ON INTEGRABILITY OF CERTAIN RANK 2 SUB-RIEMANNIAN STRUCTURES

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Abstract. We discuss rank 2 sub-Riemannian structures on low-dimensional manifolds and prove that some of these structures in dimension 6, 7 and 8 have a maximal amount of symmetry but no integrals polynomial in momenta of low degrees, except for those coming from the Killing vector fields and the Hamiltonian, thus indicating non-integrability of the corresponding geodesic flows.

Introduction

A sub-Riemannian (SR) structure on a connected smooth manifold M consists of a completely non-holonomic (or bracket-generating) vector distribution $\Delta \subset TM$ and a Riemannian metric $g \in \Gamma(S^2_+\Delta^*)$ on it. For points $x, y \in M$ denote by $\mathcal{H}(x, y)$ the space of integral (horizontal) curves $\gamma : [0, 1] \to M$, $\dot\gamma \in \Delta$, joining x to y: $\gamma(0) = x$, $\gamma(1) = y$. It is nonempty by the Rashevsky-Chow theorem.

The length functional $l_g(\gamma) = \int_0^1 \|\dot\gamma\|_g dt$ on the space of horizontal curves defines the *sub-Riemannian* distance on M by

$$d_g(x, y) = \inf_{\gamma\in\mathcal{H}(x,y)} l_g(\gamma).$$

A curve $\gamma \in \mathcal{H}$ is called *geodesic* if it locally minimizes the length between any two close points with respect to d_g. The description of most geodesics (normal ones) is given by the Euler-Lagrange variational principle. There is a Hamiltonian reformulation of this principle, called the Pontrjagin maximum principle [PMP]. It allows one to consider the sub-Riemannian geodesic flow as the usual Hamiltonian flow on T^*M with the Hamiltonian $H(x, p) = \frac{1}{2}\|p\|_g^2$ (abnormal extremals play no role in this respect and will be ignored in this paper). We will recall this together with the other relevant material in Section 1.

BK and AV were supported by the NFR and DAAD cooperation grant 2014-2015 respectively. AV is a research fellow of Istituto Nazionale di Alta Matematica and thanks GRK 1523 (DFG) and the project FIR-2013 Geometria delle equazioni differenziali for financial support. GLG was supported by the UNCE-204020 and GACR-17-06962Y grants.

As in the standard theory of Riemannian geodesics, the metric g is *integrable* if this Hamiltonian flow is integrable on T^*M in the Liouville sense, i.e. there are almost everywhere functionally independent integrals $I_1 = H, I_2, \dots, I_n$ that Poisson-commute $\{I_k, I_l\} = 0$; see [A, AKN] and also [BF] for a review of methods and problems.

In this paper we investigate certain aspects of integrability of SR-structures on vector distributions of rank 2 (the smallest rank in non-holonomic mechanics). In general, SR-structures need not be integrable. For the first time, this was illustrated with a precise example in [MSS] by Montgomery, Shapiro and Stolin. More examples can be found in [Kr]. We will focus on left-invariant SR-structures on Carnot groups, which serve as tangent cones (nilpotent approximations) for general SR-structures. In Riemannian geometry, the tangent cone is the Euclidean space and it is integrable. This integrability does not carry over to the sub-Riemannian case.

We discuss *integrability*[1] of SR-structures and particularly pose the specific question whether it is related to the amount of *symmetry* present in these structures. On Carnot groups of dimensions up to 5 the geodesic flow of all left-invariant SR-structures are Liouville integrable (see Section 2), however starting from dimension 6 we show that the final polynomial integrals, required for Liouville integrability, cease to exist at least in low degrees (up to 6), even in the maximally symmetric situations. For precise formulations in dimension 6, 7 and 8, see Theorems 1, 2 and 3 in Sections 5, 6 and 7, respectively.

In Section 8, we reduce the corresponding systems of PDEs to systems with 2 degrees of freedom in a convenient form that allows us to consider obstructions for integrability in a uniform setting. The reduced systems provide a parametric 3-components first order system of ODEs. Its dynamics is interesting in its own right, we speculate that the case corresponding to dimension 6 is similar to a forced pendulum.

In Section 9, we complement our results with the trajectory portraits that demonstrate irregular dynamics. Our computations show that the systems exhibit chaotic behavior for various values of parameters in the reduced formulation, providing more evidence of non-integrability. In dimension 8 our study agrees with the numerical observations of [Sa].

The combination of established low-degree non-integrability, the reduced formulation (the known integrable quadratic Hamiltonians with

[1] We consider the integrals that are analytic in momenta. For a quadratic Hamiltonian H, the existence of such an integral implies by [Dar, Wh] the existence of an integral that is homogeneous polynomial in momenta. Moreover, in all our cases we need only one additional integral I commuting with H and the linear integrals, so this I can always be assumed homogeneous polynomial in momenta.

2 degrees of freedom have integrals of deg ≤ 4), and the numerical evidence strongly suggests that generic SR-structures are in general not Liouville integrable with analytic in momenta integrals. In the Riemannian setting this was recently proved in [KM2].

The technique we use in sections 5, 6 and 7 is inherited from the work [KM1], where it was exploited to prove rigorously non-existence of low-degree integrals for the Zipoy-Voorhees metric from general relativity; for related work on this topic, see [LG, MPS, V1]. We will explain the method in detail in Sections 3 and 4. In short, it allows us to reduce the search of integrals of a fixed degree d to a linear algebra problem, namely to a computation of the rank of a matrix with the size polynomially growing with d. The entries of this matrix are integers, and the computer verification, solely based on evaluation of the rank, gives a rigorous proof of the result. To the best knowledge of the authors, it is at present the only method that allows one to make non-existence statements for the class of integrals under consideration.

Acknowledgment. We thank Vladimir S. Matveev for an encouragement throughout our work.

1. SR-structures as Hamiltonian systems

Let us recall some basic facts from sub-Riemannian geometry and geometry of vector distributions.

(A). *Pontrjagin maximum principle*. Consider a completely non-holonomic distribution given as an inclusion $i:\Delta\hookrightarrow TM$ and a sub-Riemannian metric $g\in\Gamma(S^2_+\Delta^*)$ on it, yielding the isomorphism $\sharp^g:\Delta^*\to\Delta$. The following composition defines a vector bundle morphism

$$\Psi_g : T^*M \xrightarrow{i^*} \Delta^* \xrightarrow{\sharp^g} \Delta \xrightarrow{i} TM.$$

We have: $\operatorname{Ker}(\Psi_g)=\operatorname{Ann}\Delta$ and $\operatorname{CoKer}(\Psi_g)=TM/\Delta$.

Define the Hamiltonian function on T^*M as the composition

$$H : T^*M \xrightarrow{i^*} \Delta^* \xrightarrow{\sharp_g} \Delta \xrightarrow{\frac12\|\cdot\|^2_g} \mathbb{R}.$$

This function is locally described via an orthonormal frame $\xi_1,\dots,\xi_k$, considered as fiber-linear functions on T^*M: $H=\frac12\sum_1^k\xi_i^2$.

The Pontrjagin maximum principle [PMP] states that trajectories of the Hamiltonian vector field X_H in the region $\{H>0\}$ of the cotangent bundle equipped with the standard symplectic structure, when projected to M are extremals of the corresponding variational problem. They are called (normal) geodesics.

Whenever the SR-geodesic flow X_H on T^*M is Liouville integrable, the level surfaces of the integrals are cylinders $\{I = \text{const}\} \simeq \mathbb{T}^{n-r} \times \mathbb{R}^r$ with a linear dynamic on them. The possibility $r > 0$ is due to either degeneracy $k < n$ or non-compactness of M.

(B). *Vector distributions.* Given a vector distribution $\Delta \subset TM$ we define its weak derived flag by bracketing the generating vector fields: $\Delta_1 = \Delta$, $\Delta_{i+1} = [\Delta, \Delta_i]$. The distribution is non-holonomic if $\Delta \subsetneq \Delta_2$ and completely non-holonomic if $\Delta_k = TM$ for some k. We will assume that the rank of the distributions Δ_i is constant throughout M, then $(\dim \Delta_1, \dim \Delta_2, \dots, \dim \Delta_k)$ is called the growth vector of Δ.

The family of graded vector spaces $\{\mathfrak{g}_i = \Delta_i/\Delta_{i-1}\}$, equipped with the natural bracket induced by the commutators of vector fields, forms a sheaf of graded nilpotent Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ over M. In this paper we consider only the strongly regular case, when it is a bundle (i.e. the structure constants in the fiber do not depend on $x \in M$). The typical fiber is then called the Carnot algebra of Δ.

