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COSHEAVES

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ABSTRACT. The categories $\mathbf{pCS}(X, \mathbf{Pro}(k))$ of precosheaves and $\mathbf{CS}(X, \mathbf{Pro}(k))$ of cosheaves on a small Grothendieck site X, with values in the category $\mathbf{Pro}(k)$ of pro-k-modules, are constructed. It is proved that $\mathbf{pCS}(X, \mathbf{Pro}(k))$ satisfies the AB4 and AB5* axioms, while $\mathbf{CS}(X, \mathbf{Pro}(k))$ satisfies AB3 and AB5*. Homology theories for cosheaves and precosheaves, based on quasi-projective resolutions, are constructed and investigated.

Contents

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1. Introduction

A *presheaf* (*precosheaf*) on a topological space X with values in a category $\mathbf{K}$ is just a contravariant (covariant) functor from the category of open subsets of X to $\mathbf{K}$, while a *sheaf* (*cosheaf*) is such a functor satisfying some extra conditions. The category of (pre)cosheaves with values in $\mathbf{K}$ is dual to the category of (pre)sheaves with values in the dual category $\mathbf{K}^{op}$.

While the theory of sheaves is well developed, and is covered by plenty of publications, the theory of cosheaves is more poorly represented. The main reason for this is that *cofiltered limits* are *not* exact in the "usual" categories like sets, abelian groups, rings, or modules. On the contrary, *filtered colimits* are exact in the above categories, which allows to construct rather rich theories of sheaves with values in "usual" categories. To sum up, the "usual" categories $\mathbf{K}$ are badly suited for cosheaf theory. Dually, the categories $\mathbf{K}^{op}$ are badly suited for sheaf theory.

The first step in building a suitable theory of cosheaves would be constructing a *cosheaf* $\mathcal{A}_{\#}$ *associated with a precosheaf* $\mathcal{A}$ (simply: *cosheafification* of $\mathcal{A}$), as a right adjoint

$$()_{\#} : Precosheaves \longrightarrow Cosheaves$$

to the inclusion

$$\iota : Cosheaves \hookrightarrow Precosheaves.$$

As is shown in [Prasolov, 2016, Theorem 3.1], it is possible in many situations, namely for precosheaves with values in an arbitrary *locally presentable* [Adámek and Rosický, 1994, Chapter 1] *category* (or a dual to such a category). See also Theorem 2.2.6 in this paper.

However, our purpose is to prepare a foundation for *homology* theory of cosheaves (see Theorems 3.2.1, 3.4.1, and Conjecture 1.0.3). In future papers, we plan to develop also the *nonabelian* homology theory (in other words, the *homotopy* theory) of (pre)cosheaves (see Conjectures 1.0.4, 1.0.5, and 1.0.7 below).

Therefore, we need a more or less *explicit* construction. Moreover, we need a construction satisfying *good exactness* properties. As is shown in [Prasolov, 2016], the most suitable categories for these purposes are the categories of (pre)cosheaves with values in the pro-category $\mathbf{Pro}(\mathbf{K})$ (Definition 2.1.4), where $\mathbf{K}$ is a cocomplete (Remark 2.0.2 (1)) category. In [Prasolov, 2016, Theorem 3.11], connections with **shape theory** have been established: it was shown that the cosheafification $G_{\#}$ of the *constant* precosheaf G^{const}, $G \in \mathbf{K}$, is isomorphic to $G \otimes_{\mathbf{Set}} pro\text{-}\pi_0$, where $pro\text{-}\pi_0$ is the pro-homotopy from Definition B.3.4 (for the pairing $\otimes_{\mathbf{Set}}$ see Definition A.1.1(4)). If $\mathbf{K} = \mathbf{Mod}(k)$ is the category of pro-modules over a commutative ring k, the cosheafification $G_{\#}$ becomes the pro-homology (Definition B.3.5):

$$G_{\#} \simeq (U \longmapsto pro\text{-}H_0(U, G)).$$

1.0.1. Remark. *An interesting attempt is made in [Schneiders, 1987] where the author sketches a cosheaf theory on topological spaces with values in a category* $\mathbf{L}$*, dual to an "elementary" category* $\mathbf{L}^{op}$*. He proposes a candidate for such a category. Let* $\alpha < \beta$ *be two inaccessible cardinals. Then* $\mathbf{L}$ *is the category* $\mathbf{Pro}_{\beta}(\mathbf{Ab}_{\alpha})$ *of abelian pro-groups* $(G_j)_{j\in\mathbf{J}}$ *such that* $card(G_j) < \alpha$ *and* $card(Mor(\mathbf{J})) < \beta$*. However, our pro-category* $\mathbf{Pro}(\mathbf{K})$ *cannot be used in the cosheaf theory from [Schneiders, 1987] because the category* $(\mathbf{Pro}(\mathbf{K}))^{op}$ *is* ***not*** *elementary.*

