Entropy via multiplicity

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Abstract

The topological entropy of piecewise affine maps is studied. It is shown that singularities may contribute to the entropy only if there is angular expansion and we bound the entropy via the expansion rates of the map. As a corollary we deduce that non-expanding conformal piecewise affine maps have zero topological entropy. We estimate the entropy of piecewise affine skew-products. Examples of abnormal entropy growth are provided.[1]

Introduction

For a smooth map f of a compact manifold the Ruelle-Margulis inequality together with the variational principle [Br, KH] tells us that the topological entropy of f is bounded by the maximal sum over positive Lyapunov exponents. For maps with singularities this result is no longer true and there are examples of piecewise smooth maps, where the topological entropy exceeds what can be predicted from the rate of expansion.

In this paper we study the class of piecewise affine maps. It follows from [B1] that for piecewise affine maps the entropy is bounded by the rate of expansion and the growth in the multiplicity of singularities. The latter was shown by J. Buzzi to be zero for piecewise isometries [B2], but his proof does not generalize to non-expanding piecewise affine maps. In fact, we exhibit an example of a piecewise affine contracting map with positive topological entropy.

We show that the growth of multiplicity is as effect caused by angular expansion that can be estimated by the expansion rates of the map f. As a corollary we obtain that for piecewise conformal maps the topological entropy can be estimated by its expansion rate as in the smooth compact case. It follows that piecewise affine non-expanding conformal maps have zero topological entropy.

In the second part of the paper we study the topological entropy of piecewise affine maps of skew-product type and obtain a formula which bounds the entropy of the skew products in terms of the entropy and multiplicity growth of the

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factors. The estimate includes a term which indicates that the entropy of a skew-product system may be greater than the sum of the maximal entropy of its factors and we give an example where this is realized.

Our main results have several corollaries for which one can calculate the topological entropy of various classes of piecewise affine maps.

1 Definitions and main results

1.1 Piecewise affine maps and topological entropy

Definition 1. *We say that* $(X, \mathcal{Z}, f)$ *is a piecewise affine map if*

1. $X \subset \mathbb{R}^n$
2. $\mathcal{Z} = \{Z\}$ *is a finite collection of open, pairwise disjoined polytopes such that* $X' := \cup_{Z\in\mathcal{Z}} Z$ *is dense in* X.
3. $f_Z := f|_Z : Z \to X$ *is affine for each* $Z \in \mathcal{Z}$

The maps f_Z are called the affine components of the map f. The linear part of f_Z is denoted f'_Z, and if $x \in Z$ we denote $d_x f = f'_Z$. Let $\mathrm{PAff}(X;X)$ be the set of piecewise affine maps on X and let $U_f = X' \cap f^{-1}(X') \cap f^{-2}(X') \cap \dots$ be the set of points in X with well-defined infinite orbits. Let $\mathcal{Z}^n$ be the continuity partition of the piecewise affine map f^n. We always assume X to be compact.

Since the maps we consider in this paper have singularities (consult [KS, ST]) we must define what we mean by topological entropy.

Let $U_n = \cap_{k=0}^{n-1} f^{-k}(X')$, then $U_f = \cap_{n\ge 0} U_n$. Take a metric d defining the standard topology on X. Let $d_n^f = \max_{0\le k<n}(f^k)^* d$, and define $S(d_n^f, \epsilon)$ to be the minimal number of (d_n^f, ϵ)-balls needed to cover U_n. Define

$$h_{\mathrm{top}}(f) = \lim_{\epsilon\to 0} \varlimsup_{n\to\infty} \frac{1}{n} \log S(d_n^f, \epsilon)\,.$$

This number is independent of the choice of metric on X and is finite because it is bounded by $d \cdot \sup_x \log(\|d_x f\| + |\mathcal{Z}|)$. It equals the (n,ϵ)-entropy $h_{\mathrm{top}}(f|_{U_f})$, which coincides with the (upper=lower) capacity entropy $Ch_{U_f}(f)$ [Pe]. This bounds the topological entropy $h_{U_f}(f)$ of non-compact subsets by Pesin and Pitskel' [Pe], so that we have $h_{U_f}(f) \le h_{\mathrm{top}}(f)$. Since we estimate $h_{\mathrm{top}}(f)$ from above, this bounds the other entropy too.

Remark 1. *We observe that even in the presence of singularities the property* $h_{\mathrm{top}}(f^T) = T h_{\mathrm{top}}(f^T)$ *holds for* $T \in \mathbb{N}$. *The proof uses the fact that for all* $\epsilon > 0$ *there is a number* $\delta(\epsilon) > 0$ *such that* $B_d(x, \delta(\epsilon)) \subset B_{d_T^f}(x, \epsilon)\,\forall x \in X'$, *cf. [KH].*

One can also measure the orbit growth of a piecewise affine map through the growth of continuity domains. The singularity entropy of f is

$$H_{\mathrm{sing}}(f) = \varlimsup_{n\to\infty} \frac{1}{n} \log |\mathcal{Z}^n|\,.$$

For non-expanding piecewise affine maps $h_{\mathrm{top}}(f) \le H_{\mathrm{sing}}(f)$. If $f_Z(x) \ne f_{Z'}(x)$ for all $x \in \partial Z \cap \partial Z'$ with $Z \ne Z'$, then $H_{\mathrm{sing}}(f) \le h_{\mathrm{top}}(f)$ [R].

1.2 Expansion rates and multiplicity entropy

Definition 2. *The multiplicity of the partition* $\mathcal{Z}^n$ *at a point* $a \in X$ *is* $\operatorname{mult}(\mathcal{Z}^n, a) = |\{Z \in \mathcal{Z}^n \mid \bar{Z} \ni a\}|$, *and the multiplicity of* $\mathcal{Z}^n$ *is* $\operatorname{mult}(\mathcal{Z}^n) = \sup_{a\in X} \operatorname{mult}(\mathcal{Z}^n, a)$. *The multiplicity entropy [B1] of* f *is defined as*

$$H_{\text{mult}}(f) = \varlimsup_{n\to\infty} \frac{1}{n} \log \operatorname{mult}(\mathcal{Z}^n)\,.$$

Definition 3. *For* $f \in \operatorname{PAff}(X, X)$ *we define*

$$\lambda^+(f) = \varlimsup_{n\to\infty} \sup_{x\in U_n} \frac{1}{n} \max_{0\le k\le d} \log \|\Lambda^k d_x f^n\|\,.$$

We also let (the second quantity can equal $-\infty$ *for non-invertible maps)*

$$\lambda_{\max}(f) = \varlimsup_{n\to\infty} \sup_{x\in U_n} \frac{1}{n} \log \|d_x f^n\| \text{ and } \lambda_{\min}(f) = -\varlimsup_{n\to\infty} \sup_{x\in U_n} \frac{1}{n} \log \|(d_x f^n)^{-1}\|\,.$$

Theorem 1. *For any* $f \in \operatorname{PAff}(X; X)$ *it holds:*

$$h_{\text{top}}(f) \le \lambda^+(f) + H_{\text{mult}}(f)\,.$$

This result is basically due to Buzzi [B1]. However, he only proves it for a special class of strictly expanding maps, and he considers the entropy of coding $H_{\text{sing}}(f)$ instead of $h_{\text{top}}(f)$. Hence we modify his proof.

