Using pathologies as starting points for inquiry-based mathematics education: The case of the palindrome

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Inquiry-based mathematics education (IBME) is an increasingly important ingredient of the mathematics education in the Nordic countries. The central principle of IBME is that the students are to work in ways similar to how professional mathematicians work. In this qualitative case study, we investigate whether mathematical pathologies induce students to work like mathematicians, and thus if pathologies are suitable starting points for IBME. We based our investigations on a little-known pathology: multiplication problems that can be mirrored about the equal sign without altering the answer.

Keywords: Inquiry-based mathematics education, mathematical pathologies, palindromes, mirror multiplication.

The national curricula are essential to the so-called Nordic model of education, and in the new national mathematics curriculum of Norway the Norwegian equivalent of the term *inquiry* is used repeatedly (69 times, to be exact). This is an indication of the rising popularity of IBME: a form of teaching whose guiding principle is that the students are supposed to work in ways similar to how professional mathematicians work (Artigue & Blomhøj, 2013; Council, 2000; Dorier & Maass, 2014). But what kind of tasks can be relied upon to impose questions on the student and thus form suitable starting points for IBME? In this paper, we suggest that one answer to that question lies in the realm of pathological objects.

A mathematical pathology is “an example specifically cooked up to violate certain almost universally valid properties.”[1] The history of mathematics abounds with pathological objects, many of which were of crucial importance to the development of one branch or another. Sriraman and Dickman (2017) argue that pathologies can be pathways to divergent thinking and creativity. The chief pedagogical value of pathologies, they claim, lies in

> their ability to challenge our perception and intuition, and the resulting advances as we equilibrate. Whether dealing with a space-filling Peano curve in the plane or a misapplied cancellation law sporadically holding among fractions, we are discontent to remain stationary in response to these phenomena. Instead, we mobilize and investigate how to reconcile our worldviews with the novel pathologies now recognized as co-occupying the same mathematical spaces in which we had long operated. (p. 141)

It seems reasonable to expect that these same question-inducing qualities make pathologies promising starting points for inquiry. In this qualitative case study, we investigate *whether mathematical pathologies are suitable as starting points for inquiry-based mathematics teaching*. To this end, we developed and tried an inquiry-activity around the little-known pathology of multiplication problems like $12 * 63$ that can be reversed ($36 * 21$) without altering the answer. The

[1] http://mathworld.wolfram.com/Pathological.html

main purpose of our study is to draw attention to a hitherto perhaps underused resource in mathematics education: the mathematical pathology.

Our main concern shall be whether using pathologies as starting points for inquiry induces students to work in ways similar to professional mathematicians. A source of information on how mathematicians work is introspective accounts from professional mathematicians, and we shall largely rely on such in our discussion.

Methods

The following research question was addressed in this study: *Are mathematical pathologies suitable as starting points for inquiry-based mathematics teaching?* The study followed a qualitative case study design.

Pathological palindromes

It is a little-known fact that for certain multiplication problems one may mirror the digits about the equal sign without altering the answer. Thus, while it is to be expected that $12 * 63 = 63 * 12$, it is more surprising that we also have $12 * 63 = 36 * 21$. To the best of our knowledge, the only previously published paper on this pathology in English is Manheim (1979), where (amongst other things) the author identifies all such two-digit "palindromes". The most straightforward way of investigating the palindromes is through algebraizing:

$$(10a + b)(10c + d) = (10d + c)(10b + a).$$

This leads to the necessary and sufficient criterion for being a palindrome that $ac = bd$. For more on the palindromes see Manheim (1979) or Roksvold (2018).

Participants and setting

The participants in this study were convenience sampled and were either teacher students (n=40) or upper secondary school students (n=7). Two of the upper secondary students are referred to by pseudonyms (Alma and Leo) as they were equipped with a GoPro camera and thus followed more closely. Upper secondary and teacher students are different in many respects, but we do not think that these differences are of much consequence to our research question. Accordingly, such differences shall not play a part in the discussion, and we shall occasionally refer to the *students* or the *participants* without further specification.

The teacher students all took a course that the first author instructed. The upper secondary students were recruited through their teacher, who is a participant in the SUM-project (Haavold & Blomhøj, in press), in which both authors are involved as researchers. The upper secondary students were in their last year of school, in what would correspond to grade 12 in the K12 system (it is the 13th school year in Norway).[2]

The first two sessions, involving the teacher students, took place at a university in Norway. The third session, involving the upper secondary school students, took place at an upper secondary school in Norway.

Procedure

All three sessions followed the same basic outline, and each had a duration of 1-2 hours. First, the students were divided into groups of 2-4. Then, the instructor wrote $12 * 63$ in the upper left corner of the blackboard (or whiteboard) and $36 * 21$ in the lower right corner. It was up to the students to take it from there; no further instruction was given, no question was posed, and no direction was suggested.

