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Integral operators commuting with dilations and rotations in generalized Morrey-type spaces

Natasha Samko

UiT The Arctic University of Norway, Narvik, 8514, Norway

Correspondence
Natasha Samko UiT The Arctic University of Norway, 8514 Narvik, Norway.
Email: natasha.g.samko@uit.no

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We find conditions for the boundedness of integral operators K commuting with dilations and rotations in a local generalized Morrey space. We also show that under the same conditions, these operators preserve the subspace of such Morrey space, known as vanishing Morrey space. We also give necessary conditions for the boundedness when the kernel is non-negative. In the case of classical Morrey spaces, the obtained sufficient and necessary conditions coincide with each other. In the one-dimensional case, we also obtain similar results for global Morrey spaces. In the case of radial kernels, we also obtain stronger estimates of Kf via spherical means of f. We demonstrate the efficiency of the obtained conditions for a variety of examples such as weighted Hardy operators, weighted Hilbert operator, their multidimensional versions, and others.

KEYWORDS

dilation invariant operators, generalized Morrey spaces, Hardy operators, Hilbert operator, operators with homogeneous kernels, rotation invariant operators, Matuszewska-Orlicz indices, vanishing generalized Morrey spaces

MSC CLASSIFICATION

46E30; 46E35

1 | INTRODUCTION

Within the frameworks of Morrey-type spaces, we study integral operators on $\mathbb{R}^n$ commuting with dilations and rotations. Commutation with dilation means that the kernel of the operator is homogeneous of degree $-n$.

In the one-dimensional case, the boundedness of operators with kernel homogeneous of degree -1 in Lebesgue spaces $L^p(\mathbb{R}_+)$ is well known: It goes back to various papers in the beginning of the previous century (G.H.Hardy, I.Schur, and others) and is reflected in the book.[1] In the multidimensional case, sufficient condition for the boundedness in Lebesgue spaces $L^p(\mathbb{R}^n)$ for operators, which commute with dilations and rotations, was obtained in Mikhailov.[2] Its necessity in the case of non-negative kernels was proved in Karapetiants.[3] We also refer to the presentation of the L^p-results for operators with homogeneous kernel in the overview[4] and the book.[5] A big variety of examples covered by this approach via operators with homogeneous kernels in the space L^p was presented in the survey paper.[6]

We study the integral operators

$$Kf(x) = \int_{\mathbb{R}^n} k(x, y)f(y)dy, \quad x \in \mathbb{R}^n, n \geq 2, \tag{1.1}$$

which commute with dilation and rotation:

$$K\Pi_t = \Pi_t K \quad \text{and} \quad \mathcal{R}_\omega K = K\mathcal{R}_\omega,$$

where

$$\Pi_t f(x) := f(tx), \quad t > 0, \quad \text{and} \quad \mathcal{R}_\omega f(x) := f[\omega(x)], \quad x \in \mathbb{R}^n,$$

and $\omega : \mathbb{R}^n \to \mathbb{R}^n, |\omega(x)| = |x|$ is a rotation in $\mathbb{R}^n$.

In terms of the kernel $k(x, y)$, such a commutation is equivalent to the homogeneity condition:

$$k(tx, ty) = t^{-n}k(x, y), t > 0 \tag{1.2}$$

and rotation invariance

$$k[\omega(x), \omega(y)] = k(x, y). \tag{1.3}$$

Any kernel $k(x, y)$ which depends only on $|x|$, $|y|$ and $x \cdot y$ satisfies the assumption (1.3).

In the one-dimensional case, we consider the following version of such operators with $k(x, y)$ homogeneous of degree-1

$$Kf(x) = \int_0^\infty k(x, y)f(y)dy, \quad x \in \mathbb{R}_+, \quad \mathbb{R}_+ := (0, +\infty), \tag{1.4}$$

known also as Mellin convolution operator.

Similarly, multidimensional operators (1.1) may be considered as Mellin-Fourier-Laplace convolutions, with the corresponding Fourier analysis with respect to the groups of dilations and rotations.

In the sequel, we also impose some integrability conditions on the kernel $k(x, y)$, see Theorems 4.2, 5.1, 5.3 and definition of constants, for instance, in (3.13) to (3.16). These conditions exclude Calderón-Zygmund singular operators. Typical examples of operators K under consideration are

1. Multidimensional Hardy- and Hilbert-type operators

$$H^\alpha f(x) = |x|^{\alpha-n} \int\limits_{|y|<|x|} \frac{f(y)}{|y|^\alpha} dy, \quad \mathcal{H}^\beta f(x) = |x|^\beta \int\limits_{|y|>|x|} \frac{f(y)}{|y|^{n+\beta}} dy, \quad \text{and}$$
$$\mathbb{H}^\gamma f(x) = |x|^\gamma \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x|^n + |y|^n} \frac{dy}{|y|^\gamma}; \tag{1.5}$$

2. Multidimensional versions

$$f \mapsto \frac{1}{|x|^\alpha} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \tag{1.6}$$

of the Hardy-Littlewood fractional operator;
hybrids of the above operators, eg,

$$f \mapsto \frac{1}{|x|^\alpha} \int\limits_{|y|<|x|} \frac{f(y)}{y^\beta |x-y|^{n-\alpha-\beta}} dy \quad \text{and} \quad f \mapsto \frac{1}{|x|^\alpha} \int\limits_{|y|>|x|} \frac{f(y)}{y^\beta |x-y|^{n-\alpha-\beta}} dy, \tag{1.7}$$

$\alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}, \quad \alpha + \beta \leq n$, etc.

Within the frameworks of generalized Morrey and vanishing generalized Morrey spaces, we study the boundedness of operators K.

Such spaces are known to be often used in partial differential equations, since Morrey spaces describe local regularity of solutions more precisely than Lebesgue spaces, and last decades they became also widely popular in harmonic analysis. For Morrey-type spaces and operators of harmonic analysis in these spaces, we refer to the books,[7-10] overviews,[11] and [12] and to the papers, for instance,[13-15] see also the references therein. We also mention recent advances on approximation and interpolation in Morrey spaces in previous studies.[16,17]

Many operators of harmonic analysis—singular-, maximal-, and potential-type operators—and their commutators have been intensively studied in Morrey-type spaces. We refer to the book[9] where a lot of references may be found.

The class of operators under consideration (1.1) in general was not studied in Morrey spaces even in the classical case $\varphi(r) = r^\lambda$ except for some particular cases, for instance Hardy operators, see Persson and Samko.[18] Meanwhile, in the study of weighted singular and potential operators in Morrey spaces, there appeared a necessity in mapping properties in

Morrey-type spaces of some other concrete examples of such operators, namely, operators of form (1.7), see, for instance Persson and Samko.[19]

In the multidimensional case $n > 1$, we deal with local Morrey spaces. In the one-dimensional case, we cover also the case of global Morrey spaces. In all the cases, we find sufficient conditions and necessary conditions for the boundedness together with upper and lower bounds for the norm of the operator K. We pay a special attention to realization of our general theorems for the case of classical Morrey spaces, see corollaries 5.1, 5.2, and 5.5. In this case, the obtained necessary and sufficient conditions coincide, and we also get a precise formula for the norm of the operator K.

We specially consider the case where the kernel is radial, ie,$k(x, y) = \ell(|x|, |y|)$, since this case allows stronger estimates of multidimensional Morrey norms of Kf via one-dimensional Morrey norms of spherical means of f.

The operator (1.4) may be rewritten in the form

$$Kf(x) = \int_0^\infty a(t)f(xt)dt, \quad x \in \mathbb{R}_+, \tag{1.8}$$

where $a(t) = k(1, t)$. In the form (1.8) the operators K are sometimes called Hausdorff operators; more generally, the operators of the form

$$Kf(x) = \int_0^\infty a(t)f(tx)dt, \quad x \in \mathbb{R}^n \tag{1.9}$$

are also referred to as Hausdorff operators. The operators (1.9) have the same nature as the operators (1.4) due to one-dimensional integration: They may be treated as one-dimensional integral operators with a homogeneous kernel of degree -1 of a function f of many variables in the direction $x' = \frac{x}{|x|}$. For Hausdorff operators we refer, for instance, to Liflyand[20] and the references therein, where the main emphasis on the study is with respect to the Hardy space H^1; references to versions of Hausdorff operators with multi-dimensional integration may be also found there. The author thanks Professor E.Liflyand for calling attention to the notion of Hausdorff operators.

We give an application of our general results to a variety of concrete operators, which includes, in particular, weighted Hardy and Hilbert operators. The main attention in these examples is paid to the case of classical Morrey spaces, where the conditions for the boundedness have a form of criterion and the constants are sharp. In the case of generalized Morrey spaces, results of the boundedness in these examples are given in terms of Matuszewska-Orlicz indices.

The paper is organized as follows: In Section 2, we provide all the necessary definitions concerning function spaces under consideration. In Section 3, we provide an estimation of dilation operator in generalized Morrey spaces and prove some facts for generalized Morrey norms of certain minimizing functions we need for obtaining lower bounds of the norm of the operator K. In Section 4, we deal with the operator K in its general form in local generalized Morrey spaces. The main result is given in Theorem 4.2. Its proof is based on a point-wise estimate of local modular given in Theorem 4.1, which itself is of interest. In Section 5, we treat the case of radial kernels. Finally, in Section 6, we give application to some concrete operators.