For the rank 2 distribution $\Delta \subset TM$ the prolongation is defined as follows [AK, Mon]. Let $\hat{M} = \mathbb{P}\Delta = \{(x, \ell) : x \in M, \ell \subset \Delta_x\}$ be the natural $\mathbb{S}^1$-bundle over M with the projection $\pi : \hat{M} \to M$. Then the prolonged distribution $\hat{\Delta} \subset T\hat{M}$ is given by $\hat{\Delta}_{x,\ell} = \pi_*^{-1}(\ell) \subset T_{x,\ell}\hat{M}$.

Example. *The prolongation of the tangent bundle of* $\mathbb{R}^2$ *is the Heisenberg SR-structure* $(\mathrm{Heis}_3, \Delta)$*, its prolongation is the Engel structure etc.*

Even though the SR-behavior can be quite different, the prolonged distribution has the geometry readable off the original distribution and, starting from dimension 5, we will assume that Δ is not a prolongation of a rank 2 distribution from lower dimensions.

2. SR-structures on Carnot groups of dimension 3 to 5

In this section we discuss left-invariant SR structures on low-dimensional Carnot groups G, which have appeared before in the literature, though in other contexts. We claim the following: for $\dim G \leq 5$ every such SR-structure is Liouville integrable, for distributions Δ of all ranks. However, since the concern of the paper is $\mathrm{rank}(\Delta) = 2$, we give a proof for this case only.

A Carnot group G is a graded nilpotent Lie group, with its Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ being bracket-generated by $\mathfrak{g}_1$, and distribution $\Delta \subset TG$ corresponding to it. Equipped with a left-invariant Riemannian metric on Δ, such a group naturally serves as a tangent cone at a chosen point of a general SR-structure, see e.g. [BR] for details.

In what follows we use the following notations. A basis e_i of $\mathfrak{g}$ generates the basis $\omega_i \in (\mathfrak{g}^*)^*$ of linear functions on $\mathfrak{g}^*$, given by

$$\omega_i(p) = \langle p, e_i \rangle, \quad p \in \mathfrak{g}^*.$$

We identify ω_i with the left-invariant linear functions on T^*G, and denote by θ_i their right-invariant analogs.

The Lie-Poisson structure ∇_{LP} on $\mathfrak{g}^*$ induces the Poisson structure $(\nabla_{LP}, -\nabla_{LP})$ on $\mathfrak{g}^* \oplus \mathfrak{g}^*$ and this yields the following commutation relation of the above functions with respect to the canonical symplectic structure on T^*G: If $[e_i, e_j] = c_{ij}^k e_k$, then

$$\{\omega_i, \omega_j\} = c_{ij}^k \omega_k, \ \{\omega_i, \theta_j\} = 0, \ \{\theta_i, \theta_j\} = -c_{ij}^k \theta_k.$$

As is well-known, every left-invariant Hamiltonian system on T^*G is reduced to a dynamical system on coadjoint orbits, arising from a Hamiltonian system on $\mathfrak{g}^*$ with respect to the Lie-Poisson structure, see [A] for the Riemannian and [BKM] for the sub-Riemannian cases.

(A). *Dimension 3: the Heisenberg SR-structure.* In dimension 3 the only Carnot group[2] is $G = \mathrm{Heis}_3$. The Carnot algebra is $\mathfrak{heis}_3 = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1 = \langle e_1, e_2 \rangle$, $\mathfrak{g}_2 = \langle e_3 \rangle$ and the only relation $[e_1, e_2] = e_3$.

The Hamiltonian $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$ has two integrals: $I_2 = \theta_1$ and the Casimir $I_3 = \theta_3 = \omega_3$. These $I_1 = H, I_2, I_3$ are involutive and functionally independent, yielding Liouville integrability.

There is also a fourth (noncommuting with I_2) integral $I_4 = \theta_2$ confining the motion to the cylinders $\mathbb{S}^1 \times \mathbb{R}^1 \subset T^*G = G \times \mathfrak{g}^*$, and making the system super-integrable (meaning the existence of more integrals).

Actually, for all systems considered in this paper whenever we establish Liouville integrability, the super-integrability (but not maximal super-integrability) will follow. Indeed, we will always indicate a right-invariant linear form (commuting with the left-invariant Hamiltonian) that is functionally independent of the other integrals.

(B). *Dimension 4: the Engel SR-structure.* In dimension 4 we also have only one SR-structure, related to the well-known Engel structure.

The graded nilpotent Lie algebra is $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = \langle e_1, e_2 \rangle \oplus \langle e_3 \rangle \oplus \langle e_4 \rangle$ with the (nontrivial) commutators:

$$[e_1, e_2] = e_3, \ [e_1, e_3] = e_4.$$

The Hamiltonian is $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$, and $I_2 = \theta_2$, $I_3 = \theta_3$, $I_4 = \theta_4$ together with $I_1 = H$ form a complete set of integrals. Adding $I_5 = \theta_1$ makes the Hamiltonian system super-integrable (notice though that I_5 does not commute with I_2, I_3).

[2]Left-invariant SR-structures on 3D Lie groups are considered in Appendix A.

In coordinates on G we have[3]:

$$2H = (p_1 + x_2p_3 + x_3p_4)^2 + p_2^2,$$

and the integrals are:

$$I_2 = \theta_2 = p_2 + x_1p_3 + \tfrac{1}{2}x_1^2p_4, \qquad I_4 = \theta_4 = p_4;$$
$$I_3 = \theta_3 = p_3 + x_1p_4, \qquad (I_5 = -\theta_1 = p_1).$$

Alternatively, to get an involutive set of integrals, we can use the integrals $J_2 = I_5$, $J_3 = I_4$ and the Casimir function $J_4 = \omega_3^2 - 2\omega_2\omega_4$:

$$J_2 = p_1,\ J_3 = p_4,\ J_4 = p_3^2 - 2p_2p_4 = I_3^2 - 2I_2I_4.$$

The obtained integrals establish Liouville integrability of H.

(C). *Dimension 5: the Cartan SR-structure.* In dimension 5 there are two SR-structures: one on the prolongation of the Engel structure (a partial case of the Goursat structure, easily seen to be integrable, so we skip it) and the other related to Cartan's famous $(2,3,5)$ distribution. The Carnot algebra is the positive part of the grading, corresponding to the first parabolic subalgebra of the exceptional Lie algebra Lie(G_2): $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = \langle e_1, e_2\rangle \oplus \langle e_3\rangle \oplus \langle e_4, e_5\rangle$ with the commutators

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5. \tag{1}$$

The Hamiltonian is $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$. In terms of right-invariant vector fields and the corresponding linear functions, we have the following involutive set of integrals: the Casimir function

$$I_2 = \theta_1\theta_5 - \theta_2\theta_4 + \tfrac{1}{2}\theta_3^2 = \omega_1\omega_5 - \omega_2\omega_4 + \tfrac{1}{2}\omega_3^2,$$

together with the linear integrals $I_3 = \theta_3,\ I_4 = \theta_4,\ I_5 = \theta_5$. Again adding $I_6 = \theta_1$ makes the Hamiltonian system super-integrable (the next obvious candidate $I_6' = \theta_2$ is already functionally dependent with the previous integrals; they do not commute with I_2).

In coordinates on G we have:

$$2H = (p_1 - \tfrac{1}{2}x_2p_3 - x_1x_2p_4)^2 + (p_2 + \tfrac{1}{2}x_1p_3 + x_1x_2p_5)^2,$$

and with the notation $J_\pm = x_1p_4 \pm x_2p_5$ the integrals are:

$$I_2 = p_1p_5 - p_2p_4 + \tfrac{1}{2}p_3^2 + \tfrac{1}{2}J_-^2 + \tfrac{1}{2}p_3J_+, \qquad I_4 = p_4,$$
$$I_3 = p_3, \qquad I_5 = p_5.$$

The additional integral is either $I_6 = p_1 + \frac{1}{2}x_2p_3 + (x_3 - \frac{1}{2}x_1x_2)p_4 + \frac{1}{2}x_2^2p_5$ or $I_6' = p_2 - \frac{1}{2}x_1p_3 - \frac{1}{2}x_1^2p_4 + (x_3 + \frac{1}{2}x_1x_2)p_5$.

