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Finite dimensional dynamics for nonlinear filtration equation

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Abstract

We construct new finite dimensional submanifolds in the solution space of nonlinear differential filtration equations and describe the corresponding evolutionary dynamics. This method is implemented in a computer program of symbolic computations Maple.

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1. Introduction

Nonlinear evolutionary partial differential equations

$$u_t = A(u)_{xx}, \tag{1}$$

or

$$u_t = A''(u)u_x^2 + A'(u)u_{xx},$$

describe many processes. Among them:

- one-dimensional motion of ground water with a free surface⁴ when $A(u) = \kappa u^2$, $\kappa \in \mathbb{R}$;
- polytropic gas filtration when $A(u) = \kappa u^n$, $\kappa \in \mathbb{R}^3$;
- distribution of heat radiation in nuclear explosions in their initial phase¹²;
- filtration in porous media³.

We call equation (1) by *filtration equation* and suppose that $A(u) \neq \text{const}$.

In this paper we construct finite dimensional submanifolds (“finite dynamics”) in the infinite dimensional solution space of equation (1). Constructed dynamics allow one to construct new numeric methods and exact solutions.

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Finite dimensional dynamics were constructed for Kolmogorov–Petrovsky–Piskunov equation⁷ and for generalized Rapoport – Leas equation^{2,11}.

The basic principles of the theory of finite dimensional dynamics⁵ are as follows.

Evolutionary differential equations

$$u_t = \phi(u, u_x, u_{xx}) \tag{2}$$

defines “dynamics”, i.e. flows on the infinite-dimensional space of functions of one variable x .

Their finite dimensional “sub-dynamics” can be viewed as a dynamics on the solution space of some ordinary differential equations. Evolution equation (2) determines symmetries for such ordinary differential equations.

Thus, the problem of construction of finite dimensional dynamics comes down to finding the ordinary differential equations

$$F(y, y', \dots, y^{(k)}) = 0 \tag{3}$$

for which the function $\phi(y_0, y_1, y_2)$ is a generating function of symmetries^{6,8}. Here $y(x) = u(t, x)$ with “frozen” coordinate t .

2. Symmetries of Ordinary Differential Equations

Let J^k be the space of k -jets of functions of one independent variable x and let x, y_0, y_1, \dots, y_k be canonical coordinates⁹ on J^k .

Equation (3) corresponds to the hypersurface

$$\mathcal{E} = \{F(y_0, y_1, \dots, y_k) = 0\}$$

in the space J^k .

Naively, by finite dynamics we mean a “finite dimensional submanifold in a function space” which is invariant with respect to the evolutionary vector field

$$S_\phi = \phi \frac{\partial}{\partial y_0} + D(\phi) \frac{\partial}{\partial y_1} + D^2(\phi) \frac{\partial}{\partial y_2} + \dots$$

where

$$D = \frac{d}{dx} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + y_2 \frac{\partial}{\partial y_1} + \dots$$

is the total derivation and $D^2 = D \circ D, \dots$

Recall a geometrical meaning of the generating functions of symmetries⁸.

Let S_ϕ be a shuffling symmetry and a solution of equation (3) respectively. Then the corresponding curve

$$L_y = (y_0 = y(x), y_1 = y'(x), \dots, y_k = y^{(k)}(x)) \subset \mathcal{E}$$

is a prolongation of the function $y = y(x)$ to the space J^k .

Let Φ_t be a one-parametric group of shifts along the trajectories of S_ϕ .

Then locally and for small t we have $\Phi_t(L_y) = L_{y_t}$

$$y_t = y + t \phi|_{L_y} + o(t),$$

and

$$y_t^{(i)} = y + D^i(\phi)|_{L_y} t + o(t)$$

for $i = 1, \dots, k$.

In other words, an action of symmetry S_ϕ on a solution y corresponds to the transformation of the form

$$y_t = y + t \phi|_{L_y} + o(t)$$

on functions.

This means that the function $h(t, x) = y_t(x)$ is a solutions of the evolutionary equation (2).

3. Finite Dimensional Dynamics

The ordinary differential equation (3) is called a *finite dimensional dynamic* or *dynamic* for evolutionary equation (2) if $\phi(y_0, y_1, y_2)$ is a generating function for shuffling symmetries of (3). The number is called the *order* of the dynamic.

