THE GAP PHENOMENON IN PARABOLIC GEOMETRIES

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ABSTRACT. The infinitesimal symmetry algebra of any Cartan geometry has maximum dimension realized by the flat model, but often this dimension drops significantly when considering non-flat geometries, so a gap phenomenon arises. For general (regular, normal) parabolic geometries of type (G, P), we use Tanaka theory to derive a universal upper bound on the submaximal symmetry dimension. We use Kostant's version of the Bott–Borel–Weil theorem to show that this bound is in fact sharp in almost all complex and split-real cases by exhibiting (abstract) models. We explicitly compute all submaximal symmetry dimensions when G is any complex or split-real simple Lie group.

CONTENTS

1. Introduction

A Riemannian metric g on a (connected) n-manifold M admits at most an $\binom{n+1}{2}$-dimensional symmetry (isometry) algebra, and equality occurs precisely for the constant curvature spaces. Below this maximum, a gap phenomenon occurs: there are forbidden symmetry dimensions. Submaximal symmetry dimensions are given in Table 1. More generally, for geometric structures which admit a finite maximal symmetry dimension, the determination of the submaximal symmetry dimension $\mathfrak{S}$ is called the *symmetry gap problem*. For various geometric structures, this problem has been studied since the late 19th century by many authors, including Lie, Tresse, Fubini, Cartan, Yano, Wakakuwa, Vranceanu, Egorov, Obata, and Kobayashi – see [46, 24, 22, 28] and references therein.

n	Max	Submax	References
2	3	1	Lie (1882) [32], Darboux (1894) [16]
3	6	4	Bianchi (1898) [2], Ricci (1898) [39]
4	10	8	Egorov (1955) [19]
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2}+1$	Wang (1947) [48]

Table 1. Maximal / submaximal symmetry dimensions for Riemannian geometry

A Riemannian structure (M, g) is equivalently described as a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type $(\mathrm{E}(n), \mathrm{O}(n))$, where $\mathrm{E}(n)$ is the Euclidean group and $\mathrm{O}(n)$ is the orthogonal group. Indeed, $\mathcal{G}$ is the orthonormal frame bundle and the Cartan connection ω is a sum of the Levi–Civita connection of g and the soldering form. Riemannian geometries are curved versions of the flat model $\mathrm{E}(n)/\mathrm{O}(n) \cong \mathbb{R}^n$, which admits the maximum $\dim(\mathrm{E}(n)) = \binom{n+1}{2}$ symmetries.

Many well-known geometric structures such as conformal, projective, CR, 2nd order ODE systems, and various bracket-generating distributions, are equivalently described as *parabolic geometries*, i.e. Cartan geometries $(\mathcal{G} \to M, \omega)$ of type (G, P), where G is a real or complex semisimple Lie group and $P \subset G$ is a parabolic subgroup. There is an equivalence of categories between *regular, normal* parabolic geometries and underlying structures [45, 35, 10]. For all of these, the infinitesimal symmetry algebra $\inf(\mathcal{G}, \omega)$ has maximum dimension $\dim(G)$ and this is (locally) *uniquely* realized by the flat model $(G \to G/P, \omega_G)$, where ω_G is the Maurer–Cartan form on G. Thus, the gap problem[1] can be phrased as: *Among all* **regular, normal** *parabolic geometries* $(\mathcal{G} \to M, \omega)$ *of type* (G, P) *that are* **not flat**, *what is the maximum* $\mathfrak{S}$ *of* $\dim(\inf(\mathcal{G}, \omega))$?

The advantage of the Cartan perspective is that despite the zoo of different geometric structures downstairs on the base manifold M, their description upstairs on $\mathcal{G}$ becomes much more uniform, based on common structural features inherent in the groups (G, P), which have been well studied in representation theory. For instance, all parabolic geometries admit a fundamental curvature quantity called *harmonic curvature* κ_H, valued in a certain Lie algebra cohomology group $H^2_+(\mathfrak{g}_-, \mathfrak{g})$, and which is a complete obstruction to flatness. (In conformal geometry, κ_H is the Weyl tensor.)

Previously known symmetry gap results for parabolic geometries are given in Table 2. These were obtained using a variety of techniques, e.g. computation of the algebra of all differential invariants of a pseudogroup (scalar 2nd order ODE), Cartan's method of equivalence ((2, 3, 5)-distributions), classification of Lie algebras of contact vector fields in the plane (3rd order ODE[2]), or studying

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[1]We study *infinitesimal* symmetries here. The problem of studying *global* symmetry dimensions which are "less than maximal" is quite different. For example, take the flat model $(G \to G/P, \omega_G)$ and remove a point on the base and the fibre over it. This yields a flat geometry with global automorphism group isomorphic to P.

[2]Sophus Lie classified all finite-dimensional irreducible Lie algebras of *contact* vector fields in plane. There are only three, L_6, L_7, L_{10}, with $\dim(L_i) = i$ with $L_6 \subset L_7$ and $L_6 \subset L_{10}$. The only 3rd order ODE invariant under L_6 (and hence L_7 and L_{10}) is the flat model $y''' = 0$. In [47], a *point* symmetry classification was carried out. Aside from

integrability conditions for the equations characterizing symmetries (projective structures). Some of these results relied heavily on the low-dimensional setup.

All results in Table 2 are correct, but strictly speaking, only Egorov's result has no further assumptions and is equivalent to the result we obtain here. For example for $(2, 3, 5)$-geometry, modelled on G_2/P_1, Cartan [12] identified a binary quartic invariant $\mathcal{A}$, i.e. κ_H, and studied the maximal symmetry *among structures with constant root type*. The largest is 7 and occurs where $\mathcal{A}$ has a single root of multiplicity 4. However, non-flatness of the geometry for us means only that $\kappa_H(u) \neq 0$ at *some* $u \in \mathcal{G}$, so the result $\mathfrak{S} = 7$ that we will obtain is a sharpening of Cartan's result.

More recently, for arbitrary parabolic geometries, Čap & Neusser [9] gave a general algebraic strategy for finding upper bounds on $\mathfrak{S}$ using Kostant's version of the Bott–Borel–Weil theorem [29]. However, the implementation of their strategy must be carried out on a case-by-case basis; moreover, their upper bounds are in general not sharp. (See also Remark 2.3.1.) For structures determined by a bracket-generating distribution (not necessarily underlying parabolic geometries), another approach based on an elaboration of Tanaka theory [44, 45] was proposed in the works [30, 31] of the first author.

Geometry	Model	Max	Submax	Reference
Scalar 2nd order ODE mod point transformations	$\mathrm{SL}_3(\mathbb{R})/P_{1,2}$	8	3	Tresse (1896) [46]
2-dim. projective structures	$\mathrm{SL}_3(\mathbb{R})/P_1$	8	3	Tresse (1896) [46]
$(2, 3, 5)$-distributions	G_2/P_1	14	7	Cartan (1910) [12]
3-dim. CR structures	$\mathrm{SU}(2, 1)/B$	8	3	Cartan (1932) [13]
Projective structures (dim. $\ell \geq 3$)	$\mathrm{SL}_{\ell+1}(\mathbb{R})/P_1$	$\ell^2 + 2\ell$	$(\ell - 1)^2 + 4$	Egorov (1951) [18]
Scalar 3rd order ODE mod contact transformations	$\mathrm{Sp}_4(\mathbb{R})/P_{1,2}$	10	5	Wafo Soh, Mahomed, Qu (2002) [47]
Pairs of 2nd order ODE	$\mathrm{SL}_4(\mathbb{R})/P_{1,2}$	15	9	Casey, Dunajski, Tod (2013) [15]

TABLE 2. Previously known submaximal symmetry dimensions for parabolic geometries

The main idea behind our approach is to combine Tanaka theory with the Čap–Neusser approach based on Kostant theory. This yields a *uniform algebraic approach* to the gap problem which is rooted in the structure theory of semisimple Lie algebras. The main results of this article are:

- We establish a universal upper bound $\mathfrak{S} \leq \mathfrak{U}$ (Theorem 2.4.6), where $\mathfrak{U}$ is algebraically determined. In arbitrary real cases, we have $\mathfrak{S} \leq \mathfrak{U} \leq \mathfrak{U}^{\mathbb{C}}$ (Corollary 2.4.8).
- In complex or *split-real*[3] cases, we:
  - exhibit models with $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) = \mathfrak{U}$ in almost all cases (Theorem 4.1.5). Thus, $\mathfrak{S} = \mathfrak{U}$ almost always (Theorem 4.1.8). Exceptions are also studied (§4.3).
  - give a Dynkin diagram recipe to efficiently compute $\mathfrak{U}$ (§3.3).
  - establish local homogeneity of all submaximally symmetric models[4] near non-flat *regular* points (Theorem 4.3.3); the set of all such points is open and dense in M (Lemma 2.4.5).
- We recover all results in Table 2; some sample new results are given in Table 3. Our complete classification when G is (complex or split-real) *simple* is presented in Appendix C.

In §2, we review some background and recent results, and define in §2.4 a graded subalgebra $\mathfrak{a}^{\phi} := \mathrm{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{ann}(\phi))$ of $\mathfrak{g}$ called the *Tanaka prolongation algebra*. The crucial new ingredient

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$y''' = 0$ and $y''' = \frac{3(y'')^2}{2y'}$ (which are *contact*-equivalent and have 7 and 6 point symmetries respectively), the maximal point symmetry dimension is at most 5. Indeed, for any $a \in \mathbb{R}$, $y''' = ay' + y$ has five point (contact) symmetries.

[3]We refer to $\mathfrak{g}$ as split-real if it is a split real form of its complexification, e.g. $\mathfrak{sl}(n, \mathbb{R})$, but not $\mathfrak{su}(n)$.

[4]This is not universally true outside the parabolic context, e.g. Killing fields for metrics on surfaces (see Table 1).

Geometry	Range	Model	Max	Submax
Signature (p,q) conformal geometry in dim. $n = p+q$	$p, q \geq 2$	$\mathrm{SO}_{p+1,q+1}/P_1$	$\binom{n+2}{2}$	$\binom{n-1}{2} + 6$
Systems of 2nd order ODE in m dependent variables	$m \geq 2$	$\mathrm{SL}_{m+2}(\mathbb{R})/P_{1,2}$	$(m+2)^2 - 1$	$m^2 + 5$
$(2,m)$-Segré structures	$m \geq 2$	$\mathrm{SL}_{m+2}(\mathbb{R})/P_2$	$(m+2)^2 - 1$	$m^2 + 5$
Generic rank ℓ distributions on $\frac{1}{2}\ell(\ell+1)$-dim. manifolds	$\ell \geq 3$	$\mathrm{SO}_{\ell,\ell+1}/P_\ell$	$\binom{2\ell+1}{2}$	$\begin{cases} \frac{\ell(3\ell-7)}{2} + 10, & \ell \geq 4; \\ 11, & \ell = 3 \end{cases}$
Lagrangean contact structures	$\ell \geq 3$	$\mathrm{SL}_{\ell+1}(\mathbb{R})/P_{1,\ell}$	$\ell^2 + 2\ell$	$(\ell-1)^2 + 4$
Contact projective structures	$\ell \geq 2$	$\mathrm{Sp}_{2\ell}(\mathbb{R})/P_1$	$\ell(2\ell+1)$	$\begin{cases} 2\ell^2 - 5\ell + 8, & \ell \geq 3; \\ 5, & \ell = 2 \end{cases}$
Contact path geometries	$\ell \geq 3$	$\mathrm{Sp}_{2\ell}(\mathbb{R})/P_{1,2}$	$\ell(2\ell+1)$	$2\ell^2 - 5\ell + 9$
Exotic parabolic contact structure of type E_8	-	E_8/P_8	248	147

TABLE 3. Sample new results of submaximal symmetry dimensions for parabolic geometries

which creates a bridge to Tanaka theory is the notion of a *regular point* (Definition 2.4.1). Such points form an open dense subset, and $\dim(\inf(\mathcal{G}, \omega)) \leq \dim(\mathfrak{a}^{\kappa_H(u)})$ at any regular point. For non-flat geometries, we can always find a regular point which is *non-flat*, i.e. $\kappa_H(u) \neq 0$. Defining $\mathfrak{U} := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\}$ quickly leads to the result $\mathfrak{S} \leq \mathfrak{U}$.

§3 is devoted to studying the Tanaka algebra $\mathfrak{a}^\phi$. Over $\mathbb{C}$, if $\mathbb{V}$ is a $\mathfrak{g}_0$-irreducible representation (irrep) and ϕ_0 is any extremal (i.e. highest or lowest) weight vector, then $\dim(\mathfrak{ann}(\phi)) \leq \dim(\mathfrak{ann}(\phi_0))$, $\forall 0 \neq \phi \in \mathbb{V}$. In Proposition 3.1.1, we show that this property persists for the Tanaka algebra, i.e. $\dim(\mathfrak{a}^\phi) \leq \dim(\mathfrak{a}^{\phi_0})$, $\forall 0 \neq \phi \in \mathbb{V}$. This leads in §3.3 to a purely combinatorial (Dynkin diagram) recipe to compute $\dim(\mathfrak{a}^{\phi_0})$ and hence $\mathfrak{U}$. (The E_8/P_8 case becomes a simple exercise – see Example 3.3.4.) In §3.4, we study *prolongation-rigidity*: $(\mathfrak{g}, \mathfrak{p})$ is PR iff $\mathfrak{a}^\phi_+ = 0$, $\forall 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})$; otherwise, it is NPR. An example of an NPR parabolic geometry is the geometry of 2nd order ODE systems. When $\mathfrak{g}$ is (complex) simple, a complete classification of NPR geometries is given, and the grading height $\tilde{\nu}$ of $\mathfrak{a}^\phi$ (for any $\phi \neq 0$) is highly constrained: $0 \leq \tilde{\nu} \leq 2$ always, and $\tilde{\nu} = 2$ occurs only for a couple of cases. In §3.5, we study the effect of correspondence and twistor spaces on the Tanaka prolongation. These provide an important tool for simplifying our calculations.

We show in §4 that $\mathfrak{S} = \mathfrak{U}$ in almost all complex and split-real cases: a model can generally be constructed by deforming the Lie algebra structure on $\mathfrak{a}^{\phi_0}$ by ϕ_0 (Theorem 4.1.5). To study exceptions, *filtration-rigidity* is introduced in §4.2. In §4.3, we find $\mathfrak{U} - 1 \leq \mathfrak{S} \leq \mathfrak{U}$ for all exceptions.

Concrete examples are considered in §5, and submaximally symmetric models are given explicitly (in coordinates) in terms of their underlying structure on the base manifold M. This includes conformal geometry, $(2,3,5)$-distributions, $(3,6)$-distributions, 2nd order ODE systems, projective structures, and $(2,m)$-Segré structures. Four-dimensional Lorentzian conformal geometry and $(2,3,5)$-distributions are investigated in finer detail: the maximum symmetry in each Petrov type and root type is identified. In doing so, we also exhibit in §5.1.2 the first known example of a Petrov type II metric with four (conformal) Killing vectors.

The gap problem is more subtle for general real forms since $\max\{\dim(\mathfrak{ann}(\phi)) \mid 0 \neq \phi \in \mathbb{V}\}$ can be difficult to determine in the absence of vectors which complexify to extremal weight vectors. Our upper bound $\mathfrak{U}$ is still valid, but its realizability must in general be checked. For instance, for conformal geometry, we exhibit local models in all signatures except Riemannian and Lorentzian which realize the complex upper bound $\mathfrak{U}^\mathbb{C}$, so $\mathfrak{S} = \mathfrak{U} = \mathfrak{U}^\mathbb{C}$ for these cases. The conformal Riemannian and Lorentzian cases are different; they have been recently settled in [17].

Conventions: We assume throughout that M is a connected manifold. When working with real and complex parabolic geometries, our results are formulated in the smooth and holomorphic categories, respectively. We use *left* cosets and *right* principal bundles. For a Lie group G, we identify its Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ with the *left*-invariant vector fields on G. Parabolic subalgebras will be denoted $\mathfrak{p}, \mathfrak{q}$, and corresponding parabolic subgroups are P, Q. We always assume that G acts on G/P *infinitesimally effectively*, i.e. the kernel of the G-action on G/P is at most discrete. Equivalently, simple ideals of $\mathfrak{g}$ are not contained in $\mathfrak{g}_0$. (This avoids situations like $G = G' \times G''$ and $P = P' \times G''$, where the G''-action is not visible on G/P.)

We write $A_\ell, B_\ell, C_\ell, D_\ell, G_2, F_4, E_6, E_7, E_8$ for the complex simple Lie algebras[5], **or** any complex Lie groups having these as their Lie algebras. (In §5, we abuse this notation further by letting it refer to real forms, and specify the precise real form as necessary.) We use the Bourbaki [3] / `LiE` [33] ordering of simple roots. This differs from the ordering used in [11] for E_6, E_7, E_8, F_4; also, their definition of the Cartan matrix is the transpose of ours.

Dynkin diagrams are drawn with open white circles. This is the same notation as the Satake diagram for the corresponding split real form, and serves to emphasize that all our results are the same in both settings. We use the "minus lowest weight" Dynkin diagram convention (see §3.2) when referring to irreducible $\mathfrak{g}_0$-modules. If $\mathfrak{g}$ is simple, we use $\lambda_\mathfrak{g}$ to denote its highest weight (root).

2. A UNIVERSAL UPPER BOUND

We give some brief background (§2.1–2.2), review recent results (§2.3), and then establish a universal (algebraic) upper bound on the submaximal symmetry dimension (§2.4).

2.1. **Tanaka theory in a nutshell.** The aim of Tanaka theory [44, 45, 52] is to study the local equivalence problem for filtered geometric structures. The given geometric data is a manifold M with a (vector) distribution $D \subseteq TM$ endowed possibly with some additional structure on it, e.g. a sub-Riemannian metric or a conformal structure. For many such structures, one can canonically associate a Cartan geometry $(\mathcal{G} \to M, \omega)$ of some type (G, K). We give an outline of these ideas.

Iterating Lie brackets of sections of D, we form the weak derived flag

$$D =: D^{-1} \subset D^{-2} \subset D^{-3} \subset \dots \ , \quad \text{where} \quad \Gamma(D^{i-1}) := [\Gamma(D^i), \Gamma(D^{-1})].$$

We assume that all D^i have constant rank, and D is bracket-generating in TM, i.e. $D^{-\nu} = TM$ for some minimal $\nu \geq 1$ (called the *depth*). The Lie bracket restricts to a map $\Gamma(D^i) \times \Gamma(D^j) \to \Gamma(D^{i+j})$, so $(M, \{D^i\})$ is a *filtered manifold*, and induces a tensorial bracket on the associated-graded,

$$\mathfrak{m}(x) = \mathfrak{g}_-(x) = \bigoplus_{i<0} \mathfrak{g}_i(x), \qquad \mathfrak{g}_i(x) = D^i(x)/D^{i+1}(x).$$

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[5]The classical Lie algebras are $A_\ell \cong \mathfrak{sl}(\ell+1, \mathbb{C})$, $B_\ell \cong \mathfrak{so}(2\ell+1, \mathbb{C})$, $C_\ell \cong \mathfrak{sp}(2\ell, \mathbb{C})$, $D_\ell \cong \mathfrak{so}(2\ell, \mathbb{C})$.

This *Levi bracket* turns $\mathfrak{m}(x)$ into a graded nilpotent Lie algebra called the *symbol algebra at x*. We further assume that D is of *constant type*, i.e. there is a fixed $\mathfrak{m} = \mathfrak{g}_-$ such that $\mathfrak{m}(x) \cong \mathfrak{m}$, $\forall x \in M$.

Consider the frame bundle $\mathcal{F}_{gr}(M) \xrightarrow{\pi} M$ with $\pi^{-1}(x)$ consisting of all graded isomorphisms $u : \mathfrak{m} \to \mathfrak{m}(x)$. The structure group $\mathrm{Aut}_{gr}(\mathfrak{m})$ of this principal bundle is the group of graded automorphisms of $\mathfrak{m}$. If D is endowed with additional structure, we can reduce to a principal subbundle $\mathcal{G}_0 \to M$ with structure group $G_0 \subseteq \mathrm{Aut}_{gr}(\mathfrak{m})$. (In conformal geometry, $D = TM$ and $G_0 = \mathrm{CO}(\mathfrak{g}_{-1})$; for $(2,3,5)$-distributions (§5.2.1), $G_0 \cong \mathrm{Aut}_{gr}(\mathfrak{m}) \cong \mathrm{GL}(\mathfrak{g}_{-1}) \cong \mathrm{GL}_2$.) From here, one constructs the *geometric prolongation* of the given structure, namely a tower of adapted bundles $... \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0 \to M$. We refer to [52] for details on this procedure. This geometric prolongation is controlled by an *algebraic prolongation* which we describe below.

The Lie algebra $\mathfrak{g}_0$ is contained in the algebra $\mathfrak{der}_{gr}(\mathfrak{m})$ of grade-preserving derivations of $\mathfrak{m}$. Given $(\mathfrak{m}, \mathfrak{g}_0)$, the axioms for Tanaka's algebraic prolongation $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0) := \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i(\mathfrak{m}, \mathfrak{g}_0)$ are:

(T.1) $\mathfrak{g}_{\leq 0}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}_0$;

(T.2)   If $X \in \mathfrak{g}_i(\mathfrak{m}, \mathfrak{g}_0)$ for $i \geq 0$ satisfies $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$;

(T.3) $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ is the maximal graded Lie algebra satisfying (T.1) and (T.2).

Write $\mathrm{pr}(\mathfrak{m}) := \mathrm{pr}(\mathfrak{m}, \mathfrak{der}_{gr}(\mathfrak{m}))$. Up to isomorphism, there is a unique graded Lie algebra satisfying (T.1)–(T.3). In fact, Tanaka gives an explicit inductive realization of $\mathfrak{g} := \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$; for $i > 0$,

$$(2.1) \qquad \mathfrak{g}_i = \left\{ f \in \bigoplus_{j<0} \mathfrak{g}_j^* \otimes \mathfrak{g}_{j+i} \mid f([v_1, v_2]) = [f(v_1), v_2] + [v_1, f(v_2)], \ \forall v_1, v_2 \in \mathfrak{m} \right\}.$$

The brackets on $\mathfrak{g}$ are: (i) The given brackets on $\mathfrak{m}$ and $\mathfrak{g}_0$; (ii) If $f \in \mathfrak{g}_i$, $i \geq 0$, and $v \in \mathfrak{m}$, then define $[f, v] := f(v)$; (iii) Brackets on the non-negative part are defined inductively: If $f_1 \in \mathfrak{g}_i$ and $f_2 \in \mathfrak{g}_j$, for $i, j \geq 0$, then define $[f_1, f_2] \in \mathfrak{g}_{i+j}$ by $[f_1, f_2](v) := [f_1(v), f_2] + [f_1, f_2(v)]$, $v \in \mathfrak{m}$.

Remark 2.1.1. Since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$, any $X \in \mathfrak{g}_i$, $i \geq 0$, is determined by its action on $\mathfrak{g}_{-1}$. By (T.2) the Lie bracket on $\mathfrak{g}$ induces $\mathfrak{g}_i \hookrightarrow \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{i-1}$, so $\mathfrak{g}_i \hookrightarrow \mathrm{Hom}(\otimes^i \mathfrak{g}_{-1}, \mathfrak{g}_0) \hookrightarrow \mathrm{Hom}(\otimes^{i+1} \mathfrak{g}_{-1}, \mathfrak{g}_{-1})$.

Note that by (T.2), if $\mathfrak{g}_r = 0$ for some $r \geq 0$, then $\mathfrak{g}_i = 0$ for all $i > r$. This case of finite termination is particularly important.

Theorem 2.1.2. [44, Thm. 8.4], [52] *Let $\mathcal{G}_0 \to M$ be a structure of constant type $(\mathfrak{m}, \mathfrak{g}_0)$ and $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$. Suppose $r \geq 0$ is minimal such that $\mathfrak{g}_{r+1} = 0$. Then there exists a canonical frame on the r-th geometric prolongation $\mathcal{G}_r$ of $\mathcal{G}_0$, and the symmetry dimension is bounded above by $\dim(\mathfrak{g})$.*

Often this framing is a Cartan connection. (This is the case for all structures in this article.)

Definition 2.1.3. *A Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, K) (or a "G/K geometry") is a principal K-bundle $\mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with defining properties:*
   (i) ω is K-equivariant;
   (ii) $\omega(\zeta_Y) = Y$, $\forall Y \in \mathfrak{k}$, where $\zeta_Y(u) = \frac{d}{dt}\big|_{t=0} u \cdot \exp(tY)$;
   (iii) $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism, $\forall u \in \mathcal{G}$.
The curvature of ω is $d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}, \mathfrak{g})$. The isomorphism (iii) identifies this 2-form with the curvature function $\kappa : \mathcal{G} \to \bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$, which is horizontal, so $\kappa : \mathcal{G} \to \bigwedge^2 (\mathfrak{g}/\mathfrak{k})^* \otimes \mathfrak{g}$. The infinitesimal symmetries are $\mathfrak{inf}(\mathcal{G}, \omega) = \{\xi \in \mathfrak{X}(\mathcal{G})^K \mid \mathcal{L}_\xi \omega = 0\}$.

A Cartan geometry is *flat* if $\kappa \equiv 0$. A fundamental fact is that $(\mathcal{G} \to M, \omega)$ is flat iff it is locally equivalent to the *flat model* $(G \to G/K, \omega_G)$ (equipped with the Maurer–Cartan form ω_G).

While calculating the Tanaka prolongation from given data $(\mathfrak{m}, \mathfrak{g}_0)$ is algorithmic, a naive application of (2.1) generally leads to a computationally intensive exercise in linear algebra. Moreover,

even if this task is completed, there still remains the general problem of understanding the structure of the resulting Tanaka algebra. However, in the context of parabolic geometries, these problems are resolved by Yamaguchi's prolongation theorem [50]; see Theorem A.1 of Appendix A.

Computing the Tanaka prolongation is much simpler when one knows $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$, and wants $\mathfrak{a} = \mathrm{pr}(\mathfrak{m}, \mathfrak{a}_0)$ for some subalgebra $\mathfrak{a}_0 \subseteq \mathfrak{g}_0$. By Remark 2.1.1, for $k > 0$, we have $\mathfrak{a}_k \hookrightarrow \mathfrak{g}_{-1}^* \otimes \mathfrak{a}_{k-1} \hookrightarrow \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{k-1}$, and $\mathfrak{a}_k \hookrightarrow \mathrm{Hom}(\otimes^k \mathfrak{g}_{-1}, \mathfrak{a}_0)$. The inclusion $\mathfrak{a}_0 \hookrightarrow \mathfrak{g}_0$ induces inclusions $\mathfrak{a}_k \hookrightarrow \mathfrak{g}_k$. Hence,

Lemma 2.1.4. *Suppose* $\mathfrak{m} = \mathfrak{g}_-$ *is generated by* $\mathfrak{g}_{-1}$. *If* $\mathfrak{a}_0 \subseteq \mathfrak{g}_0$ *is any subalgebra, then* $\mathfrak{a} :=$ $\mathrm{pr}(\mathfrak{m}, \mathfrak{a}_0) \hookrightarrow \mathfrak{g} := \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$. *Indeed for* $k > 0$,

$$\mathfrak{a}_k = \{ X \in \mathfrak{g}_k \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{k-1} \} = \{ X \in \mathfrak{g}_k \mid \mathrm{ad}_{\mathfrak{g}_{-1}}^k(X) \subset \mathfrak{a}_0 \}.$$

More precisely, $\mathfrak{a}_k = \{ X \in \mathfrak{g}_k \mid \mathrm{ad}_{u_1} \circ ... \circ \mathrm{ad}_{u_k}(X) \in \mathfrak{a}_0, \ \forall u_i \in \mathfrak{g}_{-1} \}$.

2.2. Parabolic geometry in a nutshell.

The following are standard facts from parabolic geometry – see [11, §3] for further details.

Given a real or complex semisimple Lie algebra $\mathfrak{g}$, a $\mathbb{Z}$-*grading* (or ν-grading) is a decomposition $\mathfrak{g} = \mathfrak{g}_{-\nu} \oplus ... \oplus \mathfrak{g}_\nu$ with $\mathfrak{g}_{\pm\nu} \neq 0$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $[\mathfrak{g}_j, \mathfrak{g}_{-1}] = \mathfrak{g}_{j-1}$ for $j < 0$, i.e. $\mathfrak{g}_{-1}$ is *bracket-generating* in $\mathfrak{g}_-$. The subalgebra $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$ is reductive, each $\mathfrak{g}_j$ is a $\mathfrak{g}_0$-module, and there exists a unique *grading element* $Z \in \mathfrak{z}(\mathfrak{g}_0)$, i.e. $[Z, X] = jX$ for $X \in \mathfrak{g}_j$. (The eigenvalues of Z on a given $\mathfrak{g}_0$-module will often be referred to as its *homogeneities*.) Defining $\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_j$, we have $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$, so $\mathfrak{g}$ is canonically a filtered Lie algebra with associated-graded $\mathrm{gr}(\mathfrak{g}) \cong \mathfrak{g}$. A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is *parabolic* if $\mathfrak{p} = \mathfrak{g}^0 = \mathfrak{g}_{\geq 0}$ for some ν-grading of $\mathfrak{g}$, and then $\mathfrak{p}^{\mathrm{op}} = \mathfrak{g}_{\leq 0}$ is the *opposite* parabolic. Each filtrand $\mathfrak{g}^i$ is $\mathfrak{p}$-invariant, and the quotient $\mathfrak{g}/\mathfrak{p}$ is naturally filtered. The Killing form B on $\mathfrak{g}$ induces the dualities $\mathfrak{g}_{-i} \cong (\mathfrak{g}_i)^*$ (as $\mathfrak{g}_0$-modules) and $\mathfrak{g}^i = (\mathfrak{g}^{-i+1})^\perp$ (as $\mathfrak{p}$-modules). The nilradical of $\mathfrak{p}$ is $\mathfrak{p}_+ := \mathfrak{g}^1 = \mathfrak{p}^\perp$, so $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ (as $\mathfrak{p}$-modules).

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. A subgroup $P \subset G$ is *parabolic* if it lies between the normalizer $N_G(\mathfrak{p})$ and its connected component of the identity. Under the adjoint action, P preserves the filtration on $\mathfrak{g}$. Define $G_0 \subset P$ to be the subgroup which preserves the grading on $\mathfrak{g}$; its Lie algebra is $\mathfrak{g}_0$. There is a decomposition $P = G_0 \ltimes P_+$ for some closed normal subgroup $P_+ \subset P$ with Lie algebra $\mathfrak{p}_+$.

A *parabolic geometry* is a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P). Since $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ via ω and $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$, the filtration $\mathfrak{g} = \mathfrak{g}^{-\nu} \supset ... \supset \mathfrak{g}^\nu$ induces a (P-invariant) filtration $T\mathcal{G} = T^{-\nu}\mathcal{G} \supset ... \supset T^\nu\mathcal{G}$ which projects to a filtration $TM = T^{-\nu}M \supset ... \supset T^{-1}M$. Given the principal G_0-bundle $\mathcal{G}_0 := \mathcal{G}/P_+$ over M, we have $\mathrm{gr}(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$. Let $\kappa : \mathcal{G} \to \bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ be the curvature function of ω. The parabolic geometry is:

(1) *regular* iff $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$, $\forall i, j < 0$, i.e. κ has *positive* homogeneity.
(2) *normal* iff $\partial^* \kappa = 0$. Here, ∂^* is the (P-equivariant) Kostant codifferential on $\bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}$, which is the negative of the Lie algebra homology differential.

Regularity is equivalent to: (i) M being a filtered manifold with respect to the above filtration, and (ii) the algebraic bracket on $\mathrm{gr}(TM)$ induced from $\mathfrak{g}_-$ is the same as the Levi bracket. Indeed, we obtain a *regular infinitesimal flag structure* of type (G, P) on M, i.e. a filtration of TM generated by $D = T^{-1}M$ together with a reduction of structure group of the (graded) frame bundle of $\mathrm{gr}(TM)$ to $\mathcal{G}_0$ corresponding to $\mathrm{Ad} : G_0 \to \mathrm{Aut}_{gr}(\mathfrak{g}_-)$. For many parabolic geometries[6] such as $(2, 3, 5)$-geometry (G_2/P_1), the underlying structure is a bracket-generating distribution with no reduction of structure group. For conformal structures $(B_\ell/P_1, D_\ell/P_1)$, the filtration on M is trivial, so the geometry is determined by a reduction of structure group alone. Parabolic contact structures have a non-trivial filtration as well as a reduction of structure group, e.g. CR and Lagrangean contact structures $(A_\ell/P_{1,\ell})$, and Lie contact structures $(B_\ell/P_2, D_\ell/P_2)$.

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[6]See §3.2 for an explanation of notations from parabolic geometry.

A fundamental result of Tanaka [45], Morimoto [35], Čap–Schichl [10] is that there is an equivalence of categories between regular, normal parabolic geometries and *underlying structures*. According to Yamaguchi [50] (see Appendix A), in most cases the Tanaka prolongation satisfies $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{g}$. Here, the "underlying structure" is precisely the regular infinitesimal flag structure. Two notable exceptions are projective structures (A_ℓ/P_1) and contact projective structures (C_ℓ/P_1). In these cases $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$ is infinite-dimensional, so the grading one component needs to be constrained to $\mathfrak{g}_1$ through additional structure.

Since $(\partial^*)^2 = 0$, then for regular, normal parabolic geometries there is a fundamental quantity called *harmonic curvature* $\kappa_H : \mathcal{G} \to \ker(\partial^*)/\mathrm{im}(\partial^*)$, which is much simpler than κ and is still a complete obstruction to flatness. Namely, $\kappa_H \equiv 0$ iff the geometry is locally equivalent to the flat model $(G \to G/P, \omega_G)$. Since κ_H is P-equivariant and $\ker(\partial^*)/\mathrm{im}(\partial^*)$ is completely reducible (so P_+ acts trivially), then κ_H descends to a G_0-equivariant function on $\mathcal{G}_0$. As G_0-modules $(\mathfrak{g}/\mathfrak{p})^* \cong (\mathfrak{g}_-)^* \cong \mathfrak{p}_+$, so consider the ($G_0$-equivariant) Lie algebra cohomology differential ∂ and the *Kostant Laplacian* $\square := \partial\partial^* + \partial^*\partial$ acting on co-chains $\bigwedge^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}$. By a lemma of Kostant, we have the following G_0-module isomorphisms:

$$\bigwedge\nolimits^k(\mathfrak{g}_-)^* \otimes \mathfrak{g} = \overbrace{\mathrm{im}(\partial^*) \oplus \underbrace{\ker(\square)}_{\ker(\partial)} \oplus \mathrm{im}(\partial)}^{\ker(\partial^*)}, \qquad \ker(\square) \cong \frac{\ker(\partial^*)}{\mathrm{im}(\partial^*)} \cong \frac{\ker(\partial)}{\mathrm{im}(\partial)} =: H^k(\mathfrak{g}_-, \mathfrak{g}).$$

By G_0-equivariancy, $\kappa_H : \mathcal{G}_0 \to H^2(\mathfrak{g}_-, \mathfrak{g})$ maps fibres of $\mathcal{G}_0 \to M$ to G_0-orbits in $H^2(\mathfrak{g}_-, \mathfrak{g})$. By regularity, κ_H is valued in the $\mathfrak{g}_0$-submodule $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \subseteq H^2(\mathfrak{g}_-, \mathfrak{g})$ on which the grading element Z acts with positive homogeneities. In the complex case, Kostant's theorem (Recipe 4) completely describes $H^2(\mathfrak{g}_-, \mathfrak{g})$.

2.3. Filtration and embedding of the symmetry algebra. We review some recent work.

2.3.1. *Čap and Neusser* [8, 9]. Let $(\mathcal{G} \xrightarrow{\pi} M, \omega)$ be a regular, normal geometry of type (G, P). Then:

(i) Let $u \in \mathcal{G}$, and $x = \pi(u)$. The map $\xi \mapsto \omega(\xi(u))$ is a linear injection $\widetilde{\mathcal{S}} := \mathfrak{inf}(\mathcal{G}, \omega) \hookrightarrow \mathfrak{g}$.[7]
(ii) Letting $\mathfrak{f}(u) := \mathrm{im}(\omega_u|_{\mathfrak{inf}(\mathcal{G}, \omega)}) \subset \mathfrak{g}$, the bracket on $\mathfrak{inf}(\mathcal{G}, \omega)$ is mapped to the operation

$$(2.2) \qquad [X, Y]_{\mathfrak{f}(u)} := [X, Y] - \kappa_u(X, Y), \qquad \forall X, Y \in \mathfrak{f}(u).$$

(iii) By regularity, restricting the canonical filtration on $\mathfrak{g}$ to $\mathfrak{f}(u)$ makes the latter into a filtered Lie algebra (which is generally not a filtered Lie subalgebra of $\mathfrak{g}$).
(iv) $\mathfrak{s}(u) := \mathrm{gr}(\mathfrak{f}(u))$ is a graded subalgebra of $\mathfrak{g}$. Note $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) = \dim(\mathfrak{s}(u))$.
(v) $\mathfrak{s}_0(u) \subseteq \mathfrak{ann}(\kappa_H(u)) \subseteq \mathfrak{g}_0$.
(vi) If $\kappa_H(u) \neq 0$, then (v) and Kostant's theorem (Recipe 5) yield a bound on $\dim(\mathfrak{s}_0(u))$.