1.3 The spherization and angular expansions

For any k-dimensional submanifold $N^k \subset X$ and $x \in N$ we define the spherical bundle $STN = \{v \in TN : \|v\| = 1\}$, where $\|\cdot\|$ is the Euclidian norm on every $T_xN \subset \mathbb{R}^d$. Let f be non-degenerate, i.e. each affine component is non-degenerate. The spherization of f is defined to be the piecewise smooth map $d_x^{(s)} f : ST_xX \to ST_{f(x)}X$ given at $x \in X'$ by the formula

$$d_x^{(s)} f(v) = \frac{d_x f(v)}{\|d_x f(v)\|}\,.$$

For $x \in \operatorname{Sing}(X) \overset{\text{def}}{=} X \setminus X'$ and $v \notin T_x \operatorname{Sing}(X)$ (the tangent cone) we define $d_x^{(s)} f(v) = \lim_{\epsilon\to+0} d_{x+\epsilon v}^{(s)} f(v)$. For other $(x, v) \in STX$ the map is not defined.

The angular expansion of f is exactly the expansion in its spherization.

Definition 4. *For a non-degenerate map* $f \in \operatorname{PAff}(X, X)$ *and* $i < d$ *we define*

$$\rho_i(f) = \varlimsup_{n\to\infty} \frac{1}{n} \sup_{x\in U_n} \max_{0\le k\le i} \sup_{v\in S^{d-1}} \log \|\Lambda^k d_v d_x^{(s)} f^n\|.$$

Note that

$$\rho_i(f) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sup_{N^i \subset X} \sup_{x \in U_n \cap N^i} \sup_{v \in ST_x N^i} \log \|\Lambda\, d_v d_x^{(s)} f^n\| \,.$$

The number $\rho_1(f)$ measures the maximal exponential rate with which angles can increase under the map f. The numbers $\rho_i(f)$ for $i < d$ measures the maximal rate of expansion of the restrictions to $(i-1)$-dimensional spheres. Clearly $\rho_0(f) = 0$ for any $f \in \mathrm{PAff}(X, X)$, and if f is conformal, i.e. all the affine components of f are conformal, then $\rho_i(f) = 0$ for all i.

Theorem 2. *$H_{\mathrm{mult}}(f) \le \sum_{i=1}^{d-1} \rho_i(f)$ for any non-degenerate $f \in \mathrm{PAff}(X;X)$.*

The following corollaries are direct consequences of Theorem 2.

Corollary 1. *If $f \in \mathrm{PAff}(X;X)$ is conformal, then $h_{\mathrm{top}}(f) \le \lambda^+(f)$.*

We say that a piecewise affine map is non-expanding if all its affine components are non-expanding, i.e. the eigenvalues of the linear part of each affine component have absolute values not exceeding 1.

Corollary 2. *If $f \in \mathrm{PAff}(X;X)$ is conformal non-expanding, then $h_{\mathrm{top}}(f) = 0$.*

It is shown in §3.1 that

$$\rho_i(f) \le \overline{\lim_{n\to\infty}} \sup_{x \in U_f} \max_{0 \le k \le i} \frac{1}{n} \log \|\Lambda^k d_x f^n\| - i\lambda_{\min}(f) \le i\Big(\lambda_{\max}(f) - \lambda_{\min}(f)\Big) \,,$$

This gives the following estimate:

Corollary 3. *For a non-degenerate map $f \in \mathrm{PAff}(X;X)$ it holds:*

$$H_{\mathrm{mult}}(f) \le \frac{d(d-1)}{2} \big(\lambda_{\max}(f) - \lambda_{\min}(f)\big) \,.$$

Hence we see that the topological entropy of a non-degenerate piecewise affine map f can be bounded using only its expansion rates. In fact, we have

$$h_{\mathrm{top}}(f) \le \lambda^+(f) + \frac{d(d-1)}{2} \big(\lambda_{\max}(f) - \lambda_{\min}(f)\big) \,.$$

Let us call $f \in \mathrm{PAff}(X;X)$ asymptotically conformal if $\lambda_{\max}(f) = \lambda_{\min}(f)$.

Corollary 4. *For asymptotically conformal $f \in \mathrm{PAff}(X;X)$: $h_{\mathrm{top}}(f) \le \lambda^+(f)$.*

Remark 2. *It is essential that in our definition of piece-wise affine maps we consider a finite number of continuity domains. With countable number of domains (this is related to countable Markov chains) the above theorems become wrong. In fact, according to [AOW] every aperiodic measure preserving transformation can be represented as an interval exchange with countable number of intervals. In particular, there exist piece-wise isometries with infinite number of continuity domains, which have positive topological and metric entropies.*

1.4 Piecewise affine skew products

Definition 5. *We say that* $S\tilde{\times}T \in \mathrm{PAff}(X\times Y, X\times Y)$ *is a piecewise affine skew product if it has the form* $f(x,y) = (S(x), T_x(y))$ *for some* $T_x \in \mathrm{PAff}(Y,Y)$.

The following results hold for piecewise affine skew products:

Theorem 3. *If* $S\tilde{\times}T$ *is a piecewise affine skew product, then*

$$h_{\mathrm{top}}(S) \le h_{\mathrm{top}}(S\tilde{\times}T) \le h_{\mathrm{top}}(S) + H_{\mathrm{mult}}(S) + \sup_{\mathbf{x}} \left(\lambda^+(T_{\mathbf{x}}) + H_{\mathrm{mult}}(T_{\mathbf{x}})\right).$$

where $\mathbf{x} = (x_0, x_1, \dots)$ *is an orbit of* S *in* X *and* $T_{\mathbf{x}}$ *is the dynamics along this orbit, i.e.* $T^n_{\mathbf{x}} = T_{x_{n-1}} \circ \cdots \circ T_{x_0}$ *(see §2.3 for details).*

From Theorem 3 we can deduce several simple corollaries:

Corollary 5. *If* $\dim(X) = 1$ *and* $T_x \in \mathrm{PAff}(Y;Y)$ *are non-expanding and conformal for all* $x \in X'$, *then* $h_{\mathrm{top}}(S\tilde{\times}T) = h_{\mathrm{top}}(S)$.

