Saikat Kanjilal, Lars-Erik Persson and Guldarya E. Shambilova

Equivalent integral conditions related to bilinear Hardy-type inequalities

MATHEMATICAL INEQUALITIES & APPLICATIONS

AIMS AND SCOPE

Mathematical Inequalities & Applications (MIA, Math. Inequal. Appl.) brings together original research papers in all areas of mathematics, provided they are concerned with inequalities or their role. From time to time MIA will publish invited survey articles. Short notes with interesting results or open problems will also be accepted.

MIA is published quarterly, in January, April, July and October.

SUBMISSION

Manuscripts should be submitted electronically in PostScript or Adobe Acrobat PDF format to the MIA Editorial Office through MIA page: www.ele-math.com or by e-mail mia@ele-math.com

The author who submitted the article for publication will be denoted as a corresponding author. He/She manages all communication and correspondence with the MIA regarding the article. Authors may indicate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not assured.

In order to facilitate refereeing, copies of those papers (whether by the author or someone else) which are essential and referred to in the manuscript but are not conveniently accessible, should be enclosed.

COPYRIGHT

The acceptance of the article automatically implies the copyright transfer to MIA. Manuscripts are accepted for review with the understanding that the same work has not been published (except in the form of an abstract), that it is not under consideration for publication elsewhere, that it will not be submitted to another journal, and that its submission for publication has been approved by all the authors.

PREPARATION OF MANUSCRIPT

Manuscripts should be written in English.

The publisher strongly encourages submission of manuscripts written in TeX or one of its variants LaTeX, AMSTeX or AMSLaTeX. On acceptance of the paper, authors will be asked to send final version of file(s) to the Editorial Office.

TITLE PAGE

A title page should include the title of the article, author's name (no degrees) author's affiliation, e-mail addresses, mailing address of the corresponding author and running head less than 60 characters. Authors should be listed alphabetically.

ABSTRACT, KEY WORDS, SUBJECT CLASSIFICATION

The manuscript must be accompanied by a brief abstract, no longer than 100-150 words. It should make minimal use of mathematical symbols and displayed formulas. Mathematics Subject Classification (2000) with primary (and secondary) subject classification codes and a list of 4-5 key words must be given.

FIGURES

Figures should be prepared in a digital form suitable for direct reproduction, grey scale only, preferably at resolution of 300 dpi or higher, sized to fit a journal page, and in EPS, TIFF or JPEG format. Figures should have a short descriptive title and should be numbered consecutively in order of appearance in the text. The position of each figure in the text should be indicated.

REFERENCES

Bibliographic references should be listed alphabetically at the end of the article. The author should consult Mathematical Reviews for the standard abbreviations of journal names.

PROOFS

The author will receive proofs only once. Late return of the proofs will delay the article to a later issue.

PAGE CHARGES, OFFPRINTS

There are no page charges. Authors will receive PDF file of the printed article free of charge. Additional offprints may be ordered from 'MIA' prior to publication.

FORTHCOMING PAPERS

Papers accepted and prepared for publication will appear in the forthcoming section of Journal Web page. They are identical in form as final printed papers, except volume, issue and page numbers.

EQUIVALENT INTEGRAL CONDITIONS RELATED
TO BILINEAR HARDY–TYPE INEQUALITIES

SAIKAT KANJILAL, LARS-ERIK PERSSON* AND GULDARYA E. SHAMBILOVA

Dedicated to the 70th anniversary
of Professor Josip Pečarić

(*Communicated by I. Perić*)

Abstract. Infinitely many, even scales of, equivalent conditions are derived to characterize the bilinear Hardy-type inequality under various ranges of parameters.

1. Introduction

Let $\mathfrak{M}$ denote the set of all Lebesgue measurable functions on (a,b), $-\infty \leqslant a < b \leqslant \infty$, $\mathfrak{M}^+ \subset \mathfrak{M}$ is the subset of all non-negative functions.

Let $u, v, v_i, \in \mathfrak{M}^+$, $0 < p, p_i, q \leqslant \infty, 1 \leqslant p_i, i = 1, 2$. Denote $p_i' := \frac{p_i}{p_i - 1}$ for $1 < p_i < \infty$, $p_i' := 1$ for $p_i = \infty$ and $p_i' := \infty$ for $p_i = 1, i = 1, 2$. We use the record $A \approx B$ that means $c_1 A \leqslant B \leqslant c_2 A$ with the constants c_1, c_2, depending only on irrelevant parameters.

We put

$$U(x) = \int_x^b u(t)\,dt, \quad V_i(x) = \int_a^x v_i^{1-p_i'}(t)\,dt, \quad i = 1, 2.$$

It is well known that the best constant C for the Hardy inequality

$$\left(\int_a^b \left(\int_a^x f \right)^q u(x)dx \right)^{\frac{1}{q}} \leqslant C \left(\int_a^b f^p v \right)^{\frac{1}{p}}, f \in \mathfrak{M}^+, \tag{1}$$

is equivalent to the Muckenhoupt constant in the case $1 < p \leqslant q < \infty$

$$A_M := \sup_{a < x < b} U^{\frac{1}{q}}(x) V^{\frac{1}{p'}}(x) < \infty$$

---

or to the Mazya-Rozin constant for $0 < q < p < \infty, 1 < p < \infty, \frac{1}{r} := \frac{1}{q} - \frac{1}{p}$

$$B_{MR} := \left(\int_a^b U^{\frac{r}{p}}(x) V^{\frac{r}{p'}}(x) u(x) dx \right)^{\frac{1}{r}} < \infty$$

and it can be replaced by different (equivalent) constants. For instance, the constants A_M and B_{MR} can be replaced by the following [4]:

$$A_T := \sup_{a<x<b} \left(\int_a^x uV^q \right)^{\frac{1}{q}} V^{-\frac{1}{p}}(x) < \infty,$$

$$B_{PS} := \left(\int_a^b \left(\int_a^x uV^q \right)^{\frac{r}{p}} V^{q-\frac{r}{p}}(x) u(x) dx \right)^{\frac{1}{r}} < \infty.$$

