Compatibility, multi-brackets and integrability of systems of PDEs

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Abstract

We establish an efficient compatibility criterion for a system of generalized complete intersection type in terms of certain multi-brackets of differential operators. These multi-brackets generalize the higher Jacobi-Mayer brackets, important in the study of evolutionary equations and the integrability problem. We also calculate Spencer δ-cohomology of generalized complete intersections and evaluate the formal functional dimension of the solutions space. The results are applied to establish new integration methods and solve several differential-geometric problems. [1]

Introduction and main results

In this paper we introduce multi-brackets of non-linear vector differential operators. In the case of bi-brackets they coincide with the well-known Jacobi bracket, which is a generalization of the classical Lagrange-Jacobi bracket important in the theory of 1st order differential equations. These latter brackets become the usual commutators in the case of linear differential operators and are widely used in mathematical physics and non-linear analysis. We apply multi-bracket of differential operators to establish a criterion of formal integrability of systems of PDEs.

0.1. Multi-brackets of non-linear differential operators

Our multi-bracket $\{F_1, \ldots, F_{m+1}\}$ is defined for differential operators on sections of the trivial m-dimensional bundle over a manifold M (notice that trivialization assumption is usually implicit for commutators or bi-brackets) and its value is an operator of the same kind. When F_i are linear vector differential operators $\nabla_i : m \cdot C^\infty_{\mathrm{loc}}(M) \to C^\infty_{\mathrm{loc}}(M)$, represented as rows $(\nabla^1_i, \ldots, \nabla^m_i)$ of scalar linear differential operators, the multi-bracket has the form:

$$\{\nabla_1, \ldots, \nabla_{m+1}\} = \sum_{k=1}^{m+1} (-1)^k \, \mathrm{Ndet}[\nabla^j_i]^{1 \le j \le m}_{i \ne k} \cdot \nabla_k,$$

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where Ndet is a version of non-commutative determinant and $\cdot$ is the product (one can perceive any determinant and product for a while, but we will discuss various versions in the sequel). For non-linear operators the bracket is obtained via linearization.

If π is a trivial vector bundle and $\ell(F) = (\ell_1(F), \ldots, \ell_m(F))$ is a linearization of the operator F, then:

$$\{F_1, \ldots, F_{m+1}\} = \frac{1}{m!} \sum_{\alpha \in S_m, \beta \in S_{m+1}} (-1)^\alpha (-1)^\beta \ell_{\alpha(1)}(F_{\beta(1)}) \circ \ldots \circ \ell_{\alpha(m)}(F_{\beta(m)}) \left(F_{\beta(m+1)}\right).$$

We will need restriction of this bracket to the system $\mathcal{E}$ of PDEs $F_i = 0$. Let $\mathrm{ord}(F_i) = l(i)$ be the orders of the considered operators. Denote by $\mathrm{diff}(\pi, \mathbf{1})$ the algebra of all (non-linear) scalar differential operators on the sections of π.

Define $\mathcal{J}_s(F_1, \ldots, F_k) = \langle \mathcal{D}_\tau F_i \,\big|\, l(i) + |\tau| \le s \rangle$ to be the subalgebra of the differential ideal generated by $F_1, \ldots, F_k$ in $\mathrm{diff}(\pi, \mathbf{1})$, where $\mathcal{D}_\tau$ is the total derivative operator with respect to a multi-index τ (formula in §1.1). We will explore the following reduced multi-bracket (further discussion in §3.4):

$$\{F_1, \ldots, F_{m+1}\} \mod \mathcal{J}_{l(1)+\cdots+l(m+1)-1}(F_1, \ldots, F_{m+1}).$$

The above equivalence class can be given by other multi-brackets, more convenient for calculations. In the canonical coordinates (x^i, p_σ^j) on the jet-space $J^\infty(\pi)$ of a rank m (vector) bundle π over the base M with $\dim M = n$ the reduced multi-bracket has the following representative:

$$[F_1, \ldots, F_{m+1}]' = \frac{1}{m!} \sum_{\sigma \in S_{m+1}} \mathrm{sgn}(\sigma) \sum_{\tau_i} \left| \frac{\partial(F_{\sigma(1)}, \ldots, F_{\sigma(m)})}{\partial(p_{\tau_1}^1, \ldots, p_{\tau_m}^m)} \right| \mathcal{D}_{\tau_1 + \cdots + \tau_m} F_{\sigma(m+1)},$$

where

$$\left| \frac{\partial(f_1, \ldots, f_m)}{\partial(u_1, \ldots, u_m)} \right| = \det \left\| \frac{\partial f_i}{\partial u_j} \right\|_{m \times m}.$$

is the Jacobian. In other words we have (see §3.2 for details):

$$[F_1, \ldots, F_{m+1}]' \equiv \{F_1, \ldots, F_{m+1}\} \mod \mathcal{J}_{l(1)+\cdots+l(m+1)-1}(F_1, \ldots, F_{m+1}).$$

When $m = 1$ we obtain the Mayer bracket $[F, G]$ of scalar differential operators. This bracket coincides with the classical Lie-Mayer bracket for the first order equations and is closely related to the Jacobi bracket on higher jets. We applied it in the previous works ([KL$_1$, KL$_2$, KL$_3$]) to establish a compatibility criterion for overdetermined systems of scalar equations of a certain type.

Namely, we considered a system of complete intersection type, i.e. given by $r \le n = \dim M$ equations which have transversal characteristic varieties on regular strata. In other words, the ideal generated by the symbols of the operators is an algebraic complete intersection.

0.2. Main results

In this paper we extend the compatibility result to the case of systems of PDEs with arbitrary number of unknown functions. To characterize the systems for which the criterion is sufficient (necessity holds always) we introduce a new class of systems generalizing the complete intersection for the scalar case.

The conditions informally say the system is not too overdetermined (we will also discuss the opposite case) and is of general kind (transversality condition).

Definition 1. *We say a system $\mathcal{E} \subset J^k(\pi)$ of r differential equations on m unknowns is of* generalized complete intersection type *if*

1. $m < r < n + m$;

2. *The characteristic variety has* $\dim_\mathbb{C} \operatorname{Char}^\mathbb{C}_{x_k}(\mathcal{E}) = n + m - r - 2$ *at each point $x_k \in \mathcal{E}$ (we assume $\dim \emptyset = -1$);*

3. *The characteristic sheaf $\mathcal{K}$ over $\operatorname{Char}^\mathbb{C}_{x_k}(\mathcal{E}) \subset P^\mathbb{C}T^*$ has fibers of dimension 1 everywhere (see §1.1-§1.3 for details of the involved objects).*

The case $r = m$ corresponds to determined systems, where the compatibility conditions are void, but all the statements hold for this case as well.

The class of systems, introduced above, is included into the systems of Cohen-Macaulay type, introduced in [KL₂], see also the discussion of complete intersection for PDEs there. Note that the number r of equations, called codimension of the system $\mathcal{E}$, is defined invariantly and is calculated via the Spencer δ-cohomology by the formula $r = \operatorname{codim}(\mathcal{E}) = \dim H^{*,1}(\mathcal{E})$, see [KL₂].

Define the reduced multi-bracket due to the system $\langle F_1, \ldots, F_r \rangle$ by the formula

$$[F_{i_1}, \ldots, F_{i_{m+1}}]_\mathcal{E} = \{F_{i_1}, \ldots, F_{i_{m+1}}\} \mod \mathcal{J}_{l(i_1) + \cdots + l(i_{m+1}) - 1}(F_1, \ldots, F_r).$$

Theorem A. *Consider a system of PDEs*

$$\mathcal{E} = \left\{ F_i\left(x^1, \ldots, x^n, u^1, \ldots, u^m, \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma}\right) = 0 \,\middle|\, 1 \leq i \leq r \right\}, \quad \operatorname{ord}(F_i) = l(i).$$

1. If the system $\mathcal{E}$ is formally integrable, then the multi-bracket vanishes due to the system, i.e. for every collection $1 \leq i_1 < \cdots < i_{m+1} \leq r$

$$[F_{i_1}, \ldots, F_{i_{m+1}}]_\mathcal{E} = 0.$$

2. Let $\mathcal{E}$ be a system of PDEs of generalized complete intersection type. Then the system $\mathcal{E}$ is formally integrable if and only if the multi-bracket vanishes due to the system:

$$[F_{i_1}, \ldots, F_{i_{m+1}}]_\mathcal{E} = 0.$$

In particular, we deduce the following compatibility criterion for scalar PDEs:

Corollary 1. *Let $\mathcal{E}$ be a system of complete intersection type, i.e. given by $r \leq n$ differential equations on one unknown function $F_1[u] = 0, \ldots, F_r[u] = 0$ of orders $l_1, \ldots, l_n$. Then the system $\mathcal{E}$ is formally integrable iff the Mayer bracket vanishes due to the system, i.e. the Jacobi bracket satisfies:*

$$\{F_i, F_j\} = 0 \mod \mathcal{J}_{l_i + l_j - 1}(F_1, \ldots, F_r), \quad \forall\, 1 \leq i < j \leq r. \qquad \square$$

Theorem A was announced in [KL$_4$]. The corollary was established in [KL$_3$] and its particular cases for $n = 2$ and $r = 2$ appeared in [KL$_1$] and [KL$_2$] respectively. We notice however that the technique used in these papers was different and we required an additional assumption that at least one of the equations has no multiple components in the characteristic variety. We remarked then that this condition is superfluous, but proved the claim only for the second order equations. Now we can remove this technical assumption completely.

Recall ([S, GS, Go]) that the obstructions to integrability belong to the second Spencer δ-cohomology group $H^{*,2}(\mathcal{E})$ (we recall the definition in §2.1). Thus it is important to calculate this bi-graded cohomology $H^{*,j}(\mathcal{E}) = \oplus_i H^{i,j}(\mathcal{E})$.

Theorem B. *Let $\mathcal{E}$ be a system of differential equations defined by a set of $r = \mathrm{codim}(\mathcal{E})$ differential operators $\Delta = (\Delta_1, \ldots, \Delta_r) : C^\infty(\pi) \to C^\infty(\nu)$ (can be of different orders). If $\mathcal{E}$ is a generalized complete intersection, then the only non-zero Spencer δ-cohomology are given by the formula:*

$$H^{*,j}(\mathcal{E}) = \begin{cases} \pi & \text{for } j = 0, \\ \nu & \text{for } j = 1, \\ S^{j-2}\pi^* \otimes \Lambda^{m+j-1}\nu & \text{for } 2 \leq j \leq r + 1 - m\,(\leq n). \end{cases}$$

In the above formula we describe $H^{*,j}(\mathcal{E})$ as a usual (non-graded) vector space. See §5.1 for more information about grading.

For the case of scalar systems $m = \dim \pi = 1$ we have: $H^{*,j}(\mathcal{E}) \simeq \Lambda^j \nu$, $0 \leq j \leq r$. This corresponds to the following well-known algebraic result: Algebra g^* of $\mathrm{codim}\, g^* = r$ is a complete intersection iff its Koszul homology forms the exterior algebra $H_i(g^*) = \Lambda^i H_1(g^*)$, $0 \leq i \leq r$ ([BH]).

The precise obstructions to formal integrability $W_i(\mathcal{E})$ are certain curvature-type invariants called Weyl tensors [L$_1$]. In [KL$_1$, KL$_2$] we calculated them for $\mathrm{codim}(\mathcal{E}) = 2$ complete intersections in terms of Jacobi-Mayer brackets. Now we can generalize this result in terms of our multi-brackets:

Corollary 2. *There is a basis $e_1, \ldots, e_s$ in $H^{*,2}(\mathcal{E})$, $s = \binom{r}{m+1}$, and a bijection ψ between the set of power $(m+1)$ subsets of $\{1, \ldots, r\}$ and the set $\{1, \ldots, s\}$ such that the graded Weyl tensor $W(\mathcal{E}) = \oplus_i W_i(\mathcal{E})$ of the system of equations $\mathcal{E} = \{F_i = 0 \,|\, 1 \leq i \leq r\}$ with $l(i) = \mathrm{ord}\, F_i$ equals*

$$W(\mathcal{E}) = \sum_{1 \leq i_1 < \cdots < i_{m+1} \leq r} [F_{i_1}, \ldots, F_{i_{m+1}}]_{\mathcal{E}} \cdot e_{\psi(i_1, \ldots, i_{m+1})}. \qquad \square$$

This follows directly from theorems A and B. What is more interesting is the precise form of the basis. We calculated it for the case of 2 scalar equations in [KL$_1$, KL$_2$]. The result immediately generalizes to arbitrary complete intersections. The case $m > 1$ is more involved and we do not discuss it here.

Finally we give a result on the space $\mathcal{S}_{\mathcal{E}} = \mathcal{S}ol_{\mathcal{E}}$ of local/formal solutions of the system $\mathcal{E} \subset J^k(\pi)$ of generalized complete intersection type. As before n is dimension of the base M of π and m its rank. Let r be the formal codimension (see §2.1) of the system $\mathcal{E}$, the same number as in definition 1.

In classical textbooks the solutions space is characterized as follows: a general solution (a generic point of $\mathcal{S}ol_{\mathcal{E}}$) depends on s_p functions of p variables, s_{p-1} functions of $(p-1)$ variables, ... and s_0 constants, where s_i are Cartan characters (introduced by E.Cartan [C]; we adapt notations from [BCG^3]).

Here p (called genre of $\mathcal{E}$) is the maximal number, such that $s_p \neq 0$: only this character s_p has absolute meaning (citing [C]). We call the number p *formal functional dimension* of the solutions space $\mathcal{S}_{\mathcal{E}}$ and the number $d = s_p$ *formal functional rank*.

The above numbers are well-defined in analytic category, i.e. when the PDEs and the solutions are considered analytic, see Cartan's test [BCG^3]. Cartan-Kähler theorem guarantees integrability. For smooth equations we need to impose additional requirements on the system to ensure that the space $\mathcal{S}ol_{\mathcal{E}}$ is non-empty and regular (see [Ho, S, M]). In general we take p and d to be the formal functional dimension and rank of the space of formal solutions.

In abstract terms the number d equals $P_{\mathcal{E}}^{(p-1)}(t)$, where $P_{\mathcal{E}}(t)$ is the Hilbert polynomial of the symbolic module of $\mathcal{E}$ and $p = \deg P_{\mathcal{E}}(t) + 1$ (see more in §5.4; for the detailed discussion of this subject we refer to [KL$_5$]).

Theorem C. *Let $\mathcal{E}$ be a formally integrable system of generalized complete intersection type. Denote its orders by $k_1, \ldots, k_r$ and the corresponding l-th symmetric polynomials by $S_l(k_1, \ldots, k_r) = \sum_{i_1 < \cdots < i_l} k_{i_1} \cdots k_{i_l}$. Then the space $\mathcal{S}ol_{\mathcal{E}}$ has formal functional dimension and rank equal respectively*

$$p = m + n - r - 1, \qquad d = S_{r-m+1}(k_1, \ldots, k_r).$$

Thus in our case p is dimension of the affine characteristic variety and when $r = m + n - 1$ the space $\mathcal{S}ol_{\mathcal{E}}$ is a d-dimensional smooth manifold. When $m < r < m + n - 1$ and the system $\mathcal{E}$ is analytical, the space $\mathcal{S}ol_{\mathcal{E}}$ is infinite-dimensional and a general analytic solution depends on precisely d arbitrary functions of $m + n - r - 1$ variables.

In smooth category the above formula for the functional rank d is important for formulation of well-posed boundary value problems. Note also that due to Cauchy-Kovalevskaya theorem the above theorem holds true in the case $r = m$ of determined system of PDEs.

0.3. Discussion and plan of the paper

The main result (Theorem A) provides an explicit compatibility criterion. To our knowledge there were only two such criteria before. One is a particular

case of our theorem for the first order scalar systems of PDEs – this was one of the motivations for the appearance of the brackets (see the historical note in [KL$_1$]) and the base for Lagrange-Charpit method (see [Gou] and §5.2).

Another classical result concerns the system of linear evolution equations and the compatibility is expressed via commutators, being thus also a special case of our general result. In fact, all known integrability methods use these simple compatibility criteria, see §5.3.

All other methods are algorithmic, but non-explicit, and are based on the Cartan's prolongation-projection idea. We mention two, which apply in the non-linear situation. One is the Spencer theory [Go, S] and the Weyl tensors in the 2nd δ-cohomology groups [L$_1$]. Another uses the differential Gröbner basis and is being implemented into computer algebra systems now [Ma, Hu]. However neither of them give precise formulas and from computational point of view our criterion is more effective [K$_1$].

Theorem B can be specified to bi-degrees, see §5.1. This is important, since it yields the place, where the system becomes involutive. In fact, we think that the generalized complete intersections represent the class of systems, where the amount of prolongations to achieve involutivity is maximal. This gives a possibility to reduce the estimate in the Poincaré δ-lemma (see [Sw], but this estimate is accepted to be too large).

Theorem C gives an asymptotic estimate for the Hilbert polynomial of the symbolic module of the system. The dimension formula is important for Lagrange-Charpit method of establishing exact solutions of PDEs.

The paper is organized as follows. In Section 1 we collect the background on the jet-spaces and linear differential operators and establish a machinery to check the formal integrability. In Section 2 we review the algebraic machinery and develop the commutative algebra concepts finishing with a resolvent for generalized complete intersections. In Section 3 we introduce multi-brackets and discuss their properties. Non-linear differential equations are treated geometrically (as in §1.1) and we refer the reader to [KLV] for more details.

In Section 4 we prove Theorem A for linear systems and then extend the methods to the non-linear situation. We apply our results to construct the compatibility complex and non-linear Spencer cohomology. Theorems B and C are proved in Section 5, where we also relate our results to classical integrability methods and multi-Poisson geometry.

In Section 6 we apply the compatibility criterion to solve some classical problems in differential geometry. We discuss invariant characterization of Liouville metrics on surfaces and the generalized Bonnet problem. Previously the compatibility criterion was applied to the plane web-geometry to solve the Blashke conjecture and to count Abelian relations [GL$_1$, GL$_2$]. This illustrates efficiency of our main result.

1. Jet-spaces and linear differential operators

In this sections we collect the basic knowledge of the geometric theory of differential equations required for our goals.

1.1. Systems of PDEs

Let M be a smooth n-dimensional manifold and $\pi : E_\pi \to M$ a (vector) bundle of rank m. Two local sections $s_1, s_2 \in C^\infty_{\text{loc}}(\pi)$ having tangency of order $\geq k$ are said to have the same k-jet at $x \in M$ and the equivalence class is called the k-jet $x_k = [s]^k_x$.

Thus we obtain the jet-bundle $\pi_k : J^k(\pi) \to M$ and there are natural projections $\pi_{k,l} : J^k(\pi) \to J^l(\pi)$ for $l < k$. We denote $x_l = \pi_{k,l}(x_k)$. Any smooth section $s \in C^\infty_{\text{loc}}(\pi)$ induces the local section $j_k s : x \mapsto [s]^k_x$ of the bundle π_k.

A system of PDEs $\mathcal{E}$ is represented as a collection of subsets $\mathcal{E}_k \subset J^k(\pi)$, $k \geq 0$, satisfying certain conditions. The first one, regularity, is that $\mathcal{E}_k$ with restricted map $\pi_{k,l}$ is a (fiber) bundle. To formulate the second condition let us define for a submanifold $\mathcal{E} \subset J^k(\pi)$ its ith prolongation by the formula

$$\mathcal{E}^{(i)} = \{x_{k+i} = [s]^{k+i}_x \in J^{k+i}(\pi) : j_k s(M) \text{ is tangent to } \mathcal{E} \text{ at } x_k \text{ with order} \geq i\}.$$

Thus we form $\mathcal{E}$ by a collection of some given equations $\mathcal{E}_k$ and the other $\mathcal{E}_l$ are obtained via the prolongation.

So a system of different order PDEs is the following collection of submanifolds: $\mathcal{E}_i = J^i(\pi)$ until a certain order l_0, at which we add some PDEs and get $\mathcal{E}_{l_0} \subset J^{l_0}(\pi)$, then $\mathcal{E}_i = \mathcal{E}^{(i-l_0)}_{l_0}$ for $l_0 < i < l_1$, whereupon we add new equations, obtain $\mathcal{E}_{l_1}$, prolong this system until jet-level l_2 etc.

Following Cartan's prolongation-projection scheme we consider $\pi_{i+s,i}(\mathcal{E}_{i+s})$ and if this is a proper subset of $\mathcal{E}_i$, the system becomes inconsistent in the sense that we need to add some equations not specified in the original system.

If we wish to exclude this we obtain: The system $\mathcal{E}$ is said to be *compatible* on the level k if $\emptyset \neq \mathcal{E}_{k+1} \subset \mathcal{E}^{(1)}_k$ and $\pi_{k+1,k} : \mathcal{E}_{k+1} \to \mathcal{E}_k$ is surjective. In the regular case the last map is a bundle projection (submersion).

The system $\mathcal{E}$ is said to be integrable to order k if it is compatible on every level $l \leq k$. System $\mathcal{E}$ is called *formally integrable* if it is integrable to order ∞ (we usually assume regularity).

We always assume there are no functional equations in $\mathcal{E}$, i.e. $\mathcal{E}_0 = J^0(\pi) = E_\pi$. The minimal l such that $\mathcal{E}_l \neq J^l(\pi)$ is called the *minimal order* l_0 of the system. Every number l with the property $\mathcal{E}_l \neq \mathcal{E}^{(1)}_{l-1}$ is called an *order* and codimension of $\mathcal{E}_l$ in $\mathcal{E}^{(1)}_{l-1}$ is called its *multiplicity*.

Due to Cartan-Kuranishi theorem on prolongations (in the regular case) the set of orders $\text{ord}(\mathcal{E}) \subset \mathbb{N}$ is finite, i.e. there exists a *maximal order* $l_{\max}$ starting from which $\mathcal{E}^{(1)}_l = \mathcal{E}_{l+1}$.

Every local coordinate system (x^i, u^j) on the bundle π induces coordinates (x^i, p^j_σ) on $J^k(\pi)$ (multiindex $\sigma = (i_1, \ldots, i_n)$ has length $|\sigma| = \sum^n_{s=1} i_s \leq k$),

where $p_\sigma^j\big([s]_x^k\big) = \dfrac{\partial^{|\sigma|} s^j}{\partial x^\sigma}(x)$. We call them *canonical coordinates*.

In a sequel we will need the operator of *total differential* (also denoted $\hat{d}$):

$$\mathcal{D} : C^\infty(J^k(\pi)) \to \Omega^1(M) \otimes_{C^\infty(M)} C^\infty(J^{k+1}(\pi)).$$

To define $\mathcal{D}$ we note that every function on $J^k(\pi)$ is a scalar differential operator $\Box : C^\infty(\pi) \to C^\infty(M)$ of order k. Post-composing it with a vector field $X \in \mathcal{D}(M)$ we get a differential operator $\Box' : C^\infty(\pi) \to C^\infty(M)$ of order $k+1$, producing the needed 1$^{\text{st}}$ order differential operator $\mathcal{D}_X = i_X \circ \mathcal{D} : C^\infty(J^k(\pi)) \to C^\infty(J^{k+1}(\pi))$.

If we write in local coordinates $X = \sum X^i \partial_{x^i}$, then $\mathcal{D}_X = \sum X^i \mathcal{D}_i$ in the corresponding canonical coordinates, where the operator of *total derivative* $\mathcal{D}_i = \mathcal{D}_{\partial_{x^i}}$ is given by the infinite series (when applied, only finite number of terms act non-trivially):

$$\mathcal{D}_i = \partial_{x^i} + \sum p_{\sigma+1_i}^j \partial_{p_\sigma^j}. \tag{1}$$

Similarly for $X \in S^l \mathcal{D}(M)$ we get the operator $\mathcal{D}_X : C^\infty(J^k(\pi)) \to C^\infty(J^{k+l}(\pi))$. For instance, if $\sigma = (i_1, \ldots, i_n)$ is a multiindex, we obtain $\mathcal{D}_\sigma = \mathcal{D}_1^{i_1} \cdots \mathcal{D}_n^{i_n}$.

1.2. Linear differential operators

Denote by $\mathbf{1}$ the trivial one-dimensional bundle over M. Let $\mathcal{A}_k = \mathrm{Diff}_k(\mathbf{1}, \mathbf{1})$ be the $C^\infty(M)$-module of scalar linear differential operators of order $\le k$ and $\mathcal{A} = \cup_k \mathcal{A}_k$ be the corresponding filtered algebra, $\mathcal{A}_k \circ \mathcal{A}_l \subset \mathcal{A}_{k+l}$.

Consider two linear vector bundles π, ν. Denote by $\mathrm{Diff}(\pi, \nu) = \cup_k \mathrm{Diff}_k(\pi, \nu)$ the filtered module of all differential operators from $C^\infty(\pi)$ to $C^\infty(\nu)$. We have the natural pairing

$$\mathrm{Diff}_k(\rho, \nu) \times \mathrm{Diff}_l(\pi, \rho) \to \mathrm{Diff}_{k+l}(\pi, \nu)$$

given by the composition of differential operators.

In particular, $\mathrm{Diff}(\pi, \mathbf{1})$ is a filtered left $\mathcal{A}$-module, $\mathrm{Diff}(\mathbf{1}, \pi)$ is a filtered right $\mathcal{A}$-module and they have an $\mathcal{A}$-valued $\mathcal{A}$-linear pairing

$$\Delta \in \mathrm{Diff}_l(\pi, \mathbf{1}), \ \nabla \in \mathrm{Diff}_k(\mathbf{1}, \pi) \mapsto \langle \Delta, \nabla \rangle = \Delta \circ \nabla \in \mathcal{A}_{k+l},$$

with $\langle \alpha \Delta, \nabla \rangle = \alpha \langle \Delta, \nabla \rangle$, $\langle \Delta, \nabla \alpha \rangle = \langle \Delta, \nabla \rangle \alpha$ for $\alpha \in \mathcal{A}$.

