

Potential Type Operators in PDEs and Their Applications

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Abstract. We prove the boundedness of Potential operator in weighted generalized Morrey space in terms of Matuszewska-Orlicz indices of weights and apply this result to the Helmholtz equation in \mathbb{R}^3 with a free term in such a space. We also give a short overview of some typical situations when Potential type operators arise when solving PDEs.

INTRODUCTION

It is well known that many operators of harmonic analysis such as potential type operators, singular operators and others are widely used in PDE and PDO. The present paper is aimed to show some typical situations when Potential type operators arise when solving PDE. We do an emphasis on the role of the function space used in the solving process.

It is well known that the Potential type operators arise in study for instance Poisson's and Helmholtz equations. Such equations occur quite frequently in a variety of applied problems of science and engineering. The boundary value problems for the three-dimensional Laplace and Poisson equations are encountered in such fields as electrostatics, heat conduction, ideal fluid flow, elasticity and gravitation [1, 2, 3, 4]. Nowadays there are a lot of problems in physics which are reduced to the consideration of such equations. Laplace and Poisson equations (the inhomogeneous form of Laplace equation) appear in problems involving volume charge density. Applications of Laplace and Poisson equations to the electrostatics in fractal media are discussed in [3]. Such equations are also used in constructing satisfactory theories of vacuum tubes, ion propulsion and magnetohydrodynamic energy conversion [5].

Helmholtz equation which represents time-independent form of wave equation appears in different areas of physics. It is mostly known to be used in the case of the acoustic equation and to apply to the study of waveguides (devices that transmit acoustic or electromagnetic energy), see for instance [6, 7, 8] and [9, 10, 11, 12, 13] and references therein. But it typically works at certain discrete frequencies [14]. Many other applications of Helmholtz equation involve unbounded domains. For instance (see [14]) the simplest scattering problem for the case of an inhomogeneous medium is reduced to such equation in \mathbb{R}^3 .

We do not provide any historical overview: this would lead us too far away.

To avoid burdeness of the exposition by details, and for readers convenience, we present all necessary definitions and properties of the spaces and weights in the Appendix.

Laplace, Poisson and Helmholtz equations related operators

Newton and Riesz potential operators

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and let $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ be the Laplace operator. Consider the integral operator

$$I^2 f(x) = \frac{1}{\gamma_n(2)} \int_{\mathbb{R}^n} \frac{f(t) dt}{|x-t|^{n-2}}, \quad n \geq 3,$$

see the definition of $\gamma_n(\alpha)$ below, known as Newton potential. In the planar case $n = 2$ it is replaced by the logarithmic potential

$$I^2 f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-t| f(t) dt.$$

For all $n \geq 2$ the function $u(x) = I^2 f(x)$ is related to the Laplace operator. Namely, the function $u(x) = I^2 f(x)$ is a particular solution of the Poisson equation

$$-\Delta u = f,$$

see for instance [15].

From the Sobolev theorem for potential operators there follows the well known fact that $f \in L^p(\mathbb{R})$, $1 < p < n/2$ implies that $u \in L^q(\mathbb{R}) \cap W^{2,p}(\mathbb{R})$, $1/q = 1/p - 2/n$.

It is also known that the potential operators of the form

$$I^{2k} f(x) = \frac{1}{\gamma_n(2k)} \int_{\mathbb{R}^n} \frac{f(t) dt}{|x-t|^{n-2k}}, \quad k = 1, 2, \dots, 2k < n,$$

is similarly a particular solution of the Poisson type equation generated by the power of the Laplace operator:

$$(-\Delta)^k u = f.$$

In the case $k = 2$ we have the bi-harmonic Poisson equation.

Potential operators are known to be considered of arbitrary order $\alpha \geq 0$ not only $\alpha = 2k$. In the case $0 < \alpha < n$ they are introduced as

$$I^\alpha f(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(t) dt}{|x-t|^{n-\alpha}},$$

known also as the Riesz fractional integral. Here $\gamma_n(\alpha)$ is the normalizing constant chosen so that

$$I^\alpha f = F^{-1} \frac{1}{|\xi|^\alpha} Ff,$$

where F is the Fourier transform. Such a potential $u = I^\alpha f$ serves as a solution of the pseudo-differential equation

$$\mathbb{D}^\alpha u = f.$$

The PDO \mathbb{D}^α is also known as a hyper-singular operator. (We refer to [16, 17, 18] for pseudo-differential operators in general and to [19] for hyper-singular integrals). The hyper-singular operators \mathbb{D}^α are interpreted as fractional powers of the Laplace operator:

$$\mathbb{D}^\alpha = (-\Delta)^{\alpha/2}.$$

The particular case $\alpha = 1$ leads to the case $(-\Delta)^{1/2} = \sqrt{-\Delta}$, which is widely used in mathematical physics, see for instance [20, 21].