The main results of this paper are establishing the most important properties of precosheaves (Theorem 3.1.1) and cosheaves (Theorem 3.3.1), as well as constructing homology theory for precosheaves (Theorem 3.2.1) and cosheaves (Theorem 3.4.1). We construct the *abelian* homology theory of (pre)cosheaves with values in the category

$$\mathbf{Pro}(k) = \mathbf{Pro}(\mathbf{Mod}(k))$$

(Notation 2.0.4), where k is a *quasi-noetherian* (Definition A.2.4) commutative ring. Due to Proposition A.2.5, the class of such rings is sufficiently large, and our construction includes, e.g., (pre)cosheaves with values in

$$\mathbf{Pro}(\mathbf{Ab}) \simeq \mathbf{Pro}(\mathbf{Mod}(\mathbb{Z})) = \mathbf{Pro}(\mathbb{Z}).$$

1.0.2. Remark. *A cosheaf theory with values in the category* $\mathbf{Pro}(k)$ *on topological spaces was sketched in [Sugiki, 2001]. Definition 2.2.7 of a cosheaf on a topological space* X *in [Sugiki, 2001] is dual to our definition of a cosheaf on the corresponding site* $OPEN(X)$*, see Example B.1.9 and Remark B.1.10. Theorem 2.2.8 in [Sugiki, 2001] states that the cosheafification exists. However, no proof of that theorem is given, and no explicit construction of such cosheafification is provided.*

Moreover, in [Sugiki, 2001, Definition 4.1.3] the author introduces the notion of ***c-injective*** *cosheaves which seem to be dual to our* ***quasi-projective*** *cosheaves, and claims in [Sugiki, 2001, Theorem 4.1.7] that c-injective cosheaves form a cogenerating subcategory in the category of all cosheaves. That statement seems to be dual to our Theorem 3.4.1(1). However, the proof is only sketched, and is based on several statements given without proofs. Moreover, [Sugiki, 2001] sketches a construction of cosheaf homology only for topological spaces (for the site* $OPEN(X)$*, see Example B.1.9 and Remark B.1.10). In this paper, on the contrary, we construct the cosheaf homology theory for arbitrary small sites.*

1.0.3. Conjecture.

1. *On the standard site* $OPEN(X)$ *(Example B.1.9), the left satellites of* H_0 *are naturally isomorphic to the* ***pro-homology*** *(Definition B.3.5):*

$$H_n(X, pro\text{-}H_0(\bullet, A)) = H_n(X, A_{\#}) := L_n H_0(X, A_{\#}) \simeq pro\text{-}H_n(X, A),$$

provided X *is* ***Hausdorff paracompact****.*

2. *The above isomorphisms exist for all topological spaces if we use the site $NORM(X)$ (Example B.1.12) instead of $OPEN(X)$.*

Example 4.0.1 illustrates the conjecture.

1.0.4. CONJECTURE.

1. *On the standard site $OPEN(X)$, the **nonabelian** left satellites of H_0 are naturally isomorphic to the pro-homotopy (Definition B.3.4):*

$$\begin{aligned} H_n(X, S_\#) &= H_n(X, S \times \textit{pro-}\pi_0) := L_n H_0(X, S_\#) \simeq S \times \textit{pro-}\pi_n(X), \\ H_n\left(X, (\mathbf{pt})_\#\right) &= H_n(X, \textit{pro-}\pi_0) := L_n H_0\left(X, (\mathbf{pt})_\#\right) \simeq \textit{pro-}\pi_n(X), \end{aligned}$$

provided X is Hausdorff paracompact.