[3]This and similar formulae are obtained via realization of the basis e_i as left-invariant vector fields on G. For the Engel structure: $e_1 = -(\partial_{x_1} + x_2\partial_{x_3} + x_3\partial_{x_4})$, $e_2 = \partial_{x_2}$, $e_3 = \partial_{x_3}$, $e_4 = \partial_{x_4}$. The right invariant vector fields are such fields on G that commute with e_j and have the same values at the unity of G.

3. Discussion: on detecting integrability

We want to understand whether integrability of a SR-geodesic flow is related to the amount of its symmetry. We consider and investigate two kinds of infinitesimal symmetries:

(i) vector fields preserving the underlying distribution Δ;
(ii) Killing symmetries whose flow also preserves g.

By the Noether theorem, every Killing symmetry field yields an integral linear in momenta (to be also called a *Noether integral*). In our main examples we find $D-2$ linear integrals in addition to the Hamiltonian, where $D = \dim(G)$, and ask whether these suffice for integrability, i.e. whether a *final* polynomial integral exists.

(A). *Integrability.* In dimension $D = 6$, we investigate all left-invariant SR-structures on Carnot groups and show non-existence of a final low-degree integral for the maximally symmetric distribution, and at the same time integrability for the maximally symmetric SR-structure.

In dimensions $D = 7, 8$ we focus on maximally-symmetric SR-structures (they have $D + 1$ independent non-involutive Noether integrals). To establish integrability we use the Tanaka theory, reviewed in Appendix B. The Hamiltonian and Noether integrals are realized in coordinates by exploiting the Baker-Campbell-Hausdorff formula.

(B). *Non-integrability.* This is more difficult to demonstrate. Few methods can detect it, and they depend on the integrability setup. For instance, analytic non-integrability on a compact manifold follows from positivity of the topological entropy, see [T]. Obstructions for algebraic integrability can be found by differential Galois theory [MR] or, for integrals rational in all coordinates, by the Painlevé test.

In contrast, we are interested in first integrals that are smooth in the base coordinates and polynomial in momenta; they are also known as Killing tensors. The above tests are not applicable to detect the existence of such integrals. The method we use to test non-existence of such integrals was proposed in [KM1]. Before going into detail in Section 4, let us explain the simple idea behind.

(C). *Our approach.* The condition governing existence of an integral of degree d is an overdetermined system of $\binom{d+D}{D-1}$ linear differential equations on $\binom{d+D-1}{D-1}$ unknown functions of D variables. It is of finite type, meaning the system is reducible to ODEs.

Checking the explicit compatibility conditions can be cumbersome. Instead, we compute all differential consequences (cf. [Wo, KM1]). This reduces the problem to linear algebra: the kernel of the system's matrix, evaluated at a fixed point, corresponds to degree d integrals, some of which are products of apriori known lower-degree integrals, thus giving a lower bound on the nullity of the matrix. Whether the final integral exists can be decided by computing the rank of this matrix. If its nullity is the minimal possible, no additional integral exists.

Similar to applications of the Galois theory or Painlevé test, our method can be implemented on a computer. Technical details of the method are given in the next section. We use the method to prove Theorems 1, 2 and 3 in a mathematically rigorous, computer-assisted manner (no approximations are involved). For an independent verification, our Maple code can be found as supplement to arXiv:1507.03082v2.

4. Method to check existence of the final integral

Similarly to [KM1] we prove, for certain systems, non-existence of the final integral F required for Liouville integrability. In all our cases (with $D = \dim G$ degrees of freedom) we have $D-2$ commuting Noether integrals, which we normalize to $p_3, \dots, p_D$. However, reduction will not be performed until Section 8, so we keep the momenta $p = (p_1, \dots, p_D)$, and the Hamiltonian is $H = H(x_1, x_2, p_1, \dots, p_D)$. A first integral that is smooth by the base variables $x = (x_1, x_2)$ and polynomial (of degree d) in momenta and that commutes with the Noether integrals $p_3, \dots, p_D$, has the form

$$F = \sum_{|\tau|=d} a_\tau(x_1, x_2)\, p^\tau. \tag{2}$$

($p^\tau = \prod_{i=1}^D p_i^{\tau_i}$ for a multi-index $\tau = (\tau_1, \dots, \tau_D)$, $|\tau| = \sum_{i=1}^D \tau_i$). The Poisson bracket relation $\{H, F\} = 0$ encodes the requirement that F is an integral. It is a homogeneous polynomial in momenta of degree $d+1$, and is equivalent to a linear PDE system, called S_d in the following.

(A). *The bounds on the number of integrals.* Instead of the differential system S_d we consider the associated system of linear equations, given by fixing a point $o \in G$. Denote by $S_d^{(k)}$ the k-th prolongation of S_d, i.e. the system obtained by differentiating the PDEs from S_d by x_1, x_2 up to total order $\leq k$. The total number of equations hence is $m_{d,k} = \binom{d+D}{D-1} \cdot \binom{k+2}{2}$. The unknowns are the derivatives (jets), whose collection we denote by $V = V_d^{(k)}$ (represented by a column vector of height $\#V$). Their number is $n_{d,k} = \#V = \binom{d+D-1}{D-1} \cdot \binom{k+3}{2}$.

Upper bound. The system $S_d^{(k)}$ evaluated at $o \in G$ has the form $M \cdot V = 0$ with some $m_{d,k} \times n_{d,k}$ matrix $M = M_d^{(k)}$. Let Λ_d be the number of linearly independent first integrals of degree d. The upper bound, in which the right hand side stabilizes for $k = d+1$ (cf. [Wo]), is

$$\Lambda_d \leq \delta_d^{(k)} := \# V_d^{(k)} - \operatorname{rank} M_d^{(k)} \,, \tag{3}$$

and we denote $\delta_d = \delta_d^{(d+1)}$, $V_d = V_d^{(d+1)}$ and $M_d = M_d^{(d+1)}$.

Lower bound. The system admits the quadratic integral $I_1 = H$ and linear integrals $I_2 = p_3, \dots, I_{D-1} = p_D$ (let d_i be the degree of I_i). The derived integrals $\prod I_i^{m_i}$ of degree $\sum m_i d_i = d$ are called *trivial*. Thus,

$$\Lambda_d \geq \Lambda_d^0 := \sum_{i=0}^{[d/2]} \binom{d-2i+D-3}{D-3}. \tag{4}$$

(B). *The procedure.* If the bounds in (3) and (4) coincide, $\Lambda_d^0 = \delta_d$, then $\Lambda_d = \Lambda_d^0$ and all integrals of degree d are trivial, confirming non-existence of the final integral in degree d.

There are two important differences to [KM1]: (i) Our model is homogeneous, so the choice of point o is not essential (in general stable values of $\delta_{d,k}$ require choosing a generic point). We choose $(x_1, x_2) = (0,0)$. (ii) The Hamiltonian H (rescaled by an integer) is a polynomial with integer coefficients (no rational expressions).

Complications arise in the computation of $\operatorname{rk}(M)$, since for a large matrix M the Gaussian elimination is costly. But simplifications are possible because M contains many zeros, and also:

1. All coefficients of M are integers, after multiplying with the common denominator of the entries.
2. At the point o, by combining F of (2) with the trivial integrals, some (superfluent) unknowns $V_{\text{spfl}} \subset V$ are removed.
3. Partial solution of the system: iteratively solve the *monomial* and *bimonomial* equations until no more such equations remain. Let V_{mon} and V_{bimon} be the corresponding unknowns.

We obtain from M_d a reduced system, $M_{\text{red}} \cdot V_{\text{red}} = 0$, with matrix M_{red} and $V_{\text{red}} = V_d \setminus (V_{\text{spfl}} \cup V_{\text{mon}} \cup V_{\text{bimon}})$. Then

$$\delta_d = \# V_d - \operatorname{rk}(M_d) = \# V_{\text{red}} + \# V_{\text{spfl}} - \operatorname{rk}(M_{\text{red}}).$$

(C). *The modular approach.* The procedure confirms non-existence of the final integral of degree d when $\delta_d = \Lambda_d^0$. The right hand side is (4), while the left hand side depends on $\operatorname{rk}(M_d)$ as in (3).

To improve the rank computation, we may work *modulo* p for a prime p. Denote by $\delta_d[p]$ the quantity δ_d computed as above, but with matrices mod p. In modular computation, the rank can decrease for specific values of p, but for sufficiently large primes p the modularly computed rank coincides with the usual one. Thus, if for some prime

p the equation $\delta_d[p] = \Lambda^0_d$ holds, the final integral of degree d does not exist. The main complication is to find a suitable p, which appears to grow fast with D. Our experiments suggest that searching for a decisive p successively is inefficient, while choosing a random increasing sequence of p turns out to be useful.