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[2] The students had R2-mathematics, which is geared towards future studies in the "hard sciences".

We recorded audio from four of the groups of teacher students. In addition, these sessions were observed by the second author, and we collected the teacher students' worksheets. We recorded audio from all three groups of upper secondary students. In addition, their classroom was filmed, and one of the upper secondary students (Leo) had a GoPro camera attached to his head.[3]

Results

An inductive content analysis (in accordance with Elo & Kyngas, 2008) was performed on the entire corpus of data. The audio and video recordings were transcribed and coded; the codes were then distilled into content-related themes that each relate to the research question. These themes were *engagement and familiarization through examples; collaboration, contingencies and alternative solutions;* and *conjectures and refutations.*

Engagement and familiarization through examples

The teacher's task in the initiation phase is restricted to somehow displaying $12 * 63$ and $36 * 21$ (preferably with good spacing between them). Some of the students initially seemed unsure about what was probably an unfamiliar situation:

Teacher student 1: What are we supposed to do?

Any hesitation was typically short lived, however, as the two multiplication problems triggered the participants' curiosity:

Teacher student 2: Ok, so these are opposite numbers. I mean … it's the same numbers only in reverse order. We should just calculate both and find the answers.

Seeing that the multiplication problems are mirror images of each other *and* that they have the same answer raised some questions:

Teacher student 3: We should see if this is true for other numbers as well. Two-digit ones. Let's try 28 times 73." And a bit later on: "This is not right; it's not the same.

Teacher student 4: So then we know it's not true in general …

Early on in all three sessions, the students discovered palindromes such as $11 * 33 = 33 * 11$ and $27 * 72 = 27 * 72$ and classified these as special cases fundamentally distinct from the initial example $12 * 63 = 36 * 21$. Having identified these degenerate cases, the students typically continued either by trying to find more non-trivial examples (by chance, even) or by trying to detect some underlying mechanism or pattern in the examples already at hand. Several of the groups focused on factorization and primes, suspecting that a multiplication problem being a palindrome (the students typically did not use this word) had something to do with the factors involved.

Collaboration

The participants of a session collaborated across the groups of 2-4 by sharing new palindromes, approaches or observations with the rest of the "class". New palindromes would help other groups by enabling them to further test their conjectures. The following discussion, between upper secondary students Alma and Leo, illustrates a typical case of the collaboration that took place *within* the groups:

[3] It was our experience that the unobtrusiveness of these cameras made participants less self-conscious about being filmed. Another methodological advantage that the cameras offer is bringing the researcher close to the action, thus perhaps increasing their chances of noticing details. Being attached to the forehead, the camera also conveys the direction of focus of its wearer.

Alma: We need to look for what these [palindromes] have in common.

And, after some trying and failing:

Alma: I think we need to start with one number and then multiply that number with four, for example. If you see what I mean …?

She then explained her idea algebraically by writing $a * 4b = 4a * b$ on the whiteboard and said that they needed to "insert numbers a and b."

Leo: What if we just insert something, like a equals fifteen … well, that becomes fifty-one, so that's wrong …

Alma: But we still need the answers to equal each other after the multiplications are mirrored.

Leo: If it's supposed to be four, then we can't have an odd factor.

Alma: Yes, I agree.

Leo: What if we take twenty-four … well, if we mirror that, it's forty-two, so that's no good …

Alma: But it doesn't have to be four [pointing at the equation $a * 4b = 4a * b$], it can also be two here, see? And twenty-four times two …

Contingencies and alternative solutions

The authors, who thought they knew these palindromes, were faced with three surprises. First and foremost, one of the teacher students discovered a new layer of hidden symmetry: In the palindrome $12 * 63 = 36 * 21$, the differences $63 - 12 = 51$ and $36 - 21 = 15$ are mirror images of each other. The students were clearly satisfied with having found something that the teacher (in this case the first author) did not even know was there, and proceeded by inquiring: "Is this always the case?" and, conversely, "Does having this symmetry of differences imply that the multiplication problem is a palindrome?" Testing these newly formed hypotheses on other examples (previously found by the groups' combined efforts), the teacher students soon discovered that the double layer of symmetry unfortunately fails in the case $63 * 24 = 42 * 36$.

Second, a group of teacher students found empirically that the equality $a : d = b : c$ seems to always hold and realized that this should enable them to produce new palindromes. The same students did not proceed to prove the equality algebraically, but it is tempting to suggest that this was a question of having too little time. The relation is equivalent to the product of the "ones" being equal to the product of the "tens".

Third, shortly after the discussion that we reproduced in the previous subsection Alma seemed to have an epiphany and eagerly wrote 12 followed by a multiplication sign on the whiteboard while exclaiming "Twelve! Twelve is a good number." Directly underneath the number twelve she writes its double, 24, along with another multiplication sign on its right. She then goes back up again and writes 42 – the mirror image of 24 – on the right side of the equal sign. Finally, underneath 42 and to the right of the lower multiplication sign she writes 21, the half of 42.