Corollary 6. *Let* $X = [0,1]^d$ *and* $A \in \mathrm{PAff}(X;X)$ *be defined by* $x \mapsto Ax \bmod \mathbb{Z}^d$ *for some* $A \in \mathrm{GL}_d(\mathbb{R})$. *If* $T_x \in \mathrm{PAff}(Y;Y)$ *are non-expanding and conformal for all* $x \in X'$, *then*

$$h_{\mathrm{top}}(A\tilde{\times}T) = h_{\mathrm{top}}(A) = \log \mathrm{Jac}^+ A\,,$$

where

$$\mathrm{Jac}^+ A = \prod_{\lambda\in \mathrm{Sp}(A)} \max\{|\lambda|, 1\}\,.$$

Corollary 7. *Let* $\Sigma^+_N = \{1,\dots,N\}^{\mathbb{Z}_{\ge 0}}$ *and* σ^+_N *be the right shift on* Σ^+_N. *Take* $T_1, \dots, T_N \in \mathrm{PAff}(Y;Y)$ *to be non-expanding and conformal. Then for the map* $f : \Sigma^+_N \times Y \to \Sigma^+_N \times Y$, $(\mathbf{t}, y) \mapsto (\sigma^+_N \mathbf{t}, T_{t_0}(y))$, *we have:* $h_{\mathrm{top}}(f) = \log N$.

Remark 3. *The class of piecewise affine skew-products of the form*

$$\sigma^+_N \tilde{\times} T : \Sigma^+_N \times Y \to \Sigma^+_N \times Y$$

is physically relevant and appears in the Zhang sandpile model of Self-Organized Criticality [BCK]. The maps T_i *correspond to the avalanches and are contracting [KR1], though not conformal. So we can get an estimate for the entropy.*

In general we cannot ensure existence of invariant measures for piecewise affine maps. In fact, there are examples with no invariant measure. However we can give an estimate for the metric entropy whenever such a measure exists.

Theorem 4. *Let* $S\tilde{\times}T$ *be a piecewise affine skew product and* $T_x \in \mathrm{PAff}(Y;Y)$ *be non-expanding for all* $x \in X'$. *If* μ *is a* $S\tilde{\times}T$*-invariant Borel probability measure on* $X \times Y$, *then*

$$h_{\pi_*\mu}(S) \le h_\mu(S\tilde{\times}T) \le h_{\pi_*\mu}(S) + H_{\mathrm{mult}}(S)\,,$$

where $\pi : X \times Y \to X$ *is the projection to* X.

Corollary 8. *Let* $A\tilde{\times}T$ *be as in Corollary 6 with* A *expanding. If* μ *is a measure of maximal entropy for* $S\tilde{\times}T$ *on* $X \times Y$, *then* $\pi_*\mu$ *is absolutely continuous with respect to the Lebesgue measure on* X.

2 Proof of the theorems

2.1 Proof of Theorem 1

Fix $\epsilon > 0$ and let $T = T(\epsilon) \in \mathbb{N}$ be such that $\mathrm{mult}(\mathcal{Z}^n) \le \exp((H_{\mathrm{mult}}(f)+\epsilon)n)$ for all $n \ge T$ and

$$\forall x \in U, n \ge T, 0 \le k \le d: \ \|\Lambda^k d_x f^n\| \le \exp((\lambda^+(f)+\epsilon)n)$$

We suppose that $\sqrt{d}+1 \le \exp(\epsilon T/d)$. Take $r = r(\epsilon)$ to be compatible with the partition $\mathcal{Z}^T$, i.e. any r-ball intersects maximally $\mathrm{mult}(\mathcal{Z}^T)$ partition elements.

We will prove inductively on l that each $Z \in \mathcal{Z}^{lT}$ can be covered by a family $Q_Z = \{W\}$ satisfying the following properties:

1. $\sum_{Z\in\mathcal{Z}^{lT}} \mathrm{card}\, Q_Z \le C_0 \exp((\lambda^+(f)+H_{\mathrm{mult}}(f)+3\epsilon)lT)$
2. $\mathrm{diam}(f^{lT}(W)) \le r$.
3. $\forall x, y \in W: \ d_l^{f^T}(x,y) < \epsilon$ and $d_{lT}^{f}(x,y) < \delta(\epsilon)$ with $\delta(\epsilon)$ from Remark 1.

The base of induction $l = 0$ is obvious and $C_0 \le |\mathcal{Z}|(\mathrm{diam}\, X/\min\{r,\epsilon\})^d$.

Take a partition element $W \in Q_Z$ that is used to cover the set $Z \in \mathcal{Z}^{lT}$. By the induction hypothesis it can be continued to cover an element of length $\mathcal{Z}^{(l+1)T}$ in at most $\mathrm{mult}(\mathcal{Z}^T)$ ways. So to cover the cylinders $Z \in \mathcal{Z}^{(l+1)T}$ we make a division of W:

$$W = \bigcup_{i=1}^{\gamma} W_i' \ , \gamma \le \mathrm{mult}(\mathcal{Z}^T)\,.$$

Let $W_i'' = f^{Tl-1}(W_i')$ and $W_i''' = f^T(W_i'')$. By the assumption $\mathrm{diam}(W'') \le r$, but the set W''' may have greater diameter than r. Thus we need to divide the sets W_i''' and pull this refinement back to the partition of sets W_i'.

The image $f^T(W_i'')$ is the image of one affine component of f^T. Let L^T denote the linear part of this affine component. We can assume that L^T is symmetric and take $\{e_k\}$ to be a basis of eigenvectors corresponding to eigenvalues $\lambda_1^T, \dots, \lambda_d^T$. Let $\{v_k\}$ be a basis in the vector subspace corresponding to W_i'''. We can choose this basis to be orthonormal and triangular with respect to $\{e_k\}$. Divide W_i''' by the hyperplanes

$$\psi_j(x) \stackrel{\mathrm{def}}{=} \langle v_j, x\rangle = p\frac{\min\{r,\epsilon\}}{\sqrt{d}}, \quad p \in \mathbb{Z}, \quad j = 1, \dots, d\,.$$

This defines cells $\tilde{W}$ of diameter less than $\min\{r,\epsilon\}$. Since $\psi_j(W''') = \psi_j(L^T(W''))$ has diam $\le |\lambda_i^T| \min\{r,\epsilon\}$, the number of cells $\tilde{W}$ needed to cover W''' is less than or equal to

$$\begin{aligned}(\sqrt{d}+1)^d|\lambda_1^T|^+ \dots |\lambda_d^T|^+ \quad &\le \quad (\sqrt{d}+1)^d \sup_{x\in U_T} \max_{1\le k\le d} \|\Lambda^k d_x f^T\| \\ &\le \quad \exp\big((\lambda^+(f)+2\epsilon)T\big)\,,\end{aligned}$$

where $|\lambda|^+ = \max\{|\lambda|, 1\}$.

