Abstract

We extend the definition of the Kobayashi pseudodistance to almost complex manifolds and show that its familiar properties are for the most part preserved. We also study the automorphism group of an almost complex manifold and finish with some examples.

The Kobayashi pseudodistance on almost complex manifolds

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Introduction

The Poincaré metric on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is a Riemannian metric

$$|v| = \frac{|v|_{euc}}{1 - |z|^2}$$

conformal with the Euclidean metric $|\cdot|_{euc}$, that induces a distance d on $\mathbb{D}$ with the remarkable property that every holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$ is distance nonincreasing in d. This fact, discovered in 1915 by Pick [Pi] is an invariant formulation of the Schwarz lemma. In 1967 Kobayashi used the distance d to define a pseudodistance d_M on any (connected) complex manifold M, such that any holomorphic mapping from a complex manifold L to a complex manifold M is distance nonincreasing with respect to d_L and d_M. When this Kobayashi pseudodistance d_M is a distance, it may be used to obtain information about holomorphic mappings to or from M; in this situation M is said to be (Kobayashi-) hyperbolic. Some references for Kobayashi hyperbolicity are [JP, Ko1, Ko2, La, NO].

In this paper* we extend the definition of the Kobayashi pseudodistance to almost complex manifolds and show that its familiar properties are for the most part preserved. Results whose proofs are similar to known ones for complex manifolds are merely stated. We also study the automorphism group of an almost complex manifold and finish with some examples.

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Chapter 1

Definition of the pseudometric

Let $\mathbb{C}$ denote the complex plane with its standard complex structure, $\mathbb{D}$ the open unit disk in it and $e = 1 \in T_0\mathbb{D}$. Let (M^{2n}, J) be an almost complex manifold, which in this paper will be taken to be connected and C^∞. This means that the smooth field of automorphisms $J \in T^*M \otimes TM$ satisfies $J^2 = -\mathbf{1}$. A Kobayashi chain joining two points $p, q \in M$ is a sequence of pseudoholomorphic mappings

$$f_k : \mathbb{D} \to (M^{2n}, J), \quad k = 1, ..., m,$$

and points $z_k, w_k \in \mathbb{D}$ such that $f_1(z_1) = p$, $f_m(w_m) = q$ and $f_k(w_k) = f_{k+1}(z_{k+1})$. The *Kobayashi pseudodistance* from p to q is defined by

$$d_M(p, q) = \inf \sum_{k=1}^{m} d(z_k, w_k)$$

where the infimum is taken over all Kobayashi chains joining p to q, if there exists some Kobayashi chain joining p to q, and is defined to be $+\infty$ otherwise. It is well known that $d_{\mathbb{C}} \equiv 0$ and $d_{\mathbb{D}} \equiv d$.

Proposition 1. *The function $d_M : M \times M \to \mathbb{R}$ is nonnegative, symmetric and satisfies the triangle inequality, i.e. it is a pseudodistance, called the Kobayashi pseudodistance. Pseudoholomorphic mappings between almost complex manifolds are distance nonincreasing with respect to the Kobayashi pseudodistance.*

As in the case of complex manifolds, d_M is finite on any almost complex manifold, but this requires an existence theorem for pseudoholomorphic disks due to Nijenhuis and Woolf [NW].

Theorem 1. (i). *The Kobayashi pseudodistance* d_M *is finite and continuous on* $M \times M$ *for any almost complex manifold* (M^{2n}, J).
(ii). *If* d_M *is a distance, it induces the standard topology on* M.

Proof. We must first prove that any two points in M can be joined by a Kobayashi chain. It is enough to prove that for any point $p \in M$ there is a neighborhood U_p of p such that any point q in U_p can be joined to p by a single pseudoholomorphic disk. For then the set of points that can be joined by a Kobayashi chain to some fixed point is open and nonempty, and its complement is open, so every point can be joined to the fixed point since M is connected.

