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Perspectives and reflections on teaching linear algebra

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This paper presents 'expert opinions' on what should be taught in a first-year linear algebra course at university; the aim is to gain a generic picture and general guiding principles for such a course. Drawing on a Delphi method, 14 university professors—called 'experts' in this study—addressed the following questions: What should be on a first-year linear algebra undergraduate course for engineering and/or mathematics students? How could such courses be taught? What tools (if any) are essential to these two groups of students? The results of the investigation, these experts' opinions, mainly concern what should be in a linear algebra course (e.g. problem-solving and applications) and what students should be able to do. The experts also emphasized that certain theoretical aspects (e.g. proofs, abstract structures, definitions and relationships) were more important to mathematics students. There was no real consensus among the experts on teaching methods or the use of digital tools, but this lack of consensus is interesting in itself. The results are discussed in relation to extant research.

1. Introduction

Students commonly see linear algebra courses at university level as difficult mathematics courses. The content is often abstract and formal, which may be new to students compared to what they have been used to from previous mathematics courses. This may disconnect linear algebra from students' previously learned mathematical ideas. This is a pity since linear algebra has a unifying power in mathematics and can be useful in fields outside of pure mathematics (Dorier, 1995; Lay *et al.*, 2016). The subject provides power to model real situations, and content areas like engineering and statistics utilize this (Harel, 1989). Courses in linear algebra may take different directions according to the focus to which the content is applied, pure and formal or more applicable. This makes it relevant to ask what a course in linear algebra should be about, what views there are among teachers of such courses, what content is essential to include and whether the answers to these questions vary depending on whether the students are studying mathematics or engineering. This is the motivation for the present investigation. For instance, we do not focus on lists of themes and concepts to be covered in a linear algebra course; SEFI, see Alpers (2013), has done this. Rather, we seek to gain a generic picture and general guiding principles by drawing on the Delphi method (Osborne *et al.*, 2003). This is done through three rounds of questionnaires given to 14 university professors with a variety of backgrounds but all experienced in teaching of linear algebra. In keeping with nomenclature of Delphi studies and due to their experiences as skilled workers, we call them 'experts' in our investigation. These experts offered and graded their views on four different issues about teaching linear algebra by answering online questionnaires anonymously. The first round consisted of open questions about the teaching of linear algebra. Analyses of round 1 response provided results presented in a previous paper (Rensaa *et al.*, 2019). The second questionnaire summarized the round 1 responses and the experts recorded their agreement (grades) about statements made. The third round questionnaire refined these gradings. This three-round Delphi method arguably stimulated experts to reconsider their feedback in the light of opinions given by other members in the panel. Our study was designed to address the following questions:

- What should be on a first-year linear algebra undergraduate course for engineering and/or mathematics students?
- How could such courses be taught?
- What tools (if any) are essential to these two groups of students?

We start by reviewing relevant literature on linear algebra with regard to views on what to teach and how to do this. We then present the methodology including further details of the Delphi method. The results section starts with a short presentation of outcomes from the analysis of data in the first round of the Delphi study. Results from rounds 2 and 3 are given in detail as these provide the foundation of the present paper. Finally, a discussion of the findings is presented followed by a brief conclusion.

2. Literature review

This section is presented in four sub-sections: the content of linear algebra; the teaching of linear algebra; tools (to be used, or not) in the learning/teaching of linear algebra; and engineering students.