5. Left-invariant SR-structures in dimension 6

In this section we show a certain type of non-integrability for a rank 2 left-invariant distribution on a 6D Carnot group G. Every such 2-distribution Δ is encoded as the space $\mathfrak{g}_{-1}$ in the corresponding graded nilpotent Lie algebra $\mathfrak{g}$.

In 6D the growth vector is $(2,3,5,6)$ (recall we assumed that Δ is not a prolongation of another rank 2 distribution), and every such Lie algebra $\mathfrak{g}$ is a central 1D extension of the Cartan algebra from Section 2 (C), the distribution also being an integrable extension [AK].

Thus $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4 = \langle e_1, e_2\rangle \oplus \langle e_3\rangle \oplus \langle e_4, e_5\rangle \oplus \langle e_6\rangle$ has first commutators as in (1), which should be accompanied by the brackets $\mathfrak{g}_1 \otimes \mathfrak{g}_3 \to \mathfrak{g}_4$. This leads to precisely three algebras, called elliptic, parabolic and hyperbolic[4] in [AK]. We will study them in turn.

(A). *Integrability of the maximally symmetric elliptic SR-structure.* The elliptic (2,3,5,6)-distribution has the following structure equations:

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5,\ [e_1, e_4] = e_6,\ [e_2, e_5] = e_6.$$

Its symmetry algebra has dimension 8 [AK], and it is not maximally symmetric as a 2-distribution, but it supports the maximally symmetric SR-structure. Namely, defining the SR structure by the orthonormal frame e_1, e_2, we conclude that its symmetry dimension is 7 (see Appendix B). The corresponding Hamiltonian is

$$2H = \bigl(p_1 - \tfrac12 x_2 p_3 - x_1 x_2 p_4 - \tfrac12 x_1^2 x_2 p_6\bigr)^2 + \bigl(p_2 + \tfrac12 x_1 p_3 + x_1 x_2 p_5 + \tfrac12 x_1 x_2^2 p_6\bigr)^2.$$

There are two Casimir functions $I_6 = \omega_6$ and $C = \frac12(\omega_4^2 + \omega_5^2) - \omega_3\omega_6$. For the maximally symmetric Hamiltonian $H = \frac12(\omega_1^2 + \omega_2^2)$, these together with $I_1 = H$, $I_2 = \omega_1\omega_5 - \omega_2\omega_4 + \frac12\omega_3^2$ and the right-invariant linear functions $I_3 = \theta_3$, $I_4 = \theta_4$, $I_5 = \theta_5$ and I_6 form 6 involutive integrals ($C = \frac12(I_4^2 + I_5^2) - I_3 I_6$), so this system is Liouville integrable. Notice that $I_2' = \theta_1\theta_5 - \theta_2\theta_4 + \frac12\theta_3^2$ is also an integral, and $I_2 - I_2' = I_6 \cdot K$, where K is the last Killing vector field (neither I_2' nor K commute with $I_1, \dots, I_6$, but they make the system super-integrable).

[4]The (2,3,5,6)-distributions are given by a conformal quadric on $\mathfrak{g}_1$ due to conformal identification $\mathrm{ad}_{\mathfrak{g}_2} : \mathfrak{g}_1 \simeq \mathfrak{g}_3$, whence elliptic, parabolic and hyperbolic.

In coordinates: $I_3 = p_3$, $I_4 = p_4$, $I_5 = p_5$, $I_6 = p_6$ and

$$\begin{aligned}I_2 = (p_1 - \tfrac{1}{2}x_2p_3 - x_1x_2p_4 - \tfrac{1}{2}x_1^2x_2p_6)(p_5 + x_2p_6)\\ - (p_2 + \tfrac{1}{2}x_1p_3 + x_1x_2p_5 + \tfrac{1}{2}x_1x_2^2p_6)(p_4 + x_1p_6)\\ + \tfrac{1}{2}(p_3 + x_1p_4 + x_2p_5 + \tfrac{1}{2}(x_1^2 + x_2^2)p_6)^2.\end{aligned}$$

(B). *Non-integrability of the parabolic SR-structure.* The parabolic (2,3,5,6)-distribution is given by the structure equations:

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5,\ [e_1, e_4] = e_6.$$

With its 11-dimensional symmetry algebra it is the maximally symmetric non-holonomic rank 2 distribution in 6D, see [DZ, AK].

Up to equivalence there is only one left-invariant SR-structure (this follows from the fact that the Tanaka prolongation $\hat{\mathfrak{g}}$ of the Carnot algebra $\mathfrak{g}$ has $\hat{\mathfrak{g}}_0 \subset \mathfrak{gl}(\mathfrak{g}_1)$ equal to the Borel subalgebra, and the corresponding group transforms the invariant SR-structures), and it is given by the orthonormal frame e_1, e_2 (the symmetry dimension of this SR-structure is 6, and so it is not maximally symmetric). The corresponding Hamiltonian $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$ has the coordinate form

$$2H = \left(p_1 - \tfrac{1}{2}x_2p_3 - x_1x_2p_4 - \tfrac{1}{2}x_1^2x_2p_6\right)^2 + \left(p_2 + \tfrac{1}{2}x_1p_3 + x_1x_2p_5\right)^2. \quad (5)$$

There are two Casimir functions $\omega_5 = \theta_5, \omega_6 = \theta_6$, and two additional Noether integrals $\theta_3 = p_3$, $\theta_4 = p_4$, that form an involutive family $I_2 = p_3, I_3 = p_4, I_4 = p_5, I_5 = p_6$. However no other Casimirs or commuting linear integrals exist.

In search of more complicated integrals we perform the computations for the final (6th) integral of degree d and arrive at the following result.

Theorem 1. *The final integral of degree $d \leq 6$ for the Hamiltonian (5) of the left-invariant SR-structure on the parabolic (2,3,5,6)-distribution does not exist.*

Proof. First let us note that it is enough to prove non-existence of a nontrivial integral I_6 of degree 6. Indeed, if a nontrivial integral I of degree $d < 6$ exists, then $I \cdot p_6^{6-d}$ is a non-trivial integral of degree 6.

Therefore we shall apply the procedure described in Section 4 to our system with $d = 6$ only[5]. For sextic integrals, seven prolongations need to be performed in order to achieve equality for the upper bound $\delta_6 = \delta_6^{(7)}$. Our computation gives:

# all eqns	# V_6	# eqns M_{red}	# V_{red}	rk(M_{red})	δ_6
28512	20790	11816	9155	9155	130

[5]In fact, we run the test for $1 \leq d \leq 5$ as well, confirming the same result.

The last number δ_6 coincides with the number of trivial integrals $\Lambda_6^0 = 130$, and hence by the discussion in §4 there is no integral of degree 6, which is independent of and commuting with $I_2, \dots, I_5$. $\square$

(C). *Hyperbolic and other elliptic SR-structures in 6D.* The hyperbolic rank 2 distribution with growth vector $(2, 3, 5, 6)$ has the following structure equations:

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5,\ [e_1, e_5] = e_6,\ [e_2, e_4] = e_6.$$

The Hamiltonian corresponding to orthonormal frame e_1, e_2 is

$$2H = \bigl(p_1 - \tfrac{1}{2}x_2p_3 - x_1x_2p_4 - \tfrac{1}{4}x_1x_2^2p_6\bigr)^2 + \bigl(p_2 + \tfrac{1}{2}x_1p_3 + x_1x_2p_5 + \tfrac{1}{4}x_1^2x_2p_6\bigr)^2.$$

There are two Casimir functions $I_6 = \omega_6$ and $C = \omega_4\omega_5 - \omega_3\omega_6$. For the Hamiltonian $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$ these two together with $I_1 = H$ and the integrals $I_3 = \theta_3$, $I_4 = \theta_4$, $I_5 = \theta_5$ ($\theta_6 = \omega_6$) form 6 involutive integrals, but they are functionally dependent ($C = I_4I_5 - I_3I_6$).

The most general left-invariant SR-structure on both the elliptic and the hyperbolic (2,3,5,6)-distributions can be brought into the form $2H = \omega_1^2 + (a\,\omega_1 + b\,\omega_2)^2$, $b \neq 0$, with the same 4 Noether integrals. However no other Casimirs or commuting linear integrals exist.

In all these cases (except the elliptic case with $a = 0$, $b = 1$) the system is neither SR-maximally symmetric (the symmetry algebra has $\dim = 6$), nor maximally symmetric as a distribution (the symmetry algebra has $\dim = 8$).