2. *The above isomorphisms exist for all topological spaces if we use the site $NORM(X)$ instead of $OPEN(X)$.*

For general topological spaces, however, one could not expect that cosheaf homology $H_n(X, G_\#)$ coincides with shape pro-homology $\textit{pro-}H_n(X, G)$ (unless $n = 0$, see Theorem 2.2.6 and [Prasolov, 2016, Theorem 3.11]). The thing is that general spaces may lack "good" polyhedral expansions (Definition B.3.1). See Remark 1.0.6 and Conjecture 1.0.5.

1.0.5. CONJECTURE. *Let X be a (pointed) finite (or even locally finite) topological space. Then:*

1. *The left satellites of H_0 are naturally isomorphic to the singular homology:*

$$H_n(X, G_\#) := L_n H_0(X, G_\#) \simeq H_n^{sing}(X, G).$$

2. *The **nonabelian** left satellites of H_0 are naturally isomorphic to the homotopy groups:*

$$\begin{aligned} H_n(X, S_\#) &= H_n(X, S \times \pi_0) := L_n H_0(X, S_\#) \simeq S \times \pi_n(X), \\ H_n\left(X, (\mathbf{pt})_\#\right) &= H_n(X, \pi_0) := L_n H_0\left(X, (\mathbf{pt})_\#\right) \simeq \pi_n(X). \end{aligned}$$

Example 4.0.2 illustrates the conjecture.

On the contrary, the pro-homology and pro-homotopy of such spaces are rather trivial:

1.0.6. Remark. *If X is a locally finite (pointed) topological space, then:*

$$pro\text{-}H_n(X,G) \simeq H_n\left((\pi_0(X))^\delta, G\right),$$
$$pro\text{-}\pi_n(X) \simeq \pi_n\left((\pi_0(X))^\delta\right),$$

where $(\pi_0(X))^\delta$ is the set of connected components of X, supplied with the discrete topology. Indeed, it is easy to check that the natural continuous projection

$$X \longrightarrow (\pi_0(X))^\delta$$

is a polyhedral expansion (Definition B.3.1).

Other possible applications could be in étale homotopy theory [Artin and Mazur, 1986] as is summarized in the following

1.0.7. Conjecture. *Let X^{et} be the site from Example B.1.13.*

1. *The left satellites of H_0 are naturally isomorphic to the étale pro-homology:*

$$H_n\left(X^{et}, A_\#\right) := L_n H_0\left(X^{et}, A_\#\right) \simeq H_n^{et}(X, A).$$

2. *The nonabelian left satellites of H_0 are naturally isomorphic to the étale pro-homotopy:*

$$H_n\left(X^{et}, (\mathbf{pt})_\#\right) \simeq H_n\left(X^{et}, \pi_0^{et}\right) := L_n H_0\left(X^{et}, (\mathbf{pt})_\#\right) \simeq \pi_n^{et}(X).$$

2. Preliminaries

We fix a commutative ring k. From now on, k is assumed to be **quasi-noetherian** (Definition A.2.4), e.g. **noetherian** (see Proposition A.2.5).

2.0.1. Notation.

1. *We shall denote **limits** (inverse/projective limits) by $\varprojlim$, and **colimits** (direct/inductive limits) by $\varinjlim$.*
2. *If U is an object of a category $\mathbf{K}$, we shall usually write $U \in \mathbf{K}$ instead of $U \in Ob(\mathbf{K})$.*

2.0.2. Remark.

1. *Remind that a category $\mathbf{C}$ is **complete** if it admits small limits $\varprojlim$, and **cocomplete** if it admits small colimits $\varinjlim$.*
2. *A complete category has a **terminal** object (a limit of an empty diagram). A cocomplete category has an **initial** object (a colimit of an empty diagram).*
3. *A functor $f : \mathbf{C} \to \mathbf{D}$ is called **left (right) exact** if it preserves **finite** limits (colimits). f is called **exact** if it is both left and right exact.*

2.0.3. Definition. *A subcategory* $\mathbf{C} \subseteq \mathbf{D}$ *is called* ***reflective*** *(respectively* ***coreflective****) iff the inclusion* $\mathbf{C} \hookrightarrow \mathbf{D}$ *is a right (respectively left) adjoint. The left (respectively right) adjoint* $\mathbf{D} \to \mathbf{C}$ *is called a* ***reflection*** *(respectively* ***coreflection****).*

