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Möbius and coboundary polynomials for matroids

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Abstract
We study how some coefficients of two-variable coboundary polynomials can be derived from Betti numbers of Stanley–Reisner rings. We also explain how the connection with these Stanley–Reisner rings forces the coefficients of the two-variable coboundary polynomials and Möbius polynomials to satisfy certain universal equations.

Keywords Matroids · Möbius polynomials · Coboundary polynomials · Betti numbers

Mathematics Subject Classification 05E45 · 94B05 · 05B35 · 13F55

1 Introduction

There are various ways to extract information about linear codes over finite fields. Many properties, like the length, dimension, generalized Hamming weights, and (generalized) weight spectra of a linear code are only dependent on the matroid(s) determined by the generator matrix (and the parity check matrix). The lattice of flats of the matroid derived from any generator matrix plays a key role in understanding properties of the code, and to this lattice one associates interesting two-variable polynomials; the Möbius polynomial and the coboundary polynomial. The coboundary polynomial determines the Tutte polynomial if the matroid coming from the generator matrix is simple, and it is well known how one may find much information about codes and matroids, knowing their Tutte polynomials. See for example [12] and [11]. Therefore it is interesting to investigate how one may find techniques

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that reveal the behaviour of these polynomials, in particular the two-variable coboundary polynomial.

In this article we show, using [15] and [8], how one can identify important coefficients of the Möbius polynomial with ungraded Betti numbers of the Stanley–Reisner ring of the matroid of a linear code, associated to its parity check matrices. If the minimum distance $d^{\perp}$ of the dual code is at least 3 (this is the same as saying that the generator matroid is simple), we also show how one can identify important coefficients of the coboundary polynomial with functions derived from Betti numbers of the Stanley–Reisner ring of the parity check matroid.

Regardless of whether the matroid of the generator matrices is simple or not, we use such identifications to derive "universal equations" that the coefficients of the coboundary polynomial must satisfy. Those are given in Theorem 36. To complete the picture we also list other such universal equations, both for the coefficients of the Möbius polynomial (Propositions 30 and 31), and the coboundary polynomial (Proposition 37). The identification of some coefficients of the Möbius polynomial with the mentioned ungraded Betti numbers provides a new way to prove Proposition 30. This is analogous to how one can identify other coefficients with dimensions of summands of relevant Orlik-Solomon algebras (see [13] and [2] to obtain information about such identifications, which we will not study in this paper). An important tool to give as many equations as possible, and to determine the coefficients in question, are the truncation formulas given in [3] , [11] and [12].

In general terms the purpose of this article is to demonstrate how Betti numbers of Stanley–Reisner rings, and invariants derived from them, constitute a natural ingredient in the theory of all the most commonly studied two-variable polynomials associated to codes and matroids. It might also be interesting in the future to investigate to what extent (q-)analogues of the results presented here also are valid for other types of codes, like Delsarte rank metric codes (even if the technical tools may be different).

2 Definitions and notation

2.1 Matroids

There are many equivalent definitions of a matroid. We refer to [14] for a deeper study of the theory of matroids.

Definition 1 A matroid is a pair (E, r) where E is a finite set and $r : 2^E \to \mathbb{N}$ is a function satisfying:

(R1) If $X \subset E$, then $0 \leq r(X) \leq |X|$,
(R2) If $X \subset Y$, then $r(X) \leq r(Y)$,
(R3) If X, Y are subsets of E, then

$$r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y).$$

The rank of a matroid $M = (E, r)$ is $r(E)$. The nullity function of the matroid is the function $n(X) = |X| - r(X)$. By (R1), this is a integer-valued non-negative function on 2^E.

In this paper, some subsets of the ground set of a matroid will play a central role, namely flats and cycles, that we will define now.

Definition 2 Let $M = (E, r)$ be a matroid. A flat of the matroid is a subset $F \subset E$ satisfying

$$\forall x \in EF,\ r(F \cup \{x\}) = r(F) + 1.$$

By definition E is a flat itself. If $X \subset E$, then the set $Y = \{x \in E,\ r(X \cup \{x\}) = r(X)\}$ is a flat. It is the smallest flat containing X, and moreover, $r(Y) = r(X)$. The set Y is also called the closure of X. The intersection of two flats is a flat.

Definition 3 Let $M = (E, r)$ be a matroid, and n be its nullity function. For $0 \leq i \leq n(E)$, let

$$\mathcal{N}_i = \{X \subset E,\ n(X) = i\}$$

and N_i be the inclusion minimal elements of $\mathcal{N}_i$. The elements of N_i are called cycles of nullity i. Cycles of nullity 1 are called circuits.

It is well known that cycles are union of circuits, and of course, by definition, $\emptyset$ is a cycle (of nullity 0).