2.1. *Content of linear algebra*

Linear algebra represents a process of abstraction where students move from concrete objects and methods in familiar vector spaces like $\mathbb{R}^2$ and $\mathbb{R}^3$ to generalizations at an abstract level. This is cognitively challenging, and linear algebra has a reputation of being considered difficult by many students (Dorier & Sierpinska, 2001). A duality exists in linear algebra courses, concrete processes such as calculating a determinant or doing Gaussian row eliminations on one hand and lots of definitions and the use of formalism on the other hand; this duality may conflict with students' expectation from prior mathematics courses. Carlson (1993), a highly cited paper, reflects on this duality as does Rensaa *et al.* (2019). In the latter paper, a code 'LA-split' embraces a variety of splits/dualities in a linear algebra course such as 'algebraic-geometric' and 'concepts-techniques'. From a didactical point of view, concentrating on step-by-step instructions and manipulations of mathematical objects may be linked to a procedural approach, leading to procedural knowledge with sequential relationships. This is in contrast to conceptual knowledge that is rich in relationships as pieces of knowledge are connected together. The notions stem back to Hiebert & Lefevre (1986) who defined procedural knowledge in mathematics to include familiarity with symbols and representation systems but also knowledge of rules and procedures. Conceptual knowledge on the other hand puts emphasis on connecting together pieces of knowledge in a conceptual network. Interpretations of the meaning of these constructs are debated among teachers (Rensaa & Vos, 2017), but research mainly focuses on procedural and conceptual learning and thinking among students rather than on teaching (Crooks & Alibali, 2014). Engelbrecht *et al.* (2009) adjusted the definition to better fit with courses for engineering students' and, by their definition, procedural approaches are about use and manipulations of mathematical skills while conceptual approaches are more about interpreting and applying concepts to mathematical situations, translating between different mathematical expressions and linking relationships. A challenge for course design lies in the fact that examination problems are traditionally orientated towards procedures. This may lead to students not seeing the need to learn proofs and formal theory; indeed, students may not realize why proofs are needed. Formalism itself and understanding the use of formalism in theory is difficult; Dorier *et al.* (2000) call this the 'obstacle of formalism'.

Despite students' struggle with abstraction, linear algebra is important because it introduces students to such thinking (Harel, 1989), axiomatic algebraic structures in the discipline being the most fundamental ones. Along with this comes a wide range of applications, for instance dealing with transportation problems with cost-effective routes and air flows over particular surfaces (Harel, 1989). Thus, linear algebra content is many-faced with concrete, applied and abstract parts.

2.2. *Teaching of linear algebra*

With regard to teaching mathematics, various teaching experiments are taking place to meet the obstacles that students encounter in learning linear algebra due to the discipline's abstraction level. Harel (2017) gives an example of such a teaching experiment. Students in this study solved mathematical problems in small groups, then discussed their solutions in a whole-class setting and identified the relevance of the linear algebra content. In such a setting, students' problem solving skills are important, which is also emphasized by researchers as vital (Liljedahl *et al.*, 2016). It is frequently rephrased as the ability to approach and solve non-trivial mathematical problems and stems back to Polya (1957). In an applied problem solving process, the obtained model needs to be dealt with mathematically to obtain mathematical results before re-translating into the real world. Thus, problem solving covers a diversity of activities that students may engage with to solve mathematical problems. This is illuminated by themes in the topic study group 'Problem solving in mathematics' at ICME 13 (Liljedahl *et al.*, 2016). This covers the role of heuristics and Polya's dictum 'to study the methods and rules of discovery and inventions' (Polya, 1957, p.112) as well as creative aspects of problem solving using digital technologies and problem posing. The split between concrete processes in dealing with linear algebra concepts on one hand and the dealing with abstraction and use of formalism on the other—an LA-split—provides an addendum to problem solving activities in linear algebra. Solving non-trivial problems may be done by striving to find techniques that may 'do the job' or by trying to envisage the problems as part of a general theory, two strategies with rather different approaches.

A more recent teaching technique that is now used in the teaching of linear algebra is flipped classroom arrangements. Love *et al.* (2014) report on research in which a group of students in a linear algebra course were split in two sub-groups, using traditional lecturing format in one group and a flipped format in the other. Their conclusion is that flipped classroom teaching is promising as students in this group performed better on exams than those following a traditional course in linear algebra.

