A goal programming approach for a fuzzy single-source capacitated facility location problem

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Abstract - **In this paper, we present a new mixed-integer linear goal programming (MILGP) model for a fuzzy single-source capacitated facility location problem (F-SSCFLP), in which demands are considered as triangular fuzzy values and capacity of facilities are fuzzy constraints. At first, the basic fuzzy model for the given problem is presented. Then, utilizing goal programming (GP) approach and fuzzy principals the proposed MILGP model is introduced. Accordingly, considering a certain acceptable deviation from capacities and demands, the cost of selecting facilities and serving the demand points is minimized. Finally, to investigate the effectiveness of the presented model some large-sized problems generated randomly and all of them are solved globally in a reasonable time.**

Keywords - **Facility location, Mixed-integer linear goal programming, Linearization, Triangular fuzzy number.**

I. INTRODUCTION

A single-source capacitated facility location problem (SSCFLP) deals with serving a set of demand points by a set of capacitated facilities, in which an open facility can serve one or more demand points while each demand point is served by exactly one facility under various optimization criteria [1]. This problem has extended applications in many areas, such as telecommunication, transportation and distribution systems [1-3].

To tackle with the uncertainty of the real-world situation, stochastic probabilistic theory and fuzzy theory are two main concepts. However, one of the advantages of the fuzzy theory over the stochastic probabilistic theory is its less dependency on historical data. Therefore, the fuzzy set theory, which has widely been accepted in modeling some of the vague phenomena in nature is considered in this paper.

Sakawa et al. [2] dealt with a case study of transportation planning as a facility location problem, in which both the demand and facility capacity are uncertain parameters. They considered a fuzzy goal and some constraints for the capacity of factories and demand points in the regions. Chan and Swarnkar [4] used the fuzzy goal programming model to deal with fuzzy-based multi-objective goals in a flexible manufacturing system.

In the fuzzy SSCFLP (F-SSCFLP) focused on the present work, the number of demands are triangular fuzzy values and the capacity of facilities are fuzzy constraints.

The motivation of the present work came from the study on a local delivery service of an after-sales service company which assigns its servicemen (i.e., facilities) to deliver parts and install or repair the products bought by customers in housing estates (i.e., demand points). Customers demand are fuzzy variables which must be served by just one serviceman (i.e., single-source facility). Also, the facilities should have sufficient serving capacity to cope with its demand assigned. On the other hand, servicemen have different abilities to install or repair and different capability to transport parts. Hence, it is considered as the fuzzy constraint of the capacity.

The rest of the paper is organized as follows. In Section II, after introducing notations, a fuzzy binary formulation is presented. In Section III, some principles of the fuzzy theory are reviewed. In Section IV, the proposed MILGP model for the F-SSCFLP is introduced. Section V contains some numerical examples. Finally, the paper is concluded in Section VI.

II. FUZZY BINARY FORMULATION

A. Notations
The notations used in problem formulation including indices, parameters and decision variables are described as follows:

Indices
i index of facilities ($i = 1,2,\ldots,m$)
j index of demand nodes ($j = 1,2,\ldots,n$)

Parameters
f_i fixed cost of facility i
v_{ij} variable cost of facility i while serves demand node j
d_j demand quantity of node j
$\tilde{d}_j$ fuzzy demand quantity of node j
Q_i capacity of facility i

Decision variables
Y_i 1, if facility i is used to serving any demand node; 0, otherwise
X_{ij} 1, if facility i is used to serving demand node j; 0, otherwise

B. Proposed fuzzy binary model

In order to select a subset of facilities from the set $\{i = 1,2,\ldots,m\}$ and find the best assignment of demand nodes to them, the following formulation is written, which is a fuzzy form of the binary model presented by Holmberg

et al. [3] for the SSCFLP in such a way that the total cost is minimized.