Definition 4 Let $M = (E, r)$ be a matroid. The dual matroid of M is the matroid $M^* = (E, r^*)$ with

$$r^*(X) = |X| + r(EX) - r(E).$$

Flats and cycles will be described in further detail in Sect. 2.3.

If C is a $[n, k]$-linear code given by a $k \times n$ generator matrix G, then we can associate to it a matroid $M_G = (E, r)$, where $E = \{1, \cdots, n\}$ and if $X \subset E$, then $r(X)$ is the rank of the submatrix of G consisting of the columns indexed by X. It can be shown that this matroid is independent of the choice of the generator matrix of the code, and we will therefore call it the matroid associated to the code, and denote it by M_C. Notice that $(M_C)^* = M_{C^\perp}$ where $C^\perp$ is the dual of the code C.

Definition 5 Let $M = (E, r)$ be a matroid of positive rank. The truncation of M is the matroid $tr(M) = (E, r')$ where

$$r'(X) = \min\{r(M) - 1, r(X)\}.$$

Note that the flats of $tr(M)$ are exactly the flats of M except those of rank $r(M) - 1$.

Definition 6
- A matroid is called simple when $r(\{x, y\}) = |\{x, y\}|$ for every $x, y \in E$.
- Any non-simple matroid $M = (E, r)$ can be simplified to a simple matroid $M' = (E', r')$, called a simplification of M, where E' is obtained by deleting from E all its loops, and also all elements but one, from each flat of rank one. Moreover $r'(X) = r(X)$ for all $X \subset E'$.

2.2 Stanley–Reisner resolutions

Any matroid $M = (E, r)$ gives rise to a simplicial complex Δ_M, where the faces of the complex are given by

$$\mathcal{I} = \{X \subset E :\ r(X) = |X|\}.$$

These are the independent sets of the matroid. If $\mathbb{K}$ is a field, we can associate to the underlying simplicial complex a monomial ideal $I_M \subset S = \mathbb{K}[X_e, e \in E]$ defined by

$$I_M =< X^\sigma : \sigma \notin \Delta_M >,$$

where $X^\sigma = \prod_{e \in \sigma} X_e$. We refer to [6] for the study of such ideals. The Stanley–Reisner ring of the matroid, is then the quotient $S_M = S/I_M$. This ring has minimal $\mathbb{N}^n$ and $\mathbb{N}$ graded free

resolutions, where $n = |E|$, and as described in [8], they are of the form

$$0 \leftarrow S_M \leftarrow S \leftarrow \bigoplus_{\alpha \in N_1} S(-\alpha)^{\beta_{1,\alpha}} \leftarrow \cdots \leftarrow \bigoplus_{\alpha \in N_{n-r(M)}} S(-\alpha)^{\beta_{|E|-r(M),\alpha}} \leftarrow 0$$

and

$$0 \leftarrow S_M \leftarrow S \leftarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{1,j}} \leftarrow \cdots \leftarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{|E|-r(M),j}} \leftarrow 0.$$

Here $S(-\alpha)$ and $S(-j)$ are the same rings as S, but the gradings are shifted by $\alpha \in \mathbb{N}^n$, and $j \in \mathbb{N}$, respectively (As a starting point, each α is a subset of E, and interpreted as an element of $\mathbb{N}^n$, its coordinates are 1 for those indices corresponding to elements of this subset, and 0 for the other coordinates). It is known in particular, that the numbers $\beta_{i,\alpha}$ and $\beta_{i,j}$ are independent of the minimal free resolution, and when the simplical complex comes from a matroid as in our case, also independent of the field $\mathbb{K}$. These numbers are called the $\mathbb{N}^n$-graded, and the $\mathbb{N}$-graded Betti numbers of the matroid. We have

$$\beta_{i,j} = \sum_{|\alpha|=j} \beta_{i,\alpha}.$$

Here $|\alpha|$ is the cardinality of α while interpreted as a subset of E. We also define:

$$\beta_i = \sum_{j=0}^{n} \beta_{i,j},$$

for $i = 0, \cdots, k$. By convention, we say that $\beta_{i,\emptyset} = \beta_{0,0} = 1$. It is known that the Stanley–Reisner ring of a matroid is Cohen–Macaulay. As a consequence, the $\mathbb{N}$-graded Betti numbers associated to a matroid satisfy so-called Herzog–Kühl equations [7], namely:

Theorem 7 *Let M be a matroid of rank k on a ground set with n elements. Then, for every $0 \leq d \leq n-k-1$, we have:*

$$\sum_{\substack{0 \leq i \leq n-k \\ 0 \leq j \leq n}} (-1)^i j^d \beta_{i,j} = 0,$$

where by convention, $0^0 = 1$.