Modified Newton potential operator

Let us consider the modified Newton potential operator:

$$u(x) = \frac{1}{|x|^2 \gamma_n(2)} \int_{\mathbb{R}^n} \frac{f(t) dt}{|x-t|^{n-2}}.$$

This potential operator is a particular solution of the Poisson equation:

$$\Delta(u \cdot |x|^2) = -f.$$

By the well known formula for Laplacian of the product of two functions, we then easily obtain that u satisfies the following equation:

$$|x|^2 \Delta u(x) + 4x \nabla u(x) + 2nu(x) = -f(x).$$

Weighted potential operators

Now we pass to the weighted Newton potential operators:

$$u(x) = \frac{1}{w(x) \gamma_n(2)} \int_{\mathbb{R}^n} w(t) f(t) \frac{dt}{|x-t|^{n-2}}.$$

It is a particular solution of the equation:

$$\Delta u(x) + \frac{u(x)}{w(x)} \Delta w(x) + 2 \nabla(\ln |w(x)|) \nabla u(x) = -f(x).$$

Potential operators related to Helmholtz equation

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and let $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ be the Laplace operator. The potential

$$Vf(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-ik|x-y|}}{|x-y|} f(y) dy, \quad x \in \mathbb{R}^3 \quad (1)$$

is a particular solution (see for instance [14, Paragraph 2.2]) of the inhomogeneous Helmholtz equation $\Delta u + k^2 u = f(x)$ widely used in diffraction theory, so that

$$(\Delta + k^2 I)u(x) = f(x), \quad x \in \mathbb{R}^3 \quad (2)$$

where I is the identity operator.

The function $V(x)$ is also known as Helmholtz potential.

The corresponding weighted potential

$$W(x) := -\frac{1}{4\pi w(x)} \int_{\mathbb{R}^3} \frac{e^{-ik|x-t|}}{|x-t|} f(t) w(t) dt$$

is a particular solution of the following second order differential equation

$$\Delta W + 2 \frac{\nabla w}{w} \nabla W + \left(\frac{\Delta w}{w} + k^2 I \right) W = f$$

In the case of power weights $w(x) = x^\beta := x_1^{\beta_1} \cdot x_2^{\beta_2} \cdot x_3^{\beta_3}$,

$$\frac{\nabla w}{w} = \left[\frac{\beta_1}{x_1}, \frac{\beta_2}{x_2}, \frac{\beta_3}{x_3} \right]$$

and

$$\frac{\Delta w}{w} = \frac{\beta_1}{x_1^2} (1 - \beta_1) + \frac{\beta_2}{x_2^2} (1 - \beta_2) + \frac{\beta_3}{x_3^2} (1 - \beta_3).$$

Application of weighted boundedness of potential operators to the study of Helmholtz equation

In this section we consider behavior of the particular solution $u(x) = Vf(x)$ of Helmholtz equation (2), when f is in the weighted generalized Morrey space $L^{p,\varphi}(\mathbb{R}^3, w)$ (see the definition 5 in Section Appendix).

We need the following result (Theorem 1) about the boundedness of potential operators in generalized Morrey spaces the proof of which can be found in [22]; we give its formulation under slightly modified conditions due to the assumptions on φ and w , given below.

We begin with some assumptions and the theorem.

We will consider the action of the potential operator from one Morrey space $L^{p,\varphi}$ to another $L^{q,\psi}$. Note that the reader can find a detailed survey of mapping properties of potential operators in various function spaces in [23].

Everywhere in the sequel it is assumed that the functions φ , and ψ , defining the Generalized Morrey spaces are non-negative almost increasing functions continuous in a neighborhood of the origin, such that $\varphi(0) = 0$, $\varphi(r) > 0$, for $r > 0$, and $\varphi \in \overline{W} \cap \underline{W}$, and similarly for ψ .

