On precise differentiation formula for weighted singular integrals of Sobolev functions

The paper is dedicated to professor Lars-Erik Persson, on the occasion of his 70th birthday

Staffan Lundberg

Department of Engineering Sciences and Mathematics,
Division of Mathematical Sciences, Luleå University of Technology,
SE-971 87 Luleå, Sweden

Abstract. In various applications there appear integral equations of the first kind with a kernel which has a logarithmic singularity. The corresponding classes of well-posedness for such equations are Sobolev spaces. We prove the differentiation formulas for weighted singular integrals, which appear in such study, with the goal of its further application to integral equations.

Keywords: Integral equations with logarithmic kernel, weighted singular integrals, power weights, Sobolev spaces
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INTRODUCTION

Integral equations are one of the mathematical models well adjusted for the study of problems arising in various applied sciences. A key role among them is played by singular integral equations which are known to be related to boundary value problems for analytic functions. The classical theory of singular integral equations is presented e.g. in the books [1], [2]. In applications there also often arise integral equations of the first kind mainly corresponding to unstable processes. An important class of such equations with logarithmic singularity in the kernel is less studied in comparison with other types of singularities. A class of such equations has connection with singular integral equations which enables a researcher to apply the technique and results used in the theory of singular integral equations. Various types of such equations are used in various applications in the theory of logarithmic potential in plane problems of mathematical physics, conformal mapping, elasticity and viscosity theory. There were developed various numerical methods of approximate solutions of integral equations with logarithmic singularity. However, their theoretical treatment exists only in some particular cases.

The problem treated in this paper is related to the theory of integral equations of the first kind with a kernel which has a logarithmic singularity:

$$\int_a^b [u(x,t)+v(x,t)\ln|x-t|]\,\varphi(t)\,dt = f(x), \quad a<x<b. \tag{1}$$

Such equations of the first kind are ill-posed. The corresponding classes of well-posedness in this case are Sobolev spaces for functions f. More precisely, if X is the space for solutions φ, then the class of the right hand sides f should be the Sobolev space with derivatives in X. In our studies we use weighted Lebesgue space $X = L^p(w,[a,b])$ with a power weight $w=(x-a)^{\alpha_1}(b-x)^{\alpha_2}$.

In the theory of such equations it is admitted that the function $u(x,t)$ may have a jump at the diagonal $t=x$ and the study of such equations is based on investigation of the model equations of the form

$$A(x)\int_a^x \varphi(t)\,dt + B(x)\int_x^b \varphi(t)\,dt + \frac{C(x)}{\pi}\int_a^b \ln|x-t|\,\varphi(t)\,dt = f(x). \tag{2}$$

There exists a formalism ([3, 4]) which allows to reduce the equation (2) to the ”loaded” singular integral equation.

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In justification of this formalism for the right-hand sides f in Sobolev spaces, there arises a problem of differentiation of weighted singular integrals of functions in Sobolev spaces, originated by the fact that the solution of singular integral equation in an interval contains such weighted singular integrals.

We do not dwell on the study of solvability of such integral equations of the first kind in this paper but present a solution of the first problem which arises in this study of this first problem of differentiation of weighted singular integrals. Note that the study of smoothness or regularity of weighted singular integrals usually requires special efforts because of "bad" behaviour of such integrals near the end points. In the case where the scale of Hölder spaces is used for measuring such kind of regularity, the reader can be referred to [5, 6, 7].

The weighted singular integrals under consideration in this paper have the following form:

$$(T^{\mu} f)(x) = (x-a)^{\mu_1}(b-x)^{\mu_2} \int_a^b \frac{f(t)\,dt}{(t-a)^{\mu_1}(b-t)^{\mu_2}(t-x)} \tag{3}$$

$$(U_a^{\mu} f)(x) = (x-a)^{\mu_1}(b-x)^{\mu_2} \int_a^b \frac{\ln(t-a)}{(t-a)^{\mu_1}(b-t)^{\mu_2}} \frac{f(t)\,dt}{t-x} \tag{4}$$

$$\left(U_b^{\mu} f\right)(x) = (x-a)^{\mu_1}(b-x)^{\mu_2} \int_a^b \frac{\ln(b-t)}{(t-a)^{\mu_1}(b-t)^{\mu_2}} \frac{f(t)\,dt}{t-x}, \tag{5}$$

where $a<x<b$, $\mu=(\mu_1,\mu_2)$, the numbers μ_1 and μ_2 may be complex and $\mathrm{Re}(\mu_1)<1, \mathrm{Re}(\mu_2)<1$, and where we assume that the principal value of the power functions is chosen so that $\arg((t-a)^{\mu_1}) = \mathrm{Im}(\mu_1 \ln(t-a))$, and similarly for $(b-t)^{\mu_2}$.