Each differential operator $\Delta : C^\infty(\pi) \to C^\infty(\nu)$ of order l induces an $\mathcal{A}$-homomorphism $\phi_\Delta : \mathrm{Diff}(\mathbf{1}, \pi) \to \mathrm{Diff}(\mathbf{1}, \nu)$ by the formula:

$$\mathrm{Diff}_k(\mathbf{1}, \pi) \ni \nabla \mapsto \Delta \circ \nabla \in \mathrm{Diff}_{k+l}(\mathbf{1}, \nu).$$

Its $\langle \, , \rangle$-dual is the $\mathcal{A}$-homomorphism $\phi^\Delta : \mathrm{Diff}(\nu, \mathbf{1}) \to \mathrm{Diff}(\pi, \mathbf{1})$ given by

$$\mathrm{Diff}_k(\nu, \mathbf{1}) \ni \Box \mapsto \Box \circ \Delta \in \mathrm{Diff}_{k+l}(\pi, \mathbf{1}).$$

By the very definitions of jets with $\mathcal{J}^k(\pi) = C^\infty(\pi_k)$ we have:

$$\mathrm{Diff}_k(\pi, \nu) = \mathrm{Hom}_{C^\infty(M)}(\mathcal{J}^k(\pi), C^\infty(\nu)), \qquad (2)$$

and differential operators Δ are in bijective correspondence with morphisms $\psi^\Delta : J^l(\pi) \to \nu$ via the formula $\Delta = \psi^\Delta \circ j_l$, where $j_l : C^\infty(\pi) \to \mathcal{J}^l(\pi)$ is the jet-section operator.

The prolongation $\psi_k^\Delta : J^{k+l}(\pi) \to J^k(\nu)$ of $\psi^\Delta = \psi_0^\Delta$ is conjugated to the $\mathcal{A}$-homomorphism $\phi^\Delta : \mathrm{Diff}_k(\nu, \mathbf{1}) \to \mathrm{Diff}_{k+l}(\pi, \mathbf{1})$ via isomorphism (2). This makes a geometric interpretation of the differential operator Δ as the bundle morphism.

Similarly, the homomorphism $\phi_\Delta : \mathrm{Diff}_k(\mathbf{1}, \pi) \to \mathrm{Diff}_{k+l}(\mathbf{1}, \nu)$ is conjugated to the following morphism:

$$
\begin{array}{ccc}
\mathrm{Hom}(J^k(\mathbf{1}), \pi) & \xrightarrow{\;\;\psi_\Delta^k\;\;} & \mathrm{Hom}(J^{k+l}(\mathbf{1}), \nu) \\[4pt]
\Big\downarrow{\scriptstyle\text{prolongation}} & & \Big\uparrow{\scriptstyle i^*} \\[4pt]
\mathrm{Hom}(J^l(J^k(\mathbf{1})), J^l(\pi)) & \xrightarrow{\;\;\psi^\Delta\;\;} & \mathrm{Hom}(J^l(J^k(\mathbf{1})), \nu),
\end{array}
$$

where $i : J^{k+l}(\mathbf{1}) \to J^l(J^k(\mathbf{1}))$ is the natural embedding.

1.3. Systems of differential equations as modules

A system $\mathcal{E}$ of PDEs of order l associated to Δ is, by definition, the subbundle $\mathcal{E}_l = \mathrm{Ker}(\psi^\Delta) \subset J^l(\pi)$. Its prolongation is $\mathcal{E}_{k+l} = \mathcal{E}_l^{(k)} = \mathrm{Ker}(\psi_k^\Delta) \subset J^{k+l}(\pi)$.

We define the dual $\mathcal{E}^*$ as the cokernel of the operator ϕ^Δ:

$$\mathrm{Diff}_k(\nu, \mathbf{1}) \xrightarrow{\;\phi_k^\Delta\;} \mathrm{Diff}_{k+l}(\pi, \mathbf{1}) \to \mathcal{E}_{k+l}^* \to 0.$$

So $\mathcal{E}^* = \{\mathcal{E}_i^*\}$. We have a natural map $\pi_{i+1,i}^* : \mathcal{E}_i^* \to \mathcal{E}_{i+1}^*$. Then we define the inductive limit $\mathcal{E}^\Delta = \varinjlim \mathcal{E}_i^*$. Notice that it is a filtered left $\mathcal{A}$-module. Thus we can consider a system as a module over differential operators ($\mathcal{D}$-module).

The dual $\mathcal{E}_\Delta = \mathrm{Ker}(\phi_\Delta) \subset \mathrm{Diff}(\mathbf{1}, \pi)$ is a right $\mathcal{A}$-module and we have a pairing $\mathcal{E}^\Delta \times \mathcal{E}_\Delta \to \mathcal{A}$. This pairing is however non-degenerate only for formally-integrable systems. This follows from the following statement:

Proposition 1. *A system $\mathcal{E} = \mathrm{Ker}(\psi^\Delta)$ is formally integrable iff $\mathcal{E}_i^*$ are projective $C^\infty(M)$-modules and the maps $\pi_{i+1,i}^*$ are injective.*

Proof. The projectivity condition is equivalent to regularity (constancy of rank), while invjectivity of $\pi_{i+1,i}^*$ is equivalent to surjectivity of $\pi_{i+1,i}$. $\qquad\square$

We can associate to the above modules their symbolic analogs. Namely, since $ST^*M \otimes \pi = \oplus S^i T^*M \otimes \pi$ is the graded module associated to the filtrated $C^\infty(M)$-module $\mathrm{Diff}(\pi, \mathbf{1}) = \cup \mathrm{Diff}_i(\pi, \mathbf{1})$, the bundle morphism ϕ^Δ produces the homomorphisms (symbols) $\sigma_k^\Delta : S^{k+l}T^*M \otimes \pi \to S^k T^*M \otimes \nu$ of our differential operator Δ. Its dual is the graded degree l morphism

$$\sigma_\Delta : STM \otimes \nu^* \to STM \otimes \pi^*.$$

The value $\sigma_{\Delta,x}$ of σ_Δ at $x \in M$ is a homomorphism of ST_xM-modules.

The ST_xM-module $\mathcal{M}_\Delta = \text{Coker}(\sigma_{\Delta,x})$ is called the *symbolic module* at $x \in M$. Its annihilator is the called *characteristic ideal* $I(\Delta)$ and the set of its zeros is the *characteristic variety* $\text{Char}(\Delta)$. We will always consider projectivization of this conical affine variety.

Moreover in this paper we shall complexify the symbolic module and work with complex characteristics. In particular, the characteristic variety becomes $\text{Char}_x^{\mathbb{C}}(\Delta) \subset P^{\mathbb{C}}T_x^*M$.

Proposition 2 [Go, S]. *For $p \in T_x^*M \setminus \{0\}$ let $\mathfrak{m}(p) \subset S(T_xM) = \oplus_{i \geq 0}S^iT_xM$ be the maximal ideal of homogeneous polynomials vanishing at p. Then localization $(\mathcal{M}_\Delta)_{\mathfrak{m}(p)} \neq 0$ iff the covector p is characteristic.*

The set of the localizations $(\mathcal{M}_\Delta)_{\mathfrak{m}(p)} \neq 0$ for characteristic covectors p form the *characteristic sheaf* $\mathcal{K}$ over the characteristic variety $\text{Char}_x^{\mathbb{C}}(\Delta)$.

If we have several differential operators $\Delta_i \in \text{Diff}(\pi, \nu_i)$ of different orders l_i, $1 \leq i \leq t$, then their sum is no longer a differential operator of pure order $\Delta : C^\infty(\pi) \to C^\infty(\nu)$, $\nu = \oplus\nu_i$. Then ϕ^Δ is not an $\mathcal{A}$-morphism, unless we put certain weights to the graded components ν_i. Still we have the bundle morphism ψ^Δ and the symbol map $\sigma_\Delta : STM \otimes \nu^* \to STM \otimes \pi^*$, which becomes a homomorphism after a suitable weighting (in §2.3). This will be used in the next section to pursuit the theory into the general setting of various orders systems.

2. Algebra of differential equations

Here we review the basics of symbolic theory, establish the preparatory material and extend it to the general case of non-linear differential equations.

2.1. Spencer cohomology

We consider at first the symbolic theory. Let $T = T_xM$ be the tangent space to the base and $N = T_{x_0}\pi^{-1}(x)$, $x_0 \in E_\pi$, the tangent space to the fiber of π. We can identify $F_k(x_k) = T_{x_k}[\pi_{k,k-1}^{-1}(x_{k-1})]$ with $S^kT^* \otimes N$ and let $g_k = g(x_k) = T_{x_k}\mathcal{E}_k \cap F_k(x_k)$ be the *symbol* of differential equation $\mathcal{E}$. Clearly $g_0 = N$ and $g_i = S^iT^* \otimes N$ for $i < l_1$ – the minimal order of the system.

The symbol of the de Rham operator is called Spencer δ-operator

$$\delta : S^kT^* \otimes N \to S^{k-1}T^* \otimes N \otimes T^*$$

and it maps g_k to $g_{k-1} \otimes T^*$. In other words, if

$$g_k^{(1)} = \{p \in S^{k+1}T^* \otimes N \,|\, \delta p \in g_k \otimes T^*\}$$

is the *first prolongation*, which in the regular case equals the symbol of the equation $\mathcal{E}_k^{(1)}$, then $g_k \subset g_{k-1}^{(1)}$.

Definition 2. *A sequence of subspaces* $g_k \subset S^k T^* \otimes N$, $k \geq 0$, *is called a symbolic system if* $g_{k+1} \subset g_k^{(1)}$.

Thus symbols of a PDE system form a symbolic system. With every such a system we associate its Spencer δ-complex of order k:

$$0 \to g_k \xrightarrow{\delta} g_{k-1} \otimes T^* \xrightarrow{\delta} g_{k-2} \otimes \Lambda^2 T^* \to \cdots \xrightarrow{\delta} g_{k-n} \otimes \Lambda^n T^* \to 0.$$

The cohomology group at the term $g_i \otimes \Lambda^j T^*$ is denoted by $H^{i,j}(\mathcal{E}, x_k)$, though we usually omit reference to the point and can also write $H^{i,j}(g)$.

In terms of this cohomology l is an order of the system $\mathcal{E}$ if $H^{l-1,1}(\mathcal{E}) \neq 0$ and multiplicity of this order is equal to $m(l) = \dim g_{l-1}^{(1)}/g_l = \dim H^{l-1,1}(\mathcal{E})$.

Now the number r of equations defining $\mathcal{E}$, which we called in [KL$_2$] *formal codimension*, is the total number of orders counted with multiplicities, i.e.

$$r = \mathrm{codim}(\mathcal{E}) = \sum_{k=1}^{\infty} \dim H^{k-1,1}(\mathcal{E}).$$

By Hilbert basis theorem this number is finite.

Consider a symbolic system $g = \{g_k \subset S^k T^* \otimes N \,|\, k \geq 0\}$ and let $g^* = \oplus g_k^*$ be its graded dual over $\mathbb{R}$ (or possibly $\mathbb{C}$) . Then g^* is an ST-module with the structure operation given by the formula

$$\langle w \cdot \varkappa, p \rangle = \langle \varkappa, \delta_w p \rangle, \quad w \in S^k T, \ \varkappa \in g_l^*, \ p \in g_{k+l},$$

where $\delta_w = \delta_{w_1} \cdots \delta_{w_k}$ for $w = w_1 \cdots w_k \in S^k T$, $w_j \in T$, and $\delta_{w_j} = i_{w_j} \circ \delta :$ $g_t \to g_{t-1}$. This g^* is called the *symbolic module*. It coincides with the module $\mathcal{M}_\Delta$ introduced in §1.3 in the case of linear equations of the same order.

This module is Noetherian and the Spencer cohomology of g dualizes to the Koszul homology of g^*.

2.2. Characteristic variety and Fitting ideals

Define the *characteristic ideal* by $I(g) = \mathrm{ann}(g^*) \subset ST$ in the ring of polynomials $R = ST$ and the *characteristic variety* as the set of non-zero covectors $v \in T^*$ such that for every k there exists a vector $w \in N \setminus \{0\}$ with $v^k \otimes w \in g_k$. This is a punctured conical affine variety. We projectivize its complexification and denote it by $\mathrm{Char}^{\mathbb{C}}(g) \subset P^{\mathbb{C}} T^*$. When $g = g(x_k)$ is the symbol of the system at a point $x_k \in \mathcal{E}$, we also denote the characteristic variety by $\mathrm{Char}_{x_k}^{\mathbb{C}}(\mathcal{E})$.

Another definition of characteristic variety is given via the homogeneous characteristic ideal $I(g)$ graded by the degree: $I = \oplus I_k$.

Proposition 3 [S]. $\mathrm{Char}^{\mathbb{C}}(g) = \{p \in P^{\mathbb{C}} T^* \,|\, f(p^k) = 0 \,\forall f \in I_k, \forall k\}$.

Consider the symbolic R-module g^*. Its *dimension* $\dim_R g^*$ is the minimal number d of homogeneous elements $f_1, \ldots, f_d \in R$ of positive degree such that

the quotient $g^*/(f_1, \ldots, f_d)g^*$ is a finite-dimensional vector space. Thus due to equality $\dim_R g^* = \dim[R/\operatorname{ann}(g^*) = R/I(g)]$, we can interpret

$$\dim_R g^* = \dim_{\mathbb{C}} \operatorname{Char}^{\mathbb{C}}(g) + 1$$

as dimension of the affine characteristic variety.

A sequence of elements $f_1, \ldots, f_s \in R$ is called regular if f_i is not a zero divisor in the R-module $g^*/(f_1, \ldots, f_{i-1})g^*$. *Depth* of the module g^* is the maximal number of elements in a regular sequence. The depth and dimension of a module g^* are related by the following inequality:

$$\operatorname{depth} g^* \leq \dim g^*$$

(we shall omit sometimes the subscript R in $\dim_R$). Now g^* is called a Cohen-Macaulay module if $\operatorname{depth} g^* = \dim g^*$ (see [BH] for details). In such a case we call the system $\mathcal{E}$ and the corresponding symbolic system g Cohen-Macaulay.

For an ideal $I \subset R$ and an R-module G the length of maximal G-regular sequence in I is denoted $\operatorname{depth}(I, G)$. *Depth* of the ideal I is $\operatorname{depth} I = \operatorname{depth}(I, R)$ (this quantity is also called the *grade* of the ideal I). In these terms the depth of the module is $\operatorname{depth} g^* = \operatorname{depth}(\mathfrak{m}, R/\operatorname{Ann} g^*)$, where $\mathfrak{m} = \oplus_{i>0} S^i T$ is the maximal ideal in R of homogeneous polynomials with positive degree.

We shall also use *codimension* of the ideal I, which is defined as $\operatorname{codim} I = \min \dim R_{\mathfrak{p}}$, the lower bound being taken over all primes $\mathfrak{p} \supset I$ in R [E] (in other sources it is called *height* [BH, BV]).

Both the depth and the codimension are geometric quantities, i.e. they are defined by the conical affine locus of the ideal I over $\mathbb{C}$: If $\operatorname{Rad}(I)$ is the radical of I, then $\operatorname{depth} \operatorname{Rad}(I) = \operatorname{depth} I$ and $\operatorname{codim} \operatorname{Rad}(I) = \operatorname{codim} I$. Moreover, since our ring R is polynomial, for any ideal $I \subset R$ we have the equality

$$\operatorname{depth} I = \operatorname{codim} I.$$

For a homomorphism of free R-modules $\varphi : U \to V$ with $\operatorname{rank} U = r$ and $\operatorname{rank} V = m$ denote by $J_j(\varphi)$ the image of the map $\Lambda^j U \otimes \Lambda^j V^\star \to R$ induced by the map $\Lambda^j \varphi$, where $\star$ means the functor $\operatorname{Hom}_R(\cdot, R)$. If we choose bases for U and V, i.e. identify $U \simeq R^r$ and $V \simeq R^m$, then the map φ is represented by an $m \times r$ matrix A and the ideal $J_j(\varphi) \subset R$ is generated by all $j \times j$ minors of A. For the case of pure order differential operator Δ of §1.3 we mean: $U = ST \otimes \nu^*$, $V = ST \otimes \pi^*$, $\varphi = \sigma_\Delta$.

Let $G = \operatorname{Coker}(\varphi)$. By the Fitting lemma the ideal $\operatorname{Fitt}_i(G) = J_{m-i}(\varphi)$ does not depend on representation $U \overset{\varphi}{\to} V \to G \to 0$ and is called the i-th Fitting invariant of G. It is known that $\operatorname{Fitt}_0(G) \subset \operatorname{ann}(G)$, the two terms have the same radicals and the equality $\operatorname{Fitt}_0(G) = \operatorname{ann}(G)$ holds if $\operatorname{depth} \operatorname{ann}(G) = r - m + 1$ (see [E, BV]).

We will be interested in the dual over $\mathbb{R}$ map $\varphi^* : V^* \to U^*$, which is the symbol of the collection of differential operators determining the system $\mathcal{E}$. Thus in this case $g = \operatorname{Ker}(\varphi^*) = \operatorname{Coker}(\varphi)^*$, whence $G = g^*$ and $\operatorname{ann}(G) = I(g)$.

Notice that the characteristic variety of the symbolic system g can be written via $\varphi^* : ST^* \otimes \pi \to ST^* \otimes \nu$ as $\operatorname{Char}^{\mathbb{C}}(g) = \{p \in P^{\mathbb{C}} T^* \mid \operatorname{rank} \varphi^*(p) < m\}$, where

by $\varphi^*(p) : \pi \to \nu$ we understand the value of φ^* at the covector p. Then we define also the *characteristic sheaf* (or kernel sheaf; actually it is not a sheaf, but just a family of vector spaces) over $\mathrm{Char}^{\mathbb{C}}(g)$ by associating to the covector p the subspace $\mathcal{K}_p = \mathrm{Ker}\,\varphi^*(p) \subset \pi$.

Remark 1. *The last requirement of definition 1,* $\dim K_p = 1\ \forall p \in \mathrm{Char}^{\mathbb{C}}(\mathcal{E})$, *means that the system is similar to a system of scalar PDEs and thus be treated via the usual Koszul complex as in $[KL_2]$. This, however, cannot be fully formalized, so that we use another approach with generalized Koszul complexes.*

2.3. Application of the Buchsbaum-Rim complex

Let F_i be some (not necessary linear) differential operators from a bundle π to a bundle ν_i of ord $F_i = l(i)$, $1 \le i \le t$. Denote $f_i = \sigma(F_i)$ the dual symbols (i.e. σ_Δ) of these operators. These $f_i : U_i \to V$ are ST-homomorphisms of degree $l(i)$, where $U_i = ST \otimes \nu_i^*$, $V = ST \otimes \pi^*$. Let $U = \oplus U_i$, $\dim U = \sum \dim U_i = r$.

Consider the map $\varphi^* = \oplus f_i^* : V^* \to U^*$, which is the symbol of differential operator $F = (F_1, \ldots, F_t)$. Its $\mathbb{R}$-dual $\varphi = \sum f_i : U \to V$ is a morphism of R-modules, but it is not a graded homomorphism unless the system is of pure order, i.e. $l(i) \equiv k$. However it becomes homogeneous of degree 1 if we consider the weighted grading $U \simeq \oplus U_i$, with the weight $l(i)^{-1}$ for the i-th summand.

We wish to find all R-relations between the homomorphisms f_i. In other words, we seek to determine the 1-syzygy of the map $\varphi : U \to V$. It is given by the Fitting ideal, but we better describe the whole free resolution.

This resolution belongs to the Buchsbaum-Eisenbud family of complexes $\mathcal{C}^i$ ($[E]$), from which we are interested in the Buchsbaum-Rim complex $\mathcal{C}^1$:

$$0 \to S^{r-m-1}V^\star \otimes \Lambda^r U \xrightarrow{\partial} S^{r-m-2}V^\star \otimes \Lambda^{r-1}U \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^{m+1}U \xrightarrow{\varepsilon} U \xrightarrow{\varphi} V.$$

Here ∂ is the multiplication by the trace element $e \in V \otimes V^\star \subset SV \otimes \Lambda V^\star$ ($\Lambda V^\star$ acts on ΛU via the map $\Lambda \varphi^\star$), corresponding to $1 \in R \hookrightarrow V \otimes V^\star$:

$$\partial(a^k \otimes b_1 \wedge \ldots \wedge b_t) = \sum (-1)^i k \langle \varphi(b_i), a \rangle\, a^{k-1} \otimes b_1 \wedge \ldots \check{b}_i \ldots \wedge b_t.$$

The splice map $\varepsilon : \Lambda^{m+s}U \to \Lambda^s U$ is the action of $\Lambda^m \varphi^\star \Omega \in \Lambda^m U^\star$, where $\Omega \in \Lambda^m V^\star \simeq R$ is a volume element (generator).

As proved in $[BR]$ the complex $\mathcal{C}^1$ is exact iff the map φ satisfies the condition depth $J_m(\varphi) \ge r - m + 1$.

Definition 3. *Let us call an R-module G* generalized complete intersection *if* $\mathrm{codim}\,\mathrm{ann}(G) \ge r - m + 1$ *(for a minimal free resolution $U \xrightarrow{\varphi} V \to G \to 0$).*

Note that the usual complete intersections $G = R/I$ satisfy this property.

Remark 2. *By the generalized principal ideal theorem of Macaulay ($[E]$) we have:* $\mathrm{codim}\,\mathrm{ann}(G) = \mathrm{codim}\,\mathrm{Fitt}_0(G) \le r - m + 1$, *so that in fact we have an equality above. In addition, as we shall see, the module G is Cohen-Macaulay, whenever it is a generalized complete intersection.*

Proposition 4. *If an R-module G is a generalized complete intersection, then the Buchsbaum-Rim complex is exact.*

Proof. Let $\varphi : U \to V$ be the 1-syzygy map for G, $r = \dim U$, $m = \dim V$. Then we have:

$$\operatorname{depth} J_m(\varphi) = \operatorname{depth} \operatorname{Fitt}_0(G) = \operatorname{depth} \operatorname{ann}(G) = \operatorname{codim} \operatorname{ann}(G) = r - m + 1,$$

where the first and third equalities are general properties of Noetherian modules, the last one is part of the definition and the second is a property of Fitting ideal, mentioned in §2.2. Therefore the Buchsbaum-Rim complex is exact. $\qquad\square$

Remark 3. *Since the polynomial ring R is an affine domain, we have [BH, E]:*

$$\dim\big(R/\operatorname{ann}(G)\big) = \dim R - \operatorname{codim} \operatorname{ann}(G) = n + m - r - 1.$$

Recall that a ring P is called *determinental* if $P = S/Q_s$, where S is a regular Cohen-Macaulay ring and Q_s is the ideal generated by $s \times s$ minors of an $m \times r$ matrix A such that the codimension of Q_s in S is exactly $(m - s + 1)(r - s + 1)$. By a theorem of Eagon and Hochster such rings are Cohen-Macaulay [BV]. Let us also call the ideal Q_s itself determinental, if this makes no confusion.

Theorem 5. *Let the symbolic module $G = g^*$ be a generalized complete intersection in the sense of definition 1 and $\varphi : ST \otimes (\oplus \nu_i^*) \to ST \otimes \pi^*$ be the corresponding R-homomorphism. Then we have:*

1. *The ideal $J_m(\varphi)$ is determinental;*

2. *$\operatorname{Fitt}_0(G) = I(g) = \operatorname{ann}(G)$;*

3. *G is a generalized complete intersection in the sense of definition 3.*

Proof. Let the conditions of definition 1 be satisfied. It was shown in [KL$_2$] that if the characteristic sheaf $\mathcal{K}$ over $\operatorname{Char}^{\mathbb{C}}(g)$ has fibers of constant dimension k, then $\operatorname{codim} \operatorname{Char}^{\mathbb{C}}(g) \leq l = k(r - m + k)$. When $k = 1$ we get $l = r - m + 1$ and this is exactly the codimension of the characteristic variety of g

$$\operatorname{codim} \operatorname{Char}^{\mathbb{C}}(g) = r - m + 1,$$

determined by $m \times m$ minors, or equivalently by the Fitting ideal $J_m(\varphi)$. Thus we see that the ideal $J_m(\varphi)$ is determinental and $\operatorname{codim} J_m(\varphi) = r - m + 1$.

This implies that the ring $R/J_m(\varphi)$ is Cohen-Macaulay and $\operatorname{depth} J_m(\varphi) = \operatorname{codim} J_m(\varphi)$. Since $\operatorname{Fitt}_0(g^*) = J_m(\varphi)$ and $I(g) = \operatorname{ann}(g^*)$ have the same radicals we have:

$$\operatorname{codim} I(g) = \operatorname{codim} J_m(\varphi) = \operatorname{depth} J_m(\varphi) = r - m + 1.$$

Thus by the results of §2.2 we conclude that $\operatorname{Fitt}_0(g^*) = I(g)$. $\qquad\square$

Corollary 3. *The Buchsbaum-Rim complex $\mathcal{C}^1$ is a resolution of the symbolic module g^* if the latter is a generalized complete intersection.*

Remark 4. *In [KL$_2$] we also obtained a criterion when the ideal $\operatorname{Fitt}_0(g^*) = J_m(\psi)$ is a topological complete intersection.*

2.4. Non-linear differential equations

In this section we study non-linear differential equations $\mathcal{E}$. A system of such equations can be considered as sequence of submanifolds $\mathcal{E}_k \subset J^k(M)$ with the property $\mathcal{E}_k^{(1)} \supset \mathcal{E}_{k+1}$ (we assume regularity, but do not require formal integrability of $\mathcal{E}$).

Let $\mathfrak{F} = C^\infty(J^\infty \pi)$ be the filtered algebra of smooth functions depending on finite jets of π, i.e. $\mathfrak{F} = \cup_i \mathfrak{F}_i$ with $\mathfrak{F}_i = C^\infty(J^i \pi)$.