$$\min\ Z_1 = \sum_{i=1}^{m} f_i Y_i + \sum_{i=1}^{m}\sum_{j=1}^{n} v_{ij} X_{ij} \tag{1}$$

s.t.

$$\sum_{i=1}^{m} X_{ij} = 1 \qquad\qquad \forall j \tag{2}$$
$$X_{ij} \leq Y_i \qquad\qquad \forall i,j \tag{3}$$
$$\sum_{j=1}^{n} \tilde{d}_j X_{ij} \stackrel{\leq}{\sim} Q_i Y_i \qquad \forall i \tag{4}$$
$$Y_i, X_{ij} \in \{0,1\} \qquad\quad \forall i,j \tag{5}$$

where $\tilde{d}_j$ indicates fuzzy demand of node j and $\stackrel{\leq}{\sim}$ indicates a fuzzy constraint. Equation (1) is the objective function that minimizes the fixed costs of selecting facilities and costs of serving demand points by their assigned facility. Equation (2) guarantees each demand point is served by exactly one facility. Constraint (3) states that no demand point j can be served by the facility i while this facility is not selected (i.e., X_{ij} cannot take the value 1 while $y_i = 0$). The fuzzy constraint (4) shows the fuzzy capability of a facility to service its assigned fuzzy demands. Finally, statement (5) indicates the binary variables.

III. SOME PRINCIPLES OF FUZZY THEORY

In this section, we review some fundamentals of the fuzzy theory, which are used throughout this paper [5-9].

Definition 1: Consider $\tilde{A}$ as a fuzzy set in a universe of discourse X. $\tilde{A}$ is described by a membership function $\mu_A(x)$ called the membership grade of x in $\tilde{A}$. Each element x in X is corresponded to a real number in the range [0 1]. Hence, the fuzzy set $\tilde{A}$ is clearly described by:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\} \tag{6}$$

Definition 2: A triangular fuzzy number $\tilde{d}_j$ can be defined as $d_j = (d_j^1, d_j^2, d_j^3)$, where $d_j^k (k = 1,3)$ are the lowest and highest possible values of d_j as shown in Fig. 1. Also, d_j^2 indicates the value of d_j with the highest membership degree.

Definition 3: $\mu_1(d_j)$ is the membership function of the fuzzy number $\tilde{d}_j$ that achieved by:

$$\mu_1(d_j) = \mu_1(d_j^1) + S_{j,1}(d_j - d_j^1) + \frac{S_{j,2} - S_{j,1}}{2}(|d_j - d_j^2| + d_j - d_j^2) \tag{7}$$

where $|a|$ indicates the absolute value of a and $S_{j,k}(k = 1,2)$ are the slope of left and right line piece of the triangle, respectively. The slopes are obtained by:

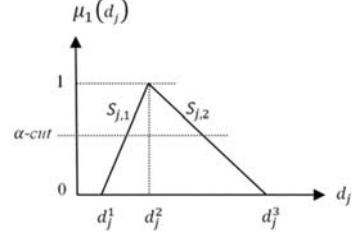$$S_{j,k} = \frac{\mu(d_j^{k+1}) - \mu(d_j^k)}{d_j^{k+1} - d_j^k} \qquad k = 1,2 \tag{8}$$

Fig. 1. Triangular membership function

Definition 4: Let $(Ax)_i \stackrel{\leq}{\sim} b_i$ as a fuzzy constraint. The linear membership function of this fuzzy constraint is as follows:

$$\mu_i(x) = \begin{cases} 1 & \text{if } Ax \leq b_i \\ 1 + \dfrac{b_i - (Ax)_i}{\Delta b_i} & \text{if } b_i < Ax \leq b_i + \Delta b_i \\ 0 & \text{if } Ax > b_i + \Delta b_i \end{cases} \tag{9}$$

where Δb_i is the tolerance related to constraint i as depicted in Fig. 2. Exceeding from the constraint leads to a reduction in membership value. Then, the maximum values of $\mu_i(x)$ with the lowest exceed from b_i are achieved by solving the following model [6]:

$$\max \sum_{i=1}^{m} \mu_i(x) \tag{10}$$

s.t.