2 | PRELIMINARIES: LOCAL AND GLOBAL GENERALIZED MORREY SPACES AND THEIR VANISHING VERSIONS

Let $\varphi(r)$ satisfy the following *a priori* assumptions. It is measurable, positive, and nondecreasing on $\mathbb{R}_+$. In this case, the values of $\varphi(r)$ outside neighborhoods $(0, \delta)$ and (N, ∞) of the origin and infinity are irrelevant provided that $0 < \inf\limits_{\delta<r<N} \varphi(r) \leq \sup\limits_{\delta<r<N} \varphi(r) < \infty$. By this reason, without essential loss of generality we assume that φ is continuous on $[0, \infty)$. To define global and local generalized Morrey spaces we introduce the modulars

$$\mathfrak{M}(f, r) := \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(r)} \int\limits_{B(x,r)} |f(y)|^p \, dy \tag{2.1}$$

and

$$\mathfrak{M}_0(f, r) := \frac{1}{\varphi(r)} \int\limits_{B(0,r)} |f(y)|^p \, dy, \tag{2.2}$$

where $1 \leq p < \infty$.

The global and local generalized Morrey spaces $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ and $\mathcal{L}_0{}^{p,\varphi}(\mathbb{R}^n)$ are defined by the norms

$$\|f\|_{\mathcal{L}^{p,\varphi}} = \sup_{r>0} (\mathfrak{M}(f,r))^{\frac{1}{p}} \tag{2.3}$$

and

$$\|f\|_{\mathcal{L}_0^{p,\varphi}} = \sup_{r>0} (\mathfrak{M}_0(f,r))^{\frac{1}{p}}.$$

We refer to the books[8,10,21] for the properties of Morrey-type spaces, see also the surveying papers,[11,12] and references therein.

The case $\varphi(0) \neq 0$ corresponds to Lebesgue spaces.

Besides the above a priori assumptions on the function φ, for different goals we will impose some additional assumptions on φ:

$$\varphi \quad \text{is doubling, ie,} \quad \varphi(2r) \leq c\varphi(r), \tag{2.4}$$

$$\frac{\varphi(r)}{r^n} \quad \text{is almost decreasing on} \quad \mathbb{R}_+, \tag{2.5}$$

$$\text{there exists an} \quad \varepsilon_0 \quad \text{such that} \quad \frac{\varphi(r)}{r^{n-\varepsilon_0}} \quad \text{is almost decreasing on} \quad \mathbb{R}_+ \tag{2.6}$$

and

$$\int_0^r \frac{\varphi(t)}{t} dt \leq c\varphi(r). \tag{2.7}$$

It is obvious that (2.6) implies (2.5) and (2.5) implies (2.4).

Note that the conditions (2.5) and (2.6) will not be used in the case of local Morrey spaces.

The condition (2.6) will be used only when we study necessity of the boundedness conditions in vanishing Morrey spaces, see Lemma 3.6 and Theorem 3.8.

Vanishing global and local generalized Morrey spaces $V\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ and $V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$ are defined as the subspaces of functions in $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ and $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$, which satisfy the condition

$$\lim_{r\to 0} \mathfrak{M}(f,r) = 0, \tag{2.8}$$

and

$$\lim_{r\to 0} \mathfrak{M}_0(f,r) = 0, \tag{2.9}$$

respectively.

All the introduced versions of generalized Morrey spaces on $\mathbb{R}^n$ are similarly defined on $\mathbb{R}_+$ with the replacement of $B(x,r)$ by $B(x,r) \cap \mathbb{R}_+$.

It is obvious that

$$V\mathcal{L}^{p,\varphi}(\mathbb{R}^n) \subset V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n).$$

Vanishing generalized Morrey spaces $V\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ and $V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$ are closed subspaces of generalized Morrey spaces $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ and $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$, respectively.

Recall that $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)\big|_{\varphi\equiv \text{cont}} = L^p(\mathbb{R}^n)$ and $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)\big|_{\varphi\equiv r^n} = L^\infty(\mathbb{R}^n)$.

In the case of classical Morrey spaces, ie, $\varphi(r) = r^\lambda$, we denote

$$L^{p,\lambda}(\mathbb{R}^n) := \mathcal{L}^{p,\varphi}(\mathbb{R}^n)|_{\varphi(r)=r^\lambda},$$

and correspondingly for local Morrey spaces, as well as for vanishing versions of both the spaces.

The Minkowski inequality

$$\left\| \int_{\mathbb{R}^n} F(x,y)dy \right\|_{\mathcal{L}^{p,\varphi}} \leq \int_{\mathbb{R}^n} \|F(\cdot,y)\|_{\mathcal{L}^{p,\varphi}} dy \tag{2.10}$$

holds since it is valid for L^p-spaces and $\sup \int_{\mathbb{R}^n} \dots dy \leq \int_{\mathbb{R}^n} \sup \dots dy$.

3 | AUXILIARY RESULTS

3.1 | On the functions φ^* and φ_*

In Lemma 3.2, we use the function φ^* defined as

$$\varphi^*(s) = \sup_{0<t<\infty} \frac{\varphi(st)}{\varphi(t)}, \quad 0 < s < \infty, \tag{3.1}$$

see Krein et al.[22]

By the definition of $\varphi^*(s)$, we have

$$\varphi(st) \leq \varphi^*(s)\varphi(t) \quad \text{and} \quad \varphi(st) \leq \varphi(s)\varphi^*(t). \tag{3.2}$$

We say that a function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is *semi-multiplicative* if $\varphi(st) \leq \varphi(s)\varphi(t), \quad s, t \in \mathbb{R}_+$ and *quasi-semi-multiplicative* if there exist a constant $C > 0$ such that $\varphi(st) \leq C\varphi(s)\varphi(t) \quad s, t \in \mathbb{R}_+$.

It is obvious that

$$\varphi \quad \text{is quasi-semi-multiplicative} \quad \Leftrightarrow \varphi^* \sim \varphi, \tag{3.3}$$

where $\varphi^* \sim \varphi$ stands for $c_1\varphi(t) \leq \varphi^*(t) \leq c_2\varphi(t), t \in \mathbb{R}_+$.

Besides φ^*, we will also use the function

$$\varphi_*(s) = \inf_{0<t<\infty} \frac{\varphi(st)}{\varphi(t)}, \quad 0 < s < \infty. \tag{3.4}$$

Lemma 3.1. *Let φ be a positive function on $\mathbb{R}_+$. Then*

$$\varphi^{**}(t) \equiv \varphi^*(t) \quad \textit{and} \quad \varphi_{**}(t) \equiv \varphi_*(t), t \in \mathbb{R}_+. \tag{3.5}$$

If φ is semi-multiplicative and $\varphi(1) = 1$, then

$$\varphi^*(t) \equiv \varphi(t), t \in \mathbb{R}_+. \tag{3.6}$$

If φ satisfies the property $\varphi(ts) \geq \varphi(t)\varphi(s)$ and $\varphi(1) = 1$, then $\varphi_*(t) \equiv \varphi(t)$.

Proof. The properties (3.5) and (3.6) should be known but it is easier to prove them, than to find a reference in literature. Since

$$\varphi(t) \leq \varphi(1)\varphi^*(t) \tag{3.7}$$

for any positive function, in particular for $\varphi^*(t)$, we have $\varphi^*(t) \leq \varphi^{**}(t)$. To obtain the inverse inequality, note that $\varphi^*(uv) \leq \varphi^*(u)\varphi^*(v)$, as is known (follows from the definition of φ^* by direct arguments) then the inequality $\varphi^{**}(t) \leq \varphi^*(t)$ becomes obvious.

For (3.6), the inequality $\varphi(t) \leq \varphi^*(t)$ follows from (3.7). To get the inverse inequality, it suffices to use semi-multiplicativity of φ in the definition of φ^*.

The proof for the function φ_* is similar. □

3.2 | On dilation operator in generalized Morrey spaces

Lemma 3.2. *Let $t > 0$. Then*

$$\mathfrak{M}(\Pi_t f, r) = \frac{\varphi(rt)}{\varphi(r)t^n}\mathfrak{M}(f, tr) \tag{3.8}$$

$$\mathfrak{M}_0(\Pi_t f, r) = \frac{\varphi(rt)}{\varphi(r)t^n}\mathfrak{M}_0(f, tr). \tag{3.9}$$

Consequently,

$$\left[\frac{\varphi_*(t)}{t^n}\right]^{\frac{1}{p}} \|f\|_{\mathcal{L}^{p,\varphi}} \leq \|\Pi_t f\|_{\mathcal{L}^{p,\varphi}} \leq \left[\frac{\varphi^*(t)}{t^n}\right]^{\frac{1}{p}} \|f\|_{\mathcal{L}^{p,\varphi}}, \tag{3.10}$$

and in the classical case $\varphi(r) = r^\lambda$

$$\|\Pi_t f\|_{L^{p,\lambda}} = \frac{\|f\|_{L^{p,\lambda}}}{t^{\frac{n-\lambda}{p}}}, \tag{3.11}$$

Proof. It suffices to consider the modular $\mathfrak{M}(\Pi_t f, r)$. We have

$$\mathfrak{M}(\Pi_t f, r) = \frac{1}{\varphi(r)} \sup_x \int\limits_{B(x,r)} |f(ty)|^p \quad dy.$$

$$\mathfrak{M}(\Pi_t f, r) = \frac{1}{t^n \varphi(r)} \sup_x \int\limits_{B(xr,tr)} |f(u)|^p \quad du = \frac{1}{t^n} \frac{\varphi(tr)}{\varphi(r)} \sup_x \frac{1}{\varphi(tr)} \int\limits_{B(xr,tr)} |f(u)|^p \quad du$$

The properties (3.10) and (3.11) obviously follow from what has been proved for the modulars. □

Corollary 3.3. *The dilation operator Π_t preserves the vanishing property (2.8)-(2.9) for every value t at which φ^* is finite. Note that $\varphi^*(t)$ is finite at any point $t > 0$, if φ^* satisfies the doubling condition.*

3.3 | Crucial constants

Recall that the kernel $k(x, y)$ is assumed to satisfy *a priori* assumptions (1.2) and (1.3) of Section 1.