More generally, it was discovered (see, e.g., [1], [5]) that these conditions are not unique and can be replaced by the scales of constants depending on a continuous parameter $s > 0$ of a form:

$$A_M(s) := \sup_{a<x<b} \left(\int_x^b uV^{q(\frac{1}{p'}-s)} \right)^{\frac{1}{q}} V^s(x) < \infty, \tag{2}$$

$$A_T(s) := \sup_{a<x<b} \left(\int_a^x uV^{q(\frac{1}{p'}+s)} \right)^{\frac{1}{q}} V^{-s}(x) < \infty, \tag{3}$$

$$B_{MR}(s) := \left(\int_a^b \left(\int_x^b uV^{q(\frac{1}{p'}-s)} \right)^{\frac{r}{q}} V^{rs-1}(x) dV(x) \right)^{\frac{1}{r}} < \infty, \tag{4}$$

$$B_{PS}(s) := \left(\int_a^b \left(\int_a^x uV^{q(\frac{1}{p'}+s)} \right)^{\frac{r}{q}} V^{-rs-1}(x) dV(x) \right)^{\frac{1}{r}} < \infty. \tag{5}$$

Moreover, it yields that (see, e.g., [1], [3], [5]) $C \approx A_M(s) \approx A_T(s)$ for $1 < p \leqslant q < \infty$ and $C \approx B_{MR}(s) \approx B_{PS}(s)$ for $0 < q < p < \infty, p > 1$ and $s > 0$. Concerning history and references about such alternative conditions, even infinite scales of equivalent conditions, we refer to the book [3], Section 7, and the references therein e.g. [1], [5] and the PhD theses [6] and [7].

Recently, M. Krepela [2] considered the following bilinear Hardy-type inequality

$$\left(\int_a^b \left(\int_a^x f \right)^q \left(\int_a^x g \right)^q u(x) dx \right)^{\frac{1}{q}} \leqslant C \left(\int_a^b f^{p_1} v_1 \right)^{\frac{1}{p_1}} \left(\int_a^b g^{p_2} v_2 \right)^{\frac{1}{p_2}}, f, g \in \mathfrak{M}^+ \tag{6}$$

and characterized it for various range of parameters p_1, p_2, q. The results can be summarized in the following modified theorem:

THEOREM A. Let $0 < q < \infty, 1 < p_1, p_2 < \infty$. Then the inequality (6) with the best constant C holds for every $f, g \in \mathfrak{M}^+$, iff the following holds:

(i) *If* $1 < \max(p_1, p_2) \leqslant q < \infty$, *then* $C \approx A_1$, *where*

$$A_1 := \sup_{a<x<b} U^{\frac{1}{q}}(x) V_1^{\frac{1}{p_1'}}(x) V_2^{\frac{1}{p_2'}}(x) < \infty.$$

(ii) *If* $1 < p_1 \leqslant q < p_2 < \infty$, $\frac{1}{r_2} := \frac{1}{q} - \frac{1}{p_2}$, *then* $C \approx A_2$, *where*

$$A_2 := \sup_{a<x<b} V_1^{\frac{1}{p_1'}}(x) \left(\int_x^b U^{\frac{r_2}{q}} V_2^{\frac{r_2}{q'}} dV_2 \right)^{\frac{1}{r_2}}.$$

(iii) *If* $1 < p_2 \leqslant q < p_1 < \infty$, $\frac{1}{r_1} := \frac{1}{q} - \frac{1}{p_1}$, *then* $C \approx A_3$, *where*

$$A_3 := \sup_{a<x<b} V_2^{\frac{1}{p_2'}}(x) \left(\int_x^b U^{\frac{r_1}{q}} V_1^{\frac{r_1}{q'}} dV_1 \right)^{\frac{1}{r_1}}.$$

(iv) *Let* $0 < q < \min(p_1, p_2) < \infty$, $\min(p_1, p_2) > 1$, $\frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i}$, $i = 1, 2$ *and* $\frac{1}{q} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$. *Then* $C \approx A_4 + A_5$, *where*

$$A_4 := \sup_{a<x<b} V_1^{\frac{1}{p_1'}}(x) \left(\int_x^b U^{\frac{r_2}{q}} V_2^{\frac{r_2}{q'}} dV_2 \right)^{\frac{1}{r_2}},$$

and

$$A_5 := \sup_{a<x<b} V_2^{\frac{1}{p_2'}}(x) \left(\int_x^b U^{\frac{r_1}{q}} V_1^{\frac{r_1}{q'}} dV_1 \right)^{\frac{1}{r_1}}.$$