$$(Ax)_i \leq b_i + \Delta b_i(1 - \mu_i(x)) \qquad \forall i \tag{11}$$
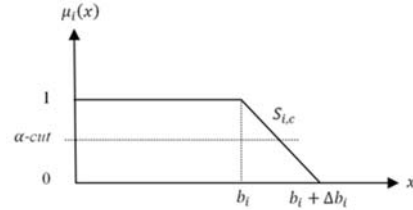$$\mu_i(x) \in [0\ 1] \qquad\qquad\qquad \forall i \tag{12}$$

Fig. 2. Membership function of the fuzzy constraints

Definition 5: The α-cut of the fuzzy set $\tilde{A}$ is all members of $\tilde{A}$ with the membership grade of equal or more than $\alpha \in$ [0 1] that is characterized by:

$$\tilde{A}_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\} \tag{13}$$

IV. CRISP MIXED-INTEGER LINEAR GOAL PROGRAMMING

In this section, we attempt to achieve a crisp mixed-integer linear goal programming (C-MILGP) for the F-SSCFLP. So, by using definitions 3 and 4, the fuzzy demands and constraints of the fuzzy binary model are transformed into their crisp values. Therefore, the crisp multi-objective mixed-integer programming model is presented below.

$$\min\ Z_1 \qquad\qquad (1)$$
$$\max\ Z_2 = \sum_{j=1}^{n}\mu_1(d_j) \qquad\qquad (14)$$
$$\max\ Z_3 = \sum_{i=1}^{m}\mu_2(q_i) \qquad\qquad (15)$$
s.t.
$$\sum_{j=1}^{n} d_j X_{ij} \leq$$
$$\left(Q_i + \Delta Q_i\big(1 - \mu_2(q_i)\big)\right) Y_i \qquad \forall i \quad (16)$$
$$\mu_1(d_j), \mu_2(q_i) \in [\alpha\ 1] \qquad\qquad \forall i,j \quad (17)$$
$$\text{Constraints (2), (3) and (5)} \qquad\qquad (18)$$

where Z_1 was defined in (1). Also, $\mu_1(d_j)$, $\mu_2(q_i)$ and ΔQ_i are membership functions of the fuzzy demands $\tilde{d}_j$, membership function of the i-th fuzzy constraint, and the tolerance related to the capacity of facility i, respectively. So, the membership level is maximized through (14) and (15). Variable q_i is the used capacity of the facility i. Equation (16) is the crisp form of (4) using definitions 2 and 4. It is worth mentioning, d_j is not a constant parameter, therefore the term $d_j X_{ij}$ in (16) is a non-linear term. Furthermore, the right-hand side of (16) is a non-linear term. Hence, the appropriate linear form of these non-linear terms will present during this section. Also, (17) indicates the range of the membership functions, where α is the lowest acceptable level for membership values.

Proposition 1: Consider the following multi-objective problem:

P1: min $\sum_{j=1}^{n} c_j x_j$
　　　max $\sum_{j=1}^{n} \mu(c_j)$
s.t.
　　$X \in F$
　　$x_j \in \{0,1\}$
　　$c_j \geq 0$

where F is a feasible set and membership functions are considered as objectives. An optimal solution for the model P1 is also optimal for the goal programming (GP) model P2 as follows:

P2: min $\sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n}\left(w_j^L \delta_j^L + w_j^R \delta_j^R\right)$
s.t.
　　$\mu(c_j) + \delta_j^L + \delta_j^R = 1$
　　$X \in F$
　　$x_j \in \{0,1\}$
　　$c_j, \delta_j^L, \delta_j^R \geq 0$

where $\mu(c_j)$ was defined in (7). δ_j^L and δ_j^R are deviations of $\mu(c_j)$ from its highest value (i.e., 1) related to left and right line segments of the membership function, respectively. Indeed, maximization of the membership functions in the model P1 is replaced by minimization of the deviations from their highest value in the model P2. According to the GP approach, w_j^L and w_j^R are the weights of the deviations δ_j^L and δ_j^R, in a row. These weights make

the appropriate change in $\mu(c_j)$ while c_j changes. The weights are calculated by (19).