For the function $\varphi(r)$, we will make use of the following conditions:

$$\varphi(r) \geq cr^n \tag{3}$$

for $0 < r \leq 1$, which makes the spaces $\mathcal{L}^{p,\varphi}(\Omega)$ non-trivial, see [22, Corollary 3.4],

$$\int_r^\infty \frac{\varphi^{\frac{1}{p}}(t)}{t^{\frac{n}{p}+1}} dt \leq C \frac{\varphi^{\frac{1}{p}}(r)}{r^{\frac{n}{p}}}. \tag{4}$$

and

$$\int_r^\infty \frac{\varphi^{\frac{1}{p}}(t)}{t^{\frac{n}{p}-\alpha+1}} dt \leq Cr^{-\frac{\alpha p}{q-p}}, \tag{5}$$

For the weights w we use the classes $\overline{W}(\mathbb{R}_+)$, $\underline{W}(\mathbb{R}_+)$ and \mathbf{V}_\pm^μ , the definition of which may be found in Section Appendix.

We will also use Zygmund classes \mathbb{Z}^β and \mathbb{Z}_γ , where $\beta, \gamma \in \mathbb{R}$, Matuszewska-Orlicz indices $M(\varphi)$ and $m(\varphi)$, of functions in such classes, see the corresponding Definitions in Appendix.

Theorem 1 [22, Theorem 5.5] *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $q > p$ and $\varphi(r)$ satisfy conditions (3) and (4)-(5). Let the weight $w \in \overline{W}(\mathbb{R}_+) \cap \underline{W}(\mathbb{R}_+)$ satisfy the conditions*

$$w \in \mathbf{V}_-^\mu \cup \mathbf{V}_+^\mu, \quad \mu = \min\{1, n - \alpha\}.$$

Then the weighted Riesz potential operator $wI^{\alpha, \frac{1}{w}}$ is bounded from $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\psi}(\mathbb{R}^n)$ under the conditions

$$\sup_{x \in \Omega, r > 0} \frac{1}{\psi(r)} \int_{B(x,r)} w^q(|y|)|y|^{q(\alpha-n)} \left(\int_0^{|y|} \frac{t^{\frac{n}{p}-1} \varphi^{\frac{1}{p}}(t)}{w(t)} dt \right)^q dy < \infty, \tag{6}$$

where $\frac{1}{p'}$ is the conjugate exponent: $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$\sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\psi(r)} \int_{B(x,r)} \left(\int_{|y|}^\infty t^{\alpha-\frac{n}{p}-1} \varphi^{\frac{1}{p}}(t) dt \right)^q dy < \infty, \tag{7}$$

in the case $w \in \mathbf{V}_+^\mu$, and the conditions

$$\sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\psi(r)} \int_{B(x,r)} |y|^{q(\alpha-n)} \left(\int_0^{|y|} t^{\frac{n}{p}-1} \varphi^{\frac{1}{p}}(t) dt \right)^q dy < \infty, \tag{8}$$

and

$$\sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\psi(r)} \int_{B(x,r)} w^q(|y|) \left(\int_{|y|}^{\infty} \frac{t^{\alpha - \frac{n}{p} - 1} \varphi^{\frac{1}{p}}(t)}{w(t)} dt \right)^q dy < \infty, \quad (9)$$

in the case $w \in \mathbf{V}_-^\mu$.

In the case when either $\varphi \in \Phi_n^0$ or $\varphi(r) = r^n$, conditions (6) - (9) are also necessary.

Note that Theorem 1 was proved in [22] for the case $\psi = \varphi$, but the analysis of the proof shows that the theorem holds in the above stated form.

We will use the above theorem to give conditions of the boundedness, more effective for possible applications. They particularly use numerical characteristics, known as Matuszewska-Orlicz indices, of weights and the function φ , which enables us to write some assumptions in terms of easily verified numerical inequalities. For the corresponding definitions and properties of such indices we refer to Appendix. Note that we admit the situation where the indices of functions at infinity are in general different from the indices at the origin.

