RIGIDITY OF 2-STEP CARNOT GROUPS

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ABSTRACT. In the present paper we study the rigidity of 2-step Carnot groups, or equivalently, of graded 2-step nilpotent Lie algebras. We prove the alternative that depending on bi-dimensions of the algebra, the Lie algebra structure makes it either always of infinite type or generically rigid, and we specify the bi-dimensions for each of the choices. Explicit criteria for rigidity of pseudo H- and J-type algebras are given. In particular, we establish the relation of the so-called J^2-condition to rigidity, and we explore these conditions in relation to pseudo H-type algebras.

1. INTRODUCTION

The study of H(eisenberg)-type algebras started in the 80's with A. Kaplan's seminal paper [24]. These graded 2-step nilpotent Lie algebras are intricately related to representations of Clifford algebras of vector spaces endowed with a positive definite inner product, and have been extensively studied for the past 35 years, see for example [14, 16, 17, 25, 26, 33, 40].

The extension of the work of Kaplan to Clifford algebras over non-degenerate scalar product spaces is more delicate and has only been treated in detail recently [6, 10, 12, 19, 21, 22]. In this paper, we will refer to such extensions as pseudo H-type algebras.

Two important algebraic results concerning the classical H-type algebras with a positive definite scalar product are a complete characterization of rigid H-type algebras, that are those with a finite Tanaka prolongation, and those H-type algebras satisfying a Clifford algebraic requirement known as the J^2-condition, see [14]. Both algebraic conditions have deep implications in other aspects of the study of H-type groups. The rigidity of most H-type algebras is an obstruction to the development of an analytic deformation theory, while the presence of the J^2-condition has profound geometric consequences on their groups, for example, they are transitive isometry subgroups of hyperbolic spaces, they appear as the nilpotent part of Iwasawa decompositions of real rank one groups $G = KAN$,

and moreover, the group AN is symmetric if and only if the Lie algebra of N satisfies the J^2-condition [14]. In this paper we discuss the analogs in the pseudo H-type context, and relate this to the split versions of the division algebras.

The fact that the Lie algebras obtained by non-degenerate indefinite bilinear forms has not been duly addressed is surprising, especially since Clifford algebras defined by non-degenerate scalar product have played a fundamental role in mathematics and physics, see [31, 32, 34]. An intimately related object to the pseudo H-type algebras that appears naturally in mathematical physics is the notion of extended (super-)Poincaré algebras, see [1, 2, 3, 4]. Some of our results have analogues in this theory.

An important problem in sub-Riemannian geometry is to detect whether the family of automorphisms of a given non-holonomic structure on a manifold is finite dimensional [35, 40, 42]. The model situation is the rigidity of Carnot groups, determined by the property whether the transformation group of a left-invariant bracket generating distribution on a Lie group is a Lie group itself. Equivalently, linearization reduces the problem of finite-dimensionality of the automorphism group of a non-holonomic geometric structure to that for the Tanaka prolongation of an associated graded nilpotent Lie algebra. This problem is non-trivial already for 2-step nilpotent algebras, which is the main subject of our work. In the present paper we give criteria of the finite dimensionality of the Tanaka prolongation for the generalizations of H-type structures discussed above, and clarify the situation with the general 2-step nilpotent Lie algebras $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ depending on their bi-dimensions $(\dim \mathfrak{n}_{-2}, \dim \mathfrak{n}_{-1})$.

The paper is structured as follows. Section 2 is devoted to the main concepts and notations that will be in use throughout the paper. In order to keep track of our hypotheses, we have introduced the notion of M-type and pseudo J-type algebras that generalize the known objects in the positive definite context. Section 3 is dedicated to proving rigidity of a class of real graded 2-step nilpotent Lie algebras with the center of dimension ≥ 3 via the so-called rank one criterion that we recall and re-interpret. This class of algebras contains some important examples previously considered in the literature, see for example [40]. This result then applies to pseudo H-type algebras and other cases. Section 4 deals with the J^2-condition of pseudo J-type algebras and their relation to rigidity.

The next two sections are devoted to investigation of the generic rigidity of graded 2-step nilpotent Lie algebras, depending on their bi-dimensions that complements the known results of P.Pansu and P.Eberlein. We discuss the moduli spaces of the graded 2-step nilpotent Lie algebras and describe the position of the pseudo H-type algebras among the rigid ones.

We study the pseudo H-type algebras that satisfy the J^2-condition in Section 7. We prove that the classical Abelian, Heisenberg, quaternionic and octonionic H-type algebras, and their split analogs that we introduce exhaust all possible pseudo H-type algebras with the J^2-condition. In Appendix A we relate these algebras to division algebras and their split versions. In Appendix B we relate them to the nilradicals of parabolics in simple Lie algebras.

2. Pseudo J-type and pseudo H-type algebras

Let $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ be a real or complex graded 2-step nilpotent Lie algebra, and let $\langle \cdot, \cdot \rangle$ be a non-degenerate real or complex symmetric bilinear form. We assume that the restriction $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}}$ of $\langle \cdot, \cdot \rangle$ to the subspace $\mathfrak{n}_{-2}$ is also non-degenerate and the decomposition $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ is orthogonal (such a choice can be made on any 2-step algebra $\mathfrak{n}$). We call the pair $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ an *M-type Lie algebra*, since such objects have been referred to as "metric Lie algebras" in the real case with a positive definite symmetric bilinear form, see, for example [17]. A complex Lie algebra may have different real forms. Each real form carries a real non-degenerate symmetric bilinear form, whose complexification coincides with the original complex non-degenerate symmetric bilinear form.

Definition 1. *Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be an M-type Lie algebra. The linear representation $J \colon \mathfrak{n}_{-2} \to \mathrm{End}(\mathfrak{n}_{-1})$ defined by*

$$(1) \qquad \langle J_z x, y \rangle_{\mathfrak{n}_{-1}} = \langle z, [x, y] \rangle_{\mathfrak{n}_{-2}} \quad \text{for all} \quad x, y \in \mathfrak{n}_{-1}, \quad z \in \mathfrak{n}_{-2}.$$

is called the J-map of $\mathfrak{n}$.

Definition 2. *A real M-type algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is of pseudo J-type if there is an orthonormal basis $\{z_1, \ldots, z_m\}$ for $\mathfrak{n}_{-2}$ such that the J-maps satisfy the condition*

$$(2) \qquad J_{z_i}^2 = \pm \mathrm{Id}_{\mathfrak{n}_{-1}}, \qquad i = 1, \ldots, m.$$

Note that this definition extends the notion of a J-type algebra in [40, Section 6]. It is relevant to note that we consider the operators J_{z_i} as complex structures for some indices $i = 1, \ldots, m$, but we also allow involutions for the rest of them. Moreover, the identities (2) are not necessarily related to the Clifford condition $J_{z_i}^2 = -\langle z_i, z_i \rangle_{\mathfrak{n}_{-2}} \mathrm{Id}_{\mathfrak{n}_{-1}}$ defined on the chosen basis for $\mathfrak{n}_{-2}$, as it has been used in [40] in the presence of a positive definite scalar product. A particular case of these pseudo J-type algebras are the pseudo H-type algebras.

Definition 3. *A real M-type algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is said to be of pseudo H-type if the J-maps satisfy the Clifford relations*

$$(3) \qquad J_{z_i} J_{z_j} + J_{z_j} J_{z_i} = -2 \langle z_i, z_j \rangle_{\mathfrak{n}_{-2}} \mathrm{Id}_{\mathfrak{n}_{-1}}, \quad i, j = 1, \ldots, m,$$

for a basis $\{z_1, \ldots, z_m\}$ of $\mathfrak{n}_{-2}$ (that can be chosen orthonormal).

Note that equation (3) implies that the J-map can be extended to a representation of the Clifford algebra $\mathrm{Cl}(\mathfrak{n}_{-2}, \langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}})$ on the space $\mathfrak{n}_{-1}$. Condition (3) implies condition (2) for a special choice of signs, but not conversely. The classical J-type and H-type algebras originated from the papers [24, 35] in which H-type algebras were defined as generalizations of the Heisenberg algebra endowed with a positive definite scalar product. Their pseudo-analogs were introduced in [10, 22].

The construction of pseudo H-type Lie algebras is delicate, so we postpone its precise description to Section 7. Nevertheless, let us describe them briefly here. We denote by $\mathbb{R}^{r,s}$ the vector space $\mathbb{R}^{r+s}$ equipped with the metric

$$\langle v, w \rangle_{r,s} = \sum_{i=1}^{r} v_i w_i - \sum_{j=r+1}^{r+s} v_j w_j, \quad v = (v_i), w = (w_i) \in \mathbb{R}^{r+s}.$$

The pseudo H-type Lie algebra $\mathfrak{n}^{r,s}(V)$ is a real graded 2-step nilpotent Lie algebra structure on the space $\mathbb{R}^{r,s} \oplus V$, where V is an admissible $\mathrm{Cl}(\mathbb{R}^{r,s})$-module (more details in Section 7). If the Clifford module V is irrelevant to the statement we will denote by $\mathfrak{n}^{r,s}$ the class of all pseudo H-type algebras with the same center. Equivalent definitions of pseudo H-type algebras can be found in [10, 22, 24].

The H-type algebras are a special case of a wider class of algebras satisfying the hypothesis (H), introduced in [35], which states that $\omega_\alpha(x, y) = \alpha([x, y])$ is a non-degenerate 2-form on $\mathfrak{n}_{-1}$ for any non-zero $\alpha \in \mathfrak{n}_{-2}^*$. In [16, 17, 33, 37] the authors established the following equivalent definitions.

Proposition 1. *The following statements are equivalent for a real graded 2-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$:*

(1) *$\mathfrak{n}$ satisfies Métivier's hypothesis (H).*
(2) *$\mathrm{ad}_x \colon \mathfrak{n}_{-1} \to \mathfrak{n}_{-2}$ is surjective for any non-zero $x \in \mathfrak{n}_{-1}$.*
(3) *$J_z \colon \mathfrak{n}_{-1} \to \mathfrak{n}_{-1}$ is a non-degenerate map for any non-zero $z \in \mathfrak{n}_{-2}$.*

3. RIGIDITY OF M-TYPE ALGEBRAS

Given a graded nilpotent Lie algebra $\mathfrak{n} = \bigoplus_{i=1}^{s} \mathfrak{n}_{-i}$ generated by $\mathfrak{n}_{-1}$, there is an algebraic procedure to compute symmetries of $\mathfrak{n}_{-1}$, called the Tanaka prolongation [47]. This is the maximal graded Lie algebra $\hat{\mathfrak{n}} = \mathfrak{n}_{-s} \oplus \cdots \oplus \mathfrak{n}_{-1} \oplus \mathfrak{n}_0 \oplus \ldots$ with $\hat{\mathfrak{n}}_{<0} = \mathfrak{n}$. There is a vast literature on this topic, and we refer the reader to [5, 8, 36, 40, 49] and the references therein for an overview.

