= c_2 \sum_{k \in \mathbb{Z}} \left(2^{\frac{k}{p}} \overline{f}(2^k, M) \right)^q. \end{aligned}$$

The proof is complete. $\square$

The statement in our Theorem 2 is related to a recent result by E. Liflyand and S. Tikhonov [7], where an extended solution of Boas' conjecture was proved (for original proof see [14]). In particular, they defined the class GM as follows:

DEFINITION 1. We say that the function f belongs to the class GM if for all $x \in (0, \infty)$ we have that

$$V_x^{2x}(f) \leqslant c \int_{\frac{x}{\beta}}^{\beta x} t^{-1} |f(t)| dt, \tag{4}$$

for some $\beta > 1$.

3. Main results

Our first main result reads as follows.

THEOREM 1. *Let* $1 < p < \infty$ *and* $f \in L_1(\mathbb{R})$. *If* f *satisfies the condition*

$$\left(\sum_{k\in\mathbb{Z}}\left(2^{\frac{k}{p'}}V_{2^k}(f)\right)^p\right)^{\frac{1}{p}} < \infty, \tag{5}$$

then $\widehat{f} \in L_p(\mathbb{R})$ *and the inequality*

$$\|\widehat{f}\|_{L_p} \leqslant c\left(\sum_{k\in\mathbb{Z}}\left(2^{\frac{k}{p'}}V_{2^k}(f)\right)^p\right)^{\frac{1}{p}} \tag{6}$$

holds. Here the constant c *does not depend on* f.

In [2] estimates of the form (6) in terms of the Fourier coefficients are obtained but these estimates are obtained with additional conditions such as GM monotonicity and non-negativity of the Fourier coefficients.

Our next main result is

THEOREM 2. *Let* $1 < p < \infty$. *Assume that the function* f *satisfy that there exists* $c > 0$ *such that*

$$V_{2^k}(f) \leqslant c \sup_{\substack{|e|\geqslant 2^k\\ e\in M}} \frac{1}{|e|}\left|\int_e f(x)dx\right|, \quad k \in \mathbb{Z}. \tag{7}$$

Then $\|\widehat{f}\|_{L_p(R)} < \infty$ *if and only if* $\|f\|_{N_{p'p}} < \infty$ *and, moreover,*

$$\|\widehat{f}\|_{L_p(\mathbb{R})} \asymp \|f\|_{N_{p'p}(M)}.$$

REMARK 1. Necessary and sufficient conditions on the Fourier transform $\widehat{f}$ for nonnegative functions from the class GM to belong to the space $L_p(\mathbb{R})$ $(1 < p < \infty)$ was proved in [7] (see also e.g. [8]). In this connection we also refer to [17]. The following proposition shows that Theorem 2 in a sense generalizes the mentioned result.

PROPOSITION 1. *(i) If* f *is a non-negative function and* $f \in GM$, *then* f *satisfies condition* (7).

(ii) The reversed implication does not hold in general, more precisely, there exists a function f *satisfying* (7) *but not* (4).

REMARK 2. In connection to statement (i) we also refer to M. Dyachenko, E. Liflyand and S. Tikhonov [1]. Note that in [1] criteria of belonging of the cosine and the sine Fourier transforms in the Lebesgue spaces with the power weights are obtained for non-negative functions $f \in GM$.

REMARK 3. The class of GM functions was important in the study both of Fourier series and Fourier transforms. In connection to the results obtained above we also refer to the recent papers [6] and [9] by E. Liflyand and S. Tikhonov. The first one surveys GM functions, while in the second one more general weighted estimates are obtained than in [7]. Moreover, these results gave rise to some multivariate extensions, see [4].

The proofs of the statements in Theorem 1, Theorem 2 and Proposition 1 are given in the next Section.

4. Proofs of the main results

Proof of Theorem 1. By using Lemma 3, we find that $V_1^{\infty}(f) < \infty$ and $V_{-\infty}^{-1}(f) < \infty$. Then, for all sequences such that $x_k \to \infty$ we have that each sequence $\{f(x_k)\}$ is a fundamental sequence (satisfying (8) below). Indeed, since $V_1^{\infty}(f) < \infty$, then there exists N such that for all $k > N$ we have $V_{x_k}^{\infty}(f) < \varepsilon$. Hence,

$$|f(x_k) - f(x_{k+p})| \leqslant \sum_{j=k}^{k+p-1} \left|f(x_j) - f(x_{j+1})\right| = V_{x_k}^{x_{k+p}}(f) \leqslant V_{x_k}^{\infty}(f) < \varepsilon. \tag{8}$$

Thus, we have that $\lim\limits_{k\to\infty} f(x_k)$ exists. Since the sequence $\{x_k\}$ is arbitrarily chosen we obtain that $\lim\limits_{x\to+\infty} f(x) = a$. Therefore, due to the fact that $f \in L_1(\mathbb{R})$, we conclude that $\lim\limits_{x\to+\infty} f(x) = 0$.