$$w_j^L = |1/S_{j,1}|, w_j^R = |1/S_{j,2}| \qquad \forall j \qquad (19)$$

Proof: To proof this proposition, we utilize the same concept used by Yu and Li [5] for a similar proposition with a maximization form of the main objective function. There is almost no single solution to optimize all the goals of the objective and membership functions simultaneously. The range of the membership functions is in the interval [0 1] while all variables and coefficients in the objective function are non-negative. Furthermore, changes in $c_j x_j$ leads to corresponding changes in left and right line segments of $\mu(c_j)$ accordance with w_j^L and w_j^R, respectively. Moreover, considering the second term of the objective function in P2, which tries to minimize the deviations and according to the (7), δ_j^L and δ_j^R never get value simultaneously. In other words, $\delta_j^L . \delta_j^R = 0;\ \forall j$. So, the membership function $\mu(c_j)$ never gets a value more than 1. Therefore, it can be concluded that the models P1 and P2 are equivalent. □

Proposition 2: Consider a problem with one or more absolute terms in the objective function or constraints as follows:

P3: min $Z = \sum_{j=1}^{n} c_j |x_j|$
s.t.
　　$X \in F$

where F is a feasible set, c_j's are given non-negative parameters and x_j's are unrestricted variables. Murty [10] transformed the model P3 to its equivalent linear programming (LP) model P4 by replacing x_j and $|x_j|$ with $x_j^+ - x_j^-$ and $x_j^+ + x_j^-$, respectively, where $x_j^+, x_j^- \geq 0$.

P4: min $Z = \sum_{j=1}^{n} c_j\left(x_j^+ + x_j^-\right)$
s.t.
　　$X \in F$
　　$x_j^+, x_j^- \geq 0$

Proof: This proposition was proven by Murty [10] and we just express the results as follows:
　(i)　if at optimal solution $x_j \geq 0$, then $x_j^- = 0$ and $x_j^+ = x_j$;
　(ii)　if at optimal solution $x_j \leq 0$, then $x_j^+ = 0$ and $x_j^- = -x_j$.
They also ensured that $x_j^+ . x_j^- = 0$. □

Now, we can rewrite the linear form of (7) considering an unrestricted variable $c_j (= d_j - d_j^2)$ as follows:

$$\mu_1(d_j) = \mu_1(d_j^1) + S_{j,1}(d_j - d_j^1)$$
$$+ \frac{S_{j,2} - S_{j,1}}{2}\left(|c_j| + c_j\right)$$

Therefore, by applying the result of Proposition 2, we have:

$$\mu_1(d_j) = \mu_1(d_j^1) + S_{j,1}(d_j - d_j^1)$$
$$+ \frac{S_{j,2} - S_{j,1}}{2}(c_j^+ + c_j^- + c_j^+ - c_j^-)$$
$$c_j^+, c_j^- \geq 0$$

Then, the linear form of (7) is as follows:

$$\mu_1(d_j) = \mu_1(d_j^1) + S_{j,1}(d_j - d_j^1) +$$
$$(S_{j,2} - S_{j,1})(c_j^+) \quad \forall j \qquad (20)$$
$$c_j^+ \geq 0 \qquad\qquad\qquad \forall j \qquad (21)$$

Proposition 3: Consider the following non-linear constraint:

$$xy \leq b \qquad (22)$$
$$x \in \{0,1\} \qquad (23)$$
$$y \geq 0 \qquad (24)$$

where x is a 0-1 variable, y is a non-negative continues variables and b is non-negative constant. Replacing the non-linear term by a new non-negative continues variable $z(= xy)$, the linear form of the above constraint is as follows:

$$z \leq b \qquad (25)$$
$$y - M(1 - x) \leq z \leq y + M(1 - x) \qquad (26)$$
$$0 \leq z \leq Mx \qquad (27)$$

where M is a large positive number.