2.0.4. Notation. $\mathbf{Pro}\,(k) = \mathbf{Pro}\,(\mathbf{Mod}\,(k))$ *is the category of pro-objects (Definition 2.1.4) in the category* $\mathbf{Mod}\,(k)$ *of* k*-modules.*

2.0.5. Remark. *Since any noetherian ring (e.g.* $\mathbb{Z}$*) is quasi-noetherian, our considerations cover a large family of pro-categories like*

$$\mathbf{Pro}\,(\mathbf{Ab}) \simeq \mathbf{Pro}\,(\mathbb{Z}),$$

$\mathbf{Pro}\,(k)$ *where* k *is a field,* $\mathbf{Pro}\,(R)$ *where* R *is a finitely generated commutative algebra over a noetherian ring, etc.*

2.0.6. Definition. *Given two categories* $\mathbf{I}$ *and* $\mathbf{K}$ *with* $\mathbf{I}$ *small, let* $\mathbf{K}^{\mathbf{I}}$ *be the category of* $\mathbf{I}$*-diagrams in* $\mathbf{K}$*.*

2.0.7. Remark. *We will also consider functors* $\mathbf{C} \to \mathbf{D}$ *where* $\mathbf{C}$ *is not small. However, such functors do* ***not*** *form a category* $\mathbf{D}^{\mathbf{C}}$*, because the morphisms* $\mathbf{D}^{\mathbf{C}}(F,G)$ *form a* ***class****, but not in general a* ***set****. Such object cannot be even called a* ***large*** *category. Probably, "a* ***huge*** *category" would be an appropriate name.*

2.0.8. Definition. *Given* $U \in \mathbf{K}$*, let*

$$h_U : \mathbf{K}^{op} \longrightarrow \mathbf{Set},\ h^U : \mathbf{K} \longrightarrow \mathbf{Set},$$

be the following functors:

$$\begin{aligned} &h_U\,(V) := Hom_{\mathbf{C}}\,(V,U)\,,\ h^U\,(V) := Hom_{\mathbf{C}}\,(U,V)\,,\\ &h_U\,(\alpha) := [(\gamma \in h_U\,(V) = Hom_{\mathbf{C}}\,(V,U)) \longmapsto (\gamma \circ \alpha \in Hom_{\mathbf{C}}\,(V',U) = h_U\,(V'))]\,,\\ &h^U\,(\beta) := \left[\left(\gamma \in h^U\,(V) = Hom_{\mathbf{C}}\,(U,V)\right) \longmapsto \left(\beta \circ \gamma \in Hom_{\mathbf{C}}\,(U,V') = h^U\,(V')\right)\right], \end{aligned}$$

where

$$\begin{aligned} (\alpha : V' \longrightarrow V) &\in Hom_{\mathbf{C}}\,(V',V) = Hom_{\mathbf{C}^{op}}\,(V,V')\,,\\ (\beta : V \longrightarrow V') &\in Hom_{\mathbf{C}}\,(V,V')\,. \end{aligned}$$

2.0.9. Remark.

1. *The functors*

$$h_\bullet : \mathbf{K} \longrightarrow \mathbf{Set}^{\mathbf{K}^{op}},\ h^\bullet : \mathbf{K}^{op} \longrightarrow \mathbf{Set}^{\mathbf{K}},$$

   are full embeddings, called the ***first*** *and the* ***second Yoneda embeddings****.*

2. *We will consider also the* ***third*** *Yoneda embedding, which is dual to the second one:*

$$(h^\bullet)^{op} : \mathbf{K} = (\mathbf{K}^{op})^{op} \longrightarrow \left(\mathbf{Set}^{\mathbf{K}}\right)^{op}.$$

2.0.10. Definition. *Let*

$$\varphi : \mathbf{C} \longrightarrow \mathbf{D}$$

be a functor, and let $d \in \mathbf{D}$.