(v) *Let* $0 < q < \min(p_1, p_2) < \infty$, $\min(p_1, p_2) > 1$, $\frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i}$, $i = 1, 2$, $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{\kappa} := \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$. *Then* $C \approx A_6 + A_7$, *where*

$$A_6 := \left(\int_a^b \left(\int_x^b U^{\frac{r_2}{q}}(y) V_2^{\frac{r_2}{q'}}(y) dV_2(y) \right)^{\frac{\kappa}{r_2}} V_1^{\frac{\kappa}{p_1'}}(x) dV_1(x) \right)^{\frac{1}{\kappa}},$$

and

$$A_7 := \left(\int_a^b \left(\int_x^b U^{\frac{r_1}{q}}(y) V_1^{\frac{r_1}{q'}}(y) dV_1(y) \right)^{\frac{\kappa}{r_1}} V_2^{\frac{\kappa}{p_2'}}(x) dV_2(x) \right)^{\frac{1}{\kappa}}.$$

Inspired by the works in [1] and [5] (see also [3]), in this paper we derive infinite many equivalent conditions to characterize the inequality (6) for each of the cases (i)-(v). More exactly, we prove that each of the Krepela constants $A_1 - A_7$ can be replaced by infinite many other equivalent constants, even scales of constants.

2. The case $1 < \max(p_1, p_2) \leqslant q < \infty$

We begin this section with the following:

THEOREM 1. *Let* $1 < \max(p_1, p_2) \leqslant q < \infty$. *Then* $C \approx A_1^{(1)}(s_1, s_2) \approx A_1^{(2)}(s_1, s_2)$, $\forall s_1, s_2 > 0$, *where*

$$A_1^{(1)}(s_1, s_2) := \sup_{a < x < b} V_1^{s_1}(x) V_2^{s_2}(x) \left(\int_x^b u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{1}{q}}$$

and

$$A_1^{(2)}(s_1, s_2) = \sup_{a < x < b} V_1^{-s_1}(x) V_2^{-s_2}(x) \left(\int_a^x u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{1}{q}}.$$

In this Theorem and also all other Theorems and proofs in this paper we let C denote the sharp constant in (6).

Proof. We have

$$C = \sup_g \sup_f \frac{\left(\int_a^b \left(\int_a^t f \right)^q \left(\int_a^t g \right)^q u(t) dt \right)^{\frac{1}{q}}}{\|f\|_{p_1, v_1} \|g\|_{p_2, v_2}}$$

$$\overset{(2)}{\approx} \sup_g \|g\|_{p_2, v_2}^{-1} \sup_{a < x < b} \left(\int_x^b \left(\int_a^y g \right)^q u(y) V_1^{q(\frac{1}{p_1'} - s_1)}(y) dy \right)^{\frac{1}{q}} V_1^{s_1}(x)$$

$$\overset{(2)}{\approx} \sup_{a < x < b} V_1^{s_1}(x) \left(\sup_{x < y < b} \left(\int_y^b u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{1}{q}} V_2^{s_2}(y) \right) = A_1^{(1)}(s_1, s_2).$$

Similarly, by using (3) twice, we find that $C \approx A_1^{(2)}(s_1, s_2)$ so the proof is complete. $\square$

REMARK 1. The Krepela constant A_1 appears just as a point $(\frac{1}{p_1'}, \frac{1}{p_2'})$ on our first scale of equivalent constants, i.e. $A_1^{(1)}(\frac{1}{p_1'}, \frac{1}{p_2'}) = A_1$.

Theorem 1 gives two scales of equivalent conditions, namely $A_1^{(1)}(s_1, s_2) < \infty$ and $A_1^{(2)}(s_1, s_2) < \infty$, characterizing the inequality (6). Below, we prove that also these scales of equivalent conditions are not unique and can in fact be replaced by several more equivalent conditions for the inequality (6) to hold. Unlike in Theorem 1, here we do not link any of the conditions with the inequality (6). Rather it is provided that each of these conditions is equivalent to the condition $A_1 < \infty$.

THEOREM 2. *Let* $1 < \max(p_1, p_2) \leqslant q < \infty$. *Then* $A_1 \approx A_1^{(3)}(s) \approx A_1^{(4)}(s)$ $\approx A_1^{(5)}(s) \approx A_1^{(6)}(s_1, s_2)$, $\forall s, s_1, s_2 > 0$, *where*

$$A_1 := \sup_{a < x < b} U^{1/q}(x) V_1^{1/p_1'}(x) V_2^{1/p_2'}(x),$$

$$A_1^{(3)}(s) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(t) V_2^{\frac{1/p_2'}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x),$$

$$A_1^{(4)}(s) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\frac{1/p_1'}{1/q-s}}(t) V_2^{\frac{1/p_2'}{1/q-s}}(t)\, dt \right)^{1/q-s} U^{s}(x), \quad 1/q > s,$$

$$A_1^{(5)}(s) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q-s}}(t) V_2^{\frac{1/p_2'}{1/q-s}}(t)\, dt \right)^{1/q-s} U^{s}(x), \quad 1/q < s$$

and

$$A_1^{(6)}(s_1,s_2) := \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'}(t) U^{\frac{1/2q}{1/p_1'+s_1}}(t)\, dt \right)^{1/p_1'+s_1} \left(\int_a^x v_2^{1-p_2'}(t) U^{\frac{1/2q}{1/p_2'+s_2}}(t)\, dt \right)^{1/p_2'+s_2}$$
$$\times V_1^{-s_1}(x) V_2^{-s_2}(x).$$