In all these cases the search for the final integral reduces to the same problem as in (B). We can apply the machinery used in Theorem 1, and the computations show the same non-existence result (in all cases except the elliptic $a = 0$, $b = 1$). This non-existence of low degree integrals suggests that these Hamiltonians are not integrable.

6. Maximally symmetric SR-structures in dimension 7

A non-integrability effect established in the previous section happens also in higher dimensions. We noted that the parabolic distribution Δ in 6D is maximally symmetric, but for the left-invariant parabolic SR-structure (Δ, g) in 6D the symmetry algebra of (Δ, g) is minimal possible: the algebra of left-translations $\mathfrak{g}$.

In general, the symmetry algebra of a left-invariant SR-structure (Δ, g) on a Carnot group G is a graded Lie algebra $\tilde{\mathfrak{g}}$ and it contains the Lie algebra of G, namely $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \mathfrak{g}_\nu \subset \tilde{\mathfrak{g}}$. The additional part is contained at most in the zero grading[6]: $\tilde{\mathfrak{g}}/\mathfrak{g} = \tilde{\mathfrak{g}}_0$ [Mo]. Clearly this piece is at most 1-dimensional $\tilde{\mathfrak{g}}_0 \subset \mathfrak{so}(\mathfrak{g}_1, g)$.

[6]We provide a simple proof of this fact in Appendix B.

Thus $\dim \mathrm{Sym}(\Delta, g) \leq \dim \mathfrak{g} + 1$. The equality is attained if the rotation endomorphism $\phi \in \mathfrak{so}(\mathfrak{g}_1, g)$ extends (uniquely) to a grading preserving derivation of $\mathfrak{g}$. Let us investigate if such a *maximally symmetric* left-invariant SR-structure on a Carnot group is integrable.

In 6D the only maximally symmetric SR-structure is the (unique up to scale) SR-structure on the elliptic (2,3,5,6)-distribution (with $\dim \mathrm{Sym} = 7$) and it is integrable. Consider the case $\dim G = 7$.

Here the only maximally symmetric SR-structure g on a rank 2 distribution Δ on a 7D Carnot group G (that is not a prolongation from lower dimensions) with $\dim \mathrm{Sym}(\Delta, g) = 8$ has growth vector (2,3,5,7) and the following structure equations[7] of the graded nilpotent Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$:

$$\begin{gathered}[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5, \\ [e_1, e_4] = -[e_2, e_5] = e_6,\ [e_1, e_5] = [e_2, e_4] = e_7.\end{gathered} \tag{6}$$

Here $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$ and $\mathfrak{g}_0 = \langle e_2 \otimes \omega_1 - e_1 \otimes \omega_2 \rangle$. There are 3 Casimir functions ω_6, ω_7 and $\omega_3(\omega_6^2 + \omega_7^2) - \frac{1}{2}(\omega_4^2 - \omega_5^2)\omega_6 - \omega_4\omega_5\omega_7$. The involutive family of integrals $\theta_3, \dots, \theta_7$ generates these Casimirs and together with the Hamiltonian they lack 1 more integral for Liouville integrability. In local coordinates, we have

$$\begin{aligned}2H = \bigl(p_1 - \tfrac{1}{2}x_2p_3 - x_1x_2p_4 - \tfrac{1}{2}x_1^2x_2p_6 - \tfrac{1}{4}x_1x_2^2p_7\bigr)^2 \\ + \bigl(p_2 + \tfrac{1}{2}x_1p_3 + x_1x_2p_5 - \tfrac{1}{2}x_1x_2^2p_6 + \tfrac{1}{4}x_1^2x_2p_7\bigr)^2,\end{aligned} \tag{7}$$

and the integrals are $I_2 = p_3, \dots, I_6 = p_7$. Looking for the final integral I_7, we again invoke the method of Section 4 to obtain:

Theorem 2. *The final integral of degree $d \leq 6$ for the Hamiltonian (7) of the left-invariant SR-structure on the (2,3,5,7)-distribution given by* (6) *does not exist.*

Proof. We perform the same computations as in the proof of Theorem 1. In this case, our computer capacities allow us to study polynomial integrals up to degree $d = 5$. We need six prolongations to arrive at a definite conclusion, which is presented in the table:

# all eqns	# V_5	# eqns M_{red}	# V_{red}	$\mathrm{rk}(M_{\text{red}})$	δ_5
25872	16632	9397	6993	6993	166

Since the number $\delta_5 = \delta_5^{(6)}$ coincides with the number of trivial integrals $\Lambda_5^0 = 166$, we conclude absence of the final integral of degree $d \leq 5$.

[7] These are obtained from the (2,3,5,6) parabolic distribution by the central extension technique of [AK].

To handle the case of degree $d = 6$, we use the modular approach, described in Section 4 (C). The computation concludes faster, but to reach a definite answer we need a suitably large prime. In our case $p = 101$ suffices and we obtain the following result:

# all eqns	# V_6	# eqns M_{red}	#V_{red}	rk(M_{red})	$\delta_6[101]$
61776	41580	19137	15848	15848	296

This computation implies $\delta_d[p] = \Lambda^0_d$, and we conclude non-existence of the final integral of degree $d \le 6$. $\square$

Remark. *The indicated p for d = 6 is not claimed to be the minimal possible. But search for the minimal p requires more computer time. For instance, with d = 5 the computation for d = 5 gives* $\delta_d[p] > \Lambda^0_d$ *for the primes* $p = 2, 3, \ldots, 29$, *and we obtain equality (implying non-existence of degree 5 integral) for the next primes* $p = 31$, 37 *and* 41.

7. On integrability of SR-structures in dimension 8

There are two SR-structures g on a rank 2 distribution Δ on a 8D Carnot group G (that is not a prolongation from lower dimensions) with $\dim \mathrm{Sym}(\Delta, g) = 9$: one with the growth vector (2,3,5,6,8) and the other with the growth vector (2,3,5,8). The distributions are obtained by central extension from 7D as in [AK], and we take the (unique up to scale) $\mathfrak{so}(2)$-symmetric metric g (in the cases, when $\mathfrak{g}_0 \supset \mathfrak{so}(2)$).

The second SR-structure (Δ, g) has a more symmetric underlying distribution (with the symmetry dimension 12 vs. 10), but it is the first one that is integrable.

(A). *The (2,3,5,6,8) SR-structure.* The structure equations of the algebra $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_5 = \langle e_1, e_2\rangle \oplus \langle e_3\rangle \oplus \langle e_4, e_5\rangle \oplus \langle e_6\rangle \oplus \langle e_7, e_8\rangle$ are the following:

$$[e_1, e_2] = e_3,\ [e_{1,2}, e_3] = e_{4,5},\ [e_1, e_4] = [e_2, e_5] = e_6,$$
$$[e_{1,2}, e_6] = e_{7,8},\ [e_3, e_{4,5}] = i\, e_{7,8},$$

where we use complex notations $e_{a,b} = e_a + i\, e_b$. In this form it is obvious that the action of SO(2) on $\mathfrak{g}$, composed of the standard action on $\mathfrak{g}_1$, $\mathfrak{g}_3$, $\mathfrak{g}_5$ and the trivial action on $\mathfrak{g}_2$, $\mathfrak{g}_4$, is an automorphism.

The left-invariant Hamiltonian $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$ has 5 commuting right-invariant Killing fields (integrals) $I_2 = \theta_4$, $I_3 = \theta_5$, $I_4 = \theta_6$, $I_5 = \theta_7$, $I_6 = \theta_8$. In addition, there are 2 Casimir functions

$$I_7 = \omega_1\omega_8 - \omega_2\omega_7 + \omega_3\omega_6 - \frac{\omega_4^2 + \omega_5^2}{2} = \theta_1\theta_8 - \theta_2\theta_7 + \theta_3\theta_6 - \frac{\theta_4^2 + \theta_5^2}{2},$$
$$C = \omega_4\omega_7 + \omega_5\omega_8 - \tfrac{1}{2}\omega_6^2 = \theta_4\theta_7 + \theta_5\theta_8 - \tfrac{1}{2}\theta_6^2,$$

of which the second is dependent on $I_2, \dots, I_7$. Yet we have one more quadratic integral

$$I_8 = \omega_1\omega_5 - \omega_2\omega_4 + \tfrac{1}{2}\omega_3^2,$$

and it is straightforward to check that the involutive integrals $I_1 = H, I_2, \dots, I_8$ are functionally independent almost everywhere on T^*G. Consequently, the considered SR-structure is Liouville integrable. Notice that $I_8' = \theta_1\theta_5 - \theta_2\theta_4 + \frac{1}{2}\theta_3^2$ is different from I_8 and is also an integral of H, which again manifests super-integrability.