1. *The* ***comma category*** $\varphi \downarrow d$ *is defined as follows:*

$$Ob(\varphi \downarrow d) := \{(\varphi(c) \to d) \in Hom_{\mathbf{D}}(\varphi(c), d)\},$$
$$Hom_{\varphi\downarrow d}((\alpha_1 : \varphi(c_1) \to d), (\alpha_2 : \varphi(c_2) \to d)) := \{\beta : c_1 \to c_2 \mid \alpha_2 \circ \varphi(\beta) = \alpha_1\}.$$

2. *Another* ***comma category***

$$d \downarrow \varphi = (\varphi^{op} \downarrow d)^{op}$$

is defined as follows:

$$Ob(d \downarrow \varphi) := \{(d \to \varphi(c)) \in Hom_{\mathbf{D}}(d, \varphi(c))\},$$
$$Hom_{\varphi\downarrow d}((\alpha_1 : d \to \varphi(c_1)), (\alpha_2 : d \to \varphi(c_2))) := \{\beta : c_1 \to c_2 \mid \varphi(\beta) \circ \alpha_1 = \alpha_2\}.$$

2.0.11. Definition. *Let* $U \in \mathbf{C}$. *The* ***comma category*** $\mathbf{C}_U$ *is defined as follows:*

$$\mathbf{C}_U = \mathbf{1_C} \downarrow U,$$

i.e.

$$Ob(\mathbf{C}_U) := \{(V \to U) \in Hom_{\mathbf{C}}(V, U)\},$$
$$Hom_{\mathbf{C}_U}((\alpha_1 : V_1 \to U), (\alpha_2 : V_2 \to U)) := \{\beta : V_1 \to V_2 \mid \alpha_2 \circ \beta = \alpha_1\}.$$

2.0.12. Definition. *Let* $F \in \mathbf{Set}^{\mathbf{C}^{op}}$. *The* ***comma category*** $\mathbf{C}_F$ *is defined as follows:*

$$Ob(\mathbf{C}_F) := \{(V, \alpha) \mid V \in \mathbf{C}, \alpha \in F(V)\},$$
$$Hom_{\mathbf{C}_U}((V_1, \alpha_1), (V_2, \alpha_2)) := \{\beta : V_1 \to V_2 \mid F(\beta)(\alpha_2) = \alpha_1\}.$$

2.0.13. Remark. *The categories* $\mathbf{C}_U$ *and* $\mathbf{C}_{h_U}$ *are equivalent.*

2.1. Pro-modules. The main reference is [Kashiwara and Schapira, 2006, Chapter 6] where the **Ind**-objects are considered. The **Pro**-objects used in this paper are dual to the **Ind**-objects:

$$\mathbf{Pro}(\mathbf{C}) \simeq (\mathbf{Ind}(\mathbf{C}^{op}))^{op}.$$

2.1.1. Definition. *A small category* $\mathbf{I}$ *is called* ***filtered*** *iff:*

1. *It is not empty.*

2. *For every two objects* $i, j \in \mathbf{I}$ *there exists an object* k *and two morphisms*

$$\begin{array}{ccc} i & \longrightarrow & k, \\ j & \longrightarrow & k. \end{array}$$

3. *For every two parallel morphisms*

$$\begin{array}{ccc} u & : & i \longrightarrow j, \\ v & : & i \longrightarrow j, \end{array}$$

there exists an object k *and a morphism*

$$w : j \longrightarrow k,$$

such that $w \circ u = w \circ v$*. A category* $\mathbf{I}$ *is called* ***cofiltered*** *if* $\mathbf{I}^{op}$ *is filtered. A diagram* $D : \mathbf{I} \to \mathbf{K}$ *is called (co)filtered if* $\mathbf{I}$ *is a (co)filtered category.*

See, e.g., [Mac Lane, 1998, Chapter IX.1] for filtered, and [Mardešić and Segal, 1982, Chapter I.1.4] for cofiltered categories.

2.1.2. Remark. *In [Kashiwara and Schapira, 2006], such categories and diagrams are called* ***(co)filtrant****.*

2.1.3. Example. *For any poset* $(X, \leq)$ *one can define the category* $\mathbf{Cat}(X)$ *with*

$$Ob(\mathbf{Cat}(X)) = X,$$

where each set $Hom_{\mathbf{Cat}(X)}(x, y)$ *consists of one object* (x, y) *if* $x \leq y$*, and is empty otherwise.*

The poset X *is called* ***directed*** *iff* $X \neq \varnothing$*, and*

$$\forall x, y \in X \ \ [\exists z\, (x \leq z \& y \leq z)].$$

The poset X *is called* ***codirected*** *iff* $X \neq \varnothing$*, and*

$$\forall x, y \in X \ \ [\exists z\, (z \leq x \& z \leq y)].$$

It is easy to see that $\mathbf{Cat}(X)$ *is (co)filtered iff* X *is (co)directed.*