Proof: this proposition can be verified as follows:
  (i) if at optimal solution $x = 0$, then (26) will be inactive and (27) causes $z = 0$ (i.e., $z = xy = 0$);
  (ii) if at optimal solution $x = 1$, then (27) will be inactive and (26) causes $z = y$ (i.e., $z = xy = y$)
The proposition is proven. □

Proposition 4: Consider the following binary model:

P5: min cx
s.t.
$$X \in F$$
$$x \in \{0,1\}$$
$$c \geq 0$$
where F is a feasible set. P5 is equivalent to P6 as follows:

P6: min $y = cx$
s.t.
$$y \geq c + M(x - 1)$$
$$X \in F$$
$$x \in \{0,1\}$$
$$y, c \geq 0$$
where M is a large positive number.

This proposition can be proven as follows:
  (i) if at optimal solution $x = 1$, then y will be forced to be $y = c$;
  (ii) if at optimal solution $x = 0$, then y will be forced to be $y = 0$
Thus, the proposition is proven. □

Hence, the above-mentioned propositions are applied and the crisp multi-objective mixed-integer programming model is reformulated as a C-MILGP with just one linear objective. The proposed C-MILGP is as follows:

$$\min Z_0 = \sum_{i=1}^m \theta_i + \sum_{i=1}^m \sum_{j=1}^n \varphi_{ij}$$
$$+ \sum_{j=1}^n (w_j^L \delta_j^L + w_j^R \delta_j^R) + \sum_{i=1}^m (w_i^c \delta_i^c) \qquad (28)$$
s.t.
$$\sum_{j=1}^n \tau_{ij} \leq Q_i + \Delta Q_i(1 - \mu_2(q_i)) +$$
$$M(1 - Y_i) \qquad\qquad \forall i \qquad (29)$$
$$\tau_{ij} \geq d_j - M(1 - X_{ij}) \qquad \forall i,j \qquad (30)$$
$$\tau_{ij} \leq d_j + M(1 - X_{ij}) \qquad \forall i,j \qquad (31)$$
$$\tau_{ij} \leq MX_{ij} \qquad\qquad \forall i,j \qquad (32)$$
$$\mu_1(d_j) + \delta_j^L + \delta_j^R = 1 \qquad \forall j \qquad (33)$$
$$\mu_2(q_i) + \delta_i^c = 1 \qquad\qquad \forall i \qquad (34)$$
$$\mu_1(d_j) \geq \alpha_1 \qquad\qquad \forall j \qquad (35)$$
$$\mu_2(q_i) \geq \alpha_2 \qquad\qquad \forall i \qquad (36)$$
$$\theta_i \geq f_i + M(Y_i - 1) \qquad \forall i \qquad (37)$$
$$\varphi_{ij} \geq v_{ij} + M(X_{ij} - 1) \qquad \forall i,j \qquad (38)$$
$$\mu_1(d_j) \leq 1 \qquad\qquad \forall j \qquad (39)$$
$$\mu_2(q_i) \leq 1 \qquad\qquad \forall i \qquad (40)$$
$$\tau_{ij}, \theta_i, \varphi_{ij}, \delta_j^L, \delta_j^R, \delta_i^c \geq 0 \qquad \forall i,j \qquad (41)$$
$$\text{Constraints (2), (3), (5) and (21)} \qquad (42)$$