2.3. *Tools*

Students' difficulties with abstractions required in linear algebra courses may be eased by the use of a variety of tools to illuminate meanings. Such tools come in different forms such as textbooks, notes or digital technology. Due to advances in technologies, a wide range of activities using computers has also been introduced to linear algebra classrooms. Dogan (2018) studies the effect of 'dynamic visual modalities' by analysing data from seven guided investigative assignments from three groups of students who used different tools. The first group made use of dynamic pictorial representations both in take-home assignments and lectures, the second group did this only in take-home assignments, while the third group were only exposed to static geometric modes, which was only used during lectures. By drawing on a framework by Sierpinska on students' thinking modes, Dogan (2018) concludes that both the first and second groups made sense of abstract and challenging ideas in linear algebra by using the geometry-based knowledge they had gained. This was in contrast to the third group, where students drew on numeric-based ideas to make sense of both abstract and geometric concepts. However, as noted by Stewart *et al.* (2005), students' mastery of mathematical software can be time-consuming.

Harel (1989) argues that students can deal with the level of abstraction required for linear algebra courses if relations between different representations are established, and Dogan (2018) points to benefits of dynamic geometric tasks in such structuring. Dogan (2018) refers to studies in which students have had difficulties with geometric representations but also refers to other investigations that document positive effects on students' cognition in using dynamic geometry software.

2.4. Engineering students

A great deal of the research on the teaching and learning of linear algebra does not distinguish between the major studies of the students but deals with students in general. Research about linear algebra for engineering students is relatively sparse in comparison. We comment on this sparse domain, which concerns linear algebra, teaching and the use of digital tools.

With regard to the content of linear algebra, Britton & Henderson (2009) draw on Dorier *et al.* (2000) with regard to engineering students' conceptual difficulties in a linear algebra course. Britton & Henderson (2009) examines about 500 students' responses to two tasks involving proofs concerning subspaces, one of the tasks within a vector space of functions. Their result show that the engineering students had severe difficulties in making a conscious shift from regarding a function f, which they view via its formula $f(x)$ and its graph $y = f(x)$, as an element in a vector space. This is an example of a task involving abstraction, and the investigation illustrates engineering students' problems in dealing with such aspects.

Turning attention to tool use, our first comment is that tool use in linear algebra courses is related to wider issues including career aspirations and expectations; this is reflected in studies that comment on tool use. Flegg *et al.* (2012) considers undergraduate mathematics in general but engineering students in particular. The paper points to the increased focus on use of data analysis tools and software packages among practising engineers and that this should be displayed in the mathematics courses. Harris *et al.* (2015) posit that students' obstacles in the learning of mathematics are due to the missing link between mathematics and engineering. Both of these studies have implications for the use of tools.

3. Methodology

A Delphi study is an established research method in which knowledgeable participants, generally referred to as 'experts', give their views anonymously on a specific matter of considered importance (in this case on teaching of linear algebra) using a sequence of questionnaires. The experts in our study were 14[1]

1 Ten is considered the minimum and 30 the maximum number of participants in Delphi studies.

TABLE 1. *Rubric presented to participants in round 2 and our numeric code*

Possible response	Meaning of response	Likert scale
YES	I strongly agree	5
yes	I agree	4
?	I am unsure	3
No	I disagree	2
NO	I strongly disagree	1

professors chosen so that, between them, they specialized in mathematics or mathematics education and had experience of teaching engineering or mathematics students. For further details on the background of the experts and the Delphi method, see (Osborne *et al.*, 2003; Rensaa *et al.*, 2019).

Our Delphi study operated in three rounds. The first round was an open-ended online questionnaire with opportunity to provide textual answers. The questionnaire had five questions:

Q1 What is important to teach in a first course in linear algebra?

Q2 Are there methods of teaching that are particularly suited or not suited to linear algebra?

Q3 Are there specific tools (techniques, software, etc.) that should or should not be used in the study of linear algebra?

Q4 Do any of your answers to (1) to (3) vary according to whether the students are studying engineering or mathematics? If so, how?