2.1.4. Definition. *Let* $\mathbf{K}$ *be a category. The pro-category* $\mathbf{Pro}(\mathbf{K})$ *(see [Kashiwara and Schapira, 2006, Definition 6.1.1], [Mardešić and Segal, 1982, Remark I.1.4], or [Artin and Mazur, 1986, Appendix]) is the full subcategory of* $\left(\mathbf{Set}^{\mathbf{K}}\right)^{op}$ *consisting of functors that are cofiltered limits of representable functors, i.e. limits of diagrams of the form*

$$\mathbf{I} \xrightarrow{\mathbf{X}} \mathbf{K} \xrightarrow{(h^{\bullet})^{op}} \left(\mathbf{Set}^{\mathbf{K}}\right)^{op}$$

where $\mathbf{I}$ *is a cofiltered category,* $\mathbf{X} : \mathbf{I} \to \mathbf{K}$ *is a diagram, and* $(h^{\bullet})^{op}$ *is the third Yoneda embedding. We will simply denote such diagrams by* $\mathbf{X} = (X_i)_{i \in \mathbf{I}}$*.*

2.1.5. Remark. *See [Kashiwara and Schapira, 2006, Lemma 6.1.2 and formula (2.6.4)]:*

1. *Let two pro-objects be defined by the diagrams* $\mathbf{X} = (X_i)_{i\in\mathbf{I}}$ *and* $\mathbf{Y} = (Y_j)_{j\in\mathbf{J}}$. *Then*

$$Hom_{\mathbf{Pro(K)}}(\mathbf{X}, \mathbf{Y}) = \varprojlim_{j\in\mathbf{J}} \varinjlim_{i\in\mathbf{I}} Hom_{\mathbf{K}}(X_i, Y_j).$$

2. $\mathbf{Pro}(\mathbf{K})$ *is indeed a category even though* $\left(\mathbf{Set}^{\mathbf{K}}\right)^{op}$ *is a "huge" category:* $Hom_{\mathbf{Pro(K)}}$ $(\mathbf{X}, \mathbf{Y})$ *is a* ***set*** *for any* $\mathbf{X}$ *and* $\mathbf{Y}$.

2.1.6. Remark. *The category* $\mathbf{K}$ *is a full subcategory of* $\mathbf{Pro}(\mathbf{K})$*: any object* $X \in \mathbf{K}$ *gives rise to the singleton*

$$(X) \in \mathbf{Pro}(\mathbf{K})$$

with a trivial index category $\mathbf{I} = (\{i\}, \mathbf{1}_i)$. *A pro-object* $\mathbf{X}$ *is called* ***rudimentary*** *[Mardešić and Segal, 1982, §I.1.1] iff it is isomorphic to an object of* $\mathbf{K}$*:*

$$\mathbf{X} \simeq Z \in \mathbf{K} \subseteq \mathbf{Pro}(\mathbf{K}).$$

The proposition below allows us to recognize rudimentary pro-objects:

2.1.7. Proposition. *Let*

$$\mathbf{X} = (X_i)_{i\in\mathbf{I}} \in \mathbf{Pro}(\mathbf{K}),$$

and $Z \in \mathbf{K}$. *Then* $\mathbf{X} \simeq Z$ *iff there exist an* $i_0 \in \mathbf{I}$ *and a morphism* $\tau_0 : X_{i_0} \to Z$ *satisfying the property: for any morphism* $s : i \to i_0$, *there exist a morphism* $\sigma : Z \to X_i$ *and a morphism* $t : j \to i$ *satisfying*

$$\begin{aligned} \tau_0 \circ X(s) \circ \sigma &= \mathbf{1}_Z, \\ \sigma \circ \tau_0 \circ X(s) \circ X(t) &= X(t). \end{aligned}$$

Proof. The statement is dual to [Kashiwara and Schapira, 2006, Proposition 6.2.1]. ∎

2.1.8. Corollary. *Let*

$$\mathbf{X} = (X_i)_{i\in\mathbf{I}} \in \mathbf{Pro}(k).$$

Then $\mathbf{X}$ *is a zero object in* $\mathbf{Pro}(k)$ *iff for any* $i \in \mathbf{I}$ *there exists a* $t : j \to i$ *with* $X(t) = 0$.