This result has an easy corollary for ungraded Betti numbers, namely

Corollary 8 *Let M be a matroid. Then*

$$\sum_i (-1)^i \beta_i = 0.$$

We recall here the main result of [8]:

Theorem 9 *Let M be a matroid. Then*

$$\beta_{i,\sigma} \neq 0 \Leftrightarrow \sigma \in N_i.$$

In particular, $\beta_{i,j} \neq 0$ if and only if there exists a cycle of cardinality j and nullity i.

The cycles described in Definition 3 appeared in an important way in the proof of Theorem 9. These cycles can also be described as non-redundant unions of circuits of the matroid in question, where the maximal number of non-redundant circuits appearing in such a decomposition is equal to the nullity of the cycle.

For a matroid M of rank k, we have the following convenient notation:

Definition 10

$$\phi_j(M) = \sum_{i=0}^{n-k} (-1)^i \beta_{i,j}.$$

By Theorem 7 these quantities satisfy the following equations:

$$\sum_{j=0}^{n} j^d \phi_j(M) = 0, \tag{1}$$

for $0 \leq d \leq n-k-1$ (with the convention that $0^0 = 1$). It is clear that these equations are independent in the variables $\phi_j(M)$, with a Vandermonde coefficient matrix. These quantities appear naturally when computing the generalized weight polynomials of the higher weight spectra of a linear code. Namely, from [10], the knowledge of all the generated weight polynomials, or all of the higher weight spectra, or of all the ϕ_j for the associated matroid and (all of) its elongations are equivalent. Moreover, from [4] it is known that the knowledge of each of these three information pieces is equivalent to knowing the Tutte polynomials, and therefore the two-variable coboundary polymomials of the associated matroid and its dual.

2.3 Lattices of flats and cycles

Definition 11 Let $(E, \mathcal{R})$ be a poset. The opposite of a poset $(E, \mathcal{R})$ is the poset $(E, \mathcal{S})$ where $x\mathcal{S}y \Leftrightarrow y\mathcal{R}x$. Some authors use the term "dual" instead of "opposite".

Definition 12
- Let $(E, \mathcal{R})$ be a finite poset. A chain in E is a totally ordered subset of E (meaning $a\mathcal{R}b$ or $b\mathcal{R}a$ for a, b in E).
- The length of a chain is equal to the cardinality of the chain minus 1. The length of a finite poset is the maximal length of chains in the poset.
- If the poset has the Jordan–Dedekind property (all maximal chains have the same length), then the rank of an element $x \in E$ is the length of the poset $([0, x], \mathcal{R})$.

Definition 13 A finite lattice is a finite poset $P = (E, \mathcal{R})$, where there exists a maximal element, denoted by 1, a minimal element, denoted by 0, and for any two elements $a, b \in E$, there exists a least upper bound (or join) $a \vee b$ and a greatest lower bound (or meet) $a \wedge b$. An atom is a minimal element of the subset $E \backslash \{0\}$.

The opposite lattice P^* of a lattice P satisfies $0_{P^*} = 1_P$, $1_{P^*} = 0_P$, $a \vee_{P^*} b = a \wedge_P b$ and $a \wedge_{P^*} b = a \vee_P b$.

Let M a matroid on the ground set E. It is well known that the set of flats of M is a lattice, where the order is the inclusion order. Moreover it is well known that this lattice has the Jordan–Dedekind property, and therefore has a well-defined rank function. The minimal element of the lattice is the closure of $\emptyset$, its maximal element is E, while the meet of two flats is their intersection, and the join is the closure of their union. We denote this lattice by $P(M)$.

We have the following well known fact:

Proposition 14 *Let M be a matroid. Then the cycles of M^* are the complements of flats of M.*

Because of this the cycles of M^* are often called open sets for M (complements of closed sets). In view of this proposition, we see that the set of cycles of M, together with the inclusion

order (meaning $X\mathcal{R}Y$ iff $X \subset Y$), is a lattice, called the lattice of cycles, which is isomorphic to the opposite of the lattice of flats of the dual matroid. We will denote by $L_F(M)$ and $L_C(M)$ the lattice of flats and the lattice of cycles respectively of the matroid. It is not difficult to see that if M is a matroid of rank k on the ground set E, and F is a flat of rank r (in the lattice $L_F(M)$), then EF is a cycle of rank $k-r$ (in the lattice $L_C(M^*)$).

3 Möbius and coboundary polynomials

3.1 Möbius polynomial of a matroid

We start this section with a result by Stanley [15], relating the Betti numbers of the resolution of a matroid, to certain Möbius functions on lattices.

Definition 15 (*Hall's Theorem,* [16, *Prop.* 3.8.5]) Let $L = (E, \mathcal{R})$ be a lattice, and $a, b \in E$ such that $a\mathcal{R}b$. Then

$$\mu_L(a, b) = \sum (-1)^{l(C)}$$

where C runs over all chains of L with minimal element a and maximal element b, and $l(C)$ denotes the length of the chain C.