The problem is now local, so we consider $(\mathbb{R}^{2n}, J)$. Let $v \in T_0\mathbb{R}^{2n} = \mathbb{R}^{2n}$. By theorem III of [NW], there is some neighborhood V of 0 in $\mathbb{R}^{2n}$ such that if $v \in V$, then there exists a pseudoholomorphic mapping $f : \mathbb{D} \to (\mathbb{R}^{2n}, J)$ with $f(0) = 0$ and $f_*(0)e = v$; this mapping could be choosen canonically. We write this mapping as $f(z; v)$ to show its dependence on v. We denote its differential with respect to z by $f_*(z; v)$. By 5.4a of [NW], $f : \mathbb{D} \times V \to \mathbb{R}^{2n}$ is a C^∞ mapping. It satisfies $f(0; v) = 0$ and $f_*(0; v)e = v$. In addition, by 5.2a of [NW] , it satisfies $f(z; 0) = 0$ on the unit disk. Since

$$\frac{\partial f_*(0; v)e}{\partial v} = I$$

there is some neighborhood W of 0 in $\mathbb{D}$ such that for every $\zeta \in W$, the matrix

$$\left.\frac{\partial f_*(\zeta; v)e}{\partial v}\right|_{v=0}$$

is of full rank. Now

$$\left.\frac{\partial f(\zeta; v)}{\partial v}\right|_{v=0} = \zeta \left.\frac{\partial f_*(\zeta; v)e}{\partial v}\right|_{v=0} + O(|\zeta|^2),$$

where multiplication is with respect to the complex structure on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ induced by the linearization. So there exists some $\zeta_0 \in W\backslash\{0\}$ such that

$$\left.\frac{\partial f(\zeta_0; v)}{\partial v}\right|_{v=0}$$

is a matrix of full rank. Then $f(\zeta_0; V)$ contains an open neighborhood of 0 in $\mathbb{R}^{2n}$, because $f(\zeta_0; 0) = 0$. So if q is in this neighborhood, there is some $v \in V$ such that $f(\cdot; v)$ is a pseudoholomorphic disk joining 0 and q.

Since

$$|d_M(p,q) - d_M(p_0,q_0)| \leq d_M(p,p_0) + d_M(q,q_0)$$

by the triangle inequality, to prove continuity of d_M it is enough to prove continuity of the mappings $d(\cdot, p_0)$ for all $p_0 \in M$. The reasoning above shows that for every $\varepsilon > 0$ there exists some neighborhood U of p_0 such that for some $\zeta_0 \in \mathbb{D}$ with $d(0,\zeta_0) = \varepsilon$ the image $f(\zeta_0; V)$ contains U and thus $d_M(p,p_0) \leq \varepsilon$ for any $p \in U$. The proof that d_M induces the standard topology on M if it is a distance can now be carried through by the method of Barth [Ba]. □

It is obvious that for any $p_1, q_1 \in (M_1, J_1)$ and $p_2, q_2 \in (M_2, J_2)$ we have

$$\max\left[d_{M_1}(p_1,q_1), d_{M_2}(p_2,q_2)\right] \leq \\ d_{M_1 \times M_2}(p_1 \times p_2, q_1 \times q_2) \leq d_{M_1}(p_1,q_1) + d_{M_2}(p_2,q_2),$$

but this inequality is much less useful than in the complex case. For if $(M_1 \times M_2, J)$ is a product manifold with an almost complex structure J, it is almost never true even locally that J decomposes as a product of complex structures $J = J_1 \times J_2$.

