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Mathematics Department

Real Plane Algebraic Curves

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Abstract

This master thesis studies several properties of real plane algebraic curves, focusing on the case of even degree. The question of the relative positions of the connected components of real plane algebraic curves originates in Hilbert's sixteenth problem which, despite its prominence, is still open in the case of higher degree curves. The goal of this thesis is an exposition of fundamental contributions to this problem, which have been obtained within the last century. The main aim of the thesis is to clarify these and to make them more accessible.

Chapter 1 gives a brief introduction into the study of real plane algebraic curves. The exposition of this chapter builds on the standard knowledge which are normally obtained in an undergraduate course of algebraic curves, which usually focus only on complex plane algebraic curves. In Chapter 2, several topological properties of real plane curves are developed. The main statements here can be mostly established from Bezout's theorem and its consequences. The main result presented in this chapter is Harnack's inequality and the classification of the curves until degree five. The goal of Chapter 3 is to prove Petrovski's inequalities using Morse theoretic results along with the original arguments which appeared in Petrovski's manuscript. Chapter 4 presents results arising from the complexification of a real plane curve. Finally, Chapter 5 mainly presents results from Smith theory. In particular, this allows to see how Smith's inequality generalizes Harnack's inequality which were presented in Chapter 2 to higher dimensions.

CHAPTER 1

Introduction

Definition 1.1. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Let $n \in \mathbb{N}$. An **affine algebraic set** in the affine space $\mathbb{K}^n$ is a set of the form

$$V_{\mathbb{K}}(f_1, \ldots, f_s) = \{(x_1, \ldots, x_n) \in \mathbb{K}^n : f_1(x_1, \ldots, x_n) = 0, \ldots, f_s(x_1, \ldots, x_n) = 0\}$$

for one $s \in \mathbb{N}$ and $f_1, \ldots, f_s \in \mathbb{K}[X_1, \ldots, X_n]$ being polynomials.

Definition 1.2. In particular, an **affine plane algebraic curve** is the set $V_{\mathbb{K}}(f)$ for a polynomial $f \in \mathbb{K}[X, Y]$.

Here are some of the properties satisfied by the affine plane algebraic curves:

i) $V_{\mathbb{K}}(f_1 \cdot f_2) = V_{\mathbb{K}}(f_1) \cup V_{\mathbb{K}}(f_2)$.

ii) $V_{\mathbb{K}}(f) = \mathbb{K}^2$ if and only if $f \equiv 0$.

iii) If $f|g$ then $V_{\mathbb{K}}(f) \subseteq V_{\mathbb{K}}(g)$.

Definition 1.3. An algebraic set Γ of $\mathbb{K}^n$ is called **irreducible** whenever $\Gamma = \Gamma_1 \cup \Gamma_2$ implies that $\Gamma = \Gamma_1$ or $\Gamma = \Gamma_2$, for Γ_1 and Γ_2 being algebraic sets in $\mathbb{K}^n$.

Definition 1.4. A curve $V_{\mathbb{K}}(f)$ is **irreducible** if it is irreducible in the sense of an algebraic set.

Definition 1.5. We say that the polynomial f is a **minimal polynomial** for the curve $V_{\mathbb{C}}(f) = \Gamma$ if every other polynomial that vanish in the curve is a multiple of f.

We will stress differences between the **real** and the **complex** plane algebraic curves. The first question is how big are these sets depending on the field we work with.

Proposition 1.6. *Every plane affine complex algebraic curve is an infinite set.*

Proof. This is taken from [3]. Let $\Gamma = V_{\mathbb{C}}(f)$. Suppose that $n = \deg_Y(f)$ where $f \in \mathbb{K}[X,Y]$, then

$$f_X(Y) = a_0(X)Y^n + a_1(X)Y^{n-1} + \ldots + a_n(X)$$

with $a_i(X) \in \mathbb{C}[X]$ for $i = 1, \ldots, n$ and $a_0(X) \neq 0$. By the fundamental theorem of algebra we have that the number of distinct roots of $a_0(X)$ is bounded by the degree of $a_0(X)$, then there are infinite values of $x \in \mathbb{C}$ that are not a root of $a_0(X)$.

We pick one of this $x \in \mathbb{C}$ and let $g(Y) = f(x,Y) \in \mathbb{C}[Y]$. Then $n = \deg(g)$ and the polynomial, by the fundamental theorem of algebra has $1 \leq k \leq n$ distinct roots in $\mathbb{C}$, lets call them $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then the points $(x,\alpha_1), (x,\alpha_2) \ldots (x,\alpha_n)$ belong to $\Gamma = V_{\mathbb{C}}(f)$. Since we have infinite possibilities for these $x \in \mathbb{C}$, we are done. $\square$

Theorem 1.7. (Bezout's Theorem). *Let $F, G \in \mathbb{C}[X,Y,Z]$ be two homogeneous polinomials i.e. with all its monomials being of the same degree, that are coprime and with degree d and e respectively. Then, counting the points with multiplicity:*

$$\operatorname{card}(V_{\mathbb{C}}(F) \cap V_{\mathbb{C}}(G)) = d \cdot e.$$

The proof of this result can be found in [3] and in [5]. Let us study now the real case.

Example.

Let $\mathbb{K} = \mathbb{R}$, $f_1 = X^2 + Y^2 - 1$, $f_2 = X^2 + Y^2$ and $f_3 = X^2 + Y^2 + 1$. Then $V_{\mathbb{R}}(f_1)$ is marked in red in figure 1.1 and $V_{\mathbb{R}}(f_2) = \{(0,0)\}$ in black. Since there is no real pair $(x,y) \in \mathbb{R}^2$ that satisfying $X^2 + Y^2 + 1 = 0$, we have that $V_{\mathbb{R}}(f_3) = \emptyset$.

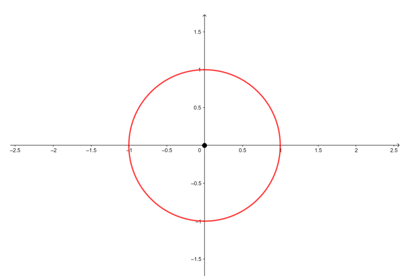


Figure 1.1: $V_{\mathbb{R}}(f_1), V_{\mathbb{R}}(f_2)$ and $V_{\mathbb{R}}(f_3)$

Thus, we have seen in this example that not only there could be only a finite number of solutions, but also that it could be none at all in the real case. This also shows that, unlike in the complex case, the curve $\Gamma \subset \mathbb{R}^2$ may have not an unique minimal polynomial f, since $\emptyset = V_{\mathbb{R}}(X^2 + Y^2 + 1) = V_{\mathbb{R}}(X^2 + Y^2 + 2)$.

Proposition 1.8. *Every algebraic set in $\mathbb{R}^2$ is a real algebraic curve. In particular, every finite set in $\mathbb{R}^2$ is a real algebraic curve.*

Proof. This is taken from [4]. We have $V_{\mathbb{R}}(f_1, \ldots, f_s) = V_{\mathbb{R}}(f_1) \cap \ldots \cap V_{\mathbb{R}}(f_s) = V_{\mathbb{R}}(f)$ for $f = f_1^2 + \cdots + f_s^2$. $\square$

Notice that $f_1 = X^2 + Y^2 - 1$ has both negative and positive values (substituing $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$), while $f_2 = X^2 + Y^2 \geq 0$ and $f_3 = X^2 + Y^2 + 1 > 0$. This leads us to consider the following definition, about the character of a polynomial:

Definition 1.9. The polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is **indefinite** if there exist $a, b \in \mathbb{R}^n$ s.t. $f(a) < 0 < f(b)$. It is called positive or negative **definite** polynomial if $f(a) > 0$ or $f(a) < 0$ for every $a \in \mathbb{R}^n$ respectively, and positive or negative **semidefinite** polynomial if $f(a) \geq 0$ or $f(a) \leq 0$ for every $a \in \mathbb{R}^n$ respectively.

The character of a polynomial does not change under an affine change of coordinates. In the former example we see that only the behaviour of real curves defined by indefinite polynomials is similar to the behaviour of complex curves, so we will start studing them.

Lemma 1.10. *If $f \in \mathbb{R}[X, Y]$ is indefinite, then $V_{\mathbb{R}}(f)$ is infinite.*

Proof. This is taken from [4]. By the definiton of indefinite polynomial and perhaps after a change of coordinates, we can assume that exist $x, y_1, y_2 \in \mathbb{R}$ s.t. $f(x, y_1) < 0 < f(x, y_2)$. If we apply Bolzano's theorem to the polynomial

$$f_x(Y) = a_0(x)Y^n + a_1(x)Y^{n-1} + \ldots + a_n(x)$$

with $a_i(x) \in \mathbb{R}[X]$ for $i = 1, \ldots, n$, we have that there exists $(x, y) \in \mathbb{R}^2$ s.t. $f(x, y) = 0$ i.e. $(x, y) \in V_{\mathbb{R}}(f)$. By the continuity of f there exists one $\delta > 0$ s.t. whenever $x_0 \in (x-\delta, x+\delta)$ we still have the inequalities $f(x_0, y_1) < 0 < f(x_0, y_2)$. Then applying Bolzano's theorem, for every fixed $x_0 \in (x - \delta, x + \delta)$ we get one y_0 such that $(x_0, y_0) \in V_{\mathbb{R}}(f)$, and thus $V_{\mathbb{R}}(f)$ is infinite. $\square$

Proposition 1.11. *Let $f \in \mathbb{R}[X, Y]$ be a polynomial of positive odd degree. Then f is indefinite.*

Proof. This is taken from [4]. Let $f = f_{(0)} + f_{(1)} + \cdots + f_{(2k+1)}$ with $k \in \mathbb{N}$ and $f_{(p)}$ being the homogeneus part of degree p and being $f_{(2k+1)} \neq 0$. The $f_{(i)}$ part (for $i = 0, 1, \ldots, 2k + 1$) looks like the following

$$f_{(i)}(X, Y) = a_0^i X^i + a_1^{i-1} X^{i-1} Y + \ldots + a_i^0 Y^i$$

so it is clear that $f_{(i)}(X, X)$ will be something of the form

$$f_{(i)}(X, X) = \lambda_i X^i$$

with $\lambda_i = a_0^i + a_1^{i-1} + \ldots + a_i^0 \in \mathbb{R}$. Then

$$f(X, X) = \lambda_0 + \lambda_1 X^1 + \ldots + \lambda_{2k+1} X^{2k+1}$$

Suppose that $\lambda_{2k+1} > 0$ (respectively $\lambda_{2k+1} < 0$). Then we can always find one $x \in \mathbb{R}^+$ big enough (respectively $x \in \mathbb{R}^-$ small enough) sattisfying $f(x, x) > 0$, and $f(-x, -x) < 0$. Thus f is indefinite.

$\square$

Lemma 1.12. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be of positive degree d. If there exists $P \in V_{\mathbb{R}}(f)$ s.t. the gradient of f at the point P does not vanish, then f is indefinite.*

Proof. This is taken from [4]. Let $P = (p_1, \dots, p_n)$, it holds that (perhaps after a change of coordinates) $\frac{\partial f}{\partial X_1}(P) \neq 0$. Then the function $f_{X_2,\dots,X_n}(X_1)$ is strictly monotonous on some neighborhood of p_1, and since $f_{p_2,\dots,p_n}(p_1) = 0$, then f changes sign in this neighborhood. □

Remark. *In the future it will be ocasionally used the following notation for the partial derivatives* $f_X = \frac{\partial f}{\partial X}$ *and* $f_Y = \frac{\partial f}{\partial Y}$.

Theorem 1.13. *Let* $f \in \mathbb{R}[X,Y]$ *be an irreducible polynomial. Then* f *is indefinite* $\iff$ $V_{\mathbb{R}}(f)$ *is infinite.*

Proof. This is taken from [4]. $\Rightarrow$ Is just lemma 1.10

$\Leftarrow$ Since f is irreducible, then the intersection of $V_{\mathbb{R}}(f)$ with any other $V_{\mathbb{R}}(g)$ with g not being a multiple of f, is finite, by Bezout's theorem 1.7. In particular, since the degree of f_X and f_Y is smaller than the degree of f, then $V_{\mathbb{R}}(f) \cap V_{\mathbb{R}}(f_X) \cap V_{\mathbb{R}}(f_Y)$ is finite. Since $V_{\mathbb{R}}(f)$ is infinite, there must be a member of this set that does not vanish the gradient, and by lemma 1.12, we have that f is indefinite. □

Theorem 1.14. (Real Study's lemma). *Given* $f, g \in \mathbb{R}[X,Y]$ *both of positive degree s.t.* f *is irreducible in* $\mathbb{R}[X,Y]$*, indefinite and* $V_{\mathbb{R}}(f) \subseteq V_{\mathbb{R}}(g)$*, then* f *divides* g.

Proof. Suppose that f does not divide g. Then f and g have no common components since f is irreducible. Also $V_{\mathbb{R}}(f) \cap V_{\mathbb{R}}(g) = V_{\mathbb{R}}(f)$ is a finite set by Bezout's theorem 1.7, but, since f is indefinite, we have by theorem 1.13 that $V_{\mathbb{R}}(f)$ is infinite, contradiction. □

Notice that the inverse inclusion direction of the last theorem is already given by the property *iii)* sattisfied by all the affine plane real algebraic curves.

Theorem 1.15. (Real Irreducibility Condition). *If* $g \in \mathbb{R}[X,Y]$ *is indefinite and irreducible in* $\mathbb{R}[X,Y]$ *then* $V_{\mathbb{R}}(g)$ *is irreducible.*

Proof. This is taken from [4]. Suppose that $V_{\mathbb{R}}(g) = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 = V_{\mathbb{R}}(h)$ and $\Gamma_2 = V_{\mathbb{R}}(f)$ by proposition 1.8. Then $V_{\mathbb{R}}(g) = V_{\mathbb{R}}(h) \cup V_{\mathbb{R}}(f) = V_{\mathbb{R}}(hf)$ and by the Real Study's lemma we get $g|hf$. Since g is irreducible, then either $g|h$ or $g|f$. Thus, either $V_{\mathbb{R}}(g) \subseteq V_{\mathbb{R}}(h)$ or $V_{\mathbb{R}}(g) \subseteq V_{\mathbb{R}}(f)$ holds, then either $V_{\mathbb{R}}(g) = \Gamma_1$ or $V_{\mathbb{R}}(g) = \Gamma_2$. □

Remark. *We can observe that the real affine curves* $V_{\mathbb{R}}(f)$ *with* f *indefinite have some similar properties to complex affine curves.*

Theorem 1.16. *Let* $f \in \mathbb{R}[X,Y]$ *be an indefinite polynomial of positive degree such that it factorizes into irreducible and indefinite polynomial factors as follows* $f = f_1^{m_1} \cdot \dots \cdot f_n^{m_n}$. *Then, the following are equivalent:*

i) $V_{\mathbb{R}}(f) \subset V_{\mathbb{R}}(g)$

ii) $f_1 \cdot \dots \cdot f_n$ *divides* g.

iii) There exist $m \in \mathbb{N}$, *s.t.* $f|g^m$.

In particular, $V(f) = V(g) \iff f$ *and* g *have the same irreducible and indefinite components.*

Proof. It will be proven in a cyclical way.

$i) \Rightarrow ii)$ Since each f_i is an indefinite polynomial, then $V_{\mathbb{R}}(f_i)$ is an infinite set. Then since $V_{\mathbb{R}}(f) \subset V_{\mathbb{R}}(g)$, it is also true that $V_{\mathbb{R}}(f_i) \subset V_{\mathbb{R}}(g)$, and by the Real Study's Lemma 1.14, then f_i divides to g. Since f_i are coprime between each other, it follows that $f_1 \cdot \ldots \cdot f_n$ divides g.

$ii) \Rightarrow iii)$ Take $m = \max\{m_1, \ldots, m_n\}$.

$iii) \Rightarrow i)$ From $f|g^m$ we obtain that $V_{\mathbb{R}}(f) \subset V_{\mathbb{R}}(g^m)$, and since it holds that $V_{\mathbb{R}}(g^m) = V_{\mathbb{R}}(g)$, then $V_{\mathbb{R}}(f) \subset V_{\mathbb{R}}(g)$.

□

Definition 1.17. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and $n \in \mathbb{N}$. A **projective algebraic set** in the projective space $\mathbb{KP}^n$ is a set of the form

$$V_{\mathbb{K}}(F_1, \ldots, F_s) = \{(x_0 : \ldots : x_n) \in \mathbb{KP}^n : F_1(x_0 : \ldots : x_n) = 0, \ldots, F_s(x_0 : \ldots : x_n) = 0\}$$

for one $s \in \mathbb{N}$ and $F_1, \ldots, F_s \in \mathbb{K}[X_0, \ldots, X_n]$ being homogeneous polynomials.

It is mandatory to work with homogeneus polynomials in the projective case because both $(x : y : z) \equiv (\lambda x : \lambda y : \lambda z)$ are homogeneus coordinates of the same point in $\mathbb{KP}^2$, then we need $F(\lambda x : \lambda y : \lambda z) = 0$ if and only if $F(x : y : z) = 0$ for every $(x : y : z) \in \mathbb{KP}^2$.

Definition 1.18. Let $f \in \mathbb{K}[X, Y]$ be a polynomial of degree d, then its **homogeneization** is defined as the polynomial

$$F(X : Y : Z) = Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right).$$

Definition 1.19. Let $F \in \mathbb{K}[X, Y, Z]$ be a form of degree d and assume that Z (respectively Y and X) does not divide F, then it is defined its **dehomogeneization** with respect to Z (respectively with respect to Y and with respect to X) $f \in \mathbb{K}[X, Y]$ as: $f(X, Y) = F(X, Y, 1)$ (respectively $f(X, Z) = F(X, 1, Z)$ and $f(Y, Z) = F(1, Y, Z)$).

Proposition 1.20. *Let $f \in \mathbb{R}[X, Y]$ be of positive degree d and $F \in \mathbb{R}[X, Y, Z]$ be its homogeneization. Then:*

i) f is indefinite $\iff$ F is indefinite.

ii) f is semi-definite $\iff$ F is semi-definite.

iii) If f is definite then F is semi-definite.

iv) If F is definite then f is definite.

Proof. $i) \Rightarrow$ Recall that $F(X : Y : Z) = Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$. There exists $(x_1, x_2) \in \mathbb{R}^2$ and $(y_1, y_2) \in \mathbb{R}^2$ s.t. $f(x_1, x_2) < 0$ and $f(y_1, y_2) > 0$, and so $F(x_1 : x_2 : 1) = f(x_1, x_2) < 0$ and $F(y_1 : y_2 : 1) = f(y_1, y_2) > 0$.

$\Leftarrow$ There exists $(x_1 : x_2 : x_3)$ s.t. $F(x_1 : x_2 : x_3) > 0$ and $(y_1 : y_2 : y_3)$ s.t. $F(y_1 : y_2 : y_3) < 0$. Suppose that $x_3, y_3 \neq 0$ (the other cases are analogous) then

$$0 < F(x_1 : x_2 : x_3) = F\left(\frac{x_1}{x_3} : \frac{x_2}{x_3} : 1\right) = f\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$$

and

$$0 > F(y_1 : y_2 : y_3) = F\left(\frac{y_1}{y_3} : \frac{y_2}{y_3} : 1\right) = f\left(\frac{y_1}{y_3}, \frac{y_2}{y_3}\right),$$

then f is also indefinite.

ii) Is similar to *i*)

iii) Suppose $f(X, Y) > 0$ for every $(X, Y) \in \mathbb{R}^2$, we can split f in $f(X, Y) = \lambda + g(X, Y)$ with $\lambda > 0$ and g is positive semi-definite. Then

$$F(X : Y : Z) = Z^d \lambda + Z^d g\left(\frac{X}{Z}, \frac{Y}{Z}\right)$$

is semidefinite, since

$$F(X : Y : 0) = 0 \quad \text{and} \quad F(X : Y : 1) = \lambda + g\left(\frac{X}{1}, \frac{Y}{1}\right) > 0.$$

iv) Let $F(X : Y : Z) > 0$ for every $(X : Y : Z) \in \mathbb{RP}^2$, then $0 < F(X : Y : 1) = f(X, Y)$ for every $(X, Y) \in \mathbb{R}^2$ and so f is definite.

□

Example.

$f = X^2 + 1$ is definite but $F = X^2 + Z^2$ is semi-definite non-definite.

Definition 1.21. Lets define V_d as the vector space of homogeneous polynomials of degree d in $\mathbb{K}[X_0, X_1, X_2]$.

Example. *The following generic projective conic is inside* V_2

$$U_{00}X_0^2 + U_{01}X_0X_1 + U_{02}X_0X_2 + U_{11}X_1^2 + U_{12}X_1X_2 + U_{22}X_2^2 \in V_2.$$

Proposition 1.22. *The vector space* V_d *has dimension* $\binom{d+2}{2}$.

Proof. This is based on [5]. A basis of V_d is a set with all the possible monomials of degree d with the variables X_0, X_1, X_2, then we need to see that there exists as many monomials of this type as $\binom{d+2}{2}$.

Any monomial of degree d can be represented as the product $X_{i_1}X_{i_2}\cdots X_{i_d}$, where $i_1, i_2, \ldots, i_d \in \{0, 1, 2\}$. The order of this product is not important of course, then there is as many monomials as combinations with repetition of the 3 elements $\{0, 1, 2\}$ being taken in groups of d, and this number is precisely

$$\binom{d+3-1}{d} = \binom{d+2}{d} = \binom{d+2}{2}.$$

□

Remark. *Since V_d has dimension $\binom{d+2}{2}$, if we dehomogeneizate a homogeneous polynomial $F \in \mathbb{K}[X_0, X_1, X_2]$ into $f \in \mathbb{K}[X, Y]$ we lose a monomial (X_0^d concretely), then f has really $\binom{d+2}{2} - 1 = \frac{d(d+3)}{2}$ monomials.*

Example. *A projective conic looks like the following*

$$F(X_0 : X_1 : X_2) = a_{00}X_0^2 + a_{01}X_0X_1 + a_{02}X_0X_2 + a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2$$

and its dehomogeneization with respect to X_0 is

$$f(X, Y) = a_{00} + a_{01}X + a_{02}Y + a_{11}X^2 + a_{12}XY + a_{22}Y^2$$

and as we noticed in the remark, we have $\frac{2(2+3)}{2} = 5$ monomials.

We can also observe that since the conic can be represented by a polynomial up to a multiplication by a constant, then we can differentiate 2 cases, when $a_{00} \neq 0$ and $a_{00} = 0$:

$$\left(\frac{a_{01}}{a_{00}}\right) X + \left(\frac{a_{02}}{a_{00}}\right) Y + \left(\frac{a_{11}}{a_{00}}\right) X^2 + \left(\frac{a_{12}}{a_{00}}\right) XY + \left(\frac{a_{22}}{a_{00}}\right) Y^2 = -1$$
$$a_{01}X + a_{02}Y + a_{11}X^2 + a_{12}XY + a_{22}Y^2 = 0$$

After this observation, we see that taking precisely $\frac{2(2+3)}{2} = 5$ points in $\mathbb{C}^2$, say (x_i, y_i) for $i = 1, 2, 3, 4, 5$ then we have, substituing in the conic equation (lets stick now to the equation with -1 on the rigth hand side, the other case is analogous), 5 equations for 5 variables ($\left(\frac{a_{01}}{a_{00}}\right) = a$, $\left(\frac{a_{02}}{a_{00}}\right) = b$, $\left(\frac{a_{11}}{a_{00}}\right) = c$, $\left(\frac{a_{12}}{a_{00}}\right) = d$, $\left(\frac{a_{22}}{a_{00}}\right) = e$):

$$ax_i + by_i + cx_i^2 + dx_iy_i + ey_i^2 = -1$$

that will give us the equation of a conic that pass by this (x_i, y_i) points:

$$aX + bY + cX^2 + dXY + eY^2 = -1.$$

Corollary 1.23. *For $\binom{d+2}{2} - 1 = \frac{d(d+3)}{2}$ points in $\mathbb{C}^2$, we can always find a curve of degree d passing through them.*

Proof. We just need to use last proposition 1.22 and generalize the procedure described for a conic above, instead of a system of $\frac{2(2+3)}{2} = 5$ equations for $\frac{2(2+3)}{2} = 5$, variables we will get $\frac{d(d+3)}{2}$ equations for $\frac{d(d+3)}{2}$ variables, with the variables being the coefficients of a curve of degree d that pass by $\frac{d(d+3)}{2}$ chosen points. □

Theorem 1.24. *Let $\Gamma = V_{\mathbb{C}}(F)$ for an irreducible homogeneous polynomial $F \in \mathbb{C}[X, Y, Z]$ of degree d, then the maximum number of singular points that Γ can have is $\frac{(d-1)(d-2)}{2}$.*

Proof. This is taken from [3] and [5]. For $d = 1$, suppose there would be one singular point in the line Γ, then any other line intersecting Γ in that point will intersect with multiplicity bigger than one, contradicting Bezout's theorem.