Alma: This should work!

She starts calculating $12 * 42$ and $21 * 24$ on the blackboard, using the standard algorithm.

Leo: But do we have a general rule for this?

Alma: Ha, ha. No! But look, it works … twelve times forty-two is equal to twenty-four times twenty-one!

Leo: We found a pattern …

Alma: This should work on other numbers too. Let's try with …

At this point, the teacher initiated a class discussion summarizing the class' findings so far. Alma's discovery was unfortunately not included in the summary, as she seemed reluctant to share it. That was a pity, because she had (with the help of Leo) just discovered a simple and game-like algorithm that was unknown to us:

1. Start with a number, let's say 41,
2. Double it: 82.
3. Then mirror the result: 28.
4. And finally, divide this last number by two: 14.

It turns out; you have just found the palindrome $41 * 28 = 82 * 14$. If the students or the teacher had fully realized this, and had been given more time, it would have been very much in the spirit of IBME to pursue the matter further: "Can you start with any number and end up with a palindrome?" (No.) "Do we get all two-digit palindromes this way?" (Sort of.[4]) And, moreover, "Why does it work?"

Conjectures and refutations

Having seemingly forgotten their recent breakthrough, or not recognized it as such, Alma and Leo began forming a series of conjectures based on the growing set of examples that the three groups of upper secondary students had found between them. They tested these conjectures against the examples, and the trials that did not end well led to the formation of a new conjecture through a modification of the old one.

The first in this series of conjectures was that *for palindromes the digit sums on each side of the equal sign must be the same*. This initial conjecture was not long-lived.

Leo: But … if we are to be completely honest with ourselves … if we just take any product and mirror it, let's say thirty-four times fifty-one and its mirror image fifteen times forty-three, these are not equal to each other, but the digit sums are still the same.

Adapting to this temporary setback and motivated by their own previously found example $12 * 42 = 24 * 21$, they form a new conjecture: *The digit sum of one of the numbers must equal three while the other one's must be divisible by three*. They write this conjecture on the whiteboard and then go about trying to falsify it, by way of counterexamples.

Leo: But forty-eight times forty-two [an example discovered by one of the other groups] does not have any digit sum equal to three – it has digit sums six and twelve.

Alma: But those are both divisible by three!

Alma then alters their conjecture by adding the sign "%". The conjecture now reads: *Digit sums must be % 3* - meaning that all digit sums must be divisible by three. Leo then notices that in both $12 * 63 = 36 * 21$ and $42 * 12 = 21 * 24$ one of the numbers has 7 as a factor.

Leo: And it changes from one number to the other [i.e. the seven is not a factor of sixty-three mirrored, but rather of twelve mirrored]. This I believe is important!

Once again, they alter their conjecture, this time by appending *and there must be a 7 which changes place.* At this point they are interrupted, since one of the other two groups of upper secondary students has discovered a promising pattern which the teacher encourages them to share, namely that the

[4] To produce all but three of the 14 non-trivial two-digit palindromes one must sometimes multiply by 3 or 4 in step two of Alma's algorithm (and divide by that same number in step four). Permitting multiplication and division by 3/2 and 4/3 yields the final three as well.

product of the ones is equal to the product of the tens. Leo immediately starts calculating $13 * 93$, etc., on the whiteboard, and soon exclaims “Wow! That actually works!” and drops the calculator to the floor.

Discussion

The first finding was that the pathology seemed to be effective at triggering the students’ curiosity and led them to pose both questions and conjectures. Typically, the students’ initial investigation was a familiarization through examples. The groups tended to alternate their efforts between looking for new examples and searching for patterns in the examples at hand; this seems to also be the way of professional mathematicians. The late Hungarian mathematician Paul Halmos expressed it like this:

> A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one. (1985, p. 63)

According to Halmos, “the examples should include, whenever possible, the typical ones and the extreme degenerate ones” (1985, p. 62). In the case of two-digit palindromes, the extreme degenerate examples are palindromes such as $11 * 33 = 33 * 11$ and $74 * 47 = 74 * 47$, which participants of all three sessions found early on.

The examples that the groups had at hand at any given time were prodded in the search for different kinds of pattern. Some groups investigated patterns in the prime factorizations, other groups looked at patterns in the digit sums, others again considered patterns of ratio, and so on. The search for and description of patterns is such a significant part of the work of a professional mathematician that mathematics itself is often referred to as the “science of patterns” (see e.g. Devlin, 2000, pp. 7, 72). The British mathematician Andrew Wiles, famous for proving Fermat’s Last Theorem, once described this inductive phase of discovery in an interview:

> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is. […] I never use a computer. I sometimes might scribble. I do doodles. I start trying to find patterns, really.[5]

Students engaging the material and pursuing their own questions are two of the pillars of IBME (Dorier & Maass, 2014, pp. 301-302). A possible explanation as to *why* the palindromes triggered the participants’ investigative spirit is that pathologies in and of themselves pose questions and problems (thus making it less necessary for the teacher to do so); besides, there is something slightly provocative about a mathematical pathology, is it not?