Although (v) is stated in [9, Cor. 5 (3)] for bracket-generating distributions, it holds in general. Indeed, given $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$, if $Y = \omega(\xi(u)) \in \mathfrak{p}$, then the curve $\varphi_t(u) = u \cdot \exp(tY)$ has vertical tangent vector $\xi(u) = \zeta_Y(u) = \frac{d}{dt}\big|_{t=0} \varphi_t(u)$. By P-equivariancy of κ_H,

$$0 = (\xi \cdot \kappa_H)(u) = \frac{d}{dt}\bigg|_{t=0} \kappa_H(\varphi_t(u)) = \frac{d}{dt}\bigg|_{t=0} \exp(-tY) \cdot (\kappa_H(u)) = -Y \cdot (\kappa_H(u)).$$

Complete reducibility of $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ implies that this action depends only on $Y \bmod \mathfrak{p}_+ \in \mathfrak{p}/\mathfrak{p}_+ \cong \mathfrak{g}_0$.

Remark 2.3.1. For non-flat geometries, Čap & Neusser bound $\dim(\mathfrak{inf}(\mathcal{G}, \omega))$ via the maximum dimension of any proper graded subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\dim(\mathfrak{b}_0)$ no larger than the Kostant bound. However, this strategy lacks finer information about the annihilating subalgebra, i.e. they use (B.3), while we will additionally use (B.2). Thus, in general, their upper bounds are not sharp.

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[7]They state this for the global automorphism algebra $\mathfrak{aut}(\mathcal{G}, \omega)$, but their proof works also for $\mathfrak{inf}(\mathcal{G}, \omega)$.

The following result is stated in [8, §2.5].

Proposition 2.3.2. *Let* $(\mathcal{G} \to M, \omega)$ *be a regular, normal parabolic geometry of type* (G, P). *If* $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) = \dim(\mathfrak{g})$, *then the geometry is flat.*

Proof. Let $u \in \mathcal{G}$, so $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) = \dim(\mathfrak{s}(u)) \leq \dim(\mathfrak{g})$. If equality holds, then the grading element has $Z \in \mathfrak{s}_0(u) \subseteq \mathfrak{ann}(\kappa_H(u))$. Since $\kappa_H(u) \in H^2_+(\mathfrak{g}_-, \mathfrak{g})$, this forces $\kappa_H(u) = 0$. $\square$

The filtration on $\mathfrak{g}$ induces (P-invariant) filtrations on $T\mathcal{G} = T^{-\nu}\mathcal{G} \supset ... \supset T^\nu \mathcal{G}$ and $\widetilde{\mathcal{S}}$:

(2.3) $$\widetilde{\mathcal{S}}(x)^i = \{\xi \in \widetilde{\mathcal{S}} \mid \xi(u) \in T^i_u\mathcal{G}, \ \forall u \in \pi^{-1}(x)\}, \qquad -\nu \leq i \leq \nu.$$

This projects to a filtration $\mathcal{S} = \mathcal{S}(x)^{-\nu} \supset ... \supset \mathcal{S}(x)^\nu$ of the symmetry algebra $\mathcal{S}$ of the underlying structure on M. Note $\omega_u(\widetilde{\mathcal{S}}(x)^i) = \mathfrak{f}(u)^i$, and $\mathfrak{s}(u) = \mathrm{gr}(\mathfrak{f}(u)) \cong \mathrm{gr}(\mathcal{S}(x))$ is a canonical isomorphism, so the latter will be denoted by $\mathfrak{s}(x)$.

2.3.2. *Kruglikov* [30]. For bracket-generating distributions (not necessarily underlying parabolic geometries), we have a filtration $\{\mathcal{S}(x)^i\}$ of $\mathcal{S}$. Let $TM = D^{-\nu} \supset ... \supset D^{-1} =: D$ generated by D with each D^i of constant rank.[8] Given $x \in M$, define $\mathfrak{g}_i(x) := D^i_x/D^{i+1}_x$ for $i < 0$. The Tanaka prolongation $\mathfrak{g}(x) = \mathrm{pr}(\mathfrak{g}_-(x))$ has $\mathfrak{g}_i(x) \hookrightarrow \mathrm{Hom}(\otimes^{i+1}\mathfrak{g}_{-1}(x), \mathfrak{g}_{-1}(x))$ for $i \geq 0$, cf. Remark 2.1.1.

Letting $\mathrm{ev}_x : \mathcal{S} \to T_x M$ be the evaluation map, define $\mathcal{S}(x)^i := \mathrm{ev}_x^{-1}(D^i)$ for $i < 0$, and $\mathcal{S}(x)^0 := \ker(\mathrm{ev}_x)$. Inductively, for $i \geq 0$, given $\mathbf{X} \in \mathcal{S}(x)^i$, there is a map $\Psi^{i+1}_{\mathbf{X}} : \otimes^{i+1} D_x \to T_x M$ given by

$$\Psi^{i+1}_{\mathbf{X}}(Y_1, \ldots, Y_{i+1}) = \big[[\ldots [[\mathbf{X}, \mathbf{Y}_1], \mathbf{Y}_2], \ldots], \mathbf{Y}_{i+1}\big](x),$$

where $\mathbf{Y}_j \in \Gamma(D)$ and $\mathbf{Y}_j(x) = Y_j$. Moreover, $\mathrm{im}(\Psi^{i+1}_{\mathbf{X}}) \subseteq D_x$. Define $\mathcal{S}(x)^{i+1} := \{\mathbf{X} \in \mathcal{S}(x)^i \mid \Psi^{i+1}_{\mathbf{X}} = 0\}$. Then $\mathcal{S} = \mathcal{S}(x)$ is a filtered Lie algebra, and $\mathfrak{s}_i(x) \hookrightarrow \mathfrak{g}_i(x)$ via $\mathbf{X} \bmod \mathcal{S}(x)^{i+1} \mapsto \Psi^{i+1}_{\mathbf{X}}$.

There is a filtration[9] on $C^\infty(M)$ by ideals μ^i_x. Let $\mu^1_x = \{f \in C^\infty(M) \mid f(x) = 0\}$ and for $i \geq 0$,

$$\mu^{i+1}_x = \{f \in C^\infty(M) \mid (\mathbf{Y}_1 \cdots \mathbf{Y}_t \cdot f)(x) = 0, \forall \mathbf{Y}_j \in \Gamma(D), 0 \leq t \leq i\}.$$

For any $\mathbf{Y} \in \Gamma(D)$, we have $\mathbf{Y} \cdot \mu^{i+1}_x \subset \mu^i_x$. Let $\{\mathbf{Z}_{jk}\}_{j=1,\ldots,\nu}$ be a local framing near x with $\mathbf{Z}_{jk} \in \Gamma(D^{-j}) \backslash \Gamma(D^{-j-1})$. Let $\mathbf{X} \in \mathcal{S}$. Then

(2.4) $$i \geq 0: \qquad \mathbf{X} = \sum_{j,k} f_{jk}\mathbf{Z}_{jk} \in \mathcal{S}(x)^i \quad \text{iff} \quad f_{jk} \in \mu^{i+j}_x, \quad \forall j, k.$$

The "only if" direction is proved in [30, Lemma 1], while an easy induction establishes the converse.

Example 2.3.3 ((2, 3, 5)-geometry). *On a 5-manifold* (x, y, p, q, z), *consider* D *locally spanned by*

$$\partial_q, \qquad \tilde{\partial}_x := \partial_x + p\partial_y + q\partial_p + (p^3 + q^2)\partial_z.$$

Then the symmetry algebra $\mathcal{S}$ *of* D *is spanned by*

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_y, \quad \mathbf{X}_3 = \partial_z, \quad \mathbf{X}_4 = x\partial_x - y\partial_y - 2p\partial_p - 3q\partial_q - 5z\partial_z.$$

At $\mathbf{u}_0 = (x_0, y_0, p_0, q_0, z_0)$, *the dimensions of* $\mathfrak{s}_i = \mathfrak{s}_i(\mathbf{u}_0)$ *are*

	$\mathfrak{s}_{-3}$	$\mathfrak{s}_{-2}$	$\mathfrak{s}_{-1}$	$\mathfrak{s}_0$
$q_0 \neq 0$ *or* $p_0 \neq 0$	2	1	1	0
$q_0 = p_0 = 0$	2	0	1	1

When $p_0 = q_0 = 0$, $\mathcal{S}(\mathbf{u}_0)^0$ *is spanned by* $\mathbf{T} = \mathbf{X}_4 - x_0\mathbf{X}_1 + y_0\mathbf{X}_2 + 5z_0\mathbf{X}_3$. *Writing* $\mathbf{T} = \sum_{j,k} f_{jk}\mathbf{Z}_{jk}$ *in the framing* $(\mathbf{Z}_{11}, \mathbf{Z}_{12}, \mathbf{Z}_{21}, \mathbf{Z}_{31}, \mathbf{Z}_{32}) = (\partial_q, \tilde{\partial}_x, \partial_p + 2q\partial_z, \partial_y, \partial_z)$ *shows* $f_{jk} \in \mu^j_x$, *so* $\mathbf{T} \notin \mathcal{S}(\mathbf{u}_0)^1$. *Note* $[\mathbf{T}, \partial_q] = 3\partial_q$, *which at* $\mathbf{u}_0$ *is not contained in* $\mathfrak{s}_{-1}$. *Thus,* $[\mathfrak{s}_0, \mathfrak{g}_{-1}] \not\subset \mathfrak{s}_{-1}$ *when* $p_0 = q_0 = 0$.

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[8]These conditions hold for underlying structures of parabolic geometries.

[9]In the complex setting, we replace $C^\infty(M)$ by the algebra of germs of holomorphic functions about the point x.

Remark 2.3.4. The above results in Kruglikov's approach still hold if D admits additional structure. In the parabolic case, Kruglikov's filtration $\{\mathcal{S}(x)^i\}$ agrees with that of Čap–Neusser, and (2.4) gives a means to compute it explicitly. Henceforth, we will mainly rely on the Čap–Neusser presentation.

2.4. **Proof of the upper bound.** Fixing (G, P), the *submaximal symmetry dimension* $\mathfrak{S}$ is the maximum of $\dim(\mathfrak{inf}(\mathcal{G}, \omega))$ among all **regular, normal** geometries $(\mathcal{G} \xrightarrow{\pi} M, \omega)$ of type (G, P) which are **not flat**, i.e. $\kappa_H \not\equiv 0$. Equivalently, $\mathfrak{S}$ maximizes $\dim(\mathcal{S})$, where $\mathcal{S}$ is the symmetry algebra of an underlying structure which is not flat.

Definition 2.4.1. *We say that $x \in M$ is a* regular point *if there exists a neighbourhood $N_x \subset M$ of x such that for $-\nu \leq j \leq \nu$, $\dim(\mathfrak{s}_j(y))$ is a constant function of $y \in N_x$. Otherwise, x is* irregular.

We have a tower of bundles $\mathcal{G} =: \mathcal{G}_\nu \to ... \to \mathcal{G}_0 \to M$, with $\mathcal{G}_i = \mathcal{G}/P_+^{i+1} = \mathcal{G}/\exp(\mathfrak{g}^{i+1})$, projections $p_i : \mathcal{G} \to \mathcal{G}_i$ and $\pi_i : \mathcal{G}_i \to M$, where $i = -1, ..., \nu$. (If $i = -1$, let $p_{-1} = \pi$, i.e. the projection to M.) Given any $\xi \in \mathfrak{X}(\mathcal{G})^P$, let $\xi^{(i)} = (p_i)_*\xi \in \mathfrak{X}(\mathcal{G}_i)$. By P-invariance, $\widetilde{\mathcal{S}}$ descends to $\mathcal{S}^{(i)} \subset \mathfrak{X}(\mathcal{G}_i)^{P/P_+^{i+1}}$. Given $x \in M$, the filtration $\widetilde{\mathcal{S}}(x)$ projects to a filtration $\mathcal{S}^{(i)}(x)$, and from (2.3),

$$\mathcal{S}^{(i)}(x)^{i+1} = \{\xi^{(i)} \in \mathcal{S}^{(i)} \mid \xi^{(i)}(u_i) = 0, \forall u_i \in \pi_i^{-1}(x)\}.$$

For any $u_i \in \pi_i^{-1}(x)$, $\mathcal{S}^{(i)}|_{u_i} \subset T_{u_i}\mathcal{G}_i$ has dimension $\dim \mathcal{S} - \sum_{j=i+1}^{\nu} \dim \mathfrak{s}_j(x)$. If x is a regular point, then $\mathcal{S}^{(i)}$ yields a constant rank distribution on a neighbourhood of $\pi_i^{-1}(x) \subset \mathcal{G}_i$. By the Frobenius theorem, there exist local rectifying coordinates and a local foliation by maximal integral submanifolds of $\mathcal{S}^{(i)}$.

Motivated by Lemma 2.1.4 and Theorem A.1, we define a variant of Tanaka prolongation:

Definition 2.4.2. *Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded semisimple Lie algebra, and $\mathfrak{a}_0 \subset \mathfrak{g}_0$ a subalgebra. Define:*

$$\text{(i)}: \quad \mathfrak{a}_{\leq 0} := \mathfrak{g}_{\leq 0}; \qquad \text{(ii)}: \quad \mathfrak{a}_i = \{X \in \mathfrak{g}_i \mid [X, \mathfrak{g}_{-1}] \subseteq \mathfrak{a}_{i-1}\}, \quad i > 0.$$

Then $\mathfrak{a} = \bigoplus_i \mathfrak{a}_i$ is a graded subalgebra of $\mathfrak{g}$ denoted by $\mathrm{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0)$. (In particular, $\mathrm{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{g}_0) = \mathfrak{g}$.)

If G is (complex) simple and G/P is not A_ℓ/P_1 (projective) or C_ℓ/P_1 (contact projective), then $\mathrm{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0) \cong \mathrm{pr}(\mathfrak{g}_-, \mathfrak{a}_0)$ by Theorem A.1 and Lemma 2.1.4.

We have the following "filtered" generalization of a result of Morimoto [34, Prop. 4.1].

Proposition 2.4.3. *Let $x \in M$ be a regular point and $u \in \pi^{-1}(x)$. Then $[\mathfrak{s}_{i+1}(u), \mathfrak{g}_{-1}] \subset \mathfrak{s}_i(u)$, $\forall i$. In particular, $\mathfrak{s}_{i+1}(u) \subseteq \mathrm{pr}_{i+1}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{s}_0(u))$.*

Proof. Suppose $i \geq -1$. For any $\xi \in \widetilde{\mathcal{S}} = \mathfrak{inf}(\mathcal{G}, \omega)$ and $\eta \in \mathfrak{X}(\mathcal{G})$, we have $0 = (\mathcal{L}_\xi \omega)(\eta) = \xi \cdot \omega(\eta) - \omega([\xi, \eta])$. Let $\xi \in \widetilde{\mathcal{S}}(x)^{i+1}$ and $\eta \in \Gamma(T^{-1}\mathcal{G})^P$, so $X = \omega(\xi(u)) \in \mathfrak{f}(u)^{i+1} \subset \mathfrak{g}^{i+1}$ and $Y = \omega(\eta(u)) \in \mathfrak{g}^{-1}$. Since $i \geq -1$, then $X \in \mathfrak{p}$, so $X = \omega(\zeta_X(u))$, where ζ_X is a fundamental vertical vector field. By P-equivariancy of $\omega(\eta)$,

$$\omega([\xi, \eta](u)) = (\xi \cdot \omega(\eta))(u) = (\zeta_X \cdot \omega(\eta))(u) = \frac{d}{dt}\bigg|_{t=0} \omega(\eta)(u \cdot \exp(tX)) = -[X, Y].$$

In particular, since $[X, Y] \in \mathfrak{g}^i$, then $[\xi, \eta](u) \in T_u^i\mathcal{G}$.

Let $u_i = p_i(u) \in \mathcal{G}_i$. Since $\xi \in \widetilde{\mathcal{S}}(x)^{i+1}$, then $\xi^{(i)}(u_i) = 0$. Since x is a regular point, there exist local (functionally independent) functions $F_1, ..., F_{t_i}$ on $\mathcal{G}_i$ (smooth in the real setting or holomorphic in the complex setting), where $t_i = \dim \mathcal{G}_i - \sum_{-\nu}^i \dim \mathfrak{s}_j(x)$, whose joint level sets define the local foliation by maximal integral submanifolds of $\mathcal{S}^{(i)}$. In particular, for our $\xi \in \widetilde{\mathcal{S}}(x)^{i+1}$, we have $\xi^{(i)} \cdot F_j = 0$ for $j = 1, ..., t_i$, and since $\xi^{(i)}(u_i) = 0$, we have $([\xi^{(i)}, \eta^{(i)}] \cdot F_j)(u_i) = 0$. Thus, $[\xi^{(i)}, \eta^{(i)}](u_i) \in \mathcal{S}^{(i)}|_{u_i} = (p_i)_*(\widetilde{\mathcal{S}}|_u)$ is tangent to the foliation. Since any element of $\widetilde{\mathcal{S}}$ is uniquely determined by its value at u, then $[\xi^{(i)}, \eta^{(i)}](u_i) = \xi'^{(i)}(u_i)$ for some $\xi' \in \widetilde{\mathcal{S}}$. Equivalently,

$$[\xi, \eta](u) = \xi'(u) + \chi(u) \in T_u^i\mathcal{G}, \qquad \xi'(u) \in \widetilde{\mathcal{S}}|_u, \quad \chi(u) \in T_u^{i+1}\mathcal{G}.$$

Hence, $\omega([\xi, \eta](u)) \in \mathfrak{f}(u)^i + \mathfrak{g}^{i+1}$. Thus, $[\mathfrak{f}(u)^{i+1}, \mathfrak{g}^{-1}] \subset \mathfrak{f}(u)^i + \mathfrak{g}^{i+1}$, so $[\mathfrak{s}_{i+1}(u), \mathfrak{g}_{-1}] \subset \mathfrak{s}_i(u)$.

Now suppose $i \leq -2$. Since x is regular, then $D_{\mathcal{S}}^j|_y := D_y^j \cap \mathcal{S}|_y$, $j \leq -1$, define constant rank distributions near x. Let $\mathbf{X} = \sum_i f_i \mathbf{X}_i \in \Gamma(D_{\mathcal{S}}^{i+1})$ where $\{\mathbf{X}_i\}$ is basis of $\mathcal{S}$. Given $\mathbf{Y} \in \Gamma(D)$,

$$[\mathbf{X}, \mathbf{Y}](x) = \sum_i f_i(x)[\mathbf{X}_i, \mathbf{Y}](x) - (\mathbf{Y} \cdot f_i)(x)\mathbf{X}_i(x) \in (D_x + \mathcal{S}|_x) \cap D_x^i.$$

Quotient by $D_x^{i+1} \supset D_x$, so with respect to the Levi bracket, $[\mathfrak{s}_{i+1}(x), \mathfrak{g}_{-1}(x)] \subset \mathfrak{s}_i(x)$, hence the first claim. The second claim then follows immediately. $\square$

If $\mathcal{S}$ is *transitive* at x, i.e. $\mathrm{ev}_x(\mathcal{S}) = T_x M$, then $[\mathfrak{s}_{i+1}(x), \mathfrak{g}_{-1}(x)] = [\mathfrak{s}_{i+1}(x), \mathfrak{s}_{-1}(x)] \subset \mathfrak{s}_i(x)$ immediately follows since $\mathfrak{s}(x)$ is a graded Lie algebra. Equivalently, $[\mathfrak{s}_{i+1}(u), \mathfrak{g}_{-1}] \subset \mathfrak{s}_i(u)$.

Example 2.4.4. *In Example 2.3.3, the regular points are precisely those with $p_0 \neq 0$ or $q_0 \neq 0$. At all irregular points, we saw that $[\mathfrak{s}_0, \mathfrak{g}_{-1}] \not\subset \mathfrak{s}_{-1}$.*

Lemma 2.4.5. *The set of regular points is open and dense in M.*

Proof. For $-\nu \leq i \leq \nu$, define $q_i^{\pm} : M \to \mathbb{Z}$ by $q_i^-(x) = \sum_{j=-\nu}^i \dim \mathfrak{s}_j(x)$ and $q_i^+(x) = \dim \mathcal{S} - q_i^-(x) = \dim \mathcal{S}(x)^{i+1} = \sum_{j=i+1}^{\nu} \dim \mathfrak{s}_j(x)$. Then q_i^- is lower semi-continuous because linear independence of $\xi_1^{(i)}, \ldots, \xi_s^{(i)} \in \mathcal{S}^{(i)}$ at $u_i \in \mathcal{G}_i$ persists near u_i. Hence, q_i^+ is upper semi-continuous. Given $U \subset M$, define $m_i(U) = \{x \in U \mid q_i^+(x) \leq q_i^+(y), \forall y \in U\}$. Upper semi-continuity implies that if $\emptyset \neq U \subset M$ is open, then $\emptyset \neq m_i(U) \subset M$ is open.

Denote $M_0 := M$, and define $N_{j+1} = m_{-\nu} \circ \cdots \circ m_\nu(M_j)$ and $M_{j+1} = M \backslash (\bar{N}_1 \cup \cdots \cup \bar{N}_j)$ for $j \geq 1$. Each $M_j \subset M$ is open, so $N_{j+1} \subset M$ is open; when the former is non-empty, so is the latter. Each $\mathrm{im}(q_i^+) \subset [0, \dim \mathcal{S}] \cap \mathbb{Z}$ is finite, so there exists a minimal $k \geq 0$ with $M_{k+1} = \emptyset$. The open set $N = N_1 \cup \cdots \cup N_k$ is the set of all regular points, which is dense in M since $\bar{N} = M$. $\square$

We are now in a position to establish our universal upper bound. For $0 \neq \phi \in H_+^2(\mathfrak{g}_-, \mathfrak{g})$, define

$$(2.5) \qquad \mathfrak{a}^\phi := \mathrm{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{ann}(\phi)), \qquad \mathfrak{U} := \max\left\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H_+^2(\mathfrak{g}_-, \mathfrak{g})\right\} < \dim(\mathfrak{g}).$$

(If $\mathfrak{U} = \dim(\mathfrak{g})$, then $\mathfrak{a}^\phi = \mathfrak{g}$ for some $\phi \neq 0$, and $Z \in \mathfrak{ann}(\phi)$, which is a contradiction.)

Theorem 2.4.6. *Let G be a semisimple Lie group and $P \subset G$ a parabolic subgroup. Let $(\mathcal{G} \xrightarrow{\pi} M, \omega)$ be a regular, normal geometry of type (G, P), $x \in M$ a regular point, and $u \in \pi^{-1}(x)$. Then $\mathfrak{s}(u) \subseteq \mathfrak{a}^{\kappa_H(u)}$ is a graded subalgebra, and $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \leq \dim(\mathfrak{a}^{\kappa_H(u)})$. Moreover, $\mathfrak{S} \leq \mathfrak{U} < \dim(\mathfrak{g})$.*

Proof. We have $\mathfrak{s}(u) \subseteq \mathfrak{g}_- \oplus \mathfrak{s}_{\geq 0}(u) \subseteq \mathrm{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{s}_0(u)) \subseteq \mathfrak{a}^{\kappa_H(u)}$, using Proposition 2.4.3 and $\mathfrak{s}_0(u) \subseteq \mathfrak{ann}(\kappa_H(u))$. Thus, $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) = \dim(\mathfrak{s}(u)) \leq \dim(\mathfrak{a}^{\kappa_H(u)})$.

If the geometry is not flat, then $N = \{x \in M \mid \kappa_H(u) \neq 0, \forall u \in \pi^{-1}(x)\}$ is non-empty and open, so by Lemma 2.4.5, N contains a non-flat regular point x. At any $u \in \pi^{-1}(x)$, the previous bound applies. Since $\kappa_H(u) \in H_+^2(\mathfrak{g}_-, \mathfrak{g})$, then $\mathfrak{S} \leq \mathfrak{U} < \dim(\mathfrak{g})$. $\square$

Remark 2.4.7. Only $[\mathfrak{s}_{i+1}(u), \mathfrak{g}_{-1}] \subseteq \mathfrak{s}_i(u)$ for $i \geq 0$ was essential for proving Theorem 2.4.6. This relied only on local constancy of $\dim(\mathfrak{s}_j(y))$ for $j > 0$, so our "regular point" definition could be weakened. However, $[\mathfrak{s}_0(u), \mathfrak{g}_{-1}] \subseteq \mathfrak{s}_{-1}(u)$ will be used in proving Theorem 5.1.3.

Let $(\mathfrak{g}, \mathfrak{p})$ be real with complexification $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{p}^{\mathbb{C}})$. Define

$$\mathfrak{U}^{\mathbb{C}} := \max\left\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H_+^2(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})\right\}.$$

Corollary 2.4.8. *For real (regular, normal) parabolic geometries of type (G, P), $\mathfrak{S} \leq \mathfrak{U} \leq \mathfrak{U}^{\mathbb{C}}$.*

Proof. We have $H_+^2(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) \cong H_+^2(\mathfrak{g}_-, \mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C}$. In Proposition 3.1.1, we show that over $\mathbb{C}$, $\dim(\mathfrak{a}^\phi)$ is maximized on some extremal vector in some sub-irrep. Since there may not exist vectors $H_+^2(\mathfrak{g}_-, \mathfrak{g})$ which complexify to an extremal vector of $H_+^2(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$, then we immediately obtain $\mathfrak{U} \leq \mathfrak{U}^{\mathbb{C}}$. $\square$

Corollary 2.4.9. *For regular, normal geometries of type (G, P), suppose that $\mathfrak{S} = \mathfrak{U}$. Then any submaximally symmetric model $(\mathcal{G} \xrightarrow{\pi} M, \omega)$ is locally homogeneous near a non-flat regular point $x \in M$.*

Proof. By Theorem 2.4.6, $\mathfrak{U} = \mathfrak{S} = \dim(\mathfrak{inf}(\mathcal{G}, \omega)) = \dim(\mathfrak{s}(u)) \leq \dim(\mathfrak{a}^{\kappa_H(u)}) \leq \mathfrak{U}$, $\forall u \in \pi^{-1}(x)$. Hence, $\mathfrak{s}(u) = \mathfrak{a}^{\kappa_H(u)} \supset \mathfrak{g}_-$, so $\mathfrak{s}(x) \supset \mathfrak{g}_-$, i.e. $\mathcal{S}$ is transitive at x. Lie's third theorem implies the result. $\qquad\square$

Remark 2.4.10. Recall that the fibres of $\mathcal{G} \to M$ are mapped by κ_H into G_0-orbits in $H^2_+(\mathfrak{g}_-, \mathfrak{g})$. Let $\{0\} \neq \mathcal{O} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$ be a G_0-invariant subset, e.g. a G_0-orbit or a G_0-submodule. Analogously defining $\mathfrak{S}_{\mathcal{O}}$ and $\mathfrak{U}_{\mathcal{O}}$ using the constraint $\mathrm{im}(\kappa_H) \subset \mathcal{O}$, we obtain $\mathfrak{S}_{\mathcal{O}} \leq \mathfrak{U}_{\mathcal{O}}$ similar to Theorem 2.4.6. If $\mathfrak{S}_{\mathcal{O}} = \mathfrak{U}_{\mathcal{O}}$, then any model realizing the equality must be locally homogeneous near a non-flat regular point.

We will use Remark 2.4.10 in §5.1.2 to study the Petrov types in 4-dimensional Lorentzian conformal geometry, and in §5.2.1 for $(2, 3, 5)$-distributions with κ_H having constant root type.

3. PROLONGATION ANALYSIS

The universal upper bound $\mathfrak{U}$ of Theorem 2.4.6 was defined in (2.5) via the Tanaka algebra $\mathfrak{a}^\phi$. Observe that if $H^2_+(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_i \mathbb{V}_i$ is the decomposition into $\mathfrak{g}_0$-irreps and $\phi = \sum_i \phi_i$, where $\phi_i \in \mathbb{V}_i$, $\forall i$, then $\mathfrak{ann}(\phi) \subset \mathfrak{ann}(\phi_i)$ and $\mathfrak{a}^\phi \subset \mathfrak{a}^{\phi_i}$, so $\mathfrak{U} = \max_i\{\max\{\dim(\mathfrak{a}^{\phi_i}) \mid 0 \neq \phi_i \in \mathbb{V}_i\}\}$. Thus, it suffices to understand $\max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in \mathbb{V}\}$, where $\mathbb{V}$ is a $\mathfrak{g}_0$-irrep. In the complex case, this maximum is realized on any extremal weight vector $\phi_0 \in \mathbb{V}$ (§3.1). The structure of $\mathfrak{a}^{\phi_0}$ (in particular, its dimension) can be efficiently deduced from a Dynkin diagram recipe (§3.3). We investigate the notion of *prolongation-rigidity* in §3.4, and the behaviour of $\mathfrak{a}^{\phi_0}$ under correspondence and twistor space constructions in §3.5.

3.1. **Maximizing the Tanaka prolongation.** In the *complex* setting, given a G_0-irrep $\mathbb{V}$, where G_0 is reductive (with non-trivial semisimple part), a well-known consequence of the Borel fixed point theorem is that $\mathbb{P}(\mathbb{V})$ has a *unique* Zariski closed G_0-orbit $\mathcal{O}$, namely the orbit of any extremal line $[\phi_0]$. This is the unique orbit of minimal dimension, so $\max\{\dim(\mathfrak{ann}(\phi)) \mid 0 \neq \phi \in \mathbb{V}\}$ is realized precisely when $[\phi] \in \mathcal{O}$. This is the $k = 0$ assertion in the proposition below.

Proposition 3.1.1. *Let G be a complex semisimple Lie group, and P a parabolic subgroup. Let $\mathbb{V}$ be an G_0-irrep, and $\phi_0 \in \mathbb{V}$ an extremal $\mathfrak{g}_0$-weight vector. Then $\forall 0 \neq \phi \in \mathbb{V}$, $\dim(\mathfrak{a}^\phi_k) \leq \dim(\mathfrak{a}^{\phi_0}_k)$, $\forall k \geq 0$. Thus,*
$$\max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in \mathbb{V}\} = \dim(\mathfrak{a}^{\phi_0}).$$
Moreover, $\dim(\mathfrak{a}^\phi) = \dim(\mathfrak{a}^{\phi_0})$ iff $[\phi] \in G_0 \cdot [\phi_0] \subset \mathbb{P}(\mathbb{V})$.

Proof. If $\mathfrak{g}_0^{ss} = 0$, then irreducibility implies $\mathbb{V} \cong \mathbb{C}$, and the result is immediate. So suppose $\mathfrak{g}_0^{ss} \neq 0$ and let $k \geq 1$. Define $\Psi_k : \mathbb{P}(\mathbb{V}) \to \mathbb{Z}_{\geq 0}$ by $\Psi_k([\phi]) = \dim(\mathfrak{a}^\phi_k)$, which is constant on G_0-orbits since $\mathfrak{ann}(g \cdot \phi) = \mathrm{Ad}_g(\mathfrak{ann}(\phi))$ and $\mathfrak{a}^{g \cdot \phi}_k = \mathrm{Ad}_g(\mathfrak{a}^\phi_k)$. From the definition (2.5) of $\mathfrak{a}^\phi$,
$$\mathfrak{a}^\phi_k = \mathrm{pr}^\mathfrak{g}_k(\mathfrak{g}_-, \mathfrak{ann}(\phi)) = \{X \in \mathfrak{g}_k \mid \mathrm{ad}^k_{\mathfrak{g}_{-1}}(X) \cdot \phi = 0\}$$
$$= \{X \in \mathfrak{g}_k \mid (\mathrm{ad}_{Y_{i_1}} \circ \dots \circ \mathrm{ad}_{Y_{i_k}}(X)) \cdot \phi = 0, \ \forall Y_{i_j} \in \mathfrak{g}_{-1}\}.$$

The function $(\mathrm{ad}_{Y_{i_1}} \circ \dots \circ \mathrm{ad}_{Y_{i_k}}(X)) \cdot \phi$ is a multilinear function of $X, Y_{i_1}, \dots, Y_{i_k}, \phi$. Its vanishing is determined by evaluating all $Y_{i_1}, \dots, Y_{i_k}$ on all tuples of basis elements $\{e_i\}$ of $\mathfrak{g}_{-1}$. So there is a bilinear function $T(X, \phi)$ such that $\mathfrak{a}^\phi_k = \{X \in \mathfrak{g}_k \mid T(X, \phi) = 0\}$. In any basis of $\mathfrak{g}_k$, there is a matrix $M(\phi)$ such that $X \in \mathfrak{a}^\phi_k$ iff its coordinate vector is in $\ker(M(\phi))$. Since the rank of a matrix is a lower semi-continuous function of its entries, and $M(\phi)$ depends linearly on ϕ, then $\mathrm{rank}(M(\phi))$

is lower semi-continuous in ϕ (and depends only on $[\phi]$). We have $\dim(\mathfrak{a}_k^\phi) = \dim(\mathfrak{g}_k) - \text{rank}(M(\phi))$, so Ψ_k is upper semi-continuous.

We know that $\mathcal{O} = G_0 \cdot [\phi_0]$ is the *unique* Zariski closed G_0-orbit in $\mathbb{P}(\mathbb{V})$, so $[\phi_0]$ is in the Zariski closure of *every* G_0-orbit in $\mathbb{P}(\mathbb{V})$. Hence, $\forall 0 \neq \phi \in \mathbb{V}$, there is a sequence $\{g_n\} \subset G_0$ such that $g_n \cdot [\phi] = [g_n \cdot \phi] \to [\phi_0]$ as $n \to \infty$. Thus, $\Psi_k([\phi]) = \Psi_k([g_n \cdot \phi]) = \Psi_k(g_n \cdot [\phi]) \leq \Psi_k([\phi_0])$ by upper semi-continuity of Ψ_k.

The final claim follows from the discussion preceding this lemma on the $k = 0$ case. $\qquad\square$

Remark 3.1.2. In general, $\mathfrak{ann}(\phi) \not\subset \mathfrak{ann}(\phi_0)$, and $\mathfrak{a}_k^\phi \not\subset \mathfrak{a}_k^{\phi_0}$.

The recipe to compute $\mathfrak{a}_0^{\phi_0} = \mathfrak{ann}(\phi_0)$ is well-known (see Recipe 5). In §3.3, we show that $\mathfrak{a}_+^{\phi_0}$ is also easily determined. First, we fix notations from representation theory.

3.2. **Representation theory notations and conventions.** Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with associated Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$, root system $\Delta \subset \mathfrak{h}^*$, and simple roots $\Delta^0 = \{\alpha_1, ..., \alpha_\ell\}$, where $\ell = \dim(\mathfrak{h}) = \text{rank}(\mathfrak{g})$. (We use the Bourbaki ordering.) Let $\{Z_1, ..., Z_\ell\} \subset \mathfrak{h}$ be the dual basis to $\Delta^0 \subset \mathfrak{h}^*$. If $S \subset \{1, ..., \ell\}$, then we let $Z_S := \sum_{k \in S} Z_k$. For any $\alpha \in \Delta$, let e_α denote a root vector and let $\mathfrak{g}_\alpha$ be its root space. If a subspace $U \subset \mathfrak{g}$ is a direct sum of root spaces and possibly some subspace of $\mathfrak{h}$, let $\Delta(U)$ denote the corresponding roots. Thus, $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\Delta(\mathfrak{b}) = \Delta^+$. If $\mathfrak{g}$ is simple, the unique highest root is $\lambda_\mathfrak{g}$. The Killing form B induces a symmetric pairing $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$, which determines the Cartan matrix $c_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \in \mathbb{Z}$ (where $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$) and Dynkin diagram $\mathcal{D}(\mathfrak{g})$. The nodes $\mathcal{N}(j) = \{i \mid c_{ij} \leq -1\}$ are the neighbours of the j-th node of $\mathcal{D}(\mathfrak{g})$. Let $\{\lambda_1, ..., \lambda_\ell\} \subset \mathfrak{h}^*$ denote the fundamental weights of $\mathfrak{g}$, i.e. $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. Given a weight $\lambda \in \mathfrak{h}^*$, we have $\lambda = \sum_i r_i(\lambda) \lambda_i$, where $r_i(\lambda) = \langle \lambda, \alpha_i^\vee \rangle$, and λ is $\mathfrak{g}$-dominant iff $r_i(\lambda) \in \mathbb{Z}_{\geq 0}$, $\forall i$. Encode λ on $\mathcal{D}(\mathfrak{g})$ by inscribing $r_i(\lambda)$ over the i-th node. The Cartan matrix is the transition matrix between $\{\alpha_i\}$ and $\{\lambda_i\}$, i.e. if $\lambda = \sum_i m_i \alpha_i = \sum_i r_i \lambda_i$, then $\sum_i m_i c_{ij} = r_j$.