In chapter 2 we state a result on hyperbolicity of almost complex manifolds which are fiber spaces. A more elementary fact is

Proposition 2. *Let $\pi : E \to B$ be an almost complex locally trivial bundle. Then for every two points $p, q \in B$ we have*

$$d_B(p,q) = d_E(\pi^{-1}(p), \pi^{-1}(q)) = \inf\{d_E(\tilde{p},\tilde{q}) \,|\, \pi(\tilde{p}) = p, \pi(\tilde{q}) = q\}.$$

Proof. Since $\pi : E \to B$ is pseudoholomorphic we have

$$d_B(p,q) \leq d_E(\tilde{p},\tilde{q}).$$

On the other hand every pseudoholomorphic disk $f : \mathbb{D} \to B$ lifts to a pseudoholomorphic disk $\tilde{f} : \mathbb{D} \to E$. □

Corollary. *The same formula holds for almost complex (unbranched) coverings. Hyperbolicity of an almost complex manifold is equivalent to hyperbolicity of any of its (unbranched) coverings.*

We note that the Kobayashi-Royden infinitesimal pseudometric F_M may also be defined on almost complex manifolds using the results of Nijenhuis and Woolf. The definition is

$$F_M(p; v) = \inf \frac{1}{r}$$

where $p \in M$, $v \in T_pM$ and the infimum is taken with respect to all pseudoholomorphic mappings $f : \mathbb{D} \to M$ with $f(0) = p$ and $f_*(0)e = rv$. This may be reformulated as another invariant $\pi F_M(p; v)^{-2} = \sup(\pi r^2)$ which looks similar to the so-called Gromov width from symplectic geometry, see [Gr, MS].

Chapter 2

Hyperbolicity

An almost complex manifold (M^{2n}, J) is *Kobayashi hyperbolic* if the Kobayashi pseudodistance d_M is a distance. It is *Brody hyperbolic* if any pseudoholomorphic mapping $f : \mathbb{C} \to M$ is constant. It is clear that Kobayashi hyperbolicity implies Brody hyperbolicity. Brody [Br] discovered that for complex manifolds the converse holds under the additional condition of compactness. As is usual, we take the term hyperbolic to mean Kobayashi hyperbolic, and denote other hyperbolic properties by a prefix.

The following is a version of Brody's theorem [Br] for almost complex manifolds:

Theorem 2. *Let (M^{2n}, J) be an almost complex manifold and $|\cdot|$ a continuous norm on TM. If there exists a constant C such that*

$$|f_*(0)e| \leq C$$

for all pseudoholomorphic mappings $f : \mathbb{D} \to M$, then M is hyperbolic.

If M is compact hyperbolic, then there exists a constant C such that

$$|f_*(z)e| \leq \frac{C}{1 - |z|^2}$$

for all pseudoholomorphic mappings $f : \mathbb{D} \to M$ and all $z \in \mathbb{D}$.

A compact almost complex manifold is hyperbolic if and only if it is Brody hyperbolic.

Proof. Suppose that a constant C exists with $|f_*(0)e| \le C$ for all pseudoholomorphic mappings $f : \mathbb{D} \to M$. Then

$$|f_*(z)e| \le \frac{C}{1 - |z|^2}$$

by precomposing with suitable disk automorphisms. Let ρ be the distance associated to $|\cdot|$. Then for every pseudoholomorphic mapping $f : \mathbb{D} \to M$ and for every pair $z_1, z_2 \in \mathbb{D}$ the following holds: $\rho(f(z_1), f(z_2)) \le Cd(z_1, z_2)$. The last inequality implies $0 < \rho(p, q) \le Cd_M(p, q)$ for every pair of points $p \ne q$, whence the first claim follows.

To prove the second claim, note that if no such constant C exists, then neither does there exist a constant C such that

$$|f_*(0)e| \le C$$

for all pseudoholomorphic mappings $f : \mathbb{D} \to M$, again by precomposing with suitable disk automorphisms. So let $f_k : \mathbb{D} \to M$ be a sequence of pseudoholomorphic mappings with

$$|(f_k)_*(0)e| \to \infty,$$

and assume that M is compact. We will show that M is not hyperbolic. Since M is compact, we may extract a subsequence to assure that $f_k(0) \to p \in M$. Let K be a compact set with p an interior point on which J is tamed by an exact symplectic form, and let r_k be the supremum of radii $r \le 1$ such that $f_k(r\mathbb{D}) \subseteq K$. Gromov's Schwarz lemma, see Corollary 4.1.4 of [Mu], implies that $r_k \to 0$, since $|(f_k)_*(0)e| \to \infty$. It is clear that there exists a sequence $z_k \in r_k\overline{\mathbb{D}}$ such that $q_k = f(z_k) \in \partial K$. Extract a subsequence to assure that $q_k \to q \in \partial K$. Then $d_M(p, q_k) \to d_M(p, q)$ by the continuity of d_M and