Proof. Note that the function U is decreasing while V_1 and V_2 are increasing. First we prove that $A_1 \approx A_1^{(3)}(s)$. We have

$$U^{1/q}(x) V_1^{1/p_1'}(x) V_2^{1/p_2'}(x) = U^{1/q+s}(x) V_1^{1/p_1'}(x) V_2^{1/p_2'}(x) U^{-s}(x)$$
$$= \left(\int_x^b u(t)\, dt \right)^{1/q+s} V_1^{1/p_1'}(x) V_2^{1/p_2'}(x) U^{-s}(x)$$
$$= \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(x) V_2^{\frac{1/p_2'}{1/q+s}}(x)\, dt \right)^{1/q+s} U^{-s}(x)$$
$$\leqslant \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(t) V_2^{\frac{1/p_2'}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x).$$

Hence,

$$\sup_{a<x<b} U^{1/q}(x) V_1^{1/p_1'}(x) V_2^{1/p_2'}(x) \leqslant \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(t) V_2^{\frac{1/p_2'}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x). \tag{7}$$

Moreover,

$$\left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(t) V_2^{\frac{1/p_2'}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x)$$
$$\leqslant \sup_{x<t<b} U^{1/q}(t) V_1^{1/p_1'}(t) V_2^{1/p_2'}(t) \left(\int_x^b u(t) U^{-\frac{1/q}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x)$$
$$= \sup_{x<t<b} U^{1/q}(t) V_1^{1/p_1'}(t) V_2^{1/p_2'}(t) \left(\left(\frac{1/q+s}{s} \right) U^{\frac{s}{1/q+s}}(x) \right)^{1/q+s} U^{-s}(x)$$

$$\leqslant \left(\frac{1/q+s}{s} \right)^{1/q+s} \sup_{a<t<b} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t). \tag{8}$$

The equivalence $A_1 \approx A_1^{(3)}(s)$ now follows by taking supremum in (8) and using (7).

Next, we prove that $A_1 \approx A_1^{(4)}(s)$. Fix $x \in (a,b)$ and define $y \in (x,b)$ such that

$$\int_x^y u(t)\,dt = \int_y^b u(t)\,dt. \tag{9}$$

Then

$$U^{1/q}(x) = \left(\int_x^b u(t)\,dt \right)^{1/q} = \left(\int_x^y u(t)\,dt + \int_y^b u(t)\,dt \right)^{1/q} = 2^{1/q} \left(\int_y^b u(t)\,dt \right)^{1/q}$$

$$= 2^{1/q} \left(\int_y^b u(t)\,dt \right)^{1/q-s} \left(\int_y^b u(t)\,dt \right)^{s} = 2^{1/q} \left(\int_x^y u(t)\,dt \right)^{1/q-s} U^s(y).$$

We have

$$U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) = 2^{1/q} \left(\int_x^y u(t)\,dt \right)^{1/q-s} U^s(y)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x)$$

$$\leqslant 2^{1/q} \left(\int_x^y u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt \right)^{1/q-s} U^s(y)$$

$$\leqslant 2^{1/q} \left(\int_a^y u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt \right)^{1/q-s} U^s(y).$$

Hence,

$$U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) \leqslant 2^{1/q} \sup_{a<y<b} \left(\int_a^y u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt \right)^{1/q-s} U^s(y), \tag{10}$$

and the estimate $A_1 < cA_1^{(4)}(s)$ follows.

Moreover, since $1/q - s > 0$, we have

$$\left(\int_a^x u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt \right)^{1/q-s} U^s(x)$$

$$\leqslant \sup_{a<t<x} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t) \left(\int_a^x u(t)U^{-\frac{1/q}{1/q-s}}(t)\,dt \right)^{1/q-s} U^s(x)$$

$$\leqslant \sup_{a<t<b} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t) \left(-\int_a^x U^{-\frac{1/q}{1/q-s}}(t)\,dU(t) \right)^{1/q-s} U^s(x)$$

$$= \sup_{a<t<b} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t)\left(\frac{1/q-s}{s}\right)^{1/q-s}\left(U^{-\frac{s}{1/q-s}}(x)-U^{-\frac{s}{1/q-s}}(a)\right)^{1/q-s}U^s(x)$$

$$\leqslant \left(\frac{1/q-s}{s}\right)^{1/q-s} \sup_{a<t<b} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t). \tag{11}$$

By now just taking supremum in (11) and using (10), we obtain the equivalence $A_1 \approx A_1^{(4)}(s)$.

Next, we prove that $A_1 \approx A_1^{(5)}(s)$. Since $1/q - s < 0$, we get

$$U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) = U^{1/q-s}(x)U^s(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x)$$

$$= \left(\int_x^b u(t)\,dt\right)^{1/q-s} U^s(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x)$$

$$\leqslant \left(\int_x^b u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(x).$$

Therefore

$$\sup_{a<x<b} U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) \leqslant \tag{12}$$

$$\sup_{a<x<b} \left(\int_x^b u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(x). \tag{13}$$

Now fix $x \in (a,b)$ and define $y \in (x,b)$ such that

$$\int_x^y u(t)\,dt = \int_y^b u(t)\,dt.$$

Then $U(x) = 2U(y)$ and using this fact, we obtain

$$\left(\int_x^b u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(x) \leqslant \left(\int_x^y u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(x)$$

$$\leqslant \left(\int_x^y u(t)\,dt\right)^{1/q-s} V_1^{1/p_1'}(y)V_2^{1/p_2'}(y)U^s(x) = \left(\int_y^b u(t)\,dt\right)^{1/q-s} V_1^{1/p_1'}(y)V_2^{1/p_2'}(y)U^s(x)$$

$$= U^{1/q-s}(y)V_1^{1/p_1'}(y)V_2^{1/p_2'}(y)2^s U^s(y) = 2^s U^{1/q}(y)V_1^{1/p_1'}(y)V_2^{1/p_2'}(y)$$

$$\leqslant 2^s \sup_{a<y<b} U^{1/q}(y)V_1^{1/p_1'}(y)V_2^{1/p_2'}(y). \tag{14}$$

We now take supremum in (14) and use (13) and the equivalence $A_1 \approx A_1^{(5)}(s)$ follows.