A graded 2-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ is called *rigid* or of *finite type* if its *Tanaka prolongation* $\hat{\mathfrak{n}} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1} \oplus \mathfrak{n}_0 \oplus \cdots$ is finite dimensional. Otherwise it is called of *infinite type*. These definitions are valid for either complex or real Lie algebras. We will assume throughout the text that $\mathfrak{n}$ is *fundamental* or equivalently *stratified*, i.e. $\mathfrak{n}_{-1}$ generates $\mathfrak{n}$, and in the 2-step case that $\mathfrak{n}_{-1}$ contains no central elements, so the center $\mathfrak{z}$ of $\mathfrak{n}$ is exactly $\mathfrak{n}_{-2}$, see [47].

A criterion to detect whether a given *complex* graded 2-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ is rigid is the *corank one criterion* (see [15, Theorem 1] and [41, 29], which are based on [47, Corollary 2 of Theorem 11.1]). It states that $\mathfrak{n}$ is of infinite type if and only if there exist $x \in \mathfrak{n}_{-1}$ and a hyperplane $\Pi \subset \mathfrak{n}_{-1}$, such that

$$[x, y] = 0 \quad \text{for all} \quad y \in \Pi.$$

A key observation is that the corank one criterion can be rewritten conveniently in the case when the adjoint map of $\mathfrak{n}$ induces endomorphisms of $\mathfrak{n}_{-1}$ through a non-degenerate symmetric bilinear form, as in Definition 1. We conclude that the above is equivalent to the existence of a non-zero vector $x \in \mathfrak{n}_{-1}$ with

$$J_{\mathfrak{n}_{-2}} x \in \Pi^\perp, \quad \text{or} \quad \dim_{\mathbb{C}} J_{\mathfrak{n}_{-2}} x = \dim(\Pi^\perp) = 1.$$

This means that $J_{\mathfrak{n}_{-2}}$ is a one dimensional complex subspace of $\mathrm{End}(\mathfrak{n}_{-1})$ and therefore there exists

$$L \subset \mathfrak{n}_{-2}, \quad \mathrm{codim}_{\mathbb{C}} L = 1, \quad \text{such that} \quad J_L x = 0.$$

The last condition is equivalent to the property $x \in \bigcap_{z \in L} \ker(J_z)$, so we conclude:

$\mathfrak{n}$ *is of infinite type if and only if there exists a subspace*

(4) $\qquad L \subset \mathfrak{n}_{-2}, \quad \text{codim}_{\mathbb{C}} L = 1, \quad \text{such that} \quad \bigcap_{z \in L} \ker(J_z) \neq \{0\}.$

Notice that for real graded 2-step nilpotent Lie algebras $\mathfrak{n}$ the criterion fails in its sufficient part and the complexification is required, i.e. condition (4) will be applied to the complexifications $\mathfrak{n}^{\mathbb{C}}$ and $J^{\mathbb{C}}$.

Remark 1. Even though this rigidity criterion is formulated in terms of M-type algebras, viz. the definition of the operators J_z involves a choice of a scalar product on $\mathfrak{n}$, the output, namely the alternative whether the algebra $\mathfrak{n}$ is of finite or infinite type, does not depend on this choice.

Since the Tanaka prolongation is a linear algebra operation, the following folklore-known statement relates the complexification and the prolongation of real Lie algebras.

Proposition 2. *Let $\mathfrak{n}$ be a real graded nilpotent Lie algebra of any step, and let $\mathfrak{n}^{\mathbb{C}}$ be its complexification. Then $\hat{\mathfrak{n}}^{\mathbb{C}} = \widehat{\mathfrak{n}^{\mathbb{C}}}$.*

This statement has the following immediate implications.

Corollary 1. *Let $\mathfrak{n}$ and $\tilde{\mathfrak{n}}$ be two graded nilpotent Lie algebras such that $\mathfrak{n}^{\mathbb{C}} \cong \tilde{\mathfrak{n}}^{\mathbb{C}}$. Then they are either simultaneously rigid or of infinite type.*

Corollary 2. *Let $\mathfrak{n}$ be a real graded nilpotent Lie algebra of any step with the complexification $\mathfrak{n}^{\mathbb{C}}$. Then $\mathfrak{n}$ and $\mathfrak{n}^{\mathbb{C}}$ are either simultaneously rigid or of infinite type in their respective categories.*

Proof. The claims follow from the equality $\dim_{\mathbb{R}} \mathfrak{n} = \dim_{\mathbb{C}} \mathfrak{n}^{\mathbb{C}}$. $\qquad\square$

The idea of the following constructions is to employ the complexification and criterion (4) in order to detect the rigidity of real 2-step nilpotent Lie algebras. The following is a special case of Theorem 4 in [15], see also [40, Lemma 6].

Lemma 1. *Let $\mathfrak{n} = \mathfrak{n}_{-1} \oplus \mathfrak{n}_{-2}$ be a real M-type algebra. If $\dim \mathfrak{n}_{-2} = 2$, then $\mathfrak{n}$ is of infinite type.*

Proof. Let us complexify $\mathfrak{n}$ and use Corollary 2. Choose a basis z_1, z_2 of $\mathfrak{n}_{-2}^{\mathbb{C}}$. For some $\lambda \in \mathbb{C}$ the vector $z = z_1 + \lambda z_2$ is null and so $\det J_{z_1 + \lambda z_2} = 0$. Then one can take $L = \text{span}\{z\}$ and the claim follows from $\ker(J_z) \neq 0$. $\qquad\square$

The aim of this section is to generalize [40, Theorem 1] to a much wider class of real Lie algebras. In order to state the result, we introduce the following condition (C): an M-type algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ satisfies this condition if

(C) $\qquad$ *There exist three linearly independent vectors $z_1, z_2, z_3 \in \mathfrak{n}_{-2}$ such that the J-maps J_{z_i} are non-degenerate, and $J_{z_i} J_{z_j} = \sigma_{ij} J_{z_j} J_{z_i}$, for all $i, j \in \{1, 2, 3\}$, where $\sigma_{ij} \in \{-1, 1\}$ and $\sigma_{12}\sigma_{13}\sigma_{23} = -1$.*

Condition (C) is not too restrictive, and several important systems satisfy it. For instance, it holds for the pseudo H-type algebras as well as in the case when J_{z_i} commutes with both J_{z_j} and J_{z_k}, but J_{z_j} and J_{z_k} anti-commute.

Theorem 1. *Let $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ be a real M-type algebra with $\dim \mathfrak{n}_{-2} \geq 3$ that satisfies condition* (C). *Then $\mathfrak{n}$ is rigid.*

Proof. Let $z_1, z_2, z_3 \in \mathfrak{n}_{-2}$ be the linearly independent vectors from condition (C) and with a slight abuse of notation define $K = \operatorname{span}_{\mathbb{C}}\{z_1, z_2, z_3\} \subset \mathfrak{n}_{-2}^{\mathbb{C}}$.

Suppose, that $\mathfrak{n}^{\mathbb{C}}$ is of infinite type and let $L \subset \mathfrak{n}_{-2}^{\mathbb{C}}$ be the codimension one subspace coming from the corank one criterion. By the dimension count, it is easy to see that $\dim_{\mathbb{C}}(K \cap L) \geq 2$.

Without loss of generality, we can assume that there are $s_1, s_2 \in \mathbb{C}$ such that $K \cap L \supset \operatorname{span}_{\mathbb{C}}\{z_1 - s_1 z_3, z_2 - s_2 z_3\}$. By the definition of L, there is a non-zero $x \in \mathfrak{n}_{-1}^{\mathbb{C}}$ such that

$$(J_{z_1} - s_1 J_{z_3})(x) = (J_{z_2} - s_2 J_{z_3})(x) = 0,$$

where we use the notation J_z for the complexification as well. By the non-degeneracy of J_{z_k}, $k \in \{1, 2, 3\}$, it holds that $s_1, s_2 \neq 0$. By condition (C)

$$J_{z_1} J_{z_2} x = s_2 J_{z_1} J_{z_3} x = \sigma_{13} s_2 J_{z_3} J_{z_1} x = \sigma_{13} s_1 s_2 J_{z_3}^2 x,$$

and analogously, $J_{z_2} J_{z_1} x = \sigma_{23} s_1 s_2 J_{z_3}^2 x$. Thus, it follows that $\sigma_{23} = \sigma_{12}\sigma_{13}$. This contradicts $\sigma_{12}\sigma_{13}\sigma_{23} = -1$ in condition (C), and so $\mathfrak{n}^{\mathbb{C}}$ is rigid. The rigidity of $\mathfrak{n}$ follows from Corollary 2. □

Observe that Theorem 1 is applicable in a broader context than just for M-type algebras. In particular, it holds when the symmetric bilinear form $\langle \cdot, \cdot \rangle$ degenerates on $\mathfrak{n}_{-2}$, as long as the latter possesses a three dimensional subspace satisfying condition (C). Such degenerate cases have been considered before in the literature, see [6, 9, 12].

As a consequence of Theorem 1 we have the following results, see also [43].

Corollary 3. *Any pseudo H-type algebra with $\dim \mathfrak{n}_{-2} \geq 3$ is rigid.*

Corollary 4. *Let $\mathfrak{n}$ be a pseudo J-type algebra, and let A be the subalgebra of $\operatorname{End}(\mathfrak{n}_{-1})$ generated by the set $\{J_z J_w \colon z, w \in \mathfrak{n}_{-2}\}$. If $\mathfrak{n}$ is of infinite type, then A has a common eigenvector over $\mathbb{C}$.*

Proof. Choose an orthonormal basis $\{z_1, \ldots, z_m\}$ of $\mathfrak{n}_{-2}$. As in the proof of Theorem 1, by complexifying, we know there exist non-zero complex numbers $s_1, \ldots, s_{m-1} \in \mathbb{C}$ such that (perhaps after re-enumeration)

$$(5) \qquad\qquad\qquad J_{z_i} - s_i J_{z_m} \in J_L,$$

where L is the subspace given in the corank one condition. If $x \in \mathfrak{n}_{-1}^{\mathbb{C}}$ is the non-zero vector corresponding to L, then equation (5) implies that there are constants $c_{ij} \in \mathbb{C}$, $i, j \in \{1, \ldots, m\}$, such that $J_{z_i} J_{z_j} x = c_{ij} x$. Consequently, since any $I \in A$ can be written as a polynomial in the variables $J_{z_i} J_{z_j}$, say $I = P(\ldots, J_{z_i} J_{z_j}, \ldots)$, we conclude that $Ix = P(\ldots, J_{z_i} J_{z_j}, \ldots)x = P(\ldots, c_{ij}, \ldots)x$. □

Remark 2. Let $z_1, z_2, z_3 \in \mathfrak{n}_{-2}$ be linearly independent. Corollary 4 implies that if $[J_{z_i}, J_{z_j}]$, $1 \leq i < j \leq 3$, have no common eigenvector over $\mathbb{C}$, then $\mathfrak{n}$ is rigid.

4. Rigidity and J^2-condition

The main goal of this section is to characterize an analogue of the so-called J^2-condition for pseudo J- and H-type algebras, studied in [11, 14]. Although in the classical situation this condition has deep geometric implications, we only focus here on those algebras that admit this algebraic property.