Theorem 2 Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $q > p$, and

$$w \in [\overline{W}(\mathbb{R}_+) \cap \underline{W}(\mathbb{R}_+)] \cap [\mathbf{V}_-^\mu(\mathbb{R}_+) \cup \mathbf{V}_+^\mu(\mathbb{R}_+)], \quad \mu = \min\{1, n - \alpha\}.$$

Suppose also that the functions φ and ψ satisfy the assumptions:

$$M(\varphi), M_\infty(\varphi) < n - \alpha p, \quad \text{and} \quad \varphi(r) \leq cr^{n - \frac{\alpha}{p} - \frac{1}{q}} \quad \text{and} \quad \frac{\varphi^{1/p}(|y|)}{|y|^{\frac{n}{p} - \alpha}} \in L^{q, \psi}. \quad (10)$$

Under the conditions

$$\alpha - \frac{n - M(\varphi)}{p} < m(w) \leq M(w) < \frac{n}{p'} + \frac{m(\varphi)}{p}, \quad (11)$$

and

$$\alpha - \frac{n - M_\infty(\varphi)}{p} < m_\infty(w) \leq M_\infty(w) < \frac{n}{p'} + \frac{m_\infty(\varphi)}{p}, \quad (12)$$

the weighted Riesz potential operator $wI^{\alpha, \frac{1}{w}}$ is bounded from $\mathcal{L}^{p, \varphi}(\mathbb{R}^n)$ to $\mathcal{L}^{q, \psi}(\mathbb{R}^n)$.

Proof We have to show that the conditions of this theorem imply the assumptions of Theorem 1.

The condition (4) means (see (22)) that $\varphi^{1/p} \in \mathbb{Z}_\gamma$, with $\gamma = n/p$. By (42) $\varphi^{1/p} \in \mathbb{Z}_\gamma \iff M(\varphi^{1/p}) < n/p, M_\infty(\varphi^{1/p}) < n/p$. Therefore, by (26) and (36), $M(\varphi), M_\infty(\varphi) < n$ which is satisfied by the first inequality in (10).

From the property (30) and the first inequality in (10), we can see that (3) is satisfied.

Integration of the second inequality in (10), implies (5).

To show the validity of (6), under our assumptions, note that interior integral in (6) is dominated, by (8), by the function $c \frac{\varphi^{1/p}(|y|)}{w(|y|)|y|^{-\frac{n}{p'}}$, which follows from the fact that $\frac{\varphi^{1/p}}{w} \in Z^\beta$, with $\beta = -\frac{n}{p'}$. The latter is implied by the right hand side inequalities (11) and (12) in view of the properties (26)-(29) and (36)-(38), (42). Consequently, the third condition in (10) implies (6).

To show the validity of (7), under our assumptions, note that interior integral in (7) is dominated by the function $c \frac{\varphi^{1/p}(|y|)}{|y|^{\frac{n}{p} - \alpha}}$, which follows from the fact that $\varphi^{1/p} \in Z_\gamma$, with $\gamma = \frac{n}{p} - \alpha$. The latter is implied by the first inequality in (10) in view of the properties (26) and (29), and (36) and (42). Consequently, the third condition in (10) implies (7).

To show the validity of (8), under our assumptions, note that interior integral in (8) is dominated by the function $c \frac{\varphi^{1/p}(|y|)}{|y|^{-\frac{n}{p'}}$, which follows from the fact that $\varphi^{1/p} \in Z^\beta$, with $\beta = -\frac{n}{p'}$. In view of the properties (26), (29), and (36), (42),

the latter holds under the condition $p > 1 - \frac{m(\varphi)}{n}$, and $p > 1 - \frac{m_\infty(\varphi)}{n}$, which always holds since $m(\varphi), m_\infty(\varphi) \geq 0$. Consequently, the third condition in (10) implies (7).

To show the validity of (9), under our assumptions, note that interior integral in (9) is dominated by the function $c \frac{\varphi^{1/p}(|y|)}{w(|y|)|y|^{\frac{n}{p}-\alpha}}$, which follows from the fact that $\frac{\varphi^{1/p}}{w} \in Z_\gamma$, with $\gamma = \frac{n}{p} - \alpha$. The latter is implied by the left hand side inequalities (11) and (12) in view of the properties (26)-(29) and (36)-(38), (42). Consequently, the third condition in (10) implies (9).

The proof is complete.