Denote $\mathfrak{F}_i^{\mathcal{E}} = C^\infty(\mathcal{E}_i)$. The projections $\pi_{i+1,i} : \mathcal{E}_{i+1} \to \mathcal{E}_i$ induce the maps $\pi_{i+1,i}^* : \mathfrak{F}_i^{\mathcal{E}} \to \mathfrak{F}_{i+1}^{\mathcal{E}}$, so that we can form the space $\mathfrak{F}^{\mathcal{E}} = \cup \mathfrak{F}_i^{\mathcal{E}}$, the points of which are infinite sequences $(f_i, f_{i+1}, \dots)$ with $f_i \in \mathfrak{F}_i^{\mathcal{E}}$ and $\pi_{i+1,i}^*(f_i) = f_{i+1}$. This $\mathfrak{F}^{\mathcal{E}}$ is a $C^\infty(M)$-algebra. If the system $\mathcal{E}$ is not formally integrable, the set of infinite sequences can be void, and the algebra $\mathfrak{F}^{\mathcal{E}}$ can be trivial. To detect formal integrability, we investigate the finite level jets algebras $\mathfrak{F}_i^{\mathcal{E}}$ via the following algebraic approach.

Let $\mathcal{E}$ be defined by a collection $F = (F_1, \dots, F_r)$ of non-linear scalar differential operators of (possibly different) orders $l(1), \dots, l(r)$. Each F_i determines a sequence of smooth maps $J^k(\pi) \to J^{k-l(i)}(\mathbf{1})$ and so their collection yields a map $J^\infty(\pi) \to J^\infty(\nu)$, where $\nu = \oplus \nu_s$ with $\dim \nu_s = m(s) = \{ \#i : l(i) = s \}$, $\sum m(s) = r$.

Pre-composition of our differential operator $F : C^\infty(\pi) \to C^\infty(\nu)$ with other non-linear differential operators gives the following exact sequence of $C^\infty(M)$-modules

$$\operatorname{diff}(\nu, \mathbf{1}) \xrightarrow{F} \operatorname{diff}(\pi, \mathbf{1}) \to \mathfrak{F}^{\mathcal{E}} \to 0 \tag{3}$$

Note that $\mathcal{J}_s(F_1, \dots, F_r) = \operatorname{Im}(F)_s \subset \operatorname{diff}_s(\pi, \mathbf{1})$ is the submodule described in Introduction, and

$$\mathfrak{F}_i^{\mathcal{E}} = \operatorname{diff}_i(\pi, \mathbf{1}) / \mathcal{J}_i(F_1, \dots, F_r). \tag{4}$$

On the level of finite jets, the map F of (3) decreases the order appropriately, but is not homogenous. However we can adjust this by imposing weights to the vector bundles ν_i as we did in §2.3. Thus we can assume for simplicity that the operator f has pure order k.

It is important that the terms of (3) are modules over the algebra of scalar $\mathscr{C}$-differential operators $\mathscr{C}\operatorname{Diff}(\mathbf{1}, \mathbf{1})$, which are total derivative operators and have the following form in local coordinates [KLV]: $\Delta = \sum f_\sigma \mathcal{D}_\sigma$, with $f_\sigma \in C^\infty(J^\infty(M))$. We can identify $\mathscr{C}\operatorname{Diff}(\mathbf{1}, \mathbf{1}) = \cup \mathfrak{F}_i \otimes \operatorname{Diff}_j(\mathbf{1}, \mathbf{1})$ with the twisted tensor product of the algebras $\mathfrak{F}$ and $\operatorname{Diff}(\mathbf{1}, \mathbf{1})$ over the action

$$\hat{\Delta} : \; \mathfrak{F}_i \to \mathfrak{F}_{i+j} \quad \text{for} \quad \Delta \in \operatorname{Diff}_j(\mathbf{1}, \mathbf{1}).$$

This $\mathscr{C}\operatorname{Diff}(\mathbf{1}, \mathbf{1})$ is a non-commutative $C^\infty(M)$-algebra. We need a more general $\mathfrak{F}$-module of $\mathscr{C}$-differential operators $\mathscr{C}\operatorname{Diff}(\pi, \mathbf{1}) = \cup \mathscr{C}\operatorname{Diff}_i(\pi, \mathbf{1})$, where

$$\mathscr{C}\operatorname{Diff}_i(\pi, \mathbf{1}) = \mathfrak{F}_i \otimes_{C^\infty(M)} \operatorname{Diff}_i(\pi, \mathbf{1}).$$

Remark that $\mathscr{C}\operatorname{Diff}(\pi, \mathbf{1})$ is a filtered $\mathscr{C}\operatorname{Diff}(\mathbf{1}, \mathbf{1})$-module, i.e. $\mathscr{C}\operatorname{Diff}_i(\mathbf{1}, \mathbf{1}) \cdot \mathscr{C}\operatorname{Diff}_j(\pi, \mathbf{1}) \subset \mathscr{C}\operatorname{Diff}_{i+j}(\pi, \mathbf{1})$.

Define now the filtered $\mathfrak{F}^{\mathcal{E}}$-module $\mathscr{C}\operatorname{Diff}^{\mathscr{E}}(\pi,\mathbf{1})$ with $\mathscr{C}\operatorname{Diff}_i^{\mathscr{E}}(\pi,\mathbf{1}) = \mathfrak{F}_i^{\mathscr{E}} \otimes \operatorname{Diff}_i(\pi,\mathbf{1})$. Since the module $\operatorname{Diff}(\pi,\mathbf{1})$ is projective and we can identify $\operatorname{diff}(\pi,\mathbf{1})$ with $\mathfrak{F}$, we have from (4) the following exact sequence

$$0 \to \mathcal{J}_s(F) \otimes \operatorname{Diff}_s(\pi,\mathbf{1}) \to \mathscr{C}\operatorname{Diff}_i(\pi,\mathbf{1}) \to \mathscr{C}\operatorname{Diff}_i^{\mathscr{E}}(\pi,\mathbf{1}) \to 0. \qquad (5)$$

Similar modules can be defined for the vector bundle ν and they determine the $\mathfrak{F}^{\mathcal{E}}$-module $\mathcal{E}^* = \cup \mathcal{E}_i^*$ by the following sequence:

$$\mathscr{C}\operatorname{Diff}_i^{\mathscr{E}}(\nu,\mathbf{1}) \xrightarrow{\ell_F} \mathscr{C}\operatorname{Diff}_{i+k}^{\mathscr{E}}(\pi,\mathbf{1}) \to \mathscr{E}_{i+k}^* \to 0, \qquad (6)$$

where $\ell : \operatorname{diff}(\pi,\nu) \to \mathfrak{F} \otimes_{C^\infty(M)} \operatorname{Diff}(\pi,\nu)$ is the operator of universal linearization [KLV], $\ell_F = \ell(F)$.

This sequence is not exact in the usual sense, but it becomes exact in the following one. The space to the left is an $\mathfrak{F}_i^{\mathcal{E}}$-module, the middle term is an $\mathfrak{F}_{i+k}^{\mathcal{E}}$-module. The image $\ell_F(\mathscr{C}\operatorname{Diff}_i^{\mathscr{E}}(\nu,\mathbf{1}))$ is an $\mathfrak{F}_i^{\mathcal{E}}$-module, but we generate by it an $\mathfrak{F}_{i+k}^{\mathcal{E}}$-submodule in the middle term. With this understanding of the image the term $\mathcal{E}_{i+k}^*$ of (6) is an $\mathfrak{F}_{i+k}^{\mathcal{E}}$-module and the sequence is exact. In other words

$$\mathcal{E}_s^* = \mathscr{C}\operatorname{Diff}_s^{\mathscr{E}}(\pi,\mathbf{1})/(\mathfrak{F}_s^{\mathscr{E}} \cdot \operatorname{Im}\ell_F).$$

Sequences (6) are nested (i.e. their union is filtered) and so we have the sequence

$$\mathcal{E}_{s-1}^* \to \mathcal{E}_s^* \to \mathcal{F}g_s^* \to 0, \qquad (7)$$

which becomes exact if we treat the image of the first arrow as the corresponding generated $\mathfrak{F}_s^{\mathcal{E}}$-module. Thus $\mathcal{F}g_s^*$ is an $\mathfrak{F}_s^{\mathcal{E}}$-module with support on $\mathcal{E}_s$ and its value at a point $x_s \in \mathcal{E}_s$ is dual to the s-symbol of the system $\mathcal{E}$:

$$(\mathcal{F}g_s^*)_{x_s} = g_s^*(x_s); \qquad g_s(x_s) = \operatorname{Ker}[T_{x_s}\pi_{s,s-1} : T_{x_s}\mathcal{E}_s \to T_{x_{s-1}}\mathcal{E}_{s-1}].$$

This is a geometric definition of the symbol. Equivalently we can use the algebraic approach as in §1.3: Graded space $g = \oplus g_s$ is dual to cokernel of the symbol σ_F of F, considered as an ST-homomorphism $ST \otimes \nu^* \to ST \otimes \pi^*$, which depends on the point of equation $\mathcal{E}$.

Our weight-convention apply here and hence we describe the situation on the level of finite jets $x_i \in \mathcal{E}_i$ for a pure order k operator F, which is the case represented by the following exact sequence ($x = \pi_i(x_i) \in M$) with the dual symbol map:

$$0 \to g_i(x_i) \to S^i T_x^* M \otimes \pi_x \xrightarrow{\sigma_i^F(x_i)} S^{i-k} T_x^* M \otimes \nu_x.$$

Remark 5. *We interpret $\mathfrak{F}^{\mathcal{E}}$ as the algebra of all smooth functions on our equation $\mathcal{E}$. Define $\mathcal{E}^* = \cup \mathcal{E}_i^*$ in the same manner as $\mathfrak{F}^{\mathcal{E}}$, taking into account that the map $\mathcal{E}_s^* \to \mathcal{E}_{s+1}^*$ in our approach is coupled with the change of rings ($\mathfrak{F}_s^{\mathcal{E}}$ to $\mathfrak{F}_{s+1}^{\mathcal{E}}$). So we can think of $\mathcal{E}^*$ as of sections of the symbolic bundle g over $\mathcal{E}$ with $\mathfrak{F}^{\mathcal{E}}$-coefficients. Thus (note linearization in (6)) we interpret $\mathcal{E}^*$ as the space $\Omega^1(\mathcal{E})$ of differential forms on $\mathcal{E}$.*

This remark gives us a way to treat formal integrability of the system $\mathcal{E}$ as possibility of augmenting exact sequence (7) with 0 from the left (injectivity of $\mathcal{E}_s^* \to \mathcal{E}_{s+1}^*$ under the change of rings). This is reduced to the question of finding a left resolution of complex (6), which will boil down onto the symbolic level as we shall show.

Thus linearization of the system of generalized complete intersection type and methods from §2.3 will lead to the proof of our compatibility criterion.

3. Multi-bracket of vector differential operators

3.1. Non-commutative determinants

Consider the algebra $\mathcal{A} = \mathrm{Diff}(\mathbf{1}, \mathbf{1})$ of linear scalar differential operators. It is non-commutative, so no direct generalization of the determinant function $\det : \Lambda^m \mathcal{A}^m \to \mathcal{A}$ exists (tensor product is taken over scalars, not over $\mathcal{A}$). We view the elements of the space $\mathcal{A}^m$ as rows $\Delta = (\Delta_1, \ldots, \Delta_m)$, which act on columns of functions $s = (s_1, \ldots, s_m)^t \in C^\infty(M)^m$.

We define non-commutative determinant $\mathrm{Ndet} : \Lambda^m \mathcal{A}^m \to \mathcal{A}$ via the standard formula, where order of multiplication of matrix elements corresponds to the order of columns:

$$\mathrm{Ndet} \begin{pmatrix} \nabla_{11} & \nabla_{12} & \ldots & \nabla_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \nabla_{m1} & \nabla_{m2} & \ldots & \nabla_{mm} \end{pmatrix} = \sum_{\alpha \in S_m} (-1)^\alpha \nabla_{\alpha(1)1} \nabla_{\alpha(2)2} \cdots \nabla_{\alpha(m)m}.$$

In other words, we define non-commutative determinant via decomposition by columns, i.e. if $C_i(B)$ is the ith column of $B \in \mathrm{Mat}_{m \times m}(\mathcal{A})$ and $M_{ij}(B)$ is the minor obtained by removing row i and column j, then we have:

$$\mathrm{Ndet}(B) = \sum_{i=1}^m (-1)^{i-1} C_1(B)_i \, \mathrm{Ndet}(M_{i1}(B)) = \sum_{j=1}^m (-1)^{n-j} \, \mathrm{Ndet}(M_{jm}(B)) \, C_m(B)_j$$

(it is however embarrassing to write decomposition via a mid-column). We obviously have skew-symmetry by rows and $\mathbb{R}$-linearity, but we lack $\mathcal{A}$-linearity and skew-symmetry by columns. Thus we can write the non-commutative determinant in the form

$$\mathrm{Ndet}(\nabla_1 \wedge \ldots \wedge \nabla_m).$$

Note that the symbol of the non-commutative determinant is the standard determinant

$$\sigma\big(\mathrm{Ndet}(\nabla_1 \wedge \ldots \wedge \nabla_m)\big) = \det\big(\sigma(\nabla_1) \wedge \ldots \wedge \sigma(\nabla_m)\big),$$

where the symbol of order l differential operator $\nabla_i = (\nabla_{i1}, \ldots, \nabla_{im}) \in \mathcal{A}_l^m$ is

$$\sigma(\nabla_i) = \sigma_l(\nabla_i) = (\sigma_l(\nabla_{i1}), \ldots, \sigma_l(\nabla_{im})).$$

Beware that since the components of the operator can have smaller order, it is not true that $\sigma(\nabla_i) = (\sigma(\nabla_{i1}), \ldots, \sigma(\nabla_{im}))$. In other words, we consider the grading of $\mathcal{A}^m$ corresponding to increasing filtration $\{\mathcal{A}_l^m\}_{l=0}^\infty$.

Denoting $U^* = \mathrm{Hom}_{\mathbb{R}}(U, \mathcal{A})$ we have an $\mathbb{R}$-linear map

$$\Xi : \Lambda^{m+1}\mathcal{A}^m \to \mathcal{A}^{(m+1)*}$$

given by the formula

$$\Xi(\nabla_1 \wedge \ldots \wedge \nabla_{m+1}) = \mathrm{Ndet} \left(\begin{array}{|c|c|} \hline \nabla_1 & \\ \hline \ldots & \\ \hline \nabla_{m+1} & \\ \hline \end{array} \right),$$

where the last column serves as a place-holder, though the result (image of Ξ) we write as as a row.

Notice that the map $\Xi(\nabla_1 \wedge \ldots \wedge \nabla_{m+1})$ is a right $\mathcal{A}$-homomorphism for all $\nabla_i \in \mathcal{A}$, i.e. $\mathrm{Im}\,\Xi \subset \mathrm{Hom}_{\mathcal{A}}^{\mathrm{right}}(\mathcal{A}^{m+1}, \mathcal{A}) \subset \mathcal{A}^{(m+1)*}$.

Remark 6. *Since our constructions are algebraic they can be generalized to other operator algebras, like pseudo-differential operators, Fourier operators etc. Then multi-brackets of the next section lead to the compatibility conditions for the corresponding overdetermined problems.*

One can use the theory of quasi-determinants by Gelfand et al [G^{2}RW] to define other multi-brackets via similar formulas. However this requires division and extends the class of differential operators to non-local operators. It could be an exciting relation between local and global aspects of compatibility.

3.2. Multi-brackets

At first we define multi-brackets in the linear case.

Let $\Upsilon : \underbrace{\mathcal{A}^m \times \cdots \times \mathcal{A}^m}_{m+1} \to \mathrm{Mat}_{(m+1)\times m}(\mathcal{A})$ denote the matrix formed by $m + 1$ vectors-rows from $\mathcal{A}^m$. Then we define the multi-bracket

$$\Lambda^{m+1}\mathcal{A}^m \to \mathcal{A}^m$$

of $m + 1$ vector differential operators $\nabla_i \in \mathcal{A}^m$ via the operation of the last section and the multiplication action $\mathcal{A}^{(m+1)*} \times \mathrm{Mat}_{(m+1)\times m}(\mathcal{A}) \to \mathcal{A}^m$ on columns of matrices:

$$\{\nabla_1, \ldots, \nabla_{m+1}\} = \Xi(\nabla_1 \wedge \ldots \wedge \nabla_{m+1}) \cdot \Upsilon(\nabla_1, \ldots, \nabla_{m+1}).$$

The ith component of the multi-bracket is given by

$$\{\nabla_1, \ldots, \nabla_{m+1}\}_i = \Xi(\nabla_1 \wedge \ldots \wedge \nabla_{m+1})\big(C_i(\Upsilon(\nabla_1, \ldots, \nabla_{m+1}))\big).$$

It is easy to check that this multi-bracket coincides with the multi-bracket defined in the introduction. This bracket is skew-symmetric by its entries and

is $\mathbb{R}$-linear. It is not however $\mathcal{A}$-linear and does not commute with $\mathbb{R}$-linear transformations of $\mathcal{A}^m$.

To formulate properties of this bracket we will need later an opposite multi-bracket $\{\cdots\}^\dagger : \Lambda^{m+1}\mathcal{A}^m \to \mathcal{A}^m$, which is defined by the same formula except that the map Ξ is changed to $\Xi^\dagger$, with the place-holder in non-commutative determinant being put to the first column.

Note that

$$\{\mathcal{A}_{l_1}, \ldots, \mathcal{A}_{l_{m+1}}\} \subset \mathcal{A}_{l_1 + \cdots + l_{m+1} - 1},$$

where $\mathcal{A}_i$ is the i-th subalgebra of the filtered algebra $\mathcal{A}$.

Let now $F_i \in \operatorname{diff}(m \cdot \mathbf{1}, \mathbf{1})$, $i = 1, \ldots, m+1$, be non-linear differential operators of orders $\operatorname{ord}(F_i) = l(i)$, which we can identify with smooth functions of the jet-space space, i.e. elements of $C^\infty(J^{l(i)}(M; \mathbb{R}^m)) \subset \mathcal{F}(J^\infty(M; \mathbb{R}^m))$. Then we can define the multi-bracket $\{F_1, \ldots, F_{m+1}\}$ as an operation

$$\Lambda^{m+1}\mathcal{F}(J^\infty(M; \mathbb{R}^m)) \to \mathcal{F}(J^\infty(M; \mathbb{R}^m))$$

via the linearization operator $\ell : \operatorname{diff}(m \cdot \mathbf{1}, \mathbf{1}) \to C^\infty(J^\infty(M; \mathbb{R}^m)) \otimes_{C^\infty(M)} \mathcal{A}^m$, see [KLV].

Namely, exploring the formula for the linear case, we let:

$$\{F_1, \ldots, F_{m+1}\} = \Xi(\ell_{F_1} \wedge \ldots \wedge \ell_{F_{m+1}}) \cdot \Upsilon(F_1, \ldots, F_{m+1}).$$

This multi-bracket is related to the multi-bracket of linear differential operators via the formula

$$\ell_{\{F_1, \ldots, F_{m+1}\}} = \{\ell_{F_1}, \ldots, \ell_{F_{m+1}}\}. \tag{8}$$

Similarly we can define the opposite multi-bracket $\{\cdots\}^\dagger$ for non-linear differential operators.

3.3. Non-commutative "Plücker identities"

The multi-bracket we introduced does not satisfy the Jacobi identity of Nambu [N] (or generalized Poisson) multi-bracket. Neither does it satisfy the axioms of SH-algebras [LS] (because the background is different: If we change the length of the multi-bracket, the functional space changes as well).

However there are certain properties, these brackets do satisfy. Later in this section we explain that they should be viewed as a kind of generalized Jacobi identity. For simplicity we begin with the formulation in the linear case.

Theorem 6. *Let $\nabla_i \in \mathcal{A}^m$ be linear vector differential operators, $1 \leq i \leq m+2$, and let $\nabla_{i,j}$ denote component j of ∇_i. Then we have the identities (where as usual check means absence of argument) relating the multi-bracket and the opposite multi-bracket for $1 \leq i \leq m$:*

$$\sum(-1)^k \{\nabla_1, \ldots, \check{\nabla}_k, \ldots, \nabla_{m+2}\}_i^\dagger \nabla_k = \sum(-1)^k \nabla_{k,i} \{\nabla_1, \ldots, \check{\nabla}_k, \ldots, \nabla_{m+2}\}.$$

Proof. Indeed let $\Upsilon_i = C_i(\hat{\Upsilon})$ be column i of the matrix $\hat{\Upsilon}(\nabla_1, \ldots, \nabla_{m+2})$, the map $\hat{\Upsilon} : \underbrace{\mathcal{A}^m \times \cdots \times \mathcal{A}^m}_{m+2} \to \mathrm{Mat}_{(m+2)\times m}(\mathcal{A})$ being given by the same rule as Υ (but with one more row).

Then the right hand side of the identity is obtained by decomposing the determinant

$$\mathrm{Ndet}\left(\boxed{\Upsilon_i} \begin{array}{c} \boxed{\nabla_1} \\ \boxed{\cdots} \\ \boxed{\nabla_{m+2}} \end{array} \boxed{\Upsilon_j} \right)$$

via the first and then the last column, while the left hand side of the identity is the result of decomposition by the last and then the first column. But these operations commute. Now we unite the results by j to a row. $\qquad\square$

Let $\sigma \in S_m$ be a permutation and $T_\sigma : \mathcal{A}^m \to \mathcal{A}^m$ be the corresponding linear transformation, $\nabla_i = (\nabla_{i,1}, \ldots, \nabla_{i,m}) \mapsto T_\sigma(\nabla_i) = (\nabla_{i,\sigma(1)}, \ldots, \nabla_{i,\sigma(m)})$.

This action leads to conjugated multi-brackets given by

$$\{\nabla_1, \ldots, \nabla_{m+1}\}^\sigma = T_\sigma^{-1}\{T_\sigma(\nabla_1), \ldots, T_\sigma(\nabla_{m+1})\}.$$

It's easy to see that

$$\{\nabla_1, \ldots, \nabla_{m+1}\}_m^\dagger = \{T_\tau(\nabla_1), \ldots, T_\tau(\nabla_{m+1})\}_1 = \{\nabla_1, \ldots, \nabla_{m+1}\}_m^\tau$$

for $\tau = \begin{pmatrix} 1 & 2 & \ldots & m \\ m & 1 & \ldots & m-1 \end{pmatrix}$. Thus we get an identity for the multi-bracket alone:

Corollary 4. *For the above cyclic permutation τ and arbitrary vector differential operators $\nabla_i \in \mathcal{A}^m$ it holds:*

$$\sum (-1)^k \{\nabla_1, \ldots, \check{\nabla}_k, \ldots, \nabla_{m+2}\}_m^\tau \nabla_k = \sum (-1)^k \nabla_{k,m}\{\nabla_1, \ldots, \check{\nabla}_k, \ldots, \nabla_{m+2}\}.$$

We readily generalize the above formulae for non-linear operators (with the help of linearization operator as in the previous section). For instance, the latter formula becomes:

$$\sum (-1)^k \left(\ell_{\{F_1, \ldots, \check{F}_k, \ldots, F_{m+2}\}_m^\tau} F_k - \ell_{F_{k,m}}\{F_1, \ldots, \check{F}_k, \ldots, F_{m+2}\} \right) = 0.$$

Notice that this formula for $m = 1$ becomes the standard Jacobi identity. In this scalar case our multi-bracket becomes bi-bracket and it coincides with the classical Jacobi bracket $\{F, G\}$ of scalar (non-linear) differential operators $F, G \in \mathrm{diff}(\mathbf{1}, \mathbf{1})$ (in the linear case $F, G \in \mathcal{A}$ it is the commutator, for the non-linear case see [KLV]). Indeed the formula is:

$$\sum_{\mathrm{cyclic}} \left(\ell_F\{G, H\} - \ell_{\{G,H\}}F \right) = \sum_{\mathrm{cyclic}} \{F, \{G, H\}\} = 0.$$

20

Thus the multi-bracket identities could be considered as generalized Jacobi identities (but not in the sense of [APB]). We however call them non-commutative Plücker identities by the following reason. Consider for simplicity the case $m = 2$.

In this case the multi-bracket of operators $\nabla_i = (\nabla_{i,1}, \nabla_{i,2})$, $i = 1, 2, 3$ correspond to the composition $\varphi_0 \varphi_1$ of the $\mathcal{A}$-homomorphisms

$$0 \to \mathcal{A} \xrightarrow{\varphi_1} \mathcal{A}^3 \xrightarrow{\varphi_0} \mathcal{A}^2,$$

where (the determinant is Ndet)

$$\varphi_1 = \left(\begin{vmatrix} \nabla_{2,1} & \nabla_{2,2} \\ \nabla_{3,1} & \nabla_{3,2} \end{vmatrix}, \begin{vmatrix} \nabla_{3,1} & \nabla_{3,2} \\ \nabla_{1,1} & \nabla_{1,2} \end{vmatrix}, \begin{vmatrix} \nabla_{1,1} & \nabla_{1,2} \\ \nabla_{2,1} & \nabla_{2,2} \end{vmatrix} \right), \quad \varphi_0 = \begin{pmatrix} \nabla_{1,1} & \nabla_{1,2} \\ \nabla_{2,1} & \nabla_{2,2} \\ \nabla_{3,1} & \nabla_{3,2} \end{pmatrix}.$$

The above sequence is not a complex (whence the multi-bracket), but its symbolic part is a complex and is actually a resolution of the module g^* corresponding to the system $\{\nabla_1[u] = 0, \nabla_2[u] = 0, \nabla_3[u] = 0\}$.