From [15, p. 59] we have:

Theorem 16 *Let $M = (E, r)$ be a matroid, $L_C(M)$ its lattice of cycles, and $L_F(M^*)$ the lattice of flats of its dual matroid. Let $X \subset E$ be a cycle of nullity i. Then*

$$\beta_{i,X} = |\mu_{L_F(M^*)}(E - X, E)| = |\mu_{L_C(M)}(\emptyset, X)|.$$

Following [12, Definition 10.8], we give the following definition:

Definition 17 The Möbius polynomial of a lattice L, which has the Jordan–Dedekind property, is

$$\mu_L(S, T) = \sum_{x \in L} \sum_{\substack{y \in L \\ x\mathcal{R}y}} \mu(x, y) S^{r(x)} T^{r(L)-r(y)}.$$

For any matroid, its Möbius polynomial is the Möbius polynomial of its lattice of flats. Note that this is a polynomial of total degree equal to the rank of the matroid.

The aim of this section is to relate some of the coefficients of the lattice of flats of a matroid to the Betti numbers of its dual matroid. In order to do so, we need a result, which is a little finer than Theorem 16:

Definition 18 Let L be a lattice with the Jordan–Dedekind property. Its Möbius function alternates in sign if

$$(-1)^{l([a,b])} \mu_L(a, b) \geq 0 \; \forall a\mathcal{R}b.$$

Definition 19 A lattice L is semi-modular if it has the Jordan–Dedekind property, and if its rank function satisfies: $r(a) + r(b) \geq r(a \wedge b) + r(a \wedge b)$ for all $a, b \in L$.

Then, we have, from [16, Proposition 3.10.1]

Proposition 20 *The Möbius function of a finite semi-modular lattice alternates in sign.*

Corollary 21 *With the notation of Theorem 16, we have*

$$\beta_{n(X),X} = (-1)^{n(X)}\mu_{L_F(M^*)}(E\backslash X, E) = (-1)^{n(X)}\mu_{L_C(M)}(\emptyset, X).$$

Proof It is well known (See for example [16, Prop. 3.3.3]) that the lattice of flats of a matroid is semi-modular. Then the Möbius function of $L_F(M^*)$ alternates in sign, by Proposition 20. It is then immediate from the definition that also its opposite, $L_C(M)$, alternates in sign. Furthermore $l_{L_F(M^*)}(E\backslash X, E) = l_{L_C(M)}(\emptyset, X) = n(X)$. Hence the rank function on the cycles $X \in L_C(M)$ is $n(X)$. Then we get by Proposition 20 that $\mu_{L_C(M)}(\emptyset, X)$ is positive if and only if $n(X)$ is even. Hence Theorem 16 gives:

$$\beta_{n(X),X} = |\mu_{L_F(M^*)}(E\backslash X, E)| = (-1)^{n(X)}\mu_{L_F(M^*)}(E\backslash X, E) = (-1)^{n(X)}\mu_{L_C(M)}(\emptyset, X).$$

□

Theorem 22 *Let $M = (E, r)$ be a matroid of rank k. Then the coefficient of S^sT^0 in its two-variable Möbius polynomial is $(-1)^{k-s}\beta_{k-s}(M^*)$.*

Proof Since E is the only subset with $r(E) = r(L)$, the coefficient of S^sT^0 is equal to

$$\sum_{\substack{x\in L_F(M)\\ r(x)=s}} \mu_{L_F(M)}(x, E) = \sum_{\substack{y\in L_C(M^*)\\ \eta^*(y)=k-s}} \mu_{L_C(M^*)}(\emptyset, y).$$

Here η^* is the nullity function of M^*. From Corollary 21, this is

$$\sum_{\substack{y\in L_C(M^*)\\ \eta^*(y)=k-s}} \mu_{L_C(M^*)}(\emptyset, y) = \sum_{\substack{y\in L_C(M^*)\\ \eta^*(y)=k-s}} (-1)^{\eta^*(y)}\beta_{\eta^*(y),y}(M^*) = (-1)^{k-s}\beta_{k-s}(M^*).$$

□

Example 23 Let C be the $[6, 3]_2$-code with generator matrix

$$G = M_C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

A straightforward computation shows that

$$\mu_{M_C}(S, T) = S^3 + S^2(4T - 4) + S(4T^2 - 9T + 5) + (T^3 - 4T^2 + 5T - 2).$$

The ungraded minimal free resolution of the Stanley–Reisner ring R of $(M_C)^*$ is

$$0 \leftarrow R \leftarrow S \leftarrow S^4 \leftarrow S^5 \leftarrow S^2 \leftarrow 0,$$

and we see that the Betti numbers correspond to the S^0 term of μ_{M_C}. Given a subset $X \subset E$ for a matroid $M = (E, r)$ we let the restriction $M|_X$ be the pair $(X, r|_X)$. The S^i terms of μ_{M_C}, for $i > 0$ correspond to Betti numbers of restrictions of the matroid M_C as the following corollary shows.