DIFFERENTIATION FORMULAS

We prove a theorem, where we use the following notation for the weighted Sobolev space:

$$W^{p,1}(w) = \{f \in L^p(w,[a,b]) : df/dx \in L^p(w,[a,b])\},$$

where the derivative is understood as usual in the weak sense.

Weighted space $L^p(w,[a,b]) =: L^p(w)$ is defined as

$$L^p(w) := \{\varphi : \int_a^b |\varphi(x)w(x)|^p dx < \infty\}.$$

We also use the notations:

$$f_\mu = \int_a^b \frac{f(t)\,dt}{(t-a)^{\mu_1}(b-t)^{\mu_2}},\ \rho_{1-\mu}(x) := \frac{1}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}} \ \text{ and } \ D = d/dx.$$

Theorem 1 *Let $f \in W^{p,1}(w,[a,b])$, where $w=(x-a)^{\alpha_1}(b-x)^{\alpha_2}$.*

Under the assumption that

$$-1/p \le \alpha_1 + \mathrm{Re}(\mu_1 - 1) \le 1/p' \ \textit{and} \ -1/p \le \alpha_2 + \mathrm{Re}(\mu_2 - 1) \le 1/p', \textit{where } 1/p + 1/p' = 1,$$

the following differentiation formula is valid:

$$\frac{d}{dx} T^{\mu} f(x) =$$

$$\frac{1}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}} \int_a^b (t-a)^{1-\mu_1}(b-t)^{1-\mu_2} \frac{f'(t)\,dt}{(t-x)} + \frac{(\mu_1+\mu_2-1) f_\mu}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}}, \tag{6}$$

or in short form

$$\left(DT^{\mu}f\right)(x)=\left(T^{\mu-1}Df\right)(x)+(\mu_1+\mu_2-1)f_{\mu}\cdot\rho_{1-\mu}(x). \tag{7}$$

Proof The proof on nice functions is more or less direct but requires some tricks because we cannot differentiate the singular integral directly under the integral sign. The extension to functions in the Sobolev space is based on the use of definition of weak derivatives.

We start with the proof of the formula (7) for nice functions and then prove it for functions in $W^{p,1}(w)$, making use of its validity for nice functions.

I: The case of nice functions.

By nice functions we may mean functions in $C^{\infty}([a,b])$ with support in (a,b). We follow ideas from [4, 3]. To prove formula (7), we represent $T^{\mu}f$ as

$$\left(T^{\mu}f\right)(x)=\frac{(x-a)^{\mu_1}(b-x)^{\mu_2}}{b-a}\left[\int\limits_a^b\frac{(b-t)^{1-\mu_2}f(t)\,dt}{(t-a)^{\mu_1}(t-x)}+\int\limits_a^b\frac{(t-a)^{1-\mu_1}f(t)\,dt}{(b-t)^{\mu_2}(t-x)}\right]. \tag{8}$$

The main trick is to make a substitution to avoid the presence of the variable x in the expression $t-x$ in the denominator. Namely, we put $t-a=s(x-a)$ in the first integral and $b-t=s(b-x)$ in the second one, which yields

$$\begin{aligned}\left(T^{\mu}f\right)(x)=\frac{(b-x)^{\mu_2}}{b-a}\int\limits_0^{(b-a)/(x-a)}\frac{[b-a-s(x-a)]^{1-\mu_2}f[a+s(x-a)]}{s^{\mu_1}(s-1)}\,ds+\\+\frac{(x-a)^{\mu_1}}{b-a}\int\limits_0^{(b-a)/(b-x)}\frac{[b-a-s(b-x)]^{1-\mu_1}f[b-s(b-x)]}{s^{\mu_2}(1-s)}\,ds\end{aligned} \tag{9}$$

In this form differentiate under the integral sign is possible.

After direct differentiation and some simple calculations we obtain formula (7). Technical details of this calculation is given in Appendix.

II: The case of functions in $W^{p,1}(w)$.

In accordance with the definition of the Sobolev space, we need to show that the operator $T^{\mu-1}$ is bounded in the space $L^p(w)$, with the weight $w=(x-a)^{\alpha_1}(b-x)^{\alpha_2}$ i.e. that

$$\|T^{\mu-1}g\|_{L^p(w)}\le C\|g\|_{L^p(w)}\ \text{ for all }\ g\in L^p(w).$$

This is equivalent to that

$$\|T^{\mu-1+\alpha}\varphi\|_{L^p}\le C\|\varphi\|_{L^p}\ \text{ for all }\ \varphi\in L^p,$$

where $\alpha=(\alpha_1,\alpha_2)$. The latter inequality is nothing else but the boundedness of the weighted singular operator in L^p-spaces, which is well known to be valid if and only if $\mu_k+\alpha_k-1\in\left(-\frac{1}{p},\frac{1}{p'}\right), k=1,2$, see for instance [1, p. 30] and the references therein.