The objective function consists of four terms: 1) the first term and (37) are the linear form of the first term of (1) by using the proposition 4 and calculate the fixed cost of selected facilities, 2) the second term and (38) are the linear form of the second term of (1) by applying the proposition 4 and compute the variable cost of the assignments, 3) the third term and (33) are joined to calculate the deviations of the membership functions related to the fuzzy demands from their highest value (i.e., 1) by employing the proposition 1, and 4) the fourth term and (34) obtain the deviations of the membership functions related to the fuzzy constraints from their highest value (i.e., 1) by utilizing the proposition 1, in which $w_i^c = |1/S_{i,c}|$ and $S_{i,c}$ is the slope of the right line segment of $\mu_2(q_i)$ as depicted in Fig. 2. The acceptable deviations of the membership functions from their highest value (i.e., 1) for the demands and capacities are restricted by (35) and (36), respectively. Furthermore, the linear form of (16) is achieved by uniting (29) to (32) based on proposition 3. Finally, $\mu_1(d_j)$ can be obtained from (20).

V. NUMERICAL EXAMPLES

In this section, to investigate the effectiveness of the proposed C-MILGP for the F-SSCFLP, a set of large-sized

test problems comprising 200 costumers and 30 potential facilities are generated randomly [3, 6]. The problems are as follows:

- The location of the depot is assumed to be at coordinate $(0, 0)$;
- The coordinates of the location of the demand points are considered to be uniformly distributed in range $(10, 200)$;
- The variable costs of facility i while serves demand node j are supposed to be $v_{ij} = \rho e_{ij}$, where e_{ij} is the Euclidian distance between the location of the facility i and the demand point j and ρ is a positive scalar (we assume $\rho = 4$);
- The fixed costs of facility i have a uniform distribution in range $(300, 700)$;
- The middle possible value of the fuzzy demands d_j^2 are uniformly distributed in the range of $(10, 50)$; also, the lowest and highest values are determined as $d_j^1 = \left(1 - \lambda_j^1\right) d_j^2$ and $d_j^3 = \left(1 + \lambda_j^3\right) d_j^2$, where λ_j^1 and λ_j^3 are selected at random from the range $(0, 1)$;
- The capacity of each facility has a uniform distribution over $(100, 500)$ and the corresponding capacity tolerances are assumed to be $\Delta Q_i = \pi_i Q_i$ where π_i is selected at random from the range $(0, 1)$;
- The lowest acceptable level for the membership functions of the demand points and the facilities ($\alpha - cut$) are assumed $\alpha_1 = \alpha_2 \in \{0.75, 0.85, 0.95\}$.

TABLE I
CPU TIMES FOR DIFFERENT LEVELS OF $\alpha - cut$

Problem	0.75	0.85	0.95
P1	17.75	9.22	40.3
P2	4.81	7.05	6.5
P3	26.78	27.98	37.36
P4	9.2	6.44	15.02
P5	4.5	7.48	8.39
P6	4.38	5.45	8.28
P7	4.95	5.27	7.53
P8	4.52	8.38	8.61
P9	4.11	4.63	11.58
P10	6.33	6.36	8.53
P11	5.75	5.92	7.83
P12	6.33	7.84	17.13

All the problems are solved globally by the CPLEX 12.9 software on a 2.40 GHz Intel® Core™ i5 CPU and 4GB of RAM. Table I indicates the CPU times in seconds for different levels of the α-cut. Accordingly, the α-cut level has a direct impact on the computational time, so reducing the α-cut level will relatively reduce the computational time. Also, the optimal values of the objective and membership functions are achieved in reasonable times which shows the strength of the proposed C-MILGP model to tackle with large-sized instances.

VI. CONCLUSION

In this paper, fuzzy principals combined with a GP approach to deal with uncertainties of the SSCFLP in real-world. Accordingly, a new MILGP model has been presented for the F-SSCFLP, in which demands and capacity constraints were considered as triangular fuzzy numbers and fuzzy constraints, respectively. Finally, the effectiveness of the presented model has been examined trough some large-sized problems generated at random. All the test problems obtained the global optimum solutions in a reasonable time by solving the model through the CPLEX 12.9 software.

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