The above theorem leads us to the following result for the Helmholtz equation, in the case $n = 3$, $\alpha = 2$. In this application we consider Morrey spaces imbedded into the corresponding weighted Lebesgue spaces, i.e. $L^{p,\varphi}(\mathbb{R}^3, w) \hookrightarrow L^p(\mathbb{R}^3, w)$. To this end, it suffices to assume that $\varphi(r)$ is a bounded function.

Theorem 3 *Let $1 < p < \frac{3}{2}$, $q > p$, and*

$$w \in [\overline{W}(\mathbb{R}_+) \cap \underline{W}(\mathbb{R}_+)] \cap [\mathbf{V}_-^1(\mathbb{R}_+) \cup \mathbf{V}_+^1(\mathbb{R}_+)].$$

Let also the functions φ and ψ satisfy the assumptions:

$$M(\varphi) < 3 - 2p, \text{ and } \varphi(r) \leq cr^{3-\frac{2}{p-\frac{1}{q}}} \text{ and } \frac{\varphi^{1/p}}{r^{\frac{3}{p}-2}} \in L^{q,\psi}. \quad (13)$$

Under the conditions

$$2 - \frac{3 - M(\varphi)}{p} < m(w) \leq M(w) < \frac{3}{p'} + \frac{m(\varphi)}{p}, \quad (14)$$

and

$$2 - \frac{3 - M_\infty(\varphi)}{p} < m_\infty(w) \leq M_\infty(w) < \frac{3}{p'} + \frac{m_\infty(\varphi)}{p}, \quad (15)$$

for every $f \in L^{p,\varphi}(\mathbb{R}^3, w)$, there exists a twice Sobolev differentiable particular solution $u \in L^{q,\psi}(\mathbb{R}^3, w)$ of the Helmholtz equation:

$$(\Delta + k^2 I)u(x) = f(x).$$

Proof The function u chosen as $u = Vf$, where Vf is the Helmholtz potential (1), is a particular solution of the Helmholtz equation (2).

Since the Helmholtz potential (1) is dominated by the Newton potential: $|Vf| \leq I^2(|f|)$, the inclusion of this solution $u = Vf$ into the space $L^{q,\psi}(\mathbb{R}^3, w)$ is guaranteed by Theorem 2.

As regards the differentiability of u , a direct differentiation of Vf leads to the sum of a Calderón-Zigmund singular operator of f and potential type operators. A justification of such a procedure for Sobolev derivatives in the case of weighted Lebesgue spaces is done for Muckenhoupt weights, see for instance [24]. The classical Morrey spaces are imbedded into the weighted Lebesgue spaces with the weight $w(x) = (1 + |x|)^{-\gamma}$, $\gamma > \lambda$, see [25]. Therefore imbedding of such a type is also valid for generalized Morrey spaces under the assumption that $\varphi(r) \leq cr^\gamma$ for all $r \in \mathbb{R}_+$ with some $\gamma \in [0, n)$. The condition of such a type is assumed in (13). Then the above mentioned procedure is valid within the frameworks of generalized Morrey spaces under the conditions of our theorem.

Therefore, the existence of the second derivatives of Vf follows from Theorem 2. For the singular operators in generalized weighted Morrey spaces we refer to [26, Theorem 3.5].

The proof is complete.

In the case of classical Morrey spaces, i.e. $\varphi(r) = r^\lambda, 0 < \lambda < n$, the statement of Theorem 3 holds in a more precise form as given in the following theorem.

Theorem 4 [27, Theorem 5.3]. *Let $1 < p < \frac{3}{2}$, $q > p, \lambda < 3 - 2p$ and*

$$w \in [\overline{W}(\mathbb{R}_+) \cap \underline{W}(\mathbb{R}_+)] \cap [\mathbf{V}_-^1(\mathbb{R}_+) \cup \mathbf{V}_+^1(\mathbb{R}_+)].$$

Under the conditions

$$2 - \frac{3 - \lambda}{p} < \min(m(w), m_\infty(w)) \quad (16)$$

and

$$\max(M(w), M_\infty(w)) < \frac{3}{p'} + \frac{\lambda}{p} \quad (17)$$

for every $f \in L^{p,\lambda}(\mathbb{R}^3, w)$, there exists a twice Sobolev differentiable particular solution $u \in L^{q,\lambda}(\mathbb{R}^3, w)$ of the Helmholtz equation:

$$(\Delta + k^2 I)u(x) = f(x),$$

where $\frac{1}{q} = \frac{1}{p} - \frac{2}{3-\lambda}$.