To perceive the properties of the multi-brackets we consider 4 vector differential operators $\nabla_i \in \mathcal{A}^2$, $1 \leq i \leq 4$, and extend the above complex to

$$0 \to \mathcal{A}^2 \xrightarrow{\phi_2} \mathcal{A}^4 \xrightarrow{\phi_1} \mathcal{A}^4 \xrightarrow{\phi_0} \mathcal{A}^2,$$

where ϕ_0 is a 4×2 matrix with rows ∇_i, ϕ_2 is the transposed matrix and ϕ_1 is a skew-symmetric 4×4 matrix with entries $\begin{vmatrix} \nabla_{i,1} & \nabla_{i,2} \\ \nabla_{j,1} & \nabla_{j,2} \end{vmatrix}$ being non-commutative determinants.

Again the symbolic sequence is exact, but the general sequence is not a complex and the composition $\phi_0 \phi_1 \phi_2$ gives us the desired properties of the multi-bracket. Clearly the above "resolution" is built on a certain determinental identity, which is exactly the Plücker identity in Grassmaninan $G(2, 4)$.

For $m > 2$ we see that our non-commutative identities model the standard Plücker identities in other Grassmaninans.

3.4. Reduced brackets and coordinates

Let $\mathcal{E} = \{F_1[u] = 0, \ldots, F_r[u] = 0\}$ be an overdetermined system of PDEs, where $F_i \in \operatorname{diff}(m \cdot \mathbf{1}, \mathbf{1})$ and $u = (u_1, \ldots, u_m)^t \in C^\infty(M, \mathbb{R}^m)$. As in the introduction we denote by $\mathcal{J}_s(\mathcal{E}) = \langle \ell_\Delta \circ F_i \mid \operatorname{ord}(\Delta) + \operatorname{ord} F_i \leq s \rangle$ the submodule generated by $F_1, \ldots, F_r$.

We let $\mathcal{E}_s^* = \mathcal{A}_s^m / \mathcal{J}_s(\mathcal{E})$ in the linear case and $\mathcal{E}_s^* = \operatorname{diff}_s(m \cdot \mathbf{1}, \mathbf{1}) / \mathcal{J}_s(\mathcal{E})$ in the non-linear. In this way we obtain the reduced bracket

$$[f_1, \ldots, f_{m+1}]_{\mathcal{E}} = \{f_1, \ldots, f_{m+1}\} \bmod \mathcal{J}_{l-1}(\mathcal{E}) \in \mathcal{E}_{l-1}^*.$$

for $l = l(f_1) + \cdots + l(f_{m+1})$.

Remark 7. *This multi-bracket appears due to the fact, that we do not have a unique non-commutative determinant. If we consider determinants with the values in the reduced (quotient) module, as it is done in the case of Dieudonne determinant, we will arrive to this reduced multi-bracket.*

Every k-th order scalar differential operator $G \in \mathrm{diff}_k(\cdot \mathbf{1}, \mathbf{1})$ induces via linearization a map $\hat{G} : \mathcal{E}_s^* \to \mathcal{E}_{s+k}^*$. With respect to this map Theorem 6 implies the reduced identities:

Theorem 7. *Let $\mathcal{E}$ be an over-determined system $(r > m)$ defined by (non-linear) differential operators $F_1, \ldots, F_r$. Then for any subset $\{i_1, \ldots, i_{m+2}\} \subset \{1, \ldots, r\}$ and any $j \in [1, m]$ we have:*

$$\sum (-1)^k \hat{F}_{i_k, j}[F_{i_1}, \ldots, \check{F}_{i_k}, \ldots, F_{i_{m+2}}]_{\mathcal{E}} = 0. \qquad \square$$

Notice that we do not assume integrability and so the multi-brackets occurring in the Main Theorem are not arbitrary, but vanishing of some of them gives certain restrictions for the rest.

Remark 8. *In general it is not true that*

$$\{\mathcal{J}_{l_1}(\mathcal{E}), \ldots, \mathcal{J}_{l_{m+1}}(\mathcal{E})\} \subset \mathcal{J}_{l_1 + \cdots + l_{m+1} - 1}(\mathcal{E}),$$

but for formally integrable $\mathcal{E}$ it is. Then we can define the bracket

$$[\mathcal{E}_s^*, \mathcal{J}_{l_1}(\mathcal{E}), \ldots, \mathcal{J}_{l_m}(\mathcal{E})]_{\mathcal{E}} \subset \mathcal{E}_{l_1 + \cdots + l_m + s - 1}^*.$$

Then elements $\theta \in \mathcal{E}_s^*$ such that $[\theta, \mathcal{J}_{l_1}(\mathcal{E}), \ldots, \mathcal{J}_{l_m}(\mathcal{E})]_{\mathcal{E}} = 0$ with respect to this bracket, can be interpreted as another generalization of the classical notion of symmetry.

Finally we can give a coordinate representation of the introduced multi-bracket. For calculational purposes it is however more convenient to work with the following multi-bracket:

$$[F_1, \ldots, F_{m+1}] = \frac{1}{m!} \sum_{\substack{\sigma \in S_{m+1} \\ \nu \in S_m}} \frac{\mathrm{sgn}(\sigma)}{\mathrm{sgn}(\nu)} \sum_{\substack{1 \leq i \leq m \\ |\tau_i| = l(\sigma(i))}} \prod_{j=1}^m \frac{\partial F_{\sigma(j)}}{\partial p_{\tau_j}^{\nu(j)}} \mathcal{D}_{\tau_1 + \cdots + \tau_m} F_{\sigma(m+1)},$$

where $F_i \in \mathrm{diff}_{l(i)}(m \cdot \mathbf{1}, \mathbf{1})$. For $m = 1$ this gives Mayer brackets instead of Jacobi brackets [KL$_1$]. The following statement is straightforward:

Proposition 8. *Restrictions of the two multi-brackets to the system $\mathcal{E}$ coincide:*

$$[F_1, \ldots, F_{m+1}]_{\mathcal{E}} \equiv [F_1, \ldots, F_{m+1}] \mod \mathcal{J}_{l-1}(\mathcal{E}),$$

where $l = \sum_{i=1}^{m+1} l(i)$ as before. $\qquad \square$

4. Compatibility criterion

In this section we prove our compatibility criterion. Its particular cases are theorems from [KL$_1$, KL$_2$, KL$_3$], where we used geometric theory of PDEs and the obstructions to compatibility were identified with certain curvatures (Weyl tensors). Here we propose an approach based on the construction of symbolic compatibility complex, which uses the dual algebraic approach.

4.1. Syzygies for modules of linear differential operators

Consider the filtered $\mathcal{A}$-module $\mathrm{Diff}(\pi, \mathbf{1})$ and let

$$\mathrm{Diff}(\nu, \mathbf{1}) \xrightarrow{\phi^{\Delta}} \mathrm{Diff}(\pi, \mathbf{1}) \to \mathcal{E}^* \to 0$$

be a representation of the dual to a system of linear equations $\mathcal{E}$. Set $\mathcal{I} = \mathrm{Im}(\phi^{\Delta})$. In other words, we let $\mathcal{I}_k \subset \mathrm{Diff}_k(\pi, \mathbf{1})$ denote the sequence of sub-modules $\mathcal{J}_k(\mathcal{E}) = \mathcal{J}_k(F_1, \ldots, F_r)$ as in Introduction, $F_i \in \mathrm{Diff}_{l(i)}(\pi, \mathbf{1})$. Notice that these $\mathcal{I}_k$ define our equations: $\mathcal{E}_k = \{x_k : h(x_k) = 0 \, \forall h \in \mathcal{I}_k\} \subset J^k(\pi)$.

Proposition 9. *The formal integrability of the system $\mathcal{E}$ is equivalent to the requirement that $\mathcal{I}_{k+1} \cap \mathrm{Diff}_k(\pi, \mathbf{1}) = \mathcal{I}_k$ for all k.*

Remark 9. *The last condition means it is not possible to get new relations of order k in $\mathcal{I}$ from the relations of order $k+1$ via linear combinations over $\mathcal{A}_{k+1}$.*

Proof. In fact, surjectivity of the projections $\pi_{k+1,k} : \mathcal{E}_{k+1} \to \mathcal{E}_k$ (formal integrability) is equivalent to injectivity of the dual maps $\pi_{k+1,k}^* : \mathcal{E}_k^* \to \mathcal{E}_{k+1}^*$. The claim follows from the natural isomorphism $\mathcal{E}_k^* \simeq \mathrm{Diff}_k(\pi, \mathbf{1})/\mathcal{I}_k$. $\qquad \square$

Now if $\mathcal{I}$ is generated by linear differential operators $F_1, \ldots, F_r$, every element in $\mathcal{I}_{k+1}$ is represented in the form $\sum A_i F_i$, where $A_i = \sum a_i^\tau \mathcal{D}_\tau$ are scalar differential operators of $\mathrm{ord}(A_i) \leq k+1 - \mathrm{ord}(F_i)$. The condition of proposition 9 is equivalent to the following relation on the k-symbols:

$$\sigma\left(\sum A_i F_i\right) = \sum \sigma(A_i)\sigma(F_i) = 0 \in S^{k+1}T \otimes N^*. \tag{9}$$

To describe all such relations in the submodule of symbolic relations $I \subset ST \otimes N^*$ we use the syzygy approach of §2.3.

We also denote $V^* = ST^* \otimes \pi$ and $U^* = ST^* \otimes \nu$, where the bundle $\nu = \oplus \nu_i$ is, in general, graded by the degrees of operators Δ_i. We have symbols $f_i = \sigma(F_i)$ (previously denoted σ^{F_i}) of our differential operators F_i, $i = 1, \ldots, r$. The map $\psi = \varphi^* : V^* \to U^*$, which can be represented in bases as $\psi : R^m \to R^r$, $(u^1, \ldots, u^m) \mapsto (f^1, \ldots, f^r)$, has the kernel $g \subset ST^* \otimes N$.

If $\Pi = \Pi(g) \subset ST \otimes N^*$ is the annihilator submodule, then $ST \otimes N^*/\Pi(g)$ is the symbolic module $g^* = \mathrm{Coker}(\varphi)$. Consider $\mathbb{R}$-dual to the Buchsbaum-Rim complex from §2.3 (as the Spencer complex is $\mathbb{R}$-dual to the Koszul complex):

$$0 \to g \to V^* \xrightarrow{\psi} U^* \xrightarrow{\omega} \Lambda^{m+1}U^* \xrightarrow{\delta} V^\times \otimes \Lambda^{m+2}U^* \xrightarrow{\delta} \ldots S^{r-m-1}V^\times \otimes \Lambda^r U^* \to 0,$$

where $\delta = \partial^*$, $\omega = \varepsilon^*$ and $V^\times = (V^\star)^*$. Choosing a basis $e_1, \ldots, e_m$ of $V^* \simeq R^m$ we can describe informally $\omega(\xi) = \xi \wedge \psi(e_1) \wedge \ldots \wedge \psi(e_m)$. However as the symbolic differential operator φ decreases degrees, we need to change it to $\omega(h_1 \ldots h_m \xi) = \xi \wedge \psi(h_1 e_1) \wedge \ldots \wedge \psi(h_m e_m)$ for certain elements $h_i \in R$ of sufficiently high degrees.

Thus all the relations between the symbolic differential operators $\sigma(\Delta_i)$ are given by the explicit formula $\omega \circ \psi = 0$ from the above complex, which we eventually call generalized Spencer δ-complex.

4.2. Proof of the main theorem for linear systems

At first let us consider two partial cases.

1. *Scalar equations.* In this case $m = 1$ and the condition of definition 1 says that $g^* = ST/I(g)$ is the usual complete intersection. So $I(g) = \langle f_1, \ldots, f_r \rangle$, where $\{f_i\}$ form a regular sequence of length $r \leq n$. Then the Koszul complex

$$0 \to I \otimes \Lambda^r R^r \xrightarrow{\partial} \cdots \to I \otimes \Lambda^2 R^r \xrightarrow{\partial} I \otimes R^r \to I \to 0$$

is exact. In particular, vanishing of the 1st homology yields:

$$\sum_{i=1}^{r} a_i f_i = 0 \implies a_i = \sum_{j=1}^{r} c_{ij} f_j, \quad c_{ij} + c_{ji} = 0 \qquad (a_i, c_{ij} \in R),$$

and so the 1-syzygy module is generated by the relations $f_i f_j - f_j f_i = 0$. Thus we need to check the condition of proposition 9 only for combinations $F_i F_j - F_j F_i$ (multiplication in algebra $\mathcal{A}$ is non-commutative), see (9).

As a consequence we obtain that the commutator $[F_i, F_j]$, which is an operator of order $l_{ij} = l(i) + l(j) - 1$ should belong to the space $\mathcal{J}_{l_{ij}}$ generated by $F_1, \ldots, F_r$ and their total derivatives up to the order l_{ij}. This is the compatibility condition for the system $\mathcal{E}$, exactly as theorem A states.

2. *Systems on two-dimensional manifolds.* In this case $n = 2$ and condition 1 of definition 1 gives $r = m + 1$. Now instead of Koszul complex we use the following approximation to a resolution

$$0 \to g \to R^m \xrightarrow{\psi} R^{m+1} \xrightarrow{\tau} \hat{I} \to 0,$$

where $\hat{I} = a J_m(\psi) \subset R$ for a non-zero divisor a. Let $f_i : R^m \to R$ be symbols of the defining equations for the system $\mathcal{E}$ and $f_i(e_j) = f_{ij}$ their values on a basis. Denote by $A(f) = \|f_{ij}\|$ the $(m+1) \times m$ matrix of the operator ψ. Then the map τ is given by the formula $\tau(\xi) = \sum_{i=1}^{m+1} (-1)^{i-1} \xi_i \det A_i(f)$, with $A_i(f)$ is obtained from $A(f)$ by deleting the i-th row (Laplace decomposition).

The Hilbert-Burch theorem states that the above complex is exact whenever depth $J_m(\psi) \geq 2$. By theorem 5 this follows from the conditions of Definition 1 (in particular $\mathrm{Char}^{\mathbb{C}}(g) = \emptyset$) and also we see that we have equality. Thus the only generator of 1-syzygy is the relation:

$$\sum_{i=1}^{m+1} (-1)^{i-1} \det A_i(f) f_i = 0.$$

Let $\mathcal{A}_i(f)$ be some differential operators with the symbols $A_i(f)$. As a consequence of the 1-syzygy description we obtain that the expression

$$\sum_{i=1}^{m+1}(-1)^{i-1}\det\mathcal{A}_i(f)F_i, \tag{10}$$

which is an operator of order $l = l(1) + \cdots + l(r) - 1$, should belong to the space $\mathcal{J}_l$ generated by $F_1, \ldots, F_r$ and their total derivatives up to the order l. Noticing that the expression in (10) is the multi-bracket $\{F_1, \ldots, F_r\}$, we get the compatibility condition from theorem A.

Now, as we have clarified the simple situations, we will study

3. *The general linear case.*

Here we should use the generalized Spencer complex described in §4.1 ($\mathbb{R}$-dualization of the complex $\mathcal{C}^1$ from §2.3). If the assumptions of definition 1 are satisfied, we conclude that the relations generating the 1-syzygy are of the type (10), namely

$$\sum_{j=1}^{m+1}(-1)^{j-1}\det A_{\varrho,i_j}(f)f_{i_j},$$

where $A_\varsigma(f)$ is obtained from the $r \times m$ matrix $A(f)$ of the map φ by deleting rows with the numbers $\varsigma \subset \{1, \ldots, r\}$ and ϱ is the complement to the subset $\{i_1, \ldots, i_{m+1}\}$, so that the resulting matrix $A_{\varrho,i}(f)$ is square of size $m \times m$.

It's clear that changing $A_{\varrho,i_j}(f) \mapsto \mathcal{A}_{\varrho,i_j}(f)$ and $f_{i_j} \mapsto F_{i_j}$ in the above expressions we obtain the multi-brackets $\{F_{i_1}, \ldots, F_{i_{m+1}}\}$ (or $[F_{i_1}, \ldots, F_{i_{m+1}}]$ depending on the manner we extend the symbol $A_{\varrho,i_j}(f)$ to a differential operator $\mathcal{A}_{\varrho,i_j}(f)$) for various (ordered) subsets $\{i_1, \ldots, i_{m+1}\} \subset \{1, \ldots, r\}$, which by proposition 9 should belong to the subspace $\mathcal{J}_{l(i_1)+\cdots+l(i_{m+1})-1}$ iff $\mathcal{E}$ is formally integrable. The claim is proved.

4.3. The proof for non-linear operators and generalization

Now consider the general case of non-linear systems $\mathcal{E}$. The main theorem in this case can be proved as follows: We linearize the system. The new linear system is of generalized complete intersection type as well. Compatibility of non-linear system $\mathcal{E}$ is equivalent to compatibility for each linearization on a jet-solution. The multi-brackets are also preserved (this is well-studied in the case of usual brackets, see [KLV] for a relation between universal linearization operators, Jacobi brackets and evolutionary differentiations), see (8).

We however will not develop these vague ideas, but show instead how to modify the proof from §4.2. Let the system $\mathcal{E}$ be given by a (matrix) differential operator $\Delta = (F_1, \ldots, F_r) : C^\infty(\pi) \to C^\infty(\nu)$. The bundle ν can be locally trivialized ($\mathbb{R}^r$ above) or more generally we can split it $\nu = \oplus\nu_s$ according to different orders of the differential operators F_i. Since we showed in §2.3 how to reduce this case to the pure order by imposing weights on ν_s, we just assume the operator is of pure order k (can be set 1) $\Delta : J^i(\pi) \to J^{i-k}(\nu)$. Its symbol we denote by φ.

Let $t = i - k(m+1)$. Denote by $\Lambda^{m+1}_{[t]}$ the t-th graded component of the $(m+1)$-st exterior degree of the corresponding module in the context of symbolic systems and the t-th filtered submodule for the case of differential operators (with the above weight-convention). Let us also use the short notations $\mathrm{Diff}_i(\pi) = \mathrm{Diff}_i(\pi, \mathbf{1})$, $\mathrm{Diff}_i(\nu) = \mathrm{Diff}_i(\nu, \mathbf{1})$ [only for large diagrams].

The following diagram commutes:

$$
\begin{array}{ccccccccc}
& 0 & & 0 & & 0 & & 0 & \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\Lambda^{m+1}_{[t]}ST \otimes \nu^* & \xrightarrow{\varepsilon} & S^{i-k}T \otimes \nu^* & \xrightarrow{\varphi^*} & S^iT \otimes \pi^* & \longrightarrow & g_i^* & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \varrho} & & \uparrow & & \\
\Lambda^{m+1}_{[t]}\mathrm{Diff}(\nu) & \xrightarrow{\nabla_t^*} & \mathrm{Diff}_{i-k}(\nu) & \xrightarrow{\Delta^*} & \mathrm{Diff}_i(\pi) & \longrightarrow & \mathcal{E}_i^* & \longrightarrow & 0 \\
\uparrow & & & & {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle (\pi^{\mathcal{E}}_{i,i-1})^*} & & \\
& & \mathrm{Diff}_{i-k-1}(\nu) & \xrightarrow{\Delta^*} & \mathrm{Diff}_{i-1}(\pi) & \longrightarrow & \mathcal{E}_{i-1}^* & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
& 0 & & 0 & & \mathrm{Ker}(\pi^{\mathcal{E}}_{i,i-1})^* & & &
\end{array}
$$

The first row is a part of the Buchsbaum-Rim complex $\mathcal{C}^1$ and the operator ∇_t^* in the second one is a differential operator with the symbol ε.

Proposition 10. *Let the system $\mathcal{E}$ be generalized complete intersection. Its compatibility on the level of $(i-1)$ jets, i.e. $\mathrm{Ker}(\pi^{\mathcal{E}}_{i,i-1})^* = 0$, is equivalent to existence of an operator ∇_t^*, such that the second row is a complex.*

Proof. By the assumption the first row is exact. The diagram chase, not involving ∇_t^*, yields a homomorphism

$$\zeta_i : \Lambda^{m+1}_{[t]}ST \otimes \nu^* / \mathrm{Ker}\,\varepsilon \simeq \mathrm{Im}\,\varepsilon \longrightarrow \mathrm{Ker}(\pi^{\mathcal{E}}_{i,i-1})^*. \tag{11}$$

It is always an epimorphism, but it is a monomorphism iff the map $\Delta^* \circ \nabla_t^*$ is a monomorphism (the precise form of ∇_t^* is inessential here, we may consider its construction to be inductive by t or refer to our variety of multi-brackets). Vanishing of our homomorphism is equivalent to vanishing of $\Delta^* \circ \nabla_t^*$. $\qquad \square$

Since $\varrho \circ \Delta^* \circ \nabla_t^* = 0$, the map $\Delta^* \circ \nabla_t^* : \Lambda^{m+1}_{[t]}\mathrm{Diff}(\nu, \mathbf{1}) \to \mathrm{Diff}_i(\pi, \mathbf{1})$ from the proof can be identified as follows:

$$j^{-1} \circ \Delta^* \circ \nabla_t^* : \Lambda^{m+1}_{[t]}\mathrm{Diff}(\nu, \mathbf{1}) \to \mathrm{Diff}_{i-1}(\pi, \mathbf{1}),$$

which can be varied by a map of the form

$$\Delta^* \circ \nabla_{t-1}^* : \Lambda^{m+1}_{[t-1]}\mathrm{Diff}(\nu, \mathbf{1}) \to \mathrm{Diff}_{i-1}(\pi, \mathbf{1}).$$

Since every differential operator from $\mathrm{Diff}(\pi, \mathbf{1})$ determining our system (i.e. vanishing on $\mathcal{E}$) can be represented by operators from $\mathrm{Diff}(\nu, \mathbf{1})$ composed with the operator Δ^*, the above operator represents our multi-bracket (indeed one of them due to non-uniqueness of ∇_t^*), or more exactly by the collection of reduced multi-brackets $[F_{i_1}, \ldots, F_{i_{m+1}}]_{\mathcal{E}} = \{F_{i_1}, \ldots, F_{i_{m+1}}\} \bmod \mathcal{J}'_{i-1}$ (with an appropriate index i).

The condition $\mathrm{Ker}(\pi^{\mathcal{E}}_{i,i-1})^* = 0\ \forall i$ is equivalent to vanishing of the maps $j^{-1} \circ \Delta^* \circ \nabla_t^*$ on $\Lambda^{m+1}_{[t]} \mathrm{Diff}(\nu, \mathbf{1})$ for all t. Thus formal integrability is equivalent to vanishing of all multi-brackets of $(m+1)$-tuples of differential operators $F_1, \ldots, F_r$ due to the system $\mathcal{E}$ (as explained in theorem A).

Now this line of arguments works well for the case of non-linear vector differential operators $F = (F_1, \ldots, F_r)$:

Proof of theorem A. The above diagram over M should be considered over the equation $\mathcal{E}$ and the $\mathrm{Diff}(\mathbf{1}, \mathbf{1})$-module $\mathrm{Diff}(\pi, \mathbf{1})$ of linear differential operators should be changed to the $\mathscr{C}\,\mathrm{Diff}^{\mathscr{E}}(\mathbf{1}, \mathbf{1})$-module $\mathscr{C}\,\mathrm{Diff}^{\mathscr{E}}(\pi, \mathbf{1})$ of $\mathscr{C}$-differential operators with coefficients in $\mathfrak{F}^{\mathcal{E}}$ (see §2.4).

From the arising diagram of non-linear complexes we also obtain the multi-brackets as the obstructions to formal integrability and observe that their vanishing due to the system ensures this integrability.

Namely when the conditions of definition 1 are satisfied, we obtain the following sequence of non-linear differential operators, which is exact at the terms $\mathscr{C}\,\mathrm{Diff}^{\mathcal{E}}(\pi, \mathbf{1})$ and $\mathcal{E}^*$ (in the sense specified in §2.4):

$$\Lambda^{m+1}\mathscr{C}\,\mathrm{Diff}^{\mathcal{E}}(\nu, \mathbf{1}) \xrightarrow{\nabla^*} \mathscr{C}\,\mathrm{Diff}^{\mathcal{E}}(\nu, \mathbf{1}) \xrightarrow{\ell_F} \mathscr{C}\,\mathrm{Diff}^{\mathcal{E}}(\pi, \mathbf{1}) \longrightarrow \mathcal{E}^* \to 0. \qquad (12)$$

The composition $\ell_F \circ \nabla^*$ corresponds to multi-brackets of non-linear differential operators and this sequence is a complex iff the brackets are zero due to the system. Thus vanishing of the multi-brackets yields formal integrability.

Let us give more details. Compatibility on the level of i-jets, which is equivalent to injectivity of the map $\pi^*_{i,i-1} : \mathfrak{F}^{\mathcal{E}}_{i-1} \to \mathfrak{F}^{\mathcal{E}}_i$, can be expressed by saying that no relation of functions from $\mathcal{J}_i(F_1, \ldots, F_r)$ over $\mathfrak{F}_i$ is an operator from $\mathrm{diff}_{i-1}(\pi, \mathbf{1})$, see (3).

Linearization reduces this to the following claim (cf. Proposition 9):

$$\mathscr{C}\,\mathrm{Diff}^{\mathscr{E}}_i(\pi, \mathbf{1}) \cap \mathscr{C}\,\mathrm{Diff}_{i-1}(\pi, \mathbf{1}) = \mathscr{C}\,\mathrm{Diff}^{\mathscr{E}}_{i-1}(\pi, \mathbf{1}).$$

Thus due to exact sequences (5) and (6) the formal integrability can be expressed via 1-syzygy of the module $\mathcal{E}^*$, which is given by sequence (12).

Indeed any non-linear relation can be evaluated at points of the equation $x_i \in \mathcal{E}_i$, which gives a relation for the symbolic system $g(x_i)$. Since all such relations are described via the Buchsbaum-Rim complex, we describe 1-syzygy of $\mathcal{E}^*$ in exactly the same way as in §4.1.

This yields multi-brackets $[F_{i_1}, \ldots, F_{i_{m+1}}]_{\mathcal{E}}$ as obstructions to compatibility and theorem A is proved. $\qquad \square$

A generalization. What happens to the compatibility criterion, when the system is not of generalized complete intersection type?