Finally, we prove that $A_1 \approx A_1^{(6)}(s_1,s_2)$. It yields that

$$U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x)$$

$$=U^{1/q}(x)V_1^{1/p_1'+s_1}(x)V_2^{1/p_2'+s_2}(x)V_1^{-s_1}(x)V_2^{-s_2}(x)$$

$$=U^{1/q}(x)\left(\int_a^x v_1^{1-p_1'}(t)\,dt\right)^{1/p_1'+s_1}\left(\int_a^x v_2^{1-p_2'}(t)\,dt\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x)$$

$$\leqslant\left(\int_a^x v_1^{1-p_1'}(t)U^{\frac{1/2q}{1/p_1'+s_1}}(t)\,dt\right)^{1/p_1'+s_1}\left(\int_a^x v_2^{1-p_2'}(t)U^{\frac{1/2q}{1/p_2'+s_2}}(t)\,dt\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x).$$
(15)

On the other hand

$$\left(\int_a^x v_1^{1-p_1'}(t)U^{\frac{1/2q}{1/p_1'+s_1}}(t)\,dt\right)^{1/p_1'+s_1}\left(\int_a^x v_2^{1-p_2'}(t)U^{\frac{1/2q}{1/p_2'+s_2}}(t)\,dt\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x)$$

$$=\left(\int_a^x U^{\frac{1/2q}{1/p_1'+s_1}}(t)\,dV_1(t)\right)^{1/p_1'+s_1}\left(\int_a^x U^{\frac{1/2q}{1/p_2'+s_2}}(t)\,dV_2(t)\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x)$$

$$\leqslant\left(U^{\frac{1/2q}{1/p_1'+s_1}}(x)V_1(x)\right)^{1/p_1'+s_1}\left(U^{\frac{1/2q}{1/p_2'+s_2}}(x)V_2(x)\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x)$$

$$=U^{1/2q}(x)V_1^{1/p_1'+s_1}(x)U^{1/2q}(x)V_2^{1/p_2'+s_2}(x)V_1^{-s_1}(x)V_2^{-s_2}(x)$$

$$=U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x).$$
(16)

The equivalence $A_1 \approx A_1^{(6)}(s_1,s_2)$ now follows by just combining (15) and (16). The proof is complete. $\square$

3. The cases $1 < p_1 \leqslant q < p_2 < \infty$ and $1 < p_2 \leqslant q < p_1 < \infty$

We begin this section with the following:

THEOREM 3. *Let* $1 < p_1 \leqslant q < p_2 < \infty$ *and* $\frac{1}{r_2} := \frac{1}{q} - \frac{1}{p_2}$. *Then* $C \approx A_2^{(1)}(s_1,s_2) \approx A_2^{(2)}(s_1,s_2) \approx A_2^{(3)}(s_1,s_2) \approx A_2^{(4)}(s_1,s_2)$, $s_1,s_2 > 0$, *where*

$$A_2^{(1)}(s_1,s_2) = \sup_{a<x<b} V_1^{s_1}(x)\left(\int_x^b\left(\int_y^b uV_1^{q(\frac{1}{p_1}-s_1)}V_2^{q(\frac{1}{p_2}-s_2)}\right)^{\frac{r_2}{q}}V_2^{r_2s_2-1}(y)\,dV_2(y)\right)^{\frac{1}{r_2}},$$

$$A_2^{(2)}(s_1,s_2) = \sup_{a<x<b} V_1^{-s_1}(x)\left(\int_a^x\left(\int_a^y uV_1^{q(\frac{1}{p_1}+s_1)}V_2^{q(\frac{1}{p_2}+s_2)}\right)^{\frac{r_2}{q}}V_2^{-r_2s_2-1}(y)\,dV_2(y)\right)^{\frac{1}{r_2}},$$

$$A_2^{(3)}(s_1,s_2) = \sup_{a<x<b} V_1^{s_1}(x)\left(\int_x^b\left(\int_x^y uV_1^{q(\frac{1}{p_1}-s_1)}V_2^{q(\frac{1}{p_2}+s_2)}\right)^{\frac{r_2}{q}}V_2^{-r_2s_2-1}(y)\,dV_2(y)\right)^{\frac{1}{r_2}}$$

and

$$A_2^{(4)}(s_1,s_2) = \sup_{a<x<b} V_1^{-s_1}(x) \left(\int_a^x \left(\int_y^x uV_1^{q(\frac{1}{p_1}+s_1)} V_2^{q(\frac{1}{p_2}-s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2-1}(y) dV_2(y) \right)^{\frac{1}{r_2}}.$$

Proof. We have

$$C = \sup_g \sup_f \frac{\left(\int_a^b \left(\int_a^t f \right)^q \left(\int_a^t g \right)^q u(t) dt \right)^{\frac{1}{q}}}{\|f\|_{p_1,v_1} \|g\|_{p_2,v_2}}$$

$$\overset{(2)}{\approx} \sup_g \|g\|_{p_2,v_2}^{-1} \sup_{a<x<b} \left(\int_x^b \left(\int_a^y g \right)^q u(y) V_1^{q(\frac{1}{p_1}-s_1)}(y) dy \right)^{\frac{1}{q}} V_1^{s_1}(x)$$

$$\overset{(4)}{\approx} \sup_{a<x<b} V_1^{s_1}(x) \left(\int_x^b \left(\int_y^b uV_1^{q(\frac{1}{p_1}-s_1)} V_2^{q(\frac{1}{p_2}-s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2-1}(y) dV_2(y) \right)^{\frac{1}{r_2}}.$$