To each (standard) parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$, there is an index set $I_\mathfrak{p} := \{i \mid \mathfrak{g}_{-\alpha_i} \not\subseteq \mathfrak{p}\} \subseteq \{1, ..., \ell\}$. Conversely, given $I \subset \{1, ..., \ell\}$, the corresponding parabolic is $\mathfrak{p}_I$, which is the sum of $\mathfrak{b}$ and the negative root spaces $\mathfrak{g}_{-\alpha}$ with $\alpha = \sum_i m_i \alpha_i$ satisfying $m_j = 0$, $\forall j \in I_\mathfrak{p}$. We encode $\mathfrak{p}$ on $\mathcal{D}(\mathfrak{g})$ by crossing all nodes in $I_\mathfrak{p}$, and denote this by $\mathcal{D}(\mathfrak{g}, \mathfrak{p})$. Thus, $I_\mathfrak{b} = \{1, ..., \ell\}$ and all nodes are crossed. The grading element is $Z = Z_{I_\mathfrak{p}}$, the $\mathbb{Z}$-grading is $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j = \mathfrak{g}_- \oplus \mathfrak{p}$ with $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [Z, x] = jx\}$, and $\mathfrak{z}(\mathfrak{g}_0)$ has basis $\{Z_i\}_{i \in I_\mathfrak{p}}$. The grading has depth $\nu = \max\{Z(\lambda_{\mathfrak{g}'}) \mid \mathfrak{g}' \subset \mathfrak{g} \text{ simple ideal}\}$. Given $\lambda \in \mathfrak{h}^*$, its *homogeneity* is $Z(\lambda)$. If $r_i(\lambda) \in \mathbb{Z}_{\geq 0}$, $\forall i \in \{1, ..., \ell\} \backslash I_\mathfrak{p}$, then λ is $\mathfrak{p}$-dominant.

Given $S \subset \{1, ..., \ell\}$, the tuple $\widetilde{Z}_S := (Z_k)_{k \in S}$ induces the multi-grading $\mathfrak{g} = \bigoplus_A \mathfrak{g}_A$, where $A = (a_k)_{k \in S} \in \mathbb{Z}^S$ is a multi-index, and $\mathfrak{g}_A = \{X \in \mathfrak{g} \mid [Z_k, X] = a_k X, \forall k \in S\}$. (Note $\mathfrak{g}_0$ with respect to Z_S agrees with $\mathfrak{g}_\mathbf{0}$ with respect to $\widetilde{Z}_S$. We use the boldface $\mathbf{0}$ when emphasizing the multi-grading.) Then $\mathfrak{z}(\mathfrak{g}_\mathbf{0})$ is spanned by $\{Z_k\}_{k \in S}$, and an important result [49, Thm.8.13.3] is that $\mathfrak{g}_A$ is a $\mathfrak{g}_\mathbf{0}$-irrep, $\forall A \neq \mathbf{0}$.

The Weyl group $W \subset \text{O}(\mathfrak{h}^*)$ preserves $\langle \cdot, \cdot \rangle$ and is generated by simple reflections σ_i, where $\sigma_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i$. We write $(ij) := \sigma_i \circ \sigma_j$. Given $w \in W$, let $|w|$ denote its length. The affine W-action is $w \cdot \lambda := w(\lambda + \rho) - \rho$, where $\rho = \sum_{i=1}^\ell \lambda_i$. The inversion set $\Phi_w = w(\Delta^-) \cap \Delta^+$ satisfies $|\Phi_w| = |w|$ and $\sum_{\alpha \in \Phi_w} \alpha = -w \cdot 0$. The *Hasse diagram* is $W^\mathfrak{p} = \{w \in W \mid \Phi_w \subset \Delta(\mathfrak{g}_+)\}$ with length r elements $W^\mathfrak{p}(r)$. (Equivalently, $w \in W^\mathfrak{p}$ sends $\mathfrak{g}$-dominant weights to $\mathfrak{p}$-dominant weights.) Every $w \in W^\mathfrak{p}$ corresponds precisely to an element of the W-orbit through $\rho^\mathfrak{p} = \sum_{i \in I_\mathfrak{p}} \lambda_i$ under the *right* action $(\rho^\mathfrak{p}, w) \mapsto w^{-1} \rho^\mathfrak{p}$. In particular, $W^\mathfrak{p}(2)$ is efficiently described by Recipe 3.

Dynkin diagram recipes will play an important role in our analysis – see Appendix B and §3.3. To compute $H^2(\mathfrak{g}_-, \mathfrak{g})$ via Kostant's theorem (see Recipe 4), we adopt the commonly used:

> **"Minus lowest weight" convention:** We let $\mathbb{V}_\mu$ denote the unique $\mathfrak{g}_0$-irrep with *lowest* weight μ, so $-\mu$ is $\mathfrak{p}$-dominant, and $\forall i \in I_\mathfrak{p}$, Z_i acts by the scalar $Z_i(\mu)$. We denote $\mathbb{V}_\mu$ by the Dynkin diagram notation for $-\mu$. The *homogeneity* of $\mathbb{V}_\mu$ is $Z(\mu)$.

By Kostant's theorem, each $\mathfrak{g}_0$-irrep in $H^2(\mathfrak{g}_-, \mathfrak{g})$ is of the form $\mathbb{V}_\mu$, where $\mu = -w \cdot \lambda$, with λ the highest weight of a simple ideal in $\mathfrak{g}$, and $w \in W^\mathfrak{p}(2)$. (In particular, $w \cdot \lambda$ is $\mathfrak{p}$-dominant.)

Given $\mu \in \mathfrak{h}^*$ where $-\mu$ is $\mathfrak{p}$-dominant, let

$$(3.1) \qquad J_\mu := \{j \notin I_\mathfrak{p} \mid \langle \mu, \alpha_j^\vee \rangle \neq 0\} \qquad \text{(uncrossed nodes with a nonzero coefficient)},$$

$$(3.2) \qquad I_\mu := \{j \in I_\mathfrak{p} \mid \langle \mu, \alpha_j^\vee \rangle = 0\} \qquad \text{(crossed nodes with a zero coefficient)}.$$

We augment our Dynkin diagram notation for $-\mu$ on $\mathcal{D}(\mathfrak{g}, \mathfrak{p})$ by putting a *square* around I_μ nodes, an *asterisk* on J_μ nodes, and denote this by $\mathcal{D}(\mathfrak{g}, \mathfrak{p}, \mu)$. Define $\mathfrak{U}_\mu := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in \mathbb{V}_\mu\}$, so that by Proposition 3.1.1, if ϕ_0 is a lowest weight vector of $\mathbb{V}_\mu$, then

$$\mathfrak{U}_\mu = \dim(\mathfrak{a}(\mu)), \qquad \mathfrak{a}(\mu) := \mathfrak{a}^{\phi_0} = \mathrm{pr}^\mathfrak{g}(\mathfrak{g}_-, \mathfrak{ann}(\phi_0)).$$

If $\mathfrak{g}$ is *simple*, then for $w \in W^\mathfrak{p}(2)$ and $\mu := -w \cdot \lambda_\mathfrak{g}$, we define $J_w := J_\mu$, $I_w := I_\mu$, $\mathfrak{a}(w) := \mathfrak{a}(\mu)$, and $\mathfrak{U}_w := \mathfrak{U}_\mu$. Let $W_+^\mathfrak{p}(2) := \{w \in W^\mathfrak{p}(2) \mid Z(-w \cdot \lambda_\mathfrak{g}) \geq 1\}$, so that $H_+^2(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{w \in W_+^\mathfrak{p}(2)} \mathbb{V}_{-w \cdot \lambda_\mathfrak{g}}$.

3.3. A Dynkin diagram recipe. In this section, we work over $\mathbb{C}$.

Lemma 3.3.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra. Let $\mathfrak{a}_0 \subset \mathfrak{g}_0$ be a subalgebra and $\mathfrak{a} := \mathrm{pr}^\mathfrak{g}(\mathfrak{g}_-, \mathfrak{a}_0)$.*

(a) For $k \geq 0$, if $\mathfrak{a}_k = 0$, then $\mathfrak{a}_{k+1} = 0$.[10]

(b) If $\mathfrak{a}_0$ is an $\mathfrak{h}$-module, then $\mathfrak{a}_k$ is a direct sum of root spaces, $\forall k > 0$.

Proof. Let $X \in \mathfrak{a}_{k+1}$, so $[X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_k = 0$. By [11, Prop. 3.1.2 (5)], $X = 0$, so we obtain claim (a).

Since $\mathfrak{a}_0$ and $\mathfrak{g}_-$ are $\mathfrak{h}$-modules, then so is $\mathfrak{a}_k$, $\forall k > 0$. Fix $k > 0$. All root spaces are 1-dimensional, so let $v = \sum_{\alpha \in \mathcal{A}} e_\alpha \in \mathfrak{a}_k$ be a sum of root vectors, where $\mathcal{A} \subset \Delta(\mathfrak{g}_k)$. We show by induction on $N = |\mathcal{A}|$ that $e_\alpha \in \mathfrak{a}_k$, $\forall \alpha \in \mathcal{A}$. The result is trivial for $N = 1$, so assume the general case. For any $h \in \mathfrak{h}$, $[h, v] = \sum_{\alpha \in \mathcal{A}} \alpha(h) e_\alpha \in \mathfrak{a}_k$. Fix $\beta \in \mathcal{A}$. Since all $\alpha \in \Delta^+$ (and hence $\mathcal{A}$) are distinct, then $\exists h \in \mathfrak{h}$ so that $\alpha(h) - \beta(h) \neq 0$, $\forall \alpha \in \mathcal{A} \backslash \{\beta\}$. Hence, $[h, v] - \beta(h) v = \sum_{\alpha \in \mathcal{A} \backslash \{\beta\}} (\alpha(h) - \beta(h)) e_\alpha \in \mathfrak{a}_k$. The induction hypothesis implies claim (b). $\qquad \square$

Consider $(\mathfrak{g}, \mathfrak{p})$ and $-\mu$ a $\mathfrak{p}$-dominant weight. The $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ is induced by $Z_{I_\mathfrak{p}}$, but in general $\mathfrak{g}_j$ are not $\mathfrak{g}_0$-irreducible. Each $\mathfrak{g}_j$ decomposes into $\mathfrak{g}_0$-irreps using $\widetilde{Z}_{I_\mathfrak{p}}$. Each of these further decomposes into $\mathfrak{g}_{0,0}$-irreps using $\widetilde{Z}_{J_\mu}$. Writing $1_i = (\delta_{ij})_{j \in I_\mathfrak{p}} \in \mathbb{Z}^{I_\mathfrak{p}}$, we observe that for $\mathfrak{g}_1$, we have the *irreducible* decompositions:

$$(3.3) \qquad \begin{aligned} &\text{As } \mathbf{\mathfrak{g}_0}\text{-modules:} \quad \mathfrak{g}_1 = \bigoplus_{i \in I} \mathfrak{g}_{1_i}, \quad \Delta(\mathfrak{g}_{1_i}) = \{\alpha \in \Delta(\mathfrak{g}_1) \mid Z_i(\alpha) = 1\}; \\ &\text{As } \mathbf{\mathfrak{g}_{0,0}}\text{-modules:} \quad \mathfrak{g}_{1_i} = \bigoplus_{A \geq \mathbf{0}} \mathfrak{g}_{1_i, A}, \quad \Delta(\mathfrak{g}_{1_i, A}) = \{\alpha \in \Delta(\mathfrak{g}_{1_i}) \mid \widetilde{Z}_{J_\mu}(\alpha) = A\}. \end{aligned}$$

Here "$A \geq \mathbf{0}$" means *all* entries are non-negative. The unique lowest root in $\Delta(\mathfrak{g}_{1_i, \mathbf{0}})$ is α_i.

Definition 3.3.2. *If $\mathcal{A}, \mathcal{B} \subset \Delta$, define $\mathcal{A} \dot{+} \mathcal{B} := \{\alpha + \beta \mid \alpha \in \mathcal{A}, \beta \in \mathcal{B}\} \cap \Delta$.*

Theorem 3.3.3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra. If $-\mu$ is a $\mathfrak{p}$-dominant weight and $\mathfrak{a} := \mathfrak{a}(\mu)$, then*

(a) $\mathfrak{a}_k$ is a direct sum of root spaces, $\forall k > 0$;

---

[10]Recall that for $k = 0$, we assume under "Conventions" that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$.

(b) $\Delta(\mathfrak{a}_1) = \bigcup_{i \in I_\mu} \Delta(\mathfrak{g}_{1i,\mathbf{o}})$. *In particular, $I_\mu = \emptyset$ iff $\mathfrak{a}_+ = 0$.*

(c) *If $I_\mu \neq \emptyset$, then $\mathfrak{a}_1$ is bracket-generating in $\mathfrak{a}_+$, and*

$$\Delta(\mathfrak{a}_k) = \{\alpha \in \Delta(\mathfrak{g}_k) \mid Z_{I_\mu}(\alpha) = k,\ Z_{J_\mu}(\alpha) = 0\}, \qquad k \geq 1.$$

In particular, $\forall \alpha \in \Delta(\mathfrak{a}_k)$, we have $Z_j(\alpha) = 0$, $\forall j \in (I_\mathfrak{p} \backslash I_\mu) \cup J_\mu$.

Proof. From Recipe 5, $\mathfrak{a}_0 = \mathfrak{ann}(\phi_0) = \{H \in \mathfrak{h} \mid \mu(H) = 0\} \oplus \bigoplus_{\gamma \in \Delta(\mathfrak{g}_{0,\leq 0})} \mathfrak{g}_\gamma$ is an $\mathfrak{h}$-module, so claim (a) follows from Lemma 3.3.1.

Recall that $\forall 0 \neq \gamma \in \mathfrak{h}^*$, $\exists! H_\gamma \in \mathfrak{h}$ such that $\gamma(H) = B(H, H_\gamma)$. Hence, $\alpha(H_\gamma) = \gamma(H_\alpha) = B(H_\alpha, H_\gamma) =: \langle \alpha, \gamma \rangle$. Any root vectors $e_\gamma, e_{-\gamma}$ satisfy $[e_\gamma, e_{-\gamma}] = n_\gamma H_\gamma$, where $n_\gamma = B(e_\gamma, e_{-\gamma}) \neq 0$.

Now $\mathfrak{a}_1 = \{X \in \mathfrak{g}_1 \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_0\}$, so test $\{e_\alpha\}_{\alpha \in \Delta(\mathfrak{g}_1)}$. Let $\beta \in \Delta(\mathfrak{g}_1)$. Using $\mathfrak{a}_0$ from (B.2),

(i) $\beta = \alpha$: $[e_\alpha, e_{-\alpha}] = n_\alpha H_\alpha$. Thus, if $H_\alpha \notin \mathfrak{a}_0$, i.e. $\mu(H_\alpha) = \langle \mu, \alpha \rangle \neq 0$, then $\alpha \notin \Delta(\mathfrak{a}_1)$.

(ii) $\beta \neq \alpha$: If $\alpha - \beta \notin \Delta$, then $[e_\alpha, e_{-\beta}] = 0$ (so no constraints). Otherwise, if $\alpha - \beta \in \Delta$, then $[e_\alpha, e_{-\beta}]$ is a nonzero multiple of $e_{\alpha-\beta}$. Thus, $\alpha - \beta \in \Delta(\mathfrak{g}_{0,+})$ iff $e_{\alpha-\beta} \notin \mathfrak{a}_0$ iff $\alpha \notin \Delta(\mathfrak{a}_1)$.

Defining $\mathcal{T}_1 = \{\alpha \in \Delta(\mathfrak{g}_1) \mid \langle \mu, \alpha^\vee \rangle \neq 0\}$ and $\mathcal{T}_2 = \{\alpha \in \Delta(\mathfrak{g}_1) \mid \exists \beta \in \Delta(\mathfrak{g}_1) \text{ with } \alpha - \beta \in \Delta(\mathfrak{g}_{0,+})\}$, we have $\Delta(\mathfrak{a}_1) = \Delta(\mathfrak{g}_1) \backslash (\mathcal{T}_1 \cup \mathcal{T}_2)$.

Fix $i \in I_\mathfrak{p}$. By Schur's lemma, for each $\mathfrak{g}_{1i,A}$ in (3.3), either $\mathfrak{g}_{1i,A} \subset \mathfrak{a}_1$ or $\mathfrak{a}_1 \cap \mathfrak{g}_{1i,A} = 0$. By $\mathfrak{g}_{0,0}$-irreducibility, it suffices to test the lowest root in $\Delta(\mathfrak{g}_{1i,A})$.

(i) $A > \mathbf{0}$ (i.e. *at least* one entry of A is positive): The lowest $\alpha \in \Delta(\mathfrak{g}_{1i,A})$ is not the lowest in $\Delta(\mathfrak{g}_{1i})$, i.e. $\alpha \neq \alpha_i$. Since $\mathfrak{g}_{1i}$ is a $\mathfrak{g}_0$-irrep, there are simple roots $\{\alpha_{j_k}\}_{k=1}^m \subset \Delta(\mathfrak{g}_0)$ such that $\beta_k := \alpha - \alpha_{j_1} - \ldots - \alpha_{j_k} \in \Delta$, $\forall k$ and $\beta_m = \alpha_i$. In particular, $\beta_1 = \alpha - \alpha_{j_1} \in \Delta(\mathfrak{g}_1)$. Since α is the lowest in $\Delta(\mathfrak{g}_{1i,A})$, then $j_1 \in J_\mu$ and $\alpha_{j_1} \in \Delta(\mathfrak{g}_{0,+})$. Hence, $\alpha \in \mathcal{T}_2$, so $\Delta(\mathfrak{g}_{1i,A}) \subset \mathcal{T}_2$.

(ii) $A = \mathbf{0}$: the lowest root of $\Delta(\mathfrak{g}_{1i,\mathbf{0}})$ is α_i, and clearly $\alpha_i \notin \mathcal{T}_2$. We have $i \notin I_\mu$ iff $\alpha_i \in \mathcal{T}_1$.

Thus, claim (b) follows. (If $I_\mu = \emptyset$, then $\mathfrak{a}_1 = 0$, so $\mathfrak{a}_+ = 0$ by Lemma 3.3.1.)

Let $\mathcal{A}_k := \{\alpha \in \Delta(\mathfrak{g}_k) \mid Z_{I_\mu}(\alpha) = k,\ Z_{J_\mu}(\alpha) = 0\}$. Note $\Delta(\mathfrak{a}_1) = \mathcal{A}_1$ follows from (b). Let $k \geq 2$ and $\gamma \in \Delta(\mathfrak{a}_k) \subset \Delta(\mathfrak{g}_k)$. Then $\gamma = \alpha + \beta$ for some $\alpha \in \Delta(\mathfrak{g}_{k-1})$ and $\beta \in \Delta(\mathfrak{g}_1)$, since $\mathfrak{g}_1$ is bracket-generating in $\mathfrak{g}_+$. By definition, $\mathfrak{a}_k = \{X \in \mathfrak{g}_k \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{k-1}\}$, so $\alpha = \gamma - \beta \in \Delta(\mathfrak{a}_{k-1})$. Since $\mathfrak{a}$ is graded, then $[\mathfrak{a}_k, \mathfrak{a}_{-(k-1)}] \subset \mathfrak{a}_1$, so $\beta = \gamma - \alpha \in \Delta(\mathfrak{a}_1)$. Thus, $\Delta(\mathfrak{a}_k) \subseteq \Delta(\mathfrak{a}_{k-1}) \dot{+} \Delta(\mathfrak{a}_1) \subseteq \mathcal{A}_k$.

Use induction on k to show that $\Delta(\mathfrak{a}_k) = \mathcal{A}_k$. Let $\gamma \in \mathcal{A}_k$ and $\beta \in \Delta(\mathfrak{g}_1)$. If $\alpha = \gamma - \beta \in \Delta$, then $\alpha \in \Delta(\mathfrak{g}_{k-1})$. Since $\alpha \in \Delta^+$, then $0 \leq Z_{I_\mathfrak{p} \backslash I_\mu}(\alpha) \leq Z_{I_\mathfrak{p} \backslash I_\mu}(\gamma) = 0$ and $0 \leq Z_{J_\mu}(\alpha) \leq Z_{J_\mu}(\gamma) = 0$, so $Z_{I_\mathfrak{p} \backslash I_\mu}(\beta) = 0$, $Z_{I_\mu}(\beta) = 1$, $Z_{J_\mu}(\beta) = 0$. Thus, $\alpha \in \mathcal{A}_{k-1}$, so $\alpha \in \Delta(\mathfrak{a}_{k-1})$ by induction. Hence, $\gamma \in \Delta(\mathfrak{a}_k)$. Thus, $\Delta(\mathfrak{a}_k) = \Delta(\mathfrak{a}_{k-1}) \dot{+} \Delta(\mathfrak{a}_1) = \mathcal{A}_k$ for $k \geq 2$. This proves claim (c). $\qquad \square$

From $\mathcal{D}(\mathfrak{g})$, remove the nodes in $I_\mathfrak{p} \backslash I_\mu$ and J_μ. In this diagram, any node $j \notin I_\mu$ that is not in the connected component of an I_μ node corresponds to a simple root $\alpha_j \in \Delta(\mathfrak{g}_{0,0})$ with $\Delta(\mathfrak{a}_1) \dot{+} \{\alpha_j\} = \emptyset$. Such nodes play no role in the description of $\Delta(\mathfrak{a}_1)$ (and hence $\Delta(\mathfrak{a}_k)$ for $k > 0$), so can be removed. Thus, to the data $(\mathfrak{g}, \mathfrak{p}, \mu)$ when $I_\mu \neq \emptyset$, we associate a *reduced geometry* $(\overline{\mathfrak{g}}, \overline{\mathfrak{p}})$ via Recipe 7, and it is clear that $\Delta(\mathfrak{a}_+(\mu))$ and $\Delta(\overline{\mathfrak{g}}_+)$ naturally correspond, so $\mathfrak{U}_\mu = \dim(\mathfrak{a}(\mu))$ is easily computable. In Theorem 4.1.6, we show that this symmetry dimension is almost always realizable.

When $\mathfrak{g}$ is simple, write $I_w := I_{-w \cdot \lambda_\mathfrak{g}}$, $\mathfrak{a}(w) := \mathfrak{a}(-w \cdot \lambda_\mathfrak{g})$, etc. For G_2/P_1, see Example B.1.

Example 3.3.4 (E_8/P_8, $w = (87)$). $\lambda_\mathfrak{g} = \lambda_8$ *and* $w \cdot \lambda_\mathfrak{g} = \lambda_6 + \lambda_7 - 4\lambda_8$, *so*

$$\mathbb{V}_{-w \cdot \lambda_\mathfrak{g}} = \overset{0}{\circ} \!\!-\!\! \overset{0}{\circ} \!\!-\!\! \overset{0}{\underset{\underset{0}{|}}{\circ}} \!\!-\!\! \overset{0}{\circ} \!\!-\!\! \overset{1}{\color{red}*} \!\!-\!\! \overset{1}{\color{red}*} \!\!-\!\! \overset{-4}{\times} , \qquad J_w = \{6, 7\}, \quad I_w = \emptyset.$$

By Recipe 1, $\dim(\mathfrak{g}_0) = 1 + \dim(E_7) = 134$, *so* $\dim(\mathfrak{g}_-) = \frac{1}{2}(\dim(E_8) - \dim(\mathfrak{g}_0)) = \frac{1}{2}(248 - 134) = 57$. *By Recipe 5,* $\mathfrak{p}_w^{\mathrm{op}} \cong \mathfrak{p}_{6,7} \subset E_7$ *with* $\dim(\mathfrak{a}_0) = \dim(\mathfrak{p}_w^{\mathrm{op}}) = \frac{1}{2}(\dim(E_7) + 2 + \dim(D_5)) = 90$. *Since* $I_w = \emptyset$, *then* $\mathfrak{a}_+ = 0$ *(Recipe 6). Thus,* $\dim(\mathfrak{a}(w)) = 57 + 90 = 147$.

Example 3.3.5 ($C_6/P_{1,2}$, $w = (21)$). $\lambda_{\mathfrak{g}} = 2\lambda_1$ and $w \cdot \lambda_{\mathfrak{g}} = -5\lambda_2 + 4\lambda_3$, so $J_w = \{3\}$, $I_w = \{1\}$:

$$\mathbb{V}_{-w\cdot\lambda_{\mathfrak{g}}} = \overset{0}{\boxtimes}\!\!\overset{-5}{-\!\!\!\times\!\!\!-}\!\!\overset{4}{\underset{*}{-\!\!-}}\!\!\overset{0}{-\!\!\circ\!\!-}\!\!\overset{0}{-\!\!\circ}\!\!\overset{0}{\Longleftarrow\!\!\circ} \quad \overset{Recipe\ 7}{\rightsquigarrow} \quad \times \qquad (\bar{\mathfrak{g}}/\bar{\mathfrak{p}} = A_1/P_1).$$

Note $\Delta(\mathfrak{a}_1) = \{\alpha_1\}$. *We have* $\dim(\mathfrak{g}_-) = \frac{1}{2}(\dim(C_6) - 2 - \dim(C_4)) = \frac{1}{2}(78 - 2 - 36) = 20$. *Since* $\mathfrak{p}_w^{op} \cong \mathfrak{p}_1 \subset C_4$, *then* $\dim(\mathfrak{a}_0) = \dim(\mathfrak{p}_w^{op}) = 30$. *Thus,* $\dim(\mathfrak{a}(w)) = 20 + 30 + 1 = 51$.

Example 3.3.6 ($C_6/P_{1,2,5}$, $w = (21)$). *As above, but now* $J_w = \{3\}$, $I_w = \{1,5\}$:

$$\mathbb{V}_{-w\cdot\lambda_{\mathfrak{g}}} = \overset{0}{\boxtimes}\!\!\overset{-5}{-\!\!\!\times\!\!\!-}\!\!\overset{4}{\underset{*}{-\!\!-}}\!\!\overset{0}{-\!\!\circ\!\!-}\!\!\overset{0}{\boxtimes}\!\!\overset{0}{\Longleftarrow\!\!\circ} \quad \overset{Recipe\ 7}{\rightsquigarrow} \quad \times \quad \circ\!\!-\!\!-\!\!\times\!\!\Longleftarrow\!\!\circ \qquad (\bar{\mathfrak{g}}/\bar{\mathfrak{p}} = A_1/P_1 \times C_3/P_2).$$

Note $\bar{\mathfrak{g}}$ *is 2-graded, with* $\dim(\mathfrak{a}_1) = \dim(\bar{\mathfrak{g}}_1) = 5$ *and* $\dim(\mathfrak{a}_2) = \dim(\bar{\mathfrak{g}}_2) = 3$. *Indeed,*

$$\Delta(\mathfrak{a}_1): \quad \alpha_1, \quad \alpha_5, \quad \alpha_4 + \alpha_5, \quad \alpha_5 + \alpha_6, \quad \alpha_4 + \alpha_5 + \alpha_6;$$
$$\Delta(\mathfrak{a}_2): \quad 2\alpha_5 + \alpha_6, \quad \alpha_4 + 2\alpha_5 + \alpha_6, \quad 2\alpha_4 + 2\alpha_5 + \alpha_6.$$

We calculate $\dim(\mathfrak{a}(w)) = \dim(\mathfrak{g}_-) + \dim(\mathfrak{a}_0) + \dim(\mathfrak{a}_+) = 32 + 11 + 8 = 51$. *By Theorem 3.5.4, it is no coincidence that this agrees with the result of the previous example. Also, the property* $\mathfrak{a}_2(w) \neq 0$ *is rather exceptional: if* $\mathfrak{g}$ *is simple and* $w \in W_+^{\mathfrak{p}}(2)$, *then* $\mathfrak{a}_2(w) = 0$ *almost always (Theorem 3.4.9).*

3.4. **Prolongation-rigidity.** For regular, normal parabolic geometries of type (G, P), harmonic curvature κ_H is valued in $H_+^2(\mathfrak{g}_-, \mathfrak{g})$. If $H_+^2(\mathfrak{g}_-, \mathfrak{g}) = 0$, then $\kappa_H \equiv 0$, so the geometry is flat: we say $(\mathfrak{g}, \mathfrak{p})$ is *Yamaguchi-rigid*; otherwise, $(\mathfrak{g}, \mathfrak{p})$ is *Yamaguchi-nonrigid*. For our study of symmetry gaps, we need only study the latter, and when G is simple these were classified by Yamaguchi [50, 51] (see Appendix A).

Definition 3.4.1. *If* $\mathfrak{g}$ *is complex semisimple and* $-\mu$ *is* $\mathfrak{p}$*-dominant, then* $(\mathfrak{g}, \mathfrak{p}, \mu)$ *(or* $(\mathfrak{g}, \mathfrak{p}, w)$ *with* $\mu = -w \cdot \lambda_{\mathfrak{g}}$ *if* $\mathfrak{g}$ *is simple) is PR (prolongation-rigid) if* $\mathfrak{a}_+^{\phi} = 0$ *whenever* $0 \neq \phi \in \mathbb{V}_{\mu}$; *otherwise,* $(\mathfrak{g}, \mathfrak{p}, \mu)$ *is NPR.*[11] *If* $\mathfrak{g}$ *is real or complex,* $(\mathfrak{g}, \mathfrak{p})$ *is PR if* $\mathfrak{a}_+^{\phi} = 0$ *whenever* $0 \neq \phi \in H_+^2(\mathfrak{g}_-, \mathfrak{g})$. *Otherwise, it is NPR.*

Remark 3.4.2. The following observations are immediate:
   (1) $(\mathfrak{g}, \mathfrak{p})$ is PR if and only if $(\mathfrak{g}, \mathfrak{p}, \mu)$ is PR for all $\mathbb{V}_{\mu} \subset H_+^2(\mathfrak{g}_-, \mathfrak{g})$.
   (2) Any Yamaguchi-rigid $(\mathfrak{g}, \mathfrak{p})$ has $H_+^2(\mathfrak{g}_-, \mathfrak{g}) = 0$, so is (vacuously) PR.
   (3) If $(\mathfrak{g}, \mathfrak{p})$ is real and is NPR, then its complexification $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{p}^{\mathbb{C}})$ will also be NPR.

Lemma 3.4.3. $(\mathfrak{g}, \mathfrak{p}, \mu)$ *is PR iff* $I_{\mu} = \emptyset$ *iff* $\mathcal{D}(\mathfrak{g}, \mathfrak{p}, \mu)$ *has no squares. (This is Recipe 6.)*

Proof. This follows immediately from part (b) of Theorem 3.3.3. $\qquad\square$

Example 3.4.4. *Since* $F_4/P_{1,2}$ *is Yamaguchi-rigid (see Theorem A.2), it is PR. However,* $w = (21) \in W^{\mathfrak{p}}(2) \backslash W_+^{\mathfrak{p}}(2)$ *gives the (non-regular) irrep* $\mathbb{V}_{-w\cdot\lambda_1} = \overset{0}{\boxtimes}\!\!\overset{-4}{-\!\!\!\times\!\!\!\Rightarrow\!\!\!-}\!\!\overset{6}{\underset{*}{-}}\!\!\overset{0}{-\!\!\circ}$, *so* $(\mathfrak{g}, \mathfrak{p}, w)$ *is NPR. Its reduced geometry is* A_1/P_1, *so* $\mathfrak{a} = \mathfrak{a}(w)$ *has 1-dimensional* $\mathfrak{a}_+ = \mathfrak{a}_1$.

In terms of the grading element $Z = Z_{I_{\mathfrak{p}}}$, we have:

Lemma 3.4.5. *Suppose* $\lambda = \sum_a r_a \lambda_a$ *is the highest weight of a simple ideal of* $\mathfrak{g}$. *Let* $w = (jk) \in W^{\mathfrak{p}}(2)$ *and* $\mu = -w \cdot \lambda$. *The regularity condition* $Z(\mu) \geq 1$ *is equivalent to:*

$$(3.4) \qquad\qquad Z(\lambda) \leq r_j + (r_k + 1)(Z(\alpha_k) - c_{kj}).$$

Independent of regularity, we have $i \in I_{\mu}$ *iff*

$$(3.5) \qquad \begin{cases} \text{(a)}: & i, j \in I_{\mathfrak{p}} \text{ distinct}, \quad k \in \mathcal{N}(j) \backslash I_{\mathfrak{p}}, \quad 0 = r_i = c_{ji} = c_{ki}; \quad \text{or,} \\ \text{(b)}: & i, j, k \in I_{\mathfrak{p}} \text{ distinct}, \quad 0 = r_i = c_{ji} = c_{ki}; \quad \text{or,} \\ \text{(c)}: & i, j \in I_{\mathfrak{p}} \text{ distinct}, \quad k = i, \quad c_{ij} = c_{ji} = -1, \quad r_j = 0. \end{cases}$$

---

[11]Here, μ is *not* assumed to have positive homogeneity.

Proof. Since $\lambda \in \Delta^+$, then $r_a = \langle \lambda, \alpha_a^\vee \rangle \geq 0$, $\forall a$. By Recipe 4, $\Phi_w = \{\alpha_j, \sigma_j(\alpha_k)\}$, so:

$$
(3.6) \qquad -\mu = w \cdot \lambda = w(\lambda) + w \cdot 0 = w(\lambda) - \sum_{\alpha \in \Phi_w} \alpha = \sigma_j(\sigma_k(\lambda)) - \alpha_j - \sigma_j(\alpha_k)
$$
$$
= \sigma_j(\lambda - r_k \alpha_k) - \alpha_j - \sigma_j(\alpha_k) = \lambda - (r_j + 1)\alpha_j - (r_k + 1)(\alpha_k - c_{kj}\alpha_j).
$$

Simplify $Z(\mu) \geq 1$ using $j \in I_{\mathfrak{p}}$ (Recipe 3) to obtain (3.4).

Now $i \in I_\mu$ iff $\langle \mu, \alpha_i^\vee \rangle = 0$, so using (3.6),

$$
(3.7) \qquad\qquad 0 = r_i - (r_j + 1)c_{ji} - (r_k + 1)(c_{ki} - c_{kj}c_{ji}).
$$

If $i = j$, then $c_{ji} = 2$ and $c_{kj} \leq 0$ (since $k \neq j$ by Recipe 3), so (3.7) implies $0 = r_j + 2 - (r_k+1)c_{kj} \geq 2$, a contradiction. Thus, we take $i \neq j$ below, i.e. $c_{ij}, c_{ji} \leq 0$.

From Recipe 3, $j \in I_{\mathfrak{p}}$ and $j \neq k \in I_{\mathfrak{p}} \cup \mathcal{N}(j)$, where $\mathcal{N}(j) = \{i \mid c_{ij} \leq -1\}$.

- If $k \in \mathcal{N}(j) \backslash I_{\mathfrak{p}}$ or $i, j, k \in I_{\mathfrak{p}}$ are distinct, then $c_{ki}, c_{kj} \leq 0$, so (3.7) implies

$$
0 = \underbrace{r_i}_{\geq 0} - \underbrace{(r_j + 1)}_{\geq 1} \underbrace{c_{ji}}_{\leq 0} - \underbrace{(r_k + 1)}_{\geq 1}\underbrace{(c_{ki} - c_{kj}c_{ji})}_{\leq 0} \quad \Rightarrow \quad r_i = c_{ji} = c_{ki} = 0.
$$

- If $k = i$, then (3.7) implies $(r_i + 1)(2 - c_{ij}c_{ji}) = r_i - (r_j + 1)c_{ji} \geq 0$. Thus, $0 \leq c_{ij}c_{ji} \leq 2$. If $c_{ij}c_{ji} = 0$, then $c_{ij} = c_{ji} = 0$, so $r_i = -2$. If $c_{ij}c_{ji} = 2$, then $r_i = c_{ji} = 0$. Both give contradictions, so $c_{ij}c_{ji} = 1$ remains, i.e. $c_{ij} = c_{ji} = -1$, and (3.7) reduces to $r_j = 0$.