2.1.9. Remark. *Remark 2.1.5 allows the following description of morphisms in the pro-category: any*

$$f \in Hom_{\mathbf{Pro(K)}}\left((X_i)_{i\in\mathbf{I}}, (Y_j)_{j\in\mathbf{J}}\right) = \varprojlim_{j\in\mathbf{J}} \varinjlim_{i\in\mathbf{I}} Hom_{\mathbf{K}}(X_i, Y_j)$$

can be represented (not uniquely!) by a triple

$$\left(\varphi, \lambda, (f_j)_{j\in\mathbf{J}}\right),$$

where

$$\begin{array}{lcl}\varphi & : & Ob(\mathbf{J}) \longrightarrow Ob(\mathbf{I}), \\ \lambda & = & \left[\alpha \longmapsto \left[\varphi(j_1) \overset{\lambda_1(\alpha)}{\longleftarrow} \Lambda(\alpha) \overset{\lambda_0(\alpha)}{\longrightarrow} \varphi(j_0)\right]\right] : Mor(\mathbf{J}) \longrightarrow Ob(\mathbf{I}) \times Mor(\mathbf{I}) \times Mor(\mathbf{I}),\end{array}$$

are functions, and

$$\left(f_j : X_{\varphi(j)} \longrightarrow Y_j\right)_{j\in\mathbf{J}}$$

is a family of morphisms, such that the following diagram

$$\begin{array}{ccccc} X_{\varphi(j_1)} & \xleftarrow{X(\lambda_1(\alpha))} & X_{\Lambda(\alpha)} & \xrightarrow{X(\lambda_0(\alpha))} & X_{\varphi(j_0)} \\ \downarrow & & & & \downarrow \\ Y_{j_1} & & \xleftarrow{\quad Y(\alpha) \quad} & & Y_{j_0} \end{array}$$

*commutes for any $\alpha : j_0 \to j_1$ in $\mathbf{J}$ (see [Mardešić and Segal, 1982, §I.1.1] and [Artin and Mazur, 1986, §A.3]). It is known that such a morphism is equivalent to a **level morphism** (Definition 2.1.10). Moreover, any **finite** diagram of pro-objects **without loops** is equivalent to a level diagram (see Definition 2.1.10 and Proposition 2.1.11). However, it is not in general possible to "levelize" the **whole** set $Hom_{\mathbf{Pro(K)}}(\mathbf{X}, \mathbf{Y})$ (or an **infinite** diagram, or a diagram **with loops**) in $\mathbf{Pro}(\mathbf{K})$.*

2.1.10. Definition.

1. *A morphism*

$$f \in Hom_{\mathbf{Pro(K)}}\left(\mathbf{X} = (X_i)_{i\in\mathbf{I}}, \mathbf{Y} = (Y_j)_{j\in\mathbf{J}}\right)$$

*is called a **level morphism** (compare to [Mardešić and Segal, 1982, §I.1.3]) iff $\mathbf{I} = \mathbf{J}$, and there is a morphism*

$$\gamma : (X_i)_{i\in\mathbf{I}} \longrightarrow (Y_i)_{i\in\mathbf{I}} : \mathbf{I} \longrightarrow \mathbf{K}$$

of functors, generating f, i.e. such that the following diagram

$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & \mathbf{Y} \\ \downarrow \simeq & & \simeq \downarrow \\ \varprojlim_{i\in\mathbf{I}} \left(h^{X_i}\right)^{op} & \longrightarrow & \varprojlim_{i\in\mathbf{I}} \left(h^{Y_i}\right)^{op} \end{array}$$

where

$$\varprojlim_{i\in\mathbf{I}} \left(h^{X_i}\right)^{op}, \varprojlim_{i\in\mathbf{I}} \left(h^{Y_i}\right)^{op} \in \left(\mathbf{Set}^{\mathbf{K}}\right)^{op},$$

is commutative. In the notations of Remark 2.1.9 it means that:

$$\begin{aligned}\varphi &= \mathbf{1}_{Ob(\mathbf{I})} : Ob\left(\mathbf{I}\right) \longrightarrow Ob\left(\mathbf{I}\right),\\ \lambda\left(\alpha : j_0 \to j_1\right) &= \left[\varphi\left(j_1\right) = j_1 \overset{\alpha}{\longleftarrow} j_0 \overset{\mathbf{1}_{j_0}}{\longrightarrow} j_0 = \varphi\left(j_0\right)\right],\\ f_i &= \gamma_i, i \in \mathbf{I}.\end{aligned}$$