Thus the right hand side of formula (7) is well defined for all functions f in the Sobolev space $W^{p,1}(w)$ and belongs to $L^p(w)$.

To prove (7) with derivative D on the left hand side in the weak sense, we have to prove that

$$\left(DT^{\mu}f,\omega\right)=\left(T^{\mu-1}Df,\omega\right)+(\mu_1+\mu_2-1)f_{\mu}\left(\rho_{1-\mu},\omega\right) \tag{10}$$

for all test functions $\omega\in C^{\infty}$ on $[0,1]$ with support in $(0,1)$, where (f,ω) is the usual bilinear form:

$$(f,\omega)=\int_a^b f(x)\omega(x)dx.$$

We start with the right hand side and work with the term $\left(T^{\mu-1}Df,\omega\right)$. We observe that the operator transposed to $T^{\mu-1}$ is $-T^{1-\mu}$ and proceed as follows:

$$\left(T^{\mu-1}Df,\omega\right)=-\left(Df,T^{1-\mu}\omega\right)=\left(f,DT^{1-\mu}\omega\right),$$

where in the last passage the notion of weak derivative works.

Now we use the fact that our formula has been proved for nice functions, so that

$$DT^{1-\mu}\omega=T^{-\mu}D\omega+(1-\mu_1-\mu_2)\omega_{1-\mu}\cdot\rho_\mu(x)$$

where $\omega_{1-\mu}=\int\limits_a^b\frac{\omega(t)dt}{(t-a)^{1-\mu_1}(b-t)^{1-\mu_2}}$. Consequently,

$$\left(T^{\mu-1}Df,\omega\right)=\left(f,T^{-\mu}D\omega\right)+(1-\mu_1-\mu_2)\omega_{1-\mu}(f,\rho_\mu)$$

and then

$$\left(T^{\mu-1}Df,\omega\right)+(\mu_1+\mu_2-1)f_\mu\left(\rho_{1-\mu},\omega\right)=\left(f,T^{-\mu}D\omega\right)+(\mu_1+\mu_2-1)\left[f_\mu\left(\rho_{1-\mu},\omega\right)-\omega_{1-\mu}(f,\rho_\mu)\right].$$

It is easy to see that

$$f_\mu\left(\rho_{1-\mu},\omega\right)-\omega_{1-\mu}(f,\rho_\mu)=0.$$

Therefore the right-hand side of (10) is equal to $(f,T^{-\mu}D\omega)$ which is nothing else but the left-hand side of (10), since the operator $(T^{-\mu})^*=-T^\mu$, and $D^*=-D$. This completes the proof.

Corollary 2 *Let $f\in W^{p,1}(w,[a,b])$, where $w=(x-a)^{\alpha_1}(b-x)^{\alpha_2}$ and $-1/p\le\alpha_k-1/2\le1/p'$, $k=1,2$. Then*

$$\frac{d}{dx}\int_a^b\sqrt{\frac{(x-a)(b-x)}{(t-a)(b-t)}}\frac{f(t)dt}{t-x}=\int_a^b\sqrt{\frac{(t-a)(b-t)}{(x-a)(b-x)}}\frac{f'(t)dt}{t-x}. \tag{11}$$

Theorem 3 *Let $f\in W^{p,1}(w)$, where $w=(x-a)^{\alpha_1}(b-x)^{\alpha_2}$.*

Under the assumption that

$$-1/p\le\alpha_1+\mathrm{Re}(\mu_1-1)\le1/p' \text{ and } -1/p\le\alpha_2+\mathrm{Re}(\mu_2-1)\le1/p', \text{where } 1/p+1/p'=1,$$

the following differentiation formulas are valid:

$$\left(DU_a^\mu f\right)(x)=\left(U_a^{\mu-1}Df\right)(x)+\left(T^{\mu-1}\left(\frac{f(t)}{t-a}\right)\right)(x)+\frac{(\mu_1+\mu_2-1)f_a}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}}, \tag{12}$$

$$\left(DU_b^\mu f\right)(x)=\left(U_b^{\mu-1}Df\right)(x)+\left(T^{\mu-1}\left(\frac{f(t)}{b-t}\right)\right)(x)+\frac{(\mu_1+\mu_2-1)f_b}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}}, \tag{13}$$

where

$$f_a=\int\limits_a^b\frac{\ln(t-a)f(t)\,dt}{(t-a)^{\mu_1}(b-t)^{\mu_2}},\quad f_b=\int\limits_a^b\frac{\ln(b-t)f(t)\,dt}{(t-a)^{\mu_1}(b-t)^{\mu_2}}. \tag{14}$$

Proof The proof follows mainly in the same way as the proof of Theorem 1, so we leave out the details.