For $d = 2$, suppose there would be a singular point. Then choosing any line going from this point to any other of the conic, the line will be intersecting the conic with multiplicity bigger than 2, contradicting again Bezout's theorem.

Now that we have it for $d = 1, 2$, we will suppose $d \geq 3$. Suppose that Γ has $1 + \frac{(d-1)(d-2)}{2} = \frac{d^2-3d+4}{2}$ singular points. Taking $\frac{(d-2)(d+1)}{2}$ points, by the last corollary 1.23, we can always find a curve C of degree $d-2$ passing by the $\frac{d^2-3d+4}{2}$ singular points and another

$$\frac{(d-2)(d+1)}{2} - \frac{d^2-3d+4}{2} = d-3$$

points of Γ, that by hypothesis they are regular points.

Since the multiplicity of intersection between C and Γ in the singular points should be at least 2, counting the multiplicities, we have at least

$$2 \cdot \left(\frac{d^2-3d+4}{2}\right) + d - 3 = d^2 - 2d + 1 = d(d-2) + 1$$

intersection points (counted with multiplicity) between C and Γ, since Γ was irreducible, this contradicts Bezout's theorem. □

Theorem 1.25. (Sign change criterion). *Let $F \in \mathbb{R}[X_0, X_1, X_2]$ be a homogeneous polynomial of positive degree without multiple irreducible factors. We have that F is indefinite $\iff$ there exists a regular point $P \in V_{\mathbb{R}}(F)$.*

Proof. Based on [4]. $\Rightarrow$ If F is indefinite, we have that at least one irreducible factor of F, say F_i, is indefinite. Thus $V_{\mathbb{R}}(F_i)$ is an infinite set by lemma 1.10 and proposition 1.20. Since the set of singular points of a homogeneous polynomial is finite by theorem 1.24, then there must be a regular point inside $V_{\mathbb{R}}(F_i)$, and thus inside $V_{\mathbb{R}}(F)$.

$\Leftarrow$ Suppose now that there exists a regular point $P \in V_{\mathbb{R}}(F)$. This point has at least one coordinate that is not zero, suppose that is the first one, thus $P = (1 : a : b)$. Taking now $f(X,Y) = F(1,X,Y)$, since $f_X(a,b)$ and $f_Y(a,b)$ are not both null, then there exists a regular point ((a,b) concretely) in the set $V_{\mathbb{R}}(f)$. Using lemma 1.12 we obtain that f is indefinite, and by proposition 1.20 we have that F is indefinite also. □

Lemma 1.26. *Suppose that $f \in \mathbb{R}[X_1, \dots, X_n]$ is irreducible in $\mathbb{R}[X_1, \dots, X_n]$. Then f is reducible in $\mathbb{C}[X_1, \dots, X_n]$ $\iff$ either f or $-f$ is a sum of two squares in $\mathbb{R}[X_1, \dots, X_n]$.*

Proof. This is taken from [4]. $\Leftarrow$ If $f = r_1^2 + r_2^2$ with $r_1, r_2 \in \mathbb{R}[X_1, \dots, X_n]$ and $i = \sqrt{-1}$ then $f = (r_1 + ir_2)(r_1 - ir_2)$ is reducible in $\mathbb{C}[X_1, \dots, X_n]$.

$\Rightarrow$ Suppose that f factors non-trivially $f = (r_1 + ir_2)(s_1 + is_2)$ with $r_1, r_2, s_1, s_2 \in \mathbb{R}[X_1, \dots, X_n]$. Taking complex conjugates we obtain $f \cdot \overline{f} = (r_1^2 + r_2^2)(s_1^2 + s_2^2)$. Since f was irreducible in $\mathbb{R}[X_1, \dots, X_n]$ then either $(r_1^2 + r_2^2)$ or $(s_1^2 + s_2^2)$ divides f, then $f = a^2(r_1^2 + r_2^2)$ for some $a \in \mathbb{C} - \{0\}$ (analogously for $(s_1^2 + s_2^2)$) and so f or $-f$ is a sum of two squares. □

Definition 1.27. Define the following function for $d \in \mathbb{N}$:

$$\alpha(d) := \max\left[\frac{d^2}{4}, \frac{(d-1)(d-2)}{2}\right].$$

For $d \leq 5$, $\alpha(d) = \frac{d^2}{4}$ and for $d \geq 6$, $\frac{(d-1)(d-2)}{2}$. Taking derivatives at d in both $\frac{d}{4}$ and $\frac{(d-1)(d-2)}{2d}$ it is obtained that $\frac{\alpha(d)}{d}$ is monotically increasing in $\mathbb{N}$. Thus

$$\frac{\alpha(d_j)}{d_j} \leq \frac{\alpha(d_1 + d_2)}{d_1 + d_2},$$

for $j = 1, 2$

$$\alpha(d_1) + \alpha(d_2) \leq \left(\frac{d_1}{d_1 + d_2} + \frac{d_2}{d_1 + d_2} \right) \alpha(d_1 + d_2) = \alpha(d_1 + d_2).$$

Theorem 1.28. *Let $F \in \mathbb{R}[X, Y, Z]$ be a semi-definite irreducible form of positive degree d. Then $|V_{\mathbb{R}}(F)| \leq \alpha(d)$.*

Proof. This is taken from [4]. First, assume that F is reducible in $\mathbb{C}[X, Y, Z]$. Then by lemma 1.26 we have that $F = R_1^2 + R_2^2$ for some forms $R_1, R_2 \in \mathbb{R}[X, Y, Z]$ both necessarily of degree $\frac{d}{2}$. Moreover, R_1 and R_2 must be coprime, since F is irreducible in in $\mathbb{R}[X, Y, Z]$. Now by Bezout's theorem, the set $V_{\mathbb{R}}(F) = V_{\mathbb{R}}(R_1) \cap V_{\mathbb{R}}(R_2)$ contains at most $\left(\frac{d}{2}\right)^2$ points counted with multiplicity.

Suppose now that F is irreducible in $\mathbb{C}[X, Y, Z]$. By 1.25, since F is a semi-definite polynomial, each point of $V_{\mathbb{R}}(F)$ is singular. Therefore by theorem 1.24, $|V_{\mathbb{R}}(F)| \leq \frac{(d-1)(d-2)}{2}$. □

Proposition 1.29. *For any semi-definite homogeneous polynomial $F \in \mathbb{R}[X, Y, Z]$ of degree $d \geq 2$, then F is divisible by the square of some indefinite (homogeneous) polynomial $\iff$ $V_{\mathbb{R}}(F)$ is infinite.*

Proof. The first inclusion is taken from [4]. $\Rightarrow$ Since F is divisible by the square of a indefinite form G, then $V_{\mathbb{R}}(G) \subset V_{\mathbb{R}}(F)$. By lemma 1.10 and proposition 1.20, $V_{\mathbb{R}}(F)$ is infinite.

$\Leftarrow$ Let $V_{\mathbb{R}}(F)$ be infinite, by theorem 1.28, F must be reducible. Suppose that $F = F_1 \cdot \ldots \cdot F_n$, then by theorem 1.13 and proposition 1.20 at least we must have one F_i indefinite irreducible polynomial, for $1 \leq i \leq n$. Since F is semi-definite, ignoring all the F_m semi-definite polynomials of $F = F_1 \cdot \ldots \cdot F_n$, there must be one F_j irreducible indefinite polynomial for $j \neq i$, s.t. it have the same negative and positive interval that F_i, making $F_i \cdot F_j \geq 0$. In oher words, they must change the sign on the same points, i.e. $V(F_i) = V(F_j)$. Therefore by the Real Study's lemma $F_i | F_j$ and $F_j | F_i$, since they are irreducible, up to a constant multiplication, they must be the same polynomial $F_i = F_j$. Thus F is divisible by the square of the indefinite homogeneous polynomial F_i. □

The following result is taken from [4]:

Corollary 1.30. *If $F \in \mathbb{R}[X, Y, Z]$ is a homogeneous polynomial of positive degree without multiple irreducible factors (in particular, if F is irreducible), then:*

1. *$V_{\mathbb{R}}(F)$ is infinite $\iff$ F is indefinite.*
2. *$V_{\mathbb{R}}(F)$ is finite non-empty $\iff$ F is semi-definite nondefinite.*
3. *$V_{\mathbb{R}}(F)$ is empty $\iff$ F is definite.*

CHAPTER 2

Topology on real algebraic curves

First it is important to notice that there are topological differences between the ambient spaces, $\mathbb{R}^2$ and $\mathbb{C}^2$, affine and projective. First $\mathbb{R}^2$ and $\mathbb{RP}^2$ are connected $\mathbb{R}$-topological manifolds of dimension 2. Moreover, $\mathbb{R}^2$ is orientable while $\mathbb{RP}^2$ is both non-orientable and compact. $\mathbb{C}^2$ and $\mathbb{CP}^2$ are both connected, orientable $\mathbb{R}$-topological manifolds of dimension 4, and $\mathbb{CP}^2$ it is also compact.

As an $\mathbb{R}$-topological manifold, every non-singular curve in $\mathbb{C}^2$ has always dimension 2. In $\mathbb{CP}^2$ the projective algebraic curves are connected since if there were 2 different connected components in a complex projective curve, by Bezout's theorem they have to intersect with multiplicity equal to the multiplication of their degrees. In $\mathbb{RP}^2$ things are very different.

Let $F \in \mathbb{R}[X, Y, Z]$ be a homogeneous polynomial such that its decomposition into coprime irreducible factors is

$$F = cF_1^{k_1} \cdot \ldots \cdot F_s^{k_s} \cdot F_{s+1}^{k_{s+1}} \cdot \ldots \cdot F_t^{k_t} \cdot F_{t+1}^{k_{t+1}} \cdot \ldots \cdot F_r^{k_r}$$

with $c \in \mathbb{R} - \{0\}$, $0 \leq s \leq t \leq r$ and $s, t, r, k_j \in \mathbb{N}$ for all $1 \leq j \leq r$. Suppose that F_j is indefinite, for all $j \leq s$, F_j is semi-definite nondefinite for all j with $s+1 \leq j \leq t$ and F_j is definite, for all j with $t+1 \leq j \leq r$. Then using last corollary 1.30 it is obtained that

$$V_{\mathbb{R}}(F) = V_{\mathbb{R}}(F_1) \cup \ldots \cup V_{\mathbb{R}}(F_s) \cup V_{\mathbb{R}}(F_{s+1}) \cup \ldots \cup V_{\mathbb{R}}(F_t)$$

where $V_{\mathbb{R}}(F_{s+1}) \cup \ldots \cup V_{\mathbb{R}}(F_t)$ is a finite set. (This is taken from [4]).

Definition 2.1. Using the same notation as above, we define the irreducible components of the curve $V_{\mathbb{R}}(F)$. There are two different types:

1. Each $V_{\mathbb{R}}(F_j)$, with $j < s$.
2. Each point in the set $\displaystyle\bigcup_{j=s+1}^{t} V_{\mathbb{R}}(F_j) \setminus \bigcup_{j=1}^{s} V_{\mathbb{R}}(F_j)$.

Note. *Each irreducible component of type 2 is an isolated point in $V_{\mathbb{R}}(F)$, and each irreducible component of type 1 is an algebraic irreducible curve.*

Then, it can be concluded that every non-singular real curve Γ is of type 1 and has dimension 1 as an $\mathbb{R}$-topological manifold. A real singular curve Γ that has only irreducible components of type 2 is of dimension 0. In a singular curve Γ that have both types of irreducible components then the subset of regular points of the curve is a manifold of dimension 1, not necessarily dense in the curve Γ. In this last case there are different dimensions at a local level of the real plane curve, lets see an example.

Example. *The cubic $V_{\mathbb{R}}(X^2 + Y^2 - Y^3)$ has dimension 1, but the local dimension at the isolated point $(0, 0)$ is 0. Nevertheless, the local dimension of this curve is 1 at any other point of this cubic. This can be seen in figure 2.1.*

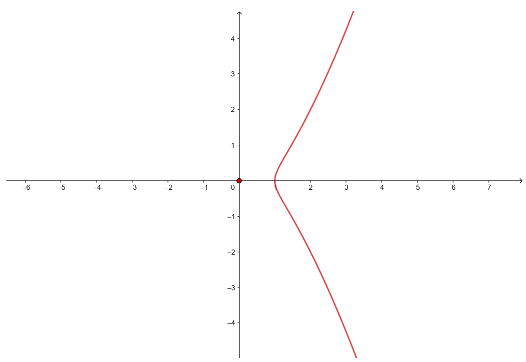


Figure 2.1: $V_{\mathbb{R}}(X^2 + Y^2 - Y^3)$

Definition 2.2. For every point x in a real plane curve Γ and every sufficient small open disk U with center at x, the set $U \setminus \{x\}$ is homeomorphic to the union of an even number of open segments, which are denoted as **half branches of Γ at the point** x.

Complex projective curves are unbounded, compact sets and furthermore, by Bezout's theorem, they are connected. Nevertheless as we have seen in the last example, in the projective case, real curves are compact sets but they can fail to be connected. It will be seen in the following results that there is a bound on the connected components that a real non-singular curve may have, and that this bound it is always the best possible.

Definition 2.3. An **oval** of a real projective curve Γ is a connected component whose complement is not connected, equivalently, an oval is a connected component bounding a disk in $\mathbb{RP}^2$.

Definition 2.4. A **pseudoline** is a curve that it is topologically equivalent to a line.

Lemma 2.5. *Let $\Gamma = V_{\mathbb{R}}(F)$ be a non-singular algebraic curve of degree d in $\mathbb{RP}^2$ with $F \in \mathbb{R}[X, Y, Z]$, then:*

i) If d is even, every connected component of Γ is an oval.

i) If d is odd, one of the connected components of Γ is a pseudoline, while all the others are ovals.

Proof. This is taken from [1]. The real projective plane can be thought as the sphere $\mathbb{S}^2$ where all the antipodal points are being identified:

$$\pi : \mathbb{S}^2 \longrightarrow \mathbb{RP}^2 = {}^{\mathbb{S}^2}\!\big/_{x \sim (-x)}.$$

Since Γ is a non-singular curve, it can not have self-intersections i.e. Γ can only have at maximum 1 pseudoline because the pseudolines in $\mathbb{RP}^2$ are seen in $\mathbb{S}^2$ topologically equivalent to maximum circles, and is clear that every two maximum circles intersects exactly at two (antipodal) points in the sphere.

Now let γ be a path in $\mathbb{S}^2$ that crosses transversaly to $\pi^{-1}(\Gamma)$ and joins two antipodal points which are not in $\pi^{-1}(\Gamma)$. If C_1 and C_2 are connected components of Γ, then γ intersects $\pi^{-1}(C_2)$ in an even number of points if C_2 is an oval and γ intersects $\pi^{-1}(C_1)$ in an odd number of points if C_1 is a pseudoline. This can be visualized in figure 2.2:

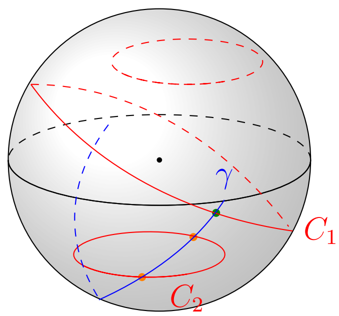


Figure 2.2: $\mathbb{S}^2$ with all its antipodals points being identified.

Counting the sign changes of F along the path γ, we see that Γ have a connected component that is a pseudoline iff F takes opposite signs near antipodal points i.e. if d is odd. If F takes tha same sign near antipodal points, then d is even, and there can not be any pseudoline, so all the connected components must be ovals. □

Now we can easily think in the restrictions for the number of connected components for the case of degree 2. Suppose that there would be more than 1 connected component in this case, due to the last lemma 2.5, this is the same as saying more than 1 oval. Suppose there would be 2 ovals, then joining one interior point of each oval, we would have a line that it is intersecting with multiplicity 4 to a curve of degree 2, which by Bezout's theorem, this is impossible. Of course for more than 2 ovals this is also impossible. Then for degree $d = 2$ we have 1 connected component as maximum. This can be seen in figure 2.3:

Figure 2.3: Counting geometric intersections between two ovals and a line.

Definition 2.6. Let Γ be a curve of degree $d \in \mathbb{N}$, then we define its **genus** as

$$g(d) = \frac{(d-1)(d-2)}{2}.$$

Theorem 2.7. (Harnack's theorem).

The number of connected components c of a non-singular projective curve of degree $d \geq 2$ is bounded:

$$c \leq \frac{(d-1)(d-2)}{2} + 1 = g(d) + 1.$$

Proof. This is taken from [1]. By the last commentary on the previous page, it is assumed that $d > 2$ since for $d = 2$ it has already been shown that $c \leq g(2) + 1 = 1$. It suffices also to consider irreducible curves, since for a reducible curve $f = g \cdot h$, with $\deg(g) = d_1$, $\deg(h) = d_2$, the following inequality holds

$$g(d_1) + 1 + g(d_2) + 1 \leq g(d_1 + d_2) + 1$$

whenever $d_1 > 1$ or $d_2 > 1$.

Suppose now that Γ is an non-singular irreducible curve of degree d with more than $g(d) + 1$ components. Then by the last lemma 2.5, Γ contains $p = g(d) + 1$ ovals $\Omega_1, \Omega_2, \ldots, \Omega_p$ and at least one other connected component (which it will be an oval if d is even or it will be a pseudoline if d is odd).

Lets pick $\frac{1}{2}d(d-1) - 1$ points on Γ. Since $\frac{1}{2}d(d-1) - 1 \geq g(d) + 1 = p$ for $d \geq 2$, it can be chosen a point on each of the ovals $\Omega_1, \Omega_2, \ldots, \Omega_p$ and the rest of the points on the other connected component left that has Γ. By corollary 1.23, since

$$\frac{(d-2)(d-2+3)}{2} = \frac{(d+1)(d-2)}{2} = \frac{1}{2}d(d-1) - 1,$$

there is a curve of degree $d-2$, say Δ, passing through all this $\frac{1}{2}d(d-1) - 1$ points. The curves Γ and Δ have no common irreducible component since Γ is irreducible of degree d and Δ is of degree $d-2$. By Bezout's theorem 1.7, the number of intersection points of Γ and Δ, counted with multiplicity can not be greater than $d(d-2)$.

If Δ intersects transversely (i.e. with multiplicity one, see [5]) an oval Ω_i in one point, then Δ necessarily intersects Ω_i in another point. Hence the number of intersection points between Δ and Γ counted with multiplicity is at least

$$\frac{d(d-1)}{2} - 1 + g(d) + 1 = \frac{d^2 - d + d^2 - 3d + 2}{2} = (d-1)^2$$

since there are at least $p = g(d) + 1$ ovals. Is clear that $(d-1)^2 > d(d-2)$ for $d > 2$, then by Bezout's theorem it is concluded that Γ can not have more than $g(d) + 1$ connected components. □

Note. *Notice that since by hypothesis the curve is non-singular, it can not have any self intersections between the different connected components.*

Then the number

$$\frac{(d-1)(d-2)}{2} + 1 = g(d) + 1$$

bounds the number of connected components that a non-singular real curve can have, now it is natural to ask if this is the best bound possible. The answer will be in fact affirmative and it is seen in the following proposition.

Proposition 2.8. (Harnack's construction). *The bound $g(d)+1$ for a non-singular real curve of degree $d \geq 2$ is the best possible i.e. for each d, there exist a non-singular curve of degree d in $\mathbb{RP}^2$ which reaches this bound.*

Proof. This is taken from [1]. Choose a line L in $\mathbb{RP}^2$. Starting with $d = 2$, it will be constructed by induction a non-singular curve Γ_d of degree d in $\mathbb{RP}^2$ with exactly $g(d)+1$ connected components and which, furthermore, sattisfies the following properties:

i) Γ_d has a connected component C_d intersecting L in d distinct points.

ii) There is an orientation of L and an orientation of C_d, such that these d distinct points are arranged in the same order on L as on C_d. Taking this order into account, we denote this points as $a_1, \ldots, a_d$.

iii) For each $i = 1, 2, \ldots, d-1$ the union of the interval $[a_i, a_{i+1}]$ on L and the segment of C_d bounded by a_i and a_{i+1} form an oval in $\mathbb{RP}^2$ (which is not smooth).

For $d = 2$ it is clear that any conic has exactly one connected component, and it can always be chosen Γ_2 to be a non-singular conic intersecting L in 2 distinct points, then *i)* holds. We can easily see in the following drawing that it is satisfied also *ii)*. The oval that is described in *iii)* is drawed in blue in figure 2.4:

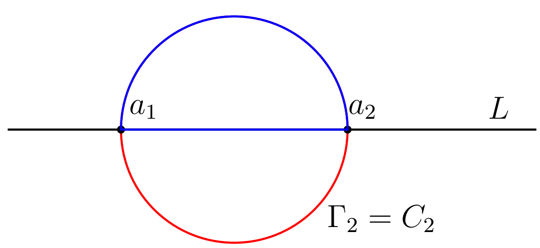


Figure 2.4: Drawing of the oval (in blue) sattisfying property *iii)* for $d = 2$.

Since for $d = 2$ holds, using induction, suppose that we have already constructed Γ_d with $g(d)+1$ components and satisfying properties *i), ii), iii)*. Then the curve Γ_{d+1} is constructed as follows, we choose distinct points $b_1, b_2, \ldots, b_{d+1}$ on L s.t. $a_1, \ldots, a_d, b_1 \ldots, b_{d+1}$ are ordered according to the orientation of L. Now choose lines $L_1, L_2, \ldots, L_{d+1}$ distinct from L and passing through $b_1, b_2, \ldots, b_{d+1}$ respectively like it is drawed in figure 2.5.

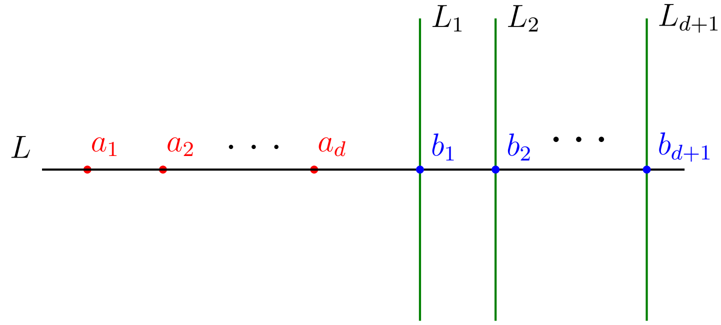


Figure 2.5: Drawing lines through points $b_1, \ldots, b_{d+1}$ different than $a_1, \ldots, a_d$.