Hence, (3.7) is equivalent to (3.5). $\qquad\square$

Example 3.4.6. *Condition* (3.4) *can be used to quickly calculate* $W_+^{\mathfrak{p}}(2)$. *Consider* $A_\ell/P_{1,2,s,t}$ *for* $4 \leq s < t < \ell$. *Since* $\lambda_{\mathfrak{g}} = \lambda_1 + \lambda_\ell = \alpha_1 + \ldots + \alpha_\ell$, *then* $Z(\lambda_{\mathfrak{g}}) = 4$. *Since* $r_j, r_k, Z(\alpha_k), -c_{kj} \in \{0, 1\}$, *then* (3.4) *forces* $r_k = Z(\alpha_k) = -c_{kj} = 1$, *so* $k = 1$, $j = 2$, *and* $W_+^{\mathfrak{p}}(2) = \{(21)\}$, *while* $|W^{\mathfrak{p}}(2)| = 11$.

Proposition 3.4.7. *Let* λ *be the highest weight of a simple ideal in* $\mathfrak{g}$. *Let* $w = (jk) \in W^{\mathfrak{p}}(2)$ *and* $\mu = -w \cdot \lambda$. *Then* $(\mathfrak{g}, \mathfrak{p}, \mu)$ *is PR if: (a)* $|I_{\mathfrak{p}}| = 1$, *or (b)* $I_{\mathfrak{p}} = \{j, k\}$ *and* $c_{jk} = 0$.

Proof. In both cases, it is impossible to satisfy (3.5), so $I_\mu = \emptyset$. Lemma 3.4.3 gives the result. $\quad\square$

Suppose $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}''$ is a product of simple ideals, $\mathfrak{p} = \mathfrak{p}' \times \mathfrak{p}''$, with $I_{\mathfrak{p}'}, I_{\mathfrak{p}''} \neq \emptyset$. Let $\lambda = \lambda_{\mathfrak{g}'}$. For any $i'' \in I_{\mathfrak{p}''}$, we have $r_{i''} := \langle \lambda, \alpha_{i''}^\vee \rangle = 0$. Pick any $w = (j'k') \in W^{\mathfrak{p}'}(2)$ and let $\mu = -w \cdot \lambda$. Then (3.5)(a) or (b) is satisfied, so $i'' \in I_\mu$, i.e. $(\mathfrak{g}, \mathfrak{p}, \mu)$ is NPR. Indeed, $\mathbb{V}_\mu \subset H^2(\mathfrak{g}_-, \mathfrak{g}') \boxtimes \mathbb{C}$ (external tensor product with trivial $\mathfrak{g}_0''$-action on $\mathbb{C}$), so $\mathfrak{a}(\mu) \supset \mathfrak{g}''$. If $(\mathfrak{g}', \mathfrak{p}')$ is Yamaguchi non-rigid, then w can be chosen so that $Z(\mu) \geq 1$. Thus, for most *semisimple* cases of interest, $(\mathfrak{g}, \mathfrak{p})$ is NPR.

If $\mathfrak{g}$ is *simple*, the NPR condition on $(\mathfrak{g}, \mathfrak{p})$ is more restrictive. In particular, the highest weight (root) $\lambda_{\mathfrak{g}} = \sum_{a=1}^\ell m_a \alpha_a$ satisfies $m_a \geq 1$, so $|I_{\mathfrak{p}}| \leq Z(\lambda_{\mathfrak{g}})$, which constrains the right side of (3.4).

Corollary 3.4.8. *Let* $\mathfrak{g}$ *be complex simple. Then* $(\mathfrak{g}, \mathfrak{p})$ *is PR if: (a)* $\mathfrak{g}$ *is 1-graded; (b)* $\mathfrak{g}$ *has a contact gradation; or (c)* $\mathfrak{g}$ *is a Lie algebra of exceptional type.*

Proof. All Yamaguchi-rigid $(\mathfrak{g}, \mathfrak{p})$ are PR. By Proposition 3.4.7, consider those Yamaguchi-nonrigid $(\mathfrak{g}, \mathfrak{p})$ with $|I_{\mathfrak{p}}| > 1$ and $\mathfrak{p} \neq \mathfrak{p}_{j,k}$ for $c_{jk} = 0$. Among 1-gradings and contact gradings, this leaves $A_2/P_{1,2}$ (see Theorem A.1), and among Lie algebras of exceptional type, this leaves $G_2/P_{1,2}$ (see Theorem A.2). Both fail (3.5). $\quad\square$

Hence, with the exception of pairs of 2nd order ODE $(A_3/P_{1,2})$, all known symmetry gap results for parabolic geometries (see Table 2) have been for PR geometries. (Note $C_2/P_{1,2}$ is also PR.) Table 4 gives a classification of NPR geometries. (See Appendix E for details.)

The *height* of (the Z-grading on) $\mathfrak{a}$ is the maximal $\tilde{\nu} \geq 0$ such that $\mathfrak{a}_{\tilde{\nu}} \neq 0$.

G	P	Range
A_ℓ	$P_{1,2}$	$\ell \geq 3$
	$P_{1,2,s}$	$3 \leq s \leq \ell$
	$P_{1,2,s,t}$	$3 \leq s < t \leq \ell$
	$P_{1,s,\ell}$	$3 \leq s \leq \ell - 2$
	$P_{i,i+1}$	$2 \leq i \leq \ell - 2$
	$P_{1,i}$	$4 \leq i \leq \ell - 1$
	$P_{2,i}$	$4 \leq i \leq \ell - 1$
B_ℓ	$P_{1,2}$	$\ell \geq 3$
	$P_{2,3}$	$\ell \geq 4$

G	P	Range
C_ℓ	$P_{1,2}$	$\ell \geq 3$
	$P_{1,\ell}$	$\ell \geq 4$
	$P_{2,\ell}$	$\ell \geq 4$
	$P_{1,2,s}$	$3 \leq s \leq \ell$
D_ℓ	$P_{1,2}$	$\ell \geq 4$
	$P_{1,\ell}$	$\ell \geq 5$
	$P_{2,3}$	$\ell \geq 5$
	$P_{1,2,\ell}$	$\ell \geq 4$

TABLE 4. Classification of NPR geometries G/P with G simple

Theorem 3.4.9. *Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra. If $(\mathfrak{g}, \mathfrak{p})$ is NPR, then the height of $\mathfrak{a}(w)$, where $w \in W_+^{\mathfrak{p}}(2)$, always satisfies $0 \leq \tilde{\nu} \leq 1$, except $\tilde{\nu} = 2$ for:*

Label	G/P	Range	w	J_w	I_w
NPR–A	$A_\ell/P_{1,2,s,t}$	$4 \leq s < t < \ell$	(21)	$\{3, \ell\}$	$\{1, s, t\}$
NPR–C	$C_\ell/P_{1,2,s}$	$4 \leq s < \ell$	(21)	$\{3\}$	$\{1, s\}$

Proof. Using Table 4, examine all NPR $(\mathfrak{g}, \mathfrak{p}, w)$, $w \in W_+^{\mathfrak{p}}(2)$, in Tables 12 and 13. The reduced geometry $(\bar{\mathfrak{g}}, \bar{\mathfrak{p}})$ is 1-graded in all cases, except for NPR-A and NPR-C where it is 2-graded. □

If we remove the regularity assumption, higher prolongations can exist:

Example 3.4.10. *For $\ell \geq 5$, consider $A_\ell/B = A_\ell/P_{1,2,\dots,\ell}$ and $w = (21) \in W^{\mathfrak{p}}(2) \backslash W_+^{\mathfrak{p}}(2)$. Then:*

$$\mathbb{V}_{-w \cdot \lambda_{\mathfrak{g}}} = \overset{0}{\boxtimes}\overset{-4}{\times}\overset{3}{\times}\overset{0}{\boxtimes}\cdots\overset{0}{\boxtimes}\overset{1}{\times}, \qquad Z(-w \cdot \lambda_{\mathfrak{g}}) = 5 - \ell.$$

Thus, $J_w = \emptyset$, $I_w = \{1, 4, 5, \dots, \ell - 1\}$, so $\bar{\mathfrak{g}}/\bar{\mathfrak{p}} \cong A_1/P_1 \times A_{\ell-4}/P_{1,\dots,\ell-4}$. Note $\nu = \ell$, while $\tilde{\nu} = \ell - 4$.

3.5. **Correspondence and twistor spaces.** We review correspondence and twistor space constructions from [7, 11]. Let G be a semisimple Lie group G, and $Q \subset P \subset G$ parabolic subgroups, so there is a projection $G/Q \to G/P$. Write the decompositions of $\mathfrak{g}$ corresponding to $\mathfrak{q}, \mathfrak{p}$ as

$$(3.8) \qquad \mathfrak{g} = \mathfrak{q}_- \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_+ = \mathfrak{p}_- \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_+,$$

where (see Lemma 4.4.1 in [11]),

$$(3.9) \qquad I_{\mathfrak{p}} \subset I_{\mathfrak{q}}, \qquad \mathfrak{p}_\pm \subset \mathfrak{q}_\pm, \qquad \mathfrak{q}_0 \subset \mathfrak{p}_0, \qquad \mathfrak{z}(\mathfrak{p}_0) \subset \mathfrak{z}(\mathfrak{q}_0).$$

Given a G/P geometry $(\mathcal{G} \to N, \omega)$, the *correspondence space* for $Q \subset P$ is $\mathcal{C}N = \mathcal{G}/Q$. This carries a canonical G/Q geometry $(\mathcal{G} \to \mathcal{C}N, \omega)$, and the fibers of $\mathcal{C}N \to N$ are diffeomorphic to P/Q. The vertical subbundle $V\mathcal{C}N \subset T\mathcal{C}N$ corresponds to the Q-submodule $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$, and the Cartan curvature $\kappa^{\mathcal{C}N}$ of $(\mathcal{G} \to \mathcal{C}N, \omega)$ satisfies $i_\xi \kappa^{\mathcal{C}N} = 0$ for any $\xi \in \Gamma(V\mathcal{C}N)$.

Conversely, given a G/Q geometry $(\mathcal{G} \to M, \omega)$, we may ask if there exists a G/P geometry $(\mathcal{G} \to N, \omega)$ such that M is locally isomorphic to $\mathcal{C}N$. If so, we call N the (local) *twistor space* for M. From above, a necessary condition is that the subbundle $VM \subset TM$ induced from $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ must satisfy $i_\xi \kappa = 0$ for all $\xi \in \Gamma(VM)$. Locally, this is sufficient, and moreover $(\mathcal{G} \to N, \omega)$ is unique when P/Q is assumed to be connected [7] (or Corollary 4.4.1 in [11]). More convenient still is that it suffices to only check that $\mathrm{im}(\kappa_H)$ vanishes on $\mathfrak{p} \cap \mathfrak{q}_-$, cf. Theorem 3.3 in [7].

Two additional nice properties of correspondence spaces include:

- $\mathfrak{inf}(\mathcal{G} \to N, \omega) = \mathfrak{inf}(\mathcal{G} \to \mathcal{C}N, \omega)$, provided P/Q is connected.
- $(\mathcal{G} \to N, \omega)$ is normal iff $(\mathcal{G} \to \mathcal{C}N, \omega)$ is normal.

In contrast to normality, regularity is not preserved in passing to a correspondence space.

Proposition 3.5.1. *Let G be a complex semisimple Lie group, and $Q \subset G$ a parabolic subgroup. Consider a G/Q geometry $(\mathcal{G} \to M, \omega)$ with $\mathrm{im}(\kappa_H) \subset \mathbb{V}_\mu$, where $\mu = -w \cdot \lambda$, with λ the highest weight of a simple ideal in $\mathfrak{g}$, and $w = (jk) \in W^{\mathfrak{q}}(2)$. Let (a) $\mathfrak{p} := \mathfrak{p}_j$ if $c_{kj} < 0$; or, (b) $\mathfrak{p} := \mathfrak{p}_{j,k}$ if $c_{kj} = 0$. Then $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ and $\mathrm{im}(\kappa_H)$ consists of cochains which vanish using $\mathfrak{p} \cap \mathfrak{q}_- \subset \mathfrak{p}_0$ insertions.*

Proof. By Recipe 3, $j \in I_{\mathfrak{q}}$ and $j \neq k \in I_{\mathfrak{q}} \cup \mathcal{N}(j)$. From Recipe 4, the lowest $\mathfrak{q}_0$-weight vector in the $\mathfrak{q}_0$-module $\mathbb{V}_\mu$ is $\phi_0 = e_{\alpha_j} \wedge e_{\sigma_j(\alpha_k)} \otimes e_{w(-\lambda)} \in \bigwedge^2 \mathfrak{q}_-^* \otimes \mathfrak{g}$.

If $c_{kj} < 0$, take $\mathfrak{p} := \mathfrak{p}_j$. Then $\sigma_j(\alpha_k) = \alpha_k - c_{kj}\alpha_j$ has positive $Z_{I_{\mathfrak{p}}}$-grading, so $\phi_0 \in \mathfrak{p}_1 \wedge \mathfrak{p}_+ \otimes \mathfrak{g}$. Now recall that the $\mathfrak{q}_0$-module $\mathbb{V}_\mu$ is generated by applying $\mathfrak{q}_0$-raising operators to ϕ_0. Since $\mathfrak{q}_0 \subset \mathfrak{p}_0$, and $\mathfrak{p}_1 \wedge \mathfrak{p}_+ \otimes \mathfrak{g}$ is $\mathfrak{p}_0$-invariant (hence $\mathfrak{q}_0$-invariant), then $\mathbb{V}_\mu \subset \mathfrak{p}_1 \wedge \mathfrak{p}_+ \otimes \mathfrak{g}$. Thus, any map in $\mathrm{im}(\kappa_H) \subset \mathbb{V}_\mu$ vanishes on $\mathfrak{p}_0$ and hence on $\mathfrak{p} \cap \mathfrak{q}_- \subset \mathfrak{p}_0$.

If $c_{kj} = 0$, take $\mathfrak{p} := \mathfrak{p}_{j,k}$. Then $\sigma_j(\alpha_k) = \alpha_k$ has positive $Z_{I_{\mathfrak{p}}}$-grading, and proceed as before. $\square$

Corollary 3.5.2. *Under the hypothesis of Proposition 3.5.1, if there exists a parabolic subgroup P with $Q \subset P \subset G$, and Lie algebra $\mathfrak{p}$ given as in (a) or (b), then $(\mathcal{G} \to M, \omega)$ admits a twistor space N with $(\mathcal{G} \to N, \omega)$ of type (G, P).*

Remark 3.5.3. If Q is connected, then by the subalgebra–subgroup correspondence, there exists a unique connected (parabolic) subgroup P such that $Q \subset P \subset G$ with $\mathrm{Lie}(P) = \mathfrak{p}$. This guarantees the existence of a twistor space, and will play a substantial simplifying step in the calculation of submaximal symmetry dimensions – see Remark 4.1.10.

Theorem 3.5.4. *Let $\mathfrak{g}$ be complex semisimple. Let $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ be parabolic subalgebras, and let λ be the highest weight of a simple ideal in $\mathfrak{g}$. Let $w \in W^{\mathfrak{p}}(2) \subset W^{\mathfrak{q}}(2)$ and $\mu = -w \cdot \lambda$. Choosing root vectors in $\mathfrak{g}$, let ϕ_0 be the Kostant lowest weight vector as in (B.1). Let $\mathfrak{a} = \mathrm{pr}^{\mathfrak{g}}(\mathfrak{p}_-, \mathfrak{ann}_{\mathfrak{p}_0}(\phi_0))$ and $\mathfrak{b} = \mathrm{pr}^{\mathfrak{g}}(\mathfrak{q}_-, \mathfrak{ann}_{\mathfrak{q}_0}(\phi_0))$, which are respectively $Z_{I_{\mathfrak{p}}}$-graded and $Z_{I_{\mathfrak{q}}}$-graded. Then as $Z_{I_{\mathfrak{p}}}$-graded Lie algebras, $\mathfrak{a} = \mathfrak{b} \subset \mathfrak{g}$.*

Proof. Write $w = (jk) \in W^{\mathfrak{p}}(2) \subset W^{\mathfrak{q}}(2)$, so by Recipe 3, $k \neq j \in I_{\mathfrak{p}}$, and if $c_{kj} = 0$, then $k \in I_{\mathfrak{p}}$. We always have $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{p}_j \subset \mathfrak{g}$, but if $c_{kj} = 0$, we also have $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{p}_{j,k} \subset \mathfrak{g}$. Hence, it suffices to consider either $\mathfrak{p} = \mathfrak{p}_j$ if $c_{kj} < 0$, or $\mathfrak{p} = \mathfrak{p}_{j,k}$ if $c_{kj} = 0$. We assume this below.

By Proposition 3.4.7, $(\mathfrak{g}, \mathfrak{p}, \mu)$ is PR, so $\mathfrak{a} = \mathfrak{p}_- \oplus \mathfrak{a}_0$. Thus, $I_\mu^{\mathfrak{p}} = \emptyset$. We know $\mathfrak{p}_- \subset \mathfrak{q}_-$, and by Recipe 5, $\ker(\mu) \subset \mathfrak{a}_0 \cap \mathfrak{b}_0$. Let us show $\Delta(\mathfrak{a}_0) \subset \Delta(\mathfrak{b})$. Given $\alpha \in \Delta(\mathfrak{a}_0)$, we have $Z_{I_{\mathfrak{p}}}(\alpha) = 0$ and $Z_{J_\mu^{\mathfrak{p}}}(\alpha) \leq 0$. Since $\mathfrak{q}_- \oplus \mathfrak{q}_{0, \leq 0} \subset \mathfrak{b}$, we may assume $\alpha \in \Delta(\mathfrak{q}_{0,+} \oplus \mathfrak{q}_+) \subset \Delta^+$, hence $Z_{J_\mu^{\mathfrak{p}}}(\alpha) = 0$. (The nonzero coefficients of a root must all have the same sign.) Since $J_\mu^{\mathfrak{p}} = J_\mu^{\mathfrak{q}} \,\dot\cup\, (I_{\mathfrak{q}} \backslash (I_{\mathfrak{p}} \cup I_\mu^{\mathfrak{q}}))$, then:

  (i) $Z_{J_\mu^{\mathfrak{q}}}(\alpha) = 0$. Thus, $\alpha \notin \Delta(\mathfrak{q}_{0,+})$, so $\alpha \in \Delta(\mathfrak{q}_+)$, i.e. $r = Z_{I_{\mathfrak{q}}}(\alpha) > 0$;

  (ii) $Z_{I_{\mathfrak{q}} \backslash (I_{\mathfrak{p}} \cup I_\mu^{\mathfrak{q}})}(\alpha) = 0$. Since $Z_{I_{\mathfrak{p}}}(\alpha) = 0$, then $Z_{I_{\mathfrak{q}} \backslash I_\mu^{\mathfrak{q}}}(\alpha) = 0$.

Therefore, $r = Z_{I_{\mathfrak{q}}}(\alpha) > 0$. By Theorem 3.3.3, $\alpha \in \Delta(\mathfrak{b}_r)$. Thus, $\mathfrak{a}_0 \subset \mathfrak{b}$. Hence, $\mathfrak{a} \subset \mathfrak{b}$.

Conversely, let $\alpha \in \Delta(\mathfrak{b})$. We show that $\alpha \in \Delta(\mathfrak{a})$. We have three cases:

  (i) $Z_{I_{\mathfrak{p}}}(\alpha) < 0$: Then $\alpha \in \Delta(\mathfrak{p}_-) \subset \Delta(\mathfrak{a})$.

  (ii) $Z_{I_{\mathfrak{p}}}(\alpha) = 0$: Assume $Z_{J_\mu^{\mathfrak{p}}}(\alpha) > 0$, so $Z_{J_\mu^{\mathfrak{q}}}(\alpha) \geq 0$ since $J_\mu^{\mathfrak{q}} \subset J_\mu^{\mathfrak{p}}$. Hence, $\alpha \in \Delta(\mathfrak{b}_{\geq 0})$, so by Theorem 3.3.3, $Z_{J_\mu^{\mathfrak{q}}}(\alpha) \leq 0$. Thus, $Z_{J_\mu^{\mathfrak{q}}}(\alpha) = 0$ and $Z_{J_\mu^{\mathfrak{p}} \backslash J_\mu^{\mathfrak{q}}}(\alpha) > 0$. Since $J_\mu^{\mathfrak{p}} \backslash J_\mu^{\mathfrak{q}} \subset I_{\mathfrak{q}} \backslash I_\mu^{\mathfrak{q}}$, then $Z_{I_{\mathfrak{q}} \backslash I_\mu^{\mathfrak{q}}}(\alpha) > 0$. But by Theorem 3.3.3, $Z_{I_{\mathfrak{q}} \backslash I_\mu^{\mathfrak{q}}}(\alpha) = 0$, a contradiction. Thus, $Z_{J_\mu^{\mathfrak{p}}}(\alpha) \leq 0$, so $\alpha \in \Delta(\mathfrak{a}_0)$.

  (iii) $Z_{I_{\mathfrak{p}}}(\alpha) > 0$: Since $I_{\mathfrak{p}} \subset I_{\mathfrak{q}} \backslash I_\mu^{\mathfrak{q}}$, then $Z_{I_{\mathfrak{q}} \backslash I_\mu^{\mathfrak{q}}}(\alpha) > 0$ and $\alpha \in \Delta(\mathfrak{b}_+)$. But the latter implies $Z_{I_{\mathfrak{q}} \backslash I_\mu^{\mathfrak{q}}}(\alpha) = 0$ by Theorem 3.3.3, a contradiction.

Thus, $\mathfrak{b} = \mathfrak{p}_- \oplus \mathfrak{a}_0 = \mathfrak{a}$. $\square$

Example 3.5.5. *For each of A_6/P_2, $A_6/P_{1,2}$, $A_6/P_{1,2,4}$, $A_6/P_{1,2,4,5}$, we have $w = (21) \in W_+^{\mathfrak{p}}(2)$, with respective gradings on $\mathfrak{a}(w) \subset A_6 = \mathfrak{sl}_7(\mathbb{C})$ given below. (Diagonal entries are suppressed.)*

*	0					
0	*					
-1	-1	*				
-1	-1	0	*	0	0	
-1	-1	0	0	*	0	
-1	-1	0	0	0	*	
-1	-1	0	0	0	0	*

A_6/P_2

*	1					
-1	*					
-2	-1	*				
-2	-1	0	*	0	0	
-2	-1	0	0	*	0	
-2	-1	0	0	0	*	
-2	-1	0	0	0	0	*

$A_6/P_{1,2}$

*	1					
-1	*					
-2	-1	*				
-2	-1	0	*	1	1	
-3	-2	-1	-1	*	0	
-3	-2	-1	-1	0	*	
-3	-2	-1	-1	0	0	*

$A_6/P_{1,2,4}$

*	1					
-1	*					
-2	-1	*				
-2	-1	0	*	1	2	
-3	-2	-1	-1	*	1	
-4	-3	-2	-2	-1	*	
-4	-3	-2	-2	-1	0	*

$A_6/P_{1,2,4,5}$

4. SUBMAXIMAL SYMMETRY DIMENSIONS

Consider the *complex* or *split-real* case. Define $\mathfrak{S}_\mu$ analogous to $\mathfrak{S}$ but with the constraint $\mathrm{im}(\kappa_H) \subset \mathbb{V}_\mu \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$. As a consequence of Theorem 2.4.6, using $\mathcal{O} = \mathbb{V}_\mu$ in Remark 2.4.10, we obtain $\mathfrak{S}_\mu \leq \mathfrak{U}_\mu = \dim(\mathfrak{a}(\mu))$. We now show that this upper bound is almost always sharp.

4.1. Establishing submaximal symmetry dimensions.
Kostant's theorem (Recipe 4) gives an explicit description of the lowest $\mathfrak{g}_0$-weight vector $\phi_0 \in \mathbb{V}_\mu$. Using the $\mathfrak{g}_0$-module isomorphism $H^2(\mathfrak{g}_-, \mathfrak{g}) \cong \ker(\square) \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$, we have ϕ_0 naturally realized as a 2-cochain. Suppose that $\mathrm{im}(\phi_0) \subset \mathfrak{a} := \mathfrak{a}^{\phi_0}$. In view of (2.2), our strategy for producing a model is to *deform the Lie bracket on $\mathfrak{a}$ by ϕ_0*. Namely, define $\mathfrak{f}$ to be the vector space $\mathfrak{a}$ with the deformed bracket

$$(4.1) \qquad [\cdot, \cdot]_\mathfrak{f} := [\cdot, \cdot] - \phi_0(\cdot, \cdot).$$

Lemma 4.1.1 guarantees when $[\cdot, \cdot]_\mathfrak{f}$ is a Lie bracket. Since ϕ_0 vanishes on $\mathfrak{p} = \mathfrak{g}_{\geq 0}$, then $\mathfrak{k} := \mathfrak{a}_{\geq 0}$ is a subalgebra of both $\mathfrak{a}$ and $\mathfrak{f}$. We construct a *local* homogeneous space $M = F/K$ (i.e. F may only be a local Lie group) and P-bundle $\mathcal{G} = F \times_K P$ with an F-invariant normal Cartan connection ω whose curvature is determined by ϕ_0. Thus, $\mathfrak{S}_\mu \geq \dim(\mathfrak{inf}(\mathcal{G}, \omega)) \geq \dim(\mathfrak{f}) = \dim(\mathfrak{a})$. (In fact, this construction does not depend on regularity.)

Lemma 4.1.1. *Let λ be the highest weight of a simple ideal in $\mathfrak{g}$ and $w \in W^\mathfrak{p}(2)$. Suppose that $w(-\lambda) \in \Delta^-$. Let ϕ_0 be the lowest weight vector of $\mathbb{V}_{-w \cdot \lambda}$ and $\mathfrak{a} := \mathfrak{a}^{\phi_0}$. Define $\mathfrak{f}$ to be the vector space $\mathfrak{a}$ with deformed Lie bracket (4.1). Then $\mathfrak{f}$ is a Lie algebra.*

Proof. Write $w = (jk)$. From (B.1), $\phi_0 = e_{\alpha_j} \wedge e_{\sigma_j(\alpha_k)} \otimes e_{w(-\lambda)}$ with $w(-\lambda) \in \Delta^- = \Delta(\mathfrak{g}_-) \cup \Delta^-(\mathfrak{g}_0)$. Root vectors from $\Delta^-(\mathfrak{g}_0)$ annihilate the lowest $\mathfrak{g}_0$-weight vector ϕ_0, so $e_{w(-\lambda)} \in \mathfrak{g}_- \oplus \mathfrak{ann}(\phi_0) \subset \mathfrak{a}$. Thus, $\mathrm{im}(\phi_0) \subset \mathfrak{a}$, so $\phi_0 \in \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{a} \hookrightarrow \bigwedge^2 \mathfrak{a}^* \otimes \mathfrak{a}$. Note $e_{\alpha_j}, e_{\sigma_j(\alpha_k)} \in \mathfrak{g}_+ \cong \mathfrak{g}_-^*$ vanish on $\mathfrak{a}_{\geq 0}$.

By Theorem 3.5.4, the Lie algebra structure of $\mathfrak{a}$ is the same if we take $\mathfrak{p} = \mathfrak{p}_j$ if $c_{kj} < 0$ or $\mathfrak{p} = \mathfrak{p}_{j,k}$ if $c_{kj} = 0$. Assuming this, then by Proposition 3.4.7, $(\mathfrak{g}, \mathfrak{p}, -w \cdot \lambda)$ is PR, i.e. $\mathfrak{a} = \mathfrak{g}_- \oplus \mathfrak{a}_0$.

It suffices to verify the Jacobi identity $\mathrm{Jac}_\mathfrak{f}(x, y, z) = 0$ for $[\cdot, \cdot]_\mathfrak{f}$. For $x, y, z \in \mathfrak{f}$,

$$[[x, y]_\mathfrak{f}, z]_\mathfrak{f} = [[x, y] - \phi_0(x, y), z]_\mathfrak{f} = [[x, y], z] - \phi_0([x, y], z) - [\phi_0(x, y), z] + \phi_0(\phi_0(x, y), z).$$

Since ϕ_0 vanishes on $\mathfrak{a}_0$, clearly $\mathrm{Jac}_\mathfrak{f}(x, y, z) = 0$ when at least two of $x, y, z \in \mathfrak{a}_0$. Suppose $x, y \in \mathfrak{g}_-$ and $z \in \mathfrak{a}_0$. Since $\mathrm{Jac}_\mathfrak{a}(x, y, z) = 0$ and $z \in \mathfrak{a}_0 = \mathfrak{ann}(\phi_0)$, then

$$\mathrm{Jac}_\mathfrak{f}(x, y, z) = [z, \phi_0(x, y)] - \phi_0([z, x], y) - \phi_0([y, z], x) = (z \cdot \phi_0)(x, y) = 0.$$

Finally, suppose $x, y, z \in \mathfrak{g}_-$. Since $\phi_0 \in \ker(\square)$, it is in particular ∂-closed. This means

$$(4.2) \qquad 0 = (\partial \phi_0)(x, y, z) = [x, \phi_0(y, z)] - [y, \phi_0(x, z)] + [z, \phi_0(x, y)]$$
$$- \phi_0([x, y], z) + \phi_0([x, z], y) - \phi_0([y, z], x), \qquad \forall x, y, z \in \mathfrak{g}_-.$$

This equation uses the bracket on $\mathfrak{g}$, but restricts to $\mathfrak{a}$ since ϕ_0 has image in $\mathfrak{a}$, and the bracket on $\mathfrak{a}$ is inherited from that of $\mathfrak{g}$ by Lemma 2.1.4. Using (4.2) and $\mathrm{Jac}_\mathfrak{a}(x, y, z) = 0$, we obtain

$$\mathrm{Jac}_\mathfrak{f}(x, y, z) = \phi_0(\phi_0(x, y), z) + \phi_0(\phi_0(y, z), x) + \phi_0(\phi_0(z, x), y).$$

Since $e_{w(-\lambda)}$ spans $\mathrm{im}(\phi_0)$, it suffices to show $w(-\lambda) \notin \{-\alpha_j, -\sigma_j(\alpha_k)\}$. If $w(-\lambda) = -\alpha_j$, then $\lambda = -\sigma_k(\alpha_j) \in \Delta^-$. If $w(-\lambda) = -\sigma_j(\alpha_k)$, then $\lambda = -\alpha_k \in \Delta^-$. These contradict $\lambda \in \Delta^+$. $\qquad\square$

Let us determine all exceptions:

Lemma 4.1.2. *Let G be complex simple. Let $w \in W^{\mathfrak{p}}$ satisfy $w(-\lambda_{\mathfrak{g}}) \in \Delta^+$.*
(a) If $|w| = 2$, then G/P is one of: $\quad A_2/P_1, \quad A_2/P_{1,2}, \quad B_2/P_1, \quad B_2/P_{1,2}.$
(b) If $|w| = 1$, then $G/P \cong A_1/P_1$.
For $B_2/P_{1,2}$, $w(-\lambda_{\mathfrak{g}}) \in \Delta^+$ if $w = (12)$, but not for $w = (21)$.

Proof. Part (b) is obvious, so we prove (a). Since $W^{\mathfrak{p}}(2) \neq \emptyset$, then $\mathrm{rank}(\mathfrak{g}) \geq 2$. Write $w = (jk)$. If $w(-\lambda_{\mathfrak{g}}) \in \Delta^+$, then $w(-\lambda_{\mathfrak{g}}) \in w\Delta^- \cap \Delta^+ = \Phi_w = \{\alpha_j, \sigma_j(\alpha_k)\}$. There are two cases:

  (i) $w(-\lambda_{\mathfrak{g}}) = \alpha_j$: Then $\lambda_{\mathfrak{g}} = \sigma_k(\alpha_j) = \alpha_j - c_{jk}\alpha_k$, which is the highest root, so $\mathrm{rank}(\mathfrak{g}) = 2$.
  (ii) $w(-\lambda_{\mathfrak{g}}) = \sigma_j(\alpha_k)$: Then $\lambda_{\mathfrak{g}} = \alpha_k$. Hence, $\mathrm{rank}(\mathfrak{g}) = 1$, a contradiction.

For B_2/P_2, $\exists! w = (21) \in W^{\mathfrak{p}}(2)$, so $w(-\lambda_{\mathfrak{g}}) = w(-2\lambda_2) = -2\lambda_1 + 2\lambda_2 = -\alpha_1 \notin \Delta^+$. We verify the final claim for $B_2/P_{1,2}$ directly. $\qquad\square$

Remark 4.1.3. Part (b) is relevant only when G is *semisimple* and G/P contains $A_1/P_1 \times G'/P'$. Let $\lambda = 2\lambda_1$ be the highest weight of A_1, and $w = (1, k')$, where the 1 comes from the first factor, $k' \in I_{\mathfrak{p}'}$ is arbitrary. Note $w(-\lambda) = \alpha_1 \in \Delta^+$, so Lemma 4.1.1 cannot be applied.

We first establish a result outside the parabolic context:

Lemma 4.1.4. *Let G be a real or complex Lie group, and $P \subset G$ a closed subgroup. Let $\mathfrak{f}$ be a Lie algebra, $\mathfrak{k} \subset \mathfrak{f}$ a subalgebra, and $\iota : \mathfrak{k} \to \mathfrak{p}$ a Lie algebra injection. Suppose there exists a linear map $\vartheta : \mathfrak{f} \to \mathfrak{g}$ such that: (i) $\vartheta|_{\mathfrak{k}} = \iota$; (ii) $\vartheta([z, x]_{\mathfrak{f}}) = [\iota(z), \vartheta(x)]$, $\forall z \in \mathfrak{k}$ and $\forall x \in \mathfrak{f}$; (iii) ϑ induces an isomorphism from $\mathfrak{f}/\mathfrak{k}$ to $\mathfrak{g}/\mathfrak{p}$. Then there exists a geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) with $\mathfrak{inf}(\mathcal{G}, \omega)$ containing a subalgebra isomorphic to $\mathfrak{f}$.*

Proof. Let K and F be the unique connected, simply-connected Lie groups with Lie algebras $\mathfrak{k}$ and $\mathfrak{f}$, respectively. There are unique Lie group homomorphisms $\Phi : K \to F$ and $\Psi : K \to P$ whose differentials $\Phi' : \mathfrak{k} \to \mathfrak{f}$ and $\Psi' : \mathfrak{k} \to \mathfrak{p}$ are the given inclusions $\mathfrak{k} \hookrightarrow \mathfrak{f}$ and $\iota : \mathfrak{k} \hookrightarrow \mathfrak{p}$.

   Case 1: Φ is injective and $\Phi(K) \subset F$ is a Lie subgroup. Define $\mathcal{G} = F \times_K P = (F \times P)/\sim_K$, where $(u, p) \sim_K (u \cdot \Phi(k), \Psi(k^{-1}) \cdot p)$ for any $k \in K$. This has a natural projection $\mathcal{G} \to M := F/\Phi(K)$ (note $\Phi(K) \subset F$ is closed). Since Φ is injective, identify K with its image $\Phi(K)$. Any F-invariant Cartan connection ω on $\mathcal{G}$ corresponds [27], [11] to a linear map $\vartheta : \mathfrak{f} \to \mathfrak{g}$ satisfying (i)–(iii). The curvature function of $\omega = \omega_\vartheta$ corresponds to

$$(4.3) \qquad\qquad \kappa_\vartheta(x, y) = [\vartheta(x), \vartheta(y)] - \vartheta([x, y]_{\mathfrak{f}}) \qquad \forall x, y \in \mathfrak{f}.$$

By construction, $\mathfrak{f} \subset \mathfrak{inf}(\mathcal{G}, \omega)$.

   Case 2: Φ is not injective or $\Phi(K) \subset F$ is not a Lie subgroup. Since Φ' is injective, then Φ has a discrete kernel. In place of F in the definition of the Cartan bundle $\mathcal{G}$ in Case 1, we will define a *semi-local* Lie group U, an inclusion $K \hookrightarrow U$ as a closed subgroup which acts on the right on U, and such that $\mathrm{Lie}(U) = \mathfrak{f}$. ("Semi-locality" will be clarified below.) Then $\mathfrak{f}$-invariant Cartan connections on $\mathcal{G} = U \times_K P \to M := U/K$ correspond to linear maps $\vartheta : \mathfrak{f} \to \mathfrak{g}$ as in Case 1.