We introduce the following constants:

$$\varkappa^*(p, \varphi) := \int_{\mathbb{R}^n} \left[\frac{\varphi^*(|y|)}{|y|^n}\right]^{\frac{1}{p}} |k(e_1, y)| dy, \tag{3.12}$$

$$\varkappa^+(p, \varphi) := \sup_{r>0} \int_{\mathbb{R}^n} |k(e_1, y)| \left[\frac{\varphi(r|y|)}{\varphi(r)|y|^n}\right]^{\frac{1}{p}} dy, \tag{3.13}$$

$$\varkappa_-(p, \varphi) := \inf_{r>0} \int_{\mathbb{R}^n} |k(e_1, y)| \left[\frac{\varphi(r|y|)}{\varphi(r)|y|^n}\right]^{\frac{1}{p}} dy \tag{3.14}$$

and

$$\varkappa_*(p, \varphi) := \int_{\mathbb{R}^n} \left[\frac{\varphi_*(|y|)}{|y|^n}\right]^{\frac{1}{p}} |k(e_1, y)| dy, \tag{3.15}$$

where $e_1 = (1, 0, \dots, 0)$, so that

$$\varkappa_*(p, \varphi) \le \varkappa_-(p, \varphi) \le \varkappa^+(p, \varphi) \le \varkappa^*(p, \varphi).$$

In the case of classical Morrey spaces, i.e. $\varphi(r) = r^\lambda$, all the above constants coincide with each other and have the form

$$\varkappa_p(\lambda) := \int_{\mathbb{R}^n} |k(e_1, y)| \frac{dy}{|y|^{\frac{n-\lambda}{p}}} = \int_{\mathbb{R}^n} |k(x, e_1)| \frac{dx}{|x|^{\frac{n}{p'} + \frac{\lambda}{p}}}, 0 \le \lambda \le n. \tag{3.16}$$

The coincidence of the integrals in the above line is known in the case $\lambda = 0$, see Karapetiants,[5] page 69, where it was proved in the study of the operators of the form K in Lebesgue space $L^p(\mathbb{R}^n)$. In the case $0 \le \lambda \le n$ this coincidence follows from its validity for $\lambda = 0$ since one may re-denote $\frac{n-\lambda}{p} = \frac{n}{q}$ with $q \in [1, \infty)$.

Remark 3.4. In the one-dimensional case, the integration in formulas of all the constants introduced above should be taken over $\mathbb{R}_+$.

3.4 | On a minimizing function and minimizing sequence

We need the following lemma which in fact is contained in statements of Samko,[23 lemma 3.1] but we give its direct proof since it is obtained under simpler conditions in this case.

Lemma 3.5. *Let $1 \le p < \infty$. If φ satisfies the Zygmund condition (2.7), then*

$$f_0(x) := \frac{\varphi(|x|)^{\frac{1}{p}}}{|x|^n} \in \mathcal{L}_0^{p,\varphi}(\mathbb{R}^n).$$

If besides this φ satisfies the assumption (2.5), then also

$$f_0 \in \mathcal{L}^{p,\varphi}(\mathbb{R}^n).$$

Proof. The case of local Morrey space is easy:

$$\mathfrak{M}_0(f_0, r) = \frac{|S^{n-1}|}{\varphi(r)} \int_0^r \frac{\varphi(t)}{t} dt \le c < \infty, \quad \text{by the Zygmund condition} \quad (2.7).$$

In the case of global Morrey space, we distinguish the cases $|x| \le 2r$ and $|x| \ge 2r$. In the first case $B(x, r) \subset B(0, 3r)$ and then

$$\frac{1}{\varphi(r)} \int_{B(x,r)} |f_0(y)|^p dy \le \frac{1}{\varphi(r)} \int_{B(0,3r)} |f_0(y)|^p dy = \frac{|S^{n-1}|}{\varphi(r)} \int_0^{3r} \frac{\varphi(t)}{t} dt$$

and it remains to refer to the Zygmund condition (2.7) and the property $\varphi(3r) \le c\varphi(r)$, the latter following from the property (2.5).

Let $|x| \ge 2r$. Then $|y| \ge |x| - |x - y| \ge r$, so that by the property (2.5), we have

$$\frac{1}{\varphi(r)} \int_{B(x,r)} \frac{\varphi(|y|)}{|y|^n} dy \le \frac{C}{\varphi(r)} \frac{\varphi(r)}{r^n} \int_{B(x,r)} dy = const.$$

□

To treat the case of vanishing space, we will use the minimizing sequence in the form:

$$f_\varepsilon(x) := f_0(x) v_\varepsilon(x) = \left[\frac{\varphi(|x|)}{|x|^n}\right]^{\frac{1}{p}} v_\varepsilon(x)$$

where $v_\varepsilon(x) = \begin{cases} |x|^{\frac{\varepsilon}{p}}, & |x| \le 1 \\ 1, & |x| \ge 1, \end{cases}$ $\varepsilon > 0.$

Lemma 3.6. *Let $1 \le p < \infty$. Then $f_\varepsilon \in V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$ under the condition (2.7), and $f_\varepsilon \in V\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ under the conditions (2.6) and (2.7).*

Proof. Since $f_\varepsilon(x) \le f_0(x)$, the inclusion $f_\varepsilon \in \mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ follows from Lemma 3.5. So we only have to check the validity of the vanishing property for f_ε. For $\mathfrak{M}_0(f_\varepsilon, r)$, the statement $\mathfrak{M}_0(f_\varepsilon, r) \to 0$ as $r \to 0$ is obvious. For $\mathfrak{M}(f_\varepsilon, r)$, we have

$$\mathfrak{M}(f_\varepsilon, r) \le \frac{1}{\varphi(r)} \sup_{|x|<2r} \int_{B(x,r)} f_\varepsilon(y)^p dy + \frac{1}{\varphi(r)} \sup_{|x|>2r} \int_{B(x,r)} f_\varepsilon(y)^p dy =: M_1(r) + M_2(r).$$

The estimation for $M_1(r)$ is easier since $B(x, r) \subset B(0, 3r)$ when $|x| < 2r$ and then for $r < \frac{1}{3}$ we have that

$$M_1(r) \le \frac{1}{\varphi(r)} \int_{B(0,3r)} \frac{\varphi(|y|)}{|y|^{n-\varepsilon}} dy \le C_\varepsilon r^\varepsilon \frac{\varphi(3r)}{\varphi(r)} \to 0 \quad \text{as} \quad r \to 0,$$

since $\frac{\varphi(3r)}{\varphi(r)} \le c$ by (2.5).

For $M_2(r)$, we have

$$M_2(r) \leq \frac{1}{\varphi(r)} \sup_{|x|>2r} \int\limits_{\substack{|x-y|<r \\ |y|<1}} \frac{\varphi(|y|)}{|y|^{n-\varepsilon}} dy + \frac{1}{\varphi(r)} \sup_{|x|>2r} \int\limits_{\substack{|x-y|<r \\ |y|>1}} \frac{\varphi(|y|)}{|y|^n} dy =: M_2^\varepsilon(r) + \tilde{M}_2.$$

For $M_2^\varepsilon(r)$, note that $\frac{\varphi(|y|)}{|y|^{n-\varepsilon}}$ is almost decreasing by (2.6) if $\varepsilon \leq \varepsilon_0$. Then for such ε, we have

$$M_2^\varepsilon(r) \leq \frac{1}{\varphi(r)} \int\limits_{|y|<r} \frac{\varphi(|y|)}{|y|^{n-\varepsilon}} dy,$$

where we used the fact that $|y| > |x| - |x-y| > r > |x-y|$. Hence,

$$M_2^\varepsilon(r) \leq c \frac{r^\varepsilon}{\varphi(r)} \int\limits_{|y|<r} \frac{\varphi(|y|)}{|y|^n} dy \leq C \frac{r^\varepsilon}{\varphi(r)} \int_0^r \frac{\varphi(\varrho)}{\varrho} d\varrho \to 0 \quad \text{as} \quad r \to 0$$

in view of (2.7).

Let ε_0 be the number from (2.6). We have

$$\tilde{M}_2(r) \leq \frac{c}{\varphi(r)} \frac{\varphi(r)}{r^{n-\varepsilon_0}} |B(x,r)| \to 0 \quad \text{as} \quad r \to 0$$

by (2.6) and the inequality $|y| > r$.

According to the estimation of $M_2^\varepsilon(r)$, the proof is completed for $\varepsilon \leq \varepsilon_0$ and then for all $\varepsilon > 0$ since $f_\varepsilon(x) \leq f_{\varepsilon_0}(x)$ when $\varepsilon > \varepsilon_0$. □

Examples of the functions f_0 and f_ε in Lemmas 3.5 and 3.6 are important in the sequel: they will provide minimizing functions in obtaining necessary conditions for the boundedness of the operator K. In these examples in the one-dimensional case $n = 1$ on $\mathbb{R}_+$, $|x|$ should be replaced by x.

Theorem 3.7. *Let $1 \leq p < \infty$ and $k(x,y) \geq 0$. Then*

$$K f_0(x) \geq \varkappa_-(p,\varphi) f_0(x) \geq \varkappa_*(p,\varphi) f_0(x), \tag{3.17}$$

and

$$K f_0(x) = \varkappa_p(\lambda) f_0(x) \tag{3.18}$$

in the case $\varphi(r) = r^\lambda$.