Then the curve Γ_{d+1} is constructed as a small perturbation of the union $\Gamma_d \cup L$. Identifying the curves with their respective homogeneous polynomials, we define

$$\Gamma_{d+1} = L \cdot \Gamma_d + \epsilon \cdot \prod_{i=1}^{d+1} L_i$$

where $\epsilon \in \mathbb{R}$ is small enough, and its sign is the opposite of the sign of $L \cdot \Gamma_d$ at the interior of the ovals that lie between a_i and a_{i+1}, this way the effect of the perturbation is to shrink these ovals towards their interior. Hence this perturbation of $C_d \cup L$ has $d-1$ ovals and an additional connected component C_{d+1} that intersects L in $b_1, b_2, \ldots, b_{d+1}$ and so we can conclude that Γ_{d+1} sattisfies $i), ii)$ and $iii)$ for $b_1, b_2, \ldots, b_{d+1}$. We can visualize this for the case $d = 5$ at figure 2.6 obtained from https://www.math.tamu.edu :

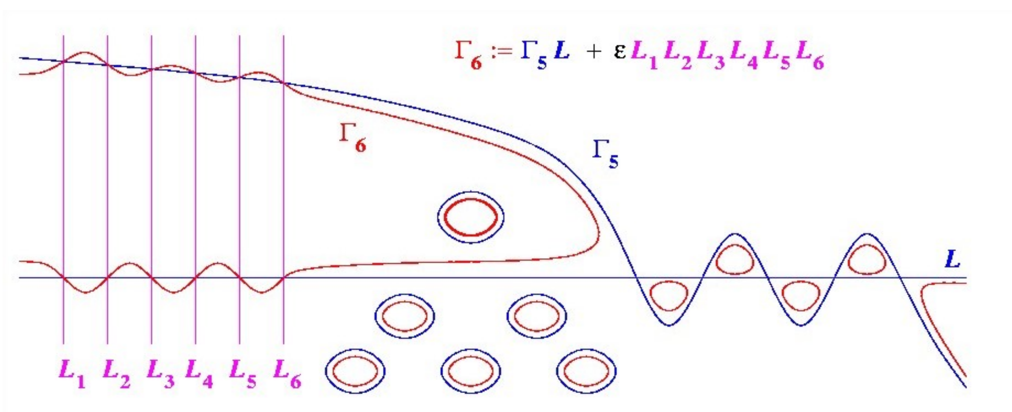


Figure 2.6: Harnack's construction from the case of degree 5 to degree 6.

Since we are taking ϵ small enough, the perturbation of the ovals of Γ_d which do not intersect with L does not erase them, therefore the curve Γ_{d+1} has precisely

$$g(d) + 1 + (d-1) = \frac{(d-1)(d-2)}{2} + d = \frac{d(d-1)}{2} + 1 = g(d+1) + 1$$

connected components. $\square$

Definition 2.9. *(M-curve).*

A real plane curve that attains the maximum number of connected components is called an **M-curve**.

Example. *An elliptic curve with 2 components, such as* $Y^2 = X^3 - X$ *is an M-curve. Nevertheless,* $Y^2 = X^3 - X + 1$ *has a single component, thus is not an M-curve. See 2.7:*

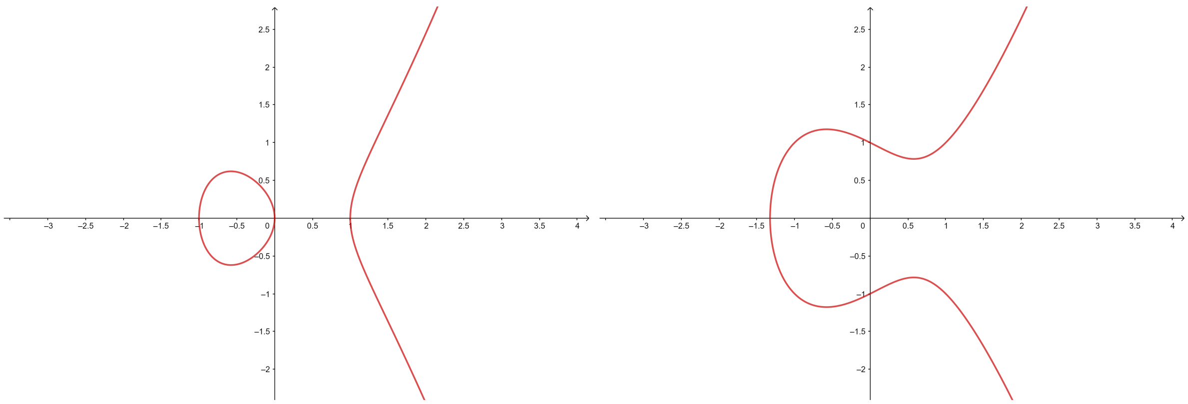


Figure 2.7: $V_{\mathbb{R}}(Y^2 - X^3 + X)$ (M-curve) and $V_{\mathbb{R}}(Y^2 - X^3 + X - 1)$ (not an M-curve).

Definition 2.10. *(Simple Harnack's curve).*

A non-singular curve $\Gamma \subset \mathbb{RP}^2$ of degree d is called a non-singular **simple Harnack's curve** if it is an M-curve and:

1. All ovals of Γ are disjoint (this means, they have disjoint interiors) if d is odd.

2. One oval of Γ contains $\frac{(k-1)(k-2)}{2}$ ovals in its interior while all other ovals are disjoint if $d = 2k$ is even.

Example. *Let's construct a simple Harnack's curve of degree* 6*. One oval is going to contain*

$$\frac{(3-1)\cdot(3-2)}{2} = 1$$

oval inside and since it is also an M-curve we have that it has $\frac{(6-1)\cdot(6-2)}{2} + 1 = 11$ *total components. Then it will look like the following schematic drawing, and staring at the image 2.6 from the proof of Harnack's construction for the case* $d = 5$ *to* $d = 6$*, we see that they are both the same curve as it is seen in the drawing below in figure 2.8.*

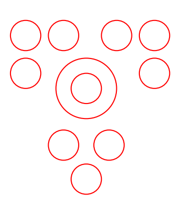


Figure 2.8: Schematic drawing of a simple Harnack's curve of degree 6.

Definition 2.11. The **depth** of an oval Ω in a non-singular real plane curve Γ is the number of ovals of Γ containing Ω in their interiors. One oval is **nested** in another if it is contained in its interior, we say that a **nest** has **complexity** $k+1$ if there is at least one oval of depth k.

Corollary 2.12. (Hilbert's theorem) *For a non-singular real plane curve of degree even ($d = 2k$), the complexity of a nest is at most k.*

Proof. This proof is based on [16].

Suppose it is true that there is a nest of complexity $k+1$, then by the figure 2.9

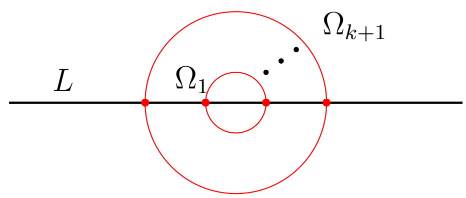


Figure 2.9: Hilbert's theorem proof schematic drawing idea.

is seen that the line L is intersecting the curve with multiplicity at least $2 \cdot (k+1)$, which is impossible by Bezout's theorem since this multiplicity of intersection can be at most $2k$. □

Notice that Harnack's restriction 2.7 is only restricting how many connected components can a non-singular real plane curve have as maximum, but it is not telling about how this ovals or pseudolines are distributed, up to homeomorphism. Bezout's theorem can help on finding out how are distributed in $\mathbb{RP}^2$ in the first cases of lower degree.

Definition 2.13. The notation $\langle P \rangle$ denotes a **pseudoline** and $\langle n \rangle$ denotes the number of **ovals** of a curve. Also to denote m ovals inside one oval, it is denoted as $\langle 1\langle m \rangle\rangle$. To express the union of this lasts different cases it is used $\amalg$.

Theorem 2.14. (Classification of non-singular real plane curves of degree $d \leq 4$).

Up to homeomorphism, the classification of non-singular real plane curves of degree $d \leq 4$ is:

i) $d = 1$ $\langle P \rangle$.

ii) $d = 2$ $\langle 0 \rangle$, $\langle 1 \rangle$.

iii) $d = 3$ $\langle P \rangle$, $\langle P \amalg 1 \rangle$.

iv) $d = 4$ $\langle 0 \rangle$, $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 1\langle 1 \rangle\rangle$, $\langle 3 \rangle$, $\langle 4 \rangle$.

Proof. Based in key ideas from [8] and [15] using original own examples.

i) By lemma 2.5, a non-singular curve of degree $d = 1$ has one pseudoline and since it can not have more connected components by Harnack's theorem 2.7, hence its clasification is $\langle P \rangle$, as example take the non-singular curve defined by $f(X, Y) = Y$.

ii) By Harnack's theorem 2.7 it can only have one connected component and if it is non-empty, by lemma 2.5 it is an oval, like in $V_{\mathbb{R}}(f)$ for $f(X,Y) = X^2+Y^2-1$, then one possibility is $\langle 1 \rangle$. Nevertheless there are non-singular empty real conics like the one defined by $f(X,Y) = X^2+Y^2+1$ then the other possibility is $\langle 0 \rangle$.

iii) By Harnack's theorem 2.7 a curve of degree 3 has at most 2 connected components. By lemma 2.5 it has at least a pseudoline and the rest is ovals. Then clearly the only two possibilities are like the ones that we study at example 2, respectively $f(X,Y) = Y^2 - X^3 + X - 1$ for $\langle P \rangle$ and $f(X,Y) = Y^2 - X^3 + X$ for $\langle P \amalg 1 \rangle$ which are both non-singular curves.

iv) By Harnack's theorem 2.7 a curve of degree 4 has at most 4 connected components. By lemma 2.5 it is only formed by ovals. It can have a nest but of complexity at most 2 by corollary 2.12. It is possible to have at most 1 nest of this type without any exterior ovals, i.e. $\langle 1\langle 1 \rangle\rangle$, for example $f(X,Y) = (X^2+Y^2-2)(2X^2+Y^2-1)-1$. Suppose that there would be an exterior oval to the nest, if it is traced a line between the interior of the nest and the interior of a exterior oval, the line intersects with multiplicity 6, wich by Bezout's theorem 1.7 it is impossible.

Other possibility is $\langle 0 \rangle$, thinking on the non-singular cuartic $f(X,Y) = X^4+Y^4+1$. The case $\langle 1 \rangle$ is represented by $f(X,Y) = X^4+Y^4-1$ for example. To contruct 2 ovals that are not one nested inside the other it can be used the non-singular curve defined by the polynomial $f(X,Y) = (2X^2+Y^2-1)(X^2+Y^2-1)+0.11$. Hence it it can be added to the classification $\langle 2 \rangle$.

Case $\langle 3 \rangle$ can be constructed out of a perturbation of the following picture which represents in the left of figure 2.10 an union between an ellipse and another ellipse, represented by the equation $f(X,Y) = (X^2+2Y^2-1)\ (2X^2+Y^2-2)$. It is represented also where f is positive and negative at figure 2.10:

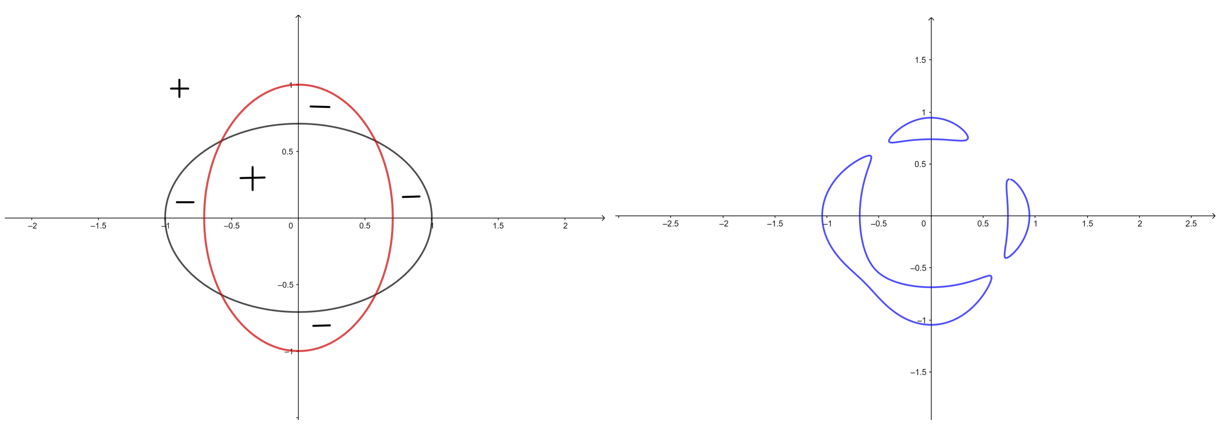


Figure 2.10: Small perturbation of two ellipses.

The perturbation drawed in blue is made adding a perturbation in two extra odd degree coefficients, in this case:

$$\overline{f}(X,Y) = (X^2+2Y^2-1)(2X^2+Y^2-2)+0.1X^3+0.1Y^3.$$

Looking at the partial derivatives and the homogeneization, it is seen that it is a non-singular complex curve.

To construct the case $\langle 4 \rangle$ it will be needed to make again a slight perturbation in the following union of 2 ellipses, $f(X,Y) = (X^2+2Y^2-1)(2X^2+Y^2-1) = 0$, where it is represented also where f is positive and negative in figure 2.11:

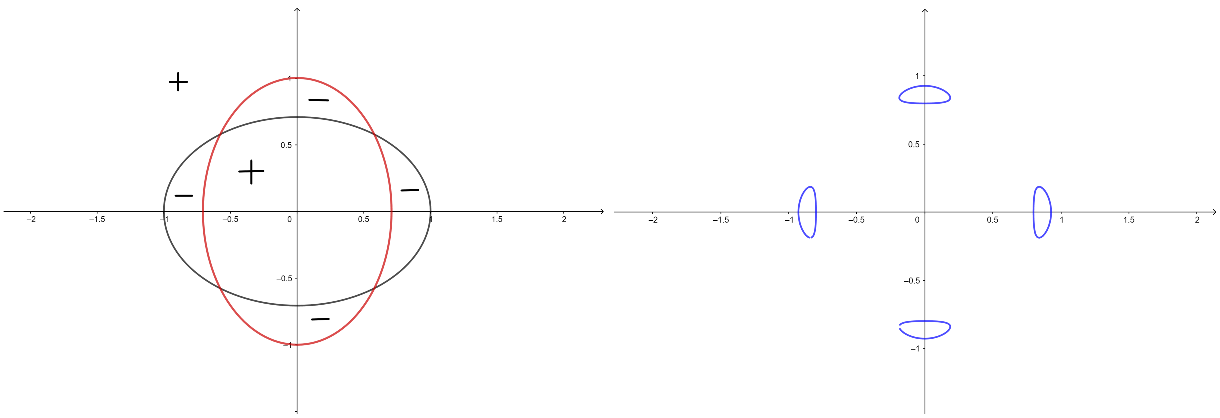


Figure 2.11: Small perturbation of two transversal ellipses.

Looking at the left picture above, the optimal perturbation to obtain 4 ovals will be shrinking slightly the negative parts towards their interior, and this can be done adding one small $\epsilon > 0$ to f, i.e. $(X^2+2Y^2-1)(2X^2+Y^2-1) = -\epsilon$. In particular for $\epsilon = 0.1$ we obtain the right above picture in blue. Notice that the partial derivatives of X and Y are the same of f and $f+\epsilon$, then the singular points that there are in the intersection between the 2 ellipses in the left picture are no longer in the right picture with the perturbation done. Doing the partial derivatives in the homogeneous polynomial of $f(X,Y) = (X^2+2Y^2-1)(2X^2+Y^2-1)+0.1$ it is seen that $V_{\mathbb{C}}(f)$ is a non-singular curve of degree 4 with 4 ovals.

$\square$

The degree 5 case relies on the clasification of degree 4, adding a pseudoline and perturbing the curve. Two extra cases must be considered also.

Theorem 2.15. (Classification of non-singular real plane curves of degree 5).
Up to homeomorphism the classification of non-singular real curves of degree 5 is

$$\langle P \rangle, \quad \langle P \amalg 1 \rangle, \quad \langle P \amalg 2 \rangle, \quad \langle P \amalg 1\langle 1 \rangle \rangle, \quad \langle P \amalg 3 \rangle, \quad \langle P \amalg 4 \rangle, \quad \langle P \amalg 5 \rangle, \quad \langle P \amalg 6 \rangle.$$

Proof. The idea of the proof can be found in [8] and [15].

By theorem 2.7 the number of connected components is bounded by $\frac{(5-1)(5-2)}{2}+1 = 7$. By lemma 2.5, the possible extra cases for degree 5 that can not be obtained from the last theorem 2.14 adding a pseudoline to the curve and perturbing it (perturbing the coefficients) in order to create a non-singular curve are $\langle P \amalg 5 \rangle$ and $\langle P \amalg 6 \rangle$.

For $\langle P \amalg 5\rangle$ can be used:

$$f(X,Y) = (3X^2+Y^2-1)\left(X^2+2\left(Y-\frac{1}{2}\right)^2-\frac{1}{3}\right)\left(Y-\frac{1}{2}-\frac{1}{\sqrt{6}}\right)$$

that it is represented with its sign in the following left picture below of figure 2.12.

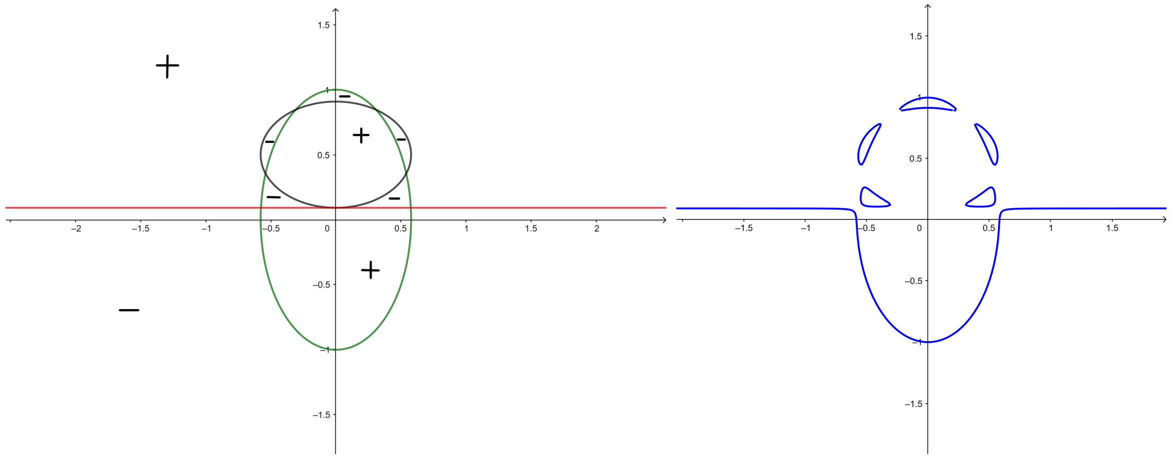


Figure 2.12: Small perturbation of two ellipses and a tangent line to one of the ellipses.

The optimal way to proceed is to shrink towards itself the negative parts, using a small $\epsilon > 0$ that makes the curve non-singular, it is obtained a real curve similar to the one at the right in figure 2.12 for $f + \epsilon = 0$. Thus it is obtained a non-singular quintic with 5 ovals. To construct $\langle P \amalg 6\rangle$ it can be obtained by the Harnack's construction 2.8 or it may be similarly used an optimal perturbation, this time in figure 2.13:

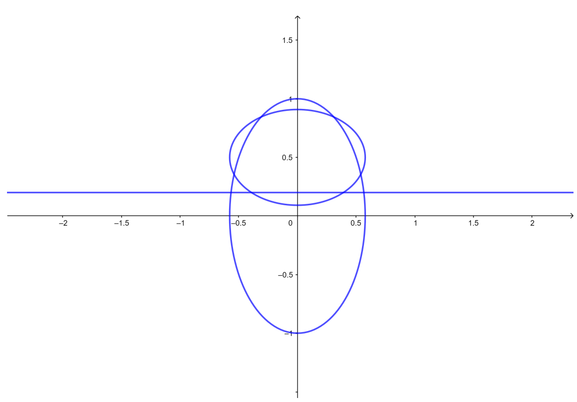


Figure 2.13: Base case to construct $\langle P \amalg 6\rangle$ by making a perturbation.

$\square$

CHAPTER 3

Petrovski inequality for the even degree

In the last chapter it was seen the complete classification of non-singular real plane curves until degree 5. Now for degree 6 it will be seek to find the M-curves using the same methods from the last chapter. By Harnack's theorem 2.7 the M-curve of degree 6 has exactly

$$\frac{(6-1)(6-2)}{2}+1=11$$

connected components. Since it is of even degree, by lemma 2.5 it will be only ovals, so 11 ovals. Using Hilbert's theorem 2.12 it is impossible to have a nest of complexity greater than 3. Nevertheless, it is also impossible to have a nest of 3 ovals since joining the center of one nest of 3 ovals with another oval by a line, this line will be intersecting with intersection at least 8 to a curve of degree 6, which it is impossible by Bezout's theorem 1.7. Also an M-curve of degree 6 can not have 2 different nest of complexity 2 each, since a line between its centers will be intersecting with multiplicity 8, also impossible by the Bezout's theorem 1.7. Then as far as we easy get with Bezout's theorem is to 2 possible types of curves, one with 11 ovals all outside each other and the other with 11 ovals with one nest of complexity 2. In fact the second type will be seen that it is in fact the only possibility by the Petrovski inequality.

This far, it seems that Bezout's theorem is not enough and that it will be needed another topological results not mainly based on this theorem to go further.

Definition 3.1. Let $f \in \mathbb{R}[X,Y]$, then the set M_t is defined as

$$M_t=\{(x,y)\in\mathbb{R}^2|f(x,y)\leq t\}$$

and the **level curve** set L_t it is defined as $L_t=\{(x,y)\in\mathbb{R}^2|f(x,y)=t\}$.

Definition 3.2. *(Critical values).*

Let $f \in \mathbb{R}[X,Y]$, it is said that a real number c_0 is a **critical value** of f if this polynomial takes the value c_0 at some critical point (x_0,y_0), i.e.

$$\frac{\partial f}{\partial X}(x_0,y_0)=\frac{\partial f}{\partial Y}(x_0,y_0)=0, \qquad f(x_0,y_0)=c_0.$$

Definition 3.3. It is said that a vector field $X = X_1 \frac{\partial}{\partial X} + X_2 \frac{\partial}{\partial Y}$ in $\mathbb{R}^2$ (see appendix A) is a **gradient like vector field** for a polynomial $f \in \mathbb{R}[X, Y]$ if in every non-critical point (x_0, y_0) sattisfies

$$X \cdot f(x_0, y_0) = X_1(x_0, y_0) \frac{\partial f}{\partial X}(x_0, y_0) + X_2(x_0, y_0) \frac{\partial f}{\partial Y}(x_0, y_0) > 0.$$

Lemma 3.4. (Topological lemma).

If f has no critical value inside the interval $[a, b]$, then $M_{[a,b]}$ is homeomorphic to the product

$$f^{-1}(a) \times [0, 1].$$

Proof. This is taken from [9]. Let X be a gradient like vector field for f. Since $X(f) > 0$ at every non-critical point of f, it can be defined a new vector field Y on $\mathbb{R}^2$ outside the critical points:

$$Y = \frac{1}{X \cdot f} X.$$

Since by hypothesis $M_{[a,b]} = \{(x, y) \in \mathbb{R}^2 | a \leq f(x, y) \leq b\}$ contains no critical points of f, this vector field Y is well-defined. Consider the integral curve $c_p(t)$ of Y which starts at point p, i.e. $c_p(0) = p$. To see how fast climbs this integral curve from $f^{-1}(a)$ to $f^{-1}(b)$, it is the same as see how fast it goes inside the height function i.e. f:

$$\frac{d}{dt} f(c_p(t)) = \frac{dc_p}{dt}(t) \cdot f = Y(c_p(t)) \cdot f = \frac{1}{X \cdot f} X \cdot f = 1.$$

Thus the integral curve climbs up from $f^{-1}(a)$ with a constant speed 1 with respect to the height that is defined by f as shown in figure 3.1. Since it starts at the level $f = a$ at $t = 0$ it will reach the level b at the time $t = b - a$.

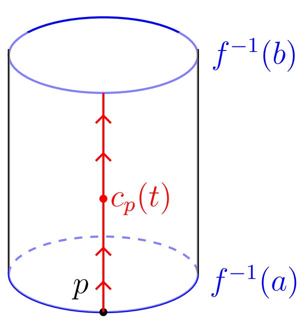


Figure 3.1: The integral curve $c_p(t)$ climbing through the height of f.