Let $\mathfrak{v}$ be a vector space complementary to $\mathfrak{k}$ in $\mathfrak{f}$. Choose two norms $\|\cdot\|_{\mathfrak{v}}, \|\cdot\|_{\mathfrak{k}}$ on these vector spaces and define the norm on the Lie algebra $\mathfrak{f} = \mathfrak{v} \oplus \mathfrak{k}$ by the formula $\|v + k\|_{\mathfrak{f}} = \max\{\|v\|_{\mathfrak{v}}, \|k\|_{\mathfrak{k}}\}$ for $v \in \mathfrak{v}$, $k \in \mathfrak{k}$. Then the balls in these spaces satisfy: $B_\epsilon^{\mathfrak{f}} = B_\epsilon^{\mathfrak{v}} \times B_\epsilon^{\mathfrak{k}}$.

We choose $\epsilon > 0$ so small that the maps $\exp : B_{5\epsilon}^{\mathfrak{k}} \to K$, $\exp : B_{5\epsilon}^{\mathfrak{f}} \to F$ are injective, $\exp(B_{5\epsilon}^{\mathfrak{k}}) \cap \Phi^{-1}(1_F) = 1_K$, and we will further specify ϵ below. One of the requirements is that $\exp \cdot \exp : B_\epsilon^{\mathfrak{v}} \times B_\epsilon^{\mathfrak{k}} \to F$ is an embedding. This holds for small ϵ because $\log\big(\exp(\epsilon v) \exp(\epsilon k) \exp(-\epsilon v - \epsilon k)\big) = o(\epsilon)$ by the Baker–Campbell–Hausdorff formula ($\|v\|_{\mathfrak{v}} \leq 1$, $\|k\|_{\mathfrak{k}} \leq 1$), implying that for $\epsilon \ll 1$, $U_0 := \exp(B_\epsilon^{\mathfrak{v}}) \cdot \exp(B_\epsilon^{\mathfrak{k}}) \subset \exp(B_{2\epsilon}^{\mathfrak{f}})$ and $U_0^{-1} = \{g^{-1} \mid g \in U_0\} = \exp(B_\epsilon^{\mathfrak{k}}) \cdot \exp(B_\epsilon^{\mathfrak{v}}) \subset \exp(B_{2\epsilon}^{\mathfrak{f}})$.

Let $K_0 = \exp(B_\epsilon^{\mathfrak{k}}) \subset K$ and $V_0 = \Phi(K_0) \subset U_0$. Denote $U_0^2 := U_0^{-1} \cdot U_0 \subset F$ and $V_0^2 = (U_0^2 \cap \operatorname{im}(\Phi))_0$, where the last zero means the connected component of 1_F. We have $V_0^2 = \Phi(K_0^2)$ for a unique connected neighborhood K_0^2 of 1_K. Denote also $K_0^4 := K_0^2 \cdot K_0^2$. Using the Baker–Campbell–Hausdorff formula as above we get $U_0^2 \subset \exp(B_{3\epsilon}^{\mathfrak{f}})$, $K_0^2 \subset \exp(B_{3\epsilon}^{\mathfrak{k}})$ and $K_0^4 \subset \exp(B_{5\epsilon}^{\mathfrak{k}})$ for small enough ϵ. This implies that $\Phi : K_0^4 \to V_0^4 = V_0^2 \cdot V_0^2 \subset F$ is an embedding and that $V_0^4 \cap U_0^2 = V_0^2$. Indeed, from $V_0^4 \subset \exp(B_{5\epsilon}^{\mathfrak{f}})$ we get: $\log(V_0^4 \cap U_0^2) \subset \mathfrak{k} \cap \log(U_0^2) = \log(V_0^2) \subset \mathfrak{f}$. Consequently $V_0^4 \cap U_0^2 \subset V_0^2$ and the reverse inclusion is obvious.

Consider the sheaf of neighborhoods $\{U_k = U_0 \cdot \Phi(k)\}_{k \in K}$ over K. We introduce an equivalence relation on it as follows. Take $x_1 \in U_{k_1}$ and $x_2 \in U_{k_2}$, i.e. $x_1 = u_1 \cdot \Phi(k_1)$, $x_2 = u_2 \cdot \Phi(k_2)$ for some $u_1, u_2 \in U_0$. We let $x_1 \sim x_2$ iff $u_2 = u_1 \cdot \Phi(k_1 k_2^{-1})$ and $k_1 k_2^{-1} \in K_0^2$.

Let us check that it is an equivalence relation. That $\sim$ is reflexive is obvious and the symmetry follows from $(K_0^2)^{-1} = K_0^2 \Leftrightarrow (U_0^2)^{-1} = U_0^2$. Let now $x_1 \sim x_2$ and $x_2 \sim x_3$. Then $u_3 = u_2 \cdot \Phi(k_2 k_3^{-1}) = u_1 \cdot \Phi(k_1 k_2^{-1}) \cdot \Phi(k_2 k_3^{-1}) = u_1 \cdot \Phi(k_1 k_3^{-1})$ and $k_1 k_3^{-1} = (k_1 k_2^{-1})(k_2 k_3^{-1}) \in K_0^4$. On the other hand $\Phi(k_1 k_3^{-1}) = u_1^{-1} u_3 \in U_0^2$, and these two imply $k_1 k_3^{-1} \in K_0^2$. Thus we get transitivity.

Define $U = \sqcup_{k \in K} U_k / \sim$. This is a manifold with the chart centered at k given by the embedding $R_{\Phi(k)} : U_0 \to U$. Overlap maps are given by the equivalence relation. The group K is embedded into U by the formula $K \ni k \mapsto R_{\Phi(k)}(1_F) \in U_k \hookrightarrow U$. Indeed, if $k \sim k'$ i.e. $1_F \cdot \Phi(k) = 1_F \cdot \Phi(k')$ and $k' k^{-1} \in K_0^2$, then $\Phi(k' k^{-1}) = 1_F$ yields $k = k'$. This implies that topologically $U \simeq B_\epsilon^{\mathfrak{v}} \times K$.

In fact, we can make this isomorphism canonical by strengthening the condition on ϵ (taking it possibly smaller). Namely let us require that $\exp \cdot \exp : B_\epsilon^{\mathfrak{v}} \times B_{5\epsilon}^{\mathfrak{k}} \to F$ is an embedding. Every element of U can be represented in the form $U_k \ni u_0 \cdot \Phi(k) = \exp(\epsilon v) \cdot \Phi(\exp(\epsilon \kappa) \cdot k)$, where $\|v\|_{\mathfrak{v}} \le 1$ and $\|\kappa\|_{\mathfrak{k}} \le 1$. We map this element to $(\epsilon v, \exp(\epsilon \kappa) \cdot k)$. Independence of the representative follows from the above condition on the map $\exp \cdot \exp$.

Moreover, U is a semi-local Lie group, which means that for any two points $k_1, k_2 \in K$ there exists two neighborhoods $U'_{k_i} \subset U_{k_i}$ containing $\exp(B_\epsilon^{\mathfrak{k}}) \cdot \Phi(k_i)$ in U such that the multiplication (induced from F) acts as $U'_{k_1} \times U'_{k_2} \to U_{k_1 \cdot k_2} \subset U$ by $(u_1 \cdot \Phi(k_1)) \cdot (u_2 \cdot \Phi(k_2)) = (u_1 \cdot Ad_{\Phi(k_1)} u_2) \cdot \Phi(k_1 k_2)$. Notice that $u_1 \cdot Ad_{\Phi(k_1)} u_2 \in U_0$ if $u_1, u_2 \in U_0$ are sufficiently close to 1_F. (This condition specifies U'_{k_i}.) Similarly, we impose locality of the inverse along K, as a map $U'_k \to U_{k^{-1}}$ for some neighborhood $U'_k \subset U_k$ of $\exp(B_\epsilon^{\mathfrak{k}}) \cdot \Phi(k)$ given by $(u \cdot \Phi(k))^{-1} = Ad_{\Phi(k^{-1})} u^{-1} \cdot \Phi(k^{-1})$.

The semi-local Lie group U is naturally homomorphically mapped to F (this is an immersion, not necessarily an embedding). The chart $U_{1_K} \hookrightarrow U$ is isomorphically mapped to $U_0 \subset F$. Therefore the Lie algebra of U is naturally identified with $\mathfrak{f}$.

In addition, $K \hookrightarrow U$ is a closed subgroup, which acts from the right on U. The orbit of the right K-action is given by: $K \ni k \mapsto u_0 \cdot \Phi(k) \in U_k \hookrightarrow U$, $u_0 \in U_0$. All orbits are isomorphic to K and the two orbits either do not intersect or coincide. Moreover (provided we adopt the stronger condition on ϵ above) we can represent the orbit uniquely via u_0 by claiming that $u_0 \in \exp(B_\epsilon^{\mathfrak{v}})$. The orbit space is then $U/K = U_0/K_0 = \exp(B_\epsilon^{\mathfrak{v}}) \simeq B_\epsilon^{\mathfrak{v}}$, and this is a neighborhood of the point $K/K \equiv 0 \in B_\epsilon^{\mathfrak{v}}$. Thus we get the local homogeneous space as the desired model.

Now define the Cartan bundle $\mathcal{G} = U \times_K P = (U \times P)/\sim_\Psi$, where $(q_1, p_1) \sim_\Psi (q_2, p_2)$ iff $q_2 = q_1 \cdot k$ and $p_2 = \Psi(k)^{-1} \cdot p_1$ for some $k \in K$. This $\mathcal{G}$ is a principal P-bundle over $U/K \simeq B_\epsilon^{\mathfrak{v}}$, in particular $\mathcal{G} \simeq B_\epsilon^{\mathfrak{v}} \times P$ topologically. We conclude as in Case 1. □

The following result is independent of the regularity condition on curvature.

Theorem 4.1.5 (Realizability). *Let G be a complex semisimple Lie group, $P \subset G$ a parabolic subgroup, λ the highest weight of a simple ideal in $\mathfrak{g}$, $w \in W^{\mathfrak{p}}(2)$, and $\mu = -w \cdot \lambda$. Suppose $w(-\lambda) \in \Delta^-$. Then there exists a non-flat normal geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) with $\operatorname{im}(\kappa_H) \subset \mathbb{V}_\mu$ and $\dim(\operatorname{inf}(\mathcal{G}, \omega)) \ge \dim(\mathfrak{a}(\mu)) = \mathfrak{U}_\mu$.*

Proof. Define $\mathfrak{f}$ to be the vector space $\mathfrak{a} := \mathfrak{a}(\mu)$ with deformed bracket (4.1). Since $w(-\lambda) \in \Delta^-$, then by Lemma 4.1.1, $\mathfrak{f}$ is a Lie algebra. Since ϕ_0 is trivial on $\mathfrak{k} := \mathfrak{a}_{\ge 0}$, then $\mathfrak{k} \subset \mathfrak{f}$ is a subalgebra.

Also, $\mathfrak{k} \subset \mathfrak{p} = \mathfrak{g}_{\geq 0}$ is a subalgebra. Since $\mathfrak{f} = \mathfrak{a}$ as vector spaces, and $\mathfrak{a} \subset \mathfrak{g}$, take $\vartheta : \mathfrak{f} \to \mathfrak{g}$ to be the induced *vector space* inclusion. Then (i)–(iii) in Lemma 4.1.4 hold, so an $\mathfrak{f}$-invariant geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) exists, i.e. $\dim(\mathfrak{f}) \leq \dim(\inf(\mathcal{G}, \omega))$, with κ determined by

$$\kappa_\theta(x, y) = [\vartheta(x), \vartheta(y)] - \vartheta([x, y]_{\mathfrak{f}}) = [x, y] - [x, y]_{\mathfrak{f}} = \phi_0(x, y), \qquad \forall x, y \in \mathfrak{f}.$$

Since $\phi_0 \in \ker(\Box)$, then $\partial^* \phi_0 = 0$, so $(\mathcal{G} \to M, \omega)$ is normal. $\qquad \square$

For *split real forms*, notions of roots and weights from complex representation theory are similarly defined. All preceding statements and proofs continue to hold. Using Theorem 4.1.5 when $Z(\mu) > 0$, we have $\mathfrak{S}_\mu = \mathfrak{U}_\mu$ almost always, c.f. Lemma 4.1.2. Combined with Corollary 2.4.9, we obtain:

Theorem 4.1.6. *Let G be a complex or split-real semisimple Lie group, $P \subset G$ a parabolic subgroup, λ the highest weight of a simple ideal in $\mathfrak{g}$, $w \in W^{\mathfrak{p}}(2)$, and $\mu = -w \cdot \lambda$ satisfies $Z(\mu) > 0$. We always have $\mathfrak{S}_\mu \leq \mathfrak{U}_\mu$. If $w(-\lambda) \in \Delta^-$ or G/P does not contain the factors A_1/P_1, A_2/P_1, $A_2/P_{1,2}$, B_2/P_1, $B_2/P_{1,2}$, then $\mathfrak{S}_\mu = \mathfrak{U}_\mu = \dim(\mathfrak{a}(\mu))$ and any submaximally symmetric model is locally homogeneous about a non-flat regular point.*

Example 4.1.7 (3rd order ODE up to contact transformations, $B_2/P_{1,2}$). *This geometry is PR, $\lambda_{\mathfrak{g}} = 2\lambda_2$, and $\dim(\mathfrak{a}(w)) = 5$ for $w \in W_+^{\mathfrak{p}}(2) = \{(12), (21)\}$, so $\mathfrak{S} \leq \mathfrak{U} = 5$. Since $(21)(-\lambda_{\mathfrak{g}}) \in \Delta^-$, then by Theorem 4.1.5, a model with 5-dimensional symmetry exists, so $\mathfrak{S} = \mathfrak{U} = 5$, and has twistor space type B_2/P_2, i.e. a 3-dimensional contact projective structure. In §4.3, we show that 5 is not realizable in the (12)-branch, having twistor type B_2/P_1, i.e. a 3-dimensional conformal structure.*

Consequently, $B_2/P_{1,2}$ is not exceptional with regard to $\mathfrak{S} = \mathfrak{U}$. Thus,

Theorem 4.1.8. *Let G be a complex or split-real semisimple Lie group, $P \subset G$ a parabolic subgroup. If G/P does not contain any of the factors A_1/P_1, A_2/P_1, $A_2/P_{1,2}$, B_2/P_1, then*

$$\mathfrak{S} = \mathfrak{U} = \max\{\mathfrak{U}_\mu \mid \mathbb{V}_\mu \subset H_+^2(\mathfrak{g}_-, \mathfrak{g})\}, \qquad \mathfrak{U}_\mu = \dim(\mathfrak{a}(\mu)),$$

and any submaximally symmetric model is locally homogeneous about a non-flat regular point.

Remark 4.1.9. For each $\mathbb{V}_\mu \subset H_+^2(\mathfrak{g}_-, \mathfrak{g})$, the corresponding $w = (jk) \in W_+^{\mathfrak{p}}(2)$ is generated from the Weyl group of at most two simple ideals in $\mathfrak{g}$. Any remaining simple factor $(\hat{\mathfrak{g}}, \hat{\mathfrak{p}})$ yields a flat factor, so $\hat{\mathfrak{g}} \subset \mathfrak{a}(\mu)$. Thus, the general (complex or split-real) semisimple (non-simple) case reduces to studying two factors $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}''$.

Remark 4.1.10. Aside from the exceptions in Theorem 4.1.6, given $(\mathfrak{g}, \mathfrak{p}, \mu)$, $\mathfrak{S}_\mu$ is the same for all (G, P) with $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(P) = \mathfrak{p}$. To simplify the calculation of $\mathfrak{S}_\mu = \mathfrak{U}_\mu$, we may assume P is *connected* and pass to the minimal twistor space. (See Corollary 3.5.2 and Remark 3.5.3.)

4.2. **Deformations.** A graded subalgebra $\mathfrak{a} \subset \mathfrak{g}$ admits a canonical filtration, i.e. $\mathfrak{a}^i = \bigoplus_{j \geq i} \mathfrak{a}_j$.

Definition 4.2.1. *A graded subalgebra $\mathfrak{a}$ is* filtration-rigid *if any filtered Lie algebra $\mathfrak{f}$ with $\mathrm{gr}(\mathfrak{f}) = \mathfrak{a}$ as graded Lie algebras has $\mathfrak{f} \cong \mathfrak{a}$ as filtered Lie algebras.*

Proposition 4.2.2. *Let G be a real or complex semisimple Lie group and $P \subset G$ a parabolic subgroup, and suppose that $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{g}$. Let $(\mathcal{G} \xrightarrow{\pi} M, \omega)$ be a regular, normal G/P geometry which is not flat. Given $u \in \mathcal{G}$ with $\kappa_H(u) \neq 0$, if $\mathfrak{g}_- \subset \mathfrak{s}(u)$, then $\mathfrak{s}(u)$ is not filtration-rigid.*

Proof. Since $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{g}$, the geometry is completely determined by its regular infinitesimal flag structure. Assume $\mathfrak{f}(u) \cong \mathfrak{s}(u)$, so $\mathfrak{f}(u)$ and hence $\mathcal{S}(x)$ is graded, where $x = \pi(u)$. Then $\mathcal{S}(x)/\mathcal{S}(x)^0 \cong \mathfrak{g}_- =: \mathfrak{m}$ acts transitively on M at x. Thus, M can be locally identified near x with the simply-connected Lie group $M(\mathfrak{m})$ with Lie algebra $\mathfrak{m}$. This is endowed with the $M(\mathfrak{m})$-invariant distribution generated by $\mathfrak{g}_{-1}$, and there is a $M(\mathfrak{m})$-invariant principal G_0-bundle $\mathcal{G}_0 \to M(\mathfrak{m})$, where $G_0 \subseteq \mathrm{Aut}_{gr}(\mathfrak{m})$, which is a reduction of the graded frame bundle over $M(\mathfrak{m})$.

The triple $(M(\mathfrak{m}), \mathfrak{g}_{-1}, \mathfrak{g}_0)$ is the standard model in Tanaka theory, and since $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{g}$ is finite-dimensional, its symmetry algebra $\mathcal{S}$ is isomorphic to $\mathfrak{g}$ [44, §6]. But since $\kappa_H(u) \neq 0$, then $\dim(\mathcal{S}) \leq \mathfrak{S} < \dim(\mathfrak{g})$, c.f. Proposition 2.3.2 or Theorem 2.4.6, a contradiction. $\square$

In the exceptional cases, we will show that $\mathfrak{a}(\mu)$ is filtration-rigid, and hence $\mathfrak{S}_\mu < \mathfrak{U}_\mu$.

Example 4.2.3. *Without the* $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{g}$ *assumption, Proposition 4.2.2 generally fails. Consider a projective structure* $[\nabla]$, *c.f. §5.4. The condition* $\mathfrak{g}_- \subset \mathfrak{s}(u)$ *implies local homogeneity, and since* $\mathfrak{g}_-$ *is abelian, then near* $\pi(u)$, *we have local coordinates* $(t, x^1, ..., x^m)$, *and symmetries* $\partial_t, \partial_{x^1}, ..., \partial_{x^m}$ *of* $[\nabla]$. *The geodesic equations (5.10) admit these symmetries, so all coefficients in these equations are constants. Hence, all connection coefficients* Γ^a_{bc} *can be made constant by a projective change. For* $m \geq 2$, *this does not imply the structure is flat, e.g. taking all* $\Gamma^a_{bc} = 0$ *except* $\Gamma^1_{01} = -\frac{1}{2}$, *the Fels invariants (5.4) satisfy* $S \equiv 0$ *and* $T \neq 0$.

However, for $m = 1$, *the Tresse invariants (5.8) vanish everywhere for* $\ddot{x} = F(\dot{x})$, *where* F *is a cubic polynomial. Thus, Proposition 4.2.2 is true for 2-dimensional projective structures.*

Consider $\mathfrak{a} = \bigoplus_i \mathfrak{a}_i$ with (graded) basis $\{e_i^\alpha\}$ and structure equations $[e_i^\alpha, e_j^\beta] = \sum_\gamma c_\gamma^{\alpha\beta} e_{i+j}^\gamma$. Any filtered Lie algebra $\mathfrak{f} \subset \mathfrak{g}$ with $\mathrm{gr}(\mathfrak{f}) = \mathfrak{a}$ has a (filtered) basis $\{f_i^\alpha\}$ with

$$(4.4) \qquad [f_i^\alpha, f_j^\beta]_\mathfrak{f} = \sum_\gamma c_{ij\gamma}^{\alpha\beta} f_{i+j}^\gamma + \sum_{k>i+j} \sum_\gamma b_{ij\gamma}^{\alpha\beta k} f_k^\gamma,$$

and we consider $\mathfrak{f}$ up to filtration-preserving automorphisms $f_i^\alpha \mapsto f_i^\alpha + \sum_{j>i} \sum_\beta \chi_{i\beta}^{\alpha j} f_j^\beta$. We refer to the terms in the double summation in (4.4) as *tails*. So establishing filtration-rigidity amounts to eliminating the presence of tails (in some basis).

Proposition 4.2.4. *Let* $\mathfrak{f}$ *be a filtered Lie algebra,* $\mathrm{gr}(\mathfrak{f}) =: \mathfrak{a} = \mathfrak{a}_{-\hat{\nu}} \oplus ... \oplus \mathfrak{a}_0 \oplus ... \oplus \mathfrak{a}_{\tilde{\nu}}$ *its associated-graded, and let* $\tau_j : \mathfrak{f}^j \to \mathfrak{a}_j$ *denote the canonical projections. Suppose there exists* $h \in \mathfrak{a}_0$ *such that* ad_h *is diagonal in some (graded) basis* $\{e_i^\alpha\}$ *of* $\mathfrak{a}$, *and let* $\mathcal{E}_i = \mathrm{Spec}(\mathrm{ad}_h|_{\mathfrak{a}_i})$. *If* $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ *for distinct* i, j, *then for any* $\tilde{h} \in \mathfrak{f}^0$ *with* $\tau_0(\tilde{h}) = h$, *there is a (filtered) basis* $\{f_i^\alpha\}$ *of* $\mathfrak{f}$, *with* $\tau_i(f_i^\alpha) = e_i^\alpha$, *and with respect to which* $\mathrm{ad}_{\tilde{h}}$ *is diagonal. Moreover, if:*[12]

(i) $\forall i \neq j$: $(\mathcal{E}_i + \mathcal{E}_j) \cap \bigcup_{k>i+j} \mathcal{E}_k = \emptyset$; *and*

(ii) $\forall i$: $\bigwedge^2 \mathcal{E}_i \cap \bigcup_{k>2i} \mathcal{E}_k = \emptyset$,

then $\mathfrak{a}$ *is filtration-rigid.*

Proof. Use (reverse) induction on r, where $-\hat{\nu} \leq r \leq \tilde{\nu}$. The $r = \tilde{\nu}$ case is trivial since $\tau_{\tilde{\nu}}|_{\mathfrak{f}^{\tilde{\nu}}} : \mathfrak{f}^{\tilde{\nu}} \to \mathfrak{a}_{\tilde{\nu}}$ is a Lie algebra isomorphism. Assume there is a basis $\{f_i^\alpha\}_{i=r+1}^{\tilde{\nu}}$ of $\mathfrak{f}^{r+1}$ with $\tau_i(f_i^\alpha) = e_i^\alpha$ such that $\mathrm{ad}_{\tilde{h}}$ is diagonal (with eigenvalues μ_i^α). We have $[\tilde{h}, f_r^\alpha] = \mu_r^\alpha f_r^\alpha + \sum_{k>r} \sum_\gamma b_\gamma^{\alpha,k} f_k^\gamma$. Since $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \neq j$, then $\mu_r^\alpha \neq \mu_{r+1}^\beta$, so letting $\hat{f}_r^\alpha = f_r^\alpha + \sum_\beta \sigma_\beta^\alpha f_{r+1}^\beta$ with $\sigma_\beta^\alpha = (\mu_r^\alpha - \mu_{r+1}^\beta)^{-1} b_\beta^{\alpha,r+1}$, we can normalize $b_\beta^{\alpha,r+1} = 0$. Iteratively, we normalize all $b_\gamma^{\alpha,k} = 0$.

We have $[\tilde{h}, [f_i^\alpha, f_j^\beta]] = (\mu_i^\alpha + \mu_j^\beta)[f_i^\alpha, f_j^\beta]$. If $i \neq j$, then by (i), there is no tail. If $i = j$, then for $\alpha \neq \beta$, (ii) implies there is no tail. Thus, $\mathfrak{a}$ is filtration-rigid. $\square$

4.3. Exceptional cases and local homogeneity.

4.3.1. *Simple exceptions.* In the complex or split-real case, by passing to twistor spaces the exceptions A_2/P_1, $A_2/P_{1,2}$, B_2/P_1, $B_2/P_{1,2}$ (Lemma 4.1.2) reduce to[13]: A_2/P_1, B_2/P_1. (Notice that

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[12]Here $T + T := \{a + b \mid a, b \in T\}$ and $\bigwedge^2 T := \{a + b \mid a, b \in T, a \neq b\}$.

[13]By Example 4.1.7, only the (12)-branch of $B_2/P_{1,2}$ is exceptional.

regularity is preserved here.) By Recipe 7, we have:

G/P	$H^2_+(\mathfrak{g}_-,\mathfrak{g})$	$\mathfrak{U}$	$\mathfrak{a} = \mathfrak{g}_{-1} \oplus \mathfrak{a}_0$	Eigenvalues for diagonal element in $\mathfrak{a}_0$
A_2/P_1	✕⎯⎯✳ $^{-5}$ ^{1}	4	$\begin{pmatrix} h & 0 & 0 \\ u & 4h & 0 \\ v & s & -5h \end{pmatrix}$	$\mathcal{E}_{-1} = \{3,-6\}$, $\mathcal{E}_0 = \{-9,0\}$
B_2/P_1	✕⟹✳ $^{-5}$ ^{4}	5	$\begin{pmatrix} 2h & 0 & 0 & 0 & 0 \\ u & 3h & 0 & 0 & 0 \\ v & s & 0 & 0 & 0 \\ w & 0 & -s & -3h & 0 \\ 0 & -w & -v & -u & -2h \end{pmatrix}$	$\mathcal{E}_{-1} = \{1,-2,-5\}$, $\mathcal{E}_0 = \{-3,0\}$

By Proposition 4.2.4, $\mathfrak{a}$ is filtration-rigid for both, and so $\mathfrak{S} \leq \mathfrak{U} - 1$. (By Example 4.2.3, this is valid for A_2/P_1.) The models (5.13) and (5.2) imply $\mathfrak{S} = \mathfrak{U} - 1$. See also (5.9) for $A_2/P_{1,2}$. Thus:

Proposition 4.3.1. *For complex or split-real (regular, normal) parabolic geometries of type A_2/P_1, $A_2/P_{1,2}$, or B_2/P_1, we have $\mathfrak{S} = \mathfrak{U} - 1$. For $B_2/P_{1,2}$ with $w = (12)$, we have $\mathfrak{S}_w = \mathfrak{U}_w - 1$.*

Among the exception list, only $A_2/P_{1,2}$ and B_2/P_1 admit two non-split-real forms:
  (1) 3-dim. CR: $\mathfrak{g} \cong \mathfrak{su}(2,1)$, $\mathfrak{g}_0 \cong \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathbb{R} \oplus \mathbb{C}$ (Heisenberg algebra), and $H^2_+(\mathfrak{g}_-,\mathfrak{g}) \cong \mathbb{C}$ (homogeneity $+4$) has action $z \cdot \phi := z\phi$. If $\phi \neq 0$, then $\mathfrak{ann}(\phi) = 0$, so $\mathfrak{S} \leq \mathfrak{U} = 3$. In [13], Cartan gave models with 3 symmetries. Thus, $\mathfrak{S} = 3$.
  (2) 3-dim. Riem. conformal: $\mathfrak{g} \cong \mathfrak{so}_{1,4}$, $\mathfrak{g}_0 \cong \mathbb{R} \oplus \mathfrak{so}(3)$, $\mathfrak{g}_- = \mathfrak{g}_{-1} \cong \mathbb{R}^3$, and $H^2_+(\mathfrak{g}_-,\mathfrak{g}) \cong \{\phi \in \mathrm{Mat}_3(\mathbb{R}) \mid \mathrm{tr}(\phi) = 0, \phi^{\mathsf{T}} = \phi\}$ (homogeneity $+3$) with $\mathfrak{so}(3)$-action $A \cdot \phi = [A,\phi]$. Any ϕ is orthogonally diagonalizable. If ϕ has three distinct eigenvalues, then $\mathfrak{ann}(\phi) = 0$. Otherwise, ϕ has two equal nonzero eigenvalues, and so $\dim(\mathfrak{ann}(\phi)) = 1$. Thus, $\mathfrak{S} \leq \mathfrak{U} = 4$, and indeed $\mathfrak{S} = 4$: see (5.2).

4.3.2. Semisimple exceptions.

Proposition 4.3.2. *For complex or split-real (regular, normal) $G/P = A_1/P_1 \times G'/P'$ geometries with $\mathrm{im}(\kappa_H) \subset \mathbb{V}_\mu$, $w = (1,k') \in W^{\mathfrak{p}}(2)$, and $\mu = -w \cdot 2\lambda_1$, we have $\mathfrak{U}_\mu - 1 \leq \mathfrak{S}_\mu \leq \mathfrak{U}_\mu$.*

Proof. Note $\mu = 2\alpha_1 + \alpha_{k'}$, $\phi_0 = e_{\alpha_1} \wedge e_{\alpha_{k'}} \otimes e_{\alpha_1} \in \mathbb{V}_\mu$. Take the standard $\mathfrak{sl}_2$-basis H, X, Y with $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$. Thus, $\mathfrak{g}_- = \mathrm{span}\{Y\} \oplus \mathfrak{g}'_-$ and $\alpha_1(H) = 2$. By Recipe 5,

$$(4.5) \qquad \mathfrak{a}_0 = \mathfrak{ann}(\phi_0) = \mathfrak{h}_0 \oplus \bigoplus_{\gamma \in \Delta(\mathfrak{g}'_{0,\leq 0})} \mathfrak{g}'_\gamma, \qquad \mathfrak{h}_0 := \left\{-\frac{1}{4}\alpha_{k'}(h')H + h' \mid h' \in \mathfrak{h}'\right\}.$$

Note $\ker(\alpha_{k'}) \subset \mathfrak{h}' \subset \mathfrak{h}_0$ defines a subalgebra $\tilde{\mathfrak{a}}_0 \subset \mathfrak{a}_0 \cap \mathfrak{g}'_0$ and $\tilde{\mathfrak{a}} = \mathrm{span}\{Y\} \oplus \mathrm{pr}^{\mathfrak{g}'}(\mathfrak{g}'_-, \mathfrak{ann}(e_{\alpha_{k'}})) \subset \mathfrak{a}$, with $\dim(\tilde{\mathfrak{a}}) = \mathfrak{U}_\mu - 1$. As in §4.1, define $\tilde{\mathfrak{f}} = \tilde{\mathfrak{a}}$ as vector spaces, but deform the $\tilde{\mathfrak{a}}$-brackets by the cochain $\phi = e_{\alpha_1} \wedge e_{\alpha_{k'}} \otimes e_{-\alpha_1}$, i.e. $[Y, e_{-\alpha_{k'}}]_{\tilde{\mathfrak{f}}} = Y$. We need to show that $\tilde{\mathfrak{f}}$ is a Lie algebra.

We make two simplifications: (i) By Remark 4.1.9, take G' to be simple, and (ii) By Theorem 3.5.4, for the Lie algebra structure of $\mathfrak{a} := \mathfrak{a}(\mu)$, take $\mathfrak{p}' = \mathfrak{p}_{k'} \subset \mathfrak{g}'$. (These are assumed only in this paragraph to check that $\tilde{\mathfrak{f}}$ is a Lie algebra.) By Proposition 3.4.7, $(\mathfrak{g},\mathfrak{p},\mu)$ is PR, so $\mathfrak{a} = \mathfrak{g}_- \oplus \mathfrak{a}_0$. Write $[Y,u]_{\tilde{\mathfrak{f}}} = T(u)Y$, $\forall u \in \tilde{\mathfrak{f}}$, where T vanishes on $\mathfrak{h}'$ and on all root vectors, except $T(e_{-\alpha_{k'}}) = 1$. For Jacobi, it suffices to check $0 = \mathrm{Jac}_{\tilde{\mathfrak{f}}}(Y,u,v)$, $\forall u,v \in \tilde{\mathfrak{a}}' := \tilde{\mathfrak{a}} \cap \mathfrak{g}'$, i.e. $T([u,v]_{\tilde{\mathfrak{f}}}) = 0$, or $[u,v]_{\tilde{\mathfrak{f}}}$ has no $e_{-\alpha_{k'}}$-component. If $u,v \in \mathfrak{g}_-$, this is clear. Otherwise, if $u \in \tilde{\mathfrak{a}}_0$, then $[u,v]_{\tilde{\mathfrak{f}}} = [u,v]$. If $u \in \ker(\alpha_{k'})$, then $[u,e_{-\alpha_{k'}}] = 0$ and so $T([u,v]) = 0$, $\forall v \in \tilde{\mathfrak{a}}'$. On the other hand, if $u = e_\gamma$, with $\gamma \in \Delta(\mathfrak{g}'_{0,\leq 0})$, then since $u \cdot \phi_0 = 0$ and $[u,e_{\alpha_1}] = 0$, then $[u,e_{\alpha_{k'}}] = 0$, i.e. $\gamma + \alpha_{k'} \notin \Delta$. Thus, if $v = e_\beta$, with $\beta \in \Delta(\mathfrak{a})$, then $\gamma + \beta \neq -\alpha_{k'}$. Hence, $T([u,v]_{\tilde{\mathfrak{f}}}) = 0$ again, and $\tilde{\mathfrak{f}}$ is a Lie algebra.

Now returning to the general setting, let $\tilde{\mathfrak{k}} = \tilde{\mathfrak{a}}_{\geq 0} \subset \tilde{\mathfrak{f}}$. Define the linear map $\vartheta : \tilde{\mathfrak{f}} \to \mathfrak{g}$ by $\vartheta = \mathrm{id} + \frac{1}{2}H \otimes e_{\alpha_{k'}} - \frac{1}{2}X \otimes e_{\alpha_1}$. Thus, $\vartheta|_{\tilde{\mathfrak{k}}}$ is the natural inclusion, $\vartheta(Y) = Y - \frac{1}{2}X$, and

$\vartheta(e_{-\alpha_{k'}}) = e_{-\alpha_{k'}} + \frac{1}{2}H$. For $u \in \tilde{\mathfrak{k}}$ and $v \in \tilde{\mathfrak{f}}$, since $e_{\alpha_1}([u,v]) = 0$ and $e_{\alpha_{k'}}([u,v]) = 0$, then $\vartheta([u,v]_{\tilde{\mathfrak{f}}}) = \vartheta([u,v]) = [u,v] = [u, \vartheta(v)] = [\vartheta(u), \vartheta(v)]$, so ϑ is $\tilde{\mathfrak{k}}$-equivariant. Also, ϑ induces a linear isomorphism $\tilde{\mathfrak{f}}/\tilde{\mathfrak{k}} \cong \mathfrak{g}/\mathfrak{p}$. The hypotheses of Lemma 4.1.4 are satisfied.

By construction, ϑ defines an $\tilde{\mathfrak{f}}$-invariant Cartan connection ω_ϑ on some $\mathcal{G} = F \times_K P$ over $M = F/K$ (where F may be a semi-local Lie group as before). Using (4.3), κ_ϑ vanishes except for

$$\kappa_\vartheta(Y, e_{-\alpha_{k'}}) = [\vartheta(Y), \vartheta(e_{-\alpha_{k'}})] - \vartheta([Y, e_{-\alpha_{k'}}]_{\tilde{\mathfrak{f}}}) = \left[Y - \frac{1}{2}X, e_{-\alpha_{k'}} + \frac{1}{2}H\right] - \vartheta(Y) = X,$$

i.e. $\kappa_\vartheta = \phi_0$. Thus, the geometry is regular and normal, and $\mathfrak{U}_\mu - 1 \leq \mathfrak{S}_\mu \leq \mathfrak{U}_\mu$. $\qquad\square$

4.3.3. *Local homogeneity.*

Theorem 4.3.3. *Among all complex or split-real (regular, normal) parabolic geometries of type (G, P), all submaximally symmetric models are locally homogeneous near a non-flat regular point. Given a $\mathfrak{g}_0$-irrep $\mathbb{V}_\mu \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$, the same conclusion holds with the restriction that $\mathrm{im}(\kappa_H) \subset \mathbb{V}_\mu$.*

Proof. It suffices to prove the second statement. For all non-exceptional cases, Theorem 4.1.6 asserts $\mathfrak{S}_\mu = \mathfrak{U}_\mu$, so the result follows from Corollary 2.4.9 (and Remark 2.4.10).