The first relation follows by the formula (2), the second relation is due to formula (4). By now using (3) and (5) we have $C \approx A_2^{(2)}(s_1,s_2)$. Moreover, using (2) and (5) we find that $C \approx A_2^{(3)}(s_1,s_2)$. Similarly, by instead using (3) and (4) we conclude that $C \approx A_2^{(4)}(s_1,s_2)$. The proof is complete. $\square$

The case $1 < p_2 \leqslant q < p_1 < \infty$ can be handled similarly to obtain the following result:

THEOREM 4. *Let* $1 < p_2 \leqslant q < p_1 < \infty$ *and* $\frac{1}{r_1} := \frac{1}{q} - \frac{1}{p_1}$. *Then* $C \approx A_3^{(1)}(s_1,s_2) \approx A_3^{(2)}(s_1,s_2) \approx A_3^{(3)}(s_1,s_2) \approx A_3^{(4)}(s_1,s_2), \forall s_1,s_2 > 0$, *where*

$$A_3^{(1)}(s_1,s_2) = \sup_{a<x<b} V_2^{s_2}(x) \left(\int_x^b \left(\int_y^b uV_1^{q(\frac{1}{p_1}-s_1)} V_2^{q(\frac{1}{p_2}-s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1-1}(y) dV_1(y) \right)^{\frac{1}{r_1}},$$

$$A_3^{(2)}(s_1,s_2) = \sup_{a<x<b} V_2^{-s_2}(x) \left(\int_a^x \left(\int_a^y uV_1^{q(\frac{1}{p_1}+s_1)} V_2^{q(\frac{1}{p_2}+s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1-1}(y) dV_1(y) \right)^{\frac{1}{r_1}},$$

$$A_3^{(3)}(s_1,s_2) = \sup_{a<x<b} V_2^{s_2}(x) \left(\int_x^b \left(\int_x^y uV_1^{q(\frac{1}{p_1}+s_1)} V_2^{q(\frac{1}{p_2}-s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1-1}(y) dV_1(y) \right)^{\frac{1}{r_1}}$$

and

$$A_3^{(4)}(s_1,s_2) = \sup_{a<x<b} V_2^{-s_2}(x) \left(\int_a^x \left(\int_y^x uV_1^{q(\frac{1}{p_1}-s_1)} V_2^{q(\frac{1}{p_2}+s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1-1}(y) dV_1(y) \right)^{\frac{1}{r_1}}.$$

Remark 2. The Krepela constants A_2 and A_3 show up just as $A_2 = A_2^{(1)}(\frac{1}{p_1'}, \frac{1}{p_2'})$ and $A_3 = A_3^{(1)}(\frac{1}{p_1'}, \frac{1}{p_2'})$.

4. The case $0 < q < \min(p_1, p_2) < \infty$

We begin this section with the following:

THEOREM 5. *Let* $0 < q < \min(p_1, p_2) < \infty$, $\min(p_1, p_2) > 1$, $\frac{1}{q} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i}$, $i = 1, 2$. *Then* $C \approx A_4^{(1)}(s_1, s_2) + A_4^{(2)}(s_1, s_2) \approx A_4^{(3)}(s_1, s_2) + A_4^{(4)}(s_1, s_2) \approx A_4^{(5)}(s_1, s_2) + A_4^{(6)}(s_1, s_2) \approx A_4^{(7)}(s_1, s_2) + A_4^{(8)}(s_1, s_2)$, *where*

$$A_4^{(1)}(s_1, s_2) := \sup_{a < x < b} V_1^{s_1}(x) \left(\int_x^b \left(\int_y^b u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{1}{r_2}},$$

$$A_4^{(2)}(s_1, s_2) := \sup_{a < x < b} V_2^{s_2}(x) \left(\int_x^b \left(\int_y^b u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{1}{r_1}},$$

$$A_4^{(3)}(s_1, s_2) := \sup_{a < x < b} V_1^{-s_1}(x) \left(\int_a^x \left(\int_a^y u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{r_2}{q}} V_2^{-r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{1}{r_2}},$$

$$A_4^{(4)}(s_1, s_2) := \sup_{a < x < b} V_2^{-s_2}(x) \left(\int_a^x \left(\int_a^y u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{1}{r_1}},$$

$$A_4^{(5)}(s_1, s_2) := \sup_{a < x < b} V_1^{s_1}(x) \left(\int_x^b \left(\int_x^y u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{r_2}{q}} V_2^{-r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{1}{r_2}},$$

$$A_4^{(6)}(s_1, s_2) := \sup_{a < x < b} V_2^{-s_2}(x) \left(\int_a^x \left(\int_y^x u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{1}{r_1}},$$

$$A_4^{(7)}(s_1, s_2) := \sup_{a < x < b} V_1^{-s_1}(x) \left(\int_a^x \left(\int_y^x u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{1}{r_2}}$$

and

$$A_4^{(8)}(s_1, s_2) := \sup_{a < x < b} V_2^{s_2}(x) \left(\int_x^b \left(\int_x^y u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{1}{r_1}}.$$