There is no harm on consider $[a, b]$ instead of $[0, b - a]$ redefining the integral curve c as $c(t) := c(t + a)$. Then it can be defined the following continuous map

$$\begin{aligned} h : f^{-1}(a) \times [a, b] &\longrightarrow M_{[a,b]} \\ (p, t) &\longrightarrow c_p(t). \end{aligned}$$

Then it can be considered the following continuous map

$$h^{-1}: M_{[a,b]} \longrightarrow f^{-1}(a) \times [a,b]$$
$$p \longrightarrow (c_p(-f(p)), f(p)).$$

Thus $M_{[a,b]}$ is homeomorphic to $f^{-1}(a) \times [a,b]$. Now since there can be constructed also the following homeomorphism

$$H: [a,b] \longrightarrow [0,1]$$
$$t \longrightarrow \frac{t-a}{b-a}$$

it is concluded that

$$M_{[a,b]} \cong f^{-1}(a) \times [0,1].$$

□

Theorem 3.5. *Let $b < c$ be real numbers such that $f \in \mathbb{R}[X,Y]$ has no critical values in the interval $[b,c]$. Then M_b and M_c are homeomorphic. In particular, L_b and L_c are homeomorphic.*

Proof. This is taken from [9]. Denote again by $M_{[b,c]}$ the portion between L_b and L_c

$$M_{[b,c]} = \{(x,y) \in \mathbb{R}^2 | b \leq f(x,y) \leq c\}.$$

By the previous definitions it is clear that $M_b \cup M_{[b,c]} = M_c$. Since the number of singular points of $f(x,y)$ is bounded, there can not be a convergent sequence of critical values $\{b_n\}_{n=1}^{\infty}$ that converges to b, which by hypothesis it is not a critical value. Thus it can be assumed that f has no critical points in $M_{[b-\epsilon,c]}$ for a small enough $\epsilon > 0$. By the topological lemma above 3.4, $M_{[b-\epsilon,c]}$ is homeomorphic to the product $L_{b-\epsilon} \times [0,1]$. Also note that $M_{[b-\epsilon,b]} \subset M_{[b-\epsilon,c]}$ and f has no critical points either in $M_{[b-\epsilon,b]}$, thus $M_{[b-\epsilon,b]} \cong L_{b-\epsilon} \times [0,1]$ too by lemma 3.4. Then we have a homeomorphism

$$h: M_{[b-\epsilon,b]} \longrightarrow M_{[b-\epsilon,c]}$$

where the restriction to the level curve $L_{b-\epsilon}$ must be the identity map. In addition, it can be defined the following homeorphism using the identity map

$$H = \mathrm{id} \cup h : M_{b-\epsilon} \cup M_{[b-\epsilon,b]} \longrightarrow M_{b-\epsilon} \cup M_{[b-\epsilon,c]}$$

Since $M_{b-\epsilon} \cup M_{[b-\epsilon,b]} = M_b$ and $M_{b-\epsilon} \cup M_{[b-\epsilon,c]} = M_c$, it can be obtained as consecuence the homeomorphism

$$H: M_b \longrightarrow M_c.$$

□

Definition 3.6. *(Positive and negative ovals).*

An oval is said to be **positive** if when crossing the oval from the inside, the function decreases and to be **negative** if when crossing the oval from the inside, the function increases.

Remark. *Now, with this definition of positive and negative ovals it will be seeked to find a restriction for the difference between the number of negative ovals and the number of positive ovals that a curve of even degree has.*

Lemma 3.7. (Lemma 1).

Let $V_{\mathbb{R}}(F)$ be a non-singular real plane curve of even order n with $F \in \mathbb{K}[X_0, X_1, X_2]$ being a homogeneous polynomial. Then it can be continuously slightly perturbed any of its coefficients without changing its topological structure.

Proof. By Morse's lemma 3.5 the coefficient a_{00} can be slightly variated in

$$F(X_0, X_1, X_2) = a_{n,0}X_2^n + a_{n-1,1}X_2^{n-1}X_1 + \ldots + a_{0,0}X_0^n,$$

without changing the topological structure of the curve.

It can be made the following change of coordinates on the axis

$$H : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$$
$$[X_0 : X_1 : X_2] \longrightarrow [X_2 : X_1 : X_0]$$

and F will still being a non-singular real curve under this kind of change of coordinates (changing the axis), since H is an homeomorphism it can be deduced that the coefficient $a_{n,0}$ can be also continuously variated without changing the topological structure of the curve. Analogously it can be deduced that the coefficient $a_{0,n}$ can be also slightly perturbed without changing the topological structure of the curve.

Every small perturbation in a coefficient $a_{i,j}$ with $i + j \leq n$ due to adding to $a_{i,j}$ a small enough (in absolute value) epsilon ϵ can be continuously controlled

$$|\epsilon||X_2^i X_1^j X_0^{n-i-j}| < |\epsilon_0|X_0^n + |\epsilon_1|X_1^n + |\epsilon_2|X_2^n \tag{3.1}$$

by $|\epsilon_0|, |\epsilon_1|$ and $|\epsilon_2|$ perturbations (respectively of $a_{0,0}$, $a_{0,n}$ and $a_{n,0}$) that do not change the topological structure of the curve. This inequality is clear whenever a point takes $X_i = 0$ for any $i = 0, 1, 2$. Now consider any point P s.t. $X_i \neq 0$ for every $i = 0, 1, 2$. Let $P = [1 : a : b]$. It can always be considered that both $|a|$ and $|b|$ are bigger than 1 since we can always divide all the coordinates by the smallest number (in absolute value) on the coordinates of P and then do a change of coordinates if it would be necessary. Taking $\mu = \max\{|a|, |b|\}$ and supposing that $\mu = |a|$ (analogously for $\mu = |b|$) is clear for $|\epsilon| < |\epsilon_1|$ ($|\epsilon| < |\epsilon_2|$ analogously) that

$$|\epsilon||b^i a^j| < |\epsilon_1|\mu^n = |\epsilon_1|a^n < |\epsilon_0| + |\epsilon_1|a^n + |\epsilon_2|b^n.$$

Thus taking $|\epsilon| < \min\{|\epsilon_0|, |\epsilon_1|, |\epsilon_2|\}$ then the inequality (3.1) holds always true.

Since the perturbation of $a_{i,j}$ is continuously made and it is controlled by another continuous perturbation that does not change the topology of the curve, it is deduced that this perturbation of $a_{i,j}$ does not change either the topology of the curve. □

Lemma 3.8. (Lemma 2).

Let $V_{\mathbb{R}}(f)$ be a non-singular real plane curve of even order n with $f \in \mathbb{R}[X, Y]$. Then it can be slightly perturbed its equation without changing its order or its topological structure s.t. the equations

$$\frac{\partial f}{\partial X} = 0, \qquad \frac{\partial f}{\partial Y} = 0, \tag{3.2}$$

will have $(n-1)^2$ different finite solutions, real or imaginary.

Proof. Based on [6]. By the last lemma 3.7, the coefficients of $f \in \mathbb{K}[X,Y]$ can be continuously slightly variated such that the topology of the curve does not change, i.e. such that no singularities will arise in the new curve. When the equations from (3.2) have an infinite number of solutions is because they both have common components. This situation can be solved by a continuous perturbation of the equation, lets see this. Suppose that $p_1, p_2, h \in \mathbb{R}[X,Y]$ are non-constant polynomials s.t.

$$\frac{\partial f}{\partial X} = p_1 \cdot h, \qquad \frac{\partial f}{\partial Y} = p_2 \cdot h. \tag{3.3}$$

Then it will happen that the partials have a common component $h(X,Y)$ and due to this, an infinite number of solutions. Since the set of irreducible polynomials in $\mathbb{R}[X,Y]$ is dense, whenever is made any continuous perturbation in f we are able to find one irreducible polynomial in any of the partials, making (3.3) impossible to be, and thus having a finite number of solutions.

If (3.2) had not all different solutions like it is being seeked, it is always possible to get to a different polynomial with a small perturbation s.t. all its solutions in (3.2) are distinct. Thus either real or imaginary, all the roots in (3.2) can be considered to be all different, and by Bezout's theorem they are going to be exactly $(n-1)^2$ solutions in the projective plane.

The finite points set in the projective plane (points that are not in the line at infinity) is dense, thus it is always possible to make all this $(n-1)^2$ solutions of (3.2) to be finite ones due to a slight continuously perturbation in the coefficients of f. □

Definition 3.9. *(Plus and minus points).*

Let (x_0, y_0) be a real finite critical point of the function $f(X,Y)$. If at this point occurs that

$$D(x_0,y_0) = \det \begin{pmatrix} \frac{\partial^2 f}{\partial X^2}(x_0,y_0) & \frac{\partial^2 f}{\partial X \partial Y}(x_0,y_0) \\ \frac{\partial^2 f}{\partial Y \partial X}(x_0,y_0) & \frac{\partial^2 f}{\partial Y^2}(x_0,y_0) \end{pmatrix} > 0 \tag{3.4}$$

then it is called a **plus point**. If it happens that $D(x_0,y_0) < 0$, then it is called a **minus point**.

Definition 3.10. *(Degenerate points).*

Let $f \in \mathbb{R}[X,Y]$ be a real polynomial and (x_0,y_0) be a critical point of f, then (x_0,y_0) is called a **degenerate point** if $D(x_0,y_0) = 0$. Analogously, (x_0,y_0) is called a **non-degenerate point** if $D(x_0,y_0) \neq 0$.

Remark. *By lemma 3.7, it is possible to make a slight perturbation in the coefficients of the polynomial $f \in \mathbb{R}[X,Y]$ without changing its topology. It is clear that this perturbation will affect also to the second derivatives, so it can be always assumed that $f \in \mathbb{R}[X,Y]$ is a polynomial without any degenerate points.*

Note. By theorem 3.5 when there is no critical value between $[b,c]$, the topology of L_b and L_c remains equal. Next lemma 3 will explain for a real polynomial of even degree how the topology changes when one of this critical values are being actually crossed. By the reasoning from the last remark it will be consider only plus and minus points, i.e. non-degenerate points.

Lemma 3.11. (Lemma 3).

Let (x_0, y_0) *be a real finite critical point of the polynomial* $f \in \mathbb{R}[X, Y]$ *of even degree. Suppose that* (x_0, y_0) *is a plus point, then when* C *varies from* $f(x_0, y_0) + \epsilon$ *to* $f(x_0, y_0) - \epsilon$ *for* $\epsilon > 0$*, then the difference* $p - m$ *between the number* p *of positive ovals of the curve* $f(X, Y) = C$ *and the number* m *of negative ovals of this curve increases by 1.*

If (x_0, y_0) *is a minus point, then when* C *varies from* $f(x_0, y_0) + \epsilon$ *to* $f(x_0, y_0) - \epsilon$ *the difference* $p - m$ *of the curve* $f(X, Y) = C$ *decreases by 1, except once when decreases 2.*

Proof. Taken from [6]. If $D(x_0, y_0) > 0$, then at the critical point (x_0, y_0) there is a maximum or a minimum of $f(X, Y)$.

Suppose that (x_0, y_0) is a maximum. Moving C from $f(x_0, y_0) + \epsilon$ to $f(x_0, y_0) - \epsilon$ it is really lowering the level curve from $f(x_0, y_0) + \epsilon$ to $f(x_0, y_0) - \epsilon$. Since (x_0, y_0) is a maximum, in a small enough neighborhood around (x_0, y_0) there is nothing above $f(x_0, y_0)$, i.e. $L_{f(x_0,y_0)+\epsilon} = \emptyset$. Nevertheless when the level decreases down the maximun suddenly an oval around (x_0, y_0) appears as is seen in figure 3.2, and since when crossing this oval towards the outside, the function f decreases, thus a positive oval appears and so $p - m$ increases by 1.

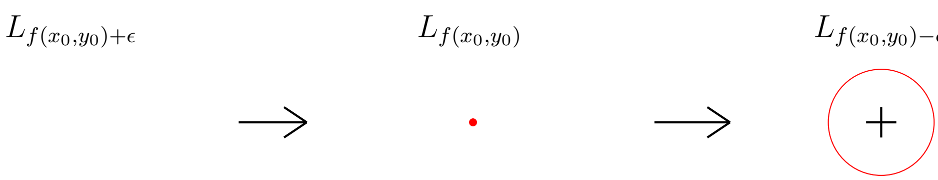


Figure 3.2: Lowering the level curve around a maximum of f.

Suppose now that (x_0, y_0) is a minimum. Then lowering the level curve from $f(x_0, y_0)+\epsilon$ to $f(x_0, y_0) - \epsilon$ it is making one oval dissaper around (x_0, y_0). Since it was a minimum this oval is negative because going towards the outside is actually increasing the function f. Then again $p - m$ increases by 1.

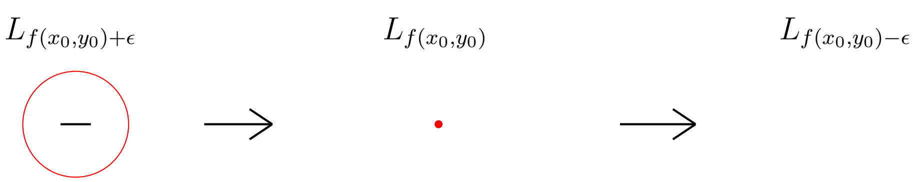


Figure 3.3: Lowering the level curve around a minimun of f.

If $D(x_0, y_0) < 0$ then there is a saddle point of the function $f(X, Y)$ in (x_0, y_0). Then there are 2 possibilities, that an oval touches another oval or that an oval touches itself.

Suppose first that a positive oval touches another positive oval in $L_{f(x_0,y_0)}$. Then lowering the level curve from $f(x_0, y_0) + \epsilon$ to $f(x_0, y_0) - \epsilon$ passes from 2 separate positive ovals to 1 positive oval created out of the union of the last 2, since the positive domain expands as shown at figure 3.4.

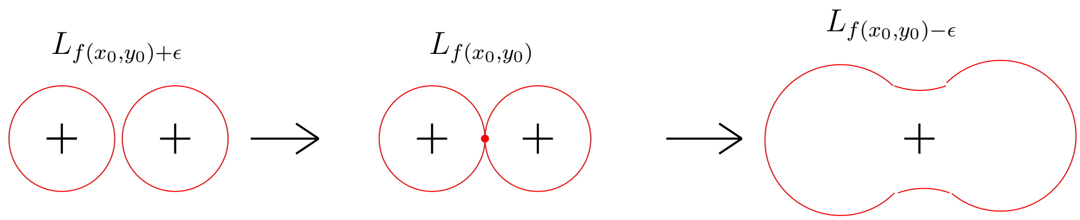


Figure 3.4: Lowering the level curve around two maximums of f.

So a positive oval disapears and $p-m$ decreases 1. When a negative oval touches itself, the drawing is the same but inverting the sign of the oval and inverting the direction of the arrows, since lowering the level makes the negative domain smaller, it shrinks the negative ovals towards themselves. Thus it is created 2 negative ovals from a negative oval, in conclusion, $p-m$ decreases 1 too.

Now consider the case when a positive oval touches a negative oval. Since all the most outer ovals are positive (up to a change of sign in f) it can only happen when a negative oval is inside a positive oval. Lowering the level curve from $f(x_0, y_0)+\epsilon$ to $f(x_0, y_0)-\epsilon$ the positive oval will get bigger since there would be more domain around it that is positive.

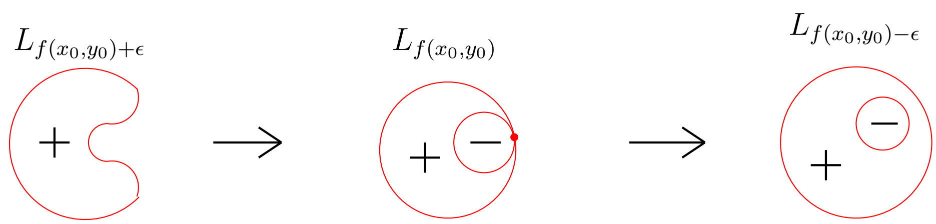


Figure 3.5: Lowering the level curve around a saddle point f.

Then it will be creating a new negative oval as in figure 3.5, so $p-m$ decreases 1. If we interchange the papers of the signs in the drawing, the arrows will invert the direction, and so a positive oval dissapears, thus $p-m$ decreases 1. A positive oval touching itself is analogous.

Now as last case, an infinite oval touches itself, and since it is an infinite oval, this case can only happens once. This occurs when a critical point looks locally as the saddle point of a horse seat (see figure 3.6, taken from https://www.wikidata.org/wiki/Q357268). Lowering the level curve, the positive oval (red one) becomes a negative oval (in black). So a positive oval becomes a negative oval, and $p-m$ decreases 2.

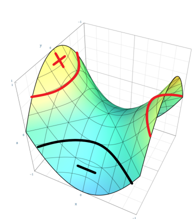


Figure 3.6: Saddle point.

□

Lemma 3.12. (Lemma 4). *Suppose that the real curve of even degree $f(X,Y) = C$ meets the line at infinity in k different points (k does not depend on C since it relies on the homogeneization) and s.t. all critical points of $f(X,Y)$ are finite and different; then the difference between the number of minus points $(-p)$ and the number of plus points $(+p)$ of the function f, is $\#(-p) - \#(+p) = k - 1$.*

Proof. Taken from [6]. Denote by C_M the maximal critical value that reaches $f(X,Y)$ and C_m the minimal critical value. Consider the projective plane seen as the upper hemisphere $\mathbb{S}^2_+$ which its equator is the line of infinity. Let M_C be the set of all points of the hemisphere which corresponds to the points (x,y) of the plane for which $f(x,y) > C$. If $k \geq 1$ then when $C > C_M$, thes set M_C consist of k open regions $G_1, G_2, \ldots, G_k$ between two different points where the curve $f(X,Y) = C$ meets the real line of infinity.

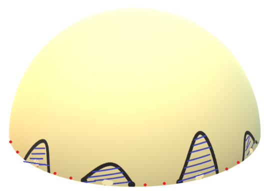


Figure 3.7: Projective real plane semisphere with regions.

When $C < C_m$ the set M_C consists already in all the semisphere region which has all been formed around the point where f takes the minimal critical value C_m, because f is a continuous function. Hence when C varies from $C > C_M$ to $C < C_m$ all the k regions $G_1, \ldots, G_k$ come together. By lemma 3.11, every 2 different regions G_i come together when C passes through a minus point, thus this gives precisely $k-1$ minus points.

Notice that every new positive oval will arise in a plus point and it will disappear in a minus point, and every new negative oval will arise in a minus point and disappear in a plus point by lemma 3.11. Perhaps, while C varies from $C_M + \epsilon$ to $C_m - \epsilon$ new ovals may arise, but also subsequently disappear in order to arrive to the final whole semisphere region, thus all this does not affect to the difference $\#(-p) - \#(+p)$ which remains $k-1$.

If $k = 0$, then either the curve $f(X,Y) = C$ when $C > C_M$ consist of a single negative oval (if there would be more than one, they would touch at some point creating another higher critical value, and C_M is the highest) and when $C < C_m$ the curve is imaginary containing no real points (as example take $f(X,Y) = X^2 + Y^2$), or conversely, it can happen also the opposite, for $C > C_M$ the curve is imaginary and when $C < C_m$ the curve consist of a single positive oval (as example take $f(X,Y) = -X^2 - Y^2$). It can not happen that both when $C < C_m$ and $C > C_M$ the curve is imaginary or that when $C < C_m$ the curve $f(X,Y) = C$ consist of a single positive oval and when $C > C_M$ the curve $f(X,Y) = C$ consist of a single negative oval, because f has only one height for each $(x,y) \in \mathbb{R}^2$.

Since in the first case the negative oval is disappearing and in the second case the positive oval is appearing, in both cases there must exist a plus point where the negative oval vanish and the positive oval arises respectively. Thus the difference in both cases is -1 because all other ovals appearing and disappearing from $C_M + \epsilon$ to $C_m - \epsilon$, give rise to equal number of plus and minus points, so the difference remains -1.

$\square$

Lemma 3.13. (Lemma 5).

Let $f_1(X,Y)$ and $f_2(X,Y)$ be 2 real polynomials of degree n vanishing simultaneously at exactly n^2 different finite points, and $f(X,Y) \in \mathbb{R}[X,Y]$ a polynomial of degree $l < n$ in $\mathbb{R}[X,Y]$ not identically zero. Then $f(X,Y)$ cannot vanish in more than $n \cdot l$ points at which $f_1(X,Y)$ and $f_2(X,Y)$ vanish simultaneously.

Proof. Taken from [6]. The polynomials f_1 and f_2 have no common factor because they vanish simultaneosly at a discrete set of points. Denote by $M_1(X,Y)$ the greatest common factor of f_1 and f, and by $M_2(X,Y)$ the greatest common factor of f_2 and f and let the degrees of $M_1(X,Y)$ and $M_2(X,Y)$ respectively n_1 and n_2 (notice this number could be both 0). Since f_1 and f_2 have no common factor, it can be written:

$$
\begin{aligned}
f_1(X,Y) &= M_1(X,Y) \cdot \overline{M}_1(X,Y) \\
f_2(X,Y) &= M_2(X,Y) \cdot \overline{M}_2(X,Y) \\
f(X,Y) &= M_1(X,Y) \cdot M_2(X,Y) \cdot M(X,Y)
\end{aligned}
$$

where $\overline{M}_1(X,Y), \overline{M}_2(X,Y), M(X,Y) \in \mathbb{R}[X,Y]$. The functions f_1,f_2 and f can only vanish at a finite number of points which are the solutions of at least one of the following systems

$$
\begin{aligned}
M_1(X,Y) &= 0, \quad f_2(X,Y) = 0 \\
M_2(X,Y) &= 0, \quad f_1(X,Y) = 0 \\
M(X,Y) &= 0, \quad f_1(X,Y) = 0 \quad (\text{or} \quad f_2(X,Y) = 0).
\end{aligned}
$$

All the left members of each of these systems of equations are relatively prime with their right members. Therefore the first system has at most $n_1 \cdot n$ solutions, the second at most $n_2 \cdot n$ and the third at most $(l - n_1 - n_2) \cdot n$. The sum of these numbers is $l \cdot n < n^2$.

$\square$

Lemma 3.14. (Lemma 6).

Let A be a fixed complex number different from 0 and $g(X,Y) \in \mathbb{R}[X,Y]$. Then the condition that the real part of the product $A \cdot g(X,Y)^2 = 0$ at a given point (x_0, y_0), real or complex s.t. $g(x_0, y_0)$ non-zero real part, can always be expressed in the form of a linear homogeneous equation, with real coefficients, precisely in the coefficients of $g(x_0, y_0)$.

Proof. Taken from [6]. Let $A = a + bi$ and $g(x_0, y_0) = c + di$ where $a, b, c, d \in \mathbb{R}$ and $c \neq 0$. Then $A \cdot g(x_0, y_0)^2 = (a + bi)(c + di)^2$ and the real part of $A \cdot g(x_0, y_0)^2$ is precisely

$$ac^2 - ad^2 - 2bcd = c^2(a - 2bx - ax^2)$$

where $x = \frac{d}{c}$.

The equation $ax^2 + 2bx - a = 0$ has for any real numbers $a, b \in \mathbb{R}$ at least one real finite root, since its discriminant is $4(b^2 + a^2)$ and therefore non-negative. Denote this root by x_0. Then $d = x_0 c$ implies that the real part of $A \cdot g(x_0, y_0)^2$ is 0, and since the equation $d = x_0 c$ is a linear homogeneous equation with real coefficients from $g(x_0, y_0)$, the proof is finished. $\square$

Theorem 3.15. (Petrovski inequalities).

Denoting by p the number of positive ovals of a non-singular real curve $f(X,Y)=0$ of even degree n and by m the number of negative ovals. Then

$$-\frac{3n^2-6n}{8}-\delta \leq p-m \leq \frac{3n^2-6n}{8}+1-\delta$$

where $\delta=0$ if the outer ovals are positive and $\delta=1$ if the outer ovals are negative.