Definition 4. *A pseudo J-type algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ satisfies the J^2-condition if for every $x \in \mathfrak{n}_{-1}$, $\langle x, x \rangle_{\mathfrak{n}_{-1}} \neq 0$, and for every orthogonal pair $z, z' \in \mathfrak{n}_{-2}$, there exists $z'' \in \mathfrak{n}_{-2}$ satisfying*

$$(6) \qquad\qquad J_z J_{z'} x = J_{z''} x.$$

Equation (6) implies that for any given $x \in \mathfrak{n}_{-1}$, $\langle x, x \rangle_{\mathfrak{n}_{-1}} \neq 0$, the space

$$(7) \qquad\qquad A_x = \mathbb{R}x \oplus J_{\mathfrak{n}_{-2}} x = \{ \alpha x + J_{z'} x \mid \alpha \in \mathbb{R}, z' \in \mathfrak{n}_{-2} \}$$

is J_z-invariant for every $z \in \mathfrak{n}_{-2}$ (note that $\mathbb{R}x \cap J_{\mathfrak{n}_{-2}} x = 0$ if $\|x\|^2_{\mathfrak{n}_{-1}} \neq 0$).

The converse statement is true for pseudo H-type algebras: if $J_z A_x \subset A_x$ for any non-null $x \in \mathfrak{n}_{-1}$ and any $z \in \mathfrak{n}_{-2}$, then the pseudo H-type algebra $\mathfrak{n}$ satisfies the J^2-condition. This implication holds thanks to the identity

$$(8) \qquad\qquad \langle J_z x, J_{z'} x \rangle_{\mathfrak{n}_{-1}} = \langle z, z' \rangle_{\mathfrak{n}_{-2}} \langle x, x \rangle_{\mathfrak{n}_{-1}}$$

that will be used in the sequel.

If $\langle x, x \rangle_{\mathfrak{n}_{-1}} \neq 0$, then A_x is a non-degenerate subspace of $(\mathfrak{n}_{-1}, \langle \cdot, \cdot \rangle_{n_{-1}})$. In the case when x is null, the restriction of $\langle \cdot, \cdot \rangle_{n_{-1}}$ to A_x is degenerate. We define the general J^2-condition by omitting the requirement $\langle x, x \rangle_{\mathfrak{n}_{-1}} \neq 0$.

Definition 5. *A pseudo J-type algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ satisfies the general J^2-condition if for any $x \in \mathfrak{n}_{-1}$, and for all orthogonal pairs $z, z' \in \mathfrak{n}_{-2}$ there exists $z'' \in \mathfrak{n}_{-2}$ such that equation (6) holds.*

Observe that for J-type algebras (more generally, when $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{n}_{-1}$) the general J^2 condition is equivalent to the usual J^2 condition. Of course, the general J^2 condition implies the usual one. Note also that the (general) J^2-condition is trivially fulfilled for $\dim(\mathfrak{n}_{-2}) = 0, 1$.

Lemma 2. *The J^2-condition is never satisfied for $\dim \mathfrak{n}_{-2} = 2$.*

Proof. Assuming the opposite we get two linear operators $J_i = J_{z_i} \in \mathrm{GL}(\mathfrak{n}_{-1})$ with $J_i^2 = \epsilon_i = \pm 1$, $i = 1, 2$, for an orthonormal basis $z_1, z_2 \in \mathfrak{n}_{-2}$.

Note that if $z = s_1 z_1 + s_2 z_2 \neq 0$, $s_i \in \mathbb{R}$, then $J_z \neq \lambda_z \cdot \mathbf{1}$ because $\lambda_z = 0$ contradicts non-degeneracy of $\langle \cdot, \cdot \rangle_{n_{-2}}$, while for $\lambda_z \neq 0$ the operator J_z is not skew-symmetric with respect to $\langle \cdot, \cdot \rangle_{n_{-1}}$.

Now the J^2-condition implies that for any non-null $x \in \mathfrak{n}_{-1}$ for some scalar functions α_x, β_x we have the equality: $J_1 J_2 x = (\alpha_x J_1 + \beta_x J_2)x$. If $\alpha_x \equiv 0$ then $J_1 \equiv \beta \cdot \mathbf{1}$, and if $\beta_x \equiv 0$ then $J_1 \equiv \alpha \cdot \mathbf{1}$. Consequently, $\alpha_x \beta_x \not\equiv 0$. Thus application of the operator J_1 to the above equality from the left yields

$$(9) \qquad\qquad J_1 x = (p_x \mathbf{1} + q_x J_2)x$$

for some (perhaps rational) scalar functions p_x, q_x. Since for generic $x \in \mathfrak{n}_{-1}$ the vectors x and $J_2 x$ are linearly independent, the linearity of the operators $J_1, \mathbf{1}, J_2$ implies that $p_x = p = \text{const}, q_x = q = \text{const}$ in formula (9), i.e. $J_1 = p\mathbf{1} + qJ_2$.

Squaring this latter identity yields $\epsilon_1 \mathbf{1} = (p^2 + \epsilon_2 q^2)\mathbf{1} + 2pqJ_2$. Then $pq = 0$, but since $q \neq 0$ we get $p = 0, q = \pm 1$, whence $J_2 = \pm J_1$ and $J_z = 0$ for $z = z_1 \mp z_2$, which is a contradiction. $\square$

Theorem 2. *A pseudo J-type algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$, $\dim(\mathfrak{n}_{-2}) \geq 3$, satisfying the general J^2-condition is rigid.*

Proof. We will work over the field $\mathbb{C}$. If necessary we use the complexification and finish the proof by applying Corollary 2.

By contradiction, let us suppose that a pseudo J-type algebra $\mathfrak{n}$ is of infinite type. Corollary 4 implies that there exists a common eigenvector $x \in \mathfrak{n}_{-1}$ for the algebra A generated by the operators $J_{z_i} J_{z_j}$, where $\{z_i\}_{i=1}^m$ is an orthonormal basis for $\mathfrak{n}_{-2}$. Thus, we obtain $J_{z_i} J_{z_j} x = \mu_{z_i, z_j} x$ for some non-vanishing $\mu_{z_i, z_j} \in \mathbb{C}$.

We claim that the same x is also an eigenvector for all $J_z, z \in \mathfrak{n}_{-2}$. Indeed, since the pseudo J-type algebra satisfies Definition 5, for any pair z_i, z_j of vectors from the orthonormal basis for $\mathfrak{n}_{-2}$, there is $z_{ij}'' \in \mathfrak{n}_{-2}$ such that $J_{z_i} J_{z_j} x = J_{z_{ij}''} x$. We get $J_{z_j} x = \epsilon_i J_{z_i} J_{z_{ij}''} x = \pm \mu_{z_i, z_{ij}''} x$ by $J_{z_i}^2 = \epsilon_i = \pm \text{Id}_{n_{-1}}$. Since j is an arbitrary index and $\{z_j\}_{i=1}^m$ is a basis for $\mathfrak{n}_{-2}$, the claim follows.

Let us fix the eigenvector x. Then $J_z x = \lambda_z x$ for a non-zero linear function $\lambda \colon \mathfrak{n}_{-2} \to \mathbb{C}$. The definition of pseudo J-type algebras leads to

$$(10) \qquad \langle z, [x, y] \rangle_{\mathfrak{n}_{-2}} = \langle J_z x, y \rangle_{\mathfrak{n}_{-1}} = \lambda_z \langle x, y \rangle_{\mathfrak{n}_{-1}} = -\langle x, J_z y \rangle_{\mathfrak{n}_{-1}}$$

for all $y \in \mathfrak{n}_{-1}, z \in \mathfrak{n}_{-2}$. This implies that $\langle x, x \rangle_{n_{-1}} = 0$, and thus $[x, \Pi] = 0$ for the co-dimension 1 subspace $\Pi \subset \mathfrak{n}_{-1}$ orthogonal to x. We can choose $y \notin \Pi$ such that $\langle x, y \rangle_{\mathfrak{n}_{-1}} = 1$. Thus, there exists a basis $e_1 = y, e_2 = x, e_3, \ldots, e_n$, $n = \dim \mathfrak{n}_{-1}, e_i \in \Pi$ for $i > 1$, such that $\langle e_1, e_2 \rangle_{\mathfrak{n}_{-1}} = 1, \langle e_2, e_i \rangle_{\mathfrak{n}_{-1}} = 0$ for $i > 1$.

From (10) with $x = e_2$ and $y = e_1$ we obtain $\lambda_z = -\langle e_2, J_z e_1 \rangle_{\mathfrak{n}_{-1}}$, whence $J_z e_1 = -\lambda_z e_1 \mod \Pi$. Similarly, substituting $y = e_i$ for $i > 2$ in (10) we obtain $\langle e_2, J_z e_i \rangle_{\mathfrak{n}_{-1}} = 0$, whence $J_z e_i = 0 \mod \Pi$ for $i > 2$. Finally, since $e_2 = x$ is an eigenvector, we obtain $J_z e_2 = \lambda_z e_2$. Hence, the matrix of the operator $J_z \in \text{End}(\mathfrak{n}_{-1})$ in the chosen basis $e_1, e_2, \ldots, e_n$ has the form

$$(11) \qquad J_z = \begin{pmatrix} -\lambda_z & 0 & 0 & \ldots & 0 \\ * & \lambda_z & * & \ldots & * \\ * & 0 & * & \ldots & * \\ * & \vdots & \ddots & \vdots & * \\ * & 0 & * & \ldots & * \end{pmatrix}.$$

Taking z' orthogonal to z and multiplying (11) by $J_{z'}$ from the right, we arrive at

$$
J_z J_{z'} = \begin{pmatrix}
\lambda_z \lambda_{z'} & 0 & 0 & \dots & 0 \\
* & \lambda_z \lambda_{z'} & * & \dots & * \\
* & 0 & * & \dots & * \\
* & \vdots & \ddots & \vdots & * \\
* & 0 & * & \dots & *
\end{pmatrix}
$$

that differs from any matrix $J_{z''}$ unless $\lambda_z \lambda_{z'} = \lambda_{z''} = 0$. Since for a generic orthogonal pair $z, z' \in \mathfrak{n}_{-2}$, it holds that $\lambda_z \neq 0$, $\lambda_{z'} \neq 0$, we obtain a contradiction. This proves the required rigidity of $\mathfrak{n}$. $\qquad\square$

5. Generic rigidity of 2-step nilpotent algebras

Let us discuss the general rigidity problem for graded 2-step nilpotent algebras $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$. Notice that if $0 \leq \dim \mathfrak{n}_{-2} \leq 2$, then the algebra $\mathfrak{n}$ is of infinite type, see Section 3. So we study the algebras with $\dim \mathfrak{n}_{-2} > 2$.

In this section we work with both real and complexified cases, so we omit specification of the field and signature for the metric $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{-1}}$ and simply write $\mathfrak{so}(n)$ below (the conclusion does not depend on this signature).