Proof. We have

$$K f_0(x) = \int\limits_{\mathbb{R}^n} k(x,y) \left[\frac{\varphi(|y|)}{|y|^n} \right]^{\frac{1}{p}} dy.$$

By $\omega_x(y)$, where $x, y \in \mathbb{R}^n$, we denote a rotation in the variable y such that $\omega_x(e_1) = \frac{x}{|x|}$. Then the change of variables $y \to |x| \cdot \omega_x(y)$ and the homogeneity of the kernel and its invariance under rotations yield

$$K f_0(x) = f_0(x) \int\limits_{\mathbb{R}^n} k(e_1,y) \left[\frac{\varphi(|x||y|)}{\varphi(|x|)|y|^n} \right]^{\frac{1}{p}} dy,$$

from which (3.17) and (3.18) follow. □

Theorem 3.8. *Let $1 \leq p < \infty$ and $k(x,y) \geq 0$. Then*

$$K f_\varepsilon(x) \geq \varkappa_-^\varepsilon(p,\varphi) f_\varepsilon(x) \geq \varkappa_*^\varepsilon(p,\varphi) f_\varepsilon(x), \tag{3.19}$$

where

$$\varkappa_-^\varepsilon(p,\varphi)=\inf_{r>0}\left[\int\limits_{|y|<\frac{1}{r}} k(e_1,y)\left(\frac{\varphi(r|y|)}{\varphi(r)|y|^{n-\varepsilon}}\right)^{\frac{1}{p}}dy+\int\limits_{|y|>\frac{1}{r}} k(e_1,y)\left(\frac{\varphi(r|y|)}{\varphi(r)|y|^{n}}\right)^{\frac{1}{p}}dy\right]$$

and

$$\varkappa_*^\varepsilon(p,\varphi)=\int\limits_{|y|<1} k(e_1,y)\left(\frac{\varphi_*(|y|)}{|y|^{n-\varepsilon}}\right)^{\frac{1}{p}}dy+\int\limits_{|y|>1} k(e_1,y)\left(\frac{\varphi_*(|y|)}{|y|^{n}}\right)^{\frac{1}{p}}dy.$$

Proof. We have

$$Kf_\varepsilon(x)=\int\limits_{|y|<1} k(x,y)\left(\frac{\varphi(|y|)}{|y|^{n-\varepsilon}}\right)^{\frac{1}{p}}dy+\int\limits_{|y|>1} k(x,y)\left(\frac{\varphi(|y|)}{|y|^{n}}\right)^{\frac{1}{p}}dy.$$

The familiar change $y\to|x|\omega_x(y)$ of variables yields

$$Kf_\varepsilon(x)=\left(\frac{\varphi(|x|)}{|x|^{n-\varepsilon}}\right)^{\frac{1}{p}}\int\limits_{|y|<\frac{1}{|x|}} k(e_1,y)\left(\frac{\varphi(|x||y|)}{\varphi(|x|)|y|^{n-\varepsilon}}\right)^{\frac{1}{p}}dy$$
$$+\left(\frac{\varphi(|x|)}{|x|^{n}}\right)^{\frac{1}{p}}\int\limits_{|y|>\frac{1}{|x|}} k(e_1,y)\left(\frac{\varphi(|x||y|)}{\varphi(|x|)|y|^{n}}\right)^{\frac{1}{p}}dy,$$

From which (3.19) easily follows by direct arguments. □

4 | MULTIDIMENSIONAL CASE; GENERAL KERNELS

Contrast to the case of Lebesgue spaces, the proof for generalized Morrey spaces is more complicated. It is based on the estimate (4.3) of Theorem 4.1. The proof of Theorem 4.1 requires some special efforts.

The proof of the main Theorem 4.2 on the boundedness of the operator K will be based on the following theorem which itself is of interest since it provides a pointwise estimate of the modular of Kf via that of the function f itself. This is due to Theorem 4.1 that we can cover also the case of vanishing generalized Morrey space. In Theorem 4.1, we use the notation

$$A(t)=\int_{|x|<\frac{1}{t}}\left[\frac{\varphi^*(|x|)}{|x|^n}\right]^{\frac{1}{p'}}k(x,e_1)\quad dx,\quad t>0. \tag{4.1}$$

and

$$\kappa^*(p,\varphi):=\int_{\mathbb{R}^n}\left[\frac{\varphi^*(|x|)}{|x|^n}\right]^{\frac{1}{p'}}\varphi^*\left(\frac{1}{|x|}\right)|k(x,e_1)|dx. \tag{4.2}$$

Theorem 4.1. *Let* $1\le p<\infty$. *If* $\varkappa^+(p,\varphi)<\infty$ *and* $\kappa^*(p,\varphi)<\infty$, *then for* $f\in\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$

$$\mathfrak{M}_0(Kf,r)\le\frac{\varkappa^+(p,\varphi)^{p-1}}{\varphi(r)}\int\limits_0^\infty\varphi(tr)\left|\frac{d}{dt}A(t)\right|\mathfrak{M}_0(f,tr)dt. \tag{4.3}$$

Proof. We may assume that $k(x,y)\ge0,\quad f(y)\ge0$ and represent the operator K in the form

$$Kf(x)=\int_{\mathbb{R}^n}k(x,y)^{\frac{1}{p'}}\left[\frac{\varphi(|y|)}{|y|^n}\right]^{\frac{1}{pp'}}k(x,y)^{\frac{1}{p}}\left[\frac{|y|^n}{\varphi(|y|)}\right]^{\frac{1}{pp'}}f(y)dy.$$

By the Hölder inequality, we obtain

$$Kf(x) \le \left(\int_{\mathbb{R}^n} k(x,y)\left[\frac{\varphi(|y|)}{|y|^n}\right]^{\frac{1}{p}} dy\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} k(x,y)\left[\frac{|y|^n}{\varphi(|y|)}\right]^{\frac{1}{p'}} f(y)^p dy\right)^{\frac{1}{p}}.$$

By the homogeneity and rotation invariance properties of $k(x, y)$ we easily obtain that the first factor on the right hand side is dominated by

$$\frac{1}{|x|^{\frac{n}{pp'}}}\left(\int_{\mathbb{R}^n} k(e_1,y)\left[\frac{\varphi(|x||y|)}{|y|^n}\right]^{\frac{1}{p}} dy\right)^{\frac{1}{p'}}$$
$$= \left(\frac{\varphi(|x|)}{|x|^n}\right)^{\frac{1}{pp'}} \left(\int_{\mathbb{R}^n} k(e_1,y)\left[\frac{\varphi(|x||y|)}{\varphi(|x|)|y|^n}\right]^{\frac{1}{p}} dy\right)^{\frac{1}{p'}} \le \varkappa^+(p,\varphi)^{\frac{1}{p'}}\left(\frac{\varphi(|x|)}{|x|^n}\right)^{\frac{1}{pp'}}.$$

Hence,

$$Kf(x) \le \varkappa^+(p,\varphi)^{\frac{1}{p'}}\left[\frac{\varphi(|x|)}{|x|^n}\right]^{\frac{1}{pp'}} \left(\int_{\mathbb{R}^n} k(x,y)\left[\frac{|y|^n}{\varphi(|y|)}\right]^{\frac{1}{p'}} f(y)^p dy\right)^{\frac{1}{p}}.$$

Then

$$\frac{1}{\varphi(r)}\int_{|y|<r} Kf(y)^p dy \le \frac{\varkappa^+(p,\varphi)^{p-1}}{\varphi(r)}\int_{|x|<r}\left[\frac{\varphi(|x|)}{|x|^n}\right]^{\frac{1}{p'}} dx \int_{\mathbb{R}^n} k(x,y)\left[\frac{|y|^n}{\varphi(|y|)}\right]^{\frac{1}{p'}} f(y)^p dy$$
$$= \frac{\varkappa^+(p,\varphi)^{p-1}}{\varphi(r)}\int_{\mathbb{R}^n}\left[\frac{|y|^n}{\varphi(|y|)}\right]^{\frac{1}{p'}} f(y)^p dy \int_{|x|<r} k(x,y)\left[\frac{\varphi(|x|)}{|x|^n}\right]^{\frac{1}{p'}} dx;$$

whence by dilation and rotation changes of variables, we obtain

$$\mathfrak{M}_0(Kf,r) \le \frac{\varkappa^+(p,\varphi)^{p-1}}{\varphi(r)}\int_{\mathbb{R}^n}\frac{f(y)^p dy}{\varphi(|y|)^{\frac{1}{p'}}}\int_{|x|<\frac{r}{|y|}} k(x,e_1)\left[\frac{\varphi(|x||y|)}{|x|^n}\right]^{\frac{1}{p'}} dx.$$

Making use of the property (3.2) again we obtain

$$\mathfrak{M}_0(Kf,r) \le \frac{\varkappa^+(p,\varphi)^{p-1}}{\varphi(r)}\int_{\mathbb{R}^n} f(y)^p A\left(\frac{|y|}{r}\right) dy = \varkappa^+(p,\varphi)^{p-1}\frac{r^n}{\varphi(r)}\int_{\mathbb{R}^n} f(ry)^p A(|y|)dy,$$

where $A(\cdot)$ is defined in (4.1).