Theorem 1. A function $\phi(y_0, y_1, y_2)$ is a generating function of symmetries of equation (3) if and only if

$$[\phi, F] = 0 \text{ mod } \langle DF \rangle, \quad (4)$$

where $\langle DF \rangle$ is a differential ideal which is generated by the function $F(y_0, y_1, \dots, y_k)$ and

$$[\phi, F] = S_\phi(F) - S_F(\phi)$$

is the Poisson–Lie bracket^{2,8}.

Note that the Poisson–Lie bracket is skew-symmetric, \mathbb{R} -bilinear, satisfies the Jacobi identity and can be calculated by the following formula⁸:

$$[\phi, F] = \sum_{i=0}^k \left(\frac{\partial \phi}{\partial y_i} D^i(F) - \frac{\partial F}{\partial y_i} D^i(\phi) \right).$$

Therefore the equation $F(y_0, y_1, \dots, y_k) = 0$ is a finite dynamics for evolutionary equation (2) if F satisfies (4). Solving equation (4) one can find F .

Theorem 2.² Equation (3) is a dynamic of evolutionary equation (2) if and only if

$$[\phi, F] = aF + bD(F),$$

where a and b are functions from the space J^{k+1} .

Conditions when dynamics F is an attractor¹⁰ of evolutionary equation (2) can be formulated in terms of the functions a and b ^{1,2}.

Assume that equation (3) is resolved with respect to the higher derivative:

$$y^{(k)} = f(y, y', \dots, y^{(k-1)}), \quad (5)$$

i.e. the hypersurface

$$\mathcal{E} = \{y_k = f(y_0, \dots, y_{k-1})\}. \quad (6)$$

Then the solution space of this equation could be identified with \mathbf{R}^k by taking the initial data at a point x_0 . In this case the dynamics is given by the vector field

$$E_\phi = \bar{\phi} \frac{\partial}{\partial y_0} + D(\bar{\phi}) \frac{\partial}{\partial y_1} + \dots + D^{k-1}(\bar{\phi}) \frac{\partial}{\partial y_{k-1}}.$$

The bar over the function ϕ denotes its restriction to hypersurface (6).

4. Dynamics of Filtration Equation

Find first order dynamics for equation (1). We find them in the form

$$F(y_0, y_1) = y_1 - f(y_0).$$

In this case the Poisson–Lie bracket has the following form

$$[\phi, F] = -(A'''(y_0)f(y_0) + A''(y_0)f'(y_0) + A'(y_0)f''(y_0))y_1^2 - A''(y_0)f(y_0)y_2.$$

Restrict this bracket to the equation $\mathcal{E} = \{y_1 - f(y_0) = 0\}$. Since

$$y_1 = f(y_0),$$

$$y_2 = y_1 f'(y_0) = f(y_0) f'(y_0),$$

equation (4) takes the form

$$f(y_0)^2 (A'''(y_0)f(y_0) + 2A''(y_0)f'(y_0) + A'(y_0)f''(y_0)) = 0.$$

The last equation can be viewed as an ordinary differential equation with respect to the function f . Its common solution is

$$f(y_0) = \frac{\alpha y_0 + \beta}{A'(y_0)},$$

where α and β are arbitrary constants.

Equation (5) has the following form:

$$y' = \frac{\alpha y + \beta}{A'(y)}. \tag{7}$$

Suppose that $A(y) = y^2$. Then equation (7) has two singular points: $y = 0$ and $y = -1$. The corresponding vector field shown in Fig. 1. A common solution of this equation can be written in terms of Lambert’s function. When $\alpha = \beta = 1$ we get:

$$y(x) = -\text{LambertW}\left(\gamma \exp\left(-1 - \frac{x}{2}\right)\right) - 1,$$

where γ is an arbitrary constant.

Theorem 3. *First order dynamics for equation (1) has the form (7), where α and β are arbitrary constants. The dynamics on the initial data is given by vector field*

$$E_\phi = \frac{\alpha(\alpha y_0 + \beta)}{A'(y_0)} \frac{\partial}{\partial y_0}.$$