Corollary 24 *Let M be a matroid of rank k. Then the coefficient of S^sT^t in the Möbius polynomial is*

$$\sum_{\substack{X \text{ flat of } M\\ r(X)=k-t}} (-1)^{k-s-t}\beta_{k-s-t}((M|_X)^*).$$

Proof If X is a flat, the lattice of flats of $M|_X$ is equal to the interval $[\emptyset, X]$. □

3.2 Coboundary polynomials

Definition 25 In a (finite) lattice L (as above) the atoms are the minimal non-zero elements. Such a lattice is atomic if every element is a join of some finite set of atoms.

We now define the coboundary polynomial of a matroid:

Definition 26 Let L be an atomic lattice. Its coboundary polynomial is

$$\chi_L(S,T) = \sum_{x\in L}\sum_{\substack{y\in L\\ x\mathcal{R}y}} \mu_L(x,y) S^{a(x)} T^{r(L)-r(y)}$$

where $a(x)$ is the number of atoms a of L such that $a\mathcal{R}x$. If M is a matroid, its coboundary polynomial $\chi_M(S,T)$ is the coboundary polynomial of its lattice of flats.

When the matroid is simple, then for any flat F, we have $a(F) = |F|$. In this case indeed, singletons are flats, and therefore form the atoms of the lattice. In this case, the coboundary polynomial is of degree $k = r(M)$ in T and n in S.

Given a matroid $M = (E, r)$. We can actually always assume that the matroid is simple, since we have the following well-known result (known at least as early as in [1], for a modern reference, see [14, p. 54]):

Proposition 27 *Let $M = (E, r)$ be a matroid and M' be a simplification of M. Then their coboundary polynomials are equal.*

A proof of the proposition can be found in [14, p. 54].

We have the following:

Proposition 28 *Let M be a simple matroid on a ground set of cardinality n. Then the coefficient of $S^s T^0$ in the coboundary polynomial is $\phi_{n-s}(M^*)$.*

Proof As in the proof of Theorem 22, the coefficient of $S^s T^0$ is

$$\sum_{\substack{x\in L_F(M)\\ a(x)=s}} \mu_{L_F(M)}(x,E) = \sum_{\substack{x\in L_F(M)\\ |x|=s}} \mu_{L_F(M)}(x,E).$$

As before, this is also

$$\begin{aligned}\sum_{\substack{x\in L_F(M)\\ |x|=s}} \mu_{L_F(M)}(x,E) &= \sum_{\substack{y\in L_C(M^*)\\ |y|=n-s}} \mu_{L_C(M^*)}(\emptyset,y) = \sum_{\substack{y\in L_C(M^*)\\ |y|=n-s}} (-1)^{\eta^*(y)}\beta_{\eta^*(y),y}\\ &= \phi_{n-s}(M^*).\end{aligned}$$

□

Example 29 We continue with Example 23. A straightforward computation shows that

$$\chi_{M_C}(S,T) = T^3 + T^2(4S-4) + T(S^3+3S^2-9S+5) + (S^4-S^3-3S^2+5S-2).$$

The matroid M_C is not simple, but a simplification M' is the restriction of M_C to the set $E' = \{1,3,5,6\}$, so M' is the matroid associated to

$$\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$$

with a orthogonal matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$$

associated to $(M')^*$.

Let R' be the Stanley–Reisner ring of $(M')^*$. Then a $\mathbb{N}$-graded minimal free resolution is

$$0 \leftarrow R' \leftarrow S \leftarrow S(-1) \oplus S(-2)^3 \leftarrow S(-3)^5 \leftarrow S(-4)^2 \leftarrow 0,$$

so that $\phi_i((M')^*) = 1, -1, -3, 5, -2$ for $i = 0, 1, 2, 3, 4$, which corresponds to the T^0 terms of χ_{M_C}.

4 Equations for coefficients of Möbius and coboundary polynomials

In this section, we shall exhibit equations that the coefficients of the two-variable Möbius and coboundary polynomials satisfy. We have seen in the previous section that some of the coefficients of these polynomials are closely related to certain Betti numbers of (Cohen–Macaulay) Stanley–Reisner rings, and it is not unexpected that this implies that the coefficients of these polynomials then will satisfy equations, which to a great extent can be derived from Herzog–Kühl equations.

4.1 The Möbius polynomial

In this subsection, we will describe linear equations satisfied by the coefficients of the Möbius polynomial of a matroid.