Below we follow a trick from Kato,[24] p132, which allows to arrive at Morrey norm or Morrey modular via integration by parts in the radial variable. Passing to polar coordinates, we have

$$\mathfrak{M}_0(Kf,r) \le \varkappa^+(p,\varphi)^{p-1}\frac{r^n}{\varphi(r)}\int_0^\infty A(\varrho)\varrho^{n-1}d\varrho \int_{\mathbb{S}^{n-1}} f(r\varrho\sigma)^p d\sigma = \frac{\varkappa^+(p,\varphi)^{p-1}}{\varphi(r)}\int_0^\infty A\left(\frac{\varrho}{r}\right) g(\varrho)d\varrho,$$

where we have denoted

$$g(\varrho) = \varrho^{n-1}\int_{\mathbb{S}^{n-1}} f(\varrho\sigma)^p d\sigma. \tag{4.4}$$

Let

$$G(\varrho) = \int_0^\varrho g(t)dt = \int_{|y|<\varrho} f(y)^p\, dy = \varphi(t)\mathfrak{M}_0(f,r). \tag{4.5}$$

Integrating by parts and noticing that the function $A(\varrho)$ is decreasing, we have

$$\mathfrak{M}_0(Kf,r) \leq \varkappa^+(p,\varphi)^{p-1}\left(\frac{1}{\varphi(r)}A\left(\frac{t}{r}\right)G(t)|_{t=0}^{\infty} - \frac{1}{\varphi(r)}\frac{1}{r}\int_0^\infty A'\left(\frac{t}{r}\right)G(t)dt\right)$$
$$= \varkappa^+(p,\varphi)^{p-1}\left(\frac{\varphi(t)}{\varphi(r)}A\left(\frac{t}{r}\right)\mathfrak{M}_0(f,r)|_{t=0}^{\infty} + \frac{1}{\varphi(r)}\int_0^\infty \left|A'(t)\right| G(tr)dt\right).$$

Denote $\tilde{A}(t) = \varphi^*(t)A(t)$. Then

$$\frac{\varphi(t)}{\varphi(r)}A\left(\frac{t}{r}\right) = C(r,t)\tilde{A}\left(\frac{t}{r}\right),$$

where the fraction $C(r,t) = \frac{\varphi(t)}{\varphi(r)\varphi^*(\frac{t}{r})}$ is bounded (less then 1 by (3.2)). Since $\varphi^*(t)$ is nondecreasing, we obtain

$$\tilde{A}(t) \leq \int_{|y|<\frac{1}{t}} |k(y,e_1)| \left[\frac{\varphi^*(|x|)}{|x|^n}\right]^{\frac{1}{p'}} \varphi^*\left(\frac{1}{|x|}\right) dx \leq \kappa^*(p,\varphi),$$

from which it follows that $\tilde{A}(\infty) = 0$ and then also $A(\infty) = 0$, since $C(r,t) \leq 1$. Hence,

$$\mathfrak{M}_0(Kf,r) \leq \varkappa^+(p,\varphi)^{p-1}\left[\frac{1}{\varphi(r)}\int_0^\infty \left|A'(t)\right| \varphi(tr)\mathfrak{M}_0(f,tr)dt - \frac{1}{\varphi(r)}\lim_{t\to 0}\varphi(t)A\left(\frac{t}{r}\right)\mathfrak{M}_0(f,t)\right]$$
$$\leq \frac{\varkappa^+(p,\varphi)^{p-1}}{\varphi(r)}\int_0^\infty \varphi(tr)\left|A'(t)\right|\mathfrak{M}_0(f,tr)dt.$$

□

Theorem 4.2. *Let $1 \leq p < \infty$. If $\varkappa^+(p,\varphi) < \infty$ and $\kappa^*(p,\varphi) < \infty$, then the operator K is bounded in the spaces $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$ and $V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$, and*

$$\|Kf\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)} \leq \varkappa^+(p,\varphi)^{\frac{1}{p'}}\kappa^*(p,\varphi)^{\frac{1}{p}}\|f\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}. \tag{4.6}$$

Let $k(x,y) \geq 0$. If (2.7) holds, the conditions $\varkappa_-(p,\varphi) < \infty$ and $\varkappa_*(p,\varphi) < \infty$ are necessary for the boundedness in the spaces $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$ and $V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$, respectively, and

$$\|K\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)} \geq \varkappa_-(p,\varphi) \quad \textit{and} \quad \|K\|_{V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)} \geq \varkappa_*(p,\varphi). \tag{4.7}$$

Proof.

Sufficiency part.
Passing to supremum in r in the inequality (4.3), we obtain

$$\|Kf\|^p_{\mathcal{L}_0^{p,\varphi}} \leq \varkappa^+(p,\varphi)^{\frac{1}{p'}}\left[\int_0^\infty \varphi^*(t)\left|A'(t)\right|dt\right]^{\frac{1}{p}}\|f\|_{\mathcal{L}_0^{p,\varphi}}. \tag{4.8}$$

Direct calculations show that

$$\int_0^\infty \varphi^*(t)\left|A'(t)\right|dt = \kappa^*(p,\varphi),$$

which proves (4.6).

To cover the boundedness in the vanishing generalized Morrey spaces, we only have to verify that

$$\lim_{r\to 0}\mathfrak{M}_0(f,r)=0 \Rightarrow \lim_{r\to 0}\mathfrak{M}_0(Kf,r)=0,$$

which follows from (4.3) by the Lebesgue dominated convergence theorem.

Necessity part. The proof of the necessity is prepared by Lemmas 3.5 and 3.6 and Theorems 3.7 and 3.8.
Suppose that

$$\|Kf\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)} \le \|K\|_{\mathcal{L}_0^{p,\varphi}\to\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}\|f\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}. \tag{4.9}$$

Choose $f=f_0(x)$ in the case of the space $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$ and $f=f_\varepsilon(x)$ in the case of the space $V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$, where f_0 and f_ε are the minimizing functions introduced in Section 3.4. By Theorems 3.7 and 3.8 (similarly to the arguments in the proof of the necessity part of the one-dimensional Theorem 5.1, where such arguments are presented with more details), with the use of Fatou lemma in case of the vanishing spaces, we obtain that the condition $\varkappa_-(p,\varphi)<\infty$ and $\varkappa_*(p,\varphi)<\infty$ are necessary for the corresponding space, and (4.7) holds. □

Corollary 4.3. *Let $1\le p<\infty$ and $\lambda>0$. The operator K is bounded in the spaces $L_0^{p,\lambda}(\mathbb{R}^n)$ and $VL_0^{p,\lambda}(\mathbb{R}^n)$ under the condition $\varkappa_p(\lambda)<\infty$. When k(x,y)≥0, this condition is also necessary for the boundedness.*

5 | THE CASE OF RADIAL KERNELS

Now we single out the case of radial kernels $k(x,y)=\ell(|x|,|y|)$. The main reason for that is that in this case, we can provide stronger estimates: We can obtain estimates of multi-dimensional Morrey norms of Kf via one-dimensional Morrey norms of spherical means of f.

The kernels in (1.5) are examples covered by theorems of this section.

We start with the one-dimensional case. In this case we can cover boundedness in global Morrey spaces. In the multidimensional case, we shall consider only local Morrey spaces.

5.1 | One-dimensional case

The constants (3.12) and (3.14) take the form

$$\varkappa^*(p,\varphi)=\int_0^\infty\left[\frac{\varphi^*(y)}{y}\right]^{\frac{1}{p}}|k(1,y)|dy \tag{5.1}$$

and

$$\varkappa_-(p,\varphi):=\inf_{r>0}\int_0^\infty|k(1,y)|\left[\frac{\varphi(ry)}{\varphi(r)y^n}\right]^{\frac{1}{p}}dy\ge\int_0^\infty|k(1,y)|\left[\frac{\varphi_*(y)}{y^n}\right]^{\frac{1}{p}}dy=:\varkappa_*(p,\varphi). \tag{5.2}$$

Theorem 5.1. *The operator K of the form (1.4) with a kernel homogeneous of degree −1 is bounded in the spaces $\mathcal{L}_0^{p,\varphi}$, $V\mathcal{L}_0^{p,\varphi}$, $\mathcal{L}^{p,\varphi}$ and $V\mathcal{L}^{p,\varphi}$ if $\varkappa^*(p,\varphi)<\infty$ and*

$$\|Kf\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}_+)}\le\varkappa^*(p,\varphi)\|f\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}_+)} \quad \text{and} \quad \|Kf\|_{\mathcal{L}^{p,\varphi}(\mathbb{R}_+)}\le\varkappa^*(p,\varphi)\|f\|_{\mathcal{L}^{p,\varphi}(\mathbb{R}_+)}.$$

Let $k(x,y)\ge 0$ and φ satisfy (2.7). Then the condition $\varkappa_-(p,\varphi)<\infty$ is necessary for the boundedness of the operator K in the space $\mathcal{L}_0^{p,\varphi}(\mathbb{R}_+)$ and also in $\mathcal{L}^{p,\varphi}(\mathbb{R}_+)$ if (2.5) holds, and the condition $\varkappa_*(p,\varphi)<\infty$ is necessary for the boundedness of the operator K in the space $V\mathcal{L}_0^{p,\varphi}(\mathbb{R}_+)$ and also in $V\mathcal{L}^{p,\varphi}(\mathbb{R}_+)$ if (2.6) holds, and

$$\|K\|_{\mathcal{L}_0^{p,\varphi}\to\mathcal{L}_0^{p,\varphi}}\ge\varkappa_-(p,\varphi) \quad \text{and} \quad \|K\|_{\mathcal{L}^{p,\varphi}\to\mathcal{L}^{p,\varphi}}\ge\varkappa_-(p,\varphi)$$

and

$$\|K\|_{V\mathcal{L}_0^{p,\varphi}\to V\mathcal{L}_0^{p,\varphi}}\ge\varkappa_*(p,\varphi) \quad \text{and} \quad \|K\|_{V\mathcal{L}^{p,\varphi}\to V\mathcal{L}^{p,\varphi}}\ge\varkappa_*(p,\varphi).$$

Proof.