Proof. We have

$$C = \sup_g \sup_f \frac{\left(\int_a^b \left(\int_a^t f \right)^q \left(\int_a^t g \right)^q u(t) dt \right)^{\frac{1}{q}}}{\|f\|_{p_1,v_1} \|g\|_{p_2,v_2}}$$

$$\overset{(4)}{\approx} \sup_g \|g\|_{p_2,v_2}^{-1} \left[\left(\int_a^b \left(\int_x^b \left(\int_a^y g \right)^q u(y) V_1^{q\left(\frac{1}{p_1'} - s_1 \right)}(y) dy \right)^{\frac{r_1}{q}} V_1^{r_1 s_1 - 1}(x) dV_1(x) \right)^{\frac{q}{r_1}} \right]^{\frac{1}{q}}$$

$$= \sup_g \|g\|_{p_2,v_2}^{-1} \left[\sup_h \frac{\int_a^b \left(\int_a^x g \right)^q \left(\int_a^x h \right) u(x) V_1^{q\left(\frac{1}{p_1'} - s_1 \right)}(x) dx}{\left(\int_a^b h^{\frac{p_1}{q}} V_1^{(r_1 s_1 - 1)\left(-\frac{p_1}{r_1} \right)} v_1^{\frac{p_1'}{r_1}} \right)^{\frac{q}{p_1}}} \right]^{\frac{1}{q}}$$

$$\overset{(4)}{\approx} \sup_h \frac{\left(\int_a^b \left(\int_x^b \left(\int_a^y h \right) u(y) V_1^{q\left(\frac{1}{p_1'} - s_1 \right)}(y) V_2^{q\left(\frac{1}{p_2'} - s_2 \right)}(y) dy \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(x) dV_2(x) \right)^{\frac{1}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}} V_1^{(r_1 s_1 - 1)\left(-\frac{p_1}{r_1} \right)} v_1^{\frac{p_1'}{r_1}} \right)^{\frac{1}{p_1}}}$$

$$\approx \sup_h \frac{\left(\int_a^b \left(\int_a^x h \right)^{\frac{r_2}{q}} \left(\int_x^b u V_1^{q\left(\frac{1}{p_1'} - s_1 \right)} V_2^{q\left(\frac{1}{p_2'} - s_2 \right)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(x) dV_2(x) \right)^{\frac{1}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}} V_1^{(r_1 s_1 - 1)\left(-\frac{p_1}{r_1} \right)} v_1^{\frac{p_1'}{r_1}} \right)^{\frac{1}{p_1}}}$$

$$+ \sup_h \frac{\left(\int_a^b \left(\int_x^b h(y) \left(\int_y^b u V_1^{q\left(\frac{1}{p_1'} - s_1 \right)} V_2^{q\left(\frac{1}{p_2'} - s_2 \right)} \right) dy \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(x) dV_2(x) \right)^{\frac{1}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}} V_1^{(r_1 s_1 - 1)\left(-\frac{p_1}{r_1} \right)} v_1^{\frac{p_1'}{r_1}} \right)^{\frac{1}{p_1}}}$$

$$=: H_1 + H_2. \tag{17}$$

The first relation follows by the formula (4), the second relation is due to duality. For the third relation we use again formula (4), the fourth relation holds by the Fubini theorem. Finally, we have Hardy's constants with $\tilde{p} = \frac{p_1}{q}, \tilde{q} = \frac{r_2}{q}$, and for the case $\tilde{p} \leqslant \tilde{q}$, we obtain that $H_1 \approx A_4^{(1)}(s_1, s_2)$ and $H_2 \approx A_4^{(2)}(s_1, s_2)$. The proof follows by just using these facts. $\square$

REMARK 3. We note that $A_4^{(1)}(\frac{1}{p_1'},\frac{1}{p_2'}) = A_4$ and $A_4^{(2)}(\frac{1}{p_1'},\frac{1}{p_2'}) = A_5$, where A_4 and A_5 are the Krepela constants defined in Theorem A (iv).

Our final results reads:

THEOREM 6. *Let* $0 < q < \min(p_1,p_2) < \infty$, $\min(p_1,p_2) > 1$, $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i}$, $i = 1,2$. *Let* $\frac{1}{\kappa} := \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$. *Then* $C \approx A_5^{(1)}(s_1,s_2) + A_5^{(2)}(s_1,s_2) \approx A_5^{(3)}(s_1,s_2) + A_5^{(4)}(s_1,s_2) \approx A_5^{(5)}(s_1,s_2) + A_5^{(6)}(s_1,s_2) \approx A_5^{(7)}(s_1,s_2) + A_5^{(8)}(s_1,s_2)$, *where*

$$(A_5^{(1)}(s_1,s_2))^\kappa := \int_a^b \left(\int_t^b \left(\int_y^b uV_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}-s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2s_2-1}(y)dV_2(y) \right)^{\frac{\kappa}{r_2}}$$
$$\times V_1^{\kappa s_1-1}(t)dV_1(t),$$

$$(A_5^{(2)}(s_1,s_2))^\kappa := \int_a^b \left(\int_t^b \left(\int_y^b uV_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}-s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1s_1-1}(y)dV_1(y) \right)^{\frac{\kappa}{r_1}}$$
$$\times V_2(t)^{\kappa s_2-1}dV_2(t),$$