Q5 Do you have any further comments?[2]

The responses were analysed using thematic analysis (Braun & Clarke, 2006). The themes resulting from this analysis were used as the basis for the round 2 questionnaire. The round 2 questionnaire presented 36 statements under six broad categories, and participants were asked to respond to these using a 5-point Likert scale. Table 1. displays the rubric presented to participants in round 2 and our numeric code. The responses to round 2 were analysed using descriptive statistics and means.[3]

The round 3 questionnaire re-presented the 16 round 2 responses with means $\geq$4; this was a convenient cut off number with the advantage of minimizing possible participant boredom and presenting only statements ranked 'yes' or 'YES' in round 2, thus opening up the possibility of greater differentiation between support for statements in round 3. Therefore, the questionnaire in round 3 used a 6-point scale: Strongly disagree (NO!), Disagree (no), Unsure (?), Agree (yes), Strongly agree (YES!), 100%. Numeric codes 1 to 6 were assigned to these responses and, again, means were calculated. The purpose of the extra response was to further differentiate 'strongly agree' responses.

The three rounds of questionnaires were sent out in May, September and November 2018. All 14 experts responded to each round of questionnaires.

4. Results

We present the results from the three rounds of this Delphi study. We only summarize the results for round 1 as these have been reported in a separate publication (Rensaa *et al.*, 2019).

2 This question did not produce further comments.

3 Strictly speaking, we should have used medians for ordinal data but this is common practice.

4.1. *Round* 1

The results of a thematic analysis of round 1 textual data produced 11 themes that we present below under the questions from which they arose.

Q1 What is important to teach in a first course in linear algebra?

- Linear algebra as a discipline
- Aspects of doing linear algebra
- Problem solving, modelling and applications
- Pedagogical issues and conditions

Q2 Are there methods of teaching that are particularly suited or not suited to linear algebra?

- Teaching differences regarding linear algebra content
- Teaching differences informed by aims and ways of working

Q3 Are there specific tools (techniques, software, etc.) that should or should not be used in the study of linear algebra?

- Specific tools to use
- The purpose of using digital technology

Q4 Do any of your answers to (1) to (3) vary according to whether the students are studying Engineering or Mathematics? If so, how?

- Linear algebra content differences between engineering students and mathematics students
- Teaching differences between engineering students and mathematics students
- Differences between types of students

It will probably come as no surprise to the reader that there are interrelations between the themes. For example, several experts commented that it is more important to give formal definitions of mathematical objects to mathematics students than it is to engineering students; this clearly links to linear algebra as a discipline and to both content and teaching differences between engineering students and mathematics students.

4.2. *Round* 2

The 11 themes isolated in round 1 were organized into six categories in the round 2 questionnaire. Each category was followed by six statements to which respondents were to apply a Likert code. The six categories are listed below. The 36 statements can be seen in the first column of Table 2.

- Linear algebra as a discipline
- Aspects of doing linear algebra
- Problem solving, modelling and applications
- The teaching of linear algebra
- Differences between students
- Tools (their place in teaching and learning linear algebra)

TABLE 2. *Summary of results from the round 2 questionnaire with notes on results for round 3*

Themes and statements	R2 Mean	R3 Mean[1]
Linear algebra as a discipline		
It is important to start a linear algebra course with teaching of vectors	3	
It is important that mathematics students focus more on proofs than engineering students	4.5	(5.29)
It is important that mathematics students are familiar with more abstract structures than engineering students	4.6	(5.21)
Applications of linear algebra are more important for engineering students than for mathematics students	3.2	
Mathematics is a tool for solving problems for engineering students rather than for mathematics students	3.1	
Students should be encouraged to appreciate the importance of formal definitions	3.9	
Aspects of doing linear algebra		
Knowing techniques and how to operate these are essential parts of learning linear algebra	4	(4.43)
Students are expected to reason on a general level in linear algebra, producing arguments that actually constitutes a proof of a statement	3.5	
A geometric understanding in visual dimensions (R^2 and R^3) is important before generalising	4.1	(5.07)
Students need to possess analytic skills in order to be able to interpret results	4.1	
It is important for students to know that topics in linear algebra can be represented in different formats like geometric, tabular, graphical etc.	4.2	(4.8)
Engineering students should concentrate on gaining mastery of techniques and exploring applications while mathematics students should concentrate on solving more abstract exercises	2.9	
Problem solving, modelling and applications		
It is important that students experience problem solving in a linear algebra course	4.6	(5.36)
It is important that students experience mathematical modelling in a linear algebra course	4	(4.36)
It is important that students encounter applications of linear algebra outside of pure mathematics in a linear algebra course	4	(4.86)
Applications and modelling in a linear algebra course are more important for engineering students than they are for mathematics students	3.3	
Students should be able to extract information from a text, formulate the problem in mathematical terms and solve the problem with techniques based on linear algebra	4.4	(5.14)
Students should have experience working with vector spaces other than R^n or C^n	3.3	
The teaching of linear algebra		
Traditional teaching of formal definitions and proofs is important	3.6	
The use of ICT should be a natural and incorporated part of teaching	3.9	
A flipped classroom approach with short video explanations of mathematical objects are suited for a linear algebra course	3.4	
For mathematics students it is more important to give formal definitions of mathematical objects than for engineering students	4	(4.71)
Teaching linear algebra to engineering students should involve more applications than for mathematics students	3.9	
It is more important to teach how to abstract and generalize to mathematics students than to engineering students	4.1	(4.57)