Therefore the total cardinality of the new partition is less than or equal to

$$\operatorname{mult}(\mathcal{Z}^T)\exp\big((\lambda^+(f)+2\epsilon)T\big)\exp\big((\lambda^+(f)+H_{\mathrm{mult}}(f)+3\epsilon)lT\big) \\ \le \exp\big((\lambda^+(f)+H_{\mathrm{mult}}(f)+3\epsilon)(l+1)T\big)\,.$$

The elements of the partition Q_Z have diameter less than ϵ in the metric $d_l^{f^T}$. By Remark 1 each partition element has diameter less than some number $\delta(\epsilon) > 0$ in the metric d_{lT}^f. This proves the statement.

2.2 Proof of Theorem 2

Define the bundles

$$S^{(k)}TX = \{(x, v_1, \dots, v_k) \,|\, x \in X, v_1 \in ST_xX, v_i \in ST_xX \cap \langle v_1, \dots, v_{i-1}\rangle^{\perp}\}.$$

They form the spherical towers:

$$S^{(d)}TX \xrightarrow{\pi_d} S^{(d-1)}TX \xrightarrow{\pi_{d-1}} \cdots \xrightarrow{\pi_2} S^{(1)}TX = STX \xrightarrow{\pi_1} X$$

with fibers $S^0, S^1, \dots, S^{d-1}$ respectively.

The spherization $d^{(s)}f : STX \to STX$ induces the maps $S^{(k)}f : S^{(k)}TX \to S^{(k)}TX$. Although defined on $S^{(k)}TX'$ they extend over the strata of $\operatorname{Sing}(X)$ as in §1.3 (modulo spherization this corresponds to the differential $f_{x,v_1,\dots,v_{k-1}}(v_k)$ of Tsujii and Buzzi [T, B2]):

$$S^{(k)}f(x, v_1, \dots, v_k) = \lim_{\epsilon_1 \to +0} \cdots \lim_{\epsilon_k \to +0} d^{(s)}_{x+\epsilon_1 v_1+\dots+\epsilon_k v_k} f(x, v_1, \dots, v_k)$$

(the r.h.s. operator is applied to each vector v_i successively). In particular, the map $S^{(d)}f$, though has singularities, is defined everywhere on $S^{(d)}TX$.

Let $\mathcal{Z}^n_{(x,v_1,\dots,v_{k-1})}(S^{(k)}f|S^{(k-1)}f)$ be the continuity partition of the piecewise smooth map $(S^{(k)}f)^n$ restricted to the fiber $\pi_k^{-1}(x, v_1, \dots, v_{k-1})$. Define

$$H_{\mathrm{sing}}(S^{(k)}f|S^{(k-1)}f) = \varlimsup_{n\to\infty} \sup_{(x,v_1,\dots,v_{k-1})} \frac{1}{n}\log|\mathcal{Z}^n_{(x,v_1,\dots,v_{k-1})}(S^{(k)}f|S^{(k-1)}f)|\,.$$

Clearly $\operatorname{mult}(\mathcal{Z}^n_{(x,v_1,\dots,v_{k-1})}(S^{(k)}f|S^{(k-1)}f), v_k) = |\mathcal{Z}^n_{(x,v_1,\dots,v_k)}(S^{(k+1)}f|S^{(k)}f)|$. So $H_{\mathrm{mult}}(S^{(k)}f|S^{(k-1)}f) = H_{\mathrm{sing}}(S^{(k+1)}f|S^{(k)}f)$.

In particular, we have: $H_{\mathrm{mult}}(f) = H_{\mathrm{sing}}(S^{(1)}f|f)$. Applying the arguments from the proof of Theorem 1 we obtain:

$$H_{\mathrm{sing}}(S^{(1)}f|f) \le \rho_{d-1}(f) + H_{\mathrm{mult}}(S^{(1)}f|f)\,.$$

Doing its once more for $H_{\mathrm{mult}}(S^{(1)}f|f) = H_{\mathrm{sing}}(S^{(2)}f|S^{(1)}f)$ we get:

$$H_{\mathrm{sing}}(S^{(2)}f|f) \le \rho_{d-2}(f) + H_{\mathrm{mult}}(S^{(2)}f|f)\,.$$

Applying the same argument $d-1$ times yields: $H_{\mathrm{mult}}(f) \le \rho_{d-1}(f)+\dots+\rho_1(f)$.

2.3 Proof of Theorem 3

The inequality $h_{\text{top}}(S) \le h_{\text{top}}(S\tilde{\times}T)$ is obvious since S is a quotient of $S\tilde{\times}T$. We will now prove the upper bound for $h_{\text{top}}(S\tilde{\times}T)$. Denote $d_Y = \dim Y$.

Let Γ be the set of all sequences $((x_0,z_0),(x_1,z_1),(x_2,z_2),\dots) \in (X \times \mathbb{R}^{d_Y})^{\mathbb{Z}_{\ge 0}}$ satisfying $S(x_i + \epsilon z_i) \to x_{i+1}$ as $\epsilon \to +0$. Define $T_{(x,z)} = \lim_{\epsilon\to+0} T_{x+\epsilon z}$ and let $\mathcal{Y}^{n,\mathbf{x}}$ be the collection of non-empty sets

$$Y_{(x_0,z_0)} \cap T^{-1}_{(x_0,z_0)}(Y_{x_1,z_1}) \cap \dots \cap (T_{(x_{n-1},z_{n-1})} \circ \dots T_{(x_0,z_0)})^{-1}(Y_{(x_n,z_n)}),$$

for $\mathbf{x} = ((x_0,z_0),(x_1,z_1),(x_2,z_2),\dots) \in \Gamma$, where the sets $Y_{(x_i,z_i)}$ are the continuity domains of the maps $T_{(x_i,z_i)}$. Let

$$P^n(x) = \{Y \in \mathcal{Y}^{n,\mathbf{x}} \,|\, \mathbf{x} = ((x_0,z_0),(x_1,z_1),\dots) \in \Gamma, x_0 = x\}$$

be the continuity partition iterated along all possible S-orbits starting from $x \in X$. Clearly

$$|P^n(x)| \le \text{mult}(S^n, x) \sup_{\mathbf{x}\in\Gamma} |\mathcal{Y}^{n,\mathbf{x}}|.$$