Then the first row of the diagram from §4.3 needs not to be exact and homomorphism (11) does not tell much. In proposition 10 we assumed the system is of generalized complete intersection type, which gives $\operatorname{Im}\varepsilon = \operatorname{Ker}\varphi^*$.

If this does not hold, we get a cohomology group $H_{\nu^*}^{i-k}$ at the term $S^{i-k}T\otimes\nu^*$ of the Buchsbaum-Rim complex $\mathcal{C}^1$. Then homomorphism (11) will be changed to

$$\hat{\zeta}_i = \zeta_i \oplus \zeta_i' : \operatorname{Ker}\varphi^* = \operatorname{Im}\varepsilon \oplus H_{\nu^*}^{i-k} \longrightarrow \operatorname{Ker}(\pi_{i,i-1}^{\mathcal{E}})^*.$$

The first component ζ_i is represented by multi-brackets of differential operators as before, but the second component ζ_i' is of completely different nature. Again the map $\hat{\zeta}_i$ is an epimorphism, but neither ζ_i nor ζ_i' needs to be a monomorphism.

Nevertheless, since the multi-brackets together with $\operatorname{Im}\zeta_i'$ span $\operatorname{Ker}(\pi_{i,i-1}^{\mathcal{E}})^*$, they give all compatibility conditions of the system $\mathcal{E}$.

The image of the projection $\pi_{i,i-1}^{\mathcal{E}} : \mathcal{E}_i \to \mathcal{E}_{i-1}$ is the locus of the Weyl tensors $W_{i-1}(\mathcal{E}) \in H^{i-2,2}(\mathcal{E})$ (we will not describe them here, see [L$_1$]; for geometric structures they are also called structural functions [St], curvature tensors and sometimes torsion tensors [BCG^3]). We let $W(\mathcal{E}) = \oplus W_i(\mathcal{E})$. Note that the relation $W(\mathcal{E}) = 0$ is precisely the set of all compatibility conditions.

Epimorphism $\hat{\zeta}_i$ leads to the decomposition

$$\operatorname{Ker}(\pi_{i,i-1}^{\mathcal{E}})^* \simeq (\zeta_i \circ \varepsilon)(\Lambda_{[t]}^{m+1}ST \otimes \nu^*) \oplus \zeta_i'(H_{\nu^*}^{i-k}).$$

and so we get:

Theorem 11. *The Weyl tensor can be decomposed $W(\mathcal{E}) = W_B(\mathcal{E}) + W_H(\mathcal{E})$. The bracket part $W_B(\mathcal{E})$ can be expressed via multi-brackets by the same formula as in corollary 2. The second part $W_H(\mathcal{E})$ is a homological term.* $\square$

4.4. An exact sequence of non-linear differential operators

We would like now to make the complex from the proof of theorem A into an exact sequence by interchanging the functors Λ^{m+1} and Diff:

Theorem 12. *Assume that the system $\mathcal{E}$ is of generalized complete intersection type and the multi-brackets vanish as in theorem A. Then there is an exact complex of differential operators, which is a resolution of the module $\mathcal{E}^*$:*

$$\cdots \to \mathscr{C}\operatorname{Diff}^{\mathcal{E}}(\Lambda^{m+3}\nu \otimes S^2\pi^*, \mathbf{1}) \xrightarrow{\square_2^*} \mathscr{C}\operatorname{Diff}^{\mathcal{E}}(\Lambda^{m+2}\nu \otimes \pi^*, \mathbf{1}) \xrightarrow{\square_1^*}$$

$$\to \mathscr{C}\operatorname{Diff}^{\mathcal{E}}(\Lambda^{m+1}\nu, \mathbf{1}) \xrightarrow{\nabla^*} \mathscr{C}\operatorname{Diff}^{\mathcal{E}}(\nu, \mathbf{1}) \xrightarrow{\ell_F} \mathscr{C}\operatorname{Diff}^{\mathcal{E}}(\pi, \mathbf{1}) \longrightarrow \mathcal{E}^* \to 0. \quad (13)$$

Proof. Consider at first the linear case, where we shall use the notation $\Delta = (F_1, \ldots, F_r)$ for a given vector-valued operator, defining $\mathcal{E}$.

We have the following diagram with the exact columns:

$$
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\xrightarrow{\delta} \Lambda^{m+1}_{[t]} ST \otimes \nu^* & \xrightarrow{\varepsilon} & S^{i-k}T \otimes \nu^* & \xrightarrow{\varphi^*} & S^i T \otimes \pi^* & \longrightarrow & g_i^* & \longrightarrow & 0 \\
q_3 \downarrow \uparrow \rho_3 & & q_2 \downarrow \uparrow \rho_2 & & q_1 \downarrow \uparrow \rho_1 & & \uparrow \\
\cdots \to \mathrm{Diff}_t(\Lambda^{m+1}\nu) & \xrightarrow{\nabla_t^*} & \mathrm{Diff}_{i-k}(\nu) & \xrightarrow{\Delta_i^*} & \mathrm{Diff}_i(\pi) & \xrightarrow{\alpha_i} & \mathcal{E}_i^* & \longrightarrow & 0 \\
\uparrow j_3 & & \uparrow j_2 & & \uparrow j_1 & & \uparrow (\pi_{i,i-1}^{\mathcal{E}})^* \\
\ldots \mathrm{Diff}_{t-1}(\Lambda^{m+1}\nu) & \xrightarrow{\nabla_{t-1}^*} & \mathrm{Diff}_{i-k-1}(\nu) & \xrightarrow{\Delta_{i-1}^*} & \mathrm{Diff}_{i-1}(\pi) & \xrightarrow{\alpha_{i-1}} & \mathcal{E}_{i-1}^* & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

The first row is an exact sequence. We will now inductively construct the operators ∇_t^*, $\Box_1^*$ etc, so that all the rows are exact and the diagram commutes.

Let us choose a connection on the base manifold M and on the bundles π and ν. Then we can canonically embed the symbol bundle into the bundle of differential operators ([P, L$_3$]) via some maps q_i, whence the splitting of the middle row of the diagram into direct sum of the first and the third rows.

Now elements of the middle complex are 2-vectors, with the upper and lower components being elements of the first and the third complex respectively, so that the maps ∇_t^* and Δ_i^* must have the following matrix form

$$
\nabla_t^* = \begin{pmatrix} \varepsilon & 0 \\ X_t & \nabla_{t-1}^* \end{pmatrix}, \quad \Delta_i^* = \begin{pmatrix} \varphi^* & 0 \\ \Gamma_i & \Delta_{i-1}^* \end{pmatrix},
$$

where the map $\Gamma_i : S^{i-k}T \otimes \nu^* \to \mathrm{Diff}_{i-1}(\pi, \mathbf{1})$ is given by the system and the splittings and the map $X_t : \Lambda^{m+1}_{[t]} ST \otimes \nu^* \to \mathrm{Diff}_{i-k-1}(\nu, \mathbf{1})$ is unknown.

By inductive assumptions, the equality $\Delta_i^* \circ \nabla_t^* = 0$ is equivalent to the following:

$$
\Gamma_i \varepsilon + \Delta_{i-1}^* X_t = 0.
$$

Thus existence of X_t is equivalent to the fact that $\Gamma_t \varepsilon$ belongs to the image of Δ_{i-1}^*. By the inductive assumption this in turn is equivalent to the equality $\alpha_{i-1} \Gamma_i \varepsilon = 0$.

The map Γ_i is given by the condition $j_1 \Gamma_i = \Delta_i^* q_2 - q_1 \varphi^*$. Thus we get the required equality:

$$
(\pi_{i,i-1}^{\mathcal{E}})^* \alpha_{i-1} \Gamma_i \varepsilon = \alpha_i j_1 \Gamma_i \varepsilon = \alpha_i \Delta_i^* q_2 \varepsilon - \alpha_i q_1 \varphi^* \varepsilon = 0
$$

because the map $(\pi_{i,i-1}^{\mathcal{E}})^*$ is a monomorphism and the the second row is exact on the level of the right three arrows. Therefore we constructed the map ∇_t^* in

the second row, so that the constructed part (right four arrows) is a complex in the commutative diagram.

Now by the same procedure we construct successively the arrows $\square_1^*, \square_2^*$ etc with the condition that the second row is a complex and the diagram commutes.

Since the first and the third complexes are exact, the diagram chase yields exactness of the middle complex. This proves the theorem in the linear case.

In the non-linear case the proof is basically the same, but we need to work over the infinitely prolonged equation $\mathcal{E}^{(\infty)} \subset J^\infty(\pi)$, use linearizations of differential operators and change the rings as in §2.4. Then we get the complex similar to the above, evaluation of which at the corresponding jet equals the complex for the case of the linearized system. $\qquad\square$

We will now turn the exact sequence from the theorem into compatibility complex for the differential operator Δ. For the linear differential operators it is known [S, KLV, V] that the cohomology of the compatibility complex coincide with the cohomology of the stable Spencer D-complex. We can generalize this now to the non-linear case, whenever the system is of generalized complete intersection type and is formally integrable.

Remark 10. *In fact, the general non-linear Spencer D-complex is unknown, but in the considered situation we can construct it via the splitting as in the proof above. Indeed, the symbolic part is given by the diagram from the following section 5.1 and then the complexes can be constructed inductively in the filtration number by the technique of the above proof.*

4.5. The compatibility complex

The homomorphisms of the spaces of differential operators as in theorem 12 are not always coming from the actual differential operators (as we treated in §1.2), but in our situation we can arrange this.

Indeed, in the proof from the preceding section we need to construct the operators $\nabla_t^*, \square_1^*, \square_2^*$ etc only when they appear for the first time (the corresponding index is zero or equivalently, when the defining differential operators are introduced). Afterwards, we can let $\nabla_t = \nabla_{t-1}^{(1)}$ to be the prolongation, $\square_{1,s} = \square_{1,s-1}^{(1)}$ and so on.

Thus we get actual differential operators between vector bundles, forming *compatibility complex* for the differential operator Δ (or the system $\mathcal{E}$).

Let us consider at first the case of linear system $\mathcal{E}$. Denote by $\mathcal{S}_\mathcal{E} = \mathcal{S}ol_\mathcal{E}$ the sheaf of local solutions of the PDE system $\mathcal{E}$. Then by the preceding discussion we constructed the complex

$$0 \to \mathcal{S}_\mathcal{E} \to C_{\mathrm{loc}}^\infty(\pi) \xrightarrow{\Delta} C_{\mathrm{loc}}^\infty(\nu) \xrightarrow{\nabla} C_{\mathrm{loc}}^\infty(\Lambda^{m+1}\nu) \xrightarrow{\square_1} C_{\mathrm{loc}}^\infty(\pi^* \otimes \Lambda^{m+2}\nu) \to \ldots,$$

which is formally exact and thus is a compatibility complex of the PDE system $\mathcal{E}$ (in some cases this complex is required to be constructed to both sides, but we will work only with right differential resolutions).

Remark 11. *In the classical geometric theory of PDEs, the compatibility resolution is constructed for any formally integrable system of PDEs (see the section about Kuranishi theorem in [S]), but then the construction is quite sophisticated (as in the second Spencer complex [S, KLV]). In our case of generalized complete intersection the spaces in the complex are explicit and simple.*

In fact, the same construction will work for any formally integrable system, where we have managed to write explicitly a resolution of the symbolic module.

What is more interesting is that we can write the whole compatibility complex explicitly. Consider at first two examples.

1. Let us write exact form of the compatibility conditions for linear non-homogeneous scalar PDEs of complete intersection type ($m = 1$ and $1 < r \le n$):

$$\Delta_1(u) = f_1, \ldots, \Delta_r(u) = f_r. \tag{14}$$

We solve non-homogeneous linear system under assumption that the linear homogeneous system is compatible. By theorem A this means certain commutation relation between operators Δ_i and their differential corollaries (we use the standard summation rule by the repeated indices):

$$[\Delta_i, \Delta_j] = C_{ij}^k \Delta_k, \tag{15}$$

$$\sum_{\text{cyclic: } ijk} [\Delta_i, [\Delta_j, \Delta_k]] = 0 \Leftrightarrow \sum_{\text{cyclic: } ijk} [\Delta_i, C_{jk}^\alpha] = \sum_{\text{cyclic: } ijk} C_{jk}^\beta C_{\beta i}^\alpha + D_{ijk}^{\beta\gamma} R_{\beta\gamma}^\alpha,$$

$$\text{where } R_{\beta\gamma}^\alpha = \Delta_\beta \delta_\gamma^\alpha - \Delta_\gamma \delta_\beta^\alpha - C_{\beta\gamma}^\alpha, \qquad \text{etc.} \tag{16}$$

Here C_{ij}^k are certain scalar differential operators on the base manifold M^n of $\operatorname{ord} C_{ij}^k < l_i + l_j - l_k$, $l_s = \operatorname{ord} \Delta_s$, and $D_{ijk}^{\beta\gamma}$ are scalar differential operators of $\operatorname{ord} D_{ijk}^{\beta\gamma} < l_i + l_j + l_k - \min[l_s + l_t : s, t \in \{\alpha, \beta, \gamma\}, s \neq t] - 1$ etc.

We may assume that C_{ij}^k is skew-symmetric by ij and $D_{ijk}^{\beta\gamma}$ is skew-symmetric by ijk and by $\beta\gamma$. Similar conditions will be imposed on other differential operators with multi-indices, which occur in the higher Jacobi identities.

Because of (15) the compatibility conditions for non-linear system (14) are

$$[\Delta_i - f_i, \Delta_j - f_j] - C_{ij}^k(\Delta_k - f_k) = \Delta_j f_i - \Delta_i f_j + C_{ij}^k f_k = 0.$$

Thus the differential 1-syzygy operator is $\nabla = (\nabla_{ij})_{1 \le i < j \le n}$, where $\nabla_{ij} = \Delta_i P_j - \Delta_j P_i - C_{ij}^k P_k$ with P_k being the projector to the k-th component. In other words, for $f = (f_1, \ldots, f_r)^t$ we have:

$$\nabla(f_1, \ldots, f_r) = (\ldots, \Delta_i f_j - \Delta_j f_i - C_{ij}^k f_k, \ldots).$$

To obtain differential 2-syzygy we write compatibility conditions for the system $\nabla_{ij}(f) = \lambda_{ij}$, which are:

$$\sum_{\text{cyclic: } ijk} (\Delta_i \nabla_{jk} - C_{ij}^\alpha \nabla_{\alpha k} + D_{ijk}^{\alpha\beta} \nabla_{\alpha\beta})$$

$$= \sum_{\text{cyclic: } ijk} (C_{ij}^\beta C_{\beta k}^\alpha - [\Delta_k, C_{ij}^\alpha] + D_{ijk}^{\beta\gamma} R_{\beta\gamma}^\alpha) P_\alpha = 0$$

due to (16). Thus the next operator in the differential resolution is $\Box = (\Box_{ijk})_{1 \leq i < j < k \leq n}$, where for $\lambda = (\ldots, \lambda_{ij}, \ldots)$ we have

$$\Box_{ijk}(\lambda) = \sum_{\text{cyclic: } ijk} (\Delta_i \lambda_{jk} - C_{ij}^{\alpha} \lambda_{\alpha k} + D_{ijk}^{\alpha\beta} \lambda_{\alpha\beta}).$$

Continuing in this way we arrive at the following compatibility complex:

$$0 \to \mathcal{S}_{\mathcal{E}} \longrightarrow C^{\infty}(\mathbf{1}) \xrightarrow{\Delta} C^{\infty}(\nu) \xrightarrow{\nabla} C^{\infty}(\Lambda^2 \nu) \xrightarrow{\Box} C^{\infty}(\Lambda^3 \nu) \to \ldots, \quad (17)$$

where ν is a trivial bundle over M with the fiber $\mathbb{R}^r$ (naturally $\Lambda^0 \nu \simeq \mathbf{1}$).

The operator $\Theta : C^{\infty}(\Lambda^k \nu) \to C^{\infty}(\Lambda^{k+1} \nu)$ in the above complex has the following form. Let $\omega = \{i_1 < \cdots < i_k\}$, $\varsigma = \{j_1 < \cdots < j_{k+1}\}$ be ordered multi-indices, so that the elements of the domain for Θ are $\theta = \{\theta_{\omega}\}$, while the elements for the range are $\Theta = \{\Theta_{\varsigma}\}$. The map is

$$\Theta_{\varsigma}(\theta) = \frac{1}{k!} \sum_{\sigma \in S_{k+1}} (-1)^{\sigma} \Big[\Delta_{\sigma(j_1)} \theta_{\sigma(j_2 \ldots j_{k+1})} + \varkappa_{1,k} C_{\sigma(j_1 j_2)}^{\alpha} \theta_{\alpha, \sigma(j_3 \ldots j_{k+1})}$$

$$+ \varkappa_{2,k} D_{\sigma(j_1 j_2 j_3)}^{\alpha_1 \alpha_2} \theta_{\alpha_1 \alpha_2 \sigma(j_4 \ldots j_{k+1})} + \cdots + \varkappa_{k,k} K_{\sigma(j_1 \ldots j_{k+1})}^{\alpha_1 \ldots \alpha_k} \theta_{\alpha_1 \ldots \alpha_k} \Big].$$

$(D, \ldots, K$ are some differential operators with multi-indices mentioned above and $\varkappa_{i,k}$ some real numbers calculated recursively, $1 \leq i \leq k$; for instance the first numbers are $\varkappa_{1,1} = -\frac{1}{2}$, $\varkappa_{1,2} = -1$, $\varkappa_{2,2} = 1$, $\varkappa_{1,3} = -\frac{3}{2}$ and so on).

Notice that on the symbolic level (17) is the usual de Rham complex, but due to non-commutativity of the components Δ_i of the operator Δ, defining the system $\mathcal{E}$, we need to compensate the syzygy maps with the tails of (lower order) differential operators. Horizontal de Rham complex ([KLV]) is a particular case of (17), when the operators Δ_i commute.

2. Consider now a vector system of $(m + 1)$ equations on m unknown functions, given by the operators $\Delta_i = (\Delta_{i1}, \ldots, \Delta_{im})$, which is of generalized complete intersection type:

$$\begin{cases} \Delta_{11}(u_1) + \cdots + \Delta_{1m}(u_m) = f_1 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \Delta_{(m+1)1}(u_1) + \cdots + \Delta_{(m+1)m}(u_m) = f_{m+1} \ . \end{cases}$$

In this case the compatibility condition for the homogeneous system is given by the condition from Theorem A

$$\{\Delta_1, \ldots, \Delta_{m+1}\} = B^i \Delta_i,$$

where $\{\ldots\}$ is the multi-bracket from §3.2 and B^i some scalar differential operators on M of $\mathrm{ord}(B^i) < \sum_{j \neq i} \mathrm{ord}(\Delta_j)$.

Assuming formal integrability for the homogeneous system, we get the compatibility complex:

$$0 \to \mathcal{S}_{\mathcal{E}} \hookrightarrow C^{\infty}(M, \mathbb{R}^m) \xrightarrow{\Delta} C^{\infty}(M, \mathbb{R}^{m+1}) \xrightarrow{\nabla} C^{\infty}(M, \mathbb{R}) \to 0,$$

where $\Delta = (\Delta_{ij})$ is the $(m+1) \times m$ matrix determining the system $\mathcal{E}$ and

$$\nabla(f_1, \ldots, f_{m+1})^{\mathrm{t}} = \sum_{k=1}^{m+1} \left((-1)^{k-1} \operatorname{Ndet}[\Delta_{ij}]_{i \neq k}^{1 \leq j \leq m} - B^k \right) f_k,$$

For $m = 2$ this formula looks as follows (the determinants are non-commutative):

$$\nabla \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \left(\begin{vmatrix} \Delta_{21} & \Delta_{22} \\ \Delta_{31} & \Delta_{32} \end{vmatrix} - B^1 \right) f_1$$

$$+ \left(\begin{vmatrix} \Delta_{31} & \Delta_{32} \\ \Delta_{11} & \Delta_{12} \end{vmatrix} - B^2 \right) f_2 + \left(\begin{vmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{vmatrix} - B^3 \right) f_3.$$

3. Now we combine the first case of arbitrary dimension of base with the second case of arbitrary dimension of the fiber of π as in §4.2. Then we can write the compatibility complex explicitly (in the generalized complete intersection case; in general it is more complicated to write the $W_H(\mathcal{E})$-component).

Let π and ν be trivial vector bundles (locally this is not a restriction). Moreover we will assume ν is graded in the various orders case for the operator $\Delta : C^\infty(\pi) \to C^\infty(\nu)$.

Theorem 13. *The compatibility complex for a collection of linear operators $\Delta = \{\Delta_i\}$, defining a system $\mathcal{E}$ of generalized complete intersection type, has the following form:*

$$0 \to \mathcal{S}_{\mathcal{E}} \to C_{loc}^\infty(\pi) \xrightarrow{\Delta} C_{loc}^\infty(\nu) \xrightarrow{\nabla} C_{loc}^\infty(\Lambda^{m+1}\nu) \xrightarrow{\square_1}$$

$$\to C_{loc}^\infty(\pi^* \otimes \Lambda^{m+2}\nu) \xrightarrow{\square_2} C_{loc}^\infty(S^2\pi^* \otimes \Lambda^{m+3}\nu) \to \ldots$$

The differential 1-syzygy operator $\nabla : C^\infty(M; \mathbb{R}^r) \to C^\infty(M; \Lambda^{m+1}\mathbb{R}^r)$ is given by the formula

$$\nabla_\tau(f) = \sum_{k=1}^{m+1} (-1)^{k-1} \operatorname{Ndet}[\Delta_{i_s j}]_{s \neq k}^{1 \leq j \leq m} f_{i_k} - \sum_{j=1}^{r} B_\tau^j f_j,$$

where $f = (f_1, \ldots, f_r)$ and $\nabla = \{\nabla_\tau\}$, τ being a multi-index $1 \leq i_1 < \cdots < i_{m+1} \leq r$. The coefficients B_τ^k are obtained from the multi-bracket relation (compatibility of the linear homogeneous system)

$$\{\Delta_{i_1}, \ldots, \Delta_{i_{m+1}}\} = B_{i_1 \ldots i_{m+1}}^j \Delta_j.$$

The higher syzygy operators are given by the method above. $\square$

Non-linear case is treated by the same formulas via the linearization operator. We need to restrict to infinitely prolonged equation $\mathcal{E}^{(\infty)} \subset J^\infty(\pi)$. Denote the projection from this equation to the base M by $\pi_{\mathcal{E}}$. Then changing

the operator Δ to its linearization ℓ_Δ (the operator itself on the equation vanishes) and coupling this to the arguments above we will arrive at the non-linear compatibility complex

$$0 \to \mathcal{S}ym_\mathcal{E} \to C^\infty_{\mathrm{loc}}\big(\pi^*_\mathcal{E}(\pi)\big) \xrightarrow{\ell_\Delta} C^\infty_{\mathrm{loc}}\big(\pi^*_\mathcal{E}(\nu)\big) \xrightarrow{\hat\nabla} C^\infty_{\mathrm{loc}}\big(\Lambda^{m+1}\pi^*_\mathcal{E}(\nu)\big) \xrightarrow{\hat\Box_1}$$

$$\to C^\infty_{\mathrm{loc}}\big(\pi^*_\mathcal{E}(\pi^*) \otimes \Lambda^{m+2}\pi^*_\mathcal{E}(\nu)\big) \xrightarrow{\hat\Box_2} C^\infty_{\mathrm{loc}}\big(S^2\pi^*_\mathcal{E}(\pi^*) \otimes \Lambda^{m+3}\pi^*_\mathcal{E}(\nu)\big) \to \dots$$

(note the change of the solution sheaf $\mathcal{S}_\mathcal{E} = \mathcal{S}ol_\mathcal{E}$ in the linear case to the symmetries sheaf $\mathcal{S}ym_\mathcal{E} = \mathrm{Ker}(\ell_\Delta)$ in the non-linear case: For a linear equation $\mathcal{E}$ each shift by a solution is a symmetry).

This complex is formally exact and its cohomology can be identified, as we have noticed in the previous section, with what we call non-linear Spencer cohomology.

This makes an effective representation for this important invariant $H^i_D(\mathcal{E})$ of the system $\mathcal{E}$ of differential equations (recall that in general non-linear case this invariant is not defined).

5. Integrability theory

In this section we consider certain topics closely related to the main subject of this paper. In fact, explicit compatibility/solvability criteria can be applied to integrability of determined PDEs. Indeed, one can find (simple Frobenius or sometimes more sophisticated) compatibility schemes in most well-known integrability approaches, in particular in the following theories:

– Symmetry calculus; Bäcklund transformations;

– Lax pairs;   Zero-curvature representations;

– Darboux integrability;   Sato theory etc.

Thus it is important to understand formal integrability and related Poisson geometry, method of differential constraints via multi-brackets or other criteria. Some of these topics will be considered in this section.

5.1. Spencer cohomology and curvature tensors

Let $\mathcal{E} \subset J^\infty(\pi)$ be a system of differential equations defined by a set of differential operators $\Delta : C^\infty(\pi) \to C^\infty(\nu)$, which we allow to be of different orders.

Proof of theorem B. Below we denote by $g = \oplus g_i$ the symbolic system associated to $\mathcal{E}$.

Consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & & g & \to & ST^*\otimes\pi & \to & ST^*\otimes\nu & \to & \Lambda_\diamond^{m+1} & \to\dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & & g\otimes T^* & \to & ST^*\otimes\pi\otimes T^* & \to & ST^*\otimes\nu\otimes T^* & \to & \Lambda_\diamond^{m+1}\otimes T^* & \to\dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & g\otimes\Lambda^2T^* & \to & ST^*\otimes\pi\otimes\Lambda^2T^* & \to & ST^*\otimes\nu\otimes\Lambda^2T^* & \to & \Lambda_\diamond^{m+1}\otimes\Lambda^2T^* & \to\dots \\
& & \vdots & & \vdots & & \vdots & & \vdots & \ddots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & g\otimes\Lambda^nT^* & \to & ST^*\otimes\pi\otimes\Lambda^nT^* & \to & ST^*\otimes\nu\otimes\Lambda^nT^* & \to & \Lambda_\diamond^{m+1}\otimes\Lambda^nT^* & \to\dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0 & & 0
\end{array}
$$

where we write $\Lambda_\diamond^{m+1} = ST^* \otimes \Lambda^{m+1}\nu$ for brevity. The rows are generalized Spencer complexes (i.e. $\mathbb{R}$-dual to Buchsbaum-Rim) and are exact.