Sufficiency part. By the Minkowski inequality (2.10) for

$$Kf(x) = \int_0^\infty k(1,t)f(xt)dt \tag{5.3}$$

we have

$$\|Kf\|_{\mathcal{L}^{p,\varphi}} \le \int_0^\infty |k(1,t)| \|\Pi_t f\|_{\mathcal{L}^{p,\varphi}} dt.$$

Then by Lemma 3.2, we obtain

$$\|Kf\|_{\mathcal{L}^{p,\varphi}} \le \varkappa^*(p,\varphi)\|f\|_{\mathcal{L}^{p,\varphi}}.$$

In the case of vanishing generalized Morrey spaces, we only have to verify that the operator K preserves the condition (2.8) to (2.9). By Minkowski inequality from (5.3) and property (3.8), we have

$$\mathfrak{M}(Kf,r)^{\frac{1}{p}} \le \int_0^\infty |k(1,t)|\ \mathfrak{M}(\Pi_t f,r)^{\frac{1}{p}} dt \le \int_0^\infty \left[\frac{\varphi^*(t)}{t}\right]^{\frac{1}{p}} |k(1,t)| \mathfrak{M}(f,rt)^{\frac{1}{p}} dt.$$

Since $\mathfrak{M}(f,rt)^{\frac{1}{p}} \le \|f\|_{\mathcal{L}^{p,\varphi}}$, we can pass to the limit as $r \to 0$ under the integral sign by the Lebesgue dominated convergence theorem.

Necessity part. Let K be bounded in $\mathcal{L}^{p,\varphi}$. Choose

$$f_0(x) = \left(\frac{\varphi(x)}{x}\right)^{\frac{1}{p}}$$

which belongs to $\mathcal{L}^{p,\varphi}$ by Lemma 3.5, so that Kf_0 should be well defined on the function f_0. Then $\varkappa_-(p,\varphi)$ must be finite in view of the equality in (3.17).

To prove the necessity of the condition $\varkappa_*(p,\varphi) < \infty$ when the inequality

$$\|Kf\|_{\mathcal{L}^{p,\varphi}} \le \|K\|_{\mathcal{L}^{p,\varphi}\to\mathcal{L}^{p,\varphi}} \|f\|_{\mathcal{L}^{p,\varphi}} \tag{5.4}$$

holds only for $f \in V\mathcal{L}^{p,\varphi}$, we choose $f = f_\varepsilon$ and by the inequality in (3.19) we obtain that $\varkappa_*^\varepsilon(p,\varphi) \le \|K\|_{V\mathcal{L}^{p,\varphi}\to V\mathcal{L}^{p,\varphi}}$. Hence $\varkappa_*^\varepsilon(p,\varphi)$ is finite for every $\varepsilon > 0$. It remains to apply Fatou lemma to justify the passage to the limit as $\varepsilon \to 0$. The arguments for local spaces follow the same lines. □

Denote

$$\varkappa_p(\lambda) := \int_0^\infty |k(1,y)| \frac{dy}{y^{\frac{1-\lambda}{p}}} = \varkappa^*(p,\varphi)|_{\varphi(r)=r^\lambda}$$

Corollary 5.2. *Let $1 \le p < \infty$. The operator K is bounded in the spaces $L_0^{p,\lambda}(\mathbb{R}_+)$, $VL_0^{p,\lambda}(\mathbb{R}_+)$, $\lambda > 0$, and in the spaces $L^{p,\lambda}(\mathbb{R}_+)$, $VL^{p,\lambda}(\mathbb{R}_+)$, $0 < \lambda < 1$, under the condition $\varkappa_p(\lambda) < \infty$. If $k(x,y) \ge 0$, this condition is also necessary and*

$$\|K\|_{L_0^{p,\lambda}(\mathbb{R}_+)\to L_0^{p,\lambda}(\mathbb{R}_+)} = \|K\|_{VL_0^{p,\lambda}(\mathbb{R}_+)\to VL_0^{p,\lambda}(\mathbb{R}_+)} = \|K\|_{L^{p,\lambda}(\mathbb{R}_+)\to L^{p,\lambda}(\mathbb{R}_+)} = \|K\|_{VL^{p,\lambda}(\mathbb{R}_+)\to VL^{p,\lambda}(\mathbb{R}_+)} = \varkappa_p(\lambda). \tag{5.5}$$

Proof. It suffices to observe the coincidence of the constants:

$$\varkappa^*(p,\varphi)|_{\varphi(r)=r^\lambda} = \varkappa_-(p,\varphi)|_{\varphi(r)=r^\lambda} = \varkappa_*(p,\varphi)|_{\varphi(r)=r^\lambda} = \varkappa_p(\lambda).$$

□

5.2 | Multidimensional case

We need analogs of constants (3.12) to (3.15), taken with respect to the spherical means. As can be seen from the proof of the theorem below, they should have the form

$$\ell^*(p,\varphi)=\int_0^\infty t^{\frac{n}{p}-1}\varphi^*(t)^{\frac{1}{p}}|\ell(1,t)|dt, \tag{5.6}$$

$$\ell_-(p,\varphi)=\inf\int_0^\infty t^{\frac{n}{p}-1}[\frac{\varphi(rt)}{\varphi(r)}]^{\frac{1}{p}}|\ell(1,t)|dt, \tag{5.7}$$

$$\ell_*(p,\varphi)=\int_0^\infty t^{\frac{n}{p'}-1}\varphi_*(t)^{\frac{1}{p}}|\ell(1,t)|dt. \tag{5.8}$$

We also denote

$$\Phi(\varrho):=\int\limits_{\mathbb{S}^{n-1}} f(\varrho\sigma)d\sigma \qquad \text{and} \qquad F(t):=t^{\frac{n-1}{p}}\Phi(t).$$

Theorem 5.3. *Let $1\le p<\infty$. The condition $\ell^*(p,\varphi)<\infty$ is sufficient for the boundedness of the operator K in the spaces $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$, and $V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$, and*

$$\|Kf\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}\le|\mathbb{S}^{n-1}|\ell^*(p,\varphi)\|f\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}. \tag{5.9}$$

Moreover, a stronger inequality

$$\|Kf\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}\le|\mathbb{S}^{n-1}|^{\frac{1}{p}}\ell^*(p,\varphi)\|F\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}_+)} \tag{5.10}$$

holds.

If $\ell(|x|,|y|)\ge 0$ and (2.7) holds, then the conditions $\ell_-(p,\varphi)<\infty$ and $\ell_*(p,\varphi)<\infty$ are necessary for the boundedness in the spaces $\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$ and $V\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)$, respectively, and also for the inequality via spherical means, and

$$\|K\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)\to\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}\ge\ell_-(p,\varphi)$$

and

$$\inf_{f\in\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}\frac{\|Kf\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}}{\|F\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}}\ge\ell_-(p,\varphi).$$

Proof. Due to the radial nature of the operator K, we can reduce the boundedness problem to the one-dimensional case with respect to the function F.

Sufficiency part. Note that (5.9) follows from (5.10) since

$$\|F\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}_+)}\le|\mathbb{S}^{n-1}|^{\frac{1}{p'}}\|f\|_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}.$$

The latter is checked directly by using the Jensen inequality. To prove (5.10), we pass to polar coordinates and get

$$\|Kf\|^p_{\mathcal{L}_0^{p,\varphi}(\mathbb{R}^n)}=|\mathbb{S}^{n-1}|\sup_{r>0}\frac{1}{\varphi(r)}\int\limits_0^r\varrho^{n-1}\left|\int\limits_0^\infty t^{n-1}\ell(\varrho,t)\Phi(t)dt\right|^p d\varrho. \tag{5.11}$$

Denote also

$$k_1(\varrho,t):=\varrho^{\frac{n-1}{p}}t^{\frac{n-1}{p'}}\ell(\varrho,t).$$

The kernel $k_1(\varrho,t)$ is homogeneous of degree -1.

With this notation, the equality (5.11) turns into

$$\|Kf\|^p_{\mathcal{L}^{p,\varphi}_0(\mathbb{R}^n)} = |\mathbb{S}^{n-1}| \sup_{r>0} \frac{1}{\varphi(r)} \int_0^r \left| \int_0^\infty k_1(\varrho,t)F(t)dt \right|^p d\varrho.$$

Therefore, we are now in a position to apply Theorem 5.1 and by that we arrive at (5.10).

In the case of vanishing generalized Morrey space, we only have to take care about the preservation of the property $\lim_{r\to 0} \mathfrak{M}_0(f,r) = 0$ by the operator K but this was already proved in a more general case in Theorem 4.2 by means of the estimate (4.3).