$$d_M(p, q_k) = d_M(f_k(0), f_k(z_k)) \le d(0, z_k) \to 0$$

and so $d_M(p, q) = 0$, hence M is not hyperbolic.

It is clear that a hyperbolic almost complex manifold is Brody hyperbolic. Assume on the other hand that M is a compact, but not hyperbolic, almost complex manifold. We shall prove that there exists a nonconstant pseudoholomorphic mapping $g : \mathbb{C} \to M$.

We put a Riemannian metric $|\cdot|$ on M. Since M is not hyperbolic, there is a sequence of pseudoholomorphic mappings $f_k : \mathbb{D} \to M$ with $|(f_k)_*(0)e| \to \infty$. The proof of Brody's reparametrization lemma [Br] goes through unchanged for almost complex manifolds. We use it in the form that if (M^{2n}, J) is an almost complex manifold, $|\cdot|$ a continuous norm on TM and $f : r\mathbb{D} \to M$ a pseudoholomorphic mapping such that $|f_*(0)e| \geq c > 0$, then there exists a pseudoholomorphic mapping $h : r\mathbb{D} \to M$ such that $|h_*(0)e| = c/2$,

$$|h_*(z)e| \leq \frac{c}{2}\frac{r^2}{r^2 - |z|^2},$$

and $h(r\mathbb{D}) \subseteq f(r\mathbb{D})$. See p. 27 of [NO] for a proof that can be adapted virtually unchanged. We apply Brody's reparametrization lemma to the sequence of pseudoholomorphic mappings

$$f_k\left(\frac{2z}{|(f_k)_*(0)e|}\right)$$

from $r_k\mathbb{D}$ to M, where $r_k = |(f_k)_*(0)e|/2 \to \infty$. This gives a sequence of pseudoholomorphic mappings $g_k : r_k\mathbb{D} \to M$ with $|(g_k)_*(0)e| = 1$ and

$$|(g_k)_*(z)e| \leq \frac{r_k^2}{r_k^2 - |z|^2}.$$

Using the last inequality and compactness theorem 4.1.3 of [MS], we conclude that g_k has a subsequence which converges uniformly with all derivatives on compact subsets of $\mathbb{C}$ to a pseudoholomorphic mapping $g : \mathbb{C} \to M$. It is nonconstant because $|(g)_*(0)e| = 1$. □

More general versions of Brody's theorem are known for complex manifolds. See theorems 2.2 and 2.3 of [La]. They have analogs for almost complex manifolds, which are easily proved by modifying the proofs in [La] along the lines of the above proof. Then one can prove the following theorem in the same way as theorem 3.1 is proved in [La]:

Theorem 3. *Let $\pi : E \to B$ be a proper pseudoholomorphic mapping between almost complex manifolds, whose fibers are manifolds. If B is hyperbolic and each fiber is hyperbolic, then E is hyperbolic. If the fiber above a point of B is hyperbolic, this point has a neighborhood such that the fibers above all points of this neighborhood are hyperbolic.*

Chapter 3

The automorphism group

In contrast to the complex case, generic almost complex structures admit no local automorphisms other than the identity and so the group $\mathrm{Aut}(M^{2n}, J)$ is usually trivial. The obstruction theory for local and formal automorphisms of almost complex manifolds was developed in [Kr]. However in practice we usually need criteria for this generic property. Here we give one connected with the notion of hyperbolicity.