Continued

TABLE 2. *Continued*

Themes and statements	R2 Mean	R3 Mean[1]
Differences between students		
For mathematics students, linear algebra is fundamental in order to see how general structures work, and be able to abstract and generalize, as basis for their pure mathematics learning. **For mathematics students, appreciating structural properties of general linear vector spaces is fundamental for their development as mathematicians.**	4.5	(4.93)[2]
For engineering students, linear algebra is about mastering techniques and gaining a practical toolbox for solving engineering problems	3.9	
For mathematics students, focus is on concepts and the relations between the concepts, theorems and proofs, as these are important mathematical knowledge for their studies	4.3	(4.21)
For engineering students, focus should be put on applications while the meaning of theorems should be understood and applied but not proved	3.6	
The inhomogeneity of engineering student backgrounds imply that teaching in these programs must start on a lower level than for mathematics students, thus less new content can be covered	3.2	
Master engineering students and mathematics students should have similar linear algebra courses focusing on formal mathematics while bachelor engineering students should have a course more focused on applications	2.9	
Tools (their place in teaching and learning linear algebra)		
It is important that students (do a fair amount of) (**know how to do**) hand calculation in a linear algebra course.	4.1	(4.79)[4]
It is important that a part of a linear algebra course is devoted to using relevant mathematical software to solve problems	4	(4.07)
It is important that a part of a linear algebra course is devoted to using relevant mathematical software to investigate mathematical structure	3.7	
The use of digital technology is more important when teaching engineering, as opposed to mathematics, students	2.2	
It is important that students are able to work with very large matrices	2.8	
Compared to most other mathematics courses, linear algebra is a course where the computers is essential	2.8	

[1]Round 3 used a 6-point scale, whereas round 2 used a 5-point scale. We insert brackets simply to keep the reader attuned to this difference.

[2]The wording in round 3 is different. Round 3 changes noted in bold font. Slight changes in some other questions (e.g. ‘this’ to ‘these’) but these are not, in our opinion, considered worthy of note.

The reason for the reduction from 11 themes to six categories was an attempt to make the round 2 questionnaire user-friendly. We are convinced, however, that the 36 statements cover all 11 themes. For example, the statements under the category ‘Tools’ cover the themes ‘Specific tools to use’ and ‘The purpose of using digital technology’. Every effort was made to emulate the language of the round 1 responses in the phasing of the 36 statements. For example, the statement ‘It is important that students encounter applications of linear algebra outside of pure mathematics in a linear algebra course’ attempts to provide a concise form of the following round 1 response ‘It is important that the textbook contain a variety of applications ... Students should be given the opportunity to see the diversity of simple applications of matrix algebra in Economics, Computer Science, Biology, Engineering etc.’

The second column of Table 2 shows the mean ranking for each statement, which goes from 2.2 to 4.6.