Necessity part. The necessity of the conditions $\ell_-(p,\varphi) < \infty$ and $\ell_*(p,\varphi) < \infty$ for the boundedness in the spaces $\mathcal{L}^{p,\varphi}_0$ and $V\mathcal{L}^{p,\varphi}_0$, respectively, was already proved in Theorem 4.2 if we take into account that $\varkappa_-(p,\varphi) = \ell_-(p,\varphi)$. This provides also the proof of the necessity for an estimate of type (5.10) via spherical means, since our minimizing functions are radial functions and the right-hand sides of (5.10) and (5.9) coincide on such functions. □

In the following corollary we use the notation:

$$\ell_p(\lambda) := \int_0^\infty k(1,t)t^{\frac{n}{p'}+\frac{\lambda}{p}-1}dt.$$

Corollary 5.4. *Let $1 \le p < \infty$, $\lambda > 0$ and $\ell(|x|,|y|) \ge 0$. The operator K is bounded in the spaces $L^{p,\lambda}_0(\mathbb{R}^n)$ and $VL^{p,\lambda}_0(\mathbb{R}^n)$, if and only if $\ell_p(\lambda) < \infty$ and $\|K\|_{L^{p,\lambda}_0(\mathbb{R}^n)\to L^{p,\lambda}_0(\mathbb{R}^n)} = |S^{n-1}|\ell_p(\lambda)$..*

Moreover, the final estimate $\|Kf\|_{L^{p,\lambda}_0(\mathbb{R}^n)} \le c\|F\|_{L^{p,\lambda}_0(\mathbb{R}_+)}$ holds if and only if $\ell_p(\lambda) < \infty$ and

$$\inf_{f\in L^{p,\lambda}_0(\mathbb{R}^n)} \frac{\|Kf\|_{L^{p,\lambda}_0(\mathbb{R}^n)}}{\|F\|_{L^{p,\lambda}_0(\mathbb{R}_+)}} = |S^{n-1}|\ell_p(\lambda).$$

5.3 | A remark on the Hausdorff operator (1.9)

From the proof of Theorem 5.1, it is seen that in the sufficiency part, it extends to the case of an arbitrary Banach space X in which the norm of the dilation operator Π_t may be calculated or estimated. We formulate the corresponding result for the Hausdorff operator (1.9). Let $X = X(\mathbb{R}^n)$ or $X = X(\mathbb{R}_+)$ be any Banach space of functions on $\mathbb{R}^n$ which admits Minkowski inequality:

$$\left\| \int_{\mathbb{R}^n} F(x,y)dy \right\|_X \le \int_{\mathbb{R}^n} \|F(\cdot,y)\|_X dy$$

and let

$$\Lambda_X(t) := \|\Pi_t\|_{X\to X}. \tag{5.12}$$

Then the following theorem holds.

Theorem 5.5. *The Hausdorff operator (1.9) is bounded in the space X if*

$$\int_0^\infty |a(t)|\Lambda_X(t)dt < \infty. \tag{5.13}$$

Proof. The proof is straightforward in view of the validity of the Minkowski inequality admitted for the space X. □

Recall that for non-negative functions $a(t)$, the necessity of the condition (5.13) was shown in the cases $X = L^{p,\lambda}(\mathbb{R}_+)$, $X = L^{p,\lambda}_0(\mathbb{R}_+)$ and $X = L^{p,\lambda}_0(\mathbb{R}^n)$, where $\Lambda_X = t^{\frac{\lambda-n}{p}}$, see Corollaries 5.2 and 5.5.

To illustrate Theorem 5.5, we consider an example of the space X different from Morrey space.

Example 5.6. Let $X = N^{p,\alpha}(\mathbb{R}^n)$ be the Nikol'sky space defined by the seminorm

$$[f]_{p,\alpha} := \sup_{h\in\mathbb{R}^n} \frac{\|f(\cdot+h)-f(\cdot)\|_{L^p(\mathbb{R}^n)}}{|h|^\alpha}, \quad 1 \le p \le \infty, 0 < \alpha \le \frac{n}{p}.$$

Treating $N^{p,\alpha}(\mathbb{R}^n)$ as the factor-space over constants, in this case, we have

$$\Lambda_X(t) = t^{\alpha-\frac{n}{p}}.$$

6 | APPLICATIONS TO CONCRETE OPERATORS

We give application to various concrete examples of operators, concentrating on the case of classical Morrey spaces $L^{p,\lambda}$. We separately consider spaces on $\mathbb{R}_+$ and $\mathbb{R}^n$ since in the one-dimensional case we can deal both with local and global Morrey spaces. In the end of this section we show how similar results may be obtained for generalized Morrey spaces in terms of Matuszewska-Orlicz indices of the function φ.

6.1 | The case of classical Morrey spaces; one-dimensional case

By

$$H^\alpha f(x) = x^{\alpha-1}\int_0^x \frac{f(t)}{t^\alpha}dt, \quad \mathcal{H}^\beta f(x) = x^\beta \int_x^\infty \frac{f(t)}{t^{1+\beta}}dt$$

and

$$\mathbb{H}^\gamma f(x) = x^\gamma \int_0^\infty \frac{f(t)}{t+x}\frac{dt}{t^\gamma},$$

we denote the weighted versions of Hardy and Hilbert operators. L et also

$$I^{\alpha,\gamma} f(x) = x^{\gamma-\alpha}\int_0^x \frac{f(t)dt}{t^\gamma(x-t)^{1-\alpha}}, \quad \alpha > 0$$

be a weighted Hardy-Littlewood fractional operator. We demonstrate the efficiency of Theorem 5.1 in Example 6.1 below. Note that contrary to Lebesgue spaces L^p, Morrey spaces $L^{p,\lambda}$ with $\lambda \ne 0$, admit a possibility $p = 1$ for the operators H^α and $\mathbb{H}^\gamma$ even in the non-weighted case, when $\alpha = 0$, or $\beta = 0$, or $\gamma = 0$.

Example 6.1. Let $0 < \lambda < 1$ and $1 \le p < \infty$. The operators H^α, $\mathcal{H}^\beta$ and $\mathbb{H}^\gamma$ are bounded in any of the spaces $L^{p,\lambda}(\mathbb{R}_+)$, $L_0^{p,\lambda}(\mathbb{R}_+)$, $VL^{p,\lambda}(\mathbb{R}_+)$ and $VL_0^{p,\lambda}(\mathbb{R}_+)$ if and only if $\alpha < \frac{1}{p'}+\frac{\lambda}{p}$, $\beta > \frac{\lambda-1}{p}$ and $\frac{\lambda-1}{p} < \gamma < \frac{1}{p'}+\frac{\lambda}{p}$, respectively, and

$$\|H^\alpha\| = \frac{1}{\frac{1}{p'}+\frac{\lambda}{p}-\alpha}, \quad \|\mathcal{H}^\beta\| = \frac{1}{\beta+\frac{1-\lambda}{p}} \quad \text{and} \quad \|\mathbb{H}^\gamma\| = \frac{\pi}{\sin\pi\left(\gamma+\frac{1-\lambda}{p}\right)}.$$

The operator $I^{\alpha,\gamma}$ is bounded in any of these spaces if and only if

$$\gamma < \frac{\lambda}{p}+\frac{1}{p'}$$

and

$$\|I^{\alpha,\gamma}\| = B\left(\frac{\lambda}{p}+\frac{1}{p'}-\gamma, \alpha\right).$$

The “if” part in this example for the Hardy operators H^α and $\mathcal{H}^\beta$ was earlier proved in Samko.[25]

6.2 | The case of classical Morrey spaces; multi-dimensional case

In this section, we consider the multidimensional Hardy operator

$$H^\alpha f(x) = |x|^{\alpha-n} \int\limits_{|y|<|x|} \frac{f(y)}{|y|^\alpha} dy$$

and a multidimensional versions

$$J^\alpha f(x) = \frac{1}{|x|^\alpha} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

of the Hardy-Littlewood fractional operator.

Example 6.2. The operator H^α is bounded in the local Morrey spaces $L_0^{p,\lambda}(\mathbb{R}^n)$ and $VL_0^{p,\lambda}(\mathbb{R}^n)$, where $\lambda > 0$ and $1 \le p < \infty$, if and only if $\alpha < \frac{n}{p'} + \frac{\lambda}{p}$ and

$$\|H^\alpha\|_{L_0^{p,\varphi}(\mathbb{R}^n)} = \frac{|\mathbb{S}^{n-1}|}{\frac{n}{p'} + \frac{\lambda}{p} - \alpha}.$$

The reader can easily derive similar results for multi-dimensional versions of the operators $\mathcal{H}^\beta$ and $\mathbb{H}^\gamma$ from our general statements.

Example 6.3. The operator J^α is bounded in the spaces $L_0^{p,\lambda}(\mathbb{R}^n)$ and $VL_0^{p,\lambda}(\mathbb{R}^n)$, where $\lambda > 0$ and $1 \le p < \infty$, if and only if $0 < \alpha < \frac{n}{p}$ and $0 \le \lambda < n - \alpha p$, and the sharp constant $\varkappa_p(\lambda)$ is equal to

$$\varkappa_p(\lambda) = \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{\frac{n}{p'}+\frac{\lambda}{p}}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\frac{n-\lambda}{p}-\alpha}{2}\right)}{\Gamma\left(\frac{n-\lambda}{2p}\right)\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{\frac{n}{p'}+\frac{\lambda}{p}+\alpha}{2}\right)}.$$

Indeed, according to (3.12)

$$\varkappa_p(\lambda) = \int_{\mathbb{R}^n} \frac{dy}{|y-e_1|^{n-\alpha}|y|^{\frac{n-\lambda}{p}}}. \tag{6.1}$$

The convergence of this integral provide the mentioned conditions on α and λ. It remains to make use of the known expression of integrals of the form $\int_{\mathbb{R}^n} |y|^{-a}|y-e_1|^{-b}$ in terms of Gamma functions, see Samko.[26], lemma 4.1