It is a result of Kobayashi [Ko1] that the automorphism group of a closed (usually called in the literature just compact but with assumption of empty boundary) hyperbolic complex manifold is finite. This is also true in the almost complex case. For closed almost complex manifolds, the group of pseudoholomorphic diffeomorphisms of the manifold to itself is a Lie transformation group, when equipped with the topology of Σ-convergence, i.e. uniform convergence on compact sets of the mappings and their derivatives through the third order, as was proved by Boothby, Kobayashi and Wang [BKW].

Theorem 4. *The automorphism group $G = \mathrm{Aut}(M^{2n}, J)$ of a closed hyperbolic almost complex manifold (M^{2n}, J) is finite.*

Proof. Since G is a Lie group we may consider its Lie algebra $\mathcal{G}$ consisting of pseudoholomorphic vector fields; i.e. fields ξ on M^{2n} such that $L_\xi J = 0$ or $[\xi, J\eta] = J[\xi, \eta]$ for every vector field η on M^{2n}. In particular, $[\xi, J\xi] = 0$. This means that if $0 \neq \xi \in \mathcal{G}$ then we have a nonconstant pseudoholomorphic curve $f : \mathbb{C} \to M^{2n}$ through a point $p \in M^{2n}$ of the form

$$z = x + iy \mapsto \exp(x\xi + y(J\xi))p.$$

The exponential mapping is globally defined since M^{2n} was assumed closed. Now (M^{2n}, J) was assumed hyperbolic and so such a nonconstant pseudo-holomorphic curve cannot exist. Hence $\mathcal{G}$ is trivial and thus G is discrete.

We must now show that G is compact. Since M is compact and second countable, G is second countable. So it is enough to prove that any sequence $\varphi_k \in G$ has a Σ-convergent subsequence. In the following, each time we extract a subsequence, we give it the same notation as the original sequence. Note that G is a subgroup of the isometry group $\mathcal{I}(M, d_M)$ of M, where d_M is the Kobayashi distance on M. According to the theorem of van Dantzig and van der Waerden [DW], see also pp. 46-50 of [KN], $\mathcal{I}(M, d_M)$ is compact in the topology induced by d_M. Moreover this theorem easy generalizes to the following statement which we also use:

Lemma. *Let (A, d_A) and (B, d_B) be compact metric spaces. Let $\mathcal{I}(A, B)$ be the set of isometries $f : A \to B$, and topologize it by the compact-open topology. Then $\mathcal{I}(A, B)$ is compact.*

Let $\varphi_k \in G$ be a sequence, and extract a subsequence converging in $\mathcal{I}(M, d_M)$ to some $\varphi \in \mathcal{I}(M, d_M)$. Fix a Riemannian metric on M with corresponding norm $|\cdot|$. We use the boundness of derivatives from theorem 2 to deduce that the sequence $(\varphi_k)_*$ of first derivatives must be bounded. Otherwise we can extract a subsequence to assure that

$$|(\varphi_k)_*(p_k)v_k| \to \infty$$

where $|v_k| = 1$ and $v_k \in T_{p_k}M$. Now extract subsequences to assure that $p_k \to p \in M$ and $v_k \to v \in T_pM$, $|v| = 1$. Let ξ be a smooth vector field extending v and such that $|\xi| = 1$ on a neighborhood of p. By 5.4a of [NW] and arguments from chapter 1, there exists a number $r > 0$ and a smooth family $f_q : r\mathbb{D} \to M$ for $q \in U$, where U is a neighborhood of p, such that $f_q(0) = q$ and $(f_q)_*(0)e = \xi_q$. Now we obtain a sequence of pseudoholomorphic mappings $\varphi_k \circ f_{p_k} : r\mathbb{D} \to M$ with the property that

$$|(\varphi_k \circ f_{p_k})_*(0)e| \to \infty,$$

which contradicts hyperbolicity.

Note also that we might obtain the boundness of the sequence of derivatives by the nonlinear Schwarz lemma of [Gr] using the compactness. Such a Schwarz lemma also holds for higher derivatives [Mu], which yields the desired convergence result.