The columns are the usual Spencer δ-complexes and all except the first one are acyclic, i.e. have cohomology in the bi-degree $(0,0)$ only. Thus by the diagram chase the cohomology of the first column at the j-th term $H^{*,j}(g)$ equals the zero cohomology group of the $(j+1)$-th column. $\square$

Denote $h^{i,j} = \dim H^{i,j}(g)$ the Betti numbers for Spencer cohomology.

Corollary 5. *For a generalized complete intersection g we have: $h^{*,0} = m$, $h^{*,1} = r$, $h^{*,j} = \binom{m+j-3}{j-2}\binom{r}{m+j-1}$ for $2 \le j \le r + 1 - m$ and $h^{*,j} = 0$ for $r + 1 - m < j \le n$.* $\square$

Note that the Euler characteristic vanishes as it should: $\chi = \sum(-1)^i h^{*,i} = 0$.

It is possible to specify the bi-grades, where the δ-cohomology does not vanish. If g is a system of pure order k, then the non-zero Betti numbers are: $h^{0,0}$, $h^{k-1,1}$ and $h^{km+kj-j-k,j}$ for $2 \le j \le r$.

For a system of different orders $l(1),\dots,l(r)$ the above complex allows to specify the bi-grades too. The precise combinatorics is straightforward from the weighted Buchsbaum-Rim complex. However, since the formulas are involved, we indicate only what happens to first and second δ-cohomology (this latter space contains curvature, i.e. Weyl tensors, see Corollary 2).

For them the only non-zero Betti numbers are: $h^{l(i)-1,1}$, $1 \le i \le r$ and $h^{l(i_1)+\dots+l(i_{m+1})-2,2}$, $1 \le i_1 < \dots < i_{m+1} \le r$.

Remark 12. *Formulas for $h^{*,j}$ from corollary 5 suggest some polynomial relations between multi-brackets. In fact, for $m = 1$ we can see the basis of $H^{*,j}$ for $j > 2$ as the set of power j subsets in $\{1,\dots,r\}$: for each such subset we associate all possible iterated Mayer-Jacobi brackets, which is an analog of j-form. The Jacobi identity and its higher analogs yield relations between these iterated brackets $\{\dots\{\{F_{i_1}, F_{i_2}\}, F_{i_3}\}, \dots, F_{i_j}\}$.*

When $m > 1$ *the number of subsets of power* $m + j - 1$ *in* $\{1, \ldots, r\}$ *is* $\binom{r}{m+j-1}$. *Another factor of* $\dim H^{*,j}$ *for* $j \geq 2$ *is related to the fact that iteration of brackets is now arranged in a multiplicative manner, so that the relations are generalized Plücker identities as in §3.3.*

5.2. Integrability of characteristics and multi-brackets

Consider the space of scalar linear differential operators $\mathrm{Diff}(\mathbf{1}, \mathbf{1})$ on the manifold M. It bears the structure of infinite-dimensional Lie algebra when equipped with the Jacobi bracket $\{,\}$ (this is also true for the space $\mathrm{Diff}(\pi, \pi)$). This bracket induces the classical Poisson bracket on T^*M as follows.

Let $\sigma : \mathrm{Diff}(\mathbf{1}, \mathbf{1}) \to C^\infty(T^*M)$ be the symbol map. It associates to an order k differential operator F a polynomial $\sigma(F) = \mathrm{smbl}_k(F)$ of degree k in momenta p in canonical coordinates (x, p) on T^*M. Then we have (with an ambiguity in notations):

$$\{\sigma(F), \sigma(G)\} = \sigma(\{F, G\}),$$

where the brackets on the left are Poisson and on the right are Jacobi brackets. In order to make a distinction we will write $\{,\}_\sigma$ for the symbolic (2-)bracket.

Consider a system of linear scalar PDEs $\mathcal{E} = \{F_1 = 0, \ldots, F_r = 0\}$, $F_i \in \mathrm{Diff}(\mathbf{1}, \mathbf{1})$. Let $\mathcal{I} = \mathcal{I}(F)$ be the differential ideal generated by F_i, $1 \leq i \leq r$.

Proposition 14. *Let the scalar system* $\mathcal{E}$ *be formally integrable. Then the corresponding ideal* $\mathcal{I}$ *is closed with respect to the higher Jacobi bracket.*

Proof. The system $\mathcal{E}$ is defined by the ideal $\mathcal{I}$ whenever formally integrable. Jacobi bracket of two operators is their differential corollary. Since the ideal $\mathcal{I}$ is differentially closed the claim follows. $\square$

Corollary 6. *The corresponding symbolic (characteristic) ideal* $I = \sigma(\mathcal{I})$ *is closed with respect to Poisson bracket.* $\square$

A stronger statement was proved in [GQS]: Namely the radical $\sqrt{I}$ is Poisson-closed too (in fact, the claim was justified only for the components of $\mathrm{Char}^{\mathbb{C}}(\mathcal{E})$ of maximal dimension). This is the celebrated integrability of characteristics: The affine characteristic variety $\mathrm{Char}^{\mathbb{C}}_{\mathrm{aff}}(\mathcal{E}) \subset T^*M$ is integrable in the Frobenius sense, i.e. if F and G vanish on it, their Poisson bracket $\{F, G\}$ vanishes on it as well.

This was applied in [GQS] to the case when $\mathcal{I}$ (resp. I) is the annihilator of the module $\mathcal{E}^*$ defining a formally integrable PDE system (resp. the symbolic module g^*). However, for the system involving several unknown functions the characteristic variety does not bear the complete information about the dynamics in generalization of Hamilton-Jacobi theory. Indeed, as we described in §1.3 the characteristic variety $\mathrm{Char}^{\mathbb{C}}(\mathcal{E})$ of the system $\mathcal{E}$ defined by a differential operator $\Delta : C^\infty(\pi) \to C^\infty(\nu)$ is the support of the characteristic sheaf $\mathcal{K} = \mathrm{Ker}\,\sigma(\Delta)$.

Dualizing this we can equally work with $\mathrm{Coker}\,\sigma(\Delta^*)$. We describe at first the picture with differential operators. The $\mathrm{Diff}(\mathbf{1},\mathbf{1})$-module $\mathcal{E}^*$ is given by the exact sequence

$$\mathrm{Diff}(\nu,\mathbf{1}) \xrightarrow{\varphi^\Delta} \mathrm{Diff}(\pi,\mathbf{1}) \to \mathcal{E}^* \to 0.$$

Denote $\mathcal{J} = \mathrm{Im}(\varphi^\Delta)$ the submodule in $\mathrm{Diff}(\pi,\mathbf{1})$.

The following statement is obtained similarly to proposition 14:

Proposition 15. *Let the vector system $\mathcal{E}$ be formally integrable. Then the corresponding module $\mathcal{J}$ is closed with respect to the multi-bracket $\{\cdot,\ldots,\cdot\}$.* $\quad\square$

This is the necessary condition for formal integrability from theorem A. It is sufficient only in special cases, see §5.5 below. We now define the symbolic multi-bracket generalizing the above Poisson 2-bracket.

Definition 4. *The symbolic multi-bracket of sections $f_i = \sigma(F_i)$ is given by the formula:*

$$\{\sigma(F_1),\ldots,\sigma(F_{m+1})\}_\sigma = \sigma(\{F_1,\ldots,F_{m+1}\})$$

(we use the index σ just to keep distinction between multi-brackets).

It is easy to see that the right-hand side does not depend on the lower order terms of F_i, so that the left-hand side is the well-defined expression $\{f_1,\ldots,f_{m+1}\}_\sigma$. However the above formula gives the symbolic multi-bracket only for vector-valued polynomials $f \in ST \otimes \pi^*$. The standard trick extends this to formal series and analytic functions, but we can define the symbolic multi-bracket $\{\cdots\}_\sigma$ to all smooth vector-valued functions on the cotangent bundle $f_i \in C^\infty(T^*M;\pi^*)$, $1 \le i \le m+1$. This follows from the following proposition.

Let e_i be a basis in the bundle π^* (which we assume trivial or make a localization). Then a vector-valued function $f \in C^\infty(T^*M;\pi^*)$ can be identified with the collection of functions $f^j \in C^\infty(T^*M)$ via the decomposition $f = f^j e_j$. We use the components f^j below.

Proposition 16. *The symbolic multi-bracket of the vector-valued functions f_i can be expressed via the product and the standard Poisson bracket of their components f_i^j ($1 \le i \le m+1$, $1 \le j \le m$).*

Proof. The multi-brackets of differential operators $F_i = (F_i^1,\ldots,F_i^m)$ has the following form in components:

$$\{F_1,\ldots,F_{m+1}\}^l = \sum_{\tau \in S_{m+1}} (-1)^\tau F_{\tau(1)}^1 F_{\tau(2)}^2 \cdots F_{\tau(m)}^m F_{\tau(m+1)}^l.$$

Taking the symbol and making elementary transformations we get (both indices j and s vary between 1 and m) the multi-brackets of $f_i = (f_i^1,\ldots,f_i^m)$:

$$\{f_1,\ldots,f_{m+1}\}_\sigma^l = (-1)^{m-l} \sum_{\tau \in S_{m+1}} \left[\frac{1}{2} \prod_{s \ne l} f_{\tau(s)}^s \cdot \{f_{\tau(l)}^l, f_{\tau(m+1)}^l\}_\sigma \right.$$
$$\left. + \sum_{j>l} \prod_{s \ne j,l} f_{\tau(s)}^s \cdot \{f_{\tau(l)}^l, f_{\tau(j)}^j\}_\sigma \cdot f_{\tau(m+1)}^l \right] \quad (18)$$

37

This represents components of the symbolic multi-bracket via the 2-bracket. $\square$

Remark that our symbolic multi-bracket differs from other multi-versions of the Poisson bracket, like Fillipov-Nambu or generalized Poisson bracket ([APB, MV, N]). For instance, the multi-bracket with $m > 1$ is not a derivation in its arguments. However it is not pretty far from this:

Corollary 7. *The symbolic multi-bracket* $\{f_1, \ldots, f_{m+1}\}_\sigma$ *is a differential operator of the first order in each of its arguments* $f_i \in C^\infty(T^*M, \pi^*)$. $\square$

Indeed, we can write for $g \in C^\infty(T^*M)$

$$\{f_1, \ldots, gf_i, \ldots,, f_{m+1}\}_\sigma - g\{f_1, \ldots, f_i, \ldots,, f_{m+1}\}_\sigma$$
$$= \sum_{j,k} \{g, f_k^j\}_\sigma \cdot c_{ij}^k(f_1, \ldots, f_i, \ldots,, f_{m+1}), \quad (19)$$

where the exact form of the functions c_{ij}^k can be obtained from (18).

Let $J = \sigma(\mathcal{J})$ be the symbol of the submodule $\mathcal{J}$. Denoting as in §2.2-2.3 $\varphi = \sigma_\Delta$ the symbol of the differential operator Δ, we can define J via the exact sequence

$$ST \otimes \nu^* \xrightarrow{\varphi} ST \otimes \pi^* \to g^* \to 0$$

as $J = \text{Im}(\varphi) \subset ST \otimes \pi^*$, so that the symbolic module is $g^* = ST \otimes \pi^*/J$.

Then proposition 15 yields:

Corollary 8. *The (characteristic) submodule* $J = \sigma(\mathcal{J})$ *is closed with respect to the symbolic multi-bracket.* $\square$

Beside identity (19) the symbolic multi-bracket satisfies the same properties as the multi-bracket of differential operators, for instance we have the generalized Plücker identity as a corollary of §3.3 (the upper index means component):

$$\sum(-1)^k(\{f_1, \ldots, \check{f}_k, \ldots, f_{m+2}\}_\sigma^\dagger)^i \cdot f_k = \sum(-1)^k\{f_1, \ldots, \check{f}_k, \ldots, f_{m+2}\}_\sigma \cdot f_k^i,$$

where $\{\cdots\}_\sigma^\dagger$ is the symbolic multi-bracket associated to the opposite multi-bracket of differential operators $\{\cdots\}^\dagger$ of §3.2 by the same rule as in definition 4.

Consider again the case (of importance to PDEs), when $f_i \in ST \otimes \pi^*$ are polynomial vector-valued functions on T^*M.

Let $I_0(J) \subset R = ST$ be the ideal generated by the functions f_i^j and C_0 its Poisson center. The following statement follows directly from (19):

Corollary 9. *The multi-bracket is a homomorphism in its arguments over the ideal* $C_0 \subset ST$: $\{\cdots\}_\sigma \in \text{Hom}_{C_0}(\Lambda^{m+1}J, J)$. $\square$

The symbolic multi-bracket $\{\ldots\}_\sigma$ can be used to formulate a Hamiltonian formalism as generalized Poisson brackets are used. Namely, let $f_1, \ldots, f_m \in J$ be Hamiltonians and $h \in ST \otimes \alpha^*$ a polynomial vector-function. A multi-Hamiltonian operator

$$X_{f_1 \wedge \ldots \wedge f_m} : g^* \to g^*, \qquad h \bmod J \mapsto \{h, f_1, \ldots, f_m\}_\sigma \bmod J$$

determines transport on $\text{Char}^{\mathbb{C}}(g)$ (symbolic module g^* is supported on characteristics). Caustics of solutions (wave fronts) develop according to it. More details will be provided elsewhere.

5.3. Applications to smooth integrability of PDEs

The classical Lagrange-Charpit method [Gou, Gu] is designed for first order scalar PDEs and is as follows. Let $F = F_1(x^1, \ldots, x^n, u, p_1, \ldots, p_n) = 0$ be a differential equation. To solve it one searches for functions $F_2, \ldots, F_n$ on $J^1(\mathbb{R}^n)$ such that $[F_i, F_j] = 0 \mod (F_1, \ldots, F_n)$, where the $[,]$ is the classical Mayer bracket. Then the system $F_1 = 0, \ldots, F_n = 0$ gives a finite dimensional family of solutions of the PDE $F = 0$ (this family is equivalent to an ODE).

If in addition $F_2, \ldots, F_n$ are symmetries of the system, $[F, F_i] = \lambda \cdot F$, then one can obtain a complete integral of the PDE $F = 0$ as the system $F_1 = 0, F_2 = c_2, \ldots, F_n = c_n$, which in addition can be found in quadratures by the symmetry method of S.Lie ([Lie]).

Example 1. *Consider the PDE* $up_1 \cdots p_n = x^1 \cdots x^n$ *on* $\mathbb{R}^n$. *It possesses a collection of* auxiliary integrals*:* $F_i = \dfrac{p_i u^{1/n}}{x^i}$. *This gives a complete integral of the differential equation.*

Now basing on our theorem A we can formulate a generalized Lagrange-Charpit method. For manifolds of dimension 2 this was done in [KL$_2$]. To obtain the general version we start with the following idea.

Definition 5. *Let* $\mathcal{E}$ *be a formally integrable system of PDEs. Call a system* $\tilde{\mathcal{E}}$ *an* auxiliary integral *(or a set of integrals) for the system* $\mathcal{E}$ *if the joint system* $\mathcal{E} \cap \tilde{\mathcal{E}}$ *is also formally integrable (= compatible).*

We proved in [KL$_2$] that classical objects, such as point symmetries, contact symmetries and intermediate integrals ([Gou, Gu, LE]) as well as higher symmetries ([KLV]) are partial cases of this notion. Moreover, some of the newly introduced generalized symmetries are also auxiliary integrals.

Another traditional method for finding exact solutions of PDEs is a method of differential constraints. Then one considers an overdetermination $\tilde{\mathcal{E}}$ on a system $\mathcal{E}$, such that the system $\mathcal{E} \cap \tilde{\mathcal{E}}$ is *solvable*. The solvability is very non-trivial to check in practice. Thus our notion of auxiliary integral is more constructive, since we can use an effective criterion from theorem A to check compatibility (see [KL$_1$, KL$_2$] for examples).

Generalized Lagrange-Charpit method is the following special form of an auxiliary integral. Consider a determined system of m PDEs $\mathcal{E} = \{F_1 = 0, \ldots, F_m = 0\}$ on m unknowns functions $u_1, \ldots, u_m$ (we can even start with an underdetermined system). We search to add to it $n - 1$ differential equations $\tilde{\mathcal{E}} = \{F_{m+1} = 0, \ldots, F_{m+n-1} = 0\}$, so that the resulting overdetermined systems $\mathcal{E} \cap \tilde{\mathcal{E}} = \{F_1 = 0, \ldots, F_{m+n-1} = 0\}$ is:

a) generalized complete intersection;

  b) compatible.

Of course, one can add less functions, but advantage of $(n-1)$ is that the system becomes of finite type (being compatible it constitutes an integrable distribution by the Frobenius theorem [St, KL_3]) and thus reduces to a system of ODEs.

  By theorem A compatibility of $\mathcal{E} \cap \tilde{\mathcal{E}}$ is given by the conditions

$$\{F_{i_1}, \ldots, F_{i_{m+1}}\} = \sum_j A^j_{i_1 \ldots i_{m+1}} \circ F_j,$$

for some differential operators A of orders $\mathrm{ord}(A^j_{i_1 \ldots i_{m+1}}) \le i_1 + \cdots + i_{m+1} - i_j - 1$.

  On the symbolic level the first step is to include $f = \sigma(F)$ into a submodule $J = \langle f_1, f_2, \ldots, f_{m+n-1} \rangle$ of $ST \otimes \pi^*$, which is closed under the symbolic multi-bracket, as integrability of characteristics from §5.2 claims:

$$\{f_{i_1}, \ldots, f_{i_{m+1}}\}_\sigma = \sum_j a^j_{i_1 \ldots i_{m+1}} \cdot f_j.$$

Then we shall adjust sub-principal symbols of F_i.

Example 2. *Consider the Cauchy-Riemann system on the plane:*

$$\mathcal{E}: \quad u_x = v_y, \ u_y = -v_x. \tag{20}$$

Then the differential equation

$$\tilde{\mathcal{E}}: \quad \det \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} = G(u, v)$$

is an auxiliary integral suitable for generalized Lagrange-Charpit method iff

$$\Delta G = \|\nabla G\|^2 / G, \tag{21}$$

where Δ is the standard Laplacian and ∇ is the standard gradient on the Euclidean plane $\mathbb{R}^2(u, v)$. Thus our transformation can be seen as a kind of Backlünd transformation, which for any solution $G(u, v)$ of equation (21) associates a 3-dimensional family of solutions of system (20) (for dimensional calculus see the next section).

5.4. Formal dimension of the solutions space

  Consider a symbolic system $g = \{g_l \subset S^l T^* \otimes N\}$ and let $V^* \subset T^*$ be a subspace. Then we can define another symbolic system $\tilde{g} = \{g_l \cap S^l V^* \otimes N\} \subset SV^* \otimes N$. It is called the V^*-*reduction* of g ([KL_2]). Denote $W^* = T^*/V^*$.

  We take g to be the symbolic system of a generalized complete intersection $\mathcal{E}$ of formal codimension r, the same number as in definition 1. Note that this definition can be reformulated for symbolic systems as well. Then the characteristic variety $\mathrm{Char}^{\mathbb{C}}(g) = \mathrm{Char}^{\mathbb{C}}(\mathcal{E})$ has codimension $r - m + 1$ in $P^{\mathbb{C}}T^*$.

Theorem 17. *Let g be a symbolic system of generalized complete intersection type and the subspace $V^* \subset T^*$ of dimension $(r - m + 1)$ be transversal to the affine characteristic variety of g: $\mathrm{Char}(g) \cap P^{\mathbb{C}}V^* = \emptyset$. Then the reduced symbolic system $\tilde{g}$ is also a generalized complete intersection, Spencer δ-cohomology of the system g and of its V^*-reduction $\tilde{g}$ are isomorphic and $g \simeq \tilde{g} \otimes SW^*$.*

Proof. This statement was proved for more general Cohen-Macaulay symbolic systems in [KL$_2$]. Since we proved in §2.3 that a generalized complete intersection is Cohen-Macaulay, the claim follows (the part that the V^*-reduction form a generalized complete intersection is straightforward). The last claim of the theorem $g_l \simeq \sum_j \tilde{g}_{l-j} \otimes S^j W^*$ is not a part of theorem A from [KL$_2$], but is contained in the proof, see remark 8 loc.cit. $\qquad\square$

Notice that due to Noether normalization lemma [E] a generic subspace $V^* \subset T^*$ is transversal to the characteristic variety over $\mathbb{C}$. The functions on $W = \mathrm{Ann}(V^*)$ are those on which a general solution of $\mathcal{E}$ depends, as is refereed to in the discussion of functional dimension before Theorem C.

Proof of theorem C. Let us consider at first the case, when $r = n+m-1$. Then the system $\mathcal{E}$ is of finite type, namely $\pi_{i,i-1} : \mathcal{E}_i \xrightarrow{\sim} \mathcal{E}_{i-1}$ for $i \geq \sum_{i=1}^r k_i$. The equation $\mathcal{E}_\infty$ is thus a finite-dimensional manifold equipped with the Cartan distribution $\mathcal{C}_{\mathcal{E}}$ ([KLV, KL$_1$]). Since the system is compatible, the local solutions of $\mathcal{E}$ are integral manifolds of the distribution $\mathcal{C}_{\mathcal{E}}$ of dimension n.

Thus the dimension of the solutions space is

$$\dim \mathcal{S}_{\mathcal{E}} = \dim \mathcal{E}_\infty - n = \sum_{i=0}^{\infty} \dim g_i$$

(the sum is indeed finite). We can calculate dimensions of the symbol spaces g_i explicitly, since the system is of generalized complete intersection type.

For brevity sake we omit the immense combinatorics and provide details of this step only for the case $n = \dim M = 2$. The general case is similar (we refer to [KL$_5$], where arbitrary systems of pure order k are scrutinized).

If $n = 2$, we get $r = m + 1$. Let $k_1 \leq \cdots \leq k_r$ be the orders of the system g. We have:

$$\dim g_i = \begin{cases} m(i+1) & \text{if } i < k_1 \\ (m-j)(i+1) + \sum_{s=1}^{j} k_s & \text{if } k_j \leq i < k_{j+1} \\ \sum_{s=1}^{r} k_s - 1 - i & \text{if } k_r \leq i < \sum_{s=1}^{r} k_s \\ 0 & \text{else.} \end{cases}$$

Thus

$$\sum \dim g_i = \sum_{i=0}^{k_1-1} m(i+1) + \sum_{i=k_1}^{k_2-1} \Big((m-1)(i+1) + k_1\Big)$$

$$+ \sum_{i=k_2}^{k_3-1} \Big((m-2)(i+1) + (k_1+k_2)\Big) + \cdots + \sum_{i=k_{r-1}}^{k_r-1} \big(k_1 + \cdots + k_{r-1}\big)$$

$$+ \sum_{i=k_r}^{k_1+\cdots+k_r-1} \big(k_1 + \cdots + k_r - 1 - i\big) = \frac{(k_1 + \cdots + k_{r-1} - 1)(k_1 + \cdots + k_{r-1})}{2}$$

$$+ \sum_{i=1}^{r-1} \frac{k_i(k_i+1)}{2} + \sum_{i=1}^{r}(k_i - k_{i-1})(k_1 + \cdots + k_{i-1}) = \sum_{i<j} k_i k_j$$

and the result follows.

Consider now the case $m < r < n + m - 1$. Then the characteristic variety $\mathrm{Char}^{\mathbb{C}}(\mathcal{E})$ is non-empty and the system $\mathcal{E}$ is of infinite type. By theorem 17 the symbolic system g has a free factor SW^* and the dimension

$$p = \dim W = n - \dim V = n + m - r - 1$$

is clearly the formal functional dimension of $\mathcal{Sol}_{\mathcal{E}}$.

Thus the quantity $\sum_{i=0}^{t} g_i$ when $t \to \infty$ grows as $\tilde{d} \cdot \dim S^t W^*$, where $\tilde{d}$ is the formal functional rank for the system $\tilde{g}$, which is of finite type (no complex characteristics). The reduction $\tilde{g}$ is of generalized complete intersection type and it has the same orders $k_1 \le \cdots \le k_r$ because the first Spencer δ-cohomology coincide (theorem 17). Thus we can use the calculations above to conclude

$$\tilde{d} = \sum \dim \tilde{g}_i = \sum_{i_1 < \cdots < i_l} k_{i_1} \cdots k_{i_l} = d, \qquad l = \dim V = r - m + 1.$$

Since $\dim W^* = n + m - r - 1$, the asymptotic of the Hilbert polynomial for the symbolic module g^* is

$$d \cdot \dim S^t W^* \sim d \frac{t^{n+m-r-2}}{(n+m-r-2)!}.$$

This proves the theorem. $\qquad\square$

Corollary 10. *If a generalized complete intersection $\mathcal{E}$ of formal codimension r is formally integrable and has equations of the same order k, then its formal functional dimension and functional rank are $p = n + m - r - 1$ and $d = \binom{n+m-1}{n}k^n$ (i.e. a general formal solution depends on d functions of p arguments).* $\qquad\square$

5.5. Generalizations of compatibility via brackets

Vanishing of multi-brackets is necessary, but not a sufficient condition for compatibility. By theorem A it is sufficient for generalized complete intersections, however this does not generalize to more general class of Cohen-Macaulay systems.