Consider the exceptional geometries. Let $(\mathcal{G} \xrightarrow{\pi} M, \omega)$ be a submaximally symmetric geometry with $\mathrm{im}(\kappa_H) \subset \mathbb{V}_\mu$. Given a regular point $x \in M$ and $u \in \pi^{-1}(x)$, we have $\mathfrak{s}(u) \subseteq \mathfrak{a}^{\kappa_H(u)}$ by Theorem 2.4.6. We show that $\mathfrak{g}_- \subset \mathfrak{s}(u)$. From §4.3.1 and §4.3.2, $\dim(\mathfrak{s}(u)) = \dim(\inf(\mathcal{G}, \omega)) \geq \mathfrak{U}_\mu - 1$. Let $\mathcal{O} \subset \mathbb{V}_\mu$ be the minimal G_0-orbit. We have two cases:

  (i) $\kappa_H(u) \notin \mathcal{O}$: By Proposition 3.1.1, $\dim(\mathfrak{a}^{\kappa_H(u)}) < \dim(\mathfrak{a}^{\phi_0}) = \mathfrak{U}_\mu$. Thus, $\mathfrak{s}(u) = \mathfrak{a}^{\kappa_H(u)} \supset \mathfrak{g}_-$.
  (ii) $\kappa_H(u) \in \mathcal{O}$: We may assume $\kappa_H(u) = \phi_0$, so $\mathfrak{s}(u) \subseteq \mathfrak{a}^{\phi_0} =: \mathfrak{a}$. Since $\mathfrak{g}_{-1} \subset \mathfrak{g}_-$ is bracket-generating, then $\mathfrak{g}_- \not\subset \mathfrak{s}(u)$ implies $\mathfrak{s}_{-1}(u) \neq \mathfrak{g}_{-1}$. Assuming this, $\mathfrak{s}(u) \subsetneq \mathfrak{a}$ must have codimension one, so $\mathfrak{s}_0(u) = \mathfrak{a}_0$. From Proposition 2.4.3, $[\mathfrak{s}_0(u), \mathfrak{g}_{-1}] \subset \mathfrak{s}_{-1}(u) \neq \mathfrak{g}_{-1}$. Let $\mathfrak{h}_0 := \mathfrak{a}_0 \cap \mathfrak{h}$. We easily confirm $[\mathfrak{h}_0, \mathfrak{g}_{-1}] = \mathfrak{g}_{-1}$ for the simple exceptions, and the same follows from (4.5) for the semisimple exceptions. Thus, $[\mathfrak{s}_0(u), \mathfrak{g}_{-1}] = \mathfrak{g}_{-1}$, a contradiction.

Thus, in all cases the submaximal symmetric structures are homogeneous near x. $\qquad\square$

5. RESULTS AND LOCAL MODELS FOR SPECIFIC GEOMETRIES

In almost all complex or split-real cases, our Theorem 4.1.6 asserts that $\mathfrak{S}_\mu = \mathfrak{U}_\mu$. Exceptions include 2-dim projective, scalar 2nd order ODE, and 3-dim conformal structures (see §4.3). Recipe 7 describes how to efficiently compute $\mathfrak{U}_\mu$. Sample new results are given in Table 3, and a complete classification (when G is simple and complex or split-real) is given in Appendix C.

In this section, we focus on a discussion of parabolic geometries in terms of underlying structures, and describe local models which realize our results on submaximal symmetry dimensions. Among the examples in this section, the only NPR geometry is 2nd order ODE systems (§5.3). All others are PR, so $\mathfrak{a}_+(w) = 0$, $\forall w \in W^{\mathfrak{p}}_+(2)$, and Recipe 7 reduces to using only Recipes 1 and 5.

5.1. **Conformal geometry.** Let $n = p + q \geq 3$. The pseudo-conformal sphere $\mathbb{S}^{p,q}$ embedded as the null projective quadric in $\mathbb{P}(\mathbb{R}^{p+1,q+1})$ is the flat model for conformal geometry in signature (p, q). This is equivalently described as a parabolic geometry of type $\mathrm{SO}_{p+1,q+1}/P_1$, where P_1 is the stabilizer of a null line. The Lie algebra $\mathfrak{g} = \mathfrak{so}_{p+1,q+1}$ is a real form of $\mathfrak{so}_{n+2}(\mathbb{C})$, which is B_ℓ or D_ℓ when $n = 2\ell - 1$ or $n = 2\ell - 2$, respectively; $\mathfrak{g}$ is 1-graded by P_1. The space $\mathbb{W} = H^2(\mathfrak{g}_-, \mathfrak{g})$ is the space of Weyl tensors for $n \geq 4$ or Cotton tensors for $n = 3$.

First work over $\mathbb{C}$, so $\mathfrak{g} = \mathfrak{so}_{n+2}(\mathbb{C})$, and $\mathfrak{g}_0^{ss} \cong B_{\ell-1}$ or $D_{\ell-1}$. For $H_+^2(\mathfrak{g}_-, \mathfrak{g})$, Recipe 4 gives

$$B_2/P_1: \overset{-5}{\times}\!\!\!\!\!\!>\!\!\!\!>\!\overset{4}{*}, \quad B_3/P_1: \overset{-4}{\times}\!\!-\!\!\overset{0}{\circ}\!\!>\!\!\!\!>\!\overset{4}{*}, \quad B_\ell/P_1 \ (\ell \geq 4): \overset{-4}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{2}{*}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!>\!\!\!\!>\!\overset{0}{\circ},$$

$$D_3/P_1: \overset{-4}{\times}\!\diagup\!\!\!\overset{4}{\underset{0}{\diagdown}}\ \oplus\ \overset{-4}{\times}\!\diagup\!\!\!\overset{0}{\underset{4}{\diagdown}}, \quad D_4/P_1: \overset{-4}{\times}\!\!-\!\!\overset{0}{\circ}\!\!\diagup\!\!\!\overset{2}{\underset{2}{\diagdown}}, \quad D_\ell/P_1 \ (\ell \geq 5): \overset{-4}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{2}{*}\!\!-\!\!\overset{0}{\circ}\cdots\!\!\diagup\!\!\!\overset{0}{\underset{0}{\diagdown}},$$

Above, $W_+^{\mathfrak{p}}(2) = \{(12)\}$ always, except $W_+^{\mathfrak{p}}(2) = \{(12),(13)\}$ for D_3/P_1. Let $n \geq 4$. The geometry is PR by Corollary 3.4.8. Then using Recipes 1 and 5,

(1) $n \geq 5$: $\lambda_{\mathfrak{g}} = \lambda_2$, $w = (12)$, $J_w = \{3\}$. For $n \neq 6$, we have $\mathfrak{p}_w^{op} \cong \mathfrak{p}_2 \subset \mathfrak{g}_0^{ss}$.

$$\dim(\mathfrak{a}_0) = \dim(\mathfrak{p}_2) = \begin{cases} \frac{1}{2}\left(\dim(B_{\ell-1}) + 1 + \dim(A_1) + \dim(B_{\ell-3})\right), & n \text{ odd}; \\ \frac{1}{2}\left(\dim(D_{\ell-1}) + 1 + \dim(A_1) + \dim(D_{\ell-3})\right), & n \text{ even}; \end{cases}$$

$$= \begin{cases} 2\ell^2 - 7\ell + 10, & n \text{ odd}; \\ 2\ell^2 - 9\ell + 14, & n \text{ even} \end{cases} \quad \Rightarrow \quad \dim(\mathfrak{a}) = \dim(\mathfrak{g}_{-1}) + \dim(\mathfrak{a}_0) = \binom{n-1}{2} + 6.$$

The $n = 6$ case agrees with this formula, but here $\mathfrak{p}_w^{op} \cong \mathfrak{p}_{2,3} \subset \mathfrak{g}_0^{ss}$.

(2) $n = 4$: D_3/P_1, $\lambda_{\mathfrak{g}} = \lambda_2 + \lambda_3$, $\mathfrak{g}_0^{ss} = A_1 \times A_1$. For $w = (12)$, $J_w = \{3\}$, $\mathfrak{p}_w^{op} \cong A_1 \times \mathfrak{p}_1 \subset \mathfrak{g}_0^{ss}$, so $\dim(\mathfrak{a}_0) = 5$ and $\dim(\mathfrak{a}(w)) = 9$. By symmetry, for $w = (13)$, $\dim(\mathfrak{a}(w)) = 9$ as well.

Thus, $\mathfrak{U}^{\mathbb{C}} = \binom{n-1}{2} + 6$ for $n \geq 4$. In any signature over $\mathbb{R}$, $\mathfrak{S} \leq \mathfrak{U} \leq \mathfrak{U}^{\mathbb{C}}$ by Corollary 2.4.8. For $n \geq 4$, $\mathfrak{a}_0$ stabilizes a null 2-plane (in the standard representation of $\mathfrak{g}_0^{ss}$). These do not exist in Riemannian and Lorentzian signatures, so $\mathfrak{S} \leq \mathfrak{U} < \mathfrak{U}^{\mathbb{C}}$ in these cases. In all other signatures, we exhibit a model realizing the upper bound $\mathfrak{U}^{\mathbb{C}}$.

5.1.1. *Non-Riemannian and non-Lorentzian signatures.* Consider the $(2,2)$ pp-wave metric:[14]

(5.1) $$\mathrm{g}_{\mathbf{pp}}^{(2,2)} = y^2 dw^2 + dwdx + dydz.$$

This has 9 conformal symmetries, of which $\mathbf{X}_1, ..., \mathbf{X}_8$ are Killing fields, and $\mathbf{T}$ is a homothety:

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_w, \quad \mathbf{X}_4 = -y\partial_x + w\partial_z, \quad \mathbf{X}_5 = 3(z + yw^2)\partial_x - 3w\partial_y - w^3\partial_z,$$

$$\mathbf{X}_6 = 2yw\partial_x - \partial_y - w^2\partial_z, \quad \mathbf{X}_7 = -x\partial_x - y\partial_y + w\partial_w + z\partial_z, \quad \mathbf{X}_8 = 2y^3\partial_x - 3y\partial_w + 3x\partial_z,$$

$$\mathbf{T} = 2x\partial_x + y\partial_y + z\partial_z.$$

Lemma 5.1.1. *Let $n = p + q + 4$, and $\mathrm{g}_{euc}^{(p,q)}$ the flat Euclidean metric of signature (p,q). Then $\mathrm{g} = \mathrm{g}_{\mathbf{pp}}^{(2,2)} + \mathrm{g}_{euc}^{(p,q)}$ has conformal symmetry algebra of dimension $\binom{n-1}{2} + 6$.*

Proof. The Weyl tensor of g is $4(dy \wedge dw)^2$, so g is not conformally flat. Writing $\epsilon_i = \pm 1$, the metric $\mathrm{g}_{euc}^{(p,q)} = \sum_{i=1}^{p+q} \epsilon_i (du_i)^2$ admits the Killing fields $\mathbf{U}_i = \partial_{u_i}$ and $\mathbf{V}_{ij} = \epsilon_i u_i \partial_{u_j} - \epsilon_j u_j \partial_{u_i}$, where $i < j$. The metric g admits the Killing fields $\mathbf{U}_i$, $\mathbf{V}_{ij}$, $\mathbf{X}_k$, as well as $\mathbf{Y}_i = 2\epsilon_i u_i \partial_z - y\partial_{u_i}$, and $\mathbf{W}_i = 2\epsilon_i u_i \partial_x - w\partial_{u_i}$. The vector field $\tilde{\mathbf{T}} = \mathbf{T} + \sum_{i=1}^{p+q} u_i \partial_{u_i}$ is a homothety for g. $\square$

Thus, we have proven (see §5.1.4 for the $n = 3$ case):

Theorem 5.1.2. *For conformal geometry in dimension $n \geq 4$, we have $\mathfrak{S} \leq \binom{n-1}{2} + 6$. Except for Riemannian and Lorentzian signatures, this upper bound is sharp. For $n = 3$, we have $\mathfrak{S} = 4$.*

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[14] In [31], the signature $(2,2)$ pp-wave metric was announced as having submaximal conformal symmetry dimension. No proof was given there, but using tools developed in this paper, we can confirm that this is indeed correct.

5.1.2. *4-dimensional Lorentzian.* Here, $\mathfrak{g} = \mathfrak{so}_{2,4}$ and $\mathfrak{g}_0 \cong \mathbb{R} \oplus \mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$, where $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}} \cong \mathfrak{so}_{1,3}$ is the *real* Lie algebra underlying $\mathfrak{sl}_2(\mathbb{C})$. The grading element $Z \in \mathfrak{z}(\mathfrak{g}_0) \cong \mathbb{R}$ acts by $+2$ on the $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$-irrep $\mathbb{W} = H_+^2(\mathfrak{g}_-, \mathfrak{g}) \cong \bigodot^4(\mathbb{C}^2)$. Weyl tensors are identified with complex binary quartics, and their classification according to root type is the well-known Petrov classification [38, 37].

To each Petrov type, there is a collection $\mathcal{O}$ of G_0-orbits. We have (by prolongation-rigidity):

$$\mathfrak{U}_{\mathcal{O}} := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in \mathcal{O}\} = 4 + \max\{\dim(\mathfrak{ann}(\phi)) \mid 0 \neq \phi \in \mathcal{O}\}.$$

It suffices to maximize $\dim(\mathfrak{ann}(\phi))$ among representative elements from $\mathcal{O}$. Then $\mathfrak{S}_{\mathcal{O}} \leq \mathfrak{U}_{\mathcal{O}}$ by Remark 2.4.10. Let $\mathbb{C}^2 = \mathrm{span}_{\mathbb{C}}\{x, y\}$, and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, so $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$ has $\mathbb{R}$-basis $\{H, iH, X, iX, Y, iY\}$. A simple case analysis yields Table 5 (except the last column).

Type	Normal form ϕ	$\mathbb{R}$-basis for $\mathfrak{ann}(\phi)$	$\dim(\mathfrak{a}^\phi)$	$\mathfrak{a}^\phi$ filtration-rigid?
N	y^4	$Y, iY, 2Z + H$	7	×
III	xy^3	$Z + H$	5	✓
D	x^2y^2	H, iH	6	×
II	$x^2y(x-y)$	·	4	×
I	$xy(x-y)(x-ky)$	·	4	×

TABLE 5. Petrov types and upper bounds on conformal symmetry dimensions

The conformal structures below establish sharpness of the upper bounds (and filtration non-rigidity) in Table 5 in all cases except type III.

N: $\mathrm{g}_{\mathbf{pp}}^{(3,1)} = dy^2 + dz^2 + dwdx + y^2 dw^2$ (signature $(3,1)$ pp-wave):

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_w, \quad \mathbf{X}_4 = -2z\partial_x + w\partial_z,$$
$$\mathbf{X}_5 = e^{-w}(2y\partial_x + \partial_y), \quad \mathbf{X}_6 = e^w(-2y\partial_x + \partial_y), \quad \mathbf{T} = 2x\partial_x + y\partial_y + z\partial_z.$$

III: $\mathrm{g} = \frac{3}{|\Lambda|}dz^2 + e^{4z}dx^2 + 4e^z dxdy + 2e^{-2z}(dy^2 + dudx)$ (Kaigorodov metric [42, (12.35)]):

$$\mathbf{X}_1 = \partial_u, \quad \mathbf{X}_2 = \partial_x, \quad \mathbf{X}_3 = \partial_y, \quad \mathbf{X}_4 = 2x\partial_x - y\partial_y - \partial_z - 4u\partial_u.$$

Another metric is $\mathrm{g} = \frac{r^2}{x^3}(dx^2 + dy^2) - 2dudr + \frac{3}{2}xdu^2$ (Siklos metric [42, (38.1)]):

$$\mathbf{X}_1 = \partial_y, \quad \mathbf{X}_2 = \partial_u, \quad \mathbf{X}_3 = 2(x\partial_x + y\partial_y) + r\partial_r - u\partial_u, \quad \mathbf{T} = u\partial_u + r\partial_r.$$

D: $\mathrm{g} = a^2(dx^2 + \sinh^2(x)dy^2) + b^2(dz^2 - \sinh^2(z)dt^2)$ (a, b constant)

$$\mathbf{X}_1 = \partial_y, \quad \mathbf{X}_2 = \partial_t, \quad \mathbf{X}_3 = e^{-t}(\partial_z + \coth(z)\partial_t), \quad \mathbf{X}_4 = e^t(\partial_z - \coth(z)\partial_t),$$
$$\mathbf{X}_5 = -\cos(y)\partial_x + \coth(x)\sin(y)\partial_y, \quad \mathbf{X}_6 = \sin(y)\partial_x + \coth(x)\cos(y)\partial_y.$$

This is a product of two spaces of constant curvature [42, (12.8)].

II: $\mathrm{g} = dz^2 + e^{-2z}(dy^2 + 2dxdu) - 35e^{4z}dx^2 + e^{-8z}du^2$

$$\mathbf{X}_1 = \partial_u, \quad \mathbf{X}_2 = \partial_x, \quad \mathbf{X}_3 = \partial_y, \quad \mathbf{X}_4 = 2x\partial_x - y\partial_y - \partial_z - 4u\partial_u.$$

This metric is not Einstein, has Ricci scalar a nonzero constant, and appears to be new.[15] Symmetry-preserving deformations of the type III Kaigorodov metric led to the ansatz

$$\mathrm{g} = dz^2 + a_1 e^{-2z}dy^2 + a_2 e^{-2z}dxdu + a_3 e^{4z}dx^2 + a_4 e^z dydx + a_5 e^{-8z}du^2, \quad a_i \in \mathbb{R}.$$

Imposing the type II condition led to the metric indicated above.

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[15]In [42], Table 38.3 incorrectly lists (12.29) as a type II metric with 4-dimensional isometry group, while (12.29) is in fact type D, as indicated at the bottom of p.179. Also, the type II metric (13.65) is indicated as having four Killing vectors (13.66), but the fourth listed vector field is incorrect. Professor Malcolm MacCallum has indicated to us that no type II metric with 4-dimensional isometry group appears to have been known in the literature.

I: $\mathrm{g} = dx^2 + e^{-2x}dy^2 + e^x \cos(\sqrt{3}x)(dz^2 - dt^2) - 2e^x \sin(\sqrt{3}x)dzdt$ (Petrov metric [42, (12.14)]):

$$\mathbf{X}_1 = \partial_y, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_t, \quad \mathbf{X}_4 = \partial_x + y\partial_y + \frac{1}{2}(\sqrt{3}t - z)\partial_z - \frac{1}{2}(t + \sqrt{3}z)\partial_t.$$

Theorem 5.1.3. *In 4-dimensional conformal Lorentzian geometry, the maximal dimension of the conformal symmetry algebra for metrics of constant Petrov type is:*

Petrov type	N	III	D	II	I
max. sym. dim.	7	4	6	4	4

All models realizing these upper bounds are locally homogeneous near a regular point.

Proof. For type III metrics, we must show that the associated (5-dimensional) $\mathfrak{a} = \mathfrak{g}_{-1} \oplus \mathfrak{a}_0$ is filtration-rigid. As a $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$-representation, $\mathfrak{g}_{-1}$ is the space of Hermitian 2×2-matrices $\mathcal{H}$ with the action of $A \in \mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$ given by $M \mapsto AM + MA^*$. For $\mathcal{H}$, take the standard basis

$$e^1_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e^2_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e^3_{-1} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e^4_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since H acts by $\mathrm{diag}(2, 0, 0, -2)$, and Z acts as -1 on $\mathfrak{g}_{-1}$, then $e_0 := Z + H$ acts diagonally with eigenvalues $\mathcal{E}_{-1} = \{1, -1, -1, -3\}$, $\mathcal{E}_0 = \{0\}$. (All other brackets on $\mathfrak{a}$ are trivial.) Let $\mathfrak{f}$ be a filtered deformation of $\mathfrak{a}$. The first hypothesis of Proposition 4.2.4 is satisfied, as is (i), but (ii) is not. Following the argument there, we still have $\tilde{e}_0 \in \mathfrak{f}^0$ which acts diagonally in some basis $\{f^\alpha_i\}$ with the same eigenvalues as above, and with the only other non-trivial brackets $[f^1_{-1}, f^2_{-1}] = a\tilde{e}_0$, $[f^1_{-1}, f^3_{-1}] = b\tilde{e}_0$. By the Jacobi identity, $0 = \mathrm{Jac}_{\mathfrak{f}}(f^1_{-1}, f^2_{-1}, f^3_{-1}) = bf^2_{-1} - af^3_{-1}$, so $a = b = 0$. Thus, $\mathfrak{a}$ is filtration-rigid. In non-type III cases where the upper bound is realized, $\mathfrak{g}_- \subset \mathfrak{s}(u)$, so local homogeneity follows. For type III, note $[\mathfrak{a}_0, \mathfrak{g}_{-1}] = \mathfrak{g}_{-1}$. By Proposition 2.4.3, $[\mathfrak{s}_0(u), \mathfrak{g}_{-1}] \subset \mathfrak{s}_{-1}(u)$ with $\mathfrak{s}_0(u) \supset \mathfrak{a}_0$. Thus, we must have $\mathfrak{s}_0(u) = 0$ and $\mathfrak{s}_{-1}(u) = \mathfrak{g}_{-1}$, and local homogeneity follows. $\square$

5.1.3. *General Lorentzian and Riemannian.* For the Lorentzian case, let $\mathrm{g} = \mathrm{g}^{(3,1)}_{\mathbf{pp}} + \mathrm{g}^{(p,0)}_{\mathrm{euc}}$, where $\mathrm{g}^{(3,1)}_{\mathbf{pp}}$ was given in §5.1.2, and $\mathrm{g}^{(p,0)}_{\mathrm{euc}} = \sum_{i=1}^p (du^i)^2$. Then g has Killing fields $\mathbf{X}_k$ (see those for $\mathrm{g}^{(3,1)}_{\mathbf{pp}}$), $\mathbf{U}_i = \partial_{u^i}$, $\mathbf{V}_{ij} = u^i \partial_{u^j} - u^j \partial_{u^i}$, $\mathbf{Y}_i = u^i \partial_z - z\partial_{u^i}$, $\mathbf{W}_i = 2u^i \partial_x - w\partial_{u^i}$, and the homothety $\widetilde{\mathbf{T}} = \mathbf{T} + \sum_{i=1}^p u_i \partial_{u_i}$, so $\dim(\mathcal{S}) = \binom{n-1}{2} + 4$. Since $\mathfrak{U}^{\mathbb{C}} = \binom{n-1}{2} + 6$ is not realizable, it remains to show that neither is $\binom{n-1}{2} + 5$. This is done in [17], from which it follows that $\mathfrak{S} = \binom{n-1}{2} + 4$.

Given a Riemannian metric g with nowhere vanishing Weyl tensor, there exists a conformally equivalent metric $\tilde{\mathrm{g}} = \lambda^2 \mathrm{g}$ such that all conformal Killing fields of g are Killing fields for $\tilde{\mathrm{g}}$ [36]. Since the symmetry algebra is determined by restriction to any neighborhood, $\mathfrak{S}$ in the conformal Riemannian case is equal to the maximal dimension of the isometry algebra among metrics which are not conformally flat. According to Egorov's final result in [21] (incorporating results in his earlier article [20]), this number is exactly $\binom{n-1}{2} + 3$. This is realized by the product of spheres $\mathbb{S}^2 \times \mathbb{S}^{n-2}$ with their round metrics which is not conformally flat for $n \geq 4$, and has $\mathrm{SO}_3 \times \mathrm{SO}_{n-1}$ symmetry. However, we caution that there are omitted exceptions to Egorov's statement when $n = 4$ and $n = 6$.[16] In particular, with its Fubini–Study metric, $\mathbb{CP}^m$ is not conformally flat, has real dimension $n = 2m$, and has $\mathrm{SU}(m+1)$ symmetry group of dimension $m^2 + 2m$. This is strictly less than $\binom{n-1}{2} + 3$ for $n > 8$, equal to it when $n = 8$, and greater than it when $n = 4$ or 6.

For an algebraic proof of $\mathfrak{S}$ in the conformal Lorentzian and Riemannian cases, see [17].

5.1.4. *3-dimensional conformal.* In §4.3, we showed $\mathfrak{S} \leq 4$. Indeed, $\mathfrak{S} = 4$ via the models:

(5.2)

$dx^2 + dy^2 + (dz - xdy)^2$	$\mathbf{X}_1 = \partial_y, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_x + y\partial_z,$ $\mathbf{X}_4 = y\partial_x - x\partial_y + \frac{1}{2}(y^2 - x^2)\partial_z$
$dxdy + (dz - xdy)^2$	$\mathbf{X}_1 = \partial_y, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_x + y\partial_z,$ $\mathbf{X}_4 = x\partial_x - y\partial_y$

---

[16]For $n = 4$, the submaximally symmetric structure is *unique*: $\mathbb{CP}^2$, cf. Egorov [19].

5.2. **Bracket-generating distributions.**

5.2.1. $(2, 3, 5)$-*geometry.* The structure underlying a G_2/P_1 geometry is a $(2, 3, 5)$-distribution D. Locally, these can always be put in Goursat form: there exist coordinates (x, y, p, q, z), such that D is spanned by ∂_q and $\frac{d}{dx} := \partial_x + p\partial_y + q\partial_p + F\partial_z$, where $F = F(x, y, p, q, z)$. We have

$$\left[\partial_q, \frac{d}{dx}\right] = \partial_p + F_q\partial_z, \quad [\partial_q, \partial_p + F_q\partial_z] = F_{qq}\partial_z, \quad \left[\frac{d}{dx}, \partial_p + F_q\partial_z\right] = -\partial_y + H\partial_z,$$

for some $H = H(F)$. Thus, D has generic growth $(2, 3, 5)$ iff $F_{qq} \neq 0$. The flat model is obtained when $F = q^2$, and this has 14-dimensional symmetry algebra isomorphic to $\mathrm{Lie}(G_2)$ [12].

We work over $\mathbb{C}$. We have a 3-grading $\mathfrak{g} = \mathfrak{g}_{-3} \oplus ... \oplus \mathfrak{g}_3$ with $\mathfrak{g}_0 = \mathfrak{gl}(\mathfrak{g}_{-1}) \cong \mathfrak{gl}_2(\mathbb{C})$ and

$$\mathfrak{g}_{-1} \cong \mathbb{C}^2, \quad \mathfrak{g}_{-2} \cong \mathbb{C}, \quad \mathfrak{g}_{-3} \cong \mathbb{C}^2, \quad \mathfrak{g}_j \cong (\mathfrak{g}_{-j})^*,$$

as $\mathfrak{g}_0$-modules. In Example B.1, we derive $H^2_+(\mathfrak{g}_-, \mathfrak{g}) = \overset{-8}{\times}\!\!\!\!\overset{4}{\Longleftarrow}\!\!\overset{}{*} \cong \bigodot^4(\mathbb{C}^2)^*$ and $\mathfrak{S} = 7$, which recovers Cartan's result [12]. (The geometry is PR.) To each root type, we have a collection of $G_0 \cong \mathrm{GL}_2(\mathbb{C})$ orbits $\mathcal{O}$ in $\bigodot^4(\mathbb{C}^2)^*$ and we analyze each (as in §5.1.2). Let $\{e_1, e_2\}$ and $\{\omega_1, \omega_2\}$ be dual bases in $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ respectively. Take $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for a $\mathfrak{g}_0 = \mathfrak{gl}(\mathfrak{g}_{-1})$ basis. (Note the grading element $Z = -I$.) Symmetry bounds for each root type are given in Table 6. Some models, found either by Cartan [12] or Strazzullo [43, §6.10], are given in Table 7. We refer to [43] for the symmetry algebras.

Type	Normal form ϕ	Basis for $\mathfrak{ann}(\phi)$	$\dim(\mathfrak{a}^\phi)$	$\mathfrak{a}^\phi$ filtration-rigid?
(4)	ω_1^4	$Y, H - I$	7	$\times$
$(3, 1)$	$\omega_1^3\omega_2$	$2H - I$	6	$\checkmark$
$(2, 2)$	$\omega_1^2\omega_2^2$	H	6	$\times$
$(2, 1, 1)$	$\omega_1^2\omega_2(\omega_1 - \omega_2)$	$\cdot$	5	$\times$
$(1, 1, 1, 1)$	$\omega_1\omega_2(\omega_1 - \omega_2)(\omega_1 - k\omega_2)$	$\cdot$	5	$\times$

TABLE 6. Root types and upper bounds on symmetry dimensions for G_2/P_1 geometries

Type	$F(x, y, p, q, z)$	$\dim(\mathcal{S})$	Reference in [43]
(4)	q^3	7	6.1.1
$(3, 1)$	$p^3 + q^2$	4	6.1.3
$(2, 2)$	$y + \ln(q)$	6	6.3.4
$(2, 1, 1)$	pq^2	5	6.1.6
$(1, 1, 1, 1)$	pq^3	5	6.1.7

TABLE 7. $(2, 3, 5)$-distributions realizing each root type

Theorem 5.2.1. *Among $(2, 3, 5)$-distributions with constant root type:*

Root type	(4)	$(3, 1)$	$(2, 2)$	$(2, 1, 1)$	$(1, 1, 1, 1)$
max. $\dim(\mathcal{S})$	7	4 *or* 5	6	5	5

Except for type $(3, 1)$, all models realizing these upper bounds are locally homogeneous near a regular point.[17]

---

[17]Using Cartan's reduction method [12], R.L. Bryant (private communication) showed that there are no type $(3, 1)$ models with 5 symmetries. Thus, 4 is the maximal symmetry dimension for root type $(3, 1)$.

Proof. We show that in type $(3,1)$, $\mathfrak{a} = \mathfrak{g}_- \oplus \mathfrak{a}_0$ is filtration-rigid. Here, $e_0 := 2H - I$ spans $\mathfrak{a}_0$. Choose root vectors such that

$$e^1_{-1} = e_{-\alpha_1}, \quad e^2_{-1} = e_{-\alpha_1-\alpha_2}, \quad e_{-2} = e_{-2\alpha_1-\alpha_2}, \quad e^1_{-3} = e_{-3\alpha_1-\alpha_2}, \quad e^2_{-3} = e_{-3\alpha_1-2\alpha_2}$$

satisfy commutator relations $[e^1_{-1}, e^2_{-1}] = e_{-2}$ and $[e^i_{-1}, e_{-2}] = e^i_{-3}$. Then ad_{e_0} is diagonal with

$$\mathcal{E}_0 = \{0\}, \quad \mathcal{E}_{-1} = \{1, -3\}, \quad \mathcal{E}_{-2} = \{-2\}, \quad \mathcal{E}_{-3} = \{-1, -5\}.$$

Let $\mathfrak{f}$ be a filtered deformation of $\mathfrak{a}$. The first hypothesis of Proposition 4.2.4 is satisfied, as is (i), but (ii) is not. Similar to the Petrov type III case, there is $\tilde{e}_0 \in \mathfrak{f}^0$ which acts diagonally in some basis $\{f^\alpha_i\}$ with the same eigenvalues as above, and with the only other non-trivial brackets

$$[f^1_{-1}, f^2_{-1}] = f_{-2}, \quad [f^1_{-1}, f_{-2}] = f^1_{-3}, \quad [f^2_{-1}, f_{-2}] = f^2_{-3}, \quad [f^1_{-1}, f^1_{-3}] = a\tilde{e}_0, \quad [f_{-2}, f^1_{-3}] = bf^2_{-1}.$$

Since $0 = \mathrm{Jac}_\mathfrak{f}(f^1_{-1}, f^1_{-3}, f^2_{-3}) = -5af^2_{-3}$ and $0 = \mathrm{Jac}_\mathfrak{f}(f^1_{-1}, f_{-2}, f^1_{-3}) = (2a - b)f_{-2}$, then $a = b = 0$ and so $\mathfrak{a}$ is filtration-rigid. Aside from type $(3,1)$, when the upper bound is realized, $\mathfrak{g}_- \subset \mathfrak{s}(u)$. $\square$

Theorem 5.2.1 recovers the bounds found by Cartan [12] using his method of equivalence.

5.2.2. $(3,6)$-*geometry.* Bryant studied $(3,6)$-distributions in [4, 5], and constructed the associated B_3/P_3 geometry, where $B_3 = \mathrm{SO}_7(\mathbb{C})$ (or $\mathrm{SO}_{3,4}$). This is a 2-graded geometry with $\mathfrak{g}_0 \cong \mathfrak{gl}_3(\mathbb{C})$, $\mathfrak{g}_{-1} = \mathbb{C}^3$, and $\bigwedge^2 \mathfrak{g}_{-1} \cong \mathfrak{g}_{-2} \cong \mathbb{C}^3$. From $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \cong \overset{2}{*}\!\!-\!\!\overset{2}{*}\!\!\Rightarrow\!\!\overset{-6}{\times} = \mathbb{V}_{-w\cdot\lambda_2}$, generated by $w = (32)$, we obtain $\mathfrak{S} = 11$. On $(x_1, x_2, x_3, y_1, y_2, y_3)$-space, we give three models $D = \{\theta_1 = \theta_2 = \theta_3 = 0\}$, with $\theta_1 = dy_1 - x_3 dx_2$, $\theta_2 = dy_2 - x_1 dx_3$, and:

(a) $\theta_3 = dy_3 - x_2 dx_1$ (21 symmetries – flat model);
(b) $\theta_3 = dy_3 - x_2 dx_1 - (x_1 x_3)^2 dx_3$ (11 symmetries – submaximally symmetric model);
(c) $\theta_3 = dy_3 - (x_2 + y_3) dx_1$ (10 symmetries).

The symmetries can explicitly be found in `Maple 17`. The following code does this for model (b):

```
with(DifferentialGeometry): with(GroupActions): DGsetup([x1,x2,x3,y1,y2,y3],M):
dist:=evalDG([dy1-x3*dx2,dy2-x1*dx3,dy3-x2*dx1-(x1*x3)^2*dx3]):
InfinitesimalSymmetriesOfGeometricObjectFields([dist],output="list"); nops(%);
```

Model (b) was obtained by following the construction in Lemma 4.1.1. Take $\mathbb{C}^7$ with the standard anti-diagonal symmetric $\mathbb{C}$-bilinear form, define $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(\mathrm{diag}(a_1, a_2, a_3, 0, -a_3, -a_2, -a_1)) = a_i$. The simple roots are $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\alpha_3 = \epsilon_3$. The lowest weight of $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ is $-w\cdot\lambda_2 = -\alpha_1 + 3\alpha_3 = -\epsilon_1 + \epsilon_2 + 3\epsilon_3$. Since $\Phi_w = \{\alpha_3, \alpha_2 + 2\alpha_3\}$, and $w(-\lambda_2) = -\lambda_1 - \lambda_2 + 2\lambda_3 = -\alpha_1 - \alpha_2$, then $\phi_0 = e_{\alpha_3} \wedge e_{\alpha_2+2\alpha_3} \otimes e_{-\alpha_1-\alpha_2}$. Letting $\mathfrak{a}_0 = \mathfrak{ann}(\phi_0)$, then $\mathfrak{a} = \mathfrak{g}_- \oplus \mathfrak{a}_0$ has general element:

$$\left(\begin{array}{ccc|c|ccc}
a_{22} + 3a_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 \\
\hline
a_{41} & a_{42} & a_{43} & 0 & 0 & 0 & 0 \\
\hline
a_{51} & a_{52} & 0 & -a_{43} & -a_{33} & 0 & 0 \\
a_{61} & 0 & -a_{52} & -a_{42} & -a_{32} & -a_{22} & 0 \\
0 & -a_{61} & -a_{51} & -a_{41} & -a_{31} & -a_{21} & -(a_{22} + 3a_{33})
\end{array}\right).$$

Let E_{ij} denote the matrix with $a_{ij} = 1$ and zeros elsewhere. If we take the root vectors

$$e_{-\alpha_3} = E_{43} \in \mathfrak{g}_{-1}; \qquad e_{-\alpha_2-2\alpha_3} = E_{52} \in \mathfrak{g}_{-2}; \qquad e_{-\alpha_1-\alpha_2} = E_{31} \in \mathfrak{g}_0,$$

then ϕ_0 yields a filtered deformation $\mathfrak{f}$ of $\mathfrak{a}$ via $[E_{43}, E_{52}] = E_{31}$. (Note that $[E_{43}, E_{52}] = 0$ in $\mathfrak{a}$.) Letting $\mathfrak{k} = \mathfrak{a}_0$, the distribution is $\mathfrak{g}_{-1}/\mathfrak{k} \subset \mathfrak{f}/\mathfrak{k}$. Integrating the structure equations gives model (b).