In coordinates, denoting $\sigma^2 = x_1^2 + x_2^2$, we have

$$\begin{gathered}
\omega_1 = p_1 - \tfrac{1}{2}x_2p_3 - x_1x_2p_4 - \tfrac{1}{2}x_1^2x_2p_6 - \tfrac{1}{5}(\sigma^2 + x_2^2)x_3p_7 + \tfrac{1}{5}x_1x_2x_3p_8,\\
\omega_2 = p_2 + \tfrac{1}{2}x_1p_3 + x_1x_2p_5 + \tfrac{1}{2}x_1x_2^2p_6 + \tfrac{1}{5}x_1x_2x_3p_7 - \tfrac{1}{5}(x_1^2 + \sigma^2)x_3p_8,\\
\omega_3 = p_3 + x_1p_4 + x_2p_5 + \tfrac{\sigma^2}{2}\,p_6 + (\tfrac{\sigma^2}{10}x_1 + x_2x_3)p_7 + (\tfrac{\sigma^2}{10}x_2 - x_1x_3)p_8,\\
\omega_4 = p_4 + x_1p_6 + \tfrac{1}{2}x_1^2p_7 + (\tfrac{1}{2}x_1x_2 - x_3)p_8,\\
\omega_5 = p_5 + x_2p_6 + (\tfrac{1}{2}x_1x_2 + x_3)p_7 + \tfrac{1}{2}x_2^2p_8,\\
\omega_6 = p_6 + x_1p_7 + x_2p_8,\ \omega_7 = p_7,\ \omega_8 = p_8,
\end{gathered}$$

and $\theta_i = p_i$ for $4 \le i \le 8$; the formulae for I_i follow.

(B). *The (2,3,5,8) SR-structure.* The free truncated graded nilpotent Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_4 = \langle e_1, e_2\rangle \oplus \langle e_3\rangle \oplus \langle e_4, e_5\rangle \oplus \langle e_6, e_7, e_8\rangle$ with the structure equations

$$\begin{gathered}
[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5,\\
[e_1, e_4] = e_6,\ [e_1, e_5] = [e_2, e_4] = e_7,\ [e_2, e_5] = e_8
\end{gathered}$$

was also studied in [Sa]. The left-invariant Hamiltonian $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$ has 6 commuting right-invariant Killing fields, leading to Noether integrals $I_2 = \theta_3$, $I_3 = \theta_4$, $I_4 = \theta_5$, $I_5 = \theta_6$, $I_6 = \theta_7$, $I_7 = \theta_8$. In addition, there is 1 cubic Casimir function, but it depends on the linear integrals.

Thus we again lack one final integral for integrability. To set up its computation we write the Hamiltonian in local coordinates:

$$\begin{aligned}
2H = \big(p_1 - \tfrac{1}{2}x_2p_3 - \tfrac{1}{2}(x_1^2 + x_2^2)p_5 - \tfrac{1}{4}x_1x_2^2p_7 - \tfrac{1}{6}x_2^3p_8\big)^2&\\
+ \big(p_2 + \tfrac{1}{2}x_1p_3 + \tfrac{1}{2}(x_1^2 + x_2^2)p_4 + \tfrac{1}{6}x_1^3p_6 + \tfrac{1}{4}x_1^2x_2p_7\big)^2.& \quad (8)
\end{aligned}$$

Theorem 3. *The final integral of degree $d \le 5$ for the Hamiltonian (8) of the left-invariant SR-structure on the (2,3,5,8)-distribution does not exist.*

Proof. We use again the procedure from Section 4 to show non-existence of a non-trivial integral of degree 5. After six prolongations of the PDE system, we arrive at the following table:

# all eqns	# V_5	# eqns $M_{\rm red}$	# $V_{\rm red}$	rk($M_{\rm red}$)	δ_5
48048	28512	4439	3514	3514	314

The upper bound $\delta_5 = 314 = \Lambda_5^0$ realizes the number of trivial integrals, and proves that no final (8th) integral of degree $d = 5$ exists. □

8. Reduction to the system with 2 degrees of freedom

In this section, we give a uniform description of several of the previously discussed systems in terms of first order ODE systems in 3D. In particular, we reformulate in this way the three systems exhibiting non-integrable behavior, namely the (2,3,5,6) parabolic, (2,3,5,7) elliptic and (2,3,5,8) free truncated SR-structures given by the Hamiltonians (5), (7) and (8). In addition, the same reduction can be performed for the general (2,3,5,6) elliptic and hyperbolic SR-structures.

First, note that in all these cases the Hamiltonian is a sum of two squares and so can be expressed as

$$2H = \rho^2 \cos^2 z + \rho^2 \sin^2 z, \tag{9}$$

and $p_i = c_i$ for $i = 3, \dots, D$ are the Noether integrals. Symplectic reduction via these integrals (fixing them and forgetting about x_i, $3 \le i \le D$, of which nothing depends) is a classical procedure, see [Wh, A]. Thus, in view of (9), Hamilton's equations can be rewritten in terms of x, y, z and ρ (as well as $c_3, \dots, c_8$).

For instance, in the case of the parabolic (2,3,5,6)-problem, we express the coordinates p_1, p_2 in terms of the coordinate z as follows:

$$\begin{aligned} p_1 &= \rho \cos z + \tfrac{1}{2} x_2 c_3 + x_1 x_2 c_4 + \tfrac{1}{2} x_1^2 x_2 c_6, \\ p_2 &= \rho \sin z - \tfrac{1}{2} x_1 c_3 - x_1 x_2 c_5. \end{aligned}$$

Next, we can confine to an energy shell, that is fix $H = \frac{1}{2}\rho^2 = \text{const}$, which reduces the dynamics to the manifold $S_1 T^* \mathbb{R}^2 = \mathbb{R}^2(x, y) \times S^1(z)$, where we let $x = x_1$, $y = x_2$. Without loss of generality we can assume $\rho = 1$. After an appropriate change of coordinates, the Hamiltonian equation $\dot\eta = \{\eta, H\}$ on the energy shell writes as the 3×3 system:

$$\dot x = \cos z, \ \dot y = \sin z, \ \dot z = Q(x, y), \tag{10}$$

where $Q = Q(x, y)$ is a quadratic polynomial. This polynomial can be brought to the following normal form ($a \ne 0$ & $b \ne 0$)

$$Q = Q_1(x, y) = a\, x^2 + b\, y \quad \text{for } D = 6 \text{ parabolic}, \tag{11}$$

$$Q = Q_2(x, y) = a\, x^2 + b\, y^2 + c \quad \text{for } D = 7, 8 \tag{12}$$

(the latter formula contains also the 6D elliptic and hyperbolic cases). Take, for instance, the $6D$ parabolic case, formula (11). In this example, we have $a = c_6/2$ and $b = c_5$, and we assume $a, b \neq 0$.

Notice that the condition $a, b \neq 0$ is important, as otherwise the system fibers over a 2D flow, which can never be chaotic.

A similar effect happens for $a = b$ and $Q = Q_2(x, y)$, where a change of variables $x = r\cos\psi$, $y = r\sin\psi$ reduces the system to a 2D flow with coordinates r and $s = z - \psi$. This latter case corresponds to the 6D elliptic maximally symmetric SR-structure. The corresponding 3D system possesses the following integral

$$F = \tfrac{1}{4} a\, r^4 + \tfrac{c}{2} r^2 - r\sin(s),$$

which corresponds exactly to the integral I_2 identified in Section 5, cf. also [V2]. However, for the general a, b, it will be shown in the next section that the system exhibits a chaotic behavior.

Remark. *One can check that in the complement to a hypersurface the following 1-form on $\mathbb{R}^2(x, y) \times S^1(z)$ is contact:*

$$\alpha = \tfrac{1}{3}(a\, x^3 dy - b\, y^3 dx) + \tfrac{c}{2}(x\, dy - y\, dx) + \cos z\, dx + \sin z\, dy.$$

In this domain its Reeb vector field R_α, given by the two conditions $\alpha(R_\alpha) = 1$ and $d\alpha(R_\alpha, \cdot) = 0$, preserves the volume form $\alpha \wedge d\alpha$ and so is divergence-free with respect to it (the Reeb field R_α plays a distinguished role in contact geometry). Our vector field, given by (10) for $Q = Q_2(x, y)$, is proportional to R_α, and so has the same trajectories. For $Q = Q_1(x, y)$ the situation is similar, if α is properly modified.