TABLE 3. *Summary of round 3 questionnaire*

	Mean	Rank	Category
Block 1			
4) It is important that students experience problem solving in a linear algebra course	5.36	1	PSMA
2) It is important that mathematics students focus more on proofs than engineering students	5.29	2	LAD
5) It is important that mathematics students are familiar with more abstract structures than engineering students	5.21	3	LAD
14) Students should be able to extract information from a text, formulate the problem in mathematical terms and solve the problem with techniques based on linear algebra	5.14	4	PSMA
3) A geometric understanding (visualizing in R^2 and R^3) is important before generalizing to other spaces	5.07	5	ADLA
Block 2			
12) For mathematics students, appreciating structural properties of general linear vector spaces is fundamental for their development as mathematicians	4.93	6	Diff-Stud
16) It is important that students encounter applications of linear algebra outside of pure mathematics in a linear algebra course	4.86	7	PSMA
13) It is important for students to know that topics in linear algebra can be represented in different formats like geometric, tabular, graphical etc.	4.80	8	ADLA
7) It is important that students know how to do hand calculations in a linear algebra course	4.79	9	Tools
6) It is more important to give formal definitions of mathematical objects to mathematics students than it is to engineering students	4.71	10	TOLA
Block 3			
9) It is more important to teach how to abstract and generalize to mathematics students than to engineering students	4.57	11	TOLA
11) Students need to possess analytic skills (e.g. procedural understanding) in order to be able to interpret results	4.57	12	ADLA
1) Knowing techniques and how to operate these are essential parts of learning linear algebra	4.43	13	ADLA
8) It is important that students experience mathematical modelling in a linear algebra course	4.36	14	PSMA
15) For mathematics students, the focus should be on concepts and the relationship between the concepts and theorems, as this is important in mathematics	4.21	15	Diff-Stud
10) It is important that a part of a linear algebra course is devoted to using relevant mathematical software to solve problems	4.07	16	Tools

4.3. Round 3

As stated above: the round 3 questionnaire re-presented the 16 round 2 statements that had a mean response of 4 or above; round 3 used a 6-point Likert-scale. The round 3 questionnaire deliberately randomized the position of the 16 statements to prevent them appearing in groups as themes/categories. The third column of Table 2 shows the mean for each statement, which goes from 4.07 to 5.36.

In anticipation of the Discussion section below we provide Table 3, to orientate the reader to the ranking (by their means) of round 3 statements and the round 2 categories that these statements appeared under. Numbers in front of the statements show which order the experts got the statements in.

Table 3 displays the 16 round 3 statements (column 1), ranked (column 3) by their means (column 2) with round 2 categories noted in column 4. The abbreviations are as follows: ADLA—aspects of doing linear algebra; Diff-Stud—differences between students; LAD—linear algebra as a discipline; PSMA—problem solving, modelling and applications; TOLA—the teaching of linear algebra; and Tools—tools (their place in teaching and learning linear algebra). The three rows marked 'Block n' are simply separators that divide the 16 statements into three ranked groups of statements. The first block has five statements with means > 5. There are two statements on problem solving, two on differences between engineering and mathematics students and one on the primacy of geometric understanding. The second block has five statements with means just below 5. These five statements span five categories. The block with the lowest means also spans five categories.

5. Discussion

The section reviews the results in the light of the literature review with regard to the three questions the study was designed to address: What should be on a first-year linear algebra undergraduate course: engineering and/or mathematics students? How might such courses be taught? What tools (if any) are essential to these two groups of students?

5.1. What should be on a first-year linear algebra undergraduate course for engineering and/or mathematics students?

We consider expert opinion regarding this question in two parts, for students in general and then for particular types of students.