5.3. Systems of 2nd order ODE.

Let $m \geq 2$. Consider a 2nd order ODE system

$$(5.3) \qquad \ddot{x}^i = f^i(t, x^j, \dot{x}^j), \qquad 1 \leq i \leq m$$

in $m = \ell - 1$ dependent variables, up to *point transformations*. Given the jet spaces $J^k = J^k(\mathbb{R}, \mathbb{R}^m)$ with contact systems $\mathcal{C}^k$, (5.3) defines a submanifold $M \subset J^2$ transverse to $\pi_1^2 : J^2 \to J^1$. Point transformations are diffeomorphisms of J^2 which preserve $\mathcal{C}^2$; all are prolongations of diffeomorphisms of J^0. On J^2, in canonical coordinates (t, x^j, p^j, q^j), $\mathcal{C}^2$ is spanned by $\{dx^j - p^j dt, \, dp^j - q^j dt\}$ and $M = \{q^i = f^i(t, x^j, p^j)\}$ fibers over J^0. This is an "external" description of the geometry. Pulling back $\mathcal{C}^2$ to M yields an "internal" description: a manifold of dimension $n = 2m + 1$, with coordinates (t, x^i, p^i), and a distribution $D = L \oplus \Pi$ consisting of:

(1) the line field L spanned by $\frac{d}{dt} := \partial_t + p^i \partial_{x^i} + f^i \partial_{p^i}$ whose integral curves correspond to solutions of (5.3), and

(2) the integrable m-dimensional distribution Π spanned by $\{\partial_{p^i}\}_{i=1}^m$. This is the fibre of the submersion $M \to J^0$, which is preserved by all point transformations.

The flat model is $\ddot{x}^i = 0$, $1 \leq i \leq m$, with symmetry algebra isomorphic to $\mathfrak{g} = A_{m+1} = \mathfrak{sl}(m+2, \mathbb{R})$.

The *Fels invariants* [23], namely the Fels curvature and torsion, are respectively:

$$(5.4) \qquad S_{jkl}^i = G_{jkl}^i - \frac{3}{m+2} G_{r(jk}^r \delta_{l)}^i, \qquad T_j^i = F_j^i - \frac{1}{m} \delta_j^i F_k^k,$$

where

$$G_{jkl}^i = \frac{\partial^3 f^i}{\partial p^j \partial p^k \partial p^l}, \qquad F_j^i = \frac{1}{2} \frac{d}{dt}\left(\frac{\partial f^i}{\partial p^j}\right) - \frac{\partial f^i}{\partial x^j} - \frac{1}{4}\frac{\partial f^i}{\partial p^k}\frac{\partial f^k}{\partial p^j}.$$

When $m \geq 3$, the structure above (also called *path geometry*) underlies an $A_{m+1}/P_{1,2}$ geometry, where $A_{m+1} = \mathrm{SL}(m+2, \mathbb{R})$. The geometry is 2-graded, with $\mathfrak{g}_0 = \mathbb{R}^2 \oplus \mathfrak{sl}(\Pi_{-1})$, $\mathfrak{g}_{\pm 1} = L_{\pm 1} \oplus \Pi_{\pm 1}$, $\mathfrak{g}_{\pm 2} = L_{\pm 1} \otimes \Pi_{\pm 1}$, and $W_+^{\mathfrak{p}}(2) = \{(21), (12)\}$ generate homogeneity $+3$ and $+2$ modules:

$$(5.5) \qquad H_+^2(\mathfrak{g}_-, \mathfrak{g}) = \;\boxtimes\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{3}{*}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{1}{*} \;\oplus\; \overset{-4}{\times}\!\!-\!\!\overset{1}{\times}\!\!-\!\!\overset{1}{*}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{1}{*}$$

$$= L_1 \otimes \left(\bigodot{}^3 \Pi_1 \otimes \Pi_{-1}\right)_0 \oplus L_1^{\otimes 2} \otimes (\Pi_1 \otimes \Pi_{-1})_0,$$

where $(\Pi_{-1})^* \cong \Pi_1$ via the Killing form, so the "0" subscripts indicate trace-free with respect to contractions. For the tensor descriptions above, we used

$$L_{-1} = \overset{2}{\times}\!\!-\!\!\overset{-1}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}, \qquad L_1 = \overset{-2}{\times}\!\!-\!\!\overset{1}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ},$$

$$\Pi_{-1} = \overset{-1}{\times}\!\!-\!\!\overset{1}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{1}{\circ}, \qquad \Pi_1 = \overset{1}{\times}\!\!-\!\!\overset{-2}{\times}\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}.$$

(e.g. The lowest roots of Π_1 and Π_{-1} are α_2 and $-\alpha_2 - \ldots - \alpha_\ell$, so by the "minus lowest weight convention", we convert their negatives into weight notation using the Cartan matrix.)

The tensors S and T correspond to $(\kappa_H)_{+3}$ and $(\kappa_H)_{+2}$ respectively. From §3.5, we have:

- $S \equiv 0$ iff $(\kappa_H)_{+3} \equiv 0$ iff the geometry admits an A_ℓ/P_1 description, i.e. (5.3) are the equations for (unparametrized) geodesics of a projective connection, cf. §5.4.
- $T \equiv 0$ iff $(\kappa_H)_{+2} \equiv 0$ iff the geometry admits an $A_\ell/P_2 \cong \mathrm{Gr}(2, \mathbb{R}^{\ell+1})$ description. This is a 1-graded geometry with underlying structure a manifold N^{2m}, endowed with a field $\mathcal{V} \subset \mathbb{P}(TN)$ of type $(2, m)$-Segré varieties. This is a *Segré* (or *almost Grassmannian*) *structure*.[18]

In [26], Grossman studies *torsion-free* path geometries, i.e. (5.3) satisfying $T \equiv 0$. If the geometry is also "geodesic", i.e. $S \equiv 0$, then $\kappa_H \equiv 0$, and so the geometry is flat, cf. [26, Thm. 1].

---

[18]When $m = 2$, this is equivalent to a signature $(2, 2)$ conformal structure. Note $A_3/P_2 \cong D_3/P_1$.

Remark 5.3.1. For $m = 2$, i.e. $A_3/P_{1,2}$, the underlying structure is a rank 3 distribution D on a 5-manifold with a splitting $D = L \oplus \Pi$, but now Π (rank 2) may not be integrable; instead $[\Pi, \Pi] \subseteq D$ [11, 6]. Indeed, $w = (23) \in W_+^{\mathfrak{p}}(2)$ generates $\overset{4}{\times}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{0}{\circ} = \bigwedge^2 \Pi_1 \otimes L_{-1}$ (homogeneity $+1$). For pairs of 2nd order ODE, the same discussion above holds provided $(\kappa_H)_{+1} \equiv 0$. Having nowhere vanishing $(\kappa_H)_{+1}$ is equivalent to $[\Pi, \Pi] = D$, so Π becomes a $(2,3,5)$-distribution (§5.2.1). In this case, our results establish 9 as an upper bound on the symmetry dimension. On a 5-manifold (x, y, p, q, z), this is realized by $\Pi = \{\partial_q, \partial_x + p\partial_y + q\partial_p + q^2\partial_z\}$ (i.e. the flat model for $(2,3,5)$-distributions) and $L = \{\partial_p + 2q\partial_z\}$. Abstractly, on $\mathrm{Lie}(G_2)/\mathfrak{p}_1$, $\Pi = \{e_{-\alpha_1}, e_{-\alpha_1-\alpha_2}\}$ and $L = \{e_{-2\alpha_1-\alpha_2}\}$ (mod $\mathfrak{p}_1$). Only $\mathfrak{gl}_2(\mathbb{C}) \cong \mathfrak{g}_0 \subset \mathfrak{p}_1$ stabilizes both Π and L. Hence, $\mathcal{S} \cong \mathfrak{p}_1^{\mathrm{op}}$.

When $m \geq 2$, $\mathfrak{S}_{(21)} = m^2 + 5$ and $\mathfrak{S}_{(12)} = m^2 + 4$, e.g. for $w = (21)$, (5.5) yields by Recipe 7,

$$J_w = \{3, m+1\}, \quad I_w = \{1\}, \quad \bar{\mathfrak{g}}/\bar{\mathfrak{p}} \cong A_1/P_1 \quad \Rightarrow \quad \dim(\mathfrak{a}_+) = 1 \quad [\text{NPR}],$$

$$\mathfrak{p}_w^{\mathrm{op}} \cong \mathfrak{p}_{1,m-1} \subset A_{m-1} \quad \Rightarrow \quad \dim(\mathfrak{a}_0) = 1 + \dim(\mathfrak{p}_w^{\mathrm{op}}) = m^2 - 2m + 3,$$

$$\mathfrak{S}_w = \mathfrak{U}_w = \dim(\mathfrak{a}(w)) = 2m + 1 + (m^2 - 2m + 3) + 1 = m^2 + 5.$$

Proposition 5.3.2. *Any submaximally symmetric 2nd order ODE system is torsion-free (not geodesic). A model, along with its symmetries, is:*

$$(5.6) \quad \begin{cases} \ddot{x}^1 \quad = 0, \\ \quad \vdots \\ \ddot{x}^{m-1} = 0, \\ \ddot{x}^m \quad = (\dot{x}^1)^3 \end{cases} \quad \left| \begin{array}{l} \mathbf{T} = \partial_t, \quad \mathbf{X}_i = \partial_{x^i}, \quad \mathbf{A}_j = t\partial_{x^j}, \quad \mathbf{B}_j^k = x^k\partial_{x^j}, \\ \quad (1 \leq i \leq m, \quad 2 \leq j \leq m, \quad 1 \leq k < m) \\ \mathbf{C} = 2t\partial_{x^1} + 3(x^1)^2\partial_{x^m}, \quad \mathbf{D}_1 = x^1\partial_{x^1} + 3x^m\partial_{x^m}, \\ \mathbf{D}_2 = t\partial_t - x^m\partial_{x^m}, \quad \mathbf{S} = t^2\partial_t + t\sum_{i=1}^m x^i\partial_{x^i} + \frac{1}{2}(x^1)^3\partial_{x^m}. \end{array} \right.$$

The symmetries are stated as vector fields $\mathbf{V} = \xi\partial_t + \varphi^i\partial_{x^i}$ on J^0, but these admit unique prolongations $\mathbf{V}^{(1)} = \mathbf{V} + \varphi_{(1)}^i\partial_{p^i}$ on J^1 and $\mathbf{V}^{(2)} = \mathbf{V}^{(1)} + \varphi_{(2)}^i\partial_{q^i}$ on J^2 via the formula

$$(5.7) \quad \varphi_{(1)}^i = \frac{d\varphi^i}{dt} - p^i\frac{d\xi}{dt}, \qquad \varphi_{(2)}^i = \frac{d\varphi_{(1)}^i}{dt} - q^i\frac{d\xi}{dt}.$$

Externally, $\mathbf{V}^{(2)}$ are everywhere tangent to $M \subset J^2$. Internally, $\mathbf{V}^{(1)}$ are identified with vector fields on M which preserve the geometric data described earlier.

Remark 5.3.3. The model (5.6) was found in the $m = 2$ case by Casey et al. [15, (3.5)].

For the scalar case $\ddot{x} = f(t, x, \dot{x})$, i.e. $m = 1$ and $A_2/P_{1,2}$, we have $H_+^2(\mathfrak{g}_-, \mathfrak{g}) = \overset{-5}{\times}\!\!-\!\!\overset{1}{\times} \oplus \overset{1}{\times}\!\!-\!\!\overset{-5}{\times}$. The two 1-dimensional irreps correspond to the Tresse relative invariants [46]:

$$(5.8) \qquad I_1 = \left(\frac{d}{dt}\right)^2(f_{pp}) - f_p\frac{d}{dt}(f_{pp}) - 4\frac{d}{dt}(f_{xp}) + 4f_p f_{xp} - 3f_x f_{pp} + 6f_{xx}, \qquad I_2 = f_{pppp},$$

where $p = \dot{x}$, and $\frac{d}{dt} = \partial_t + p\partial_x + f\partial_p$. From §4.3, $\mathfrak{S} \leq 3$, and $\mathfrak{S} = 3$ [46] is established by:

$$(5.9) \quad \begin{array}{c|c|c} \ddot{x} = \dfrac{\dot{x} - (\dot{x})^3 + a(1-(\dot{x})^2)^{3/2}}{t}, \quad a \neq 0 & \begin{array}{l} \mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = t\partial_t + x\partial_x, \\ \mathbf{X}_3 = 2tx\partial_t + (t^2 + x^2)\partial_x, \end{array} & I_1 I_2 \neq 0 \\ \hline \ddot{x} = \dfrac{\varepsilon(\dot{x})^3 - \frac{1}{2}\dot{x}}{t}, \quad \varepsilon = \pm 1 & \begin{array}{l} \mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = t\partial_t + x\partial_x, \\ \mathbf{X}_3 = 2tx\partial_t + x^2\partial_x \end{array} & I_2 = 0 \end{array}$$

5.4. **Projective structures.** A projective connection $[\nabla]$ is an equivalence class of torsion-free affine connections, where $\nabla \sim \widehat{\nabla}$ iff their unparametrized geodesics are the same. These structures underlie an A_ℓ/P_1 geometry, or equivalently an $A_\ell/P_{1,2}$ geometry with $S \equiv 0$ (see §5.3).

Locally, on an ℓ-manifold N with coordinates $(x^a)_{a=0}^{\ell-1}$, define ∇ via its Christoffel symbols

$$\nabla_{\partial_{x^a}}\partial_{x^b} = \Gamma_{ab}^c\partial_{x^c}, \qquad \Gamma_{ab}^c = \Gamma_{(ab)}^c, \qquad 0 \leq a, b, c \leq \ell - 1 =: m.$$

A curve $\sigma(t)$ has geodesic equation $(x^a)'' + \Gamma^a_{bc}(x^b)'(x^c)' = 0$. Locally, we may assume $(x^0)' \neq 0$ and solve for $t = t(x^0)$. For $i > 0$, regard x^i as functions of x^0, so the geodesic equations become

$$(5.10) \qquad \ddot{x}^i = \Gamma^0_{ab}\dot{x}^i\dot{x}^a\dot{x}^b - \Gamma^i_{ab}\dot{x}^a\dot{x}^b,$$

$$= \Gamma^0_{jk}\dot{x}^i\dot{x}^j\dot{x}^k + 2\Gamma^0_{0j}\dot{x}^i\dot{x}^j - \Gamma^i_{jk}\dot{x}^j\dot{x}^k + \Gamma^0_{00}\dot{x}^i - 2\Gamma^i_{0j}\dot{x}^j - \Gamma^i_{00}, \qquad 1 \leq i,j,k \leq m.$$

Projective equivalence $\nabla \mapsto \widehat{\nabla}$ is given operationally by $\Gamma^a_{bc} \mapsto \widehat{\Gamma}^a_{bc} := \Gamma^a_{bc} + \delta^a_b \Upsilon_c + \delta^a_c \Upsilon_b$, where Υ is an arbitrary 1-form, and (5.10) is invariant under these changes.

Now A_ℓ/P_1 gives $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $W^{\mathfrak{p}}_+(2) = \{(12)\}$, and for $\ell \geq 3$,

$$H^2_+(\mathfrak{g}_-, \mathfrak{g}) = \overset{-4}{\times}\!\!\overset{1}{\ast}\!\!\overset{1}{\ast}\!\!\overset{0}{\circ} \cdots \overset{0}{\circ}\!\!\overset{1}{\ast} = \left(\bigwedge{}^2 \mathfrak{g}_1 \otimes \mathfrak{sl}(\mathfrak{g}_{-1})\right)_0, \quad J_w = \{2,3,\ell\}, \quad I_w = \emptyset,$$

$$\mathfrak{p}^{\mathrm{op}}_w \cong \mathfrak{p}_{1,2,\ell-1} \subset A_{\ell-1}, \quad \dim(\mathfrak{a}_0) = \ell^2 - 3\ell + 5, \quad \mathfrak{U}_w = \dim(\mathfrak{g}_{-1}) + \dim(\mathfrak{a}_0) = (\ell-1)^2 + 4,$$

which recovers Egorov's result $\mathfrak{S} = (\ell-1)^2 + 4$ for $\ell \geq 3$ in [18]. Egorov also gave the model:

$$(5.11) \qquad \Gamma^0_{12} = \Gamma^0_{21} = x^1, \quad \Gamma^c_{ab} = 0 \text{ otherwise} \quad \leftrightarrow \quad \ddot{x}^i = 2x^1\dot{x}^1\dot{x}^2\dot{x}^i, \quad 1 \leq i \leq m,$$

which has the following $m^2 + 4$ point symmetries (writing $t := x^0$):

$$(5.12) \qquad \begin{cases} \partial_t, \quad \partial_{x^2}, \ldots, \partial_{x^m}, \quad x^i\partial_t, \quad x^i\partial_{x^j} \ (i \geq 1, \ j \geq 3), \\ 2t\partial_t + x^1\partial_{x^1}, \quad t\partial_t + x^2\partial_{x^2}, \quad x^1x^2\partial_t - \partial_{x^1}, \quad (x^1)^3\partial_t - 3x^1\partial_{x^2}. \end{cases}$$

On (t, x^j, p^j)-space M, the vertical subspace for the projection $M \to N$ is spanned by $\{\partial_{p^i}\}^m_{i=1}$, so (5.12) are also the symmetries of $[\nabla]$ determined by (5.11). The Fels torsion $T = (T^i_j)$ of (5.11) has matrix rank 1 with nontrivial components $T^i_1 = -p^1p^2p^i$ and $T^i_2 = (p^1)^2p^i$.

When $m = 1$, i.e. A_2/P_1, we have $\mathfrak{S} \leq 3$ from §4.3. Writing $(x^0, x^1) = (x, y)$, $\varepsilon = \pm 1$, the model

$$(5.13) \qquad \Gamma^0_{00} = -\frac{1}{2x}, \quad \Gamma^0_{11} = \frac{\varepsilon}{x}, \quad \Gamma^c_{ab} = 0 \text{ otherwise}, \qquad \begin{aligned} \mathbf{X}_1 &= \partial_x, \quad \mathbf{X}_2 = t\partial_t + x\partial_x, \\ \mathbf{X}_3 &= 2tx\partial_t + x^2\partial_x \end{aligned}$$

confirms $\mathfrak{S} = 3$ [46]. Indeed, the second model in (5.9) is the geodesic equation of $[\nabla]$.

5.5. $(2,m)$-Segré structures.

A *Segré* (or *almost Grassmannian*) structure is a $k(\ell + 1 - k)$-manifold N with a field $\mathcal{V} \subset \mathbb{P}(TN)$ of $(k, \ell + 1 - k)$-Segré varieties. These arise as underlying structures of $A_\ell/P_k \cong \mathrm{Gr}(k, \mathbb{R}^{\ell+1})$ geometries (for $k \neq 1, \ell$). We focus on the $(k, \ell) = (2, m+1)$ case. The structure is encoded by a $2 \times m$ matrix of 1-forms $\Theta = \{\Theta^A_i\}$ on N^{2m}: its 2×2 minors

$$(5.14) \qquad \Theta^A_i\Theta^B_j - \Theta^A_j\Theta^B_i = 0, \quad 1 \leq i,j \leq m, \quad 1 \leq A,B \leq 2,$$

cut out the image $\mathcal{V}|_z \subset \mathbb{P}(T_zN)$ of the Segré embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{2m-1} \cong \mathbb{P}(T_zN)$, $[u] \times [v] \mapsto [u \otimes v]$. We have $W^{\mathfrak{p}}_+(2) = \{(21), (23)\}$, and for $m \geq 2$,

$$\mathfrak{S}_{(21)} = m^2 + 5, \qquad \mathfrak{S}_{(23)} = \begin{cases} m^2 - m + 8, & m \geq 3; \\ 9, & m = 2. \end{cases}$$

Thus, $\mathfrak{S} = \mathfrak{S}_{(21)}$, but when $m = 2$ or 3, we also have $\mathfrak{S} = \mathfrak{S}_{(23)}$.

The (21)-branch for 2nd order ODE systems corresponds to the (21)-branch for Segré structures. We exhibit a Segré structure in this branch. The twistor space N of the ODE (5.6) on M is its solution space, i.e. the quotient by the integral curves of $\frac{d}{dt}$. Any solution $c(t)$ of (5.6) is given by

$$x^i = a_it + b_i + \frac{1}{2}(a_1)^3t^2\delta_{im}, \qquad p^i = a_i + (a_1)^3t\,\delta_{im},$$

where $z = (a_1, ..., a_m, b_1, ..., b_m)$ are parameters (local coordinates on N). The map $\Psi : M \to N$ is

$$a_i = p^i - (p^1)^3t\,\delta_{im}, \qquad b_i = x^i - p^it + \frac{1}{2}(p^1)^3t^2\delta_{im}.$$

This induces a curve $\zeta(t) = \Psi_*(\Pi|_{c(t)})$ in $\mathrm{Gr}(m, T_z N)$ with $\zeta(t)$ spanned by:

$$(5.15) \qquad \mathbf{Z}_i(t) = \Psi_*(\partial_{p^i}|_{c(t)}) = \partial_{a_i} - t\partial_{b_i} + \delta_{i1}\left(-3(a_1)^2 t\partial_{a_m} + \frac{3}{2}(a_1)^2 t^2 \partial_{b_m}\right).$$

This curve satisfies (5.14), where $\Theta = \{\Theta_i^A\}$ is given by

$$(5.16) \qquad \Theta_i^1 = da_i - \frac{3}{2}(a_1)^2 \delta_{im} db_1, \qquad \Theta_i^2 = db_i.$$

Thus, $\zeta(t) \subset \mathcal{V}|_z$ as t varies (and moreover fills it when regarding t as a projective parameter).

Given any symmetry $\mathbf{V}$ of (5.6), $\widetilde{\mathbf{V}} = \Psi_*(\mathbf{V}^{(1)})$ is a symmetry of the Segré structure (5.16), i.e. the ideal generated by (5.14) is preserved under $\mathcal{L}_{\widetilde{\mathbf{V}}}$. We obtain:

$$(5.17) \quad \begin{cases} \widetilde{\mathbf{T}} = -(a_1)^3 \partial_{a_m} - \sum_{i=1}^m a_i \partial_{b_i}, \quad \widetilde{\mathbf{X}}_i = \partial_{b_i}, \quad \widetilde{\mathbf{A}}_j = \partial_{a_j}, \quad \widetilde{\mathbf{B}}_j^k = a_k \partial_{a_j} + b_k \partial_{b_j}, \\ \qquad (1 \le i \le m, \quad 2 \le j \le m, \quad 1 \le k < m) \\ \widetilde{\mathbf{C}} = 2\partial_{a_1} + 6a_1 b_1 \partial_{a_m} + 3(b_1)^2 \partial_{b_m}, \quad \widetilde{\mathbf{D}}_1 = a_1 \partial_{a_1} + 3a_m \partial_{a_m} + b_1 \partial_{b_1} + 3b_m \partial_{b_m}, \\ \widetilde{\mathbf{D}}_2 = -\sum_{k=1}^{m-1} a_k \partial_{a_k} - 2a_m \partial_{a_m} - b_m \partial_{b_m}, \quad \widetilde{\mathbf{S}} = \sum_{i=1}^m b_i \partial_{a_i} + \frac{3}{2} a_1 (b_1)^2 \partial_{a_m} + \frac{1}{2} (b_1)^3 \partial_{b_m}. \end{cases}$$

When $m = 2$, $\mathrm{g} := \Theta_1^1 \Theta_2^2 - \Theta_2^1 \Theta_1^2$ is (after changing coordinates) the $(2,2)$ pp-wave (5.1). Thus,

Proposition 5.5.1. *If $m \ne 3$, any submaximally symmetric $(2, m)$-Segré structure is necessarily of ODE type, i.e. in the (21)-branch (possibly after using duality in the $m = 2$ case). For $m \ge 2$, a submaximally symmetric model $\Theta = \{\Theta_i^A\}$ is given by (5.16) with symmetries (5.17).*

Any Segré variety admits two *rulings*: writing $\sigma_u(v) = \sigma(u, v) = \sigma^v(u)$, we have for (5.16):

- m-ruling: $\sigma_u : \mathbb{P}^{m-1} \to \mathbb{P}^{2m-1}$ for $[u] \in \mathbb{P}^1$. Each m-plane in $\mathcal{R}|_z \subset \mathrm{Gr}(m, T_z N)$ has basis

$$u_1 \partial_{a_i} + u_2 \left(\partial_{b_i} + \delta_{i1} \frac{3}{2}(a_1)^2 \partial_{a_m}\right), \quad 1 \le i \le m.$$

- 2-ruling: $\sigma^v : \mathbb{P}^1 \to \mathbb{P}^{2m-1}$ for $[v] \in \mathbb{P}^{m-1}$. Each 2-plane in $\mathcal{P}|_z \subset \mathrm{Gr}(2, T_z N)$ has basis

$$v_1 \partial_{a_1} + ... + v_m \partial_{a_m}, \quad v_1 \left(\frac{3}{2}(a_1)^2 \partial_{a_m} + \partial_{b_1}\right) + v_2 \partial_{b_2} ... + v_m \partial_{b_m}.$$

This gives submanifolds $\mathcal{R} \subset \mathrm{Gr}(m, TN)$ and $\mathcal{P} \subset \mathrm{Gr}(2, TN)$. From (5.15), we see that $\Psi_*(\Pi) \subset \mathcal{R}$. The image under Ψ of each fibre of $M \to J^0$ is an m-dimensional submanifold $\Sigma \subset N$, with $T_z \Sigma \in \mathcal{R}|_z \subset T_z N$ for any $z \in \Sigma$. Conversely, every element of $\mathcal{R}$ is tangent to such a submanifold, so $\mathcal{R}$ is *integrable*. In fact, Grossman [26] shows that integrability of the m-ruling $\mathcal{R}$ is a general feature of Segré structures arising from 2nd order ODE systems by establishing an isomorphism of the bundles $M \to N$ and the $\mathbb{P}^1$-bundle $P_\Delta \to N$ whose fibres parametrize the m-ruling. The 2-ruling $\mathcal{P}$ is never integrable for such structures, except for the flat model. For Segré structures in the (23)-branch, $\mathcal{R}$ is non-integrable while $\mathcal{P}$ is integrable.

Appendix A. Yamaguchi's prolongation and rigidity theorems

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{p}$ a parabolic subalgebra. In [50], Yamaguchi proved:[19]

Theorem A.1 (Prolongation theorem). *We have $\mathfrak{g} \cong \mathrm{pr}(\mathfrak{g}_-)$ except for:*

(a) 1-gradings: A_ℓ/P_k, B_ℓ/P_1, C_ℓ/P_ℓ, D_ℓ/P_1, D_ℓ/P_ℓ, E_6/P_6, E_7/P_7.

(b) contact gradings, i.e. 2-gradings with $\dim(\mathfrak{g}_{-2}) = 1$, and non-degenerate bracket $\bigwedge^2 \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$:

$$A_\ell/P_{1,\ell}, \quad B_\ell/P_2, \quad C_\ell/P_1, \quad D_\ell/P_2, \quad G_2/P_2, \quad F_4/P_1, \quad E_6/P_2, \quad E_7/P_1, \quad E_8/P_8.$$

(c) $(\mathfrak{g}, \mathfrak{p}) \cong (A_\ell, P_{1,i})$ with $\ell \geq 3$ and $i \neq 1, \ell$, or $(C_\ell, P_{1,\ell})$ with $\ell \geq 2$.

Moreover, we always have $\mathfrak{g} \cong \mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$ except when $(\mathfrak{g}, \mathfrak{p}) \cong (A_\ell, P_1)$ or (C_ℓ, P_1).

Theorem A.2 (Rigidity theorem). *We have $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \neq 0$ if and only if $(\mathfrak{g}, \mathfrak{p})$ is: (i) 1-graded, (ii) a contact gradation, or (iii) listed in Table 8.*

G	Range	2-graded	3-graded	4-graded	5-graded	6-graded
A_ℓ	$\ell \geq 4$	$P_{1,s}, P_{2,s}, P_{s,s+1}$	$P_{1,2,s}, P_{1,s,\ell}$	$P_{1,2,s,t}$	$-$	$-$
	$\ell = 3$	$P_{1,2}$	$P_{1,2,3}$	$-$	$-$	$-$
B_ℓ	$\ell \geq 4$	P_3, P_ℓ	$P_{1,2}$	$P_{2,3}$	$-$	$-$
	$\ell = 3$	P_3	$P_{1,2}, P_{1,3}$	$P_{2,3}$	$P_{1,2,3}$	$-$
	$\ell = 2$	$-$	$P_{1,2}$	$-$	$-$	$-$
C_ℓ	$\ell \geq 4$	$P_2, P_{\ell-1}$	$P_{1,\ell}, P_{2,\ell}, P_{\ell-1,\ell}$	$P_{1,2}$	$P_{1,2,\ell}$	$P_{1,2,s}\ (s < \ell)$
	$\ell = 3$	P_2	$P_{1,3}, P_{2,3}$	$P_{1,2}$	$P_{1,2,3}$	$-$
D_ℓ	$\ell \geq 5$	$P_3, P_{1,\ell}$	$P_{1,2}$	$P_{2,3}, P_{1,2,\ell}$	$-$	$-$
	$\ell = 4$	$P_{1,4}$	$P_{1,2}$	$P_{1,2,4}$	$-$	$-$
G_2		$-$	$-$	P_1	$-$	$P_{1,2}$	$-$

Table 8. Yamaguchi-nonrigid geometries, excluding 1-graded and parabolic contact geometries

Appendix B. Dynkin diagram recipes

We use the notations of §3.2. The following Dynkin diagram recipes are well-known [1, 11].

Recipe 1. *Deleting the $I_\mathfrak{p}$ nodes from $\mathcal{D}(\mathfrak{g}, \mathfrak{p})$ yields $\mathcal{D}(\mathfrak{g}_0^{ss})$, and $\dim(\mathfrak{z}(\mathfrak{g}_0)) = |I_\mathfrak{p}|$. Also,*

$$\dim(\mathfrak{g}_-) = \frac{1}{2}(\dim(\mathfrak{g}) - \dim(\mathfrak{g}_0)), \qquad \dim(\mathfrak{p}) = \frac{1}{2}(\dim(\mathfrak{g}) + \dim(\mathfrak{g}_0)).$$

Recipe 2. *Let $\lambda \in \mathfrak{h}^*$ be the weight with coefficient $r_i = \langle \lambda, \alpha_j^\vee \rangle$ inscribed on the i-th node of $\mathcal{D}(\mathfrak{g})$. To compute $\sigma_j(\lambda)$, add r_j to adjacent coefficients, with multiplicity if there is a multiple edge directed towards the adjacent node; then replace r_j by $-r_j$.*

Recipe 3. $w = (jk) \in W^\mathfrak{p}(2)$ *iff $j \in I_\mathfrak{p}$ and $j \neq k \in I_\mathfrak{p} \cup \mathcal{N}(j)$, where $\mathcal{N}(j) = \{i \mid c_{ij} \leq -1\}$.*

By the "minus lowest weight" convention, the $\mathfrak{g}_0$-irrep $\mathbb{V}_\mu$ with lowest weight μ is denoted by the Dynkin diagram notation for $-\mu$. Below is Kostant's theorem for $H^2(\mathfrak{g}_-, \mathbb{U})$.

Recipe 4. *Let $\mathbb{U}$ be a $\mathfrak{g}$-irrep with minus lowest weight λ. Then $H^2(\mathfrak{g}_-, \mathbb{U}) \cong \bigoplus_{w \in W^\mathfrak{p}(2)} \mathbb{V}_{-w \cdot \lambda}$ as $\mathfrak{g}_0$-modules. Let $w = (jk) \in W^\mathfrak{p}(2)$, so $\Phi_w = \{\alpha_j, \sigma_j(\alpha_k)\}$. Via the isomorphism $H^2(\mathfrak{g}_-, \mathbb{U}) \cong \ker(\square) \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathbb{U}$, the $\mathfrak{g}_0$-module $\mathbb{V}_{-w \cdot \lambda}$ has the unique (up to scale) lowest weight vector*

(B.1) $\phi_0 := e_{\alpha_j} \wedge e_{\sigma_j(\alpha_k)} \otimes v,$

where $e_\gamma \in \mathfrak{g}_\gamma$ are root vectors, and $v \in \mathbb{U}$ is a weight vector with weight $w(-\lambda)$.

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[19]The rigidity list in [50] contains some minor errors, which were corrected in [51].

Let $-\mu$ be $\mathfrak{p}$-dominant, and define $J_\mu := \{j \notin I_\mathfrak{p} \mid \langle \mu, \alpha_j^\vee \rangle \neq 0\}$.

Recipe 5. *Let $\phi_0 \in \mathbb{V}_\mu$ a lowest weight vector, $\mathfrak{a}_0 = \mathfrak{ann}(\phi_0)$, $\mathfrak{p}_\mu^{\mathrm{op}} := (\mathfrak{g}_0^{ss})_{\leq 0}$ be the (opposite) parabolic in $\mathfrak{g}_0^{ss}$ in the Z_{J_μ} grading. Then $\dim(\mathfrak{a}_0) = \max\{\dim(\mathfrak{ann}(\phi)) \mid 0 \neq \phi \in \mathbb{V}_\mu\}$, with:*

$$(\text{B.2}) \qquad \mathfrak{a}_0 = \{H \in \mathfrak{h} \mid \mu(H) = 0\} \oplus \bigoplus_{\gamma \in \Delta(\mathfrak{g}_{0,\leq 0})} \mathfrak{g}_\gamma,$$

$$(\text{B.3}) \qquad \dim(\mathfrak{a}_0) = \dim(\mathfrak{p}_\mu^{\mathrm{op}}) + |I_\mathfrak{p}| - 1.$$

The notations and recipes below are new (see §3). Define $I_\mu := \{j \in I_\mathfrak{p} \mid \langle \mu, \alpha_j^\vee \rangle = 0\}$.

- Denote by $\mathcal{D}(\mathfrak{g}, \mathfrak{p}, \mu)$ the Dynkin diagram notation for $-\mu$ marked with an *asterisk* on un-crossed nodes with a nonzero coefficient, and a *square* on crossed nodes with a zero coefficient.
- Let $\mathfrak{a}(\mu) := \mathrm{pr}^\mathfrak{g}(\mathfrak{g}_-, \mathfrak{ann}(\phi_0))$, where ϕ_0 is a lowest weight vector in $\mathbb{V}_\mu$.
- If $\mathfrak{g}$ is simple, $w \in W^\mathfrak{p}(2)$, and $\mu = -w \cdot \lambda_\mathfrak{g}$, write $J_w := J_\mu$, $I_w := I_\mu$, $\mathfrak{a}(w) = \mathfrak{a}(\mu)$, etc.