Let $x > 0$. Then

$$\int_x^{+\infty} df(y) = \lim_{b\to+\infty} (f(b) - f(x)) = -f(x).$$

By appling the duality representation of the norm of a function in the space $L_p(\mathbb{R})$, we obtain that

$$\|\widehat{f}\|_{L_p(\mathbb{R})} = \sup_{\|g\|_{L_{p'}(\mathbb{R})}=1} \int_{-\infty}^{\infty} \widehat{f(x)}\overline{g(x)}dx = \sup_{\|g\|_{L_{p'}(\mathbb{R})}=1} \int_{-\infty}^{\infty} f(x)\overline{\widehat{g(x)}}dx,$$

where $\overline{\widehat{g(x)}}$ is the conjugate function of $\widehat{g(x)}$.

This integral can be represented as follows

$$\|\widehat{f}\|_{L_p(\mathbb{R})} = \sup_{\|g\|_{L_{p'}(\mathbb{R})}=1} \left(\left| \int_{-\infty}^{0} f(x)\overline{\widehat{g(x)}}dx \right| + \left| \int_0^{\infty} f(x)\overline{\widehat{g(x)}}dx \right| \right).$$

By using the fact that $f(x) = -\int_x^{\infty} df(y)$ we can consider the following integral

$$I := \left| \int_0^{\infty} f(x)\overline{\widehat{g(x)}}dx \right| = \left| \int_0^{\infty} \left(\int_x^{\infty} df(y) \right) \overline{\widehat{g(x)}}dx \right|.$$

Hence, by interchanging the order of integration and using Lemma 1, we find that

$$\begin{aligned} I &= \left| \int_0^\infty \int_0^y \overline{\widehat{g(x)}}\, dx\, df(y) \right| \\ &= \left| \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_0^y \overline{\widehat{g(x)}}\, dx\, df(y) \right| \\ &\leqslant \sum_{k\in\mathbb{Z}} \sup_{2^k \leqslant y \leqslant 2^{k+1}} \left| \int_0^y \widehat{g(x)}\, dx \right| \cdot V_{2^k}^{2^{k+1}}(f). \end{aligned}$$

Furthemore, taking into account that

$$\left| \int_0^y \widehat{g(x)}\, dx \right| \leqslant 2^{k+1} \sup_{\substack{|e|\geqslant 2^k \\ e\in M}} \frac{1}{|e|} \left| \int_e \widehat{g}(x)\, dx \right| = 2^{k+1}\overline{(\widehat{g})}(2^k;M) \text{ for } 2^k \leqslant y \leqslant 2^{k+1},$$

where $\overline{(\widehat{g})}(2^k;M)$ is the average function of $\widehat{g}$ on the set M of the segments $[a,b]$, we obtain that

$$I \leqslant \sum_{k\in\mathbb{Z}} 2^{k+1}\overline{(\widehat{g})}(2^k,M) V_{2^k}^{2^{k+1}}(f) = c_1 \cdot \sum_{k\in\mathbb{Z}} 2^{\frac{k}{p}}\overline{(\widehat{g})}(2^k,M) V_{2^k}^{2^{k+1}}(f) 2^{\frac{k}{p'}}.$$

Next, by using Hölder's inequality, we get that

$$I \leqslant c_1 \left(\sum_{k\in\mathbb{Z}} \left(2^{\frac{k}{p}} \overline{(\widehat{g})}(2^k,M) \right)^{p'} \right)^{\frac{1}{p'}} \cdot \left(\sum_{k\in\mathbb{Z}} \left(2^{\frac{k}{p'}} V_{2^k}^{2^{k+1}}(f) \right)^{p} \right)^{\frac{1}{p}}.$$

Hence, by using Lemma 4 and Lemma 2, we obtain the following estimate:

$$I \leqslant c_2 \left(\sum_{k\in\mathbb{Z}} \left(2^{\frac{k}{p'}} V_{2^k}^{2^{k+1}}(f) \right)^{p} \right)^{\frac{1}{p}} \|g\|_{L_{p'}(\mathbb{R})}. \tag{9}$$