Second order dynamics for equation (1) we find in the form

$$F(y_0, y_1, y_2) = y_2 - a(y_0)y_1 - b(y_0).$$

In this case the Poisson–Lie bracket has the following form

$$[\phi, F] = -A^{(4)}(y_0)y_1^4 - 5\left(y_2 + \frac{1}{5}b(y_0)\right)A^{(3)}(y_0)y_1^2 -$$

$$- \left(a'(y_0)y_1^3 + b'(y_0)y_1^2 + 2y_1y_3 + 2y_2^2 + y_2b(y_0)\right)A^{(2)}(y_0) -$$

$$- y_1A'(y_0)\left(a''(y_0)y_1^2 + b''(y_0)y_1 + 2a'(y_0)y_2\right).$$

Since

$$y_2 = a(y_0)y_1 - b(y_0),$$

$$y_3 = a'(y_0)y_1^2 + a(y_0)(a(y_0)y_1 - b(y_0)) + b'(y_0)y_1,$$

equation (4) takes the form

$$-A^{(4)}(y_0)y_1^4 - \left(5a(y_0)A^{(3)}(y_0) + 3a'A^{(2)}(y_0) + a^{(2)}(y_0)A'(y_0)\right)y_1^3 +$$

$$+ \left(4b(y_0)A^{(3)}(y_0) - (4a^2(y_0) + 3b'(y_0))A^{(2)}(y_0) -$$

$$- (2a(y_0)a'(y_0) + b''(y_0))A'(y_0)\right)y_1^2 +$$

$$+ \left(5a(y_0)b(y_0)A^{(2)}(y_0) + 2a'(y_0)b(y_0)A'(y_0)\right)y_1 -$$

$$- b^2(y_0)A^{(2)}(y_0) = 0.$$

The last equation is polynomial with respect to y_1 . Therefore, it is equivalent to a system of five ordinary differential equations with respect to the functions a, b and A . Solving this system, we find all second order dynamics.

Theorem 4. *Second order dynamics for equation (1) have the following forms:*

- *if*

$$A(y_0) = \alpha y_0^3 + \beta y_0^2 + \gamma y_0 + \delta,$$

then

$$a(y_0) = b(y_0) = 0;$$

- *if*

$$A(y_0) = \alpha y_0^2 + \beta y_0 + \gamma,$$

then

$$a(y_0) = \frac{\delta}{(2\alpha y_0 + \beta)^2}, \quad b(y_0) = 0;$$

- *if*

$$A(y_0) = \alpha y_0 + \beta,$$

then

$$a(y_0) = \gamma, \quad b(y_0) = \delta y_0 + \zeta,$$

where $\alpha, \beta, \gamma, \delta$ and ζ are arbitrary constants.

Third order dynamics for equation (1) we find in the form

$$F(y_0, y_1, y_2) = y_3 - a(y_0)y_2 - b(y_0)y_1 - c(y_0).$$

Theorem 5. *Suppose that $A''(y_0) \neq 0$.*

Then there exists a third order dynamics when

$$A(y_0) = \alpha y_0^2 + \beta y_0 + \gamma,$$

where $\alpha \neq 0, \beta, \gamma$ are arbitrary constants. *This dynamic has the form*

$$F = y_3.$$

5. Finte Dynamics in Maple

Note that calculations in jet spaces are very cumbersome. Therefore, the calculations we have carried out in the system of symbolic computations Maple-17.

We have used packages *DifferentialGeometry* and *JetCalculus* which were created by I. Anderson.

A fragment of Maple programm for calculation of first order dynamics is bellow:

```
> with(DifferentialGeometry): with(JetCalculus):
with(Tools): with(PDETools):

> Preferences("JetNotation", "JetNotation2");
> DGsetup([x], [y], DYN1, 5, verbose):
```

```

> phi := convert(convert(diff(A(y(x)),x$2), DGjet), diff);

# Procedure of calculation of Poisson-Lie brackets:

> com := proc (A, B)
(diff(A, y[0]))*B+(diff(A, y[1]))*TotalDiff(B, [1])+
(diff(A, y[2]))*TotalDiff(B, [2])+(diff(A, y[3]))*TotalDiff(B, [3])+
(diff(A, y[4]))*TotalDiff(B, [4])+(diff(A, y[5]))*TotalDiff(B, [5])+
(diff(A, y[6]))*TotalDiff(B, [6])+(diff(A, y[7]))*TotalDiff(B, [7])-
(diff(B, y[0]))*A-(diff(B, y[1]))*TotalDiff(A, [1])-
(diff(B, y[2]))*TotalDiff(A, [2])-(diff(B, y[3]))*TotalDiff(A, [3])-
(diff(B, y[4]))*TotalDiff(A, [4])-(diff(B, y[5]))*TotalDiff(A, [5])-
(diff(B, y[6]))*TotalDiff(A, [6])-(diff(B, y[7]))*TotalDiff(A, [7])
end proc:

# First order dynamics:
> F:=y[1]-f(y[0]):

# Calculation of Poisson-Lie bracket:
> ur := simplify(com(phi,F),size);

# Restriction y[1] and y[2] to the equation F=0:
sub1:={y[1]=f(y[0])};
sub2:={y[2]=eval(solve(TotalDiff(F, [1]), y[2]), sub1)};

# The main equation:
pol:=simplify(eval(eval(ur, sub1), sub2), size);

# Solution of the main equation:
dsolve(pol, f(y[0]));

```

Here $\text{TotalDiff}(B, [k])$ is the k -th the total derivation of a function B .