Proof. This is taken from [6] and [13]. By lemma 3.8 it can always be assumed that the following system

$$\frac{\partial f}{\partial X}=0, \quad \frac{\partial f}{\partial Y}=0 \tag{3.5}$$

posseses $(n-1)^2$ different finite solutions (x_i,y_i) for $i=1,\dots,(n-1)^2$. In this case the following theorem of Euler-Jacobi holds (see lemma 3.16):

$$\sum_{i=1}^{(n-1)^2}\frac{P(x_i,y_i)}{D(x_i,y_i)}=0; \tag{3.6}$$

where $D(x_0,y_0)=\begin{vmatrix} \frac{\partial^2 f}{\partial X^2}(x_0,y_0) & \frac{\partial^2 f}{\partial X\partial Y}(x_0,y_0) \\ \frac{\partial^2 f}{\partial Y\partial X}(x_0,y_0) & \frac{\partial^2 f}{\partial Y^2}(x_0,y_0)\end{vmatrix}$ and $P(X,Y)\in\mathbb{R}[X,Y]$ is an arbitrary polynomial of degree strictly lower than $(n-1)+(n-1)-2=2n-4$.

Again by lemma 3.8 the system (3.5) is not having multiple solutions, and thus the denominator is never zero of any member of the sum, so is well-defined.

In particular, the equation (3.6) holds taking $P(X,Y)=F(X,Y)\cdot g(X,Y)^2$ where $g(X,Y)$ is an arbitrary polynomial of degree $\frac{1}{2}(n-4)$ and

$$F(X,Y)=nf(X,Y)-X\frac{\partial f}{\partial X}(X,Y)-Y\frac{\partial f}{\partial Y}(X,Y)$$

which is at most of degree $n-1$ by its form. Notice that since f is non-singular, there is no solution of the system (3.5) that satisfies $f(X,Y)=0$ also.

The polynomial $g(X,Y)$ can be chosen with real coefficients and s.t. vanishes in

$$\frac{\left(\frac{1}{2}(n-4)+3\right)\left(\frac{1}{2}(n-4)\right)}{2}=\frac{(n-4+6)(n-4)}{8}=\frac{n^2-2n-8}{8}=\frac{n(n-2)}{8}-1 \tag{3.7}$$

arbitrarly chosen critical points of $f(X,Y)$ using corollary 1.23.

Lemma 3.13 tells that $g(X,Y)$ can not vanish in all critical points of $f(X,Y)$, since it is bounded by

$$(n-1)\cdot\frac{1}{2}(n-4)=\frac{n^2-5n+4}{2}, \quad n\in\mathbb{N}, \tag{3.8}$$

which is smaller than $\frac{n^2}{2}-2n+2$ and thus is smaller than $n^2-2n+1=(n-1)^2$, $n\in\mathbb{N}$.

After these preliminary constructions of the auxiliary polynomials, it can be introduced the kernel of the proof. Denote by k the number of points in which the curve $f(X,Y)=C$ meets the line of infinity. If $k>0$ and $C>C_M$ the curve consists of $\frac{1}{2}k$ positive infinite ovals. When C decreases from $C_M+\epsilon$ to 0 the curve obtains:

$$
\begin{aligned}
p+\alpha-\frac{k}{2} &\quad \text{new} \quad \text{positive} \quad \text{ovals,}\\
m+\beta &\quad \text{new} \quad \text{negative} \quad \text{ovals,}
\end{aligned}
$$

and loses α positive ovals and β negative ovals. While C passes through the critical values of $f(X,Y)$ there are

$$
\begin{aligned}
p+\alpha+\beta-\frac{k}{2} &\quad \text{plus-points}\\
m+\alpha+\beta-\delta &\quad \text{minus-points}
\end{aligned}
$$

since every positive oval arises in a plus point and every negative is erased in a plus point too and every negative oval is created in a minus point and every positive oval is erased in a minus point. Also $\delta=0$ if the outer ovals are positive and $\delta=1$ if the outer ovals are negative, since it means that it have crossed a saddle point (see lemma 3.11) where an infinite positive oval touches itself.

This reasoning also holds when the number of plus points and minus points are being counted for $k=0$. Recall the case $k=0$ from lemma 3.12. When there was at $C>C_M$ an imaginary curve, it is $\delta=0$, and lowering from C_M it appears an outer oval which is positive, thus all the outer ovals are positive. When there was at $C>C_M$ a negative oval, all the outers ovals are negative, and $\delta=1$ since we are counting an extra minus point taking into acount this negative oval at $C>C_M$ (without a starting minus point).

Then this far it has been seen that $f(X,Y)>0$ at

$$
\begin{aligned}
p+\alpha+\beta-\frac{k}{2} &\quad \text{plus-points} \quad \text{and} \quad \text{at}\\
m+\alpha+\beta-\delta &\quad \text{minus-points} \quad \text{of} \quad f.
\end{aligned}
$$

Is clear since $f(X,Y)$ is non-singular, no plus points or minus points are counted while $f(X,Y)=0$, so this counting holds true too for $f(X,Y)\geq 0$. By lemma 3.12 it is sattisfied that

$$
\#(-p)-\#(+p)=(\#(-p)_{\geq}+\#(-p)_{<})-(\#(+p)_{\geq}+\#(+p)_{<})=k-1 \tag{3.9}
$$

where $\#(-p)_{\geq}$ and $\#(-p)_{<}$ denotes the number of minus points that sattisfies respectively $f(X,Y)\geq 0$ and $f(X,Y)<0$. And analogously it is defined $\#(-p)_{\geq}$ and $\#(-p)_{<}$ changing minus points for plus points in the last sentence. Recall that in 3.9 it was defined plus and minus points as **real** critical points of the function $f(X,Y)$. Thus

$$
\#(-p)+\#(+p)=(n-1)^2-2\gamma \tag{3.10}
$$

where γ denotes the number of complex and non real solutions of the system (3.5). Adding (3.9) and (3.10) it is obtained

$$
\begin{aligned}
2\#(-p)=2\#(-p)_{\geq}+2\#(-p)_{<} &= (n-1)^2-2\gamma+k-1\\
2\#(-p)_{<} &= -2\left(m+\alpha+\beta-\delta\right)+(n-1)^2-2\gamma+k-1\\
\#(-p)_{<} &= \frac{(n-1)^2+k-1}{2}-\gamma-m-\alpha-\beta+\delta.
\end{aligned}
$$

Now substracting (3.9) to (3.10)

$$2\#(+p) = 2\#(+p)_{\geq} + 2\#(+p)_{<} = (n-1)^2 - 2\gamma - k + 1$$
$$2\#(+p)_{<} = (n-1)^2 - 2\gamma - k + 1 - 2\left(p + \alpha + \beta - \frac{k}{2}\right)$$
$$\#(+p)_{<} = \frac{(n-1)^2+1}{2} - \gamma - p - \alpha - \beta.$$

Notice that at $(x_i, y_i) \in (+p)_>$ and $(x_i, y_i) \in (-p)_<$ points it holds that $\frac{f(x_i,y_i)}{D(x_i,y_i)} > 0$ with $D(x_i, y_i)$ defined as at the beginning of the proof. Analogously for $(x_i, y_i) \in (-p)_>$ and $(x_i, y_i) \in (+p)_<$ points it holds $\frac{f(x_i,y_i)}{D(x_i,y_i)} < 0$. It will be denoted due to this as **A-points** to the points of $(+p)_> \cup (-p)_<$ and **B-points** to the points of $(-p)_> \cup (+p)_<$. There are

$$\#(A - points) = \#(+p)_> + \#(-p)_< = \tag{3.11}$$
$$p + \alpha + \beta - \frac{k}{2} + \frac{(n-1)^2 + k - 1}{2} - \gamma - m - \alpha - \beta + \delta = \tag{3.12}$$
$$p - m + \delta - \gamma + \frac{(n-1)^2 - 1}{2} \tag{3.13}$$

A-points and

$$\#(B - points) = \#(-p)_> + \#(+p)_< = \tag{3.14}$$
$$m + \alpha + \beta - \delta + \frac{(n-1)^2 + 1}{2} - \gamma - p - \alpha - \beta = \tag{3.15}$$
$$m - p - \delta - \gamma + \frac{(n-1)^2 + 1}{2} \tag{3.16}$$

B-points.

Now in order to see that

$$\#(A - points) \geq \frac{n(n-2)}{8} - \gamma \quad \text{and} \quad \#(B - points) \geq \frac{n(n-2)}{8} - \gamma$$

suppose the opposite, that

$$\#(A - points) < \frac{n(n-2)}{8} - \gamma \quad \text{and} \quad \#(B - points) < \frac{n(n-2)}{8} - \gamma. \tag{3.17}$$

If $\#(A - points) < \frac{n(n-2)}{8} - \gamma$ choose $\frac{1}{8}n(n-2)$ coefficients of the polynomial $g(X, Y)$ s.t. it vanishes in all the A-points (this last thing can be done by (3.7)). Also choose this $\frac{1}{8}n(n-2)$ coefficients s.t. the real parts of the components $\frac{P(x_i,y_i)}{D(x_i,y_i)}$ of the sum (3.6) corresponding to imaginary solutions of the system (3.5) vanish also, this can be done by lemma 3.14. The polynomial $g(X, Y)$ differs from 0 in at least one real critical point of $f(X, Y)$ because using 3.17 it is deduced that

$$\gamma < \frac{n(n-2)}{8},$$

since this is the most extreme case that we could obtain from $\#(A-points) < \frac{n(n-2)}{8} - \gamma$ i.e. making $\#(A-points) = 0$. Then the number of real critical points of $f(X,Y)$ which is $(n-1)^2 - 2\gamma$ exceeds the number

$$(n-1)^2 - 2 \cdot \frac{n(n-2)}{8} + 1 = \frac{3}{4}n^2 - \frac{3}{2}n + 2.$$

The number of critical points of $f(X,Y)$ where $g(X,Y)$ vanishes can not exceed by (3.8)

$$\frac{(n-1)(n-4)}{2} = \frac{n^2}{2} - \frac{5n}{2} + 2,$$

thus $f(X,Y)$ has at least one real critical point where $g(X,Y)$ does not vanishes since $\frac{n^2}{2} - \frac{5n}{2} + 2 < \frac{3}{4}n^2 - \frac{3}{2}n + 2$. Recall that

$$0 = \sum_{i=1}^{(n-1)^2} \frac{P(x_i,y_i)}{D(x_i,y_i)} = \sum_{i=1}^{(n-1)^2} \frac{F(x_i,y_i) \cdot g(x_i,y_i)^2}{D(x_i,y_i)};$$

since $F(X,Y) = nf(X,Y) - X\frac{\partial f}{\partial X}(X,Y) - Y\frac{\partial f}{\partial Y}(X,Y)$, in a real critical point (x_j,y_j) this last expression is not zero, since f is not singular and there is one real critical A-point, lets say (x_j,y_j), where $g(X,Y)$ does not vanishes and $F(x_j,y_j) = nf(x_j,y_j)$

$$0 = \sum_{i=1}^{(n-1)^2} \frac{P(x_i,y_i)}{D(x_i,y_i)} = \sum_{i=1}^{(n-1)^2} \frac{nf(x_i,y_i)}{D(x_i,y_i)} \cdot g(x_i,y_i)^2 > 0;$$

thus it is a contradiction. Then

$$\#(A-points) \geq \frac{n(n-2)}{8} - \gamma. \tag{3.18}$$

Analogously, by a similar reasoning it can be seen that if $\#(B-points) < \frac{n(n-2)}{8} - \gamma$, then

$$0 = \sum_{i=1}^{(n-1)^2} \frac{P(x_i,y_i)}{D(x_i,y_i)} = \sum_{i=1}^{(n-1)^2} \frac{nf(x_i,y_i)}{D(x_i,y_i)} \cdot g(x_i,y_i)^2 < 0;$$

which is another contradiction, so

$$\#(B-points) \geq \frac{n(n-2)}{8} - \gamma. \tag{3.19}$$

From (3.18) and (3.11), it is obtained that

$$\begin{aligned}
p - m + \delta - \gamma + \frac{(n-1)^2 - 1}{2} &\geq \frac{n(n-2)}{8} - \gamma \\
p - m &\geq \frac{n(n-2)}{8} - \frac{(n-1)^2 - 1}{2} - \delta \\
p - m &\geq \frac{n^2 - 2n}{8} - \frac{n^2 - 2n}{2} - \delta \\
p - m &\geq \frac{-3n^2 + 6n}{8} - \delta.
\end{aligned}$$

From (3.19) and (3.14), it is obtained that

$$\begin{aligned} m - p - \delta - \gamma + \frac{(n-1)^2+1}{2} &\geq \frac{n(n-2)}{8} - \gamma \\ p - m + \delta + \frac{-(n-1)^2-1}{2} &\leq \frac{-n(n-2)}{8} \\ p - m &\leq \frac{-n^2+2n}{8} + \frac{4n^2-8n}{8} + 1 - \delta \\ p - m &\leq \frac{3n^2-6n}{8} + 1 - \delta. \end{aligned}$$

Thus

$$-\frac{3n^2-6n}{8} - \delta \leq p - m \leq \frac{3n^2-6n}{8} + 1 - \delta. \tag{3.20}$$

□

Lemma 3.16. (Jacobi-Euler Theorem).

Let $f, g \in \mathbb{R}[X,Y]$ with n =deg$(f) \in \mathbb{N}$ and m =deg$(g) \in \mathbb{N}$. If $\Gamma_1 = V_{\mathbb{R}}(f)$ and $\Gamma_2 = V_{\mathbb{R}}(g)$. If Γ_1 and Γ_2 have no common component and no multiple intersection points, then for any $P \in \mathbb{R}[X,Y]$ with deg$(P) < n+m-2$ the following holds

$$\sum_{(x,y)\in\Gamma_1\cap\Gamma_2} \frac{P(x,y)}{J(x,y)} = 0, \tag{3.21}$$

where J is the Jacobi determinant of the function $F = (f,g) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Corollary 3.17. (Restriction on M-curves of even degree $2k \geq 6$).

There can not be an M-curve of degree $n = 2k \geq 6$ with all its ovals lying outside each other.

Proof. Taken from [6]. Suppose that all the outer ovals are positive

$$\frac{(2k-1)(2k-2)}{2} + 1 - \left(\frac{12k^2-12k}{8} + 1\right) = \frac{(2k-1)(8k-8)}{8} - \frac{12k^2-12k}{8}$$

where the right hand side is

$$\frac{16k^2-24k+8}{8} + \frac{-12k^2+12k}{8} = \frac{4k^2-12k+8}{8} = \frac{k^2-3k+2}{2} = \frac{(k-1)(k-2)}{2} \geq 0,$$

thus for $k > 2$, $k \in \mathbb{N}$ i.e. $2k \geq 6$ the last inequality is strict, which by the last theorem 3.15 is a contradiction. □

In particular, for degree 6 it can not happen that the 11 ovals lie outside each other. This shows that for higher degrees, M-curves do not have this particular simple topology case like at the lower degree cases.

CHAPTER 4

Real algebraic curves from a complex point of view

Now it will be seeked to find more information about the real plane algebraic curves using the fact that it is a subset of a complex plane algebraic curve defined by the same polynomial but taking also the complex roots into account. This complex curve defines a subset of the complex plane $\mathbb{C}^2$, and remember this last space has complex dimension 2 and real dimension 4. Notice that we have 2 real equations in a complex polynomial equation:

$$\begin{aligned} f(x,y) = 0, \quad x, y \in \mathbb{C}, \\ \mathrm{Re}(f(x,y)) = 0, \\ \mathrm{Im}(f(x,y)) = 0. \end{aligned}$$

for 4 real variables, since $x = a + bi$ and $y = c + di$ for $a, b, c, d \in \mathbb{R}$.

Proposition 4.1. *Let $f \in \mathbb{R}[X,Y]$ be a polynomial and let α be a complex root of this polynomial f. Then $\overline{\alpha}$ is also a root of f.*

Proof. Suppose that $f(X,Y) = a_{n,0}Y^n + a_{n-1,1}Y^{n-1}X + a_{n-1,0}Y^{n-1} + \ldots + a_{0,0}$, then letting $\alpha = (\alpha_1, \alpha_2) \in V_{\mathbb{C}}(f) \subset \mathbb{C}^2$:

$$f(\alpha_1, \alpha_2) = a_{n,0}\alpha_2^n + a_{n-1,1}\alpha_2^{n-1}\alpha_1 + a_{n-1,0}\alpha_2^{n-1} + \ldots + a_{0,0} = 0.$$

Now for $\overline{\alpha} = (\overline{\alpha_1}, \overline{\alpha_2})$ and using that $a_{i,j} \in \mathbb{R}$ for every $i, j = 1, 2, \ldots, n$ and the conjugate property $\overline{(z+w)} = \overline{z} + \overline{w}$:

$$\begin{aligned} f(\overline{\alpha_1}, \overline{\alpha_2}) &= a_{n,0}\overline{\alpha_2}^n + a_{n-1,1}\overline{\alpha_2}^{n-1}\overline{\alpha_1} + a_{n-1,0}\overline{\alpha_2}^{n-1} + \ldots + \overline{a_{0,0}} \\ &= \overline{a_{n,0}\alpha_2}^n + \overline{a_{n-1,1}\alpha_2}^{n-1}\overline{\alpha_1} + \overline{a_{n-1,0}\alpha_2}^{n-1} + \ldots + \overline{a_{0,0}} \\ &= \overline{a_{n,0}\alpha_2^n + a_{n-1,1}\alpha_2^{n-1}\alpha_1 + a_{n-1,0}\alpha_2^{n-1} + \ldots + a_{0,0}} \\ &= \overline{f(\alpha_1, \alpha_2)} = 0. \end{aligned}$$

$\square$

The real algebraic curve Γ lying inside the real projective plane $\mathbb{RP}^2$ remains unchanged under the conjugation $\sigma : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^2$, $\sigma(x : y : z) \mapsto (\overline{x} : \overline{y} : \overline{z})$. Due to proposition 4.1, if the real projective plane curve divides the complex algebraic curve, then it divides in equal halves. Figure 4.1 is just a fake sketch to understand better what is happening, since $\mathbb{CP}^2$ has real dimension 4.

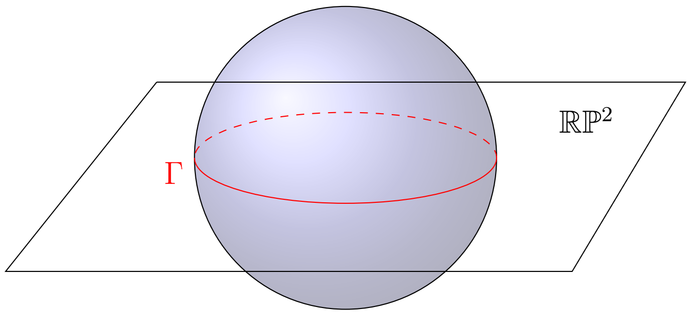


Figure 4.1: The real projective plane dividing a complex algebraic curve (visual sketch).

Since a complex projective curve is connected, the number of parts into which the curves fixed by conjugation divides the non-singular complex curve is at most two.

Definition 4.2. It is said that a real curve $V_{\mathbb{R}}(F)$ is of **type I** if the curve splits the complex curve $V_{\mathbb{C}}(F)$ into 2 halves. A real curve $V_{\mathbb{R}}(F)$ is of **type II** if the complement $V_{\mathbb{C}}(F) \setminus V_{\mathbb{R}}(F)$ is connected, i.e. the curve $V_{\mathbb{R}}(F)$ is not dividing $V_{\mathbb{C}}(F)$.

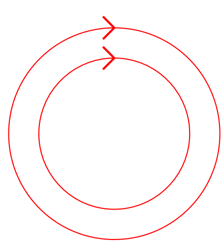
Example. *A circle, $V_{\mathbb{R}}(f)$ for $f(X,Y) = X^2 + Y^2 - 1$ is a real plane curve of type I.*

Curves of type I have an additional structure that comes from the complex domain. Since the real part of a curve of type I divides the complex curve in 2 halves, each of the halves has the orientation defined by the complex structure and defines an orientation on the real curve, which is the common boundary of each half. The set of real points get 2 exactly opposite orientations that is named **complex orientation** of the real curve, one from the upper part and the other from the lower part. When an orientation is chosen for one of the ovals, it determines the orientations of all the other ovals.

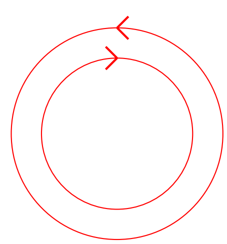
Definition 4.3. *(Positive and negative injective pair).*

Any pair of ovals s.t. one surrounds the other is called an **injective pair**. An injective pair of oriented ovals is **negative** if the ovals are both oriented clockwise or counterclockwise. Otherwise is said to be **positive**. Please see figure 4.

The number of positive injective pairs of ovals is denoted by Π^+ and the number of negative injective pairs by Π^-.



Negative pair



Positive pair

Since the real curve is precisely the set of fixed points under the complex conjugation, it will be interesting to study more properties about the conjugation map σ. Thinking in this map together with the map composition, it turns out to have the structure of an abelian group, furthermore, there is a natural group isomorphism between $\{\mathrm{id}, \sigma\} = \{\sigma^2, \sigma\}$ and the finite additive group $\mathbb{Z}_2$.

Definition 4.4. *(Euler characteristic).*
Let X be a topological space, then the **Euler characteristic** of X is defined as

$$\mathcal{X}(X) = \sum_{n=0}^{\infty}(-1)^n \mathrm{dim}(H_n(X)) = \sum_{n=0}(-1)^n \mathrm{card}(C_n^X)$$

where H_n denotes the n-homology group (see appendix C) and $\mathrm{card}(C_n^X)$ the number of n-cells of X. The dimension of the n-homology group gives the number of $\mathbb{Z}$ that appear at the direct sum of the n-homology group (the $\mathbb{Z}$-rank of the n-homology group).

Definition 4.5. *(Even and odd ovals).*
An oval of a real curve Γ is said to be **odd** if its depth is even. Analogously an oval is said to be **even** if its depth is odd. It is denoted by **p** the number of even ovals and by **n** the number of odd ovals of Γ.

Notice that **p** is the number of **positive** ovals and **n** is the number of **negative** ovals if all the outermost ovals are positive.

Definition 4.6. Let $F \in \mathbb{R}[X, Y, Z]$ be a homogeneous polynomial, then it is defined

$$B_+ = \left\{(x : y : z) \in \mathbb{RP}^2 | \quad F(x : y : z) \geq 0\right\}.$$

Proposition 4.7. *Let p and n denote respectively the number of even and odd ovals of a non-singular real curve of even degree Γ and $\mathcal{X}$ denote the Euler characteristic. Then*

$$\mathcal{X}(B_+) = p - n.$$

Proof. Please see C. Using that

$$\mathcal{X}(B_+) = \sum_{n=0}(-1)^n \dim(H_n(B_+)) = \sum_{n=0}(-1)^n \mathrm{card}(C_n^{B_+}),$$

let D be the disk in $\mathbb{R}^2$, that is one 2-cell with one 1-cell and one 0-cell, then is clear that $\mathcal{X}(D) = 1 - 1 + 1 = 1$ since $\dim(H_0(D)) = \dim(H_1(D)) = \dim(H_2(D)) = 1$ because $H_0(D) \cong H_1(D) \cong H_2(D) \cong \mathbb{Z}$. With celullar homology is easy to compute it, there is one 0-cell (a vertex v), one 1-cell (an edge e) and one 2-cell (U) as shown in figure 4.2

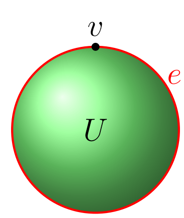


Figure 4.2: Cellular decomposition of a disk.

Now lets take a disk with a hole in it, D'. Then $H_0(D') = \mathbb{Z}$ since it is path connected. By Hurewicz's theorem C.20 (from C), we deduce that $H_1(D') \cong H_1(\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, since a punctured disk has the same homotopy type as a circle $\mathbb{S}^1$. There is no 2-cells on a circle, so $H_2(D') \cong H_2(\mathbb{S}^1) \cong 0$ again by the homotopy invariance of the homology groups. Thus $\mathcal{X}(D') = 1 - 1 + 0 = 0$.