Denote by $N(m, n)$ the space of 2-step graded nilpotent Lie algebras with bi-dimensions (m, n), where $m = \dim \mathfrak{n}_{-2}$, $n = \dim \mathfrak{n}_{-1}$, $0 \leq m \leq \binom{n}{2}$. This space is an algebraic manifold of dimension md, $d = \binom{n}{2} - m$, with the isomorphism to the Grassmanian $\mathrm{Gr}_d(\Lambda^2 \mathfrak{n}_{-1})$ given by associating $Z = \ker(\Lambda^2 \mathfrak{n}_{-1} \to \mathfrak{n}_{-2})$ to the bracket on $\mathfrak{n}$. Reciprocally, $\mathfrak{n}$ is restored by letting $\mathfrak{n}_{-2} = \Lambda^2 \mathfrak{n}_{-1} / Z$. In particular, the notion of a generic Lie algebra structure $\mathfrak{n} \in N(m, n)$ is given by the notion of a Zariski generic point $Z \in \mathrm{Gr}_d(\Lambda^2 \mathfrak{n}_{-1})$.

Theorem 3. *A generic algebra $\mathfrak{n} \in N(m, n)$ is rigid for $m \geq 3$, $n \geq 3$.*

In other words, for a generic Lie bracket on $\mathfrak{n}$ with the bi-dimensions in the range $m, n \geq 3$ the automorphism group of the Carnot structure $(\exp \mathfrak{n}, \mathfrak{n}_{-1})$ is a Lie group. Before we give a proof of this, let us notice that several authors have studied automorphisms of generic 2-step Carnot structures. In particular, let us mention the result by P. Pansu [42, Proposition 13.1], asserting that in general, the automorphism group is generated by translations ad_x, $x \in \mathfrak{n}_{-1}$, and the standard homothety, namely the action by the grading element $e \in \mathfrak{n}_0 \subset \hat{\mathfrak{n}}$, provided that $n \in 2\mathbb{Z}$, $n \geq 10$, $3 \leq m < 2n - 3$. The restrictions on bi-dimensions were relaxed by P. Eberlein [18, Proposition 3.4.3], where his assumption is that $\bar{d} = \min\{m, \binom{n}{2} - m\} \geq 3$ excluding the cases $n \leq 6$ for $\bar{d} = 3$. See also [26, 44, 46].

When the stability subgroup of the automorphism group is only scaling due to the grading element, the positive part of the Tanaka prolongation vanishes. Indeed, if $\mathfrak{n}_0 = \langle e \rangle$, where e is the grading element and $v \in \mathfrak{n}_1$ is a non-zero element, then we can choose a codimension 1 subspace $\Pi \subset \mathfrak{n}_{-1}$ and a vector $x \in \mathfrak{n}_{-1} \setminus \Pi$ such that $[x, v] = e$, $[\Pi, v] = 0$. Then $[[x, y], v] = -y$, $[[[x, y_1], y_2], v] = -[y_1, y_2] = 0$ and $[[[x, y], x], v] = -3[x, y] = 0$ for all $y, y_1, y_2 \in \Pi$, so that $\mathfrak{n}_{-2} = [\mathfrak{n}_{-1}, \mathfrak{n}_{-1}] = 0$. This contradiction yields the claim of our theorem.

However, in order to overcome the restrictions assumed by Pansu and Eberlein, we will give the proof valid for all pairs (n, m) with the range specified. Outside this range all the algebras $\mathfrak{n} \in N(m, n)$ are of infinite type.

Notice that absence of rank 1 elements in the family $\{J_z : 0 \neq z \in \mathfrak{n}_{-2}\} \subset \text{End}(\mathfrak{n}_{-1})$ is a Zariski open condition, so proving there exists one rigid Lie algebra structure on $\mathfrak{n}$ in bi-dimensions (m, n) implies the same for a generic one.

Proof. We will give two proofs of Theorem 3. First, let us remind that we work in the complexification. Using criterion (4), we have to show that generically, for a basis $z_1 \ldots, z_m$ of $\mathfrak{n}_{-2}$, the vectors $J_{z_1} x, \ldots, J_{z_m} x$ span the space of dimension greater than 1 for all non-zero vectors $x \in \mathfrak{n}_{-1}$. In other words, choosing a basis in $\mathfrak{n}_{-1}$, the $n \times m$ matrix $M = [J_{z_1} x | \ldots | J_{z_m} x]$ has rank less than or equal to 1 only if $x = 0$. In the case $x = 0$ the rank is indeed 0.

The first proof is constructive. The condition $\text{rank}\,(M) \leq 1$ means that all 2×2 minors vanish. Take independent minors $M_{1,i}^{1,j}$ for $1 < i \leq n$, $1 < j \leq m$. The entries (linear in x) are at our disposal since we are free to perturb the operators J_{z_j}, so we get $(n-1)(m-1) > n$ quadratic conditions on $x \in \mathfrak{n}_{-1}$ whose common solution is generically only zero, thus proving our claim.

For example, in the case $(m, n) = (3, 5)$ we can have the following explicit matrices giving the structure of $\mathfrak{n}$:

$$
J_{z_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \; J_{z_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \; J_{z_3} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}
$$

Then for $0 \neq x = (x_1, x_2, x_3, x_4, x_5)^t \in \mathfrak{n}_{-1}$ the 5×3 matrix

$$
[J_{z_1} x | J_{z_2} x | J_{z_3} x] = \begin{bmatrix} x_2 + x_5 & 0 & x_4 \\ -x_1 & x_3 + x_5 & x_5 \\ x_4 & -x_2 & 0 \\ -x_3 & 0 & x_5 - x_1 \\ -x_1 & -x_2 & -x_2 - x_4 \end{bmatrix}
$$

has rank greater or equal than 2, whence this Lie algebra, as well as generic nilpotent algebras $\mathfrak{n}$ with $(m, n) = (3, 5)$, are of finite type.

The second proof uses the fact that there exist three linearly independent operators $J_1, J_2, J_3 \in \mathfrak{so}(n)$ such that any 2-dimensional subspace of their span generates the whole Lie algebra $\mathfrak{so}(n)$. Note that any 3 generic skew-symmetric operators satisfy this property. Statements of this kind can be found in [7]. Assume that $J_{\mathfrak{n}_{-2}}$ contains three operators $J_{z_1}, J_{z_2}, J_{z_3}$ of the indicated type. Then the intersection $L^\dagger$ of the three-dimensional span$\{z_1, z_2, z_3\} \subset \mathfrak{n}_{-2}$ with the hyperplane L, used in criterion (4), has dimension 2 or 3. Since $L^\dagger \subset L$, there exists a non-zero vector $x \in \mathfrak{n}_{-1}$ satisfying $J_z x = 0$ for all $z \in L^\dagger$. However $L^\dagger$ generates the Lie algebra $\mathfrak{so}(n)$ and thus we get $Ax = 0$ for all $A \in \mathfrak{so}(n)$, hence $x = 0$. This contradiction proves the result. $\square$

It is natural to investigate the moduli space of nilpotent Lie structures on $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$, i.e., the quotient of $N(m,n)$ by the natural action of $\mathrm{GL}(n)$ on $\mathfrak{n}$, induced by the action on $\mathfrak{n}_{-1}$. This is no longer a manifold due to existence of singular orbits. However since the action is algebraic, it allows a rational quotient on a Zariski open stratum by Rosenlicht's theorem [45]. Thus the quotient is a rational space and it has positive dimension in the following cases:

- $\bar{d} = \min\{m, \binom{n}{2} - m\} \geq 3$, $n \geq 6$, because $\dim \mathfrak{sl}(n) < \dim \mathrm{Gr}_m(\mathfrak{so}(n))$. The standard homothety acts trivially on the Grassmanian, so we consider its quotient by the projective group $\mathrm{PSL}(n)$.
- $\bar{d} = 2$, $n \in 2\mathbb{Z}$, $n > 6$, because of the following. The algebra structure on $\mathfrak{n}$ is given by two skew-symmetric operators J_{z_1}, J_{z_2} on $\mathfrak{n}_{-1}$, which are generically non-degenerate. They are however not invariants of the Lie algebra structure, but only of the chosen M-algebra: under the change of metrics on $\mathfrak{n}_{-2}$ and $\mathfrak{n}_{-1}$ given by symmetric matrices B and A of sizes 2×2 and $n \times n$ respectively, the operator J_z changes to AJ_{Bz}. Passage to $I = J_{z_1}^{-1} J_{z_2}$ eliminates dependence on A, and the action of the Möbius group $I \mapsto \frac{a+bI}{c+dI}$ eliminates dependence on B. Since the spectrum of I generically consists only of double eigenvalues with totality $|\mathrm{Sp}(I)| = n/2$, we obtain a continuous invariant for $n > 6$.

In all other cases, there is an open orbit, and thus, no moduli for 2-step structures $\mathfrak{n} \in N(m,n)$. This is obvious if $\bar{d} \leq 1$. In the remaining cases, the codimension of the orbit of $\mathrm{PSL}(n)$ on $\mathrm{Gr}_m(\mathfrak{so}(n))$ is

$$(12) \quad \dim \mathrm{Gr}_m(\mathfrak{so}(n)) - \dim \mathfrak{sl}(n) + d(m,n) = m \cdot \binom{n}{2} - m^2 - n^2 + 1 + d(m,n),$$

where $d(m,n)$ is the dimension of the stabilizer of a generic point Z in the Grassmannian, or equivalently, the minimum of dimensions of stabilizers of all points, which was computed in [18, Section 4.3]:

$$d(2, 2k+1) = 2k+4, \; d(2,4) = 7, \; d(2,6) = 9, \; d(3,4) = 6, \; d(3,5) = 3.$$

It is straightforward to check that the value in (12) is zero for these bi-dimensions, and that it is positive for $(2, 2k)$ since $d(2, 2k) = 3k$ for $k > 3$.

We conclude generic rigidity for the moduli on the strata of highest dimension in the quotient of $N(m,n)$ by $\mathrm{PSL}(n)$ in the case $m > 2$.

6. Digression: rigidity vs. pseudo H-type

Let us at first discuss the rigidity problem in lowest dimensions. The first non-trivial case is thus $\dim \mathfrak{n}_{-2} = 3$. The fundamental property implies that $\dim \mathfrak{n}_{-1} \geq 3$ and in the case of equality $\dim \mathfrak{n}_{-2} = \dim \mathfrak{n}_{-1} = 3$ the bracket identifies $\mathfrak{n}_{-2} = \Lambda^2 \mathfrak{n}_{-1}$. It is well-known, see [8, 48], that the Tanaka prolongation of $\mathfrak{n}$ in this case is the simple Lie algebra of type B_3 with the 2-grading induced by the parabolic subalgebra $\mathfrak{p}_3$. Namely, it is $\mathfrak{so}(3,4)$ in the real case or $\mathfrak{so}(7, \mathbb{C})$ in the complex case. Thus the algebra $\mathfrak{n}$ with $\dim \mathfrak{n}_{-2} = \dim \mathfrak{n}_{-1} = 3$ is rigid.