Appendix

Morrey space

$$\mathcal{L}^{p,\lambda} = \{f \in L^p_{loc}(\Omega) : \|f\|_{p,\lambda} < \infty\}, \quad 1 \leq p < \infty, \quad 0 \leq \lambda < n, \quad (18)$$

where $\Omega \subseteq \mathbb{R}^n$. Equipped with the norm

$$\|f\|_{p,\lambda} = \sup_{x \in \Omega, r > 0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} = \sup_{x \in \Omega, r > 0} \frac{\|f\|_{L^p(B(x,r))}}{r^{\frac{\lambda}{p}}} \quad (19)$$

where $B(x, r) = \{y \in \Omega : |y - x| < r\}$, it is a Banach space.

Generalized Morrey space

Definition 5. Let $\varphi(r)$ be a non-negative function on $[0, \ell]$, positive on $(0, \ell]$, and $1 \leq p < \infty$. The generalized Morrey space $\mathcal{L}^{p,\varphi}(\Omega)$ is defined as the space of functions $f \in L^p_{loc}(\Omega)$ such that

$$\|f\|_{p,\varphi} := \sup_{x \in \Omega, r > 0} \left(\frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty. \quad (20)$$

The classical Morrey space

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$$

corresponds to the case $\varphi(x, r) \equiv r^\lambda$, $0 < \lambda < n$.

The **weighted Morrey spaces** are treated in the usual sense:

$$\mathcal{L}^{p,\varphi}(\Omega, w) := \{f : wf \in \mathcal{L}^{p,\varphi}(\Omega)\}, \quad \Omega \subseteq \mathbb{R}^n, \quad \|f\|_{\mathcal{L}^{p,\varphi}(\mathbb{R}^n, w)} := \|wf\|_{\mathcal{L}^{p,\varphi}(\mathbb{R}^n)}.$$

On some classes of quasi-monotone functions

Below we give the known definitions and properties of some classes of quasi-monotone functions. For more details and proofs we refer for instance to [28, 29, 30] and references therein.

Definition 6.

1) By $W = W([0, 1])$ we denote the class of continuous and positive functions φ on $(0, 1]$ such that there exists finite

or infinite limit $\lim_{r \rightarrow 0} \varphi(r)$;

2) by $W_0 = W_0([0, 1])$ we denote the class of almost increasing functions $\varphi \in W$ on $(0, 1)$;

3) by $\overline{W} = \overline{W}([0, 1])$ we denote the class of functions $\varphi \in W$ such that $r^a \varphi(r) \in W_0$ for some $a = a(\varphi) \in \mathbb{R}^1$;

4) by $\underline{W} = \underline{W}([0, 1])$ we denote the class of functions $\varphi \in W$ such that $\frac{\varphi(t)}{t^b}$ is almost decreasing for some $b \in \mathbb{R}^1$.

Definition 7.

1) By $W_\infty = W_\infty([1, \infty])$ we denote the class of functions φ which are continuous and positive and almost increasing on $[1, \infty)$ and which have the finite or infinite limit $\lim_{r \rightarrow \infty} \varphi(r)$,

2) by $\overline{W}_\infty = \overline{W}_\infty([1, \infty))$ we denote the class of functions $\varphi \in W_\infty$ such that $r^a \varphi(r) \in W_\infty$ for some $a = a(\varphi) \in \mathbb{R}^1$.

By $\overline{W}(\mathbb{R}_+)$ we denote the set of functions on \mathbb{R}_+ whose restrictions onto $(0, 1)$ are in $\overline{W}([0, 1])$ and restrictions onto $[1, \infty)$ are in $\overline{W}_\infty([1, \infty))$. Similarly, the set $\underline{W}(\mathbb{R}_+)$ is defined.