6.3 | The case of generalized Morrey spaces

We use the notion of Matuszewska-Orlicz indices, see, for instance, other works.[23,27] For a function φ positive on $\mathbb{R}_+$ these indices are defined as follows:

$$m_0(\varphi) = \lim_{x\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} \qquad M_0(\varphi) = \lim_{x\to \infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x}; \tag{6.2}$$

$$m_\infty(\varphi) = \lim_{x\to 0} \frac{\ln \left[\limsup\limits_{h\to \infty} \frac{\varphi(xh)}{\varphi(h)}\right]}{\ln \quad x} \qquad M_\infty(\varphi) = \lim_{x\to \infty} \frac{\ln \left[\limsup\limits_{h\to \infty} \frac{\varphi(xh)}{\varphi(h)}\right]}{\ln \quad x}. \tag{6.3}$$

It is known that the condition that there exist $\alpha, \beta \in \mathbb{R}_+$, such that

$$\frac{\varphi(t)}{t^\alpha} \text{ is almost decreasing and } \frac{\varphi(t)}{t^\beta} \text{ is almost increasing,} \quad t \in \mathbb{R}_+$$

imply that the numbers $m_0(\varphi), M_0(\varphi), m_\infty(\varphi)$ and $M_\infty(\varphi)$ are finite:

$$-\infty < m_0(\varphi) \le M_0(\varphi) < \infty \qquad \text{and} \qquad -\infty < m_\infty(\varphi) \le M_\infty(\varphi) < \infty.$$

From the definition of these indices and their properties there follows that for any arbitrary small $\varepsilon > 0$

$$c_1 t^{\max\{M_0(\varphi),M_\infty(\varphi)\}+\varepsilon} \le \frac{\varphi(xt)}{\varphi(x)} \le c_2 t^{\min\{m_0(\varphi),m_\infty(\varphi)\}-\varepsilon}, \quad \text{for} \quad 0 < t < 1, \tag{6.4}$$

and

$$c_1 t^{\min\{m_0(\varphi),m_\infty(\varphi)\}-\varepsilon} \le \frac{\varphi(xt)}{\varphi(x)} \le c_2 t^{\max\{M_0(\varphi),M_\infty(\varphi)\}+\varepsilon}, \quad \text{for} \quad t > 1, \tag{6.5}$$

where c_1 and c_2 in general depend on ε, see, for instance, Samko,[23] formulas 6.22 and 6.23 The formulas (6.4) and (6.5) provide estimates for the functions $\varphi^*(t)$ and $\varphi_*(t)$ considered in Section 3.1 via power-type functions with different, in general, exponents at the origin and infinity.

Making use of the estimates (6.4) and (6.5), from Theorem 5.1 after easy calculations, we arrive at the following result:

Theorem 6.4. *Let $1 \le p < \infty$ and φ satisfy a priori assumptions of Section 2. The operators H^α, $\mathcal{H}^\beta$ and $\mathbb{H}^\gamma$ are bounded in the space $\mathcal{L}^{p,\varphi}(\mathbb{R}_+)$ if*

$$\alpha < \frac{1}{p'} + \frac{\min\{m_0(\varphi), m_\infty(\varphi)\}}{p}, \quad \beta > \frac{\max\{M_0(\varphi), M_\infty(\varphi)\} - 1}{p}$$

and

$$\frac{\max\{M_0(\varphi), M_\infty(\varphi)\} - 1}{p} < \gamma < \frac{1}{p'} + \frac{\min\{m_0(\varphi), m_\infty(\varphi)\}}{p}.$$

Let also φ satisfy the conditions (2.5) and (2.7). Then the following conditions

$$\alpha \le \frac{1}{p'} + \frac{\max\{M_0(\varphi), M_\infty(\varphi)\}}{p}, \quad \beta \ge \frac{\min\{m_0(\varphi), m_\infty(\varphi)\} - 1}{p}$$

and

$$\frac{\min\{m_0(\varphi), m_\infty(\varphi)\} - 1}{p} \le \gamma \le \frac{1}{p'} + \frac{\max\{M_0(\varphi), M_\infty(\varphi)\}}{p}.$$

are necessary for the boundedness.

One can similarly derive analogous corollaries from our general theorems for the spaces $\mathcal{L}_0^{p,\varphi}$, $V\mathcal{L}_0^{p,\varphi}$ and $V\mathcal{L}^{p,\varphi}$, including also upper and lower bounds for the horns of the operators H^α, $\mathcal{H}^\beta$ and $\mathbb{H}^\gamma$, as in Theorem 5.1. Other examples of operators including multidimensional ones are also easily covered by our general theorems.

7 | CONCLUSION

We have proved several general statements on mapping properties of a wide class of integral operators in generalized Morrey spaces on R^n, which play an important role in applications to PDEs. This class of operators is defined by the conditions that its kernel $k(x, y)$ is homogeneous of degree $-n$ and invariant with respect to rotations. The obtained results are all new even for the case of classical Morrey spaces. More precisely:

1. By using various methods of functional analysis, we obtained integral conditions on the kernel $k(x, y)$ and the parameters of the generalized Morrey spaces, sufficient for the boundedness in these spaces.
2. Via the use of the method of minimizing sequences, we found some necessary conditions for the boundedness.
3. We show that in the case of classical Morrey spaces, the obtained necessary and sufficient conditions coincide.
4. In the case of radial kernels, we prove a stronger result in terms of spherical means of functions.

The obtained results allow to consider mapping properties of a big variety of concrete operators often used in applications.

CONFLICT OF INTEREST

This work does not have any conflict of interest.

ORCID

Natasha Samko https://orcid.org/0000-0002-8595-4326

REFERENCES

1. Hardy GH, Littlewood JE, Pólya G.. *Inequalities*. Cambridge: Cambridge University Press; 1934.
2. Mikhailov LG. On some multi-dimensional integral equations with homogeneous kernels (Russian). *Dokl Akad Nauk SSSR*. 1967;176(2):263-265.
3. Karapetiants NK. On necessary boundedness conditions of operators with a nonnegative quasihomogenous kernel (Russian). *Mat Zametki*. 1981;30(5):787-794.
4. Karapetiants NK, Samko SG. Multidimensional integral operators with homogeneous kernels. *Fract Calc Appl Anal*. 1999;2(1):67-96.
5. Karapetiants NK, Samko SG. *Equations with Involutive Operators*. Boston: Birkhäuser; 2001.
6. Lukkassen D, Persson LE, Samko S. Some sharp inequalities for integral operators with homogeneous kernel. *J Inequal Appl*. 2016;2016.
7. Adams DR. *Morrey Spaces Series: Applied and Numerical Harmonic Analysis*: Birkhöuser; 2015.
8. Giaquinta M. *Multiple integrals in the calculus of variations and non-linear elliptic systems*. Princeton: Princeton University Press; 1983.
9. Kokilashvili V, Meskhi A, Rafeiro H., Samko S.. *Integral Operators in Non-standard Function Spaces, I and II*. Switzerland: Springer-Birkhäser; 2016.
10. Triebel H. *Hybrid Function Spaces, Heat and Navier—Stokes Equations*, volume 22 of EMS Tracts in Mathematics: European Mathematical Society Publishing House Zürich; 2015.
11. Rafeiro H, Samko N, Samko S. *Morrey-Campanato Spaces: An Overview*, volume 228 of Operator Theory: Advances and Applications. Basel: Springer-Birkhäuser; 2013;293-324.
12. Sawano Y. A thought on generalized Morrey spaces. https://arxiv.org/abs/1812.08394; 2018.
13. Guliyev V, Sawano Y. Linear and sublinear operators on generalized Morrey spaces with non-doubling measures. *Publicationes Mathematicae Debrecen*. 2013;83(3):1-17.
14. Haroske DD, Skrzypczak L. Embeddings of weighted Morrey spaces. *Math Nachr*. 2016;290:1066-1086.
15. Lemarié-Rieusset PG, spaces Morrey. Multipliers and Morrey spaces. 2013;38:741-752.
16. Almeida A, Samko S. Approximation in Morrey spaces. *J Funct Anal*. 2017;272(6):2392-2411.
17. Hakim DI, Sawano Y. A note on the complex interpolation of Morrey spaces. *Linear Nonlinear Anal*. 2016;2(2):215-228.
18. Persson LE, Samko N. Weighted Hardy and potential operators in the generalized Morrey spaces. *Journal of Mathematical Analysis and Applications*. 2011;377:792-806.
19. Persson LE, Samko N. Calderó,n-Zygmund type singular operators in weighted generalized Morrey spaces. *J Fourier Anal Appl*. 2016;22(2):413-426.
20. Liflyand E. Hausdorff operators on Hardy spaces. *Eurasian Math*. 2013;4(4):101-141.
21. Kufner A., John O., Fučik S.. *Function Spaces*. Leyden: Noordhoff International Publishing; 1977;454 + XV pages.
22. Krein SG, Petunin Y. u. I., Semenov E. M.. *Interpolation of Linear Operators*, Volume 54 of Translations of Mathematical Monographs. Providence, R.I.: American Mathematical Society; 1982.
23. Samko N. Weighted hardy operators in the local generalized vanishing Morrey spaces. *Positivity*. 2013;17(3):683-706.
24. Kato T. Strong solutions of the Navier-Stokes equation in Morrey spaces. *Bol Soc Brasil Mat*. 1992;22(2):127-155.
25. Samko N. Weighted HArdy and singular operators in Morrey spaces. *Journal of Mathematical Analysis and Applications*. 2009;350(1):56-72.
26. Samko S. Best constant in the weighted Hardy inequality: the spatial and spherical version. *Fract Calc Appl Anal*. 2005;8(1):39-52.
27. Karapetiants NK, Samko NG. Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^\omega(\rho)$ via the indices m_ω and M_ω. *Fract Calc Appl Anal*. 2004;7(4):437-458.

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