Proposition 30 *Let M be a matroid of rank k. Write*

$$\mu_M(S, T) = \sum_{s,t} a_{s,t} S^s T^t.$$

Then, for every $0 \leq t \leq k-1$,

$$\sum_{s=0}^{k-t} a_{s,t} = 0.$$

Proof By Corollary 24, we have

$$\sum_{s=0}^{k-t} a_{s,t} = \sum_{\substack{X \subset E \\ r(X)=k-t}} \sum_{s=0}^{k-t} (-1)^{k-s-t} \beta_{k-s-t}((M|_X)^*,$$

and we conclude by Corollary 8. Here the sum can be taken to be only over those X that are flats of M. □

The following result is in some sense "transversal" to Proposition 30:

Proposition 31 *Let M be a matroid of rank k. Write*

$$\mu_M(S, T) = \sum_{s,t} a_{s,t} S^s T^t.$$

Then, for every $0 \leq s \leq k-1$,

$$\sum_{t=0}^{k} a_{s,t} = 0.$$

Proof We will prove the result by induction on k. For $k = 1$, it is trivial, since

$$\mu_M(S, T) = S + T - 1$$

Assume now that $k \geq 2$. The truncation formula in [11, Theorem 16] gives:

$$\mu_M(S, T) = T\mu_{tr(M)}(S, T) + (1 - T)\mu_M(S, 0) - S^{k-1}T + S^kT. \tag{2}$$

Set

$$\mu_{tr(M)}(S, T) = \sum_{s,t} a'_{s,t} S^s T^t.$$

Then $a_{s,t} = a'_{s,t}$ when $t \geq 2$, since $(1 - T)\mu_M(S, 0) + S^{k-1}T - S^kT$, is linear with respect to T.

Furthermore, by (2) we have: $a_{s,1} = a'_{s,0} - a_{s,0}$ if $s < k - 1$. Hence, if $s < k - 1$,

$$\sum_{t=0}^{k} a_{s,t} = \sum_{t=2}^{k} a'_{s,t-1} + a'_{s,0} - a_{s,0} + a_{s,0} = 0.$$

Moreover, by (2) $a_{k-1,1} = a'_{k-1,0} - a_{k-1,0} - 1$.

Hence

$$\sum_{t=0}^{k} a_{k-1,t} = a_{k-1,0} + a_{k-1,1} = a_{k-1,0} + a'_{k-1,0} - a_{k-1,0} - 1 = 0$$

because $a'_{k-1,0} = 1$. □

Remark 32 This result, for $s = 0$, apart from using direct lattice-theoretical arguments, also follows from studying the Orlik-Solomon algebra [2,13] of the lattice of flats of M, in a way analogous to, and "transversal to", the way the case $t = 0$ follows in Proposition 30. Moreover, for higher s one may analyse the Orlik-Solomon algebras of various deletions of M to obtain the result of Proposition 31.

Example 33 In [10], one describes so-called Veronese codes for all prime powers q. For $q = 5$ such a code is a linear $[31, 6]_5$-code, and one describes its generalized Hamming weights and higher weight spectra in detail. Let M be the matroid associated to any generator matrix. Its two-variable Möbius polynomial is

$$\begin{aligned}\mu_M(S, T) = {} & T^6 + T^5(S - 1) + T^4(16{,}275S^2 - 97{,}650S + 81{,}375)\\ & + T^3(3906S^3 - 12{,}090S^2 + 12{,}369S - 4185)\\ & + T^2(16{,}275S^4 - 74{,}400S^3 + 123{,}225S^2 - 88{,}350 + 23{,}250)\\ & + T(3565S^5 - 97{,}650S^4 + 32{,}534S^3 - 437{,}100S^2 + 267{,}840S - 62{,}000)\\ & + (S^6 - 3565S^5 + 81{,}375S^4 - 254{,}851S^3 + 325{,}500S^2 - 190{,}960S + 42{,}500).\end{aligned}$$

One checks that the sums of coefficients in each of the parenthesis are zero, which verifies Proposition 30.

The Möbius polynomial can also can be written as:

$$\begin{aligned}\mu_M(S, T) = {} & S^6 + 3565S^5(T - 1) + S^4(16{,}275T^2 - 97{,}650T + 81{,}375)\\ & + S^3(3906T^3 - 74{,}400T^2 + 325{,}345T - 254{,}851)\\ & + S^2(465T^4 - 12{,}090T^3 + 123{,}225T^2 - 437{,}100T + 325{,}500)\end{aligned}$$

$$+ \; S(31T^5 - 930T^4 + 12{,}369T^3 - 88{,}350T^2 + 267{,}840T - 190{,}960)$$
$$+ \; (T^6 - 31T^5 + 465T^4 - 4185T^3 + 23{,}250T^2 - 62{,}000T + 42{,}500).$$

One checks that the sums of coefficients in each of the parenthesis are zero, which verifies Proposition 31.