Similarly, we can estimate the integral $\int_{-\infty}^{0} f(x)\overline{\widehat{g(x)}}\, dx$:

$$\left| \int_{-\infty}^0 f(x)\overline{\widehat{g(x)}}\, dx \right| \leqslant c_3 \left(\sum_{k\in\mathbb{Z}} \left(2^{\frac{k}{p'}} V_{-2^{k+1}}^{-2^k}(f) \right)^{p} \right)^{\frac{1}{p}} \|g\|_{L_{p'}(\mathbb{R})}. \tag{10}$$

By combining (9) and (10), we find that

$$\|\widehat{f}\|_{L_p(\mathbb{R})} \leqslant c \left(\sum_{k\in\mathbb{Z}} \left(2^{\frac{k}{p'}} V_{2^k}(f) \right)^{p} \right)^{\frac{1}{p}}.$$

The proof is complete. □

Proof of Theorem 2. By using Lemma 2, we have that

$$\|f\|_{N_{p'p}(M)} \leqslant c_2 \|\widehat{f}\|_{L_p(\mathbb{R})}.$$

On the other hand, by using inequality (6) of Theorem 1, we find that

$$\|\widehat{f}\|_{L_p(\mathbb{R})} \leqslant c_3 \left(\sum_{k\in\mathbb{Z}} \left(2^{\frac{k}{p'}} V_{2^k}(f) \right)^p \right)^{\frac{1}{p}}.$$

Therefore, in view of the fact that

$$V_{2^k}(f) \leqslant c \sup_{\substack{|e|\geqslant 2^k \\ e\in M}} \frac{1}{|e|} \left| \int_e f(x)dx \right|, \quad k\in\mathbb{Z},$$

it yields that

$$\|\widehat{f}\|_{L_p(\mathbb{R})} \leqslant c_4 \left(\sum_{k\in\mathbb{Z}} \left(2^{\frac{k}{p'}} \sup_{\substack{|e|\geqslant 2^k \\ e\in M}} \frac{1}{|e|} \left| \int_e f(x)dx \right| \right)^p \right)^{\frac{1}{p}} = c_4 \|f\|_{N_{p'p}(M)}.$$

The proof is complete. □

Proof of Proposition 1. (i) Let f be a non-negative function and $f\in GM$. Then

$$\begin{aligned} V_{2^k}(f) = V_{-2^{k+1}}^{-2^k}(f) + V_{2^k}^{2^{k+1}}(f) &\leqslant c \left(\int_{-\frac{2^k}{\beta}}^{-\beta 2^k} t^{-1}|f(t)|dt + \int_{\frac{2^k}{\beta}}^{\beta 2^k} t^{-1}|f(t)|dt \right) \\ &\leqslant c \left(\int_{-\frac{2^k}{\beta^*}}^{-\beta^* 2^k} t^{-1}|f(t)|dt + \int_{\frac{2^k}{\beta^*}}^{\beta^* 2^k} t^{-1}|f(t)|dt \right), \end{aligned}$$

where $\beta^* = \max\{2,\beta\}$. Therefore,

$$\begin{aligned} V_{2^k}(f) &\leqslant c \left(\frac{\beta^*}{2^k} \int_{-\beta^* 2^k}^{-\frac{2^k}{\beta^*}} f(t)dt + \frac{\beta^*}{2^k} \int_{\frac{2^k}{\beta^*}}^{\beta^* 2^k} f(t)dt \right) \\ &\leqslant 2c(\beta^{*^2} - 1) \sup_{|e|\geqslant 2^k} \frac{1}{|e|} \int_e f(x)dx, \end{aligned}$$

i.e. f satisfies (7).

(ii) Let

$$f(x) = \begin{cases} 1, & -1 \leqslant x \leqslant 1, \\ \frac{\sin x}{x^2}, & 1 < x, \\ \frac{1}{x^2}, & x < -1. \end{cases}$$

We note that

$$V_{2^k}(f) \asymp \begin{cases} 2^{-k}, & k \in \mathbb{Z}^+ \\ 0, & -k \in \mathbb{N}, \end{cases}$$

$$\sup_{|e| \geqslant 2^k} \frac{1}{|e|} \int_e f(x)dx \asymp \begin{cases} 2^{-k}, & k \in \mathbb{Z}^+ \\ 1, & -k \in \mathbb{N}, \end{cases}$$

so f satisfies (7). On the other hand when $k \in \mathbb{N}$, we have

$$V_{2^k}^{2^{k+1}}(f) \asymp 2^{-k},$$

$$\int_{\frac{2^k}{\beta}}^{\beta 2^k} t^{-1}|f(t)|dt \asymp 2^{-2k},$$

which means that f does not satisfy (4). The proof is complete. $\square$

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