To simplify notations we will write elements of Γ as $\mathbf{x} = (x_0, x_1, x_2, \dots)$, where x_i consists of a point in X and a vector in $\mathbb{R}^{d_Y}$. If $x_i \in X'$ the vector z_i is not essential. Denote $T^n_{\mathbf{x}} = T_{x_{n-1}} \circ \dots \circ T_{x_0}$ for $\mathbf{x} = (x_0, x_1, x_2, \dots) \in \Gamma$ and let

$$H_{\text{mult}}(T_{\mathbf{x}}) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \text{mult}(T^n_{\mathbf{x}}).$$

Lemma 5. *For a piecewise affine skew product $S\tilde{\times}T$ with T_x non-expanding for all $x \in X'$, it holds:*

$$h_{\text{top}}(S\tilde{\times}T) \le h_{\text{top}}(S) + H^{\text{fiber}}_{\text{sing}}(T|S),$$

where

$$H^{\text{fiber}}_{\text{sing}}(T|S) = \sup_{x\in X} \overline{\lim_{n\to\infty}} \frac{1}{n} \log |P^n(x)|.$$

Remark 4. *The statement of the lemma is similar to Bowen's Theorem 17 [Bo], but the direct generalization fails, see Example 2.*

Proof. Let $\epsilon > 0$ be arbitrary. Denote $a = H^{\text{fiber}}_{\text{sing}}(T|S)$, and fix $\alpha > 0$ and $m_\alpha = [1/\alpha] \in \mathbb{N}$. For all $x \in X$ we let

$$n_\alpha(x) = \min\{n \ge m_\alpha \,|\, \frac{1}{n} \log |P^n(x)| \le a + \alpha\}.$$

Since the function $\text{mult}(\mathcal{Z}, x)$ is upper semi-continuous, the same is true for the function $n_\alpha(x)$. So $n_\alpha := \sup_{x\in X} n_\alpha(x)$ is finite. Then we have $0 < m_\alpha \le n_\alpha(x) \le n_\alpha < +\infty$ for all $x \in X$.

Let $r_\alpha > 0$ be compatible with all the partitions $\mathcal{Z}^n$, $m_\alpha \le n \le n_\alpha$, i.e. any ball of radius r_α can intersect at most $\text{mult}(\mathcal{Z}^n)$ different elements of the partition $\mathcal{Z}^n$ for $m_\alpha \le n \le n_\alpha$. We can assume that $r_\alpha < \epsilon$.

Let E_n denote a (n, r_α)-spanning set of minimal cardinality for S in X. For each $x \in X$ consider the singularity partition $\{Z \cap (\{x\} \times Y) \mid Z \in \mathcal{Z}^{n_\alpha(x)}\}$. Each element of this partition is an open polytope in the fiber $\{x\} \times Y$, and hence we can subdivide the partition so that each element has diameter no greater than ϵ. Denote the resulting partition of $\{x\} \times Y$ by F_x. The refinement can be done in such a way that $|F_x| \le C_0/\epsilon^d \cdot |P^{n_\alpha}(x)|$ for some constant $C_0 \in \mathbb{R}_+$.

For $x \in X$ we let $t_0(x) = 0$ and define recursively

$$t_{k+1}(x) = t_k(x) + |F_{S^{t_k(x)}(x)}|\,.$$

Let $q(x) = \min\{k > 0 \mid t_{k+1}(x) \ge n\}$. We will denote $q = q(x)$. For $x \in E_n$, $z_0 \in F_x$, $z_1 \in F_{S^{t_1(x)}(x)}, \ldots, z_q \in F_{S^{t_q(x)}(x)}$ denote

$$\begin{aligned} V(x; z_0, \ldots, z_q) = \big\{ w \in X \times Y \,\big|\, & d((S\tilde{\times}T)^{t+t_k(x)}(w), (S\tilde{\times}T)^t(z_k)) < 2\epsilon \\ & \forall\, 0 \le t \le |F_{S^{t_{k-1}(x)}(x)}|,\ 0 \le k \le q(x) \big\}\,. \end{aligned}$$

Then $\cup_{x,z_0,\ldots,z_q} V(x; z_0, \ldots z_q) = U_n \times Y$ and for any $(n, 4\epsilon)$-separating set $K \subset X \times Y$ for $S\tilde{\times}T$ we have $|K \cap V(x; z_0, \ldots z_q)| \le 1$. Thus if K is a maximal $(n, 4\epsilon)$-set, then the cardinality of K is bounded by the number of ways we can choose $x, z_0, \ldots, z_q$ modulo the partitions specified above. For fixed $x \in X$ the number Π_x of such admissible combinations satisfies

$$\Pi_x \le \prod_{k=0}^{q(x)} |F_{S^{t_k(x)}(x)}|\,.$$

Since $q(x) \le n/m_\alpha$ we have:

$$\begin{aligned} \log \Pi_x &\le (q(x)+1) \log \frac{C_0}{\epsilon^d} + \sum_{k=0}^{q(x)} \log |P^{n(S^{t_k(x)}(x))}(S^{t_k(x)}(x))| \\ &\le \frac{n+m_\alpha}{m_\alpha} \log \frac{C_0}{\epsilon^d} + (a+\alpha) \sum_{k=0}^{q(x)} n(S^{t_k(x)}(x))) \\ &\le \frac{n+m_\alpha}{m_\alpha} \log \frac{C_0}{\epsilon^d} + (a+\alpha)(n+n_\alpha) \end{aligned}$$

Let $Q(S\tilde{\times}T, n, 4\epsilon)$ denote the cardinality of a maximal $(n, 4\epsilon)$-separating set for $S\tilde{\times}T$ in $X \times Y$. We have that

$$\frac{1}{n} \log Q(S\tilde{\times}T, n, 4\epsilon) \le \frac{1}{n} \log |E_n| + (\frac{1}{m_\alpha} + \frac{1}{n}) \log \frac{C_0}{\epsilon^d} + (a+\alpha)(1 + \frac{n_\alpha}{n})\,.$$

This yields

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q(S\tilde{\times}T, n, 4\epsilon) \le h_{\text{top}}(S) + \frac{1}{m_\alpha} \log \frac{C_0}{\epsilon^d} + a + \alpha\,.$$

Since the left hand side does not depend on α we can let $\alpha \to 0$. Then $m_\alpha \to \infty$ and we get

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q(S\tilde{\times}T, n, 4\epsilon) \le h_{\text{top}}(S) + a\,.$$

Finally let $\epsilon \to 0$. □

Lemma 6. $\forall \mathbf{x} \in \Gamma: \ \overline{\lim_{n\to\infty}} \frac{1}{n} \log |\mathcal{Y}^{n,\mathbf{x}}| \le \lambda^+(T_\mathbf{x}) + H_{\text{mult}}(T_\mathbf{x})\,.$

The proof of Lemma 6 is similar to the proof of Theorem 1 and will be omitted.