ZBS-classes and MO-indices at the origin

Definition 8. We say that a function $\varphi \in W_0$ belongs to the Zygmund class \mathbb{Z}^β , $\beta \in \mathbb{R}^1$, if

$$\int_0^r \frac{\varphi(t)}{t^{1+\beta}} dt \leq c \frac{\varphi(r)}{r^\beta}, \quad r \in (0, 1), \quad (21)$$

and to the Zygmund class \mathbb{Z}_γ , $\gamma \in \mathbb{R}^1$, if

$$\int_r^1 \frac{\varphi(t)}{t^{1+\gamma}} dt \leq c \frac{\varphi(r)}{r^\gamma}, \quad r \in (0, 1). \quad (22)$$

We also denote

$$\Phi_\gamma^\beta := \mathbb{Z}^\beta \cap \mathbb{Z}_\gamma,$$

the latter class being also known as Bary-Stechkin-Zygmund class [31].

It is known that the property of a function to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the notion of the so called Matuszewska-Orlicz indices. We refer to [32, 33, 34, 30, 35, 36, 29] for the properties of the indices of such a type.

For a function $\varphi \in \overline{W}$:

$$m(\varphi) = \sup_{0 < r < 1} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{\varphi(hr)}{\varphi(h)} \right)}{\ln r} = \lim_{r \rightarrow 0} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{\varphi(hr)}{\varphi(h)} \right)}{\ln r} \quad (23)$$

and

$$M(\varphi) = \sup_{r > 1} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{\varphi(hr)}{\varphi(h)} \right)}{\ln r} = \lim_{r \rightarrow \infty} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{\varphi(hr)}{\varphi(h)} \right)}{\ln r} \quad (24)$$

The following properties of the indices of functions $u, v \in \underline{W} \cup \overline{W}$ are known, see for instance [37, Section 6] and references therein.

$$m[r^a u(r)] = a + m(u), \quad M[r^a u(r)] = a + M(u), \quad a \in \mathbb{R}^1, \quad (25)$$

$$m[(u)^a] = am(u), \quad M[(u)^a] = aM(u), \quad a \geq 0 \quad (26)$$

$$m\left(\frac{1}{u}\right) = -M(u), \quad M\left(\frac{1}{u}\right) = -m(u). \quad (27)$$

$$m(uv) \geq m(u) + m(v), \quad M(uv) \leq M(u) + M(v). \quad (28)$$

$$u \in \mathbb{Z}^\beta \iff m(u) > \beta \quad \text{and} \quad u \in \mathbb{Z}_\gamma \iff M(u) < \gamma. \quad (29)$$

$$c_1 r^{M(u)+\varepsilon} \leq u(r) \leq c_2 r^{m(u)-\varepsilon}, \quad 0 < r < 1, \quad (30)$$

hold with an arbitrarily small $\varepsilon > 0$ and $c_1 = c_1(\varepsilon)$, $c_2 = c_2(\varepsilon)$.

ZBS-classes and MO-indices of weights at infinity

The indices $m_\infty(u)$ of functions $u \in \overline{W}_\infty$ and $M_\infty(u)$ of functions $u \in \underline{W}_\infty$ responsible for the behavior of functions u at infinity are introduced in the way similar to (23) and (24):

$$m_\infty(u) = \sup_{r>1} \frac{\ln \left[\liminf_{h \rightarrow \infty} \frac{u(rh)}{u(h)} \right]}{\ln r}, \quad M_\infty(u) = \inf_{r>1} \frac{\ln \left[\limsup_{h \rightarrow \infty} \frac{u(rh)}{u(h)} \right]}{\ln r}. \quad (31)$$

The corresponding classes $\mathbb{Z}^{\beta_\infty}([1, \infty))$ of functions $u \in \overline{W}_\infty$ and $\mathbb{Z}_{\gamma_\infty}([1, \infty))$ of functions $u \in \underline{W}_\infty$ are introduced by the conditions

$$\int_1^r \frac{\varphi(t)}{t^{1+\beta}} dt \leq c \frac{\varphi(r)}{r^\beta}, \quad r \in (1, \infty), \quad (32)$$

$$\int_r^\infty \frac{\varphi(t)}{t^{1+\gamma}} dt \leq c \frac{\varphi(r)}{r^\gamma}, \quad r \in (1, \infty), \quad (33)$$

respectively

In view of the following equivalences

$$u \in \mathbb{Z}^\beta([1, \infty)) \iff u_* \in \mathbb{Z}_{-\beta}([0, 1]), \quad u \in \mathbb{Z}_\gamma([1, \infty)) \iff u_* \in \mathbb{Z}^{-\gamma}([0, 1]), \quad (34)$$