Remark 34 For certain matroids M, the Stanley–Reisner ring of M^* has a pure resolution. See [5] and [9] for examples of codes giving rise to such pure resolutions. That the resolution is pure, means that for each i, there is only one $j = f(i)$, for an injective f, such that $\beta_{i,j} \neq 0$. Hence we have, for each i, that

$$\beta_i = \beta_{i,f(i)} = \phi_{f(i)},$$

and there are no other non-zero ϕ_j. Then we obviously have, from Theorem 7:

$$\sum_i (-1)^i f(i)^d \beta_i = 0, \tag{3}$$

for the relevant i, d appearing. This implies that for $t = 0$ the $a_{s,t}$ appearing in Proposition 30 satisfy $k - 1$ weighted sum equations in addition to the single one appearing in that result (Here k is the rank of the M as in Proposition 30).

Moreover, if M is a matroid such that M^* has a pure resolution, and if F is a flat of M, then the contraction $M^*/(E - F) = (M|_F)^*$ has a pure resolution too. Combining with Corollary 24, and using it in the same way as Proposition 22 is used in the case $t = 0$, we obtain that for each fixed t appearing in Proposition 30 the $a_{s,t}$ satisfy $k - 1 - t$ weighted sum equations in addition to the single one appearing in that result.

To illustrate Remark 34 we give the following:

Example 35 Let C be the dual of a $(q, 3)$ Hamming code. The columns of any generator matrix of C are in $1-1$ correspondence with the points of $\mathbb{P}^2_q$. It is a $[q^2+q+1, 3]_q$ constant weight code, where the Stanley–Reisner ring of $(M_C)^*$ has a pure minimal resolution ([9]). The flats of $M = M_C$ correspond to $\emptyset$, the $q + 1$ points of $\mathbb{P}^2_q$, the $q + 1$ lines of $\mathbb{P}^2_q$ and $\mathbb{P}^2_q$ itself. One then obtains

$$\begin{aligned}\mu_{M_C}(S, T) = {} & T^3 + T^2\left((q^2 + q + 1)S - (q^2 + q + 1)\right) \\ & + T\left((q^2 + q + 1)S^2 - (q + 1)(q^2 + q + 1)S + q(q^2 + q + 1)\right) \\ & + S^3 - (q^2 + q + 1)S^2 + q(q^2 + q + 1)S - q^3.\end{aligned}$$

Keeping the same notation as in Propositions 30 and 31, the coefficients satisfy the 6 following equations:

$$\begin{aligned} a_{3,0} + a_{2,0} + a_{1,0} + a_{0,0} &= 0 \\ q^2 a_{2,0} + (q^2 + q)a_{1,0} + (q^2 + q + 1)a_{0,0} &= 0 \\ q^4 a_{2,0} + (q^2 + q)^2 a_{1,0} + (q^2 + q + 1)^2 a_{0,0} &= 0 \\ a_{2,1} + a_{1,1} + a_{0,1} &= 0 \\ q a_{1,1} + (q + 1)a_{0,1} &= 0 \\ a_{1,2} + a_{0,2} &= 0. \end{aligned}$$

Equations nr.1,4,6 here follow from Proposition 30 for $t = 0, 1, 2$, respectively, and also from (3) for $d = 0$, for M and its first two truncations. Equations nr. 2,3 follow from (3)

applied to M for the $d = 1, 2$ cases, since $(f(1), f(2), f(3)) = (d_1(C), d_2(C), d_3(C)) = (q^2, q^2+q, q^2+q+1)$, for the generalized Hamming weights $d_i(C)$. Equation nr. 5 follows from (3) in the $d = 1$ case, but applied to the first truncation of the matroid M. This example generalizes to duals of (q, r)-Hamming codes.

4.2 The coboundary polynomial

In this section, we will describe Herzog–Kühl equations satisfied by the coefficients of the coboundary polynomial.

Theorem 36 *Let M be a matroid of rank $k \geq 2$ on a set with n elements. Write*

$$\chi(S, T) = \sum_{s,t} b_{s,t} S^s T^t.$$

Then, for every $0 \leq t \leq k-1$ and for every $0 \leq d \leq k-t-1$,

$$\sum_{s=0}^{n} (n-s)^d b_{s,t} = 0.$$

This gives a total number of $k(k+1)/2$ linearly independent linear equations in the coefficients of $\chi(S, T)$.

Proof We may assume that M is a simple matroid, since if it not simple, we replace it by its simplification, which has the same coboundary polynomial. The proof uses induction on k. Since M is simple, necessarily $k \geq 2$. The unique simple matroid of rank 2 is the uniform matroid $U_{2,n}$. For this matroid, the only flats are the empty set, all singletons, and the whole ground set. Using Hall's formula (Definition 15), we have

$$\chi(S, T) = S^n + nST - nS + T^2 - nT + n - 1$$

and it is not difficult to see that the coefficients of this polynomial satisfy the Theorem.