Combining Lemmata 5 and 6 we obtain:

$$\begin{aligned} h_{\text{top}}(S\tilde{\times}T) &\le h_{\text{top}}(S) + \sup_{\mathbf{x}\in\Gamma} \overline{\lim_{n\to\infty}} \frac{1}{n} \log\Big[\text{mult}(S^n, x_0)|\mathcal{Y}^{n,\mathbf{x}}|\Big] \\ &\le h_{\text{top}}(S) + H_{\text{mult}}(S) + \sup_{\mathbf{x}\in\Gamma} \overline{\lim_{n\to\infty}} \frac{1}{n} \log|\mathcal{Y}^{n,\mathbf{x}}| \\ &\le h_{\text{top}}(S) + H_{\text{mult}}(S) + \sup_{\mathbf{x}\in\Gamma} \Big(\lambda^+(T_\mathbf{x}) + H_{\text{mult}}(T_\mathbf{x})\Big)\,. \end{aligned}$$

2.4 Proof of Theorem 4

Let μ be an f-invariant Borel probability measure on $X \times Y$. Denote the projection of μ to X by $\mu_X = \pi_*\mu$ and let $\{\nu_x\}$ be the canonical family of conditional measures on the fibers $\pi^{-1}(x)$. By the generalized Abramov-Rokhlin formula [BC] (Bogenschtz and Crauel removed restrictions on the the maps S and T_x in the original formula [AR]) we have:

$$h_\mu(S\tilde{\times}T) = h_{\mu_X}(S) + h_\mu(T|S)\,,$$

where

$$h_\mu(T|S,\xi) = \lim_{n\to\infty} \frac{1}{n} \int H_{\nu_x}\Big(\bigvee_{k=0}^{n-1} (T_{S^{k-1}(x)} \circ \dots \circ T_x)^{-1}(\xi)\Big)\, d\mu_X(x)\,,$$

for a measurable partition ξ of Y and $h_\mu(T|S) = \sup_\xi h_\mu(T|S,\xi)$, where the supremum is taken over all finite measurable partitions ξ with finite entropy and the refinement $\bigvee_{k=0}^{n-1}(T_{S^{k-1}(x)} \circ \dots \circ T_x)^{-1}(\xi)$ is understood with respect to all orbits $(x, Sx, \dots, S^{k-1}x)$ starting at x (as in §2.3).

For $\epsilon > 0$ we choose ξ such that $h_\mu(T|S) \le h_\mu(T|S,\xi) + \epsilon$. Clearly

$$\begin{aligned} \frac{1}{n} H_{\nu_x}\Big(\bigvee_{k=0}^{n-1} (T_{S^{k-1}(x)} \circ \dots \circ T_x)^{-1}(\xi)\Big) &\le \frac{1}{n} \log \text{mult}(S^n, x) \\ &\quad + \sup_{\mathbf{x}\in\Gamma_x} \frac{1}{n} H_{\nu_x}\Big(\bigvee_{k=0}^{n-1} (T_{(x_{n-1},z_{n-1})} \circ \dots \circ T_{(x_0,z_0)})^{-1}(\xi)\Big)\,. \end{aligned}$$

where $\Gamma_x \subset \Gamma$ is the set of sequences $((x_0, z_0), (x_1, z_1)\dots)$ with $x_0 = x$. If the maps T_x are non-expanding it follows from the Ruelle-Margulis inequality [KH] that the last term tends to $h_{\nu_x}(T_\mathbf{x}) = 0$ as $n \to \infty$. Moreover this happens μ_X-uniformly and so $h_\mu(T|S) \le H_{\text{mult}}(S) + \epsilon$. Let $\epsilon \to 0$.

3 Proof of the corollaries

Corollaries 1, 2 and 7 are obvious. Corollaries 3, 4 follow from the following section. Corollary 5 is implied by the fact that $H_{\mathrm{mult}}(S) = 0$ if $\dim(X) = 1$ and Corollary 8 follows from our Theorem 4 and Theorem 3 of Buzzi [B1].

3.1 Estimate for the angles expansion rates

Define

$$\lambda^+_{[i]}(f) = \overline{\lim_{n\to\infty}} \sup_{x\in U_n} \max_{0\le k\le i} \frac{1}{n} \log \|\Lambda^k d_x f^n\|.$$

Obviously $\lambda^+_{[i]}(f) \le i\lambda^+_{[1]}(f) = i \cdot \lambda_{\max}(f)$.

Theorem 7. *The following estimate holds:* $\rho_i(f) \le \lambda^+_{[i]}(f) - i\lambda_{\min}(f)$.

Proof. Let us first calculate the differential of the spherical transformation ${}^{(s)}\!A : S^{d-1} \to S^{d-1}$, ${}^{(s)}\!A(x) = \dfrac{Ax}{\|Ax\|}$, corresponding to $A : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 8. *If* $A \in \mathrm{GL}(\mathbb{R}^d)$, *then* $d\big({}^{(s)}\!Ax\big)(v) = P^\perp_w\left(\dfrac{Av}{\|Ax\|}\right)$, *where* $P^\perp_w$ *is the orthogonal projection along* $w = {}^{(s)}\!A(x)$.

In fact, $d\|Ax\| = \langle {}^{(s)}\!Ax, d(Ax)\rangle$ and so

$$d\big({}^{(s)}\!Ax\big)(v) = \frac{Av}{\|Ax\|} - {}^{(s)}\!A(x) \cdot \left\langle {}^{(s)}\!A(x), \frac{Av}{\|Ax\|} \right\rangle.$$

From this lemma we get: $\|d\big({}^{(s)}\!Ax\big)(v)\| \le \dfrac{\|Av\|}{\|Ax\|} \le C \cdot \dfrac{\max|\mathrm{Sp}(A)|}{\min|\mathrm{Sp}(A)|}$ for $\|x\| = \|v\| = 1$, where the constant C depends on the eigenbasis of A (in non semi-simple case – normal basis) only. Since this eigenbasis is the same for all iterates of A and we have a finite number of pieces in f, we get for all v: $\|d_v d^{(s)}_x f^n\| \le C_n \cdot \dfrac{\max|\mathrm{Sp}(d_x f^n)|}{\min|\mathrm{Sp}(d_x f^n)|}$ (with sub-exponentially growing C_n) and so the maximal vertical (spherical) Lyapunov exponent for the map $d^{(s)}f$ at the point $(x, v) \in STX$ does not exceed the difference $\overline{\chi}_{\max}(x) - \underline{\chi}_{\min}(x)$.