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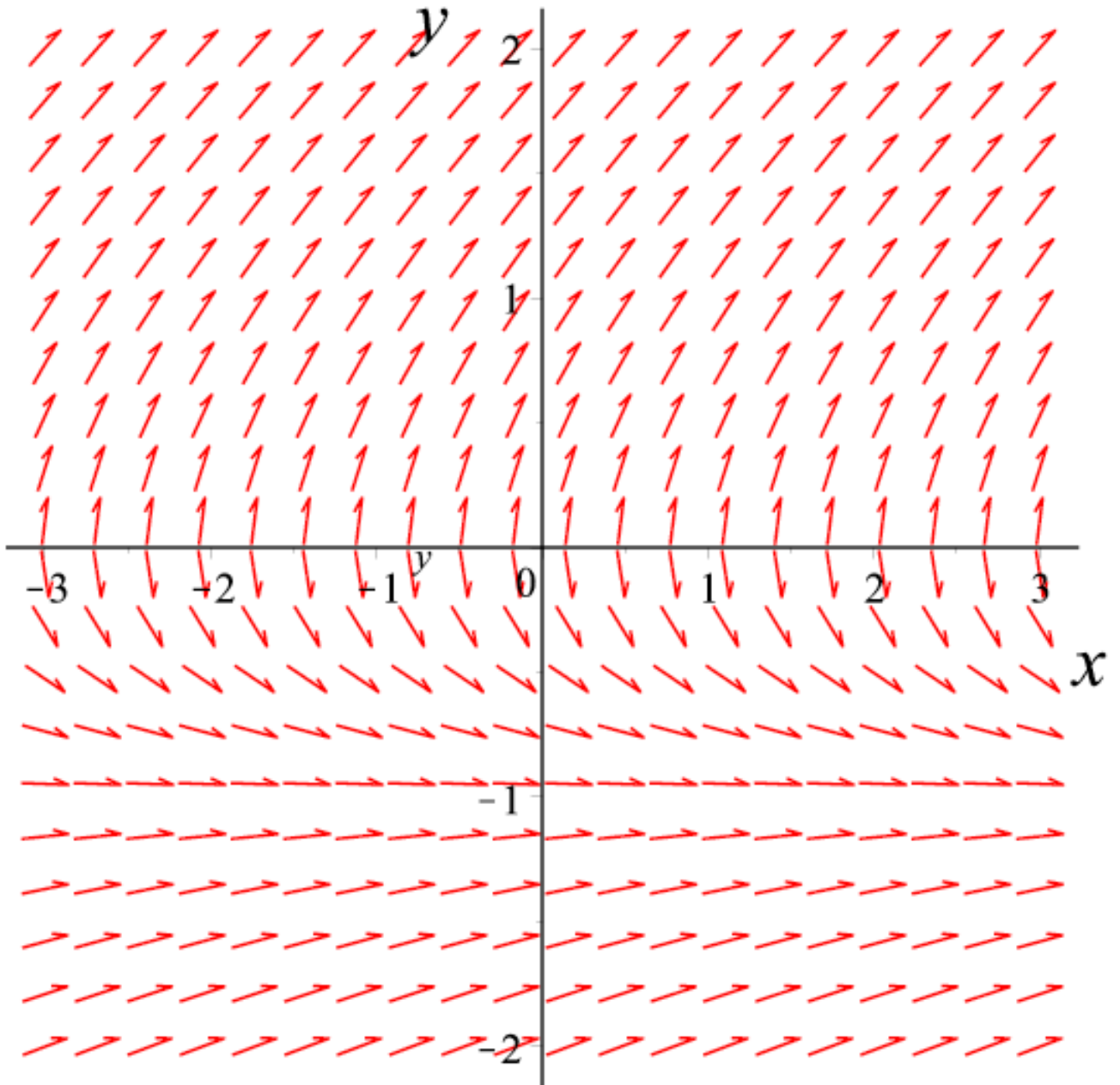


Fig. 1. First order dynamics.