To see it consider a system of finite type. It is a Cohen-Macaulay system. In fact, the finite type condition means $\mathrm{Char}^{\mathbb{C}}(g) = \emptyset$, so that $\dim g^* = 0$. But this condition also implies $\mathrm{depth}\, g^* = 0$ (alternatively $0 \leq \mathrm{depth}\, g^* \leq \dim g^*$).

Now for a completely determined system of order k (Frobenius type), i.e. $g_k = 0$, the compatibility conditions do not coincide with these of our theorem. Actually, let us write the equation in the orthonomic form:

$$\mathcal{E} = \left\{ \frac{\partial^{|\sigma|} u^i}{\partial x^\sigma} = F^i_\sigma \left(x, \frac{\partial^{|\tau|} u^j}{\partial x^\tau} \right) \,\middle|\, 1 \leq i,j \leq m,\ |\sigma| = k,\ 0 \leq |\tau| < k \right\}.$$

The compatibility condition is $\Phi^i_{a,b,\tau} = \mathcal{D}_a F^i_{\tau+1_b} - \mathcal{D}_b F^i_{\tau+1_a} = 0 \mod \mathcal{J}_k(\mathcal{E})$, $|\tau| = k - 1$, while the multi-brackets are:

$$\{ p^{i_1}_{\sigma_1} - F^{i_1}_{\sigma_1}, \ldots, p^{i_{m+1}}_{\sigma_{m+1}} - F^{i_{m+1}}_{\sigma_{m+1}} \} = H_T + (\text{smaller order terms}),$$

where the higher order term H_T (with order equal to $\sum_{j=1}^{m+1} |\sigma_j| - 1$) is non-zero iff for some indices α, β we have (in the display below the braces mean a set):

$$i_\alpha = i_\beta = k \in \{1, \ldots, n\} \text{ and } \{i_1, \ldots, \check{i}_\alpha, \ldots, \check{i}_\beta, \ldots, i_{m+1}\} = \{1, \ldots, \check{k}, \ldots, n\},$$

in which case $H^i_T = \delta^i_k (\pm \Pi_{j \neq \alpha, \beta} \mathcal{D}_{\sigma_j})[\mathcal{D}_{\sigma_\alpha} F^k_{\sigma_b} - \mathcal{D}_{\sigma_\beta} F^k_{\sigma_a}]$. This provides more conditions and they are of higher order.

Remark 13. *It is possible however to give explicit compatibility conditions for some systems different from generalized complete intersection. Let $\mathcal{E}$ be a Cohen-Macaulay system and g its symbolic module. Choosing a subspace $V^* \subset T^*$ not meeting the characteristic variety $\mathrm{Char}^{\mathbb{C}}(g)$ and of complimentary dimension, we get the reduction $\tilde{g}$, which is also Cohen-Macaulay by theorem A of [KL_2].*

If we have a compatibility criterion for systems $\tilde{\mathcal{E}}$ of type $\tilde{g}$, we can transform it to get a criterion for the system $\mathcal{E}$ of type g (cf. theorem 17). For instance, if the system $\tilde{g}$ is completely determined (Frobenius type), we can use the above formulas to get similar compatibility conditions for the system $\mathcal{E}$ (which is not of Frobenius type!).

Thus usually for systems different from generalized complete intersections, multi-brackets do not provide a basis of compatibility conditions (though multi-brackets are part of them). Indeed, in these other cases the obstructions to formal integrability (Weyl tensors) belong to different Spencer cohomology groups.

However in some cases the system not of generalized complete intersection type can have compatibility conditions in a form of multi-brackets. Usually this happens when the Spencer cohomology is of the type described in theorem B.

For instance, for a (skew-)product of a generalized complete intersection and some involutive system. We give two examples.

1. Let J be an almost complex structure on a manifold M. It defines the Cauchy-Riemann operator $\bar{\partial}_J : \Omega^{p,0}(M) \to \Omega^{p,1}(M)$, so that the system $\mathcal{E} = \mathrm{Ker}(\bar{\partial}_J)$ is not a generalized complete intersection. However we can represent the CR-operator locally as the product of the operator $\bar{\partial}_J : C^\infty(M) \to \Omega^{p,1}(M)$ and the identity on $\Omega^{p,0}(M)$. The compatibility condition (which is equivalent to integrability of the structure J) has now the form of vanishing multi-brackets.

2. Let ∇ be a connection on the bundle $\pi : E \to M$ and $\Omega^{p,q}(E)$ the bundle of p-vertical, q-horizontal forms. The horizontal de Rham operator $d_\nabla : \Omega^{p,0}(E) \to \Omega^{p,1}(E)$ is locally a product of $\Omega^{p,0}(E)$ and the horizontal de Rham operator $d_\nabla : C^\infty(\pi) \to \Omega^{0,1}(E)$. The latter has compatibility condition (which is equivalent to flatness of the connection ∇) again in the form of vanishing multi-brackets as in theorem A.

6. Applications

In this section we apply the compatibility criterion to solve certain problems arising in differential geometry. Many of them address solvability of overdetermined systems of PDEs.

To decide if the system is solvable we add compatibility conditions to the system (here we use theorem A), investigate the new system, add its compatibility conditions etc. In other words we apply prolongation-projection scheme and either close up the system or get a contradiction (empty equation $\mathcal{E} \subset J^k(\pi)$).

For example, linearization problem of 3-webs on the plane is equivalent to solvability of a system of 2 scalar second order equations of two variables. The system is a complete intersection and the compatibility condition is given by vanishing of Jacobi-Mayer bracket, as was sketched in [KL$_1$]. This condition is equivalent to vanishing of the Chern curvature and yields parallelizable webs.

To linearize the web one adds the bracket to the system and further investigates compatibilities. This was done in [GL$_1$] and thus the long standing Blaschke problem was solved.

Another example is the problem of finding the number of Abelian relations, which is equivalent to solvability of a system of $(m + 1)$ differential equations on m unknown functions on the plane. The system is a generalized complete intersection and the compatibility conditions is given by vanishing of multi-brackets. This method was applied in [GL$_2$] and the rank problem, addressed by Lie, Poincaré and Bol, was solved.

In this section we solve with our technique some other pending problems of classical differential geometry.

6.1. Killing vector fields on the plane

If a Riemannian metric g on a surface M^2 possesses a Killing vector field, it has the following local form: $ds^2 = g_{11}(x)dx^2 + 2g_{12}(x)dxdy + g_{22}(x)dy^2$ (near

the point, where the field does not vanish) and vise versa, so that this is a surface of revolution.

Now we address the following question: How to recognize if a metric can be brought to such a form? This classical question was studied by Darboux in [D]. We however did not find a clearly formulated answer in the literature. Here we give a criterion in differential invariants using our compatibility technique.

The problem is equivalent to solvability of the equation $L_\xi ds^2 = 0$, where $\xi = u\partial_x + v\partial_y$ is the required vector field and $ds^2 = g_{ij}(x,y)dx^i dx^j$ is the metric, $x^1 = x, x^2 = y$. The tensor equation is equivalent to the following 3 first order linear PDEs on the functions $u(x,y), v(x,y)$:

$$2u_x g_{11} + 2v_x g_{12} + u(g_{11})_x + v(g_{11})_y = 0, \quad 2u_y g_{12} + 2v_y g_{22} + u(g_{22})_x + v(g_{22})_y = 0,$$

$$u_y g_{11} + u_x g_{12} + v_y g_{12} + v_x g_{22} + u(g_{12})_x + v(g_{12})_y = 0.$$

Denote them by E_1, E_3, E_2 respectively. We get the linear system $\mathcal{E} = \mathcal{E}_1 \subset J^1(2;2)$ of codimension 3 (we shall write $J^k(n,m)$ instead of $J^k(\mathbb{R}^n, \mathbb{R}^m)$), so that $\dim g_1 = 1$, $\dim g_2 = 0$, whence the isomorphism $\pi_{2,1} : \mathcal{E}_2 \xrightarrow{\sim} \mathcal{E}_1$.

The compatibility condition is equivalent to the Frobenius condition on the corresponding distribution $L(\pi_{2,1}^{-1})$ on $\mathcal{E}_1$ and is given by the condition $E_4' = [E_1, E_2, E_3]_\mathcal{E} = 0 \in \mathcal{E}_1^*$. This differential operator E_4' has order 2, but due to the above isomorphism can be considered as a function on $J^1(2;2)$. However if we consider it modulo $E_1 = 0$, $E_2 = 0$, $E_3 = 0$, it becomes a function on $J^0(2;2)$ (this is not automatical and is a peculiarity of the system) and has the form:

$$E_4 = E_4'(\mathrm{mod}\, E_1, E_2, E_3) = 4|g|^2 (K_x u + K_y v),$$

where $|g| = g_{11}g_{22} - g_{12}^2$ is the determinant of the metric and K is the Gaussian curvature. Thus compatibility condition is equivalent to the claim that (M^2, g) is a spacial form: $K = $ const. Note that this is the case, when the solutions space has dimension 3.

Let us study solvability, then we need to add the equation $E_4 = 0$ to the system. This means $u = K_y w$, $v = -K_x w$ and we obtain the following system on one function $w(x,y)$:

$$\begin{pmatrix} 2\alpha & 0 & \gamma_1 \\ 0 & 2\beta & \gamma_2 \\ \beta & \alpha & \gamma_3 \end{pmatrix} \cdot \begin{bmatrix} w_x \\ w_y \\ w \end{bmatrix} = 0,$$

where $\alpha = g_{11}K_y - g_{12}K_x$, $\beta = g_{12}K_y - g_{22}K_x$, $\gamma_1 = (g_{11})_x K_y - (g_{11})_y K_x + 2g_{11}K_{xy} - 2g_{12}K_{xx}$, $\gamma_2 = (g_{22})_x K_y - (g_{22})_y K_x + 2g_{12}K_{yy} - 2g_{22}K_{xy}$, $\gamma_3 = (g_{12})_x K_y - (g_{12})_y K_x + g_{11}K_{yy} - g_{22}K_{xx}$. Note that α, β do not vanish simultaneously unless $K_x = K_y = 0$.

Denoting by S_1 the determinant of the above matrix we obtain two necessary and sufficient conditions for non-trivial solvability ($w = 0$ is always a solution): $S_1 = 0$ and $S_2 = 0$, where:

  – If $\alpha\beta \neq 0$, then $S_2 = (\alpha(\gamma_1)_y - \alpha_y \gamma_1)\beta^2 - (\beta(\gamma_2)_x - \beta_x \gamma_2)\alpha^2$.

- If $\alpha = 0$, $\beta \neq 0$, $S_2 = \beta(\gamma_2)_x - \beta_x\gamma_2 - \beta(\gamma_3)_y + \beta_y\gamma_3$ ($S_1 = 0 \Rightarrow \gamma_1 = 0$).

- If $\alpha \neq 0$, $\beta = 0$, $S_2 = \alpha(\gamma_1)_y - \alpha_y\gamma_1 - \alpha(\gamma_3)_x + \alpha_x\gamma_3$ ($S_1 = 0 \Rightarrow \gamma_2 = 0$).

Thus the criterion for existence of Killing vector field becomes the following two non-linear differential relations $S_1 = 0$ and $S_2 = 0$, having orders 4 and 5 in the coefficients of the metric g respectively.

For a tensor T denote $d_\nabla^{\otimes 2} T = d_\nabla(d_\nabla T)$ the covariant derivative of the tensor $d_\nabla T$ ($d_\nabla^{\otimes 2}$ differs from d_∇^2, which is equal to multiplication by the curvature tensor). In particular, we obtain the forms $d_\nabla^{\otimes i} K \in C^\infty(\otimes^i T^*M)$. Note that the form $d_\nabla^{\otimes 2}$ is symmetric, but the higher covariant derivatives lack this property.

Let also grad K be the g-gradient of the curvature and sgrad $K = J$ grad K be its rotation by $\pi/2$ (fix orientation). The preceding calculations imply:

Theorem 18. *The space of local Killing vector fields can have dimension 3, 1 or zero. A Riemannian metric g possesses a local Killing vector field iff*

$$d_\nabla^{\otimes 2} K(\mathrm{grad}\, K, \mathrm{sgrad}\, K) = 0 \text{ and } d_\nabla^{\otimes 3} K(\mathrm{sgrad}\, K, \mathrm{sgrad}\, K, \mathrm{sgrad}\, K) = 0.$$

There are 3 independent Killing fields iff K is constant.

Remark 14 *The main claim of the theorem is the formula (sufficiency of which is obvious). Other statements were known to Darboux [D]. In fact, even formulas can be attributed to him, though no precise statement was made in [D]; see [K₂] for details.*

Note also that global implications are straightforward, but the dimension of the space of Killing vector fields can differ. For instance, for the standard flat torus it is 2.

Proof. Let us note that if $K \neq$ const, then the system

$$\mathcal{E}' = \{E_1 = E_2 = E_3 = E_4 = 0\}$$

has symbol dimensions: $\dim g_0' = 1, \dim g_1' = 0$ and so in the compatible case the solution space is one-dimensional. Thus we need only to prove the existence part of the theorem.

For this we express the above S_1 and S_2 via differential invariants. Direct calculation shows:

$$S_1 = -4|g|^5 d_\nabla^{\otimes 2} K(\mathrm{grad}\, K, \mathrm{sgrad}\, K) \quad \text{and}$$

$$S_2 = A\, d_\nabla^{\otimes 3} K(\mathrm{sgrad}\, K, \mathrm{sgrad}\, K, \mathrm{sgrad}\, K) + \frac{2|g|^2}{|\mathrm{grad}\, K|^4}\Box(|g|^3 S_1),$$

where in isothermal coordinates, when $ds^2 = e^\lambda(dx^2 + dy^2)$, we have:

$$A = \frac{-2K_x^2 K_y^2 |g|^5}{|\mathrm{grad}\, K|^2} \quad \text{and} \quad \Box = -K_x K_y^4 D_x - K_x^4 K_y D_y +$$
$$+ \left(K_y^4 K_{xx} + K_x^4 K_{yy} + 2K_x K_y(K_x^2 + K_y^2)K_{xy} - 2K_x^2 K_y^2(\lambda_x K_x + \lambda_y K_y)\right).$$

46

Thus $|g|^{-5}S_1$ is a differential invariant, while $A^{-1}S_2$ is a differential invariant relative the condition $S_1 = 0$. $\qquad\square$

In the next section we'll need to enumerate differential invariants of a Riemannian metric on a surface. It is a known fact (see [T]), that the space of scalar differential invariants of order k of a Riemannian metric on a surface is generated by $(k-1)$ differential invariants for all $k > 0$ except $k = 3$, where there is only one invariant.

The first invariants are: $I_2 = K$ and $I_3 = |\nabla K|^2$ (the index refers to the order of differential invariant).

To fix a basis in invariants of order $i = 2 + k$ we consider the form $d_\nabla^{\otimes i}K$ and substitute $\operatorname{grad} K$ as first $(i-j)$ arguments and $\operatorname{sgrad} K$ as the next j arguments $(0 \le j \le i)$. We denote the resulting function I_{ij} and enumerate the index j by letters (so we write I_{4b} instead of I_{41}, I_{5d} instead of I_{53} etc.).

In these invariants the criterion of Theorem 18 writes: $I_{4b} = 0, I_{5d} = 0$.

6.2. Higher order integrals of plane metrics

Killing vector field on a surface M can be represented as a linear (in momenta) integral of the geodesic flow on T^*M. It is important to know when the flow admits a polynomial integrals. Locally geodesic flows are integrable, but the corresponding integrals are usually analytic only on $T^*M \setminus M$. So in general polynomial integrability requires certain conditions even locally (here and throughout the standard regularity assumption should be imposed).

Let (x, y) be local coordinates on M^2 and p_x, p_y be the corresponding momenta on T^*M. Since every homogeneous term of the integral is obviously an integral, we consider a function $F_d = \sum_{i+j=d} a_{ij}(x, y)p_x^i p_y^j$ of degree d on T^*M (i in p_z^i is a power, not index).

The Hamiltonian of the geodesic flow is $H = g^{11}p_x^2 + 2g^{12}p_x p_y + g^{22}p_y^2$ (matrix g^{ij} is inverse to the matrix g_{ij} of the metric). Let $\{H, F_d\}$ be the Poisson bracket of H and F_d. It is a polynomial in momenta of degree $d + 1$.

Thus involutivity condition $\{H, F_d\} = 0$ is equivalent to $(d + 2)$ equations $E_1 = 0, \ldots, E_{d+2} = 0$ on $(d+1)$ unknown function $a_{d0}(x, y), \ldots, a_{0d}(x, y)$. These equations form the first order system $\mathcal{E}$ of generalized complete intersection type and so the compatibility condition can be expressed via the multi-bracket

$$E_{d+3} = [E_1, \ldots, E_{d+2}]_{\mathcal{E}} = 0.$$

If this condition is not satisfied we add E_{d+3} to the system and continue with investigation of solvability.

In this section we consider the case $d = 2$. This is the classical case, studied since Darboux. It is known ([D, Ko], see also [Bi]) that existence of an additional integral, quadratic in momenta, is locally equivalent to the possibility of transforming the metric to the Liouville form

$$ds^2 = (f(x) + h(y))(dx^2 + dy^2).$$

However, no effective criterion for recognizing Liouville metric was obtained despite many attempts. The only visible success was a note [Su]. The solution to the problem was sketched there, but the answer was not written in invariant terms (notwithstanding the title), and the number of differential invariants characterizing Liouville surfaces was not given (in fact, it is difficult to pursue what the proposed set of compatibility conditions actually is and why it is complete, so that we choose another approach below).

We describe a criterion basing on our compatibility criterion. Let us write the metric in isothermal coordinates: $ds^2 = e^{\lambda(x,y)}(dx^2 + dy^2)$ (the approach works with the general form as well, but the expressions become too complicated; However since the answer will be given in differential invariants, the method plays no role).

The function $I = u(x,y)dx^2 + 2v(x,y)dxdy + w(x,y)dy^2$ is a quadratic integral of the geodesic flow iff the following system $\mathcal{E}$ (coefficients of $\{H, I\} = 0$) is satisfied:

$$u_x + \lambda_x u + \lambda_y v = 0, \qquad u_y + 2v_x + \lambda_x v + \lambda_y w = 0,$$
$$2v_y + w_x + \lambda_x u + \lambda_y v = 0, \qquad w_y + \lambda_x v + \lambda_y w = 0.$$

Denoting the equations by E_1, E_2, E_3, E_4 we obtain their compatibility condition

$$E_5 = \frac{1}{2}[E_1, E_2, E_3, E_4]_{\mathcal{E}} = 0.$$

From the general theory it might be expected that E_5 has order 2 (3 in non-reduced form), but in fact it is of the first order and has the following form (after cancelation by $2\sqrt{\det(g)}$):

$$E_5 = 5K_x v_x - 5K_y v_y$$
$$- (K_{xy} + 2\lambda_y K_x + 2\lambda_x K_y)(u - w) + (K_{xx} - K_{yy} + 4\lambda_x K_x - 4\lambda_y K_y)v.$$

Thus the system $\mathcal{E}$ is integrable iff $K = $ const. In this case dimension of the solutions space is $\sum \dim g_k = 6$ ($\dim g_i = \max\{3 - i, 0\}$) and the space of quadratic integrals is the symmetric square of the 3-dimensional space of linear integrals (a basis of the former is the pair-wise product of a basis of the latter).

Suppose that $K \neq $ const, so that at least one of the functions K_x, K_y is not zero. We add the equation $E_5 = 0$ and get a system $\mathcal{E}' \subset J^1(2,3)$ of formal codimension 5.

Its symbols $g_i \subset S^i T^* \otimes \mathbb{R}^3$ have $\dim g'_0 = 3$, $\dim g'_1 = 1$, $\dim g'_2 = 0$ and thus the only non-zero second δ-cohomology are $H^{0,2}(\mathcal{E}') \simeq \mathbb{R}^1$, $H^{1,2}(\mathcal{E}'') \simeq \mathbb{R}^1$. There are two obstructions to compatibility – Weyl tensors W'_1 and W'_2. The tensor W'_1 is proportional to

$$E'_6 = K_x E_{5x} + K_y E_{5y} - \frac{5}{2}K_x^2(E_{2x} - E_{1y}) + \frac{5}{2}K_y^2(E_{3y} - E_{4x})$$

Dividing this by $5K_y$ and simplifying modulo E_1, E_2, E_3, E_4, E_5 we obtain the following expression:

$$E_6 = -35\,|g|\,I_{4b} \cdot v_x + Q_1 \cdot (u - w) + Q_2 \cdot v,$$

where Q_1, Q_2 are certain differential expression of 5th order in the coefficients of the metric. The coefficients of E_6, as well as others E_i, are not invariant, but the condition of their vanishing is invariant, and so can be expressed in terms of differential invariants. Indeed,

$$Q_1 = J_5 \cdot I_3^{-3} \sqrt{|g|}(-K_x)(I_3\sqrt{|g|} - K_x^2),$$
$$Q_2 = J_5 \cdot I_3^{-3} \sqrt{|g|}(I_3\sqrt{|g|} - K_x^2)^{1/2}(I_3\sqrt{|g|} - 2K_x^2),$$

where

$$J_5 = 5I_3(I_{5a} - I_{5c}) + (I_{4a} - I_{4c})(I_{4c} - 6I_{4a}) - 25I_2I_3^3$$

is a differential invariant. Thus the equations $Q_1 = Q_2 = 0$ are equivalent to one condition $J_5 = 0$.

It is possible to show that this condition together with $I_{4b} = 0$ implies $I_{5d} = 0$, which gives another proof of Darboux theorem [D] (proof in [Ko]) that a Riemannian surface with 4 quadratic integrals is a surface of revolution.

The second obstruction to existence of 4 integrals – tensor W_2' – can be calculated similarly. Its vanishing is given by a scalar differential invariant of order 6 in metric, but it can be simplified modulo the conditions $I_{4b} = I_{5d} = J_5 = 0$ to the following expression:

$$J_4 = 3(I_{4a} - I_{4c})(I_{4a} + 4I_{4c})I_{4c} - 15I_2I_3^3(I_{4a} + 4I_{4c}) + 25I_3^5.$$

Thus we obtain the following statement:

Theorem 19. *The condition of exactly 4 quadratic integrals can be expressed as 3 differential conditions on the metric: $I_{4b} = 0$, $J_5 = 0$, $J_4 = 0$.*

If the compatibility condition $E_6 (\mathrm{mod}\, E_1, E_2, E_3, E_4, E_5) = 0$ is satisfied, then the system $\mathcal{E}'$ is integrable. Otherwise we add this new equation and get (again in generic case, when the corresponding matrix of coefficients of derivatives is non-degenerate) the system $\mathcal{E}''$ with symbol $g_1'' = 0$, i.e. it is of Frobenius type.

Its Spencer cohomology group $H^{0,2}(\mathcal{E}'') \simeq \mathbb{R}^3$, so the obstruction to integrability – curvature tensor – W_1'' has 3 components, represented by 3 linear equations relations on $J^0(2,3)$:

$$E_{7j} = A_{1j}(u - w) + A_{2j}v = 0, \qquad j = 1, 2, 3.$$

The expressions A_{ij} are not invariant, but their vanishing is invariant and can be expressed via four differential invariants of order 6 of the metric:

$$J_{6k} = I_{6k} - P(I_2, I_3, I_{4i}, I_{5j}), \qquad k = a, b, c, d,$$

where P is a quadratic function in I_{5j} with rational coefficients in other variables (note that I_{6e} does not enter the formulae). All the expressions are rather long and shall be provided elsewhere. Let us indicate only equation $J_{6a} = 0$:

$$I_{6a} = \frac{1}{175 I_3^2 I_{4b}} \big(700 I_3^5 I_{4b} - 825 I_2 I_3^4 I_{5b} + 50 I_2 I_3^3 I_{4b} (31 I_{4a} - 18 I_{4c})$$

$$+ 6 I_{4b} (I_{4a} - I_{4c})(6 I_{4a}^2 + 49 I_{4b}^2 - 37 I_{4a} I_{4c} + 6 I_{4c}^2) - 25 I_3^2 I_{5b} (-8 I_{5a} + I_{5c})$$

$$- 5 I_3 (48 I_{4a}^2 I_{5b} - 27 I_{5b} I_{4c}^2 + 2 I_{4b} I_{4c} (-11 I_{5a} + 46 I_{5c})$$

$$+ I_{4a} (-43 I_{5a} I_{4b} - 21 I_{5b} I_{4c} + 8 I_{4b} I_{5c}) + 7 I_{4b}^2 (4 I_{5b} - 11 I_{5d}))\big).$$

Theorem 20. *The condition of exactly 3 quadratic integrals can be expressed as 4 differential conditions on the metric: $J_{6a} = J_{6b} = J_{6c} = J_{6d} = 0$.*

Finally if E_{7j} are non-zero, we add these equations to the system. Compatibility condition of the new system $\mathcal{E}'''$ are $(E_{7j})_x = 0$, $(E_{7j})_y = 0$, when expressed as linear functions on $J^0(2,3)$ via the system $\mathcal{E}''$:

$$E_{8l} = B_{1l} u + B_{2l} v + B_{3l} w = 0.$$

Consider the matrix of coefficients of equations E_{7j}, E_{8l}: $U = U(A, B)$. It always satisfies the condition $\operatorname{rank}(U) < 3$, because H is an integral of the geodesic flow. Also $\operatorname{rank}(U) > 0$ if conditions of Theorem 20 are not fulfilled.

Thus we have only two possibilities: If $\operatorname{rank} U = 2$, the flow does not possesses local quadratic in momenta integrals. Otherwise $E_{8l} (\operatorname{mod} E_{7j}) = 0$. This means $\operatorname{rank} U = 1$ and expressing this condition in differential invariants we get one condition of order 6 and four conditions of order 7 in coefficients of the metric (these long expressions will be omitted; note though that the above 4 scalar differential invariants of order 7 involve 6 basic invariants I_{7k}, but the last of them I_{7f} does not enter the formulae): $\tilde{J}_6 = 0$, $J_{7i} = 0$.