The situation changes when $\dim \mathfrak{n}_{-2} = 3$ and $\dim \mathfrak{n}_{-1} = 4$. The classification of such graded 2-step Lie algebras, which can be extracted from [30], is as follows. Let e_1, e_2, e_3, e_4 be a basis of $\mathfrak{n}_{-1}$, and let f_1, f_2, f_3 be that of $\mathfrak{n}_{-2}$. The non-trivial

brackets of $\mathfrak{n}$ in every of six non-isomorphic cases are given below. These cases are numerated in loc.cit. as $m7\_2\_2$, $m7\_2\_3$, $m7\_2\_4$, $m7\_2\_2r$, $m7\_2\_5$ and $m7\_2\_5r$ respectively:

$$(13) \qquad\qquad [e_1, e_4] = f_1, \ [e_2, e_4] = f_2, \ [e_3, e_4] = f_3;$$

$$(14) \qquad\qquad [e_1, e_4] = f_1, \ [e_2, e_4] = f_2, \ [e_3, e_4] = f_3, [e_2, e_3] = f_1;$$

$$(15) \qquad\qquad [e_1, e_4] = f_1, \ [e_2, e_3] = f_2, \ [e_3, e_4] = f_3;$$

$$(16) \qquad [e_1, e_3] = [e_4, e_2] = f_1, \ [e_1, e_4] = [e_2, e_3] = f_2, \ [e_3, e_4] = f_3;$$

$$(17) \qquad\quad [e_1, e_2] = [e_3, e_4] = f_1, \ [e_1, e_4] = f_2, \ [e_2, e_3] = f_3;$$

$$(18) \qquad [e_1, e_2] = [e_3, e_4] = f_1, \ [e_1, e_3] = [e_4, e_2] = f_2, \ [e_1, e_4] = [e_2, e_3] = f_3.$$

Clearly in cases (13–15) $\mathrm{rank}(\mathrm{ad}_{e_1}) = 1$, and for (16) we have to use the complexification $\mathrm{rank}(\mathrm{ad}_{e_1+ie_2}) = 1$, so the corresponding algebras $\mathfrak{n}$ are of infinite type by the rank 1 criterion [15, 39]. On the contrary, in cases (17) and (18) the algebras $\mathfrak{n}$ are rigid. Indeed, in these cases they are isomorphic to the pseudo H-type algebras $\mathfrak{n}^{1,2}$ and $\mathfrak{n}^{3,0}$ respectively. Thus, we conclude the following.

Proposition 3. *The algebra $\mathfrak{n}$ with $(\dim \mathfrak{n}_{-2}, \dim \mathfrak{n}_{-1}) = (3,4)$ is of finite type if and only if it is of pseudo H-type.*

The corresponding statement does not hold for $n = \dim \mathfrak{n}_{-1} > 4$. Already for $(m,n) = (3,5)$ we have rigid algebras $\mathfrak{n} \in N(m,n)$ that are not of pseudo H-type. A criterion for 2-step nilpotent Lie algebras to be of H-type was obtained in [33]. Indeed, there are no 5-dimensional representations without trivial modules for the Clifford algebra generated by a 3-dimensional scalar product space, see Table 1 in Section 7. More generally, the following holds

Theorem 4. *Generic (resp. generic rigid) algebras $\mathfrak{n} \in N(m,n)$ in the range $m > 1$ (resp. $m > 2$) except for $(m,n) \in \{(2,4),(3,4)\}$ are not of pseudo H-type.*

Proof. Any representation of the Clifford algebra $\mathrm{Cl}(\mathfrak{n}_{-2}, \langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}})$ is decomposed into a direct sum of irreducible modules. We have to exclude trivial submodules which lead to infinite type. Thus, first of all, not every pair of bi-dimensions (m,n) can be realized for a pseudo H-type algebra.

Second, if $\mathfrak{n}_{-1} = \oplus_{i=1}^{k} U_i$ is the sum of k irreducible modules, then k scalings contribute to the choice of module, and hence, to fixing the scalar product on $\mathfrak{n}$: $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}} + \langle \cdot, \cdot \rangle_{\mathfrak{n}_{-1}}$.

Next, changing the scalar product we keep the same Lie structure of $\mathfrak{n}$ but a different J-representation as an M-type algebra. Namely, any other scalar product, having decomposition $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ orthogonal, can be obtained via two symmetric, not necessarily positive definite, operators $A \in \mathrm{End}(\mathfrak{n}_{-1})$ and $B \in \mathrm{End}(\mathfrak{n}_{-2})$: $\langle \cdot, \cdot \rangle_{\mathrm{new}} = \langle A\cdot, \cdot \rangle_{\mathfrak{n}_{-1}} + \langle B\cdot, \cdot \rangle_{\mathfrak{n}_{-2}}$. The J-representation of the Lie algebra structure changes so $J_z \rightsquigarrow A \circ J_{Bz}$. The change of the basis $z \mapsto Bz$ in $\mathfrak{n}_{-2}$ does not influence the dimension of the space of pseudo H-type algebras in $N(m,n)$, while the symmetric operator $A \in \mathrm{End}(\mathfrak{n}_{-1})$ does contribute to it.

Alternatively, we can think of $\mathrm{GL}(n)$ changing the basis in $\mathfrak{n}_{-1}$, inducing the change in $\mathfrak{n}_{-2}$. But since the orthogonal group preserves $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{-1}}$, we obtain only

$\binom{n+1}{2}$ transformations, including the above k scalings, and this number bounds from above the dimension of the space of H-type algebras.

For $n > 4$, $1 < m < \binom{n}{2} - 1$, we have $\binom{n+1}{2} < md = \dim \mathrm{Gr}_d(\mathfrak{so}(n))$, whence a generic $\mathfrak{n} \in N(m, n)$ in this range of bi-dimensions is not of pseudo H-type.

For $n > 2$ it can be checked that in the cases $m = \binom{n}{2}$ the only Lie structure (that is the modelled on the simple algebra B_n with the grading corresponding to $\mathfrak{p}_n$) is not of pseudo H-type, as well as all finite number of structures $\mathfrak{n}$ for $m = \binom{n}{2} - 1$ are not such. This follows by dimensional reasons similarly to the bi-dimension $(3,5)$ considered before the theorem. Thus, we only have to study the cases (m, n) with $3 \leq n \leq 4$, which do not satisfy the above inequalities.

They are precisely the cases $(2, 4)$, $(3, 4)$ and $(4, 4)$. The last one may not have any pseudo H-type as it follows from Table 1 in the next section. But the first two both admit a pseudo H-type structure and they are two exceptions: in the first case $\mathfrak{n}$ is always of infinite type and in the second case it is generically of finite type. This finishes the proof. $\qquad\square$

An alternative approach to the proof of the above theorem for $\dim \mathfrak{n}_{-2} > 2$ is as follows. For pseudo H-type algebras, almost all operators J_{z_i} are invertible and one can consider the operators $T_{ij} = J_{z_i}^{-1} J_{z_j}$ for some generic choice of the basis $z_i \in \mathfrak{n}_{-2}$. Similarly to how it was done in [33] for H-type algebras, one can show that T_{ij} generate a subalgebra of the even part $\mathrm{Cl}_0(\mathfrak{n}_{-2}, \langle .,. \rangle_{\mathfrak{n}_{-2}})$ of the Clifford algebra. In particular, this subalgebra has dimension at most $p = 2^{m-1}$. Since the minimal dimension of the Clifford module is $2^{\frac{m-1}{2}} \leq n$, we conclude that $p \leq n^2 = \dim \mathrm{End}(\mathfrak{n}_{-1})$, where the inequality is strict unless $n \equiv 7 \bmod 8$. Due to the results of [7] the operators T_{ij}, obtained from generic operators J_k as above, generate the whole endomorphism algebra $\mathrm{End}(\mathfrak{n}_{-1})$. Therefore, the claim follows from the inequality $p < n^2$ for $n \not\equiv 7 \bmod 8$, and by comparison of the structures of $\mathrm{Cl}_0(\mathfrak{n}_{-2}, \langle .,. \rangle_{\mathfrak{n}_{-2}})$ with $\mathrm{End}(\mathfrak{n}_{-1})$ otherwise.

7. PSEUDO H-TYPE ALGEBRAS WITH J^2-CONDITION

In this section we clasify pseudo H-type algebras satisfying the J^2-condition.

Definition 6. *Let $J \colon \mathrm{Cl}(U, \langle \cdot, \cdot \rangle_U) \to \mathrm{End}(V)$ be a Clifford algebra representation. The module V is called admissible if there is a bilinear form $\langle \cdot, \cdot \rangle_V$ such that the endomorphisms J_z are skew-symmetric for any $z \in U$, that is*

$$(19) \qquad \langle J_z x, y \rangle_V = -\langle x, J_z y \rangle_V.$$

This scalar product $\langle \cdot, \cdot \rangle_V$ is called admissible for the module V.

It is well-known [23, 31] that if U is endowed with a positive definite bilinear form $\langle \cdot, \cdot \rangle_U$, then the module V is admissible with respect to some positive definite bilinear form $\langle \cdot, \cdot \rangle_V$. In particular, any irreducible module is admissible. In the case when the bilinear form $\langle \cdot, \cdot \rangle_U$ is indefinite, the module V is not always admissible and sometimes only the direct sum $V \oplus V$ is admissible. As a consequence, in this case not all irreducible modules are admissible. Notice also that if $(U, \langle \cdot, \cdot \rangle_U)$ is an indefinite scalar product space, then any admissible module

will necessarily be a neutral space [10]. We call an admissible module of minimal possible dimension the *minimal admissible module.*

In Table 1 we give the dimensions of the minimal admissible modules $V^{r,s}$, $r,s \leq 8$. Dimensions of other minimal admissible modules can be obtained by Bott's periodicity, see [32]. The bold integers are used for the minimal admissible modules which are direct sums of two irreducible Clifford modules. Others denote the dimensions of minimal admissible modules, that are also irreducible Clifford modules. The notation $_{\times 2}$ means that there are two minimal admissible modules, related to non-isomorphic irreducible modules.

TABLE 1. Dimensions of minimal admissible modules

8	16	32	64	$64_{\times 2}$	128	128	128	$128_{\times 2}$	256
7	16	32	**64**	64	**128**	**128**	**128**	128	256
6	16	$16_{\times 2}$	32	32	**64**	$\mathbf{64}_{\times 2}$	**128**	128	256
5	**16**	16	16	16	**32**	64	128	128	**256**
4	8	8	8	$8_{\times 2}$	16	32	64	$64_{\times 2}$	128
3	**8**	**8**	**8**	8	16	32	**64**	64	**128**
2	**4**	$\mathbf{4}_{\times \mathbf{2}}$	**8**	8	16	$16_{\times 2}$	32	32	**64**
1	**2**	**4**	**8**	8	**16**	16	16	16	**32**
0	1	2	4	$4_{\times 2}$	8	8	8	$8_{\times 2}$	16
s/r	0	1	2	3	4	5	6	7	8

As it was mentioned before, pseudo H-type Lie algebras are closely related to Clifford algebras. Namely, for a pseudo H-type Lie algebra $\mathfrak{n} = (\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}, \langle \cdot, \cdot \rangle_{\mathfrak{n}_{-1}} + \langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}})$ one has the representation $J \colon \mathrm{Cl}(\mathfrak{n}_{-2}, \langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}}) \to \mathrm{End}(\mathfrak{n}_{-1})$. Conversely, for an admissible $\mathrm{Cl}(U, \langle \cdot, \cdot \rangle_U)$-module $(V, \langle \cdot, \cdot \rangle_V)$, the representation induces a graded 2-step nilpotent Lie algebra structure on $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1} = U \oplus V$ defining the Lie bracket by equation (1).