Now consider the space $J^1_{PH}(M, M)$ of 1-jets of pseudoholomorphic mappings of M to itself, i.e. the set of points (p, q, Φ) with $p, q \in M$, $\Phi \in T^*_pM \otimes T_qM$, satisfying the equality $\Phi \circ J_p = J_q \circ \Phi$. This space carries a canonical almost complex structure [Gau] for which the standard projection $\pi : J^1_{PH}(M, M) \to M \times M$ is a pseudoholomorphic mapping. Let $\sigma : M \times M \to M$ be the projection on the second factor. Denote the composition by $\rho = \sigma \circ \pi$. Let $J^1_{PH}(M, M)_r$ denote the set of points $(p, q, \Phi) \in J^1_{PH}$ such that the element $\Phi \in T^*_pM \otimes T_qM$ which also could be considered as a tangent vector to J^1_{PH}, satisfies $|\Phi| \le r$. Thus we have pseudoholomorphic mappings $\rho_r : J^1_{PH}(M, M)_r \to M$ and $\pi_r : J^1_{PH}(M, M)_r \to M \times M$. The bounded ball $\pi_r^{-1}(p \times q)$ carries the standard complex structure and hence is hyperbolic for every p, $q \in M$ and M is also hyperbolic. Thus by theorem 3 the almost complex manifold $J^1_{PH}(M, M)_r$ is hyperbolic.

Now by the arguments above for some r we have the following commutative diagram of pseudoholomorphic mappings:

$$\begin{array}{ccc} & & J^1_{PH}(M,M)_r \\ & {}^{j^1\psi}\nearrow & \downarrow{}^{\rho_r} \\ M & \xrightarrow{\psi} & M. \end{array}$$

For $p, q \in M$ and an isometry $\psi \in G$ we have

$$d_M(p, q) \ge d_{(J^1_{PH})_r}(j^1\psi(p), j^1\psi(q)) \ge d_M(\psi(p), \psi(q)) = d_M(p, q),$$

and so for every $\psi \in G$, $j^1\psi$ is an isometry with respect to d_M and $d_{(J^1_{PH})_r}$. Using the generalization of the result of van Dantzig and van der Waerden quoted above, we conclude that we may extract a subsequence to assure that φ_k converges uniformly in J^1_{PH}, i.e. the sequence of derivatives converges uniformly. Exactly the same arguments above show that the derivatives of the mappings $j^1\varphi_k$ are bounded and we may consider the hyperbolic almost complex manifold $J^1_{PH}(M, J^1_{PH}(M, M)_r)_r$ to extract a subsequence for which we have uniform convergence in the C^2 sense. Applying the argument one more time, we obtain a Σ-convergent subsequence, thus $\varphi_k \to \varphi \in G$. □

Note that if a vector field $\xi \in \mathcal{G}$ is an element of the Lie algebra of G then it is complete, i.e. globally integrable. So if also $j\xi \in \mathcal{G}$ then the beginning of the proof does not use any other completeness and we obtain the following statement:

Proposition 3. *No almost complex Lie group of positive dimension acts effectively as a pseudoholomorphic transformation group on a hyperbolic almost complex manifold.*

This is an analog of the third statement of theorem 9.1 of [Ko2], see also [Ko1]. The first two are also valid: $\dim \mathrm{Aut}(M^{2n}, J) \le 2n + n^2$ for a hyperbolic almost complex manifold with the equality iff there is an isomorphism with the standard ball $(M^{2n}, J) \simeq (B^{2n}, J_0)$ and the isotropy subgroup $\mathrm{Aut}(M^{2n}, J)_p$ is compact for every $p \in M$ (actually we may modify the corresponding proof that G is a closed subgroup of $\mathcal{I}(M)$ using the ideas from the proof of theorem 4; we may get rid of using the lemma and change $(J^1_{PH})_r$ to $(J^1_{PH})_C$, where $C : M \times M \to \mathbb{R}$ is some bounded on compact sets function and the PH-jets $\Phi \in T^*_p M \otimes T_q M$ satisfy $|\Phi| \le C(p,q)$). It turns out that the statement of the proposition is still true for another wide class of almost complex manifolds. The following is a weak version of the general position property from [Kr].