We conclude this section with a note on the resemblance of system (10) to a driven pendulum in the case $Q = Q_1(x, y)$. Let us eliminate x, y from (10). Differentiating $\dot{z}$ and replacing $\dot{x}$ and $\dot{y}$ via ODE (10), we get the following 3rd order ODE on $z = z(t)$, where $\Delta = \frac{d}{dt} \circ \frac{1}{\cos z}$:

$$\Delta\,(z'' - b\sin z) = 2a\cos z,$$

which can be written in non-local form as:

$$z'' - b\sin z = \Delta^{-1}(2a\cos z) = 2a\cos z\, D_t^{-1}\cos z. \tag{13}$$

In this form it resembles the driven pendulum $z'' - b\sin z = a\cos kt$ without dissipation. For $a = 0$ system (13) is the simple pendulum when $b < 0$, while for $b > 0$ the second term on the left hand side describes a repulsive power[8]. However, contrary to the driven pendulum, where the right hand side is an external force, system (13) seems to be self-driven. The evolution of this system is shown in Fig. 1 for three different parameter combinations. The orbital dynamics in Fig. 1 is

[8]For instance, when $z \ll 1$ the solutions are hyperbolic.

quite complex and resembles the dynamics of the damped driven pendulum (see, e.g., Fig. 9, 10 in [H] and references therein), indicating non-integrability. This resemblance appeals for a more systematic numerical analysis of system (10), which is provided in the next section.

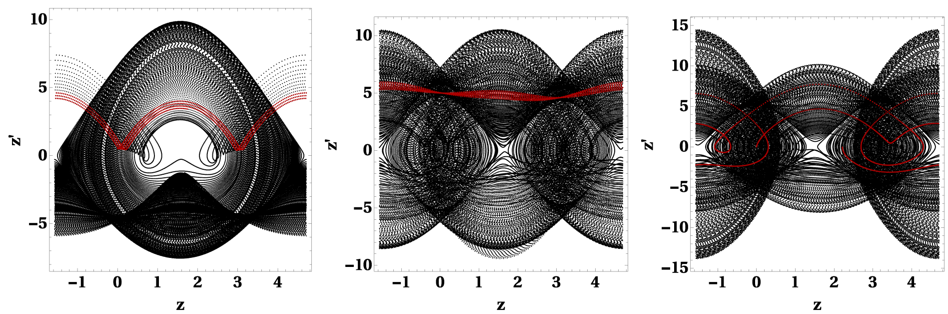

FIGURE 1. The orbital evolution of variables z, z' for the $D = 6$ parabolic case with the parameters $a = 10$ and $b = -0.1$ (left panel), $b = -1$ (middle panel), $b = 1$ (right panel). The initial conditions for (x, y, z) are $(0, -5, 0)$ in the left and middle panels and $(0, 0, 0)$ in the right panel. The red curves show the evolution in the time interval $0 \leq t \leq 1$, while the black dots continue it to time 1000.

9. Numerical evidence of non-integrability

In this section, we provide numerical evidence showing the non-integrability of systems (11)-(12) (corresponding to SR-geodesic flows with $D = 6, 7, 8$) by evolving the equation of motion of the reduced system (10). In Section 8 we have already claimed that system (11) resembles the dynamics of a driven pendulum that is chaotic. However, this resemblance can be a mere coincidence. In order to put forward a thorough investigation of whether in the above systems chaos appears or not, we need a more standardized method.

One of the most classical methods for finding chaos is given by investigating the dynamics of the return map on the surfaces of section (Poincaré map). We compute this numerically by evolving the equations of motion with the Cash-Karp-Runge-Kutta scheme. The accuracy of the numerical results is checked by reducing the integration step size by an order of magnitude and testing whether this reduction changes the trajectory of the orbit.

Surfaces of section were employed in [Sa] for the $D = 8$ case as well[9]. There the surface $z = 0$, $z' > 0$ has been chosen as the Poincaré

[9]Equations (80)-(82) of [Sa] with $q = 0$ correspond to our (10) with $Q = Q_2(x, y)$.

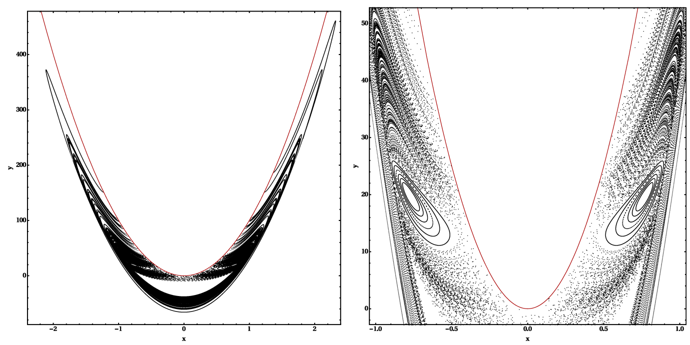

FIGURE 2. The surface of section on $z = 0$, $z' > 0$ (black dots) for system (11) with parameters $a = 10, b = -0.1$ and the orbit starting from $x = y = 0$. The red curve $Q_1(x, y) = 0$ shows the limit of this section.

section. However, if we employ the same surface of section for the $D = 6$ parabolic case we encounter a problem. Namely, the surface of section $z = 0$, $z' > 0$ does not meet all the trajectories in the phase space, because of the parabolic form of $Q_1(x, y)$. For example, in Fig. 2 the red curve $Q_1(x, y) = 0$ sets a limit for the section we can plot, and creates an obstacle to study the whole phase space. In other words, the surface $z = 0$, $z' > 0$ for system (11) is not a good choice for the Poincaré section. Moreover, the oscillations across the x-axis indicate that the system is non-compact. Namely, as an orbit evolves it tends to reach larger and larger values of $|y|$ and $|x|$.

Because of the above mentioned oscillations across the x-axis shown in Fig. 2, we assumed that a good surface of section for system (11) would be the surface $x = 0$, $x' > 0$. This assumption has proven to be correct and we show the results on Fig. 3. In both panels of Fig. 3 we can see a region of concentric closed curves (black curves), which represent regular orbits. The center of these regular orbits lies around the point $(z, y) = (0, 20)$ in the left panel, and around the point $(z, y) = (0, -20)$ in the right panel. The concentric curves indicate that the central point corresponds to a stable periodic orbit.

In both cases around these concentric orbits lie an irregular orbit (red dots), which tends to cover all the available phase space in the complement to the regular orbits. The irregular orbit apparently stems from a point around $(z, y) = (0, 30)$ in the left panel, and around $(z, y) = (0, -30)$ in the right panel. Both these points match the appearance of unstable periodic orbits.

These irregular orbits in Fig. 3 indicate that the $D = 6$ parabolic system is non-integrable. Note that the plots of Fig. 3 do not show

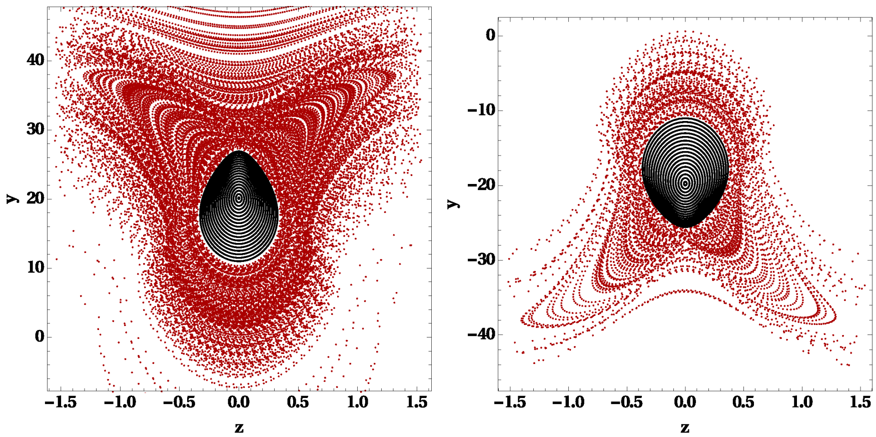

FIGURE 3. Details of the surface of section on $x = 0$, $x' > 0$ for system (11) with parameters $a = 10$ and $b = -0.1$ (left panel), $b = 0.1$ (right panel). The black closed curves represent regular orbits, while the red dots correspond to one irregular orbit.

the whole phase space, because the system is non-compact. Instead we focus our plots on the region around the regular orbits and near the unstable point, where the irregular features are more prominent.