The categories *problem solving, applications and modelling* and *aspects of doing linear algebra* had the most round 2 statements taken into round 3, four statements from each category. The statement 'It is important that students experience problem solving in a linear algebra course' had the highest ranking of all round 3 statements. A related statement, which also used the word 'problem', had a high round 3 rank (fourth), 'Students should be able to extract information from a text, formulate the problem in mathematical terms and solve the problem with techniques based on linear algebra'. Both statements deal with problem solving, one emphasizing its importance, the other how to accomplish solutions. The two statements using the words 'applications' and 'modelling' had lower round 3 ranks: 'It is important that students encounter applications of linear algebra outside of pure mathematics in a linear algebra course' (ranked seventh); 'It is important that students experience mathematical modelling in a linear algebra course' (ranked 14th). It appears that there is a difference between 'problem solving' and 'applications'/'modelling' in the views of our experts as the former is ranked much higher than the latter. We do not have data to address this apparent disparity, but we conjecture factors involved in the mix. Problem solving can be considered as a wide-ranging activity embracing applications and modelling (Liljedahl *et al.*, 2016). As such, problem solving is relevant both for engineering and mathematics, which is also mirrored in the experts' responses. The lower rankings of 'applications' and 'modelling' may indicate that the experts either share the opinion about problem solving having wider aspects and more relevance to students than the more restricted applications/modelling activities have, or they are more familiar with problem solving as a term.

With regard to *aspects of doing linear algebra*, representations in linear algebra (and geometric representations in particular) are considered important. The statement 'A geometric understanding (visualizing in $\mathbb{R}^2$ and $\mathbb{R}^3$) is important before generalizing to other spaces' had a high round 3 rank (fifth) and the statement 'It is important for students to know that topics in linear algebra can be represented in different formats like geometric, tabular, graphical etc.' had a mid-level round 3 rank (eighth). The other two ADLA round 3 statements, 'Students need to possess analytic skills in order to be able to interpret results' and 'Knowing techniques and how to operate these are essential parts of learning linear algebra', had lower ranks, 11th and 13th, respectively. 'Analytic skills' and 'knowing techniques' may be interpreted with regard to the long-standing debate on conceptual versus procedural knowledge (Hiebert & Lefevre, 1986) also in an engineering setting (Engelbrecht *et al.*, 2009), though there are those who question the meanings of these constructs (Crooks & Alibali, 2014; Rensaa & Vos, 2017). The experts' close ranking between the analytic skills and the techniques statements may indicate that experts still see the need to combine both procedural and conceptual approaches in teaching and that students need practice in both to be able to succeed in a linear algebra course.

The rankings of the round 3 statements, 'It is important that mathematics students focus more on proofs than engineering students' (ranked second) and 'It is important that mathematics students are familiar with more abstract structures than engineering students' (ranked third), provide evidence that proof and familiarity with abstract structure are considered, by experts, as particularly important for mathematics students. The round 3 statement, 'For mathematics students, the focus should be on concepts and the relationship between the concepts and theorems, as this is important in mathematics' expresses a similar opinion but had a much lower rank (15th). This difference in rankings appears strange, but we do not have data to explore this difference. The two round 3 statements that compared engineering and mathematics students 'It is more important to give formal definitions of mathematical objects to mathematics students than it is to engineering students' and 'It is more important to teach how to abstract and generalize to mathematics students than to engineering students' had similar ranks (10th and 11th, respectively). The two statements concern the culture of mathematics (definitions and abstraction) and can be viewed as stating that this culture is not so important for engineering students, though the rankings are not high within the round 3 statements. As discussed above, however, experts find both procedural and conceptual knowledge in linear algebra as important for students in general.

What does the literature say about differences between types of students? Answering this question is difficult because the literature on teaching and learning linear algebra primarily attends to students' difficulties with formalism, abstractions and generalizations (Dorier *et al.*, 2000; Dorier & Sierpinska, 2001); different types of students are rarely discussed and undergraduate students are usually dealt with as a generic group. But more recent literature that focuses on engineering students expresses opinions in line with our experts' opinions. Britton & Henderson (2009) argues that engineering students find abstractions difficult, and Harris *et al.* (2015) point out that the relation between mathematics and the career goals of engineers are important—which may be interpreted as putting more focus on applications. This resonance in the opinions and research on engineering students may suggest that, when the focus is on the types of students as opposed to linear algebra per se, linear algebra courses for engineering students should be less abstract and more applicable.