Definition. An almost complex manifold (M^{2n}, J) is of slightly general position if for a dense set of points $p \in M^{2n}$ the Nijenhuis tensor N_J at every of these points satisfies

$$\mathrm{Ker}[(N_J)_p(\xi, \cdot)] \neq T_p M^{2n}$$

for all vectors $\xi \in T_p M \backslash \{0\}$.

Theorem 5. *Let an almost complex manifold (M^{2n}, J) be of slightly general position. Then the Lie algebra of the automorphism group $G = \mathrm{Aut}(M^{2n}, J)$ has the property $\mathcal{G} \cap (J\mathcal{G}) = 0$, i.e. the tangent space of G contains no complex lines: $\xi \in \mathcal{G} \Rightarrow J\xi \notin \mathcal{G}$.*

Proof. If $\xi \in \mathcal{G} = \mathrm{aut}(M^{2n}, J)$ then $[\xi, J\eta] = J[\xi, \eta]$ for any vector field η. If in addition $J\xi \in \mathcal{G}$ then $[J\xi, J\eta] = J[J\xi, \eta]$. These two equations give $N_J(\xi, \eta) = 0$ at p for every η and hence $\xi_p = 0$. Since the set of such points p is dense we have: $\xi = 0$. □

Chapter 4

Examples

Numerous examples of hyperbolic and nonhyperbolic complex manifolds are considered in the literature. Here we consider examples with non-integrable almost complex structures.

1°. We start with an example of a nonhyperbolic almost complex manifold. The Kobayashi pseudodistance on a homogeneous almost complex manifold is invariant (for definitions and examples of homogeneous almost complex manifolds see [KN, Y]). Consider $S^6 = G_2/SU(3)$ with its well-known nonintegrable almost complex structure J that is defined by means of the octonions (or Cayley numbers). The definition (see [KN] for the full details) goes like this: let $\mathbb{R}^7 = \operatorname{Im} \boldsymbol{Ca}$ be the purely imaginary octonions and $S^6 \subset \mathbb{R}^7$ the unit sphere. On $\mathbb{R}^7$ there is defined a vector product $\times$ and we define $J : T_w S^6 \to T_w S^6$ by $\eta \mapsto \eta \times w$ where $\eta \in \mathbb{R}^7$ and $\eta \perp w$.

Let $\boldsymbol{Ca} = \mathbb{H}^2$ be the usual identification of octonions as pairs of quaternions, and consider $\mathbb{H} = \mathbb{H} \times \{0\} \subset \mathbb{H}^2 = \boldsymbol{Ca}$ as a subspace, $\mathbb{R}^3 = \operatorname{Im} \mathbb{H} \subset \boldsymbol{Ca}$ as the purely imaginary quaternions and S^2 as the unit sphere in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is closed under the vector product on $\mathbb{R}^7$, the 2-sphere $S^2 = S^2 \times \{0\} \subset \boldsymbol{Ca}$ is invariant under J and hence pseudoholomorphic. Thus S^6 is not Brody hyperbolic and hence is not hyperbolic.

2°. An almost complex structure J is called tame if there exists a symplectic form ω such that $\omega(X, JX) > 0$ for any nonzero vector X, see [Gr]. For tame almost complex structures on bounded domains in $\mathbb{R}^{2n}$ we have a sufficient condition for hyperbolicity.