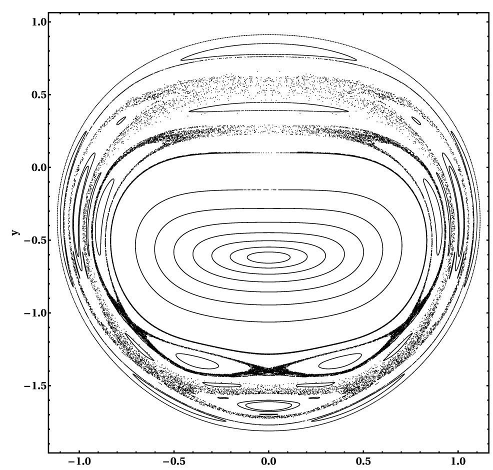

FIGURE 4. Dynamics on the surface of section $z = 0$, $z' > 0$ for system (12) with parameters $a = 2$, $b = 1$, $c = 0$ (elliptic case).

System (12) can be separated in two categories: the elliptic ones ($ab > 0$), and the hyperbolic ones ($ab < 0$). In the elliptic case the surface of section in Fig. 4 tells straightforwardly that the system is non-integrable. Namely, in Fig. 4 we can discern the characteristic features of a non-integrable system like chaotic regions and islands of stability belonging to Birkhoff chains. In the particular case of the system corresponding to $D = 8$, the indicated non-integrability is in agreement with the non-integrability conjecture of [Sa].

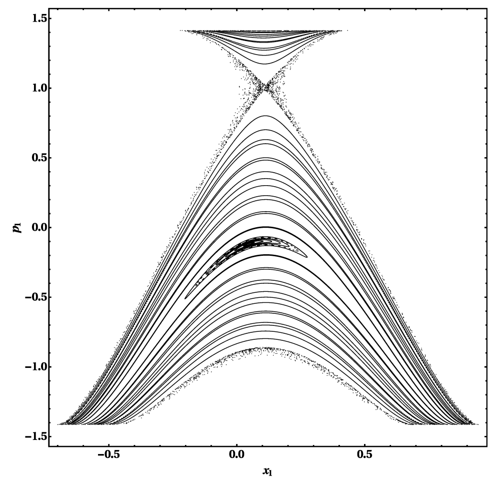

FIGURE 5. The surface of section on $z = 0$, $z' > 0$, corresponding to the hyperbolic case of system (12) with parameters $a = -1/2$, $b = 1/2$, $c = 10^2$.

We can assert non-integrability also for the hyperbolic case on the ground of analytic dependence on the parameters a, b of our system (assuming the integrals should share the same property). However, we can confirm this numerically as well, and we do it in Fig. 5: the hyperbolic orbit is shown on two different surfaces of section, and both of these surfaces indicate that the orbit is irregular, and therefore, the systems (12), corresponding to $D = 7,\ 8$, are non-integrable.

APPENDIX A. SR-STRUCTURES ON 3D LIE GROUPS

Every left-invariant SR structure on a 3-dimensional Lie group G is determined by a 2-dimensional subspace (not subalgebra) of the Lie algebra $\mathfrak{g}$ and a metric on it. The classification of such is due to [VG], and this reference also contains the integration of the equations of geodesics in terms of a semi-direct product.

Liouville integrability of left-invariant SR structures on 3D Lie groups G was proven in the preprint arXiv:math/0105128 of [Kr]. It was later re-visited in [MS]. We provide a short proof here for completeness.

Theorem 4. *Non-holonomic geodesic flows of left-invariant SR-metrics on 3-dim Lie groups are Liouville integrable with polynomial integrals.*

Proof. The left-invariant Hamiltonian $2H = \omega_1^2 + \omega_2^2$ commutes with all right-invariant forms θ_i. Every 3-dimensional Lie algebra $\mathfrak{g}$ has a Casimir function $C \in C^\infty(\mathfrak{g}^*)$ (because G has odd dimension), so the involutive set of integrals is: $I_2 = C$, $I_3 = \theta_i$, where the number i is chosen such that $I_1 = H, I_2, I_3$ are functionally independent.

Moreover this C is linear for the Heisenberg algebra and quadratic for simple Lie algebras $\mathfrak{sl}(2)$, $\mathfrak{so}(3)$, but it can be non-algebraic (depending on parameters) in the remaining semi-direct cases $\mathfrak{g} = \mathbb{R}^1 \ltimes \mathbb{R}^2$. In these cases, $\mathfrak{h} = \mathbb{R}^2$ is an Abelian subalgebra. The right-invariant forms I_2, I_3 associated to a basis in $\mathfrak{h}$ are integrals in involution. The Hamiltonian H is algebraically (and functionally) independent of those, because otherwise it would be bi-invariant. This completes the proof. $\square$

Appendix B. Prolongation of Killing symmetries

Let $\mathfrak{g} = \mathfrak{g}_{-\nu} \oplus \cdots \oplus \mathfrak{g}_{-1}$ be a (finite-dimensional) graded nilpotent Lie algebra[10], such that $\mathfrak{g}_{-1}$ generates $\mathfrak{g}$. The Tanaka prolongation is a graded Lie algebra $\hat{\mathfrak{g}}$ such that $\hat{\mathfrak{g}}_- = \oplus_{i<0}\hat{\mathfrak{g}}_i = \mathfrak{g}$ and it is the maximal graded Lie algebra with this property (its construction is outlined below). In particular, $\hat{\mathfrak{g}}_0 = \mathfrak{der}_0(\mathfrak{g})$ is the algebra of grading preserving derivations of the Lie algebra $\mathfrak{g}$.

Given a subalgebra $\mathfrak{g}_0 \subset \hat{\mathfrak{g}}_0$, the Tanaka prolongation $\mathrm{pr}(\mathfrak{g}, \mathfrak{g}_0) = \mathfrak{g}_{-\nu} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \hat{\mathfrak{g}}_1 \oplus \dots$ is naturally defined if we restrict the non-positive part to $\mathfrak{g} \oplus \mathfrak{g}_0$. Constructively $\hat{\mathfrak{g}}_1$ consists of the homomorphisms $\varphi : \mathfrak{g}_{-i} \to \mathfrak{g}_{1-i}$, $i > 0$, satisfying the Leibniz rule $\varphi([x,y]) = [\varphi(x), y] + [x, \varphi(y)]$, then we similarly define $\hat{\mathfrak{g}}_2$ etc. If some $\hat{\mathfrak{g}}_r = 0$, then also $\hat{\mathfrak{g}}_i = 0$ for $i > r$ and the algebra $\hat{\mathfrak{g}}$ is finite-dimensional.

An example of reduction of $\mathfrak{g}_0$ is given by a left-invariant SR-structure.

Theorem 5. *Let g be a Riemannian metric on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0 = \mathfrak{der}_0(\mathfrak{g}) \cap \mathfrak{so}(\mathfrak{g}_{-1}, g)$. Then $\mathrm{pr}_+(\mathfrak{g}, \mathfrak{g}_0) = 0$, i.e. $\hat{\mathfrak{g}}_i = 0$ $\forall i > 0$.*

This theorem is due to Morimoto [Mo]. His proof is based on a result due to Yatsui. In the case of our interest we give a simpler argument.

Proof in the case $\dim \mathfrak{g}_{-1} = 2$**.** Clearly the only possibility for non-zero $\mathfrak{g}_0$ is $\mathbb{R} = \mathfrak{so}(2) = \langle e_0 \rangle$ that acts on $\mathfrak{g}_{-1} = \langle e'_{-1}, e''_{-1} \rangle$ as a complex structure: $[e_0, e'_{-1}] = e''_{-1}$, $[e_0, e''_{-1}] = -e'_{-1}$.

For $0 \neq \varphi \in \hat{\mathfrak{g}}_1$ there is a basis of $\mathfrak{g}_{-1}$ such that $\varphi(e'_{-1}) = e_0$, $\varphi(e''_{-1}) = 0$. Then for $e_{-2} = [e'_{-1}, e''_{-1}]$ we have $\varphi(e_{-2}) = -e'_{-1}$. Let $\tilde{e}_{-2} = e_{-2}$

[10] It is customary in Tanaka theory to use negative gradation in the basic part, so we switch here from the notations used in the main body of the paper.

and define recursively $\tilde{e}_{-s} = [e''_{-1}, \tilde{e}_{1-s}]$, $s > 2$. We have $\varphi(\tilde{e}_{-s}) = \tilde{e}_{1-s}$ so by induction $\tilde{e}_{-s} \neq 0$ $\forall s > 2$, implying that $\dim \mathfrak{g} = \infty$. This is a contradiction. □

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