Define $A \vee B$ as the union by a point between A and B. A disk with n holes D'^n is of the same homotopy type as $\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1$ which is path-connected, then $H_0(\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1) \cong \mathbb{Z}$. For any family of topological spaces X_i for $i = 1, \ldots, p$ it holds:

$$H_n(X_1 \vee \ldots \vee X_p) \cong H_n(X_1) \oplus \ldots \oplus H_n(X_p),$$

then $H_1(\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1) \cong H_1(\mathbb{S}^1) \oplus \ldots \oplus H_1(\mathbb{S}^1) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} = \mathbb{Z}^n$ and also it is obtained that $H_2(\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1) \cong H_2(\mathbb{S}^1) \oplus \ldots \oplus H_2(\mathbb{S}^1) \cong 0$, concluding

$$\mathcal{X}(D'^n) = 1 - n + 0 = 1 - n. \tag{4.1}$$

Using that for every $n \in \mathbb{N}$:

$$H_n(X) = \bigoplus_{\alpha} H_n(X_\alpha) \tag{4.2}$$

where X_α are the path-components of X i.e. path-connected components. Then for any space X with path-components $D_1, \ldots, D_k$ we have that

$$\mathcal{X}(X) = \sum_{\alpha=1}^{k} \mathcal{X}(D_\alpha). \tag{4.3}$$

It can always be assumed (up to a sign change in $F \in \mathbb{R}[X, Y, Z]$) that F is negative on the region outside all the ovals of $V_{\mathbb{R}}(F)$ i.e. the outermost ovals are positive and even.

Assume that there is only one even/positive oval, then by the reasoning above with D as a disk in $\mathbb{R}^2$, $\mathcal{X}(B_+) = \mathcal{X}(D) = 1$. If there is an odd oval inside an even oval, then B_+ is homeomorphic to a disk with a hole, then $\mathcal{X}(B_+) = \mathcal{X}(D') = 0$.

If there are $2k + 1$ ovals in a nest ($k \in \mathbb{Z}$), the deepest oval will be even and since the rest would be unions of k pairs of one odd oval inside one even oval that have Euler characteristic 0, by (4.3) it is deduced that it is like the case when there is only one even oval case, i.e. $\mathcal{X}(B_+) = 1 = k+1-k = p-n$ since B_+ is made out of $k-1$ annulus (of one even oval and one odd oval inside) and one oval that are different connected components.

Now if there is a nest of order $2k$, then the deeper oval is odd and the nest is made out of k rings that are precisely pairs of an even oval with one odd oval inside, since this rings have Euler characteristic 0, it is deduced by (4.3) that $\mathcal{X}(B_+) = 0 = k - k = p - n$.

Perhaps this can be better understood with the following picture with the cases for a nest of order 4 (left) and a nest of order 3 (right) with the domain B_+ colored in red.

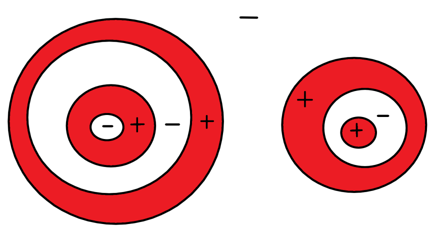


Figure 4.3: The domain B_+ in the projective plane in red. In this figure $p - n = 1$.

The last case would be to study what happens with the characteristic when we have more than one oval inside another oval. Suppose we have an even oval with 2 odd ovals lying inside. This is equivalent to a disk with 2 holes on it, thus by the reasoning of the beginning (4.1) $\mathcal{X}(B_+) = 1 - 2 = p - n$. In general, if there is an even oval with n odd different ovals lying in its interior, then again by (4.1), $\mathcal{X}(B_+) = 1 - n = p - n$.

Any other even and odd ovals distribution that may arise are combinations of all the cases it have been already studied with different connected components configurations $B_+^1, \ldots, B_+^m$, but since all our primarly cases sattisfied $\mathcal{X} = p - n$, the sum (4.3) of the different connected components would also sattisfy in general

$$\mathcal{X}(B_+) = \mathcal{X}(B_+^1 \cup \ldots \cup B_+^m) = p - n.$$

□

Using this homological definition of the Euler characteristic it can be seen due to homological properties that for a g-torus M_g (see [7]) the Euler characteristic is

$$\mathcal{X}(M_g) = 2 - 2g.$$

Proposition 4.8. *Any non-singular M-curve of even degree is of type I.*

Proof. This proof is based on [15]. Let Γ be a complex non-singular M-curve of degree d, thus of genus $g = \frac{(d-1)(d-2)}{2}$ (this result is explained in [5]). Then $\mathbb{R}\Gamma$ is the union of $g + 1 = \frac{(d-1)(d-2)}{2} + 1$ disjoint ovals lying in $\mathbb{RP}^2$. Since Γ is homeomorphic to a sphere with $g = \frac{(d-1)(d-2)}{2}$ handles ([5]) then that many disjoint ovals of the M-curve $g + 1$, make each oval necessarily divide each handle of the sphere. This can be seen in 4.4:

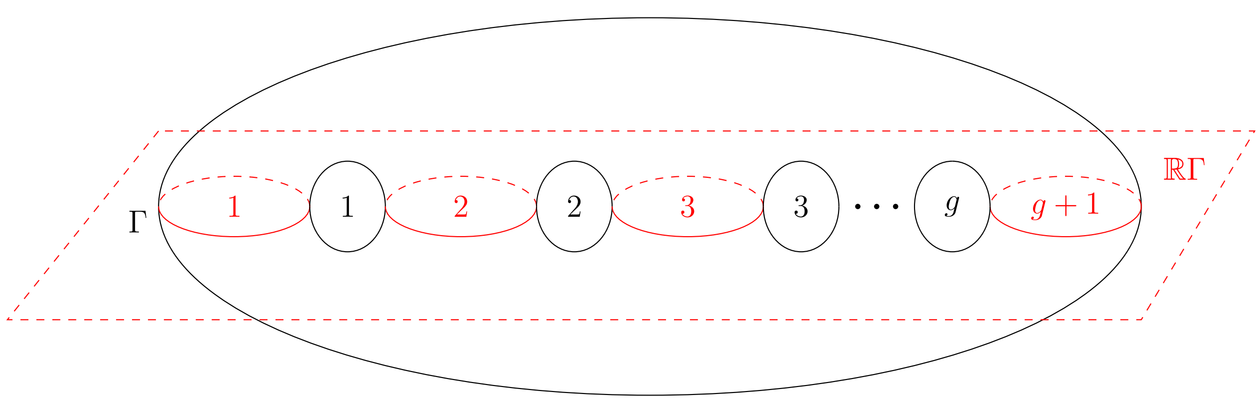


Figure 4.4: Counting ovals in a sphere with g handles.

For a nested ovals case, using that the non-singular curve Γ is homeomorphic to a g-torus, its Euler characteristic is $2 - 2g = 2 - (d-1)(d-2)$. Now cutting Γ along $\mathbb{R}\Gamma$, since the Euler characteristic behaves like a measure, it is clear that $\mathbb{R}\Gamma$ has measure zero with respect to Γ, thus the Euler characteristic under this cutting remains equal.

Now cap every boundary circle with a disk. Each component of $\mathbb{R}\Gamma$ is giving now rise to two boundary disks, thus the number of the boundary circles is $2(g+1) = (d-1)(d-2)+2$.

The Euler characteristic is additive under the union, thus the Euler characteristic of the surface is

$$2 - (d-1)(d-2) + (d-1)(d-2) + 2 = 4,$$

using $\mathcal{X}(D) = 1$ for every D being a disk. Any closed connected surface is homeomorphic to a sphere with g handles for some $g \geq 0$ which has Euler characteristic $2 - 2g \leq 2 < 4$. Since there is no connected closed surface with Euler characteristic 4, $\mathbb{R}\Gamma$ must divide Γ. Thus $\mathbb{R}\Gamma$ is a dividing type curve and Γ is a curve of type I.

□

Let $\sigma : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^2$ denote the complex conjugation. If Γ is a real curve of type I, its complexification $\mathbb{C}\Gamma$ is divided by Γ into 2 components $\mathbb{C}\Gamma_{\pm}$ that are interchanged by the complex conjugation σ. The orientation of $\mathbb{C}\Gamma_+$ orients the ovals of its boundary and it induces an orientation between 2 related ovals in a common nest. Let

$$\varepsilon_{ij} = \left\{ \begin{array}{cc} \pm 1 & \text{if the orientations of the } i^{th} \text{ and } j^{th} \text{ ovals agree or disagree} \\ 0 & \text{if the orientations of the } i^{th} \text{ and } j^{th} \text{ ovals cannot be compared} \end{array} \right\}.$$

It will be said that 2 different ovals can not be compared when they are not in the same oval nest.

Lemma 4.9. *Let Γ be a real curve of even degree $2k$ with l ovals of type I, then*

$$2 \sum_{1 \leq i < j \leq l} \varepsilon_{ij} = k^2 - l.$$

Proof. The idea of this proof is taken from [14]. See appendix C. Let $D = \bigcup_{i=1}^{l} D_i$ be the disjoint union on $\mathbb{RP}^2$ made up of the discs bounded by the l ovals of Γ. Visualizing it in figure 4.5 for the upper case (the lower case is analogous) in the case where every oval lies outside every other oval:

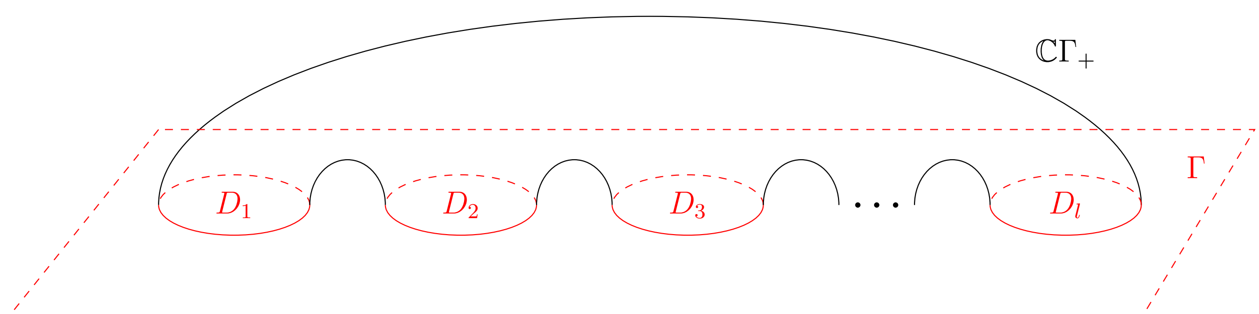


Figure 4.5: Capping with l discs half of type I complex curve $\mathbb{C}\Gamma$.

Lets define
$$E_+ = \mathbb{C}\Gamma_+ \cup D^+ \quad \text{and} \quad E_- = \mathbb{C}\Gamma_- \cup D^-$$
where D^+ is D together with the orientation of $\mathbb{C}\Gamma_+$ and D^- is D together with the orientation of $\mathbb{C}\Gamma_-$.

Let $\sigma : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^2$ denote the complex conjugation and $H_2(\mathbb{CP}^2;\mathbb{Z}) \simeq \mathbb{Z}$ the second homology group with integer coefficients, and lets denote the homology conjugation map as $\sigma_* : H_2(\mathbb{CP}^2;\mathbb{Z}) \longrightarrow H_2(\mathbb{CP}^2;\mathbb{Z})$.

Denote now by $[E_+]$ and $[E_-]$ respectively the homology clases in $H_2(\mathbb{CP}^2;\mathbb{Z})$ of E_+ and E_-. The conjugation takes one to another with a minus sign due to their respectively opposite orientation $\sigma_*([E_+]) = -[E_-]$, now
$$[E_+] \oplus [E_-] = [E_+ \cup E_-] = [\mathbb{C}\Gamma] = 2k[L] \tag{4.4}$$
where $[L]$ is the homological class of a line and $\mathbb{C}\Gamma$ being the complexification Γ. Thus by (4.4) it has to be that $[E_+] = k[L']$ and $[E_-] = k[L'']$ since the curve is of type I and the real projective plane is dividing the upper and the lower part in 2 equal halves. Then
$$[E_+] \bullet [E_-] = k^2 \tag{4.5}$$
where $\bullet$ denotes the number of intersections in the complex projective plane, which is exactly k^2 by Bezout's theorem 1.7.

Now, it will be counted the intersection $[E_+]\bullet[E_-]$ using properties of the 2 homological classes $[E_+]$ and $[E_-]$ of the second homology group $H_2(\mathbb{CP}^2;\mathbb{Z})$.
$$[E_+] \bullet [E_-] = [E_+] \bullet (-\sigma_*[E_+]) = \sum_{i,j=1}^{l} \varepsilon_{ij} = l + 2\sum_{i<j}^{l} \varepsilon_{ij}. \tag{4.6}$$

This count comes out taking into account the order or the orientation of the disks that are in $[E_+]$ and in $-\sigma_*[E_+]$. Every disk of $[E_+]$ intersects with its own projection in $-\sigma_*[E_+]$ counting $+1$. In the case of having a nest with one disk of $[E_+]$ having opposite orientation with another disk of the nest, this opposite orientation will be the same in $-\sigma_*[E_+]$, and it can be changed at the cost of a minus sign making it the same homolgy class, in order to count this intersection as a $+1$. Taking the sign into account $-(+1) = -1$. When one disk of the nest in $[E_+]$ shares the same orientation with another disk of its nest, the projection in $-\sigma_*[E_+]$ shares this orientation and it is counted this intersection as $+1$. This last 2 cases can be both visualised in 4.6.

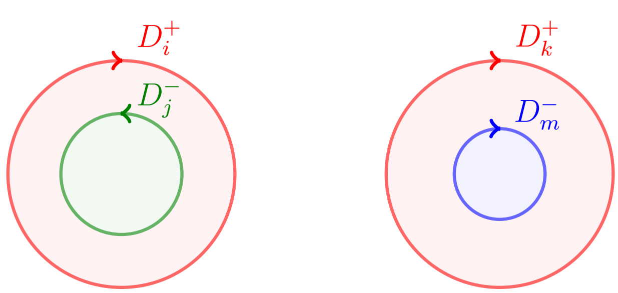


Figure 4.6: Two nests with 2 discs with opposite (left) and same (right) orientation.

Then the count from (4.6) holds. Equaling now the sums of $[E_+] \bullet [E_-]$ from (4.6) and (4.5) the result is finally obtained.

$\square$

Corollary 4.10. (Complexity of a nest). Result from [14].

The sum of the complexities of all the possible nests (i.e. $\sum_{i<j} |\varepsilon_{ij}|$) in a real non-singular algebraic curve of type I and even degree $2k$ with l ovals sattisfies the following inequality

$$\frac{k^2 - l}{2} \leq \sum_{i<j} |\varepsilon_{ij}|$$

Proof. Using lemma 4.9

$$\frac{k^2 - l}{2} = \sum_{i<j} \varepsilon_{ij} \leq \sum_{i<j} |\varepsilon_{ij}|.$$

$\square$

Theorem 4.11. (Rokhlin complex orientation formula).

For any curve of type I and degree $m = 2k$ with l ovals, then

$$2(\Pi^+ - \Pi^-) = l - k^2.$$

Proof. By the definition of Π^+, Π^- and ε_{ij} it holds that

$$\Pi^- - \Pi^+ = \sum_{i<j}^{l} \varepsilon_{ij}$$

then

$$2(\Pi^+ - \Pi^-) = -2\sum_{i<j}^{l} \varepsilon_{ij}.$$

Using lemma 4.9, $2(\Pi^+ - \Pi^-) = l - k^2$. $\square$

Corollary 4.12. (Gudkov congruence mod 4).

Let p and n denote respectively the number of positive and negative ovals of a non-singular real algebraic curve Γ of type I and of even degree $2k$, then

$$p - n \equiv k^2 \mod 4.$$

Proof. Let p and n denote respectively the number of positive ovals and negative ovals. Since l is the number of total ovals $l = p + n$. By the definition of Π^+ and Π^- it holds that

$$\Pi^+ + \Pi^- \equiv n \mod 2, \tag{4.7}$$

from where it can be deduced that

$$\Pi^+ - \Pi^- \equiv n \mod 2. \tag{4.8}$$

By the Rokhlin complex orientation formula from theorem 4.11 and the equation of (4.8), it holds that

$$\begin{aligned}\frac{l-k^2}{2} &\equiv n \mod 2\\ l-k^2 &\equiv 2n \mod 4\\ p+n &\equiv k^2+2n \mod 4\\ p-n &\equiv k^2 \mod 4.\end{aligned}$$

$\square$

Note. *By the last results it was seen that an M-curve is of type I, so in particular it should sattisfy for degree* $2 \cdot 3 = 6$ *that*

$$\mathcal{X}(B_+) = p - n \equiv 9 \mod 4 \equiv 1 \mod 4.$$

Using all this information, the possible candidates that we have this far for an M-curve of degree 6 are the following pairs of positive and negative ovals (p, n)*:*

$$(10,1) \quad (2,9) \quad (6,5) \quad (8,3) \quad (4,7). \tag{4.9}$$

In fact, there is a stronger restriction for the M-curves of even degree $2k$ *in particular, named as the Gudkov-Rokhlin congruence stating that*

$$\mathcal{X}(B_+) \equiv k^2 \mod 8$$

which by proposition 4.7 is the same as $p - n \equiv k^2 \mod 8$*. Thus the latter list (4.9) is reduced just to*

$$(10,1) \quad (2,9) \quad (6,5),$$

which are in fact the only M-curves for degree 6.

CHAPTER 5

Smith's inequality and Harnack's formula generalization

Definition 5.1. It is said that K is a **simplicial complex** when K is a set whose elements are called vertices, together with a collection of finite non-empty substets of the set of vertices called simplices such that:

1. Every vertex is contained in some simplex.
2. Every non-empty subset of a simplex is a simplex.

Let $v_1 - v_0, \ldots, v_n - v_0$ be linearly independent vectors of a vector space V. The points v_i are vertices of the n-simplex $[v_0, v_1, \ldots, v_n]$.

A proper and non-empty subset of a simplex is called a face of that simplex. A **simplicial map** from one simplicial complex to another is a map of the set of vertices which carries simplices into simplices. We denote $|K|$ the topological space of a simpicial complex K together with the weak topology for K i.e. a subset of $|K|$ it is closed if and only if its intersection with each $|s|$ is closed (using the usual topology at each $|s|$), being $|s|$ a simplex of K.

Definition 5.2. Let G be a finite group, then it is said that G **acts simplicially on** K if each trasformation of K under the action of any member of G is a simplicial map. The simplicial complex K, together with such an action is called a **simplicial G-complex**.

A simplicial action of G on K satisfying for $g_0, g_1, \ldots, g_n \in G$ that $[v_0, \ldots, v_n]$ and $[g_0 v_0, \ldots, g_n v_n]$ are both n-simplices s.t. there exists an element $g \in G$ sattisfying $g(v_i) = g_i(v_i)$ for every i it is said that this simplical action is **regular**. It can be seen that up to homeomorphism at $|K|$, any simplical action G on K is regular, it will be assumed in the sequel that the simplical action it is always regular (this result is in chapter 3 of [2]).

Definition 5.3. The vertices of a **regular G-complex K**, denoted by K/G, are orbits $v^* = Gv = \{\ gv \mid g \in G\}$ of the action of G on the vertices of K, and we take the simplices of K/G to be

$$[v_0^*, \ldots, v_n^*].$$

The simplex $[v_0, \ldots, v_n]$ of K is said to be **over** the simplex $[v_0^*, \ldots, v_n^*]$. Since K is a regular G-complex it is deduced that K/G is a well defined simplicial complex. Let $[v_0, \ldots, v_n]$ and $[w_0, \ldots, w_n]$ be simplices of K over the same simplex $[v_0^*, \ldots, v_n^*] = [w_0^*, \ldots, w_n^*]$ of K/G, then by regularity, there exists an element $g \in G$ s.t. $[w_0, \ldots, w_n] = g[v_0, \ldots, v_n] = [gv_0, \ldots, gv_n]$. Thus all the simplices of K over a simplex of K/G form an orbit of the action G on the simplices of K.

Definition 5.4. The complex K^G is the subcomplex of K consisting of all simplices which are pointwise fixed under G (i.e. every element of the group G).

An **oriented n-simplex** of K is a n-simplex s together with an equivalence class of total orderings of the vertices of s. Two orderings are **equivalent** if they differ by an even of permutations of the vertices.

If $v_0, v_1, \ldots, v_n$ are the vertices of the simplex s, then $[v_0, \ldots, v_n]$ denotes the oriented n-simplex of K consisting of the simplex s together with the equivalence class of the ordering $v_0 < \ldots < v_n$ of its vertices represented by $<$.

Example. $[v_0, v_1, v_2] \equiv [v_2, v_0, v_1] \equiv [v_1, v_2, v_0]$.

Definition 5.5. Let $C_n(K)$ be the **free abelian group generated by the oriented n-simplices** s^n sattisfying the relations

$$s_1^n + s_2^n = 0$$

whenever s_1^n and s_2^n are opposite oriented n-simplices corresponding to the same n-simplex of K (for a definition of a free group, please see appendix C).

The rank of the free abelian group $C_n(K)$ is equal to the number of n-simplices of K. Define the homomorphisms $\partial_n : C_n(K) \longrightarrow C_{n-1}(K)$ for every $n \geq 1$ as follows:

$$\partial_n[v_0, v_1, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$$

where $[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$ denotes the oriented $(n-1)$-simplex obtained by omitting v_i. If $s_1^n + s_2^n = 0$ in $C_n(K)$ then it can be verified, since ∂_n is a homomorphism:

$$\partial_n(s_1^n) + \partial_n(s_2^n) = \partial_n(s_1^n + s_2^n) = \partial_n(0) = 0.$$

In addition, by the definition of ∂_n, it holds that $\partial_n \circ \partial_{n+1} = 0$ (see [7]).

Therefore there is a **free non-negative chain complex** $C(K) = \{C_n(K), \partial_n\}$ with its correspondant **oriented chain complex of** K:

$$\cdots \to C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K).$$

The **n-th oriented homology group of** K is

$$H_n(K) = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}).$$

Since K is a simplicial G-complex, then $C(K)$ inherits also the action of G putting

$$g[v_0, \ldots, v_n] = [gv_0, \ldots, gv_n].$$

Due to this, it is a module over the group ring $\mathbb{Z}G$, whose elements are of the form:

$$\sum_{g\in G} n_g \cdot g,$$

where all the n_g are integers.

Definition 5.6. The **norm** $\sigma \in \mathbb{Z}G$ is defined to be the sum

$$\sigma = \sum_{g\in G} g$$

that can act on any chain $c \in C(K)$. The image $\sigma C(K) \subset C(K)$ of $\sigma : C(K) \longrightarrow C(K)$ is a subcomplex. As a map, σ is defined as

$$\sigma([v_0, v_1, \cdots, v_n]) = \sum_{g\in G} [gv_0, gv_1, \cdots, gv_n].$$

Let $L \subset K$ be a subcomplex which is invariant under the action of G, then G acts on the chain $C(K, L) = C(K)/C(L)$ and $\sigma C(K, L)$ is a subcomplex of $C(K, L)$.

Restrict now the attention to a multiplicative group G of prime order p (G cyclic) and to homology groups with coefficients in $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$. Let g be a fixed generator of G, then it is defined $\mathbb{Z}_pG$ as the group ring which elements are of the form

$$x = a_0 + a_1 g + \cdots + a_{p-1} g^{p-1} \quad \text{with} \quad a_0, a_1, \cdots, a_{p-1} \in \mathbb{Z}_p. \tag{5.1}$$

Notice that $\mathbb{Z}_pG \cong \mathbb{Z}_p[g]/(g^p - 1)$. For $g \in G$ being a fixed generator of G lets define:

$$\begin{aligned} \tau &= 1 - g \in \mathbb{Z}_pG, \\ \sigma &= 1 + g + g^2 + \cdots + g^{p-1} \in \mathbb{Z}_pG. \end{aligned}$$

There are several relations between this 2 elements τ, σ from $\mathbb{Z}_pG$.