Let us assume that $\mathfrak{n}$ satisfies the J^2-condition. If $x \in \mathfrak{n}_{-1}$ and $z, z' \in \mathfrak{n}_{-2}$ is any orthogonal pair satisfying

(20) $$\langle J_{\tilde{z}} x, J_z J_{z'} x \rangle_{\mathfrak{n}_{-1}} = 0, \qquad \forall \ \tilde{z} \in \mathfrak{n}_{-2},$$

then $\langle x, x \rangle_{\mathfrak{n}_{-1}} = 0$. Indeed using (8) we get

$$0 = \langle J_{\tilde{z}} x, J_z J_{z'} x \rangle_{\mathfrak{n}_{-1}} = \langle J_{\tilde{z}} x, J_{z''} x \rangle_{\mathfrak{n}_{-1}} = \langle \tilde{z}, z'' \rangle_{\mathfrak{n}_{-2}} \langle x, x \rangle_{\mathfrak{n}_{-1}}.$$

Thus, to show that a pseudo H-type Lie algebra $\mathfrak{n}$ does not satisfy the J^2-condition it is enough to find a vector $x \in \mathfrak{n}_{-1}$, $\langle x, x \rangle_{\mathfrak{n}_{-1}} \neq 0$ and an orthogonal pair $z, z' \in \mathfrak{n}_{-2}$ such that (20) holds.

For a minimal admissible module V let us call the modules $V^{\oplus k}$ isotypic.

Theorem 5. *Only the following pseudo H-type algebras satisfy the J^2-condition:*
  (1) $\dim \mathfrak{n}_{-2} = 0$: $\mathbb{R}^n$ – *any module (vector space) over* $\mathbb{R}$.
  (2) $\dim \mathfrak{n}_{-2} = 1$: $\mathfrak{n}^{1,0}$ *and* $\mathfrak{n}^{0,1}$ *for any admissible module.*
  (3) $\dim \mathfrak{n}_{-2} = 3$: $\mathfrak{n}^{3,0}$ *and* $\mathfrak{n}^{1,2}$ *for any isotypic module.*
  (4) $\dim \mathfrak{n}_{-2} = 7$: $\mathfrak{n}^{7,0}$ *and* $\mathfrak{n}^{3,4}$ *for the minimal admissible modules.*

In Appendix A we will explain these dimensions via a relation to the division algebras. Explicit descriptions of the admissible modules are given in the proof. They are also realized via simple Lie algebras as described in Appendix B.

Proof. We start by presenting a dimensional argument related to Table 1, which shows that all pseudo H-type algebras $\mathfrak{n}^{r,s}$ with $r+s \neq 0,1,3,7$, do not satisfy the J^2-condition. Let us start by pointing out that if $\mathfrak{n}^{r,s}(V)$, with a minimal admissible module V, does not satisfy the J^2 condition, then no $\mathfrak{n}^{r,s}$ satisfies it. Indeed, for any pseudo H-type algebra $\mathfrak{n}^{r,s}(V)$, with a minimal admissible module V, there exists an element $x \in \mathfrak{n}_{-1} = V$, $\langle x, x \rangle_{\mathfrak{n}_{-1}} = 1$, such that the set

$$\{x, J_{z_i}x, J_{z_i}J_{z_j}x, J_{z_i}J_{z_j}J_{z_k}x, \cdots\}$$

contains an orthonormal basis of $\mathfrak{n}_{-1}$, see [19].

Since $\mathbb{R}x \oplus J_{\mathfrak{n}_{-2}}x$ is an admissible module, it cannot have dimension less than the dimension of the minimal admissible module listed in Table 1. From this we see that the Clifford algebra $\mathrm{Cl}(\mathfrak{n}_{-2}, \langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}})$ can possess an admissible module of the form $\mathbb{R}x \oplus J_{\mathfrak{n}_{-2}}x$ of dimension $r+s+1$ only when $r+s = 0,1,3,7$. In other words, if $r+s \neq 0,1,3,7$, no pseudo H-type algebra $\mathfrak{n}^{r,s}$ admits the J^2-condition.

Henceforth we focus on pseudo H-type algebras $\mathfrak{n}^{r,s}$ with $m = r+s = 0,1,3,7$. Fix an orthonormal basis $\{z_1, \ldots, z_{r+s}\}$ of the center $\mathfrak{n}_{-2}$ of Lie algebra $\mathfrak{n}^{r,s}$ with $\langle z_i, z_i \rangle_{\mathfrak{n}_{-2}} = 1$, $1 \leq i \leq r$ and $\langle z_i, z_i \rangle_{\mathfrak{n}_{-2}} = -1$, $r < i \leq r+s$.

The J^2-condition is trivially satisfied when $m = \dim \mathfrak{n}_{-2} = 0,1$. Notice that for $\mathfrak{n}^{1,0}$ or $\mathfrak{n}^{0,1}$ the admissible modules are isotypic $V^{\oplus k}$, where V is the minimal admissible module $\mathbb{C}$ or $\mathbb{R} \oplus \mathbb{R}$ respectively.

Let $\dim \mathfrak{n}_{-2} = 3$. A similar dimensional argument as before, using Table 1, shows that the cases $(r,s) \in \{(2,1),(0,3)\}$ do not satisfy the J^2-condition. For $(r,s) \in \{(3,0),(1,2)\}$, we obtain $(J_{z_1}J_{z_2}J_{z_3})^2 = \mathrm{Id}_{\mathfrak{n}_{-1}}$. In these cases there are two non-isomorphic irreducible 2-dimensional Clifford modules V_+ and V_- of $\mathfrak{n}_{-1}$, where the endomorphism $\Omega^{r,s} = J_{z_1}J_{z_2}J_{z_3}$ acts as the identity or minus the identity, respectively. In other words, the spaces $V_\pm$ are the eigenspaces of $\Omega^{r,s}$ with the eigenvalues ± 1. In both cases $(r,s) \in \{(3,0),(1,2)\}$ the dimension of the minimal admissible module is 4.

Case $(r,s) = (1,2)$. The minimal admissible module is either $V_+ \oplus V_+$ or $V_- \oplus V_-$, see [19, 20]. It is necessary to point out that each module V_+ and V_- is a null space and by this reason we need to double them to guarantee admissibility. It follows that neither of these direct sums is orthogonal. For an orthonormal basis $\{z_1, z_2, z_3\}$ of $\mathfrak{n}_{-2}$, it holds

(21)
$$\begin{cases} J_{z_1}x = -J_{z_2}J_{z_3}x, \\ J_{z_2}x = -J_{z_1}J_{z_3}x, \\ J_{z_3}x = J_{z_1}J_{z_2}x, \end{cases}$$

for all $x \in V_+$. The signs in (21) change to the opposite ones for $x \in V_-$. Consider $\mathfrak{n}^{1,2}(V_+ \oplus V_+) = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$. Note that we could have chosen $\mathfrak{n}^{1,2}(V_- \oplus V_-)$ instead, but we would have obtained an isomorphic algebra. Choose any $z = \sum_{i=1}^{3} a_i z_i$

and $z' = \sum_{i=1}^{3} b_i z_i$ such that $\langle z, z' \rangle_{\mathfrak{n}_{-2}} = 0$. Then

$$(22) \qquad J_z J_{z'} x = \sum_{i,j} a_i b_j J_{z_i} J_{z_j} x = \langle z, z' \rangle_{\mathfrak{n}_{-2}} \operatorname{Id}_{\mathfrak{n}_{-1}} x + \sum_{i \neq j} a_i b_j J_{z_i} J_{z_j} x = J_{z''} x,$$

where z'' is a linear combination of the orthonormal basis with coefficients $\pm a_i b_j$, $i \neq j$ with the signs obtained by using (21). We conclude that the pseudo H-type algebra $\mathfrak{n}^{1,2}(V_+ \oplus V_+)$ satisfies the J^2-condition.

In the next step we allow the admissible module $\mathfrak{n}_{-1}$ to be non-minimal. There are (potentially several) non-isomorphic pseudo H-type algebras $\mathfrak{n}^{1,2}(V^{p,q})$ where $\mathfrak{n}_{-1} = V^{p,q} = (V_+ \oplus V_+)^{\oplus p} \oplus (V_- \oplus V_-)^{\oplus q}$ is an orthogonal sum of minimal admissible modules. It is not hard to see that the Lie algebras $\mathfrak{n}^{1,2}(V^{p,0})$ and $\mathfrak{n}^{1,2}(V^{0,q})$ satisfy the J^2-condition. We refer to these cases as *isotypic*.

At last we claim that for $p, q > 0$ the pseudo H-type Lie algebra $\mathfrak{n}^{1,2}(V^{p,q})$ does not satisfy the J^2-condition. Since this property for (p, q) implies the one for (p', q') with $p' \geq p, q' \geq q$, it is enough to check this for $p = q = 1$.

Choose $x_\pm \in V_\pm \oplus V_\pm$ such that $\langle x_+, x_+ \rangle_{\mathfrak{n}_{-1}} = \langle x_-, x_- \rangle_{\mathfrak{n}_{-1}} \neq 0$. Then $x = x_+ + x_- \in \mathfrak{n}_{-1}$ is such that $\langle x, x \rangle_{\mathfrak{n}_{-1}} = 2\langle x_+, x_+ \rangle_{\mathfrak{n}_{-1}} \neq 0$ and we get

$$\begin{aligned}
\langle J_{z_1} J_{z_2} x, J_z x \rangle_{\mathfrak{n}_{-1}} &= a_3 \langle J_{z_1} J_{z_2} x, J_{z_3} x \rangle_{\mathfrak{n}_{-1}} \\
&= a_3 \langle J_{z_3} x_+, J_{z_3} x_+ \rangle_{\mathfrak{n}_{-1}} - a_3 \langle J_{z_3} x_-, J_{z_3} x_- \rangle_{\mathfrak{n}_{-1}} = 0.
\end{aligned}$$

The case $(r, s) = (3, 0)$ is well known, see [10, 14], but still can be treated in a similar manner as before, changing (21) to the following ordered product

$$\pm J_{z_i} x = (-1)^i \prod_{j \neq i} J_{z_j} x \quad \text{for all} \quad x \in V_\pm.$$

Let $\dim \mathfrak{n}_{-2} = 7$. A similar dimensionality argument as before, using Table 1, shows that the cases $(r, s) \notin \{(7, 0), (3, 4)\}$ do not satisfy the J^2-condition.

Case $(r, s) = (3, 4)$. There are two irreducible Clifford modules V_+ and V_- of dimension 8 that, in this case, are also admissible. They are isometric to $\mathbb{R}^{4,4}$ and generate isomorphic pseudo H-type algebras, see [19, 20]. The following operators are mutually commuting symmetric involutions:

$$P_1 = J_{z_1} J_{z_2} J_{z_4} J_{z_5}, \quad P_2 = J_{z_1} J_{z_2} J_{z_6} J_{z_7}, \quad P_3 = J_{z_1} J_{z_3} J_{z_5} J_{z_7}.$$

Table 2 shows the commutation relations of the operators P_j and J_{z_k}, where 1 appears if they commute and -1 if they anti-commute.