5.2. *How might such courses be taught?*

We preface our discussion here by noting the range of the means for the six categories in round 2, which are, in ascending order: 0.7, 1.3, 1.3, 1.6, 1.6 and 1.9. The category *the teaching of linear algebra* had the lowest range of these means, 0.7 (4.1–3.4). The means were clustered just below 4 but only two statements were taken through to round 3, the two statements considered immediately above with ranks 10 and 11. Although these statements do concern the teaching of linear algebra (teaching definitions and teaching how to abstract and generalize), they are not, *per se*, about 'teaching methods'. There were three round 2 statements explicitly about teaching methods: 'Traditional teaching of formal definitions and proofs is important'; 'The use of ICT should be a natural and incorporated part of teaching'; and 'A flipped classroom approach with short video-explanations of mathematical objects are suited for a linear algebra course'. These statements are in line with research cited in the Literature Review above: rigour and formality, often associated with a traditional teaching format, are important in teaching of engineering students (Alpers, 2013); dynamic geometry software may help students to make sense of the challenging abstract theory (Dogan, 2018); and flipped classroom arrangements in linear algebra may be valuable (Love *et al.*, 2014). But these statements in our investigation did not go through to round 3. The upshot of these considerations, for us, is that methods for teaching linear algebra are not considered unimportant by our experts but that there is no consensus on what methods are important; this may stem from the challenges these experts experienced in teaching linear algebra. When entering a linear algebra course, students' previous mathematical knowledge is often computationally oriented, making the shift to more abstract and formal content difficult and disconnected to previous mathematical ideas. Carlson (1993) describes this as 'the fog' rolling in. Educational efforts are directed to give opportunities for students to engage in theoretical thinking with varying results.

5.3. *What tools (if any) are essential to these two groups of students?*

In round 2, the mean marks for statements in the categories were similar with the exception of *tools*, which was low. The means for statements other than *tools* were 3.72, 3.80, 3.93, 3.82 and 3.73, and the mean for the *tools* statements was 3.27. Indeed, the highest mark in the *tools* category concerned hand tools, 'It is important that students do a fair amount of hand calculation in a linear algebra course' (ranked ninth), suggesting that digital tools are not that important. This may suggest that procedural knowledge in terms of manipulations of mathematical skills by hand is regarded as being valuable by the experts. The result, however, appears to be at odds with round 1 statements: 'The use of computers will free time that can be used to focus on understanding and principles'; 'Programming (MAPLE, MathLab or PYTHON) could be a big part of the engineers' learning but also be part of mathematics students' tools for investigating how structures work'; and 'When students become proficient in the use of, say, MatLab, they can be given 'realistic' problems in linear algebra to address (they do not have to be confined to systems of 3 equations in 3 unknowns). This presents opportunities for group work in problem solving.' We do not have data to address this disparity but, similar to our consideration above, of how could such courses might be taught, it may be the case that the use of digital tools in the teaching of linear algebra is not considered unimportant by our experts, but that there is no consensus on what tools are important. There is literature to support this interpretation. Stewart *et al.* (2005), which focuses on the use of computer algebra systems (CAS) in university mathematics, comment that the use of technology in university courses other than calculus has been slow and add that 'While potential is one thing, finding a route to CAS benefits often provides to be quite another' (Stewart *et al.*, 2005, p.741). In summary, there is evidence that no consensus exists on the use of digital tools in teaching linear algebra.

6. Conclusion

To sum up our experts' preferences, the data provide a picture of a linear algebra course with a focus on problem solving within a more traditional teaching format and with both procedural and conceptual parts but with more focus on abstractions and generalizations for mathematics students than for engineering students. There is no consensus in statements on how to teach such a course, but it is noteworthy that hand calculations are ranked 9 while use of software is the least ranked statement, number 16. The results are based solely on the experts' opinions as expressed in this Delphi method (Osborne *et al.*, 2003). Some of the results resonate with extant research, as highlighted in the Discussion section, while other results point to areas for which further research is needed.

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