Proposition 5.7. *The kernel of the map*

$$\begin{aligned} \tau : \mathbb{Z}_pG &\longrightarrow \mathbb{Z}_pG \\ x &\longmapsto \tau x \end{aligned}$$

is 1-dimensional and it is spanned by σ. In particular, $\sigma \cdot \mathbb{Z}_pG$ is 1-dimensional.

Proof. Proof taken from [12]. Let x be a general element from $\mathbb{Z}_pG$ like in (5.1)

$$\begin{aligned} \tau x &= a_0 + a_1 g + \cdots + a_{p-1} g^{p-1} - a_0 g - a_2 g^2 - \cdots - a_{p-2} g^{p-1} - a_{p-1} \\ &= (a_0 - a_{p-1}) + (a_1 - a_0) g + \cdots + (a_{p-1} - a_{p-2}) g^{p-1}. \end{aligned}$$

This is zero if and only if all the a_i are equal, which is true whenever x is precisely a multiple of σ, $x = k \cdot \sigma$ for any $k \in \mathbb{Z}_p$ i.e. it is spanned by σ. Thus $\mathrm{Ker}(\tau) = \sigma \cdot \mathbb{Z}_pG$ and since it is spanned only by one element σ, the kernel is one dimensional. □

Proposition 5.8. $\tau^{p-1} = \sigma$

Proof. Taken from [12]. Using the binomial Newton's theorem

$$\tau^{p-1} = (1-g)^{p-1} = \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i g^i.$$

It must be proven that

$$\binom{p-1}{i} (-1)^i \equiv 1 \mod p$$

for $1 \leq i \leq p-1$. Thus, developing the combinatorial number:

$$\binom{p-1}{i}(-1)^i = \frac{(p-1)(p-2)\cdots(p-i)}{(i)(i-1)\cdots(1)}(-1)^i \equiv_p (-1)^i \frac{1\cdot 2\cdots i}{i\cdot(i-1)\cdots 1}(-1)^i = 1.$$

Then

$$\tau^{p-1} = \sum_{i=0}^{p-1} g^i = \sigma.$$

□

Proposition 5.9. *For every $0 \leq i \leq p-1$ it holds that $\sigma \in \tau^i \cdot \mathbb{Z}_p G$.*

Proof. Taken from [12]. By lemma 5.8 $\sigma = \tau^{p-1} = \tau^i \cdot \tau^{p-1-i}$. □

Lemma 5.10. *For every $0 \leq i \leq p-1$, the following exact sequence holds*

$$0 \longrightarrow \sigma \cdot \mathbb{Z}_p G \xrightarrow{f} \tau^i \cdot \mathbb{Z}_p G \xrightarrow{g} \tau^{i+1} \cdot \mathbb{Z}_p G \longrightarrow 0.$$

Proof. Proof from [12]. Because of lemma 5.9, the first map is just an inclusion, which is injective. $\mathrm{Im}(f)$ =$\mathrm{Ker}(g)$ by lemma 5.7 and since $g = \tau$, it is clear that g is surjective. □

Lemma 5.11. *Consider the following exact sequence*

$$D \to A \xrightarrow{f} B \xrightarrow{g} C \to E$$

with f and g being respectively injective and surjective homomorphisms, then the following holds

$$dim(B) = dim(A) + dim(C).$$

In particular, it holds for the last sequence being an exact sequence.

Proof. Since g is a surjectice homomorphism, $\mathrm{Im}(g) = C$, then:

$$\dim(\mathrm{Ker}(g)) + \dim(C) = \dim(\mathrm{Ker}(g)) + \dim(\mathrm{Im}(g)) = \dim(B).$$

Using the fact of being an exact sequence, $\mathrm{Ker}(g) = \mathrm{Im}(f)$ and because f is an injectice homomorphism $\mathrm{Ker}(f) = 0$ thus:

$$\dim(A) + \dim(C) = \dim(\mathrm{Im}(f)) + \dim(C) = \dim(\mathrm{Ker}(g)) + \dim(C) = \dim(B).$$

since $\dim(A) = \dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f)) = \dim(\mathrm{Im}(f))$. □

Lemma 5.12. *For all $0 \leq i \leq p-1$, the subspace $\tau^i \cdot \mathbb{Z}_pG$ of $\mathbb{Z}_pG$ is $(p-i)$-dimensional.*

Proof. Result from [12]. The space $\tau^0 \cdot \mathbb{Z}_pG = \mathbb{Z}_pG$ is p-dimensional (with respect to the coefficient field $\mathbb{Z}_p$), and the last lemmas 5.10, 5.11 and 5.7 imply for $0 \leq i \leq p-1$ that:

$$\dim\left(\tau^i \cdot \mathbb{Z}_pG\right) = \dim\left(\sigma \cdot \mathbb{Z}_pG\right) + \dim\left(\tau^{i+1} \cdot \mathbb{Z}_pG\right) = 1 + \dim\left(\tau^{i+1} \cdot \mathbb{Z}_pG\right).$$

Now the lemma follows since

$$\begin{aligned}
\dim\left(\tau \cdot \mathbb{Z}_pG\right) &= \dim\left(\mathbb{Z}_pG\right) - 1 = p-1 \\
\dim\left(\tau^2 \cdot \mathbb{Z}_pG\right) &= \dim\left(\tau \cdot \mathbb{Z}_pG\right) - 1 = p-2 \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\
\dim\left(\tau^{p-2} \cdot \mathbb{Z}_pG\right) &= \dim\left(\tau^{p-3} \cdot \mathbb{Z}_pG\right) - 1 = 3-1 = 2 \\
\dim\left(\tau^{p-1} \cdot \mathbb{Z}_pG\right) &= \dim\left(\tau^{p-2} \cdot \mathbb{Z}_pG\right) - 1 = 2-1 = 1.
\end{aligned}$$

Thus for every $0 \leq i \leq p-1$

$$\dim\left(\tau^i \cdot \mathbb{Z}_pG\right) = p - i.$$

□

Lemma 5.13. *Let $\rho = \tau^i$ with $1 \leq i \leq p-1$. Set $\overline{\rho} = \tau^{p-i}$. Then there is a short exact sequence:*

$$0 \longrightarrow \overline{\rho} \cdot \mathbb{Z}_pG \longrightarrow \mathbb{Z}_pG \xrightarrow{\rho=\tau^i} \rho \cdot \mathbb{Z}_pG \longrightarrow 0.$$

Proof. Taken from [12]. The third map $\rho = \tau^i$ is surjective since $\mathrm{Im}(\tau^i : \mathbb{Z}_pG \longrightarrow \tau^i \cdot \mathbb{Z}_pG) = \tau^i \cdot \mathbb{Z}_pG$. The second map is an inclusion, so it is injective. It is clear that $\mathrm{Ker}(\rho) = \mathrm{Im}(\overline{\rho})$. □

By lemma 5.8 holds that $\overline{\tau} = \tau^{p-1} = \sigma$ and that $\overline{\sigma} = \tau$. Now, it will be considered chain subcomplexes

$$\rho C(K; \mathbb{Z}_p)$$

from the chain complex of $C(K; \mathbb{Z}_p)$ for $\rho = \tau^i$, $1 \leq i \leq p-1$ with coefficients in $\mathbb{Z}_p$. One of the most important results is the following:

Theorem 5.14. *For each $\rho = \tau^j$, $1 \leq j \leq p-1$, then*

$$0 \longrightarrow \overline{\rho}C(K; \mathbb{Z}_p) \oplus C(K^G; \mathbb{Z}_p) \xrightarrow{i} C(K; \mathbb{Z}_p) \xrightarrow{\rho} \rho C(K; \mathbb{Z}_p) \longrightarrow 0$$

is a short exact sequence of chain complexes, where i is the sum of the inclusions from $\overline{\rho}C(K; \mathbb{Z}_p)$ to $C(K; \mathbb{Z}_p)$ and from $C(K^G; \mathbb{Z}_p)$ to $C(K; \mathbb{Z}_p)$ and ρ is just the map

$$\begin{aligned}
\rho : C(K; \mathbb{Z}_p) &\longrightarrow C(K; \mathbb{Z}_p) \\
s &\longmapsto \rho \cdot s.
\end{aligned}$$

Proof. Proof taken from [2]. There are two cases depending of the n-simplex s being in K^G or not.

If $s \in K^G$, then $\tau s = 0$, and as a consecuence $\rho s = 0 = \overline{\rho} s$ no matter what $1 \leq j \leq p-1$, and the sequence is exact:

$$0 \longrightarrow C(K^G; \mathbb{Z}_p) \xrightarrow{\text{id}} C(K^G; \mathbb{Z}_p) \xrightarrow{\rho} 0.$$

Let now $s \notin K^G$. Any n-chain in the orbit of s, i.e. $G(s) = \{gs | g \in G\}$, has the form:

$$c = \sum_{i=0}^{p-1} n_i g^i s, \quad n_i \in \mathbb{Z}_p$$

with $\sum_{i=0}^{p-1} n_i g^i$ being an unique element in $\mathbb{Z}_p G$. By the last lemma 5.13 it holds that:

$$0 \longrightarrow \overline{\rho} \cdot \mathbb{Z}_p G \hookrightarrow \mathbb{Z}_p G \xrightarrow{\rho = \tau^i} \rho \cdot \mathbb{Z}_p G \longrightarrow 0$$

is an exact sequence. Thus it holds for any n-chain $c \notin C(K^G; \mathbb{Z}_p)$ that:

$$0 \longrightarrow \overline{\rho} C(K; \mathbb{Z}_p) \xrightarrow{i} C(K; \mathbb{Z}_p) \xrightarrow{\rho} \rho C(K; \mathbb{Z}_p) \longrightarrow 0.$$

□

Definition 5.15. *(Smith's homology groups).*

For $\rho = \tau^i$, $1 \leq i \leq p-1$ being as always $\tau = 1 - g$ with g a generator of G, lets define:

$$H_n^{\rho}(K, L; \mathbb{Z}_p) = H_n(\rho C(K, L; \mathbb{Z}_p)).$$

Corollary 5.16. (Smith's sequence).

Let G be a cyclic group of prime order p and let K be a regular simplicial complex. Fixing $\rho = \tau^i$ with $1 \leq i \leq p-1$ and letting $\overline{\rho} = \tau^{p-i}$. Then by theorem 5.14, there exists the following long exact sequence (see appendix C):

$$\cdots \to H_n^{\overline{\rho}}(K; \mathbb{Z}_p) \oplus H_n(K^G; \mathbb{Z}_p) \to H_n(K; \mathbb{Z}_p) \to H_n^{\rho}(K; \mathbb{Z}_p) \to H_{n-1}^{\overline{\rho}}(K; \mathbb{Z}_p) \oplus H_{n-1}(K^G; \mathbb{Z}_p) \to \cdots$$

Lemma 5.17. *Consider the following exact sequence*

$$D \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} E$$

with f,g and h being homomorphisms, then the following holds

$$dim(B) \leq dim(A) + dim(C).$$

Proof. Since g it is an homomorphism: $\dim(\text{Ker}(g)) + \dim(\text{Im}(g)) = \dim(B)$. Using the fact of being an exact sequence, $\text{Ker}(g) = \text{Im}(f)$ and $\text{Im}(g) = \text{Ker}(h)$:

$$\dim(\text{Im}(f)) + \dim(\text{Ker}(h)) = \dim(\text{Ker}(g)) + \dim(\text{Im}(g)) = \dim(B).$$

Thus:

$$\dim(\text{Im}(f)) + \dim(\text{Ker}(f)) + \dim(\text{Ker}(h)) + \dim(\text{Im}(h)) \geq \dim(\text{Im}(f)) + \dim(\text{Ker}(h)) = \dim(B)$$
$$\dim(A) + \dim(C) = \dim(\text{Im}(f)) + \dim(\text{Ker}(f)) + \dim(\text{Ker}(h)) + \dim(\text{Im}(h)) \geq \dim(B)$$
$$\dim(A) + \dim(C) \geq \dim(B).$$

□

Theorem 5.18. (Smith-Floyd I).

Let G be a cyclic group of order p prime and let X be a regular finite dimensional simplicial G-complex such that all the homology groups with coefficients in $\mathbb{Z}_p$ are finite dimensional. For some $1 \leq i \leq p-1$, set $\rho = \tau^i$. Then all the Smith's homology groups $H_k^\rho(X;\mathbb{Z}_p)$ are finite-dimensional. In addition for $n \geq 0$, we have:

$$\sum_{k=n}^{\infty} \dim(H_k(X^G;\mathbb{Z}_p)) \leq \left(\sum_{k=n}^{\infty} \dim(H_k(X;\mathbb{Z}_p))\right) - \dim(H_n^\rho(X;\mathbb{Z}_p)). \tag{5.2}$$

In particular, all the $H_k(X^G;\mathbb{Z}_p)$ homology groups are finite-dimensional.

Proof. Proof taken from [12]. Let $\overline{\rho} = \tau^{p-i}$. For all k, by corollary 5.16, there is a long exact sequence containing in particular the following segment

$$H_{k+1}^\rho(X;\mathbb{Z}_p) \longrightarrow H_k^{\overline{\rho}}(X;\mathbb{Z}_p) \oplus H_k(X^G;\mathbb{Z}_p) \longrightarrow H_k(X;\mathbb{Z}_p). \tag{5.3}$$

Set

$$a_i = \dim(H_i^\rho(X;\mathbb{Z}_p)) \quad \text{and} \quad \overline{a}_i = \dim(H_i^{\overline{\rho}}(X;\mathbb{Z}_p)).$$

Now it is deduced from the short exact sequence (5.3) and from lemma 5.17 that

$$\overline{a}_k + \dim(H_k(X^G;\mathbb{Z}_p)) \leq a_{k+1} + \dim(H_k(X;\mathbb{Z}_p)). \tag{5.4}$$

Interchanging the roles of $\rho = \tau^i$ and $\overline{\rho} = \tau^{p-i}$ in corollary 5.16, it is obtained

$$a_k + \dim(H_k(X^G;\mathbb{Z}_p)) \leq \overline{a}_{k+1} + \dim(H_k(X;\mathbb{Z}_p)). \tag{5.5}$$

Let X be an N-dimensional simplicial complex, then for every $k \geq N+1$ it holds that

$$H_k^\rho(X;\mathbb{Z}_p) = H_k^{\overline{\rho}}(X;\mathbb{Z}_p) = 0.$$

For the case $k = N$ of (5.4) and (5.5)

$$\begin{aligned}\overline{a}_N + \dim(H_N(X^G;\mathbb{Z}_p)) &\leq \dim(H_N(X;\mathbb{Z}_p)) < \infty\\ a_N + \dim(H_N(X^G;\mathbb{Z}_p)) &\leq \dim(H_N(X;\mathbb{Z}_p)) < \infty,\end{aligned}$$

so $\overline{a}_N, a_N < \infty$. Now, for the case $k = N-1$ of (5.4) and (5.5)

$$\begin{aligned}\overline{a}_{N-1} + \dim(H_{N-1}(X^G;\mathbb{Z}_p)) &\leq a_N + \dim(H_{N-1}(X;\mathbb{Z}_p)) < \infty\\ a_{N-1} + \dim(H_{N-1}(X^G;\mathbb{Z}_p)) &\leq \overline{a}_N + \dim(H_{N-1}(X;\mathbb{Z}_p)) < \infty,\end{aligned}$$

thus $\overline{a}_{N-1}, a_{N-1} < \infty$. Going backwards it is seen that $\overline{a}_k, a_k < \infty$ for every k as claimed in the first conclusion of the proposition.

Now, for the second conclusion, rearranging (5.4) and (5.5)

$$\dim(H_k(X^G;\mathbb{Z}_p)) \leq a_{k+1} - \overline{a}_k + \dim(H_k(X;\mathbb{Z}_p)), \tag{5.6}$$

$$\dim(H_k(X^G;\mathbb{Z}_p)) \leq \overline{a}_{k+1} - a_k + \dim(H_k(X;\mathbb{Z}_p)). \tag{5.7}$$

Using (5.6) in an alternate way

$$\sum_{k=n}^{N} \dim(H_k(X^G;\mathbb{Z}_p)) \leq \sum_{k=n}^{N} \dim(H_k(X;\mathbb{Z}_p)) + (a_{n+1} - \overline{a}_n) + (\overline{a}_{n+2} - a_{n+1}) +$$
$$+ (a_{n+3} - \overline{a}_{n+2}) + \cdots + (\overline{a}_{N+1} - a_N)$$

it results a telescopic sum that eventually turns out into

$$\sum_{k=n}^{N} \dim(H_k(X^G;\mathbb{Z}_p)) \leq \sum_{k=n}^{N} \dim(H_k(X;\mathbb{Z}_p)) + \overline{a}_{N+1} - \overline{a}_n,$$

or into

$$\sum_{k=n}^{N} \dim(H_k(X^G;\mathbb{Z}_p)) \leq \sum_{k=n}^{N} \dim(H_k(X;\mathbb{Z}_p)) + a_{N+1} - \overline{a}_n.$$

Anyway, since X is N-dimensional, both terms $\overline{a}_{N+1} = a_{N+1} = 0$, thus

$$\sum_{k=n}^{N} \dim(H_k(X^G;\mathbb{Z}_p)) \leq \sum_{k=n}^{N} \dim(H_k(X;\mathbb{Z}_p)) - \overline{a}_n, \tag{5.8}$$

as the second conclusion claimed. □

Corollary 5.19. (Smith-Floyd inequality mod p.). *For any $p \in \mathbb{N}$:*

$$\sum_{k=0}^{\infty} \dim(H_k(X^G;\mathbb{Z}_p)) \leq \sum_{k=0}^{\infty} \dim(H_k(X;\mathbb{Z}_p)).$$

Corollary 5.20. (Harnack's inequality generalization). Result from [15].

Consider a smooth complex algebraic manifold Γ and its fixed space under the conjugate map i.e. $\mathbb{R}\Gamma$ obtained like in the last chapter, then the Smith's inequality above 5.19 generalizes Harnack's theorem 2.7.

$$\sum_{i=0}^{\infty} \dim(H_i(\mathbb{R}\Gamma;\mathbb{Z}_2)) \leq \sum_{i=0}^{\infty} \dim(H_i(\Gamma;\mathbb{Z}_2)).$$

Proof. Let's consider Γ a complex non-singular projective curve and $\mathbb{R}\Gamma$ its real part. It will be seek to arrive to Harnack's inequality using Smith's inequality.

The dimension of the group $H_0(X;\mathbb{Z}_2)$ is the number of connected components, say L. This holds since $H_0(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$ for every X being a path-connected topological space (see appendix B and C). By Hurewic's theorem C.20, every oval in the real projective plane X_i for $i = 1, \ldots, L$ sattisfies that

$$\mathbb{Z}_2 \cong \pi_1(X_i;\mathbb{Z}_2) \cong H_1(X_i;\mathbb{Z}_2)$$

and by the equality (C.1), $H_1(\mathbb{R}\Gamma;\mathbb{Z}_2) = \bigoplus_{\alpha=1}^{L} H_1(X_\alpha) = \mathbb{Z}_2^L$. Thus

$$\dim(H_0(\mathbb{R}\Gamma;\mathbb{Z}_2)) + \dim(H_1(\mathbb{R}\Gamma;\mathbb{Z}_2)) = L + L = 2L.$$

Now, go back to the complex curve Γ of degree d, then its genus is $g(d) = \frac{(d-1)(d-2)}{2}$. Thus topologically is just like a $g(d)$-torus 5.1. Every n-torus is path connected, thus $H_0(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$, and using Hurewic's C.20 together with the multiplicative property of π_1, it holds that $\pi_1(\Gamma;\mathbb{Z}_2) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \overset{g(d)}{\cdots} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \cong H_1(\Gamma;\mathbb{Z}_2)$. Since there is only one 2-cell in its identification space of the $g(d)$-torus 5.2, it results that $H_2(\Gamma;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

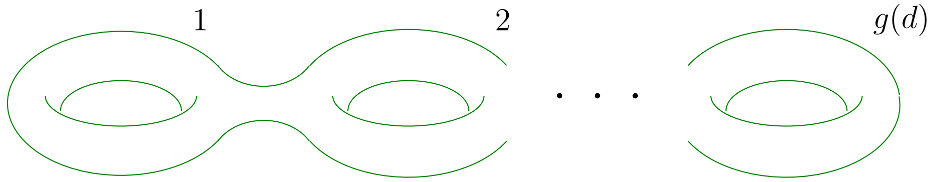


Figure 5.1: $g(d)$-torus.

Now computing the dimensions

$$\dim(H_0(\Gamma;\mathbb{Z}_2)) + \dim(H_1(\Gamma;\mathbb{Z}_2)) + \dim(H_2(\Gamma;\mathbb{Z}_2)) = 1 + 2g(d) + 1 = 2 + 2g(d)$$

and thus by corollary 5.19:

$$\sum_{i=0}^{1} \dim(H_i(\mathbb{R}\Gamma;\mathbb{Z}_2)) \leq \sum_{i=0}^{\infty} \dim(H_i(\Gamma;\mathbb{Z}_2)) \implies 2L \leq 2 + 2g(d) \implies L \leq 1 + g(d).$$

In the last step it has been used again that the dimension of the identification space of a $g(d)$-torus is up to dimension 2, thus

$$\sum_{i=0}^{\infty} \dim(H_i(\Gamma;\mathbb{Z}_2)) = \sum_{i=0}^{2} \dim(H_i(\Gamma;\mathbb{Z}_2)).$$

It is drawed the identification space 5.2 of a $g(d)$-torus:

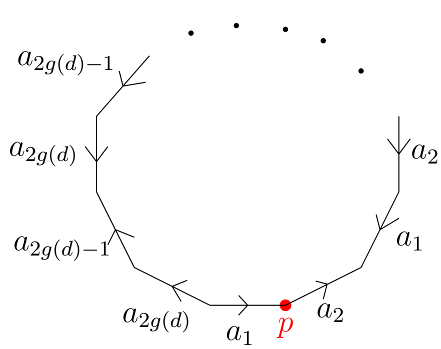


Figure 5.2: Identification space of a $g(d)$-torus.

□

Definition 5.21. *(Euler characteristic with coefficients).*

Let X be a complex, then it is defined the **Euler characteristic with coefficients** to

$$\mathcal{X}(X;\mathbb{Z}_p) = \sum_{n=0}^{\infty}(-1)^n \dim(H_n(X;\mathbb{Z}_p)).$$

Theorem 5.22. (Smith-Floyd II).

Let G be a cyclic group of order p prime and let X be a regular finite dimensional simplicial G-complex such that all the homology groups with coefficients in $\mathbb{Z}_p$ are finite dimensional (by the last theorem 5.18, it also holds that $H_k(X^G;\mathbb{Z}_p)$ homology groups are finite-dimensional). Thus

$$\mathcal{X}(X^G;\mathbb{Z}_p) \equiv \mathcal{X}(X;\mathbb{Z}_p)(mod\ p).$$

Proof. Taken from[2] and [12].