Similarly, we have: $\|d\big({}^{(s)}\!Ax\big)\| \le \dfrac{\|\Lambda^k A\|}{\|Ax\|^k}$ and

$$\|\Lambda^k d_v d^{(s)}_x f^n\| \le C_n \cdot \frac{\max\{|\lambda_1 \cdots \lambda_k| : \lambda_j \in \mathrm{Sp}(d_x f^n), \lambda_i \ne \lambda_j\}}{\min|\mathrm{Sp}(d_x f^n)|^k}$$

with $\overline{\lim\limits_{n\to\infty}} \frac{1}{n} \log C_n = 0$ ($\lambda_i \ne \lambda_j$ means that the eigenvalues are different, though in the multiple case they can be equal), whence the claim. □

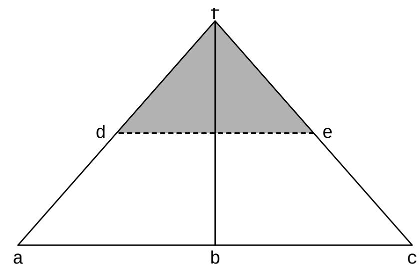

Figure 1: Shows the domains of continuity and their image in Example 1.

Note that $\rho_i(f)$ is conformally invariant in the cocycle sense: The cocycle $\mathcal{A} : X \to GL(\mathbb{R}^n)$, $x \mapsto d_x f$, can be changed by any cocycle $\alpha : X \to \mathbb{R}$, $\mathcal{A} \mapsto \alpha \cdot \mathcal{A}$. Then the Lyapunov-type characteristics $\rho_i(f)$ do not change.

However if the cocycle α has different upper and lower Lyapunov exponents, then the quantity $\lambda_{\max}(f) - \lambda_{\min}(f)$ used in the bound is not invariant.

3.2 Proof of Corollary 6

It was shown by Buzzi [B1] that $H_{\text{mult}}(A) = 0$. This was proven for strictly expanding maps, but the proof extends literally for any non-degenerate A. Thus the bound from above follows from Theorem 1.

Let's prove that $h_{\text{top}}(A) \ge \text{Jac}^+ A$. Assume A is semi-simple. Let λ be an eigenvalue with $|\lambda| > 1$ and v the corresponding unit eigenvector. We divide $[0,1]^d$ into domains $k\epsilon/|\lambda|^n \le \langle x, v\rangle \le (k+1)\epsilon/|\lambda|^n$. A (d_n^f, ϵ)-ball intersects no more than two such domains, the total number of which is less than $\sqrt{d}\cdot|\lambda|^n/2\epsilon$.

The same holds for other $\lambda \in \text{Sp}(A)$, so the number of (d_n^f, ϵ)-balls to cover $[0,1]^d$ is at least $C_0(\sqrt{d}/2\epsilon)^m(\text{Jac}^+ A)^n$, where m is the number of eigenvalues with absolute value greater than 1 and C_0 some n-independent constant.

If $A \in \text{GL}_d(\mathbb{R})$ is not semi-simple, the estimates change sub-exponentially, implying the same result. Note that the formula of the theorem holds true even in the case, when A is degenerate, though arguments should be modified.

4 Examples

Example 1: Let X be a triangle with vertices in $a = (-1, 0)$, $c = (1, 0)$ and $f = (0, 1)$. Divide this triangle in two by taking X_1 to be the left triangle with vertices $a = (-1, 0)$, $b = (0, 0)$ and $f = (0, 1)$. Let X_2 be the right triangle with vertices $b = (0, 0)$, $c = (1, 0)$ and $f = (0, 1)$. Let X be compact, i.e. the sides are contained in X, and let X_1 and X_2 be open. Then $\mathcal{Z} = \{X_1, X_2\}$ is a finite collection of open disjoined polytopes in X, and $X_1 \cup X_2$ is dense in X.

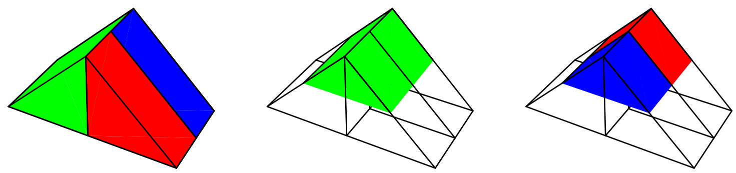

Figure 2: The left figure shows the three domains of continuity for the skew-product $S\tilde{\times}T$ in Example 2. Two right figures show the images of the continuity domains.

We define a map S on $X' = X_1 \cup X_2$ by the formula

$$S(x) = \begin{cases} A_1x + B_1 \text{ if } x \in X_1 \\ A_2x + B_2 \text{ if } x \in X_2 \end{cases} \quad \text{where } A_1 = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix},\ A_2 = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

and $B_1 = (1/2, 1/2)$, $B_2 = (-1/2, 1/2)$. This maps both X_1 and X_2 to the triangle with vertices d, e and f. Observe that $S^n(x)$ tends to the point $(0,1)$ with exponential speed for all $x \in U_f$. This implies that $h_{\text{top}}(S) = 0$. However the multiplicity of $\mathcal{Z}^n$ at the point $f = (0,1)$ is 2^n, whence $H_{\text{mult}}(S) = \log 2$.

It is also easy to see that $H_{\text{sing}}(S) = \log 2$. The eigenvalues of A are $\frac{1}{2}, 1$, so the map is non-expanding and $\lambda^+(f) = 0$. Changing $S(x) \mapsto \frac{1}{2}(S(x) + f)$ we obtain a strictly contracting piecewise affine map with positive H_{sing} and H_{mult}.

The growth of multiplicity is produced by angular expansion: on $ST_fX \simeq S^1$ the spherization is conjugated to the map $\theta \mapsto 2\theta$, whence $\rho(S) = \log 2$.

Example 2: The map in Example 1 can be modified to obtain positive topological entropy. We give here an example of a piecewise affine non-expanding map with positive topological entropy in dimension 3, but it is also possible to construct such an example in dimension 2, see [KR2].

Let $Y = [0,1]$ and let $S : X_1 \cup X_2 \to X$ be as in Example 1. For each $x \in X_1 \cup X_2$ we take a piecewise affine map $T_x \in \text{PAff}(Y,Y)$. For $x \in X_1$ we let $T_x = Id_Y$ and for $x \in X_2$ we let T_x be the interval exchange

$$T_x(y) = \begin{cases} y + 1/2 \ \text{ if } y \in (0, 1/2), \\ y - 1/2 \ \text{ if } y \in (1/2, 1). \end{cases}$$

The map $f = S\tilde{\times}T \in \text{PAff}(X \times Y, X \times Y)$ has three domains of continuity Z_1, Z_2 and Z_3. These domains and the images are shown in Figure 2.