Recipe 6. *$(\mathfrak{g}, \mathfrak{p}, \mu)$ is PR iff $I_\mu = \emptyset$ iff $\mathcal{D}(\mathfrak{g}, \mathfrak{p}, \mu)$ has no squares.*

Recipe 7. *If $I_\mu \neq \emptyset$, then on $\mathcal{D}(\mathfrak{g}, \mathfrak{p}, \mu)$, remove all nodes corresponding to $I_\mathfrak{p} \backslash I_\mu$ and J_μ, and any edges connected to these nodes. In the resulting diagram, remove any connected components which do not contain an I_μ node. This is $\mathcal{D}(\overline{\mathfrak{g}}, \overline{\mathfrak{p}})$ for the reduced geometry $(\overline{\mathfrak{g}}, \overline{\mathfrak{p}})$, where $\overline{\mathfrak{p}}$ corresponds to crosses on the I_μ nodes, and $\dim(\mathfrak{a}_k(\mu)) = \dim(\overline{\mathfrak{g}}_k)$, $\forall k > 0$. Combine with Recipes 1 and 5 to compute $\mathfrak{U}_\mu = \dim(\mathfrak{a}(\mu)) = \dim(\mathfrak{g}_-) + \dim(\mathfrak{a}_0) + \dim(\mathfrak{a}_+)$.*

Example B.1 *(G_2/P_1). The highest weight of $\mathfrak{g} = \mathrm{Lie}(G_2)$ is $\lambda_\mathfrak{g} = \lambda_2 = 3\alpha_1 + 2\alpha_2 = $ [Dynkin diagram $\overset{0}{\circ}\!\Lleftarrow\!\overset{1}{\circ}$] . For G_2/P_1, i.e. [Dynkin diagram $\times\!\Lleftarrow\!\circ$] , the grading element is $Z = Z_1$, $\mathfrak{g}$ is 3-graded, $\mathfrak{z}(\mathfrak{g}_0) = \mathrm{span}\{Z_1\}$, $\mathfrak{g}_0^{ss} \cong \mathfrak{sl}_2(\mathbb{C})$, and $W^\mathfrak{p}(2) = \{(12)\}$. Given $w = (12)$, we compute $w \cdot \lambda_\mathfrak{g} = w(\lambda_\mathfrak{g} + \rho) - \rho$ using Recipe 2:*

$$w \cdot \lambda_\mathfrak{g} = (12)(\,\overset{1}{\times}\!\Lleftarrow\!\overset{2}{\circ}\,) - \rho = (1)(\,\overset{7}{\times}\!\Lleftarrow\!\overset{-2}{\circ}\,) - \rho = \overset{-8}{\times}\!\Lleftarrow\!\overset{4}{\circ} = -8\lambda_1 + 4\lambda_2 = -4\alpha_1,$$

so by the minus lowest weight convention, $\mathbb{V}_{-w \cdot \lambda_\mathfrak{g}} = \overset{-8}{\times}\!\Lleftarrow\!\overset{4}{\circ}$ has homogeneity $+4$, and $W^\mathfrak{p}(2) = W_+^\mathfrak{p}(2)$. Since the lowest root of $\mathfrak{g}_1$ is $\alpha_1 = 2\lambda_1 - \lambda_2$, then as a $\mathfrak{g}_0 \cong \mathfrak{gl}_2(\mathbb{C})$ module,

$$H^2(\mathfrak{g}_-, \mathfrak{g}) = \mathbb{V}_{-w \cdot \lambda_\mathfrak{g}} = \overset{-8}{\times}\!\Lleftarrow\!\overset{4}{\circ} = \bigodot^4(\,\overset{-2}{\times}\!\Lleftarrow\!\overset{1}{\circ}\,) = \bigodot^4(\mathfrak{g}_1) = \bigodot^4(\mathfrak{g}_{-1})^* = \bigodot^4(\mathbb{C}^2)^*.$$

This recovers Cartan's result [12] that the fundamental (harmonic) curvature tensor for $(2, 3, 5)$-geometries is a binary quartic field defined on the 2-plane distribution. Furthermore,

$$\Phi_w = \{\alpha_1, 3\alpha_1 + \alpha_2\}, \qquad w(-\lambda_\mathfrak{g}) = 3\lambda_1 - 2\lambda_2 = -\alpha_2.$$

Thus, $H^2(\mathfrak{g}_-, \mathfrak{g}) \cong \ker(\Box) \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ has lowest weight vector $\phi_0 = e_{\alpha_1} \wedge e_{3\alpha_1 + \alpha_2} \otimes e_{-\alpha_2}$.

For $\overset{-8}{\times}\!\Lleftarrow\!\overset{4}{*}$, $J_w = \{2\}$, $I_w = \emptyset$, i.e. no squares. Thus, G_2/P_1 is PR, so $\mathfrak{a}(w) = \mathfrak{g}_- \oplus \mathfrak{a}_0$. Hence, $\mathfrak{p}_w^{\mathrm{op}} \cong \mathfrak{p}_1 \subset A_1$, so $\dim(\mathfrak{a}_0) = \dim(\mathfrak{p}_w^{\mathrm{op}}) = 2$. Thus, $\mathfrak{S} = \dim(\mathfrak{a}(w)) = \dim(\mathfrak{g}_-) + \dim(\mathfrak{a}_0) = 5 + 2 = 7$.

See the examples in §3.3 for applications of Recipe 7.

Appendix C. Submaximal symmetry dimensions

The tables to follow summarize data associated to the gap problem for all *regular, normal* parabolic geometries of *complex* or *split-real* type (G, P) with G *simple*. Let $\mathfrak{S}$ be the submaximal symmetry dimension, and define $\mathfrak{S}_w$ similarly but require $\mathrm{im}(\kappa_H) \subset \mathbb{V}_{-w \cdot \lambda_\mathfrak{g}}$. Then $\mathfrak{S} = \max_{w \in W_+^\mathfrak{p}(2)} \mathfrak{S}_w$. For each $w \in W_+^\mathfrak{p}(2)$, compute $\mathfrak{U}_w = \dim(\mathfrak{a}(w))$ by Recipe 7. Other than the exceptional A_2/P and B_2/P case (Table 9), we always have $\mathfrak{S}_w = \mathfrak{U}_w$, c.f. Theorem 4.1.6. Also, by Remark 4.1.10, we may simplify the calculation of $\mathfrak{S}_w$ by always passing to the minimal twistor space.

Note that $\mathfrak{a}_1 = 0$ in Tables 9, 10, 11.

G	P	$\dim(G/P)$	w	J_w	Twistor space type	$\mathfrak{S}_w$	$\mathfrak{U}_w$
A_2	P_1	2	(12)	$\{2\}$	–	3	4
	$P_{1,2}$	3	(12)	$\emptyset$	A_2/P_1	3	4
			(21)	$\emptyset$	A_2/P_2	3	4
B_2	P_1	3	(12)	$\{2\}$	–	4	5
	P_2	3	(21)	$\{1\}$	–	5	5
	$P_{1,2}$	4	(12)	$\emptyset$	B_2/P_1	4	5
			(21)	$\emptyset$	B_2/P_2	5	5

TABLE 9. Submaximal symmetry dimensions for geometries of type A_2/P and B_2/P

G	P	Range	$\dim(G/P)$	w	J_w	$\mathfrak{S}_w = \mathfrak{U}_w$
A_ℓ	P_1	$\ell \geq 3$	ℓ	(12)	$\{2,3,\ell\}$	$(\ell-1)^2 + 4$
	P_2	$\ell \geq 3$	$2(\ell-1)$	(21)	$\{3,\ell\}$	$(\ell-1)^2 + 5$
				(23)	$\begin{cases}\{1,4,\ell\}, \ell \geq 4;\\ \{1\}, \quad \ell = 3\end{cases}$	$\begin{cases}\ell^2 - 3\ell + 10, \ell \geq 4;\\ 9, \qquad \ell = 3\end{cases}$
	P_k	$3 \leq k \leq \lceil\frac{\ell}{2}\rceil$	$k(\ell+1-k)$	$(k, k+1)$	$\{1, k-1, k+2, \ell\}$	$\ell(\ell-1) - k(\ell-k) + 6$
				$(k, k-1)$	$\{1, k-2, k+1, \ell\}$	$\ell(\ell-k) + (k-1)^2 + 6$
B_ℓ	P_1	$\ell \geq 3$	$2\ell - 1$	(12)	$\{3\}$	$2\ell^2 - 5\ell + 9$
C_ℓ	P_ℓ	$\ell \geq 3$	$\binom{\ell+1}{2}$	$(\ell, \ell-1)$	$\{1, \ell-2, \ell-1\}$	$\frac{\ell(3\ell-5)}{2} + 5$
D_ℓ	P_1	$\ell \geq 4$	$2\ell - 2$	(12)	$\begin{cases}\{3\}, \quad \ell \geq 5;\\ \{3,4\}, \ell = 4\end{cases}$	$2\ell^2 - 7\ell + 12$
	P_ℓ	$\ell \geq 5$	$\binom{\ell}{2}$	$(\ell, \ell-2)$	$\{2, \ell-3, \ell-1\}$	$\frac{\ell(3\ell-11)}{2} + 16$
E_6	P_6	–	16	(65)	$\{2, 4\}$	45
E_7	P_7	–	27	(76)	$\{1, 5\}$	76

TABLE 10. Submaximal symmetry dimensions for 1-graded geometries

G	P	Range	$\dim(G/P)$	w	J_w	$\mathfrak{S}_w = \mathfrak{U}_w$
A_ℓ	$P_{1,\ell}$	$\ell \geq 3$	$2\ell - 1$	$(1, \ell)$	$\{2, \ell-1\}$	$(\ell-1)^2 + 4$
				(12)	$\begin{cases}\{2,3\}, \ell \geq 4;\\ \{2\}, \quad \ell = 3\end{cases}$	$(\ell-1)^2 + 4$
				$(\ell, \ell-1)$	$\begin{cases}\{\ell-2, \ell-1\}, \ell \geq 4;\\ \{2\}, \qquad \ell = 3\end{cases}$	$(\ell-1)^2 + 4$
B_ℓ	P_2	$\ell \geq 3$	$4\ell - 5$	(21)	$\{1, 3\}$	$2\ell^2 - 5\ell + 8$
				(23)	$\begin{cases}\{1,3,4\}, \ell \geq 4;\\ \{1,3\}, \quad \ell = 3\end{cases}$	$\begin{cases}2\ell^2 - 7\ell + 15, \ell \geq 4;\\ 11, \qquad \ell = 3\end{cases}$
C_ℓ	P_1	$\ell \geq 2$	$2\ell - 1$	(12)	$\begin{cases}\{2,3\}, \ell \geq 3;\\ \{2\}, \quad \ell = 2\end{cases}$	$\begin{cases}2\ell^2 - 5\ell + 8, \ell \geq 3;\\ 5, \qquad \ell = 2\end{cases}$
D_ℓ	P_2	$\ell \geq 4$	$4\ell - 7$	(21)	$\begin{cases}\{1,3\}, \quad \ell \geq 5;\\ \{1,3,4\}, \ell = 4\end{cases}$	$2\ell^2 - 7\ell + 11$
				(23)	$\begin{cases}\{1,3,4\}, \quad \ell \neq 5;\\ \{1,3,4,5\}, \ell = 5\end{cases}$	$2\ell^2 - 9\ell + 19$
		$\ell = 4$		(24)	$\{1, 3, 4\}$	15
G_2	P_2	–	5	(21)	$\{1\}$	7
F_4	P_1	–	15	(12)	$\{2, 3\}$	28
E_6	P_2	–	21	(24)	$\{3, 4, 5\}$	43
E_7	P_1	–	33	(13)	$\{3, 4\}$	76
E_8	P_8	–	57	(87)	$\{6, 7\}$	147

TABLE 11. Submaximal symmetry dimensions for parabolic contact geometries

G	P	Range	$\dim(G/P)$	w	J_w	I_w	$(\overline{\mathfrak{g}}, \overline{\mathfrak{p}})$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	Twistor space type	$\mathfrak{S}_w = \mathfrak{U}_w$
A_ℓ	$P_{1,2}$	$\ell \geq 3$	$2\ell-1$	(21)	$\{3,\ell\}$	$\{1\}$	A_1/P_1	✓	−	A_ℓ/P_2	$(\ell-1)^2+5$
				(12)	$\{3,\ell\}$	$\emptyset$	−	−	−	A_ℓ/P_1	$(\ell-1)^2+4$
		$\ell=3$		(23)	$\emptyset$	$\emptyset$	−	−	−	A_3/P_2	9
	$P_{1,3}$	$\ell \geq 4$	$3\ell-4$	(31)	$\{2,4,\ell\}$	$\emptyset$	−	−	−	−	$\ell^2-3\ell+9$
				(12)	$\{2,\ell\}$	$\emptyset$	−	−	−	A_ℓ/P_1	$(\ell-1)^2+4$
		$\ell=4$		(34)	$\{2\}$	$\emptyset$	−	−	−	A_4/P_3	14
	$P_{1,s}$	$4 \leq s \leq \ell-1$	$\ell s-(s-1)^2$	(12)	$\{2,3,\ell\}$	$\{s\}$	$A_{\ell-4}/P_{s-3}$	✓	−	A_ℓ/P_1	$(\ell-1)^2+4$
				(1,s)	$\{2,s-1,s+1,\ell\}$	$\emptyset$	−	−	−	−	$(\ell-2)s+(\ell-s)^2+6$
		$4 \leq s = \ell-1$		$(\ell-1,\ell)$	$\{\ell-2\}$	$\emptyset$	−	−	−	$A_\ell/P_{\ell-1}$	$(\ell-1)^2+5$
	$P_{2,s}$	$4 \leq s \leq \ell-1$	$s(\ell+3-s)-4$	(21)	$\{3,\ell\}$	$\{s\}$	$A_{\ell-4}/P_{s-3}$	✓	−	A_ℓ/P_2	$(\ell-1)^2+5$
		$4 \leq s = \ell-1$		$(\ell-1,\ell)$	$\{1,\ell-2\}$	$\{2\}$	$A_{\ell-4}/P_1$	✓	−	$A_\ell/P_{\ell-1}$	$(\ell-1)^2+5$
	$P_{s,s+1}$	$2 \leq s \leq \ell-2$	$(\ell-s)(s+1)+s$	(s,s+1)	$\{1,s-1,s+2,\ell\}$	$\{s+1\}$	A_1/P_1	✓	−	A_ℓ/P_s	$\ell(\ell-1)-s(\ell-s)+6$
				(s+1,s)	$\{1,s-1,s+2,\ell\}$	$\{s\}$	A_1/P_1	✓	−	A_ℓ/P_{s+1}	$\ell(\ell-1)-s(\ell-s)+6$
		$s=2 \leq \ell-2$		(21)	$\{\ell\}$	$\emptyset$	−	−	−	A_ℓ/P_2	$(\ell-1)^2+5$
	$P_{1,s,\ell}$	$3 \leq s \leq \ell-2$	$\ell-1+(\ell+1-s)s$	(1,ℓ)	$\{2,\ell-1\}$	$\{s\}$	$A_{\ell-4}/P_{s-2}$	✓	−	$A_\ell/P_{1,\ell}$	$(\ell-1)^2+4$
	$P_{1,2,3}$	$\ell \geq 3$	$3(\ell-1)$	(21)	$\{\ell\}\backslash\{3\}$	$\{1\}$	A_1/P_1	✓	−	A_ℓ/P_2	$(\ell-1)^2+5$
				(12)	$\{\ell\}\backslash\{3\}$	$\emptyset$	−	−	−	A_ℓ/P_1	$(\ell-1)^2+4$
		$\ell=3$	6	(23)	$\emptyset$	$\{3\}$	A_1/P_1	✓	−	A_3/P_2	9
				(32)	$\emptyset$	$\emptyset$	−	−	−	A_3/P_3	8
				(13)	$\emptyset$	$\emptyset$	−	−	−	$A_3/P_{1,3}$	8
	$P_{1,2,s}$	$4 \leq s < \ell$	$s(\ell+3-s)-3$	(21)	$\{3,\ell\}$	$\{1,s\}$	$A_1/P_1 \times A_{\ell-4}/P_{s-3}$	✓	−	A_ℓ/P_2	$(\ell-1)^2+5$
				(12)	$\{3,\ell\}$	$\{s\}$	$A_{\ell-4}/P_{s-3}$	✓	−	A_ℓ/P_1	$(\ell-1)^2+4$
	$P_{1,2,\ell}$	$\ell \geq 4$	$3(\ell-1)$	(21)	$\{3\}$	$\{1\}$	A_1/P_1	✓	−	A_ℓ/P_2	$(\ell-1)^2+5$
				(12)	$\{3\}$	$\emptyset$	−	−	−	A_ℓ/P_1	$(\ell-1)^2+4$
				(1,ℓ)	$\{\ell-1\}$	$\emptyset$	−	−	−	$A_\ell/P_{1,\ell}$	$(\ell-1)^2+4$
	$P_{1,2,3,s}$	$4 \leq s < \ell$	$3(\ell-1)+(\ell+1-s)(s-3)$	(21)	$\{\ell\}$	$\{1,s\}$	$A_1/P_1 \times A_{\ell-4}/P_{s-3}$	✓	−	A_ℓ/P_2	$(\ell-1)^2+5$
	$P_{1,2,3,\ell}$	$\ell \geq 4$	$4\ell-6$	(21)	$\emptyset$	$\{1\}$	A_1/P_1	✓	−	A_ℓ/P_2	$(\ell-1)^2+5$
		$\ell=4$		(34)	$\emptyset$	$\{4\}$	A_1/P_1	✓	−	A_ℓ/P_3	14
	$P_{1,2,s,t}$	$4 \leq s < t < \ell$	$2\ell-1+(\ell+1-s)(s-2)$ $+(\ell+1-t)(t-s)$	(21)	$\{3,\ell\}$	$\{1,s,t\}$	$A_1/P_1 \times A_{\ell-4}/P_{s-3,t-3}$	✓	✓	A_ℓ/P_2	$(\ell-1)^2+5$
	$P_{1,2,s,\ell}$	$4 \leq s < \ell$	$3(\ell-1)+(\ell-s)(s-2)$	(21)	$\{3\}$	$\{1,s\}$	$A_1/P_1 \times A_{\ell-4}/P_{s-3}$	✓	−	A_ℓ/P_2	$(\ell-1)^2+5$
		$4 \leq s = \ell-1$		$(\ell-1,\ell)$	$\{\ell-2\}$	$\{2,\ell\}$	$A_{\ell-4}/P_1 \times A_1/P_1$	✓	−	$A_\ell/P_{\ell-1}$	$(\ell-1)^2+5$
B_ℓ	P_3	$\ell \geq 4$	$6\ell-12$	(32)	$\{1,4\}$	$\emptyset$	−	−	−	−	$2\ell^2-7\ell+16$
		$\ell=3$		(32)	$\{1,2\}$	$\emptyset$	−	−	−	−	11
	P_ℓ	$\ell \geq 4$	$\binom{\ell+1}{2}$	$(\ell,\ell-1)$	$\{2,\ell-2,\ell-1\}$	$\emptyset$	−	−	−	−	$\frac{\ell(3\ell-7)}{2}+10$
	$P_{1,2}$	$\ell \geq 3$	$4\ell-4$	(12)	$\{3\}$	$\{2\}$	A_1/P_1	✓	−	B_ℓ/P_1	$2\ell^2-5\ell+9$
				(21)	$\{3\}$	$\emptyset$	−	−	−	B_ℓ/P_2	$2\ell^2-5\ell+8$
	$P_{2,3}$	$\ell \geq 4$	$6\ell-10$	(32)	$\{1,4\}$	$\{2\}$	A_1/P_1	✓	−	B_ℓ/P_3	$2\ell^2-7\ell+16$
		$\ell=3$		(32)	$\{1\}$	$\emptyset$	−	−	−	B_3/P_3	11
	$P_{1,3}$	$\ell=3$	8	(32)	$\{2\}$	$\emptyset$	−	−	−	B_3/P_3	11
	$P_{1,2,3}$	$\ell=3$	9	(32)	$\emptyset$	$\emptyset$	−	−	−	B_3/P_3	11

TABLE 12. Submaximal symmetry dimensions for Yamaguchi-nonrigid geometries in type A and B, excluding 1-graded & parabolic contact geometries

G	P	Range	$\dim(G/P)$	w	J_w	I_w	$(\bar{\mathfrak{g}},\bar{\mathfrak{p}})$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	Twistor space type	$\mathfrak{S}_w = \mathfrak{U}_w$
C_ℓ	P_2	$\ell \geq 3$	$4\ell-5$	(21)	$\{3\}$	$\emptyset$	−	−	−	−	$2\ell^2-5\ell+9$
		$\ell = 3$		(23)	$\{1,3\}$	$\emptyset$	−	−	−	−	11
	$P_{\ell-1}$	$\ell \geq 4$	$\frac{(\ell+4)(\ell-1)}{2}$	$(\ell-1,\ell)$	$\{1,\ell-2,\ell\}$	$\emptyset$	−	−	−	−	$\frac{\ell(3\ell-5)}{2}+5$
	$P_{1,2}$	$\ell \geq 3$	$4\ell-4$	(21)	$\{3\}$	$\{1\}$	A_1/P_1	✓	−	C_ℓ/P_2	$2\ell^2-5\ell+9$
				(12)	$\{3\}$	$\emptyset$	−	−	−	C_ℓ/P_1	$2\ell^2-5\ell+8$
	$P_{1,\ell}$	$\ell \geq 3$	$\frac{\ell^2+3\ell-2}{2}$	$(1,\ell)$	$\{2,\ell-1\}$	$\emptyset$	−	−	−	−	$\frac{\ell(3\ell-5)}{2}+5$
		$\ell \geq 4$		(12)	$\{2,3\}$	$\{\ell\}$	$C_{\ell-3}/P_{\ell-3}$	✓	−	C_ℓ/P_1	$2\ell^2-5\ell+8$
		$\ell = 3$		(12)	$\{2\}$	$\emptyset$	−	−	−	C_3/P_1	11
	$P_{2,\ell}$	$\ell \geq 4$	$\frac{\ell^2+5\ell-8}{2}$	(21)	$\{3\}$	$\{\ell\}$	$C_{\ell-3}/P_{\ell-3}$	✓	−	C_ℓ/P_2	$2\ell^2-5\ell+9$
		$\ell = 3$		(21)	$\emptyset$	$\emptyset$	−	−	−	C_3/P_2	12
		$\ell = 3$		(23)	$\{1\}$	$\emptyset$	−	−	−	C_3/P_2	11
	$P_{\ell-1,\ell}$	$\ell \geq 4$	$\frac{\ell^2+3\ell-2}{2}$	$(\ell-1,\ell)$	$\{1,\ell-2\}$	$\emptyset$	−	−	−	$C_\ell/P_{\ell-1}$	$\frac{\ell(3\ell-5)}{2}+5$
	$P_{1,2,3}$	$\ell \geq 3$	$6\ell-9$	(21)	$\emptyset$	$\{1\}$	A_1/P_1	✓	−	C_ℓ/P_2	$2\ell^2-5\ell+9$
	$P_{1,2,s}$	$4 \leq s < \ell$	$\frac{-6-3s^2+5s+4s\ell}{2}$	(21)	$\{3\}$	$\{1,s\}$	$A_1/P_1 \times C_{\ell-3}/P_{s-3}$	✓	✓	C_ℓ/P_2	$2\ell^2-5\ell+9$
	$P_{1,2,\ell}$	$\ell \geq 4$	$\frac{\ell^2+5\ell-6}{2}$	(21)	$\{3\}$	$\{1,\ell\}$	$A_1/P_1 \times C_{\ell-3}/P_{\ell-3}$	✓	−	C_ℓ/P_2	$2\ell^2-5\ell+9$
D_ℓ	P_3	$\ell \geq 5$	$6\ell-15$	(32)	$\begin{cases}\{1,4\}, \ell\geq 6;\\ \{1,4,5\}, \ell=5\end{cases}$	$\emptyset$	−	−	−	−	$2\ell^2-9\ell+20$
	$P_{1,2}$	$\ell \geq 4$	$4\ell-6$	(12)	$\begin{cases}\{3\}, \ell\geq 5;\\ \{3,4\}, \ell=4\end{cases}$	$\{2\}$	A_1/P_1	✓	−	D_ℓ/P_1	$2\ell^2-7\ell+12$
				(21)	$\begin{cases}\{3\}, \ell\geq 5;\\ \{3,4\}, \ell=4\end{cases}$	$\emptyset$	−	−	−	D_ℓ/P_2	$2\ell^2-7\ell+11$
	$P_{1,\ell}$	$\ell \geq 5$	$\frac{(\ell+2)(\ell-1)}{2}$	(12)	$\{3\}$	$\{\ell\}$	$D_{\ell-3}/P_{\ell-3}$	✓	−	D_ℓ/P_1	$2\ell^2-7\ell+12$
		$\ell = 4$		(12)	$\{3\}$	$\emptyset$	−	−	−	D_4/P_1	16
				(42)	$\{3\}$	$\emptyset$	−	−	−	D_4/P_4	16
	$P_{2,3}$	$\ell \geq 5$	$6\ell-13$	(32)	$\begin{cases}\{1,4\}, \ell\geq 6;\\ \{1,4,5\}, \ell=5\end{cases}$	$\{2\}$	A_1/P_1	✓	−	D_ℓ/P_3	$2\ell^2-9\ell+20$
	$P_{1,2,\ell}$	$\ell \geq 5$	$\frac{\ell^2+3\ell-6}{2}$	(12)	$\{3\}$	$\{2,\ell\}$	$A_1/P_1 \times D_{\ell-3}/P_{\ell-3}$	✓	−	D_ℓ/P_1	$2\ell^2-7\ell+12$
		$\ell = 4$		(12)	$\{3\}$	$\{2\}$	A_1/P_1	✓	−	D_4/P_1	16
				(42)	$\{3\}$	$\{2\}$	A_1/P_1	✓	−	D_4/P_4	16
G_2	P_1	−	5	(12)	$\{2\}$	$\emptyset$	−	−	−	−	7
	$P_{1,2}$	−	6	(12)	$\emptyset$	$\emptyset$	−	−	−	G_2/P_1	7

TABLE 13. Submaximal symmetry dimensions for Yamaguchi-nonrigid geometries in type C, D, G, excluding 1-graded & parabolic contact geometries

Appendix D. Direct proof of prolongation-rigidity for conformal geometry

Consider $\mathbb{C}^{n+2}$ with an anti-diagonal symmetric $\mathbb{C}$-bilinear form g, i.e. $g_{ij} = \delta_{i,j'}$, where $j' := n - j + 1$. Then

$$\mathfrak{so}(n+2,\mathbb{C}) = \left\{ \left(\begin{array}{c|c|c} a & \omega & 0 \\ \hline v & A & -\omega' \\ \hline 0 & -v' & -a \end{array} \right) : a \in \mathbb{C}, A \in \mathfrak{so}(n,\mathbb{C}), \omega \in (\mathbb{C}^n)^*, v \in \mathbb{C}^n \right\}$$

consists of matrices skew with respect to the anti-diagonal. We have $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss} = \mathbb{C} \oplus \mathfrak{so}(n,\mathbb{C})$. Symbolically, the brackets are:

(1) $\mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$: $[(a,A),(b,B)] = (0,[A,B])$.
(2) $\mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$: $[(a,A),\omega] = \omega(a-A)$.
(3) $\mathfrak{g}_0 \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$: $[(a,A),v] = (A-a)v$.
(4) $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0$: $[\omega,v] = (\omega v, \omega'v' - v\omega)$.

The $\mathfrak{g}_0$-module decomposition of $H^2(\mathfrak{g}_-,\mathfrak{g})$ is given by

$$H^2(\mathfrak{g}_-,\mathfrak{g}) \cong \begin{cases} \left(\bigwedge^2 \mathfrak{g}_1 \otimes \mathfrak{g}_0^{ss} \right)_0, & n \geq 4 \quad \text{(Weyl tensors)}; \\ \begin{array}{c}\boxed{}\boxed{}\\ \boxed{}\end{array}(\mathfrak{g}_1) & n = 3 \quad \text{(Cotton tensors)}. \end{cases}$$

If $n \neq 4$, $H^2(\mathfrak{g}_-,\mathfrak{g})$ is $\mathfrak{g}_0$-irreducible; if $n = 4$, it splits into self-dual and anti-self-dual submodules.

Let $\{e_i\}$ and $\{\omega^j\}$ be the standard (dual) bases of $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$ respectively. Define $E_i{}^j = e_i \otimes \omega^j$ and $R_\ell{}^k = E_\ell{}^k - E_{k'}{}^{\ell'} \in \mathfrak{g}_0^{ss}$. Note $R_\ell{}^k = -R_{k'}{}^{\ell'}$, and we require $k \neq \ell'$. Let $\omega^{ijk}{}_\ell = \omega^i \wedge \omega^j \otimes R_\ell{}^k$. An element $W = W_{ijk}{}^\ell \omega^{ijk}{}_\ell \in \bigwedge^2 \mathfrak{g}_1 \otimes \mathfrak{g}_0^{ss}$ has the index symmetries $W_{ijk}{}^\ell = -W_{jik}{}^\ell$, $W_{ijk}{}^\ell = -W_{ij\ell}{}^{k'}$.

Letting $A = \sum_{i<j'} A_j^i R_i{}^j$ and using the bracket relations above, the $\mathfrak{g}_0$-action on $\bigwedge^2 \mathfrak{g}_1 \otimes \mathfrak{g}_0^{ss}$ is

$$\widetilde{W} := (a,A) \cdot W = \left(2aW_{ijk}{}^\ell - W_{rjk}{}^\ell A_i^r - W_{irk}{}^\ell A_j^r - W_{ijr}{}^\ell A_k^r + W_{ijk}{}^r A_r^\ell \right) \omega^{ijk}{}_\ell$$

$$\widetilde{W}_{ijk}{}^\ell = 2aW_{ijk}{}^\ell - W_{rjk}{}^\ell A_i^r - W_{irk}{}^\ell A_j^r + \frac{1}{2}\left(-W_{ijr}{}^\ell A_k^r + W_{ijr}{}^{k'} A_{\ell'}^r + W_{ijk}{}^r A_r^\ell - W_{ij\ell'}{}^r A_r^{k'} \right),$$

where we have accounted for "anti-diagonal skew-symmetry" in the last line above.

Similarly when $n = 3$, given $C = C_{ijk}\omega^i \wedge \omega^j \otimes \omega^k$, and $\widetilde{C} = (a,A) \cdot C$, we have

$$\widetilde{C}_{ijk} = 3aC_{ijk} - C_{rjk}A_i^r - C_{irk}A_j^r - C_{ijr}A_k^r.$$

Lemma D.0.1. *Conformal geometry in dimension ≥ 3 is prolongation-rigid.*

Proof. Let $\omega \in \mathfrak{g}_1$ and $\mathfrak{k} = [\omega, \mathfrak{g}_{-1}]$. Use the $G_0 = CO(\mathfrak{g}_{-1})$ action to normalize ω (over $\mathbb{C}$) to one of two values, namely a null vector or a non-null vector. Suppose first that $n \geq 4$.

Case 1: ω is null. Take $\omega = \omega^1$. Then $\mathfrak{k} = \{(v^1, -v^i R_i{}^1) : v \in \mathfrak{g}_{-1}\}$ and $\dim(\mathfrak{k}) = n - 1$. (Note $R_n{}^1 = 0$.) Letting $(a,A) = (1, E_n{}^n - E_1{}^1)$, we obtain

$$0 = \widetilde{W}_{ijk}{}^\ell = W_{ijk}{}^\ell \left(2 - A_i^i - A_j^j + \frac{1}{2}\left(-A_k^k + A_\ell^\ell - A_{\ell'}^{\ell'} + A_{k'}^{k'} \right) \right).$$

Since $A_n^n - A_1^1 = 2$, then the only possible nonzero components are $W_{njn}{}^\ell = -W_{nj\ell'}{}^1 = -W_{jnn}{}^\ell$ for $1 < j, \ell < n$, i.e. $W = W_{njn}{}^\ell \omega^{njn}{}_\ell$. Now let $(a,A) = (0, R_c{}^1)$, $1 < c < n$. Note $A_1^c = 1 = -A_{c'}^n$. Then for $1 < j, \ell < n$ with $j \neq c'$, we have $0 = \tilde{W}_{c'jn}{}^\ell = -W_{rjn}{}^\ell A_{c'}^r = W_{njn}{}^\ell$. Thus, $W = 0$.

Case 2: ω is non-null. Take $\omega = \omega^1 + \omega^n$. Then $\mathfrak{k} = \{(v^1 + v^n, v^{i'}(R_1{}^i + R_n{}^i)) : v \in \mathfrak{g}_{-1}\}$ and $\dim(\mathfrak{k}) = n$. In particular, note that $(1, 0_{n \times n}) \in \mathfrak{k}$, and $0 = \widetilde{W} = (1, 0_{n \times n}) \cdot W = 2W$.

Now consider $n = 3$. If ω is null, then letting $(a,A) = (1, E_n{}^n - E_1{}^1)$ yields $0 = \widetilde{C}_{ijk} = C_{ijk}(3 - A_i^i - A_j^j - A_k^k)$, so $C_{ijk} = 0$ since $i \neq j$. If ω is not null, then as above $(1, 0_{n \times n}) \in \mathfrak{k}$, so again $C = 0$. $\square$

APPENDIX E. NPR GEOMETRIES

The classification of NPR geometries in Table 4 follows immediately from the results in Appendix C, but as we do not show the extensive calculations there, we sketch here an independent proof.

By Proposition 3.4.7 and Corollary 3.4.8, if G is of exceptional type or $|I_\mathfrak{p}| = 1$, then G/P is PR. Thus, we henceforth assume that G/P is a classical complex flag variety, with $|I_\mathfrak{p}| \geq 2$ (and G simple). Regularity $Z(-w \cdot \lambda_\mathfrak{g}) \geq 1$ is equivalent to (3.4) which simplifies in each case of (3.5) to:

$$(E.1) \qquad \begin{cases} (a): & Z(\lambda_\mathfrak{g}) \leq r_j - (r_k + 1)c_{kj}; \\ (b): & Z(\lambda_\mathfrak{g}) \leq r_j + (r_k + 1)(1 - c_{kj}); \\ (c): & Z(\lambda_\mathfrak{g}) \leq 2(r_i + 1). \end{cases}$$

The proof is simply an analysis of (3.5) and (E.1). We leave the A_ℓ and C_ℓ cases to the reader.

$\boxed{B_\ell \text{ or } D_\ell \text{ case}}$: Here, $c_{kj} \in \{0, -1\}$ iff $\mathfrak{g} = D_\ell$. Also, $0 \leq r_j + r_k \leq 1$ (since $j \neq k$) using:

$$(E.2) \qquad \lambda_\mathfrak{g} = \lambda_2 = \begin{cases} \alpha_1 + 2\alpha_2 + ... + 2\alpha_\ell, & \mathfrak{g} = B_\ell \quad (\ell \geq 3); \\ \alpha_1 + 2\alpha_2 + ... + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell, & \mathfrak{g} = D_\ell \quad (\ell \geq 4). \end{cases}$$

(a) Assume $c_{kj} = -2$, so $\mathfrak{g} = B_\ell$ and $(k,j) = (\ell-1, \ell)$. From (3.5), $c_{ki} = 0$, so $\ell \geq 4$, $r_j = r_k = 0$. From (E.1), $|I_\mathfrak{p}| = Z(\lambda_\mathfrak{g}) = 2$, which contradicts (E.2). Thus, $c_{kj} \in \{0, -1\}$. Then $r_j - (r_k + 1)c_{kj} \leq 2$, so from (E.1), $|I_\mathfrak{p}| = Z(\lambda_\mathfrak{g}) = 2$ and exactly one of j, k is 2. If $\mathfrak{g} = B_\ell$ or $j = 2$, then $Z(\lambda_\mathfrak{g}) \geq 3$, so $\mathfrak{g} = D_\ell$ and $k = 2 \notin I_\mathfrak{p}$. Hence, $\boxed{D_\ell/P_{1,\ell}, \ell \geq 5; w = (12); i = \ell}$

(b) Assume $\mathfrak{g} = B_\ell$. From (E.1), $|I_\mathfrak{p}| \geq 3$, so $Z(\lambda_\mathfrak{g}) \geq 5$. But since $0 \leq r_j + r_k \leq 1$ and $c_{kj} \leq -1$, then $r_j + (r_k + 1)(1 - c_{kj}) \geq 5$ only if $r_j = 0$, $r_k = 1$, $c_{kj} = -2$, and hence $(k,j) = (\ell-1, \ell)$. From (3.5), $c_{ki} = 0$, so $\ell \geq 4$, $r_j = r_k = 0$. From (E.1), $Z(\lambda_\mathfrak{g}) = 3$, a contradiction.

Thus, $\mathfrak{g} = D_\ell$. Assuming $r_j = 1$, then $r_k = 0$ and (E.1) implies $|I_\mathfrak{p}| = Z(\lambda_\mathfrak{g}) = 3$. But $j = 2 \in I_\mathfrak{p}$, so by (E.2), $Z(\lambda_\mathfrak{g}) \geq 4$, a contradiction. Hence, $r_j = 0$, and (E.1) implies $r_k = 1$, $c_{kj} = -1$, and $3 \leq |I_\mathfrak{p}| \leq Z(\lambda_\mathfrak{g}) \leq 4$. From (E.2), $I_\mathfrak{p} = \{1, 2, \ell\}$, so: $\boxed{D_\ell/P_{1,2,\ell}, \ell \geq 5; w = (12); i = \ell}$

(c) Assume $r_i = 0$, so from (E.1), $|I_\mathfrak{p}| = Z(\lambda_\mathfrak{g}) = 2$. Using (E.2), this is impossible given $c_{ij} = -1$ by (3.5). Thus, $r_i = 1$, and $2 \leq |I_\mathfrak{p}| \leq Z(\lambda_\mathfrak{g}) \leq 4$. Keeping in mind $c_{ij} = -1$, we have:

- $\boxed{B_\ell/P_{1,2}, \ell \geq 3; w = (12); i = 2}$ and $\boxed{D_\ell/P_{1,2}, \ell \geq 4; w = (12); i = 2}$
- $\boxed{B_\ell/P_{2,3}, \ell \geq 4; w = (32); i = 2}$ and $\boxed{D_\ell/P_{2,3}, \ell \geq 4; w = (32); i = 2}$
- $\boxed{D_\ell/P_{1,2,\ell}, \ell \geq 5; w = (12); i = 2}$. Also, $\boxed{D_4/P_{1,2,4}; w = (12) \text{ or } (42); i = 2}$

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