These conditions give solvability of the system $\mathcal{E}$, which yields us a 2-dimensional linear space of solutions generated by H and I – an independent integral of degree 2. We set $\square = (\tilde{J}_6, J_{7a}, J_{7b}, J_{7c}, J_{7d})$.

Denote by $\mathcal{S}$ the singular locus of functions $\lambda(x, y)$, that are non-generic w.r.t. at least one one of the above steps (it consists of functions of one variable – metrics with non-zero Killing fields, and certain finite-dimensional families), which corresponds to zero denominators in J_{7i}.

Let also $\mathcal{L}_+$ be the set of λ corresponding to the metrics with more than one additional quadratic integral (constant curvature or the conditions of Theorems 19 and 20). We have proved:

Theorem 21. *There exists a polynomial vector-valued differential operator of order 7 $\square : C_{loc}^\infty(\mathbb{R}^2) \to C_{loc}^\infty(\mathbb{R}^2; \mathbb{R}^5)$ and a residual subset $\mathcal{S} \subset C_{loc}^\infty(\mathbb{R}^2)$ such that for $\lambda \in C_{loc}^\infty(\mathbb{R}^2) \setminus \mathcal{S}$ the metric $e^{\lambda(x,y)} ds_{Eucl}^2$ is Liouville (has quadratically integrable geodesic flow) iff $\lambda \in \mathcal{L}_+$ or $\square(\lambda) = 0$. Moreover, Liouville metrics in $\mathcal{S}$ are residual among all Liouville metrics.*

The singular locus $\mathcal{S}$ in the space of germs of Riemannian metrics (which we identified with $C_{loc}^\infty(\mathbb{R}^2)$ only for convenience) is given by the condition $I_{4b} = 0$. The expressions of the above invariants together with a more detailed argumentation have appeared now in [K₂].

6.3. Gaussian curvature of minimal surfaces

Consider a minimal surface $M^2 \subset \mathbb{R}^3$. The Gauss map defined on it depends on the curvature function and this function is unrestricted (i.e. can be arbitrary in a certain open domain; of course, it is non-positive, but there are no equality-restrictions) if the surface is considered abstractly (non-parametrized). But it's quite known that the Gauss map is not arbitrary, which is manifested by the fact, that the Gaussian curvature on the immersed (parametrized) surface is not arbitrary. We will describe precisely, which functions K on $M^2 \subset \mathbb{R}^3$ are realized locally.

So let M^2 be given as the graph $z = u(x,y)$. Then $\nabla_1(u) = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1+u_x^2+u_y^2)^2}$ is the Gaussian curvature operator and $\nabla_2(u) = \frac{(1+u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1+u_y^2)u_{xx}}{(1+u_x^2+u_y^2)^{3/2}}$ is the operator of mean curvature. Let $\mathcal{H}^\infty = \{u \in C^\infty_{\mathrm{loc}}(\mathbb{R}^2) \,|\, \nabla_2(u) = 0\}$ be the sheaf of minimal surfaces. We define $\nabla_1 : \mathcal{H}^\infty \to C^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ and denote the image by $\mathcal{K}^\infty$. Now we want to resolve this term:

Theorem 22. *There exists an algebraic differential operator $\square : C^\infty_{\mathrm{loc}}(\mathbb{R}^2) \to C^\infty_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ of order 4 and a finite-dimensional stratified submanifold $\mathcal{S} \subset C^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ with $\mathcal{K}^\infty \setminus \mathcal{S} = \mathrm{Ker}(\square)$.*

More precisely, there exist 4 polynomials F_3, F_6, F_7, F_8 on the plane with coefficients depending differentially on K such that the function is realized as the curvature of a minimal surface iff they have a common root.

The form of the operator $\square = (\square_1, \square_2)$ will be clear from the proof, though we suppress the formulas because of their size. Since the operator has singularities, we obtain cases and $K \in \mathcal{K}^\infty \setminus \mathcal{S}$ iff $\square_1(K) = \square_2(K) = 0$, while description of $\mathcal{K}^\infty \cap \mathcal{S}$ is given by some other operators, which we omit.

Proof. Let us find the solvability criterion for the system $\nabla_1(u) = K(x,y)$, $\nabla_2(u) = 0$. Denote

$$F_1 = u_{xx}u_{yy} - u_{xy}^2 - K \cdot (1+u_x^2+u_y^2)^2, \quad F_2 = (1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x)^2 u_{yy}.$$

The Mayer bracket of these operators is

$$F_3 = [F_1, F_2] = a_{11}u_x^2 + 2a_{12}u_x u_y + a_{22}u_y^2 + b,$$

where $a_{11} = K^2(\ln|K|)_{yy} - 4K^3$, $a_{12} = -K^2(\ln|K|)_{xy}$, $a_{22} = K^2(\ln|K|)_{xx} - 4K^3$, $b = K^2((\ln|K|)_{xx} + (\ln|K|)_{yy}) - 4K^3$. Notice that F_3 has order 1, while generically the bracket of 2 second order operators after reduction is also of 2nd order. Therefore the system is compatible if $F_3 = 0$, which is equivalent to $K = 0$, i.e. the surface is a plane.

Denote $F_4 = \mathcal{D}_x(F_3)$, $F_5 = \mathcal{D}_y(F_3)$. The system $F_2 = F_4 = F_5 = 0$ has the form:

$$\begin{pmatrix} (1+u_y^2) & -2u_x u_y & (1+u_x^2) \\ 2(a_{11}u_x + a_{12}u_y) & 2(a_{12}u_x + a_{22}u_y) & 0 \\ 0 & 2(a_{11}u_x + a_{12}u_y) & 2(a_{12}u_x + a_{22}u_y) \end{pmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \\ b_2 \end{bmatrix},$$

where $b_1 = -b_x - (a_{11})_x u_x^2 - 2(a_{12})_x u_x u_y - (a_{22})_x u_y^2$, $b_2 = -b_y - (a_{11})_y u_x^2 - 2(a_{12})_y u_x u_y - (a_{22})_y u_y^2$.

Resolving this for the second derivatives and substituting to F_1 we get after multiplication by the square of the determinant of the above matrix a first order polynomial differential operator F_6'. It has degree 8 by variables u_x, u_y, but reduction due to the system is of degree 6:

$$F_6 = -F_6' + 4K(1 + u_x^2 + u_y^2)^2 (2bF_3 - F_3^2)$$
$$= b_1^2 + b_2^2 + (b_2 u_x - b_1 u_y)^2 + 4K(1 + u_x^2 + u_y^2)^2 \left(b^2 + (a_{11}u_x + a_{12}u_y)^2 + (a_{12}u_x + a_{22}u_y)^2 \right).$$

Denote $E_7 = [E_3, E_6]$ the Mayer bracket. It is the third 1st order PDE, which can be equivalently written as

$$\frac{\partial(F_3, F_6)}{\partial(x, u_x)} + \frac{\partial(F_3, F_6)}{\partial(y, u_y)} = 0.$$

Let us prolong the equation $F_6 = 0$: $\mathcal{D}_x(F_6) = 0$, $\mathcal{D}_y(F_6) = 0$. Resolving these equations coupled with $F_4 = F_5 = 0$ with respect to second derivatives ($F_7 = 0$ guarantees compatibility) and substituting to $F_2 = 0$ we get:

$$F_8 = (1 + u_y^2)\frac{\partial(F_3, F_6)}{\partial(x, u_y)} + u_x u_y \left(\frac{\partial(F_3, F_6)}{\partial(x, u_x)} - \frac{\partial(F_3, F_6)}{\partial(y, u_y)} \right) - (1 + u_x)^2 \frac{\partial(F_3, F_6)}{\partial(y, u_x)}.$$

Thus we obtain four 1st order PDEs $F_3 = 0$, $F_6 = 0$, $F_7 = 0$, $F_8 = 0$, which are polynomials by u_x, u_y of degrees 2, 6, 7, 9 (they do not depend on u) and are differential operators by K of orders 2, 3, 4, 4 respectively.

Solvability of the system $F_1 = F_2 = 0$ is equivalent to the claim that polynomial by u_x, u_y system $F_3 = F_6 = F_7 = F_8 = 0$ has a solution. The latter is equivalent to 2 conditions $\square_1 = 0$, $\square_2 = 0$, algebraic by the corresponding coefficients, which are differential operators of K. Therefore we set $\square = (\square_1, \square_2)$.

The set $\mathcal{S}$ is formed by functions K for which some of F_3, F_6, F_7, F_8 become dependent or singular. This set is given by a collection of overdetermined systems of PDEs of finite type and hence is stratified finite-dimensional.

Let us demonstrate a geometric idea behind this proof. F_1 and F_2 are Monge-Ampère operators on the plane, which means ([L$_2$]) that they are given by 2-forms Ω_1, Ω_2 on $J^1(\mathbb{R}^2)$: Equations $F_i(u) = 0$ can be rewritten as $\Omega_i|_{j_1(u)} = 0$, where $j_1(u) \subset J^1(\mathbb{R}^2)$ is the jet-section determined by u and

$$\Omega_1 = du_x \wedge du_y - K(1 + u_x^2 + u_y^2)^2 dx \wedge dy,$$
$$\Omega_2 = (1 + u_y^2)du_x \wedge dy + u_x u_y(dx \wedge du_x + du_y \wedge dy) + (1 + u_x^2)dx \wedge du_y.$$

Since Ω_i do not depend on u, the construction descends onto $T^*\mathbb{R}^2 = J^1(\mathbb{R}^2)/\mathbb{R}^1$, where we have in addition the canonical symplectic 2-form Ω_0.

We search for a Lagrangian surface, which is isotropic w.r.t. Ω_1, Ω_2 and lies in the hypersurface $\Sigma^3 = \{F_3 = 0\} \subset T^*\mathbb{R}^2$. Let ξ_i be the kernels of the 2-forms Ω_i restricted to Σ^3. They fail to be in general position only along a surface $\Sigma^2 = \{F_3 = F_6 = 0\}$, which is equivalently determined by the equation

$$\Sigma^2 = \{x \in \Sigma^3 \mid \mathrm{rank}(\xi_0(x), \xi_1(x), \xi_2(x)) < 3\}.$$

This surface is ramified over $\mathbb{R}^2$ with the fiber consisting of no more than 12 points (precisely this number if counted over $\mathbb{C}$ and with multiplicities).

Local solvability means that over a neighborhood in $\mathbb{R}^2$ at least one component Σ_α^2 of Σ^2 is isotropic for all Ω_i. By characteristic property of this surface we need to require only two restrictions $\Omega_i|_{\Sigma_\alpha^2}$ to vanish. For instance, we take the conditions $\xi_0, \xi_2 \subset T\Sigma_\alpha^2$. They are given by the equations $\Omega_0 \wedge dF_3 \wedge dF_6 = 0$ and $\Omega_2 \wedge dF_3 \wedge dF_6 = 0$, which correspond to the operators F_7 and F_8. $\square$

Example 3. *Consider the family $K = ke^{\alpha x + \beta y}$. Then it realizes the curvature of a minimal surface iff $k = 0$. In fact, the quadric $F_3 - b$ is definite and $a_{11}b \geq 0$. So $F_3 k \leq 0$ and the inequality is strict if $k \neq 0$.*

Notice that F_3 is non-singular if $K \neq 0$. The set $\mathcal{K}^\infty$ is a subset of the solutions to $F_3 = F_6 = 0$. Let us describe non-holonomic solutions to this system (ignoring the compatibility condition $F_7 = 0$).

Let's parameterize the quadric $F_3 = 0$: Consider a line $u_x = t, u_y = \lambda t$. It meets $F_3 = 0$ at the points with $t^2 = -b/\rho(\lambda)$, $\rho(\lambda) = a_{11} + 2a_{12}\lambda + a_{22}\lambda^2$. Substituting this into $\rho(\lambda)^3 F_6 = 0$ we get a polynomial of degree 6:

$$q_6(\lambda) = \rho(\lambda)^3 b^4 \left[\left(\frac{\rho(\lambda)}{b}\right)_x^2 + \left(\frac{\rho(\lambda)}{b}\right)_y^2\right] - b^5 \left[\lambda \left(\frac{\rho(\lambda)}{b}\right)_x - \left(\frac{\rho(\lambda)}{b}\right)_y\right]^2$$
$$+ 4K \left[\rho(\lambda) - b(1 + \lambda^2)\right]^2 \left[b^2 \rho(\lambda) - b\left((a_{11} + a_{12}\lambda)^2 + (a_{12} + a_{22}\lambda)^2\right)\right].$$

Thus $\{F_3 = 0\} \cap \{F_6 = 0\}$ corresponds (2-to-1) to the roots of $q_6(\lambda) = 0$.

Example 4. *Consider the family $K = \varphi(x)$. For an open set of such functions the equation $q_6(\lambda) = 0$ has 6 positive roots and so the system $F_3 = F_6 = 0$ has 12 non-holonomic solutions u_x, u_y. None of them satisfies the other two equations $F_7 = F_8 = 0$, save for the case $u_x = \mathrm{const}_1$, $u_y = \mathrm{const}_2$. Actually, the solutions depend on x only and so $u_{xy} = u_{yy} = 0$, whence $u_{xx} = 0$ and $K = 0$.*

Let $\mathcal{S}_0 \subset \mathcal{S}$ be the set of functions K, such that $\{F_3 = 0\} \equiv \{F_6 = 0\}$. It is given by the condition $q_6(\lambda) \equiv 0$, which is a system of 7 third order PDEs on K. As a by-product of the proof we get the following statement:

Proposition 23. *For $K \in \mathcal{K}^\infty \setminus \mathcal{S}_0$ the set of minimal surfaces through the origin with this curvature $\nabla_1^{-1}(K) \cap \mathcal{H}^\infty \cap \{u(0,0) = 0\}$ has cardinality at most 12, though generically this number is 2 (respectively 6 and 1 if we the function u is considered up to the sign).*

Proof. In fact, all the equations are symmetric to the change $u \mapsto -u$. The sub-system $F_3 = F_6 = 0$ with respect to u_x, u_y has as maximum $\mathrm{ord}(F_3) \cdot \mathrm{ord}(F_6) = 12$ algebraic solutions. Some of them may not satisfy the other constraints $F_7 = F_8 = 0$ and generically (in $\mathcal{K}^\infty \setminus \mathcal{S}_0$) only 2 do satisfy. $\square$

The above statement does not hold for $K = 0 \in \mathcal{S}_0$. But we suggest there are no other examples. To see the reason let us consider the overdetermined

system of 7 polynomial differential equations on K of order 3: $q_6(0) = 0, q'_6(0) = 0, \ldots, q_6^{(6)}(0) = 0$. One can expect that it has no other solutions than $K = 0$, but this is not so.

For instance, $q_6(\lambda) \equiv 0$ follows from $b = 0$ and $\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = 0$. This latter is equivalent to the system $\det \mathrm{Hess}(\ln |K|) = 0$, $\Delta \ln |K| = 4K$, which though incompatible is solvable and has the solution $K = \varphi(x \pm y)$, where $\varphi' = 2\varphi\sqrt{\varphi + c}$.

But for this choice of K the other two equations $F_7 = F_8 = 0$ are not satisfied unless $\varphi = 0$. In fact, then $F_3 = 0$ and $F_6 = 0$ are both equivalent to $u_x + u_y = 0$, which coupled with $F_1 = F_2 = 0$ gives $K = 0$.

The computer programs do not give us other solutions to the above system of 7 third order PDEs on K and so *we conjecture that*

$$\mathcal{K}^\infty \cap \mathcal{S}_0 = 0.$$

Remark 15. *This our conjecture that except the plane case a minimal surface is restored (up to $\pm$) from its Gauss image in maximal 6 different ways is similar to the known Gronwall conjecture about webs on the plane. It says that a linearizable 3-web has maximally 11 linearizations (and generically only 1), cf. [GL$_1$]. Both problems are basically algebraic and have equal complexities.*

Similarly one investigates the problem, when two functions K and H on a parametrized surface M^2 can be realized as Gaussian and mean curvatures, with the surface realized as a graph (projection $\mathbb{R}^3 \to \mathbb{R}^2$ yields the parametrization).

This is an analog of the classical Bonnet problem of realizing two quadrics as the first and the second quadratic forms on a surface. Bonnet theorem states that compatibility and solvability of this problem is the system of one Gauss and two Kodazzi equations.

For realization of the curvatures K, H compatibility is equivalent to the condition $K = H = 0$, i.e. the surface is plane (notice that the solutions space is 3-dimensional, not 4-dimensional as one can expect after §5.4, but this is due to non-genericity of the condition). Solvability leads to an operator of order 4, similar to the above $\square$. Thus we get solution to generalized Bonnet problem.

6.4. Quantum integration

Consider the algebra of scalar linear differential operators $\mathcal{A} = \mathrm{Diff}(\mathbf{1}, \mathbf{1})$ on the manifold M filtered by the $C^\infty(M)$-modules $\mathcal{A}_k$ of order $\leq k$ differential operators. Let $\mathcal{P} = \oplus_{k \geq 0} \mathcal{P}_k = \mathrm{gr}(\mathcal{A})$ be the corresponding graded module. Here $\mathcal{P}_k = \mathcal{A}_k / \mathcal{A}_{k-1}$ consists of degree k homogeneous in momenta polynomials on T^*M. Thus $\mathcal{P} = S\mathcal{D} = \oplus S^k \mathcal{D}$, where $\mathcal{D}$ is the $C^\infty(M)$-module of vector fields on M. The canonical Poisson structure on $\mathcal{P}$ is given by the formula

$$\{\nabla_1 \bmod \mathcal{A}_k, \nabla_2 \bmod \mathcal{A}_l\} = [\nabla_1, \nabla_2] \bmod \mathcal{A}_{k+l-1},$$

where the bracket in the r.h.s. is the usual commutator (or Jacobi bracket). In other words, the mapping $\sigma = \mathrm{smbl} : \mathcal{A} \to \mathcal{P}$ is a homomorphism of Lie algebras.

By quantization one understands an inverse map $q_* : \mathcal{P} \to \mathcal{A}$, i.e. a collection of morphisms $q_k : \mathcal{P}_k \to \mathcal{A}_k$ splitting the sequence

$$0 \to \mathrm{Diff}_{k-1}(\mathbf{1}, \mathbf{1}) \hookrightarrow \mathrm{Diff}_k(\mathbf{1}, \mathbf{1}) \xleftarrow{\,--\,} S^k \mathcal{D} \to 0.$$

This map allows to introduce a new non-commutative associative product on $\mathcal{P}$:

$$a \star b = q_*^{-1}(q_*(a) \circ q_*(b)).$$

These kinds of products are important in the deformation quantization. Moyal [Mo] and other star-products [L₃] are obtained by specifying the morphism q_*.

Denoting by p_k the homogeneous $\mathcal{P}_k$-component of a polynomial $p \in \mathcal{P}$ we observe the relation

$$\{a, b\} = (a \star b - b \star a)_{k+l-1} = \mathrm{smbl}_{k+l-1}([q_k(a), q_l(b)]), \quad a \in \mathcal{P}_k, b \in \mathcal{P}_l$$

between the commutator, Poisson bracket and the star-product.

Consider a mechanical system with a Hamiltonian $h \in \mathcal{P}$. Its quantization is given by $H = q_*(h)$. If the choice of q_* is subject to certain connections ([L₃]), then Riemannian metric produces the Laplace operator, choice of potential – Shrödinger operator etc.

Suppose the classical system is integrable in Liouville sense, i.e. there exist functions $f_1 = h, f_2, \ldots, f_n \in \mathcal{P}$ ($n = \dim M$) functionally independent a.e. which Poisson-commute $\{f_i, f_j\} = 0$. We wish to quantize this picture.

Definition 6. *Differential operator H is called quantum completely integrable if there exist commuting differential operators $F_1 = H, F_2, \ldots, F_n \in \mathcal{A}$, which are independent a.e.*

Clearly then the system $h = \mathrm{smbl}(H)$ is Liouville-integrable with integrals $f_i = \mathrm{smbl}(F_i)$. The quantization poses the inverse problem: To find quantum integrable system (H, F_i) by the given classical (h, f_i).

This problem was solved for many classically integrable Hamiltonian systems ([Pe]) using different approaches: analytical, Dunkl's differential-difference operator [Du], Moyal quantization, via geodesic equivalence [MT] and others.

We discuss one of them, which is closely related to our integration method. It was proposed in [He] and is based on universal enveloping algebras.

Consider the rigid body equations, which is the Hamiltonian system on $T^*SO(3)$ with Hamiltonian $h = \frac{1}{2} \sum \tau_i^{-1} p_i^2 + \sum \gamma_i x_i$, where x_i are the base coordinates and p_i are the corresponding momenta.

Let $X_i = q_*(x_i)$ at $R \in SO(3)$ be equal to (Re_i, e), where e_i is an orthonormal basis and e some unit vector in $\mathbb{R}^3$, and $P_i = q_*(p_i)$ be the left-invariant fields $\exp(E_i)$ generated by the basis $E_i \in so(3)$ given by the relations $E_i(e_i) = 0$, $E_i(e_{i\pm1}) = \pm e_{i\mp1}$, $i \in \mathbb{Z}_3$. Then the subalgebra of $\mathcal{A}(SO(3))$ generated by P_i, X_i is isomorphic to the universal enveloping algebra of $so(3) \ltimes \mathbb{R}^3$: $[P_i, P_j] = \epsilon_{ijk} P_k$, $[P_i, X_j] = \epsilon_{ijk} X_k$, $[X_i, X_j] = 0$.

The quantized Hamiltonian has the form $H = q_*(h) = \frac{1}{2} \sum \tau_i^{-1} P_i^2 + \sum \gamma_i X_i$ and we denote it also by F_1. We have two Casimir functions in $U(so(3) \ltimes \mathbb{R}^3)$:

$F_2 = \sum X_i^2$, $F_3 = \sum P_i X_i$. To achieve complete quantization of all known integrable Euler equations we must quantize the forth integral. It is as follows:

Euler case: $\gamma_i = 0$. Then $F_4 = \sum P_i^2$.

Lagrange case: $\tau_1 = \tau_2$, $\gamma_1 = \gamma_2 = 0$. Then $F_4 = P_3$.

Kovalevskaya case: $\tau_1 = \tau_2 = \tau_3/2$, $\gamma_3 = 0$. Then $F_4 = K\bar{K} + \bar{K}K - g_4$, where $K = \tau_1(P_1 + iP_2)^2 - 2(\gamma_1 + i\gamma_2)(X_1 + iX_2)$ and $g_4 = 8\tau_1^2(P_1^2 + P_2^2)$. In all these cases $[H, F_4] = 0$.

Goryachev-Chaplygin case (conditional integrability): $\tau_1 = \tau_2 = \tau_3/4$, $\gamma_3 = 0$. Then $F_4 = \tau_1(P_1^2 + P_2^2)P_3 - X_3(\gamma_1 P_1 + \gamma_2 P_2) - g_3$, where $g_3 = \frac{1}{2}(\gamma_2 X_1 - \gamma_1 X_2) + \frac{1}{4}\tau_1 P_3$. In this case $[H, F_4] = \tau_1(\gamma_2 P_1 - \gamma_1 P_2)F_3$.

Note that according to [L$_3$] any quantization q_* is determined by two linear connections (electromagnetic field and gravity) and a collection of tensors g_k: $S^k \mathcal{D} \to \mathcal{A}_{k-2}$. The first two of the above integrable cases are obtained from the classical scheme via the operator q_* and trivial tensors g_2 and g_1. In the Kovalevskaya case one needs a second order correction g_4 and in the Goryachev-Chaplygin case a first order g_3 (these corrections were found previously in [He], but the explanation of actual orders meaning was lacking).

Theorem 24. *In each of the classical integrable cases the obtained quantum integrals allow to integrate the Shrödinger operator $L[u] = u_t - H(u) - \lambda_1 u$ (where λ_1 is the spectral parameter) classically: The system*

$$L[u] = 0,\; F_2(u) = \lambda_2 u,\; F_3(u) = \lambda_3 u,\; F_4(u) = \lambda_4 u$$

is of finite type and compatible ($\lambda_3 = 0$ in the Goryachev-Chaplygin case).

Proof. In the first three cases, where we have a commutative collection of integrals F_1, F_2, F_3, F_4, the statement is rather known. However, the Goryachev-Chaplygin case seems to be quantization by analogy and its meaning is given by the above statement, which follows from our compatibility criterion. $\qquad\square$

Remark 16. *The above result is constructive, not mere an existence statement. In fact, the considered system is of Frobenius type and one reduces its integration to a certain system of ODEs, which may be integrated via symmetry approach.*

So far we obtained only local solutions, but if we are given a global Shrödinger equation on a manifold M, then we get a topological restriction for a compatibility/solvability. Namely the monodromy operator determines which spectral parameters λ are admissible. For them and only for them we get a closed leaf of the foliation corresponding to the above Frobenius system.

Note that in the definition of quantum integrability we used smbl-map, not q_*^{-1}. In fact, $q_*^{-1}(F_i)$ need not to commute, only their top-components. Denote by $\rho_k : \mathcal{P} \to \mathcal{P}_k$ the natural projection. Then commutation of $q_k(a)$, $q_l(b)$ implies that $\{a, b\} = \rho_{k+l-1}q_*^{-1}([q_*(a), q_*(b)]) = 0$. So the quantization philosophy suggests to make the construction so that the functions commute w.r.t. the deformed bracket

$$\{a, b\}_q = q_*^{-1}([q_*(a), q_*(b)]).$$

This is equivalent to the quantum integrability problem: given a system $f_1, \ldots, f_n$ involutive w.r.t. the Poisson bracket describe the quantizations q_* such that the set is still involutive w.r.t. the new bracket $\{,\}_q$.

However our compatibility result suggest that it is equally important to search for deformations giving sub-algebras, i.e. $\{f_i, f_j\}_q = \sum c_{ij}^k f_k$. In the classical case they correspond to invariant submanifolds of the Hamiltonian system. Thus our main result interprets as quantization of conditional integrability in the classical mechanics.

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