As in Example 1 the cardinality of the continuity partition of S grows like 2^n, but in this example we see that if $x_1, x_2 \in X$ are elements of different continuity domains for S, then $d(f(x_1,y), f(x_2,y)) \geq 1/2$ for all $y \in Y$. Hence the number of (d_n^f, ϵ)-balls needed to cover $X \times Y$ is at least $\mathcal{Z}^n$, where $\mathcal{Z}^n$ is the continuity partition of f^n. This implies $h_{\text{top}}(f) \geq \log 2$. The opposite inequality follows from Theorem 2, whence $h_{\text{top}}(f) = \log 2$.

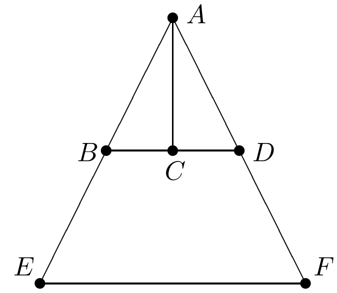

Figure 3: Example of system with zero Lyapunov exponents and positive entropy.

This example has several interesting aspects. First of all it is an example of a non-expanding piecewise affine map with positive entropy. We can easily modify f to make it strictly contracting without changing its topological entropy.

A second important point is that $h_{\text{top}}(S) = h_{\text{top}}(T_x) = 0$ for all $x \in X'$, so the the entropy of the skew product $S\tilde{\times}T$ exceeds the combined entropy of its factors. This shows that the term $H_{\text{mult}}(S)$ on the right hand side of the inequality in Theorem 3 cannot be removed. Also this justifies Remark 4.

Example 3: Let us consider a system with continuity domains as in Figure 4. The dynamics f is piece-wise affine and is given by the rules:

$$ABC \longrightarrow AEF, \quad ADC \longrightarrow AFE, \quad BDFE \longrightarrow BDFE,$$

the last map being the identity. Since every point eventually comes into the domain $BDFE$, all the Lyapunov exponents vanish. But the multiplicity of the point A is $\log 2$ and this easily yields $h_{\text{top}}(f) = \log 2$.

This shows that in the estimates of the main theorems we cannot change the difference $\lambda_{\max}(f) - \lambda_{\min}(f)$ to the maximal difference of upper and lower Lyapunov exponents $\sup_{x\in U_f} \max_{i,j}(\overline{\chi}_i(x) - \underline{\chi}_j(x))$. However we suggest that the estimate in Theorem 1 can be refined by changing $\lambda^+(f)$ to the maximal sum of positive upper Lyapunov exponents $\sup_{x\in U_f} \sum \overline{\chi}_i^+(x) \le \lambda^+(f)$.

Example 4: Consider the following map f of T^2 to itself. We represent the torus as a glued square, which is partitioned into countable number of rectangles $\Pi_i = I_i \times [0,1]$. We define $f(x,y) = (x+\alpha, y+\beta_i)$ if $x \in I_i$. Thus the map is a piece-wise isometry with countable number of continuity domains Π_i.

The number α is chosen irrational. The intervals I_i (with their lengths $l_i = |I_i|$) and the shift lengths β_i are supposed to be sufficiently generic. The value of $h_{\text{top}}(f)$ (note that singularity entropy $H_{\text{sing}}(f)$ does not have sense here) depends on the speed of convergence of the series $\sum_{i=1}^\infty l_i$.

Consider, for instance, the case of rapid convergence, when l_i decrease exponentially or at least polynomially, namely $l_i \le Ci^{-r}$ for some $r > 1$ and $C \in \mathbb{R}_+$. Then the frequency with which an interval of length ϵ meets a singularity of the

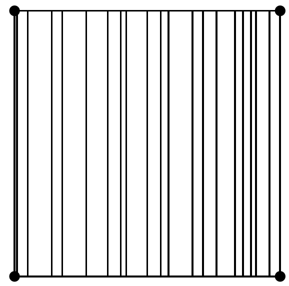

Figure 4: A countable piece-wise isometry consisting of random vertical shifts.

base $\cup\partial I_i \subset S^1$ under rotation by the angle α is at most $p_\epsilon \sim \epsilon^{\frac{r-1}{r}}$ (for exponential convergence $p_\epsilon \sim \epsilon \log \frac{1}{\epsilon}$). The number of (d_n^f, ϵ)-balls to cover the torus satisfies: $S(d_n^f, \epsilon) \le c \cdot (\frac{1}{\epsilon})^{p_\epsilon \cdot (n+2)}$. Consequently $h_{\rm top}(f) = 0$.

On the other hand, if the series $\sum_{i=1}^{\infty} l_i$ converges slowly, then the entropy may become positive and even infinite. For example, if $l_i \sim 1/i \log i (\log\log i)^2$, then the frequency with which an ϵ-interval meets $[1/\epsilon]$ different intervals I_i under rotation by the angle α has asymptotic $\sigma_\epsilon \sim 1/\log\log\frac{1}{\epsilon}$. Thus choosing the shifts β_i and geometry of the decomposition $I = \cup I_i$ appropriately we may arrive to $S(d_n^f, \epsilon) \sim (\frac{1}{\epsilon})^{\sigma_\epsilon \cdot (n+2)}$, which yields $h_{\rm top}(f) = \infty$ (note that f is a skew-product with vanishing entropies of the base and the fibers).

Note that $h_{\rm top}(f) = 0$ for a piecewise isometry f with finite number of continuity domains [B2] and the same holds for conformal non-expanding maps [KR2] (and Corollary 2). For infinite number of domains this fails.

Example 5: In Corollary 4 we stated that if a piecewise affine map f is asymptotically conformal, i.e. $\lambda_{\rm max}(f) = \lambda_{\rm min}(f)$, then $H_{\rm mult}(f) = 0$. This result does not generalize to piecewise smooth maps. This is shown by the following example due to Buzzi [B2]:

Let $X = [-1, 1]^2$ and define $f : X \to X$ by

$$(x, y) \mapsto (x/2, y/2 - \mathrm{sgn}(y)x^2)\,.$$

We observe that

$$\mathrm{Jac}(f) = \begin{bmatrix} 1/2 & 0 \\ -2x\,\mathrm{sgn}(y) & 1/2 - \delta(y)x^2 \end{bmatrix},$$

so $\lambda_{\rm max}(f) = \lambda_{\rm min}(f) = 1/2$. However it is easy to verify that multiplicity of the origin grows like 2^n, whence $H_{\rm mult}(f) = \log 2$.

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