The second finding was that all three sessions were characterized by near constant collaboration, both on a class-level across groups and within each group. This is reminiscent of two levels of collaboration found in professional mathematics.[6] The collaboration that took place on class-level mimics the collaboration of the mathematics community as a whole: When a group discovers and shares a new palindrome that fails to conform to another groups’ conjecture, it is similar to what is happening when a counterexample to a mathematician’s conjecture is found and published by an unknown colleague on the other side of the world. Likewise, the within-group collaboration mimics mathematicians knowingly collaborating on the proof of a theorem, for example at a congress.

[5] https://www.pbs.org/wgbh/nova/transcripts/2414proof.html

[6] For a description of the mathematics community see e.g. Davis, Hersh, and Marchisotto (2012, pp. 9-12). A personal account by a current mathematician is https://cameroncounts.wordpress.com/2009/10/28/collaboration-in-mathematics/.

Amongst professional mathematicians this type of direct collaboration is the rule rather than the exception. A vivid illustration of this is the collaboration graph,[7] the study of which has uncovered that in the period 1985-2009 the average number of mathematicians per paper was 1.75 (Brunson et al., 2014). Students collaborating is another pillar of IBME (Dorier & Maass, 2014, pp. 301-302).

The third finding was that several groups formulated conjectures, which they subsequently tested against the examples at hand. Representative of the collaboration that took place within the different groups was the chain of conjectures and refutations that Leo and Alma engaged in. Their dialogue is reminiscent of the fictitious classroom discussion on the Eulerian characteristic of polytopes found in Lakatos's classic *Proofs and Refutations*. Leo and Alma making repeated adjustments to their conjecture to rule out nasty counterexamples is precisely what Lakatos referred to as "monster-barring" – a phenomenon he saw as crucial to the progress and growth of informal mathematics.

On several occasions, students responded with enthusiasm to the ideas and findings of their fellow students – on their own group or otherwise. The learning environment deemed most suitable for IBME is one that values mistakes and contributions (Dorier & Maass, 2014, pp. 310-302). The professional mathematician's appreciation of the necessity of making mistakes is implicit in the words of George Pólya: "Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science" (1990, p. 117).

The fourth finding was that using pathologies as starting points is likely to increase the number of contingencies, especially so if the pathology is a peripheral one such as the palindromes. The problems most suitable for IBME are open problems with multiple solution strategies: problems that are experienced as real and/or scientifically relevant (Dorier & Maass, 2014, pp. 301-302). Contingencies, although slightly terrifying to the teacher, must be said to add authenticity to the students' mathematical experience. We suggest that students discovering something that the teacher did not know, as was the case on two different occasions with the palindromes, are the ones *most* likely to feel as if they are doing "real" science.

Limitations

We did not interview any of the students. How would *they* have described the way in which they tackled the palindromes? And did they perceive the palindromes as being – in some sense – pathological? Answers to these questions and others would have contributed a new perspective to our study.

Conclusions

Two conclusions can be drawn from this study. The first conclusion is that pathologies can form the basis of teaching that corresponds to the characteristics of IBME as these are described in Dorier and Maass (2014). Most importantly, the pathology induced students to pose and pursue their own questions.

The second conclusion is that pathologies seem to have qualities that help students work in ways similar to those of professional mathematicians. Thus, using pathologies as a starting point is one way to lend an aura of authenticity to the students' endeavor. The students are faced with something that simultaneously awakens their curiosity and causes a cognitive conflict, which they will try to equilibrate from.

The chief practical implication of these conclusions is that teachers doing IBME might have a useful and largely unmined resource in the mathematical pathology. *Useful* because what constitutes

[7] See https://oakland.edu/enp/ for information on the The Erdös Number Project and the collaboration graph.

a pathology partly depends upon one's mathematical sophistication (see Sriraman & Dickman, 2017, p. 138), which means that mathematical pathologies are plentiful at every level of mathematics.

On the theoretical level, this could call for an investigation into the scope of using pathologies in the inquiry-based teaching of specific topics; for example, it does not seem unreasonable to hope that IBME starting from a suitable pathology could play a role in dealing with specific student misconceptions. It is also possible that using pathologies as starting points for IBME can strengthen the students' cognitive flexibility by inducing them to circumscribe and (re)consider the characteristics of the pathology at hand, thereby providing an opportunity to overcome what Haylock (1997) refers to as *content-universe fixation.*

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