Theorem 6. *Let (D^{2n}, J) be an almost complex domain with J tame. If there exists a ball $D' \Supset D$ such that J is the restriction of some tame almost complex structure on D', then (D^{2n}, J) is hyperbolic.*

Proof. It was shown by Lafontaine [Laf] that if D is a bounded domain in $(\mathbb{R}^{2n}, J)$, where the almost complex structure J is tamed by a symplectic form with bounded coefficients, then (D^{2n}, J) is Brody hyperbolic. We may suppose the coefficients of the taming symplectic form on D' are bounded, shrinking this ball a little otherwise. Now consider a diffeomorphism of D' onto $\mathbb{R}^{2n}$ which maps D onto a bounded domain. The coefficients of the transported symplectic form remain bounded. Thus we have exhibited D as a bounded domain in a tame $(\mathbb{R}^{2n}, J')$, since the diffeomorphism does not disturb the taming condition. Now apply theorem 2 to the closure of D. $\square$

The theorem above is the analog of the known sufficient condition for a domain in $\mathbb{C}^n$ to be hyperbolic; that it be bounded. For this can be formulated as the statement that a bounded domain in $\mathbb{R}^{2n}$ with complex structure is hyperbolic if the complex structure can be extended to the standard complex structure on an ambient ball. A statement of this kind is however false for a general almost complex domain where some extra condition such as tameness or standard integrability is not imposed. For $S^6 \backslash \{p_0\}$ is diffeomorphic to $\mathbb{R}^6$ so we can essentially impose on $\mathbb{R}^6$ the restriction of the almost complex structure on S^6 discussed above. Choosing p_0 not to lie on the pseudoholomorphic 2-sphere that we found, and removing a closed ball around p_0, we obtain a bounded domain in $\mathbb{R}^6$ (actually an open ball), such that the almost complex structure extends to the whole of $\mathbb{R}^6$. Yet this domain is not hyperbolic.

Note that the reasoning above shows that the almost complex structure on S^6 cannot be tamed even on the complement of a small closed ball (globally it is evident since S^6 is not symplectic).

3°. In example 2 we saw that not every bounded domain in an almost complex $\mathbb{R}^{2n}$ is hyperbolic (except maybe $\mathbb{R}^4$). However theorem 6 implies that every point possesses a neighborhood which is hyperbolic. Thus it follows from theorem 3 that for every almost complex manifold M with finite dimensional (C^∞-)parametrized almost complex structure $J^{(\tau)}$, $\tau \in \mathbb{R}^p$, the property of $(M, J^{(\tau)})$ being hyperbolic is open with respect to (the usual) topology on $\mathbb{R}^p$. This can be generalized.

The following statement gives examples of non-integrable hyperbolic almost complex structures on compact manifolds, for example on the product of orientable Riemann surfaces each of genus $g_i \geq 2$ or on a closed Hermitian manifold with holomorphic sectional curvature bounded above by a negative constant.

Theorem 7. *Let (M, J) be a compact hyperbolic almost complex manifold. Then for a small neighborhood $\mathcal{O}$ of J in C^∞ topology every almost complex manifold (M, J') with $J' \in \mathcal{O}$ is hyperbolic.*

Proof. Assume that the statement is false. Then for every sequence of neighborhoods $\mathcal{O}_k$ of J shrinking to J there exists a sequence $J_k \in \mathcal{O}_k$ of almost complex structures such that the manifold (M, J_k) is not hyperbolic. For example one may take a sequence of neighborhoods $\mathcal{O}_k^{(m)}$ to be the ball of radius $\frac{1}{k}$ around J in C^m-norm (or $W^{m,p}$-norm), intersect it with $\mathcal{O}$ and then take the diagonal subsequence. According to theorem 2 for every J_k there exists a nonconstant pseudoholomorphic curve $f_k : \mathbf{C} \to (M, J_k)$. The sequence of almost complex structures J_k tends to J in C^∞ topology. As in the proof of theorem 2 we may apply Brody's reparametrization lemma to obtain uniform boundness of derivatives on compact sets and the condition $|(f_k)_*(0)e| = 1$, and thus due to the compactness theorem B.4.2. of [MS] there exists a subsequence converging uniformly with all derivatives on compact sets to some pseudoholomorphic curve $f : \mathbf{C} \to (M, J)$. Since $|f_*(0)e| = 1$ this curve is nonconstant which contradicts hyperbolicity of (M, J). $\square$

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