where $u_*(t) = u\left(\frac{1}{t}\right)$, properties of functions in the above introduced classes are easily derived from those of functions in $\Phi_\gamma^\beta([0, 1])$:

$$m_\infty[r^a u(r)] = a + m_\infty(u), \quad M_\infty[r^a u(r)] = a + M_\infty(u), \quad a \in \mathbb{R}^1, \quad (35)$$

$$m_\infty[(u^a)] = a m_\infty(u), \quad M_\infty[(u^a)] = a M_\infty(u), \quad a \geq 0 \quad (36)$$

$$m_\infty\left(\frac{1}{u}\right) = -M_\infty(u), \quad M_\infty\left(\frac{1}{u}\right) = -m_\infty(u). \quad (37)$$

$$m_\infty(uv) \geq m_\infty(u) + m_\infty(v), \quad M_\infty(uv) \leq M_\infty(u) + M_\infty(v). \quad (38)$$

$$c_1 t^{m_\infty(u)-\varepsilon} \leq u(t) \leq c_2 t^{M_\infty(u)+\varepsilon}, \quad t \geq 1, \quad u \in \overline{W}_\infty, \quad (39)$$

We say that a continuous function u in $(0, \infty)$ is in the class $\overline{W}_{0,\infty}(\mathbb{R}_+)$, if its restriction to $(0, 1)$ belongs to $\overline{W}([0, 1])$ and its restriction to $(1, \infty)$ belongs to $\overline{W}_\infty([1, \infty))$.

Without confusion of notation, by the same symbols $\mathbb{Z}^{\beta_0}([0, 1])$ and $\mathbb{Z}^{\beta_\infty}([1, \infty))$ we also denote the set of measurable functions on \mathbb{R}_+ such that their restrictions onto $[0, 1]$ and $(1, \infty)$ belong to $\mathbb{Z}^{\beta_0}([0, 1])$ and $\mathbb{Z}^{\beta_\infty}([1, \infty))$, respectively, and then we define

$$\mathbb{Z}^{\beta_0, \beta_\infty}(\mathbb{R}_+) = \mathbb{Z}^{\beta_0}([0, 1]) \cap \mathbb{Z}^{\beta_\infty}([1, \infty)), \quad \mathbb{Z}_{\gamma_0, \gamma_\infty}(\mathbb{R}_+) = \mathbb{Z}_{\gamma_0}([0, 1]) \cap \mathbb{Z}_{\gamma_\infty}([1, \infty)). \quad (40)$$

In the case where the indices coincide, i.e. $\beta_0 = \beta_\infty := \beta$, we will simply write $\mathbb{Z}^\beta(\mathbb{R}_+)$ and similarly for $\mathbb{Z}_\gamma(\mathbb{R}_+)$. We also denote

$$\Phi_\gamma^\beta(\mathbb{R}_+) := \mathbb{Z}^\beta(\mathbb{R}_+) \cap \mathbb{Z}_\gamma(\mathbb{R}_+). \quad (41)$$

Similarly to the case of the interval $[0, 1]$ the following properties

$$u \in \mathbb{Z}^\beta \iff m(u) > \beta, m_\infty(u) > \beta \text{ and } u \in \mathbb{Z}_\gamma \iff M(u) < \gamma, M_\infty(u) < \gamma. \quad (42)$$

hold for $u \in \overline{W}(\mathbb{R}_+)$ and $u \in \underline{W}(\mathbb{R}_+)$, respectively.

Definition 9. Let $0 < \mu \leq 1$. By \mathbf{V}_\pm^μ , we denote the classes of functions w non-negative on $[0, \infty)$ and positive on $(0, \infty)$, defined by the conditions:

$$\mathbf{V}_+^\mu : \quad \frac{|w(t) - w(\tau)|}{|t - \tau|^\mu} \leq C \frac{w(t_+)}{t_+^\mu}, \quad (43)$$

$$\mathbf{V}_-^\mu : \quad \frac{|w(t) - w(\tau)|}{|t - \tau|^\mu} \leq C \frac{w(t_-)}{t_-^\mu}, \quad (44)$$

where $t, \tau \in (0, \infty)$, $t \neq \tau$, and $t_+ = \max(t, \tau)$, $t_- = \min(t, \tau)$.

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