We now utilize a formula from [11, Theorem 12], where the authors give a truncation formula for the coboundary polynomial of a geometric lattice. Since the poset of flats of the truncated matroid is equal to the truncation of the poset of flats of a matroid, [11, Theorem 12] gives:

$$\chi_M(S, T) = T\chi_{tr(M)}(S, T) - (T-1)\chi_M(S, 0). \tag{4}$$

Since the truncation of a simple matroid of rank $k \geq 3$ is a simple matroid of rank $k-1$, we can apply this formula. From the formula, if

$$\chi_{tr(M)}(S, T) = \sum_{s,t} b'_{s,t} S^s T^t,$$

then we have that $b_{s,t} = b'_{s,t-1}$ if $t \geq 2$, $b_{s,1} = b'_{s,0} - b_{s,0}$. We also have $b_{s,0} = \phi_{n-s}(M^*)$ by Proposition 28. For $t \geq 2$, the result follows directly from the induction hypothesis. For $t = 0$, this is exactly Eq. (1). Finally, for $t = 1$, this is a combination of Eq. (1) and the induction hypothesis. □

We also have

Proposition 37 *Let M be a matroid of rank k on a set with $n \geq 2$ elements. Write*

$$\chi(S,T) = \sum_{s,t} b_{s,t} S^s T^t.$$

Then, for every $0 \leq s < n$,

$$\sum_{t=0}^{n} b_{s,t} = 0.$$

Proof As in the proof of Theorem 36, we may assume that M is simple. We follow the same notation as the proof of Theorem 36. It is easily checked that the theorem applies when $k = 2$. Otherwise, by induction when $k \geq 3$, we have

$$\sum_{t=0}^{n} b_{s,t} = \sum_{t=2}^{n} b'_{s,t-1} + b'_{s,0} - b_{s,0} + b_{s,0} = 0.$$

□

Example 38 We continue with Example 33. The two-variable coboundary polynomial of M is

$$\begin{aligned}\chi_M(S,T) = {} & S^{31} + S^{11}(T-1) + S^7(775T^2 - 4650T + 3875) \\ & + S^6(31T^3 - 775T^2 + 6820T - 6076) \\ & + S^4(15500T^2 - 93000T + 77500) \\ & + S^3(3875T^3 - 73625T^2 + 321625T - 251875) \\ & + S^2(465T^4 - 12090T^3 + 123225T^2 - 437100T + 325500) \\ & + S(31T^5 - 930T^4 + 12369T^3 - 88350T^2 + 267840T - 190960) \\ & + T^6 - 31T^5 + 465T^4 - 4185T^3 + 23250T^2 - 62000T + 42500.\end{aligned}$$

One checks that this polynomial satisfies Theorem 36 and Proposition 37.

5 A recursion formula for the ϕ_j

The idea of this paper has been to use well-known identities of Betti numbers of Stanley–Reisner rings of matroids to deduce identities for the coefficients of the two-variable Möbius and coboundary polynomials. One may also reverse this thinking and deduce identities for the Betti numbers, or their "derived" functions, the $\phi_j(M)$, from well-known properties of (in this case) the two-variable coboundary polynomials:

From Eq. (4) we obtain:

$$T\chi_{tr(L)}(S,T) = \chi(S,T) + (T-1)\chi_L(S,0),$$

where L is the lattice of flats of a matroid M, and $tr(L)$ is its truncation. The (first) elongation of a matroid is the dual of the truncation of its dual matroid.

If we view both sides of the last equation as polynomials in T with coefficients from $\mathbb{Z}[S]$, then, comparing the coefficients of the T^1-term we get:

$$\chi_{tr(L)}(S,0) = \chi(S,T)|_1 + \chi_L(S,0).$$

The left side involves Betti numbers of the first elongation of the dual M^*. The right side involves Betti numbers of M^*, through the term $\chi_L(S, 0)$, and of its contractions at its circuits, through the terms $\chi(S, T)|_1$.

Let W denote a cycle of M^*. Let $\phi_j = \phi_j(M^*)$ in the sense of Definition 10, and let $\phi_j^{(l)}$ and ϕ_j^W be the corresponding invariant ϕ_j for the l'th elongation matroid of M_H, and the contraction of M^* in W, respectively. After a short analysis, rewriting the last recursion formula in terms of the ϕ_j for the matroids involved, we obtain:

$$\phi_j^{(1)} = \phi_j + \Sigma_W \phi_{j-|W|}^W,$$

where in this formula we sum over cycles W of nullity 1 in M^*. Having found this, one may proceed:

$$\phi_j^{(l+1)} = \phi_j^{(l)} + \Sigma_W \phi_{j-|W|}^W, \tag{5}$$

where in this formula, as an analogue of Corollary 24, but now for the couboundary polynomial, we sum over cycles of nullity $l + 1$ in M^*. Formula (5) is interesting in view of the last paragraph of Sect. 2.2.

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