Let $\tau = 1 - g \in \mathbb{Z}_pG$, with $g \in G$ being a generator of the cyclic group G as usual. By the last theorem 5.18 all the Smith special homology groups $H_k^{\tau^i}(X)$ are finite-dimensional and since X is finite-dimensional itself only finitely many of them are non-zero, making also finite its Euler characteristic

$$\mathcal{X}^{\tau^i}(X;\mathbb{Z}_p) = \sum_{k=0}^{\infty}(-1)^k \dim(H_k^{\tau^i}(X;\mathbb{Z}_p)).$$

In particular, for $\rho = \tau$ in corollary 5.16 it is given the following long exact sequence

$$\cdots \to H_k^{\tau^{p-1}}(X) \oplus H_k(X^G) \xrightarrow{i} H_k(X) \xrightarrow{\rho=\tau} H_k^{\tau}(X) \to H_{k-1}^{\tau^{p-1}}(X) \oplus H_{k-1}(X^G) \to \cdots$$

Using that i is a double inclusion and thus injective and that $\rho = \tau$ is surjective, by last lemma 5.11 it is deduced that

$$\mathcal{X}(X;\mathbb{Z}_p) = \mathcal{X}^{\tau}(X;\mathbb{Z}_p) + \mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p) + \mathcal{X}(X^G;\mathbb{Z}_p).$$

Thus to see that $\mathcal{X}(X;\mathbb{Z}_p)$ and $\mathcal{X}(X^G;\mathbb{Z}_p)$ are equal modulo p, it is enough to prove that

$$\mathcal{X}^{\tau}(X;\mathbb{Z}_p) + \mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p) \equiv 0 \ (\text{mod } p).$$

In order to see that last statement it is necessary to prove for every $1 \leq i \leq p-1$ that

$$0 \to \tau^{p-1}C_k(X) \xrightarrow{\sigma} \tau^i C_k(X) \xrightarrow{\tau} \tau^{i+1}C_k(X) \to 0 \tag{5.9}$$

is a short exact sequence. Notice that it is well defined because for every $1 \leq i \leq p-1$ it holds that $\sigma \in \tau^i \cdot \mathbb{Z}_pG$. In addition τ is clearly surjective and $\tau^{p-1} = \sigma$ (lemma 5.8) injective by the definition of σ. By lemma 5.7 also Im(σ)=Ker(τ), thus (5.9) holds. In conclusion, for $1 \leq i \leq p-1$ it follows that

$$\cdots \to H_{k+1}^{\tau^{i+1}}(X;\mathbb{Z}_p) \to H_k^{\tau^{p-1}}(X;\mathbb{Z}_p) \to H_k^{\tau^i}(X;\mathbb{Z}_p) \to H_k^{\tau^{i+1}}(X;\mathbb{Z}_p) \to H_{k-1}^{\tau^{p-1}}(X;\mathbb{Z}_p) \to \cdots$$

and using lemma 5.11 it is obtained:

$$\mathcal{X}^{\tau^i}(X;\mathbb{Z}_p) = \mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p) + \mathcal{X}^{\tau^{i+1}}(X;\mathbb{Z}_p).$$

Thus, since $\tau^p = 0$:

$$\sum_{i=1}^{p-1} \mathcal{X}^{\tau^i}(X;\mathbb{Z}_p) = \left(\sum_{i=1}^{p-2}(\mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p) + \mathcal{X}^{\tau^{i+1}}(X;\mathbb{Z}_p))\right) + \mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p)$$
$$= \left(\sum_{i=2}^{p-1} \mathcal{X}^{\tau^i}(X;\mathbb{Z}_p)\right) + (p-1)\mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p).$$

Reordering the summands:

$$\mathcal{X}^{\tau}(X;\mathbb{Z}_p) = (p-1)\mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p)$$
$$\mathcal{X}^{\tau}(X;\mathbb{Z}_p) + \mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p) = p \cdot \mathcal{X}^{\tau^{p-1}}(X;\mathbb{Z}_p).$$

Then

$$\mathcal{X}(X^G;\mathbb{Z}_p) \equiv \mathcal{X}(X;\mathbb{Z}_p)(\text{mod } p).$$

□

Corollary 5.23. (Smith-Floyd II).
Consider a smooth complex algebraic manifold Γ *and its fixed space under the conjugate map i.e.* $\mathbb{R}\Gamma$*, by theorem 5.22 it holds that* $\mathcal{X}(\mathbb{R}\Gamma;\mathbb{Z}_2)$ *and* $\mathcal{X}(\Gamma;\mathbb{Z}_2)$ *have the same parity.*

Definition 5.24. *(Mod-p acyclic).*
The simplicial complex X is said to be **mod-p acyclic** if

$$H_k(X;\mathbb{Z}_p) = \left\{ \begin{array}{cc} \mathbb{Z}_p & \text{if } k=0 \\ 0 & \text{else} \end{array} \right\}.$$

Definition 5.25. *(Mod-p homology n-sphere).*
The simplicial complex X is said to be **mod-p homology n-sphere** if

$$H_k(X;\mathbb{Z}_p) = \left\{ \begin{array}{cc} \mathbb{Z}_p & \text{if } k=0 \\ \mathbb{Z}_p & \text{if } k=n \\ 0 & \text{else} \end{array} \right\}.$$

Corollary 5.26. (Smith-Floyd).
Let p *be a prime, let* G *be a finite ciclic group of order* p *and let* X *be a finite-dimensional simplicial* G*-complex. Then*

1. *If* X *is mod-p acyclic, then so is* X^G*. In particular,* X^G *is non-empty.*
2. *If* X *is a mod-p homology* n*-sphere, then* X^G *is either empty or a mod-p homology* m*-sphere for some* $0 \le m \le n$*.*

Proof. Proof taken from [12].

1. Since X is mod-p acyclic then by theorem 5.18

$$\sum_{k=0}^{\infty} \dim(H_k(X^G;\mathbb{Z}_p)) \le \sum_{k=0}^{\infty} \dim(H_k(X;\mathbb{Z}_p)) = 1.$$

Since X is path connected, either X^G is empty or X^G is mod-p acyclic. But by theorem 5.22

$$\mathcal{X}(X^G;\mathbb{Z}_p) \equiv \mathcal{X}(X;\mathbb{Z}_p)(\text{mod } p) \equiv 1(\text{mod } p),$$

thus X^G is mod-p acyclic.

2. Now let X be a mod-p homology n-sphere. By theorem 5.18, it holds that

$$\sum_{k=0}^{\infty} \dim(H_k(X^G;\mathbb{Z}_p)) \leq \sum_{k=0}^{\infty} H_k(X;\mathbb{Z}_p) = 2.$$

Then the left hand side is $0, 1$ or 2. If at the left there is a 0, then it would be the case when X^G is empty. If in the left there is a 1 it necessarily would be due to the 0^{th} homology group, so $\mathcal{X}(X^G) = 1$, which contradicts theorem 5.22 since

$$\mathcal{X}(X^G) \equiv \mathcal{X}(X)(\text{mod } p) = \left\{ \begin{array}{ll} 2 & \text{if } n = 2\lambda \\ 0 & \text{if } n = 2\lambda + 1 \end{array} \right\}$$

for any $\lambda \in \mathbb{N}$, thus it can be discarded the case when is 1.

Then let the left hand side be 2, so X^G is a mod-p homology m-sphere for one $m \in \mathbb{N}$. Now there is only left to check that $0 \leq m < n$, thus applying theorem 5.18

$$\sum_{k=n+1}^{\infty} \dim(H_k(X^G;\mathbb{Z}_p)) \leq \sum_{k=n+1}^{\infty} \dim(H_k(X;\mathbb{Z}_p)) = 0,$$

then $H_k(X^G;\mathbb{Z}_p) = 0$ for all $k \geq n+1$, thus in this case X^G is a mod-p homology m-sphere for $0 \leq m \leq n$.

□

APPENDIX A

Vector fields

Definition A.1. *(Manifold).*

A set $M \subset \mathbb{R}^n$ is called a **manifold** if every point $x \in M$ has an open set U in M s.t. is homeomorphic to an open set V inside $\mathbb{R}^m$. It will be denoted a **parametrization** to

$$\wp = (x_1, \dots, x_n) : V \longrightarrow U \subset M,$$

and a **coordinate system** of M to the following inverse map

$$\varphi = (\varphi_1, \dots, \varphi_m) : U \longrightarrow V \subset \mathbb{R}^m.$$

Definition A.2. A vector field on a manifold M is a continuous correspondence that assgins to each point $p \in M$ a tangent vector in T_pM i.e.

$$\begin{aligned}\mathfrak{X} : M &\longrightarrow \mathrm{TM}\\ p &\longrightarrow \mathfrak{X}(p) = (p, X(p)) \in M \times T_pM\end{aligned}$$

where TM denotes the tangent bundle. It shall be refered to $X : U \longrightarrow \mathbb{R}^n$ as **vector field**.

Definition A.3. *(Coordinate vector fields).*

Using the same notation above, given a coordinate system $\varphi : U \longrightarrow \mathbb{R}^m$ it can be defined the following special vector fields

$$\begin{aligned}\frac{\partial}{\partial x_i} : U &\longrightarrow \mathbb{R}^n\\ p &\longmapsto \frac{\partial}{\partial x_i}(p) := \frac{\partial \wp}{\partial x_i}(\varphi(p)).\end{aligned}$$

Every vector field in U can be defined then as $X(p) = \sum_{i=1}^{m} X_i(p) \frac{\partial}{\partial x_i}(p)$.

Definition A.4. A continuous curve $c : \mathbb{R} \longrightarrow M$ is called an **integral curve** of a vector field X iff for every $t \in \mathbb{R}$ it sattisfies that

$$X(c(t)) = c'(t).$$

APPENDIX B

Algebraic topology

Definition B.1. *(Homotopy).*
Let X and Y be topological spaces and let $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ be continuous applications. It is said that f is a **homotopy** to g if there exists a continuous map $H : X \times [0,1] \longrightarrow Y$ s.t.

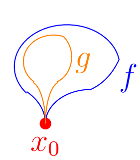
$$H(x,0) = f(x)$$
$$H(x,1) = g(x).$$

Note. *As notation it is used $f \simeq g$ and this homotopy relation is in fact an equivalence relation.*

Definition B.2. *(Homotopically equivalent).*
Let X and Y be topological spaces. It is said that X and Y are **homotopically equivalent** if there exist a continuous map $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ s.t. $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$ where 1_X and 1_Y denotes respectively the identity maps in the topological spaces X and Y.

Definition B.3. *(Loop).*
Let X be a topological space and let $x_0 \in X$. It is called a **loop** in X with basepoint on x_0 to every $f : [0,1] \longrightarrow X$ continuous map that $f(0) = f(1) = x_0$. Two loops are said to be **homotopic** if $f \simeq_{[0,1]} g$ where $\simeq_{[0,1]}$ denotes that $f(0) = g(0) = f(1) = g(1) = x_0$ i.e. f and g are loops with the same basepoint $x_0 \in X$ and this relation, is an equivalence relation.

Example. *Let $X = \mathbb{R}^2$, then $f \simeq_{[0,1]} g$ can be visualised as:*



Note. *The set $\pi_1(X, x_0)$ denotes the quotient set of the loops of basepoint $x_0 \in X$ with respect to the homotopy relation between loops defined before i.e. $\simeq_{[0,1]}$. The quotient class of any loop f is denoted as $[f]$. This notions and some of the definitions above can be found in [11].*

Definition B.4. *(Product of two loops).*

Let X be a topological space and let f and g be two loops in the topological space X. It is called **product** of the loops f and g to:

$$(f * g)(t) = \left\{ \begin{array}{cc} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{array} \right\}$$

Lemma B.5. *Let $[f], [g] \in \pi_1(X, x_0)$, then there is a well defined path class product*

$$[f]*[g] = [f * g].$$

Proof. Taking another representant of the equivalence class of f and lets call it f' i.e. $[f] = [f']$, and the same for g and lets call it g' ($[g] = [g']$). All f, f', g, g' are loops with base in x_0.

Consider

$$H_f : [0,1] \times [0,1] \longrightarrow X$$

the homotopy map relative to f to f', and

$$H_g : [0,1] \times [0,1] \longrightarrow X$$

the homotopy map relative to g to g'. Then applying the definition of the product of two loops $*$ for the first coordinate of H_f and H_g:

$$(H_f * H_g)(t, t') = \left\{ \begin{array}{cc} H_f(2t, t'), & 0 \leq t \leq \frac{1}{2} \\ H_g(2t-1, t'), & \frac{1}{2} \leq t \leq 1 \end{array} \right\}$$

and this construction is actually a homotopy between $(f * g)$ and $(f' * g')$

$$(H_f * H_g)(t, 0) = \left\{ \begin{array}{cc} H_f(2t, 0) = f(2t), & 0 \leq t \leq \frac{1}{2} \\ H_g(2t-1, 0) = g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{array} \right\} = (f * g)(t)$$

$$(H_f * H_g)(t, 1) = \left\{ \begin{array}{cc} H_f(2t, 1) = f'(2t), & 0 \leq t \leq \frac{1}{2} \\ H_g(2t-1, 1) = g'(2t-1), & \frac{1}{2} \leq t \leq 1 \end{array} \right\} = (f' * g')(t)$$

and in conclusion $[f * g] = [f' * g']$, thus the product

$$[f]*[g] = [f * g]$$

is well-defined. □

Theorem B.6. (First fundamental group).

Let X be a topological space and $x_0 \in X$. Then $\pi_1(X, x_0)$ is a group with respect to the operation $*$ defined in the last lemma B.5.

Proof. The proof of this theorem can be found at [10] in page 371 in theorem 51.2. □

Definition B.7. Let $\gamma : [0,1] \longrightarrow X$ be a path in X from x_0 to x_1 i.e. $\gamma(0) = x_0$ and $\gamma(1) = x_1$ with γ being a continuous map. It is defined

$$\begin{aligned} \hat{\gamma} : \pi_1(X, x_0) &\longrightarrow \pi_1(X, x_1) \\ [f] &\longrightarrow [\overline{\gamma}] * [f] * [\gamma] \end{aligned}$$

where $\overline{\gamma}(t) = \gamma(1-t)$ i.e. the reverse path going from x_1 to x_0. It is well-defined since $[\overline{\gamma}] * [f] * [\gamma]$ is in $\pi_1(X, x_1)$.

Theorem B.8. *The map $\hat{\gamma}$ is a group isomorphism.*

Proof. The proof of this theorem can be found at [10] in page 377 in theorem 52.1. □

Corollary B.9. *If X is a path connected i.e. there is always a path between every 2 points and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

After this last corollary it makes sense to denote the first fundamental group of a path-connected topological space X as $\pi_1(X)$ instead of $\pi_1(X, x_0)$ for any $x_0 \in X$.

Definition B.10. *(Contractible).*

If any loop in a path-connected topological space X can be continuously retracted into its base point it is said that the space X is **contractible**. Thus $\pi_1(X)$ has only one element, so it must be the identity:

$$\pi_1(X) = e.$$

Definition B.11. Let $h : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map. Then it is defined the following map

$$\begin{aligned} h_* : \pi_1(X, x_0) &\longrightarrow \pi_1(Y, y_0) \\ [f] &\longmapsto h_*([f]) = [h \circ f] \end{aligned}$$

Theorem B.12. *If X and Y are homeomorphic and path-connected, then $h_* : \pi_1(X) \longrightarrow \pi_1(Y)$ it is an isomorphism.*

Proof. The proof is in [10], results 52.4 and 52.5 in page 379. □

Theorem B.13. *(First fundamental group of a circumference $\mathbb{S}^1$).*

$$\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$$

Proof. This proof is in [10], theorem 54.5 in page 392. □

Theorem B.14. (Multiplicative property of the first fundamental group).

$\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. The proof is in [10], theorem 60.1 in page 421. □

Corollary B.15. *The first fundamental group of the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ is $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.*

APPENDIX C

Homology

Definition C.1. Let G be an abelian group. If $\{G_\alpha\}_{\alpha\in J}$ is a family of subgroups of G it is said that these groups **generate** G if every element $x \in G$ can be written as a finite product of elements of the subgroups $\{G_\alpha\}_{\alpha\in J}$. This subgroups will be known as the basis of G.

This definition means that there is a finite sequence $(x_1, x_2, \dots, x_n)$ of elements of the groups G_α s.t. $x = x_1 \cdot x_2 \cdot \dots \cdot x_n$. Such a sequence is called a **word** of length n. Due to the commutativity it can be rearranged the factors in the expression for x to group together in $(x_1, x_2, \dots, x_n)$ the factors that belong to a single subgroups G_α of the set $\{G_\alpha\}_{\alpha\in J}$. Furthermore, if any $x_i = 1$ or it happens to be in a word x_i together with $x_j = x_i^{-1}$ it can be always be both deleted from the original word making it shorter. Applying all this reductions to the original word it is obtained what is called a **reduced word**.

Definition C.2. *(Free product).*

Let G be a group and $\{G_\alpha\}_{\alpha\in J}$ be a family of subgroups of G that generates G. Let also that $G_\alpha \cup G_\beta$ consist only on the identity element alone whenever $\alpha \neq \beta$. Then it is said that G is the **free product** of the groups G_α if for every $x \in G$ there is only one reduced word in the groups G_α that represents x. In this case it is written

$$G = G_1 * \dots * G_n.$$

Definition C.3. A **standard n-simplex** is defined as the following set

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n+1} t_i = 1, \quad t_i \leq 0\}.$$

Definition C.4. A **singular n-simplex** in a space X is a continuous map

$$\sigma : \Delta^n \longrightarrow X.$$

Definition C.5. *(N-chains).*

Let $C_n(X)$ denote the free abelian group with basis the set of singular n-simplices in X. The elements of $C_n(X)$ called **n-chains** or singular n-chains, are finite formal sums $\sum_i n_i\sigma_i$ where n_i is a coefficient of a field (as for example $\mathbb{Z}$ or $\mathbb{Z}_2$) and where $\sigma_i : \Delta^n \longrightarrow X$.

Definition C.6. *(Boundary map).*

The **boundary map** is defined as

$$\begin{aligned}\partial_n : C_n(X) &\longrightarrow C_{n-1}(X)\\ \sigma &\longmapsto \sum_i^n (-1)^i \cdot \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.\end{aligned}$$

Using this last definition it can be seen that $\partial_n \circ \partial_{n-1} = 0$, thus $\mathrm{Im}(\partial_{k+1}) \subset \mathrm{Ker}(\partial_k)$ and it is obtained the following **chain complex**

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots$$

Definition C.7. *(Homology group).*

It is defined the **n-singular homology group** as:

$$H_n(X) = \mathrm{Ker}(\partial_n) \Big/ \mathrm{Im}(\partial_{n+1}).$$

Proposition C.8. *Let X_α all the connected components which conform X, then*

$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha). \tag{C.1}$$

Proof. Since the n-simplices are path connected, then the singular simplex ($\sigma : \Delta^n \longrightarrow X$ being continuous) has a path connected image and in conclusion the image always fall in a single one of the path connected components, so $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$. Thus for every $\sigma_\alpha \in C_k(X_\alpha)$ for $k \in \mathbb{N}$, using the boundary map of the complex $(C_n(X_\alpha), \partial_n^\alpha)$ it holds for every α that $\partial_k^\alpha(\sigma_\alpha) \in C_{k-1}(X_\alpha)$. Defining the following map

$$\partial_k = \bigoplus_\alpha \partial_k^\alpha : \bigoplus_\alpha C_k(X_\alpha) \longrightarrow \bigoplus_\alpha C_{k-1}(X_\alpha)$$

it holds that $\mathrm{Ker}(\partial_k)=\bigoplus_\alpha \mathrm{Ker}(\partial_k^\alpha)$ and that $\mathrm{Im}(\partial_k)=\bigoplus_\alpha \mathrm{Im}(\partial_k^\alpha)$, thus

$$H_k(X) = \mathrm{Ker}(\partial_k) \Big/ \mathrm{Im}(\partial_{k+1}) = \bigoplus_\alpha \mathrm{Ker}(\partial_k^\alpha) \Big/ \bigoplus_\alpha \mathrm{Im}(\partial_{k+1}^\alpha) = \bigoplus_\alpha H_k(X_\alpha)$$

where the last equal holds since for every α and $k \in \mathbb{N}$

$$\mathrm{Im}(\partial_{k+1}^\alpha) \subset \mathrm{Ker}(\partial_k^\alpha).$$

$\square$

Proposition C.9. *If X is path connected then $H_0(X) \cong \mathbb{Z}$.*

Proof. The proof is in [7], proposition 2.7 in page 109. □

Corollary C.10. *$H_0(X)$ is a direct sum of $\mathbb{Z}$, one for each path component of X.*

Proof. Use the last 2 propositions C.8 and C.9. □

Definition C.11. Let $f : X \longrightarrow Y$ be a continuous map between 2 topological spaces, then a **map of chain complexes** is defined as the following $f_\sharp$

$$f_\sharp : C_n(X) \longrightarrow C_n(Y), \qquad \begin{tikzcd} \Delta^n \arrow[rr, "f_\sharp(\sigma)=f\circ\sigma"] \arrow[dr, "\sigma"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

defined for every $\sigma \in C_n(X)$ i.e. $\sigma : \Delta^n \longrightarrow X$ as $f_\sharp(\sigma) = f \circ \sigma : \Delta^n \longrightarrow Y$.

Let $(C_n(X), \partial^X)$ and $(C_n(Y), \partial^Y)$ denote the singular complexes with its boundaries of X and Y. In page 111 in [7] is explained that this 2 boundary maps are related by $f_\sharp \circ \partial^X = \partial^Y \circ f_\sharp$. It is deduced from here that $f_\sharp(\mathrm{Ker}\partial^X) \subseteq \mathrm{Ker}\partial^Y$ and $f_\sharp(\mathrm{Im}\partial^X) \subseteq \mathrm{Im}\partial^Y$. Then $f_\sharp$ induces a homomorphism

$$H_n(f) = f_* : H_n(X) \longrightarrow H_n(Y).$$

Theorem C.12. *If $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are homotopic maps, then for every $n \in \mathbb{N}$:*

$$H_n(f) = H_n(g) : H_n(X) \longrightarrow H_n(Y).$$

Proof. The proof is in [7], theorem 2.10 in page 111. □

Corollary C.13. *The map $f_* : H_n(X) \longrightarrow H_n(Y)$ induced by a homotopy equivalence map $f : X \longrightarrow Y$ is an isomorphism for all $n \in \mathbb{N}$, i.e. $H_n(X) \cong H_n(Y)$.*

Definition C.14. *(Exact sequence).*

A sequence of abelian groups $\{A_i\}_{i\in I}$ and homomorphisms $\{\alpha_i\}_{i\in I}$ such

$$\cdots \xrightarrow{\alpha_{n+2}} A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \cdots$$

is said to be **exact** at $\{A_i\}_{i\in I}$ if Ker(α_n) =Im(α_{n+1}).

Definition C.15. *(Short exact sequence).*

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Note. ***Exactness at A***: *implies that Ker(f) =Im(0) and since f is a group homomorphism Ker$(f) = 0 \iff f$ is injective.*

Exactness at B: *implies that Ker(g) =Im(f), i.e. $g \circ f = 0$.*

Exactness at C: *Im(g)=Ker$(0) = C$, then g is surjective.*

Definition C.16. Given a topological space X and a subspace $A \subset X$, lets define

$$C_n(X, A) = C_n(X) \big/ C_n(A),$$

thus any chain in A is trivial in $C_n(X, A)$. The boundary map $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, thus it induces a quotient boundary map

$$\partial : C_n(X, A) \longrightarrow C_{n-1}(X, A)$$

and with it, a complex $\{C_n(X, A), \partial\}$ which it's homology groups $H_n(X, A)$ are denoted as **relative homology groups**.

Lemma C.17. *Lets consider 3 chain complexes like these $\{A_n, \partial\}$, $\{B_n, \partial\}$ and $\{C_n, \partial\}$ also having that for every $n \in \mathbb{N}$ it holds the following exact sequence*

$$0 \longrightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \longrightarrow 0.$$

Then the map $\partial : H_n(C) \longrightarrow H_{n-1}(A)$ is well-defined and it is an homomorphism.

Proof. The idea of this proof is in [7], page 116. □

Theorem C.18. *The following sequence of homology groups*

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} \cdots$$

is exact.

Proof. This proof is in theorem 2.16 in [7]. □

It is possible to compute the n-homology group $H_n(X)$ when X is a cell complex using what is called **celullar homology**. First are defined the n-chains. Let X be a cell complex, then we define $C_n^{CW}(X)$ as the free abelian group with basis the set of n-cells of X, this is denoted as

$$C_n^{CW}(X) = \langle n\text{-cells of } X\rangle.$$

With this chains it is possible to find a boundary map ∂ as it is explained in section 'Cellular homology' at page 137 on [7]. Thus there is a complex chain $\{C^{CW}(X), \partial\}$ and its n-homology group is denoted as $H_n^{CW}(X)$. In fact, it holds:

Theorem C.19.

$$H_n^{CW}(X) \cong H_n(X).$$

Proof. This proof is in theorem 2.35 at page 139 in [7]. □

Theorem C.20. (Hurewicz's theorem).
If X is a path-connected topological space, then

$$H_1(X) \cong \pi_1(X)^{ab}$$

where $\pi_1(X)^{ab}$ is the abelianization of the group $\pi_1(X)$.

Proof. This result can be found in [7]. □

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