TABLE 2. Commutation relations of operators

	J_{z_1}	J_{z_2}	J_{z_3}	J_{z_4}	J_{z_5}	J_{z_6}	J_{z_7}
P_1	-1	-1	1	-1	-1	1	1
P_2	-1	-1	1	1	1	-1	-1
P_3	-1	1	-1	1	-1	1	-1

We observe that V_+ decomposes into the orthogonal sum of the common eigenspaces for the involutions P_1, P_2, P_3. Denote $E^{1,1,1} = \{w \in \mathfrak{n}_{-1} \mid P_1 w = P_2 w = P_3 w = w\}$. Then all $J_{z_i} E^{1,1,1}$ are different mutually orthogonal eigenspaces

for P_1, P_2, P_3. Therefore, for any pair $J_{z_i}J_{z_j}$, $i \neq j$, there is $k \neq i,j$ such that $J_{z_i}J_{z_j}(w) = \pm J_{z_k}(w)$. Table 3 shows these eight different eigenspaces.

TABLE 3. Eigenspace decomposition: $\mathrm{Cl}_{3,4}$ case

Involutions	Eigenvalues							
P_1	$+1$				-1			
P_2	$+1$		-1		$+1$		-1	
P_3	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
Eigenvectors	w	$J_{z_3}w$ $J_{z_1}J_{z_2}w$ $J_{z_4}J_{z_5}w$ $J_{z_6}J_{z_7}w$	$J_{z_6}w$ $J_{z_1}J_{z_5}w$ $J_{z_2}J_{z_4}w$ $J_{z_3}J_{z_7}w$	$J_{z_7}w$ $J_{z_1}J_{z_4}w$ $J_{z_2}J_{z_5}w$ $J_{z_3}J_{z_6}w$	$J_{z_4}w$ $J_{z_1}J_{z_7}w$ $J_{z_2}J_{z_6}w$ $J_{z_3}J_{z_5}w$	$J_{z_5}w$ $J_{z_1}J_{z_6}w$ $J_{z_2}J_{z_7}w$ $J_{z_3}J_{z_4}w$	$J_{z_2}w$ $J_{z_1}J_{z_3}w$ $J_{z_4}J_{z_6}w$ $J_{z_5}J_{z_7}w$	$J_{z_1}w$ $J_{z_2}J_{z_3}w$ $J_{z_4}J_{z_7}w$ $J_{z_5}J_{z_6}w$

Choosing the basis

$$x_1 = w, \qquad x_2 = J_{z_3}w, \qquad x_3 = J_{z_6}w, \qquad x_4 = J_{z_7}w,$$
$$x_5 = J_{z_4}w, \qquad x_6 = J_{z_5}w, \qquad x_7 = J_{z_2}w, \qquad x_8 = J_{z_1}w,$$

for any x_p and any indices $i \neq j$, there is an index k such that $J_{z_i}J_{z_j}(x_p) = \pm J_{z_k}(x_p)$. Therefore, for any $x = \sum_{p=1}^8 \lambda_p x_p$, we again obtain $J_{z_i}J_{z_j}(x) = J_{\tilde{z}}(x)$ for some $\tilde{z} \in \mathfrak{n}_{-2}$. Now for any $x \in \mathfrak{n}_{-1}$ and for any pair of orthogonal vectors $z, z' \in \mathfrak{n}_{-2}$, we argue as in (22) and conclude that $\mathfrak{n}^{3,4}$ with the minimal admissible module satisfies the J^2-condition.

Let us show that $\mathfrak{n}^{3,4}(W)$ with a non-minimal admissible module W does not satisfy the J^2-condition. It is clear that if the J^2-condition holds for $\mathfrak{n}^{3,4}(W)$ it also holds for $\mathfrak{n}^{3,4}(W')$, where $W' \subset W$ is a submodule. Thus it suffices to verify the case of two summands $W = V_1 \oplus V_2$, where $V_i = V_+, V_-$. All three cases are similar and we consider only the first of them: $V_1 = V_2 = V_+$.

Let $x \in \mathfrak{n}_{-1}$ be of the following form $x = w_1 + J_{z_k}w_2$, where $w_1 \in E^{1,1,1} \cap V_1$ and $w_2 \in E^{1,1,1} \cap V_2$ and $k = 1$ or 2. We assume that the admissible metric is such that $\langle w_\alpha, w_\alpha \rangle_{V_+} > 0$, $\alpha = 1, 2$, and normalizing we assume that w_1, w_2 are unit vectors. This implies that $\langle x, x \rangle_{\mathfrak{n}_{-1}} \neq 0$. As a pair of orthogonal vectors from $\mathfrak{n}_{-2}$ we take the basis vectors z_3 and z_4. For an arbitrary vector $z = \sum_{i=1}^7 a_i z_i$ the following holds

$$\langle J_{z_3}J_{z_4}x, J_z x \rangle_{\mathfrak{n}_{-1}} = \left\langle J_{z_3}J_{z_4}(w_1 + J_{z_k}w_2), \sum_{i=1}^7 a_i J_{z_i}(w_1 + J_{z_k}w_2) \right\rangle_{\mathfrak{n}_{-1}}$$

$$= a_5 \langle J_{z_3}J_{z_4}w_1, J_{z_5}w_1 \rangle_{\mathfrak{n}_{-1}} + \left\langle J_{z_k}J_{z_3}J_{z_4}w_2, -\sum_{i \neq k} a_i J_{z_k}J_{z_i}w_2 \right\rangle_{\mathfrak{n}_{-1}}$$

$$= a_5 \langle J_{z_3}J_{z_4}w_1, J_{z_5}w_1 \rangle_{\mathfrak{n}_{-1}} - a_5 \langle z_k, z_k \rangle_{\mathfrak{n}_{-2}} \langle J_{z_3}J_{z_4}w_2, J_{z_5}w_2 \rangle_{\mathfrak{n}_{-1}} = 0$$

If the admissible metric $\langle .,. \rangle_{\mathfrak{n}_{-1}}$ is such that $\langle w_1, w_1 \rangle_{\mathfrak{n}_{-1}} = -\langle w_2, w_2 \rangle_{\mathfrak{n}_{-1}}$, then we modify the previous argument by changing the vector $x \in \mathfrak{n}_{-1}$ to $x = w_1 + J_{z_k}w_2$, $k = 5, 6$ or 7. Thus only the algebra $\mathfrak{n}^{3,4}(V_+) \simeq \mathfrak{n}^{3,4}(V_-)$ satisfies the J^2-condition.

Case $(r, s) = (7, 0)$. Though this case is known in the literature [14], it can be treated similarly to the case $(3, 4)$ starting from the involutions

$$P_1 = J_{z_1}J_{z_2}J_{z_3}J_{z_4}, \qquad P_2 = J_{z_1}J_{z_2}J_{z_5}J_{z_6}, \qquad P_3 = J_{z_1}J_{z_3}J_{z_5}J_{z_7}$$

acting on the minimal admissible module $\mathfrak{n}_{-1} = V_{\pm}$ isometric to $\mathbb{R}^8$. The conclusion is very similar: only $\mathfrak{n}^{7,0}(V_+) \simeq \mathfrak{n}^{7,0}(V_-)$ satisfies the J^2-condition. $\qquad\square$

Appendix A. Split versions of the division algebras

Recall that real rank of a Lie group G is the dimension of the Abelian factor A in the Iwasawa decomposition $G = KAN$. Real rank 1 simple Lie algebras are $|1|$- or $|2|$-graded (i.e. $\mathfrak{g} = \mathfrak{g}_{-\nu} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\nu$, $\nu = 1 \vee 2$), and the Lie algebra $\mathfrak{n} = \mathrm{Lie}(N) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{z} \oplus V$ is of H-type satisfying the J^2 condition [28].

It was noticed by B. Kostant [27] that in this case $\mathbb{R} \oplus \mathfrak{z}$ is necessarily a division algebra (so one of $\mathbb{R}$, $\mathbb{C}$ $\mathbb{H}$, or $\mathbb{O}$). The multiplication structure is as follows.

For any nonzero $x \in V$ the space A_x of (7) is the direct sum $\mathbb{R}x \oplus J_{\mathfrak{n}_{-2}}x = \mathbb{R} \oplus \mathfrak{z}$. Using this identification, supply the space A_x with the product

$$(a, z) \cdot (a', z') = (a'', z''), \quad \text{where} \quad (a + J_z)(a' + J_{z'})x = (a'' + J_{z''})x.$$

This formula also makes sense for pseudo H-type algebras with the J^2 condition, provided that x is non-null (in which case we identify $A_x = \mathbb{R} \oplus \mathfrak{z}$).

The above product $\cdot$ is bilinear, but not necessary associative in general. If $\langle z, z \rangle_{\mathfrak{z}} \neq -a^2$, then the inverse element to (a, z) is equal to

$$(a, z)^{-1} = \left(\frac{a}{a^2 + \langle z, z \rangle_{\mathfrak{z}}}, \frac{-z}{a^2 + \langle z, z \rangle_{\mathfrak{z}}} \right).$$

This shows, that for positive-definite metric $\langle \cdot, \cdot \rangle_{\mathfrak{z}}$, $A_x = \mathbb{R} \oplus \mathfrak{z}$ is a division algebra, while in the sign-indefinite og negative definite cases it is not.

This agrees with Theorem 5, where $\dim \mathfrak{z} \in \{0, 1, 3, 7\}$. The case $\mathfrak{z} = 0$ is trivial, then $A_x = \mathbb{R}$. In the other cases, when $\langle \cdot, \cdot \rangle_{\mathfrak{z}}$ is not positive-definite, we obtain the split versions of the division algebras.

Proposition 4. *The algebra $A_x = \mathbb{R} \oplus \mathfrak{z}$ in the cases of pseudo H-type Lie algebras $\mathfrak{n}^{0,1}(V)$, $\mathfrak{n}^{1,2}(V)$, and $\mathfrak{n}^{3,4}(V)$ from Theorem 5 is isomorphic to the algebra of split-complex $\mathbb{C}_s$, split-quaternion $\mathbb{H}_s$ or split-octonion numbers $\mathbb{O}_s$, respectively.*

Proof. The minimal admissible module $A_x = \mathbb{R}x \oplus J_{\mathfrak{n}_{-2}}x$ (with x non-null) from the pseudo H-type Lie algebra $\mathfrak{n}^{0,1}$ has the structure of split-complex numbers $\mathbb{C}_s$ due to the presence of a linear transformation J_z, $\langle z, z \rangle_{\mathfrak{n}_{-2}} = -1$, with $J_z^2 = \mathrm{Id}$.

The minimal admissible module A_x of $\mathfrak{n}^{1,2}$ has the structure of split-quaternion numbers $\mathbb{H}_s$, since the Clifford representations $J_{z_1}, J_{z_2}, J_{z_3}$, where z_1, z_2, z_3 are orthonormal basis for $\mathfrak{n}_{-2}$, satisfy relations (21).