$$(A_5^{(3)}(s_1,s_2))^\kappa := \int_a^b \left(\int_a^t \left(\int_a^y uV_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}+s_2)} \right)^{\frac{r_2}{q}} V_2^{-r_2s_2-1}(y)dV_2(y) \right)^{\frac{\kappa}{r_2}}$$
$$\times d\left(-\left(\int_t^b V_1^{-r_1s_1-1}dV_1 \right)^{\frac{\kappa}{r_1}} \right),$$

$$(A_5^{(4)}(s_1,s_2))^\kappa := \int_a^b \left(\int_a^t \left(\int_a^y uV_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}+s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1s_1-1}(y)dV_1(y) \right)^{\frac{\kappa}{r_1}}$$
$$\times d\left(-\left(\int_t^b V_2^{-r_2s_2-1}dV_2 \right)^{\frac{\kappa}{r_2}} \right),$$

$$(A_5^{(5)}(s_1,s_2))^\kappa := \int_a^b \left(\int_t^b \left(\int_t^y uV_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}+s_2)} \right)^{\frac{r_2}{q}} V_2^{-r_2s_2-1}(y)dV_2(y) \right)^{\frac{\kappa}{r_2}}$$
$$\times d\left(\left(\int_a^t V_1^{r_1s_1-1}dV_1 \right)^{\frac{\kappa}{r_1}} \right),$$

$$(A_5^{(6)}(s_1,s_2))^\kappa := \int_a^b \left(\int_a^t \left(\int_y^t u V_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}+s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1 -1}(y) dV_1(y) \right)^{\frac{\kappa}{r_1}}$$
$$\times d\left(-\left(\int_t^b V_2^{-r_2 s_2 -1} dV_2 \right)^{\frac{\kappa}{r_2}} \right),$$

$$(A_5^{(7)}(s_1,s_2))^\kappa := \int_a^b \left(\int_a^t \left(\int_y^x u V_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}-s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 -1}(y) dV_2(y) \right)^{\frac{\kappa}{r_2}}$$
$$\times d\left(-\left(\int_t^b V_1^{-r_1 s_1 -1} dV_1 \right)^{\frac{\kappa}{r_1}} \right)$$

and

$$(A_5^{(8)}(s_1,s_2))^\kappa := \int_a^b \left(\int_t^b \left(\int_t^y u V_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}-s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1 -1}(y) dV_1(y) \right)^{\frac{\kappa}{r_1}}$$
$$\times d\left(\left(\int_a^t V_2^{r_2 s_2 -1} dV_2 \right)^{\frac{\kappa}{r_2}} \right).$$

Proof. The proof is absolutely analogous to that of Theorem 5 so we do not give the details. We only note as in this proof we have $C \approx H_1 + H_2$, where H_1 and H_2 are defined in (17). Next for the case $\tilde{q} < \tilde{p}$ the following holds $H_1 \approx A_5^{(1)}(s_1,s_2)$ and $H_2 \approx A_5^{(2)}(s_1,s_2)$. □

REMARK 4. In this case the corresponding Krepela constants A_6 and A_7 are just recovered as $A_6 = A_5^{(1)}(\frac{1}{p_1'},\frac{1}{p_2'})$ and $A_7 = A_5^{(2)}(\frac{1}{p_1'},\frac{1}{p_2'})$.

REMARK 5. There are not many results concerning bilinear Hardy type inequalities then that by M. Krepela presented in Theorem A. However, some other results of this type can be found in the recent PhD thesis [8].

REFERENCES

[1] A. GOGATISHVILI, A. KUFNER, L.-E. PERSSON AND A. WEDESTIG, *An equivalence theorem for integral conditions related to Hardy's inequality*, Real Analysis Exchange, **29**, 2 (2003), 1–14.

[2] M. KREPELA, *Iterating bilinear Hardy inequalities*, Proc. Edinb. Math. Soc. **60**, 4 (2017), 955–971.

[3] A. KUFNER, L.-E. PERSSON AND N. SAMKO, *Weighted Inequalities of Hardy Type*, Second Edition, World Scientific New Jersey, 2017.

[4] L.-E. PERSSON AND V. D. STEPANOV, *Weighted integral inequalities with the geometric mean operator*, J. Inequal. Appl. **7**, (2002), 727–746.

[5] L.-E. PERSSON, V. D. STEPANOV AND P. WALL, *Some scales of equivalent weight characterizations of Hardy's inequality: the case $q < p$*, Math. Inequal. Appl. **10**, 2 (2007), 267–279.

[6] A. WEDESTIG, *Weighted Inequalities of Hardy Type and their Limiting Inequalities*, Doctoral Thesis, Department of Mathematics, Luleå, University of Technology, Sweden, 2003.

[7] E. P. USHAKOVA, *Norm Inequalities of Hardy and Pólya-Knopp Types*, Doctoral Thesis, Department of Mathematics, Luleå, University of Technology, Sweden, 2006.

[8] G. E. SHAMBILOVA, *Some New Hardy-Type Inequalities on the Cone of Monotone Functions*, Doctoral Thesis, Department of Mathematics, Luleå, University of Technology, Sweden, 2018.

Saikat Kanjilal
Department of Mathematics
South Asian University
New Delhi, India
e-mail: saikatkanjilal@students.sau.ac.in

Lars-Erik Persson
UIT
The Arctic University of Norway
Narvik, Norway
Department of Mathematics and Computer Science
Karlstad University
65188, Karlstad, Sweden
e-mail: larserik6pers@gmail.com

Guldarya E. Shambilova
Moscow Institute of Physics and Technology
141701 Dolgoprudnyi, Moscow Region, Russia
e-mail: shambilova@mail.ru

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com