APPENDIX

We present some technical details of the proof of the formula (6) for nice functions. By using the substitutions $t-a=s(x-a)$ and $b-t=s(b-x)$, we obtain

$$(T^{\mu}f)(x)=I_1+I_2,$$

where

$$I_1=\frac{(b-x)^{\mu_2}}{b-a}\int_0^{(b-a)/(x-a)}\frac{[b-a-s(x-a)]^{1-\mu_2}f[a+s(x-a)]}{s^{\mu_1}(s-1)}\,ds$$

and

$$I_2=\frac{(x-a)^{\mu_1}}{b-a}\int_0^{(b-a)/(b-x)}\frac{[b-a-s(b-x)]^{1-\mu_1}f[b-s(b-x)]}{s^{\mu_2}(1-s)}\,ds.$$

Next, differentiation with respect to x yields

$$\frac{d}{dx}(T^{\mu}f)(x)=\frac{d}{dx}I_1+\frac{d}{dx}I_2 \tag{15}$$

By using the Leibniz rule for differentiating under the integral sign, we obtain after some calculations that

$$\frac{d}{dx}I_1=\frac{(-\mu_2)}{b-a}(b-x)^{\mu_2-1}\int_0^{\frac{b-a}{x-a}}\frac{[b-a-s(x-a)]^{1-\mu_2}f[a+s(x-a)]}{s^{\mu_1}(s-1)}\,ds+$$

$$+\frac{(b-x)^{\mu_2}}{b-a}\left\{\int_0^{\frac{b-a}{x-a}}\frac{s}{s^{\mu_1}(s-1)(b-a-s(x-a))^{\mu_2}}\left[(\mu_2-1)f(a+s(x-a))+(b-a-s(x-a))\cdot f'(a+s(x-a))\right]ds\right\}$$

Following the same steps, we get that

$$\frac{d}{dx}I_2=\frac{\mu_1}{b-a}(x-a)^{\mu_1-1}\int_0^{\frac{b-a}{b-x}}\frac{[b-a-s(b-x)]^{1-\mu_1}f[b-s(b-x)]}{s^{\mu_2}(1-s)}\,ds+$$

$$+\frac{(x-a)^{\mu_1}}{b-a}\left\{\int_0^{\frac{b-a}{b-x}}\frac{s}{s^{\mu_2}(1-s)(b-a-s(b-x))^{\mu_1}}\left[(1-\mu_1)f(b-s(b-x))+(b-a-s(b-x))\cdot f'(b-s(b-x))\right]ds.\right\}$$

We now re-substitute $s\to t$ in the integrals. By adding the derivatives of I_1 and I_2 and collecting terms containing f and f' respectively, we obtain that the part containing $f'(t)$ is equal to

$$\frac{(b-x)^{\mu_2}(x-a)^{\mu_1}}{(b-a)(x-a)}\int_a^b\frac{(t-a)^{1-\mu_1}(b-t)^{1-\mu_2}}{t-x}f'(t)\,dt+$$

$$+\frac{(x-a)^{\mu_1}(b-x)^{\mu_2}}{(b-a)(b-x)}\int_a^b\frac{(b-t)^{1-\mu_2}(t-a)^{1-\mu_1}}{t-x}f'(t)\,dt,$$

which, after some simplifications, can be written as

$$\frac{1}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}} \int_a^b \frac{(t-a)^{1-\mu_1}(b-t)^{1-\mu_2}}{t-x} f'(t)\,dt \tag{16}$$

Furthermore, the part containing $f(t)$ is equal to

$$\frac{1}{(b-a)(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}} \int_a^b \frac{1}{(t-a)^{\mu_1}(b-t)^{\mu_2}(t-x)} \Big((\mu_1+\mu_2-1)((t-a)(b-x)-(b-t)(x-a))\Big) f(t)\,dt,$$

which, after some simplifications, can be written as

$$\frac{\mu_1+\mu_2-1}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}} \int_a^b \frac{f(t)\,dt}{(t-a)^{\mu_1}(b-t)^{\mu_2}}. \tag{17}$$

Finally, according to (15), by adding (16) and (17), we obtain formula (6). The proof is complete. ∎

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