The split-octonion structure $\mathbb{O}_s$ on the minimal admissible module A_x of $\mathfrak{n}^{3,4}$ is formed by the Clifford representations J_{z_j}, $j = 1, \ldots, 7$ for an orthonormal basis for $\mathfrak{z} = \mathfrak{n}_{-2}$. The split-octonionic relations among J_{z_j} can be verified by using the involutions P_1, P_2, P_3 and $P_4 = J_{z_5} J_{z_6} J_{z_7}$ from the proof of Theorem 5. $\qquad\square$

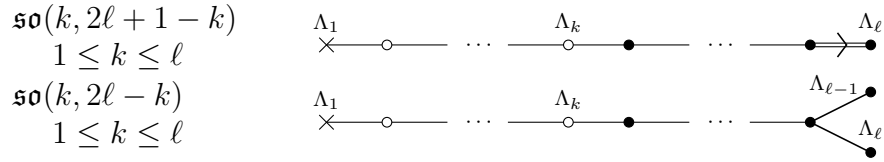## Appendix B. Realization of J^2 condition via simple Lie algebras

Gradings of simple Lie algebras are enumerated by a choice of parabolic subalgebra, or in combinatorial terms by a choice of crossed nodes on the Dynkin (in the complex case) or the Satake (in the real case) diagrams. Note that in the

real case crosses can be put only on white nodes, and those connected by arrows shall be crossed simultaneously, see [38, 8].
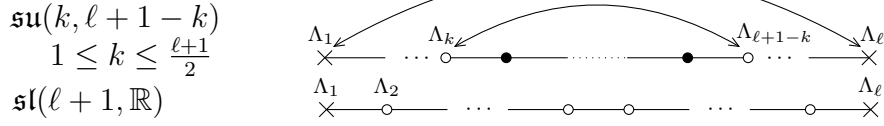
Real rank one simple Lie algebras have a unique grading (up to an internal automorphism), but the other non-compact simple Lie algebras with the same complexification have higher ranks and more choices of grading. There is however a unique choice in each case giving a pseudo H-type algebra with the J^2-condition.

Theorem 6. *Every pseudo H-type algebra satisfying the J^2-condition, is the negative graded part (the nilradical of the opposite parabolic) of one of the graded simple Lie algebras $\mathfrak{g}$ from the following list, numerated as in Theorem 5.*
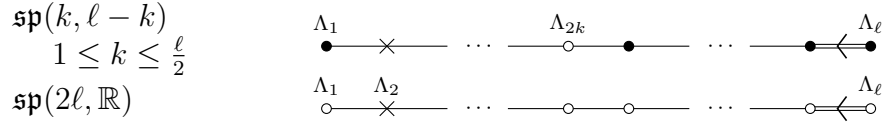
(1) BD_ℓ/P_1

$\mathfrak{so}(k, 2\ell + 1 - k)$
$1 \le k \le \ell$

$\mathfrak{so}(k, 2\ell - k)$
$1 \le k \le \ell$

$$\Lambda_1 \quad\cdots\quad \Lambda_k \quad\cdots\quad \Lambda_\ell$$

(2) $A_\ell/P_{1,\ell}$

$\mathfrak{su}(k, \ell + 1 - k)$
$1 \le k \le \frac{\ell+1}{2}$

$\mathfrak{sl}(\ell + 1, \mathbb{R})$

$$\Lambda_1 \quad \Lambda_k \quad\cdots\quad \Lambda_{\ell+1-k} \quad \Lambda_\ell$$

(3) C_ℓ/P_2

$\mathfrak{sp}(k, \ell - k)$
$1 \le k \le \frac{\ell}{2}$

$\mathfrak{sp}(2\ell, \mathbb{R})$

$$\Lambda_1 \quad \Lambda_{2k} \quad\cdots\quad \Lambda_\ell$$

(4) F_4/P_4

FII $\qquad\bullet\!\!-\!\!\bullet\!\!\Rightarrow\!\!\bullet\!\!-\!\!\times$ $\qquad$ FI $\qquad\circ\!\!-\!\!\circ\!\!\Rightarrow\!\!\circ\!\!-\!\!\times$

Note that the Satake diagrams for C_ℓ and D_ℓ cases shall be accordingly modified when $k = \frac{\ell}{2}$ and $k = \ell - 1, \ell$ respecively. The real rank of the Lie algebras is k when this is applicable, and in the other cases it is ℓ, ℓ, 1 and 4, respectively. The cases of real rank one conicide with the H-type Lie algebras studied in [14].

Proof. In case (1), $\mathfrak{n} = \mathfrak{g}_{-1}$ is simply a vector space with an inner product, and the M-type algebra is given by the signature of this inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{-1}}$, which is encoded by k and $\dim \mathfrak{n}$ (or ℓ and the choice of diagram B or D).

In case (2), $\mathfrak{n} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the Heisenberg algebra. When $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}}$ is positive, the algebra $\mathfrak{n}^{1,0}(V)$ is given by $\mathbb{C}$-module V, which is the sum of $\ell = \frac{1}{2} \dim \mathfrak{n}_{-1}$ irreducible modules $\mathbb{C}$, and then $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{-1}}$ is uniquely specified by the signature that is encoded by k in $\mathfrak{su}(k, \ell + 1 - k)$. The algebra $\mathfrak{n}^{0,1}(V)$ is unique, since V is obtained by doubling and the signature is split. This corresponds to $\mathfrak{sl}(\ell + 1, \mathbb{R})$.

In case (3), the series $\mathfrak{sp}(k, \ell - k)$ is similar to the rank $k = 1$ algebra: all cases have type $\mathfrak{n}^{3,0}(V)$ and differ by the signature of the metric on V, which is encoded by k. The algebra $\mathfrak{sp}(2\ell, \mathbb{R})$ corresponds to $\mathfrak{n}^{1,2}(V)$, with the unique

module V that is obtained by doubling and has a split signature metric. Let us give details of this latter identification.

Recall that $\mathfrak{sp}(2\ell, \mathbb{R}) = \left\{ \begin{pmatrix} D & B \\ C & -D^t \end{pmatrix} : B, C, D \in \mathfrak{gl}_\ell(\mathbb{R}), B^t = B, C^t = C \right\}$. The grading corresponding to the parabolic $\mathfrak{p}_2$ has the following negative part:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 \\ Z & X^t & 0 & -Y^t \\ X & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} \mathfrak{g}_{-1} &= \mathbb{R}^{4(\ell-2)}(X,Y) \\ & X, Y \text{ are } (\ell-2) \times 2 \text{ matrices} \\ \mathfrak{g}_{-2} &= \mathbb{R}^3(Z) \\ & Z^t = Z \text{ is a } 2 \times 2 \text{ matrix} \end{aligned}$$

If we write $Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$, where $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^{\ell-2}$ are row-vectors, then the brackets of $\mathfrak{n} = \mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ are given by $[A', A''] = A$, where $z_{ij} = \langle X'_i, Y''_j \rangle + \langle X'_j, Y''_i \rangle - \langle X''_i, Y'_j \rangle - \langle X''_j, Y'_i \rangle$ with respect to the standard scalar Euclidean product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{\ell-2}$ (we use the same notations for coordinates of A', A'', A marking them with prime, double prime or nothing).

This means that $\mathfrak{n}_{-1} = \oplus_{i=1}^{\ell-2} \mathbb{S}_i$ is the sum of standard (doubled) 4D modules over the Clifford algebra of $\mathfrak{z} = \mathfrak{n}_{-2}$ and the brackets are $[\mathbb{S}_i, \mathbb{S}_j] = 0$ for $i \neq j$, as well as the same nontrivial map $\Lambda^2 \mathbb{S}_i \to \mathfrak{z}$ for all i.

Let us describe these latter brackets for one module $\mathbb{S}$. Denote by $e_1, e_2, f_1, f_2 \in \mathbb{S}$, $h_{11}, h_{12}, h_{22} \in \mathfrak{z}$ the bases corresponding to the above coordinates. The only non-trivial brackets are $[e_1, f_1] = 2h_{11}$, $[e_1, f_2] = [e_2, f_1] = h_{12}$, $[e_2, f_2] = 2h_{22}$.

The metrics of $\mathfrak{n}_{-1}$ and $\mathfrak{n}_{-2}$ are given by $\langle e_2, f_1 \rangle_{\mathfrak{n}_{-1}} = -\langle e_1, f_2 \rangle_{\mathfrak{n}_{-1}} = 1$ and $\langle h_{11}, h_{22} \rangle_{\mathfrak{n}_{-2}} = -\frac{1}{2}$, $\langle h_{12}, h_{12} \rangle_{\mathfrak{n}_{-2}} = 1$ (these are the only non-trivial scalar products), whence the representaiton $J : \mathfrak{z} \to \text{End}(\mathbb{S})$:

$$J_{h_{11}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_{h_{12}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad J_{h_{22}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

It is straightforward now to check the pseudo H-type and the J^2-conditions.

We can explain this more representation-theoretically. Let V be the standard $\mathfrak{sl}_2$ module. The module $\mathfrak{z} = \mathfrak{ad} = S^2 V$ has a unique up to constant factor invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{-2}}$ of signature $(1, 2)$, which is indeed the Killing form of $\mathfrak{sl}_2$. The module $\mathbb{S} = V \oplus V^* \simeq V \oplus V$ has also unique up to constant factor invariant metric coming from the pairing between V and V^*, and it is $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{-1}}$ (this doubling of V is in agreement with Table 1).

With these choices the universal envelopping algebra $U(\mathfrak{sl}_2)$ action on $\mathbb{S}$ agrees with the Clifford relations, and it is easy to see that this M-type algebra is isomorphic to the one coming from the Lie algebra $\mathfrak{sp}(6, \mathbb{R})$ above, which corresponds to the partial case $\ell = 3$ of the split form of C_ℓ with one irreducible module $\mathbb{S}$.

Finally, in case (4) we have only two algebras. The algebra $\mathfrak{n}^{7,0}(V)$ corresponds to the real rank one case FII considered in [14], and completely analogously the algebra $\mathfrak{n}^{3,4}(V)$ corresponds to FI. □

Note that in both cases (1) and (2) the Lie algebra $\mathfrak{n}$ is of infinite type: it is the algebra of all formal vector fields on $\mathfrak{n}_-$ in the $|1|$-graded case and the algebra of formal contact vector fields in the case of contact grading.

On the contrary, cases (3) and (4) are rigid: the Tanaka prolongation of $\mathfrak{n} = \mathfrak{g}_-$ coincides with $\mathfrak{g}$. This follows from Yamaguchi's theorem [48] and agrees with the classification [4]. Thus these cases are of finite type.

Remark 3. The Lie algebras of all cases in (1) have the same Abelian structure for all k, it is the metric structure that varies. Similarly, all cases in (2) correspond to the same Heisenberg algebra (the theorem makes a distinction of M-type).

On the other hand, all cases in (3) and (4) are pairwise different as Lie algebras. Indeed, by the observation before this remark the Tanaka prolongations of these 2-step nilpotent Lie algebras coincide with simple Lie algebras $\mathfrak{g}$ specified in Theorem 6. Since these latter are different, so are the nilpotent Lie algebras $\mathfrak{n}$.

This finishes realization and classification of all pseudo H-type algebras satisfying the J^2 condition via nilradicals of parabolics in simple Lie algebras.

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