2. *A family*

$$\left(f_s : \mathbf{X}_s = \left(X_{si}\right)_{i\in\mathbf{I}_s} \longrightarrow \mathbf{Y}_s = \left(Y_{sj}\right)_{j\in\mathbf{J}_s}\right)$$

of morphisms in $\mathbf{Pro}\left(\mathbf{K}\right)$ *is called a* ***level family*** *iff for some* $\mathbf{H}$ *and for all* s*,*

$$\mathbf{I}_s = \mathbf{J}_s = \mathbf{H},$$

and there is a family of functors

$$\alpha_s : \left(X_{si}\right)_{i\in\mathbf{H}} \longrightarrow \left(Y_{si}\right)_{i\in\mathbf{H}},$$

such that α_s *generates* f_s *for all* s*.*

3. *A diagram*

$$D : \mathbf{G} \longrightarrow \mathbf{Pro}\left(\mathbf{K}\right)$$

in $\mathbf{Pro}\left(\mathbf{K}\right)$ *is called a* ***level diagram*** *iff for some* $\mathbf{H}$ *and for all* $g \in Ob\left(\mathbf{G}\right)$*,*

$$D\left(g\right) = \left(X_{gi}\right)_{i\in\mathbf{H}},$$

and there is a diagram

$$\alpha : \mathbf{G} \times \mathbf{H} \longrightarrow \mathbf{K},$$

such that for each

$$\left(\beta : g_1 \longrightarrow g_2\right) \in Hom_{\mathbf{G}}\left(g_1, g_2\right)$$

the morphism

$$\alpha\left(\beta\right) : \alpha\left(g_1 \times \bullet\right) \longrightarrow \alpha\left(g_2 \times \bullet\right) : \mathbf{K}^{\mathbf{H}} \longrightarrow \mathbf{K}^{\mathbf{H}}$$

generates the morphism

$$f_\alpha : D\left(g_1\right) \longrightarrow D\left(g_2\right).$$

2.1.11. PROPOSITION. *Let*

$$D : \mathbf{G} \longrightarrow \mathbf{Pro}\left(\mathbf{K}\right)$$

be a diagram in $\mathbf{Pro}\left(\mathbf{K}\right)$*, where* $\mathbf{G}$ *is finite, and* ***does not have loops****. Then the diagram is isomorphic to a level diagram, i.e.* $D \simeq D'$*, where*

$$D' : \mathbf{G} \longrightarrow \mathbf{Pro}\left(\mathbf{K}\right)$$

is a level diagram.

PROOF. See [Artin and Mazur, 1986, Proposition A.3.3] or [Kashiwara and Schapira, 2006, dual to Proposition 6.4.1]. ∎

2.1.12. Remark. *See examples of such "levelization" for one morphism [Kashiwara and Schapira, 2006, dual to Corollary 6.1.14], and for a pair of parallel morphisms [Kashiwara and Schapira, 2006, dual to Corollary 6.1.15].*

Below are other useful properties of pro-objects.

2.1.13. Proposition. *Let $\mathbf{K}$ be a cocomplete category. In (3-4) below assume that $\mathbf{K}$ admits finite limits.*

1. *For any $\mathbf{Y} \in \mathbf{Pro}(\mathbf{K})$ the functor $Hom_{\mathbf{Pro}(\mathbf{K})}(\bullet, \mathbf{Y})$ converts cofiltered limits into filtered colimits: for a diagram $(\mathbf{X}_i)_{i \in \mathbf{I}}$ in $\mathbf{Pro}(\mathbf{K})$, where $\mathbf{I}$ is cofiltered,*

$$Hom_{\mathbf{Pro}(\mathbf{K})}\left(\varprojlim_{i\in\mathbf{I}} \mathbf{X}_i, \mathbf{Y}\right) \simeq \varinjlim_{i\in\mathbf{I}^{op}} \left(Hom_{\mathbf{Pro}(\mathbf{K})}(\mathbf{X}_i, \mathbf{Y})\right).$$

2. *$\mathbf{Pro}(\mathbf{K})$ is cocomplete.*
3. *$\mathbf{Pro}(\mathbf{K})$ is complete.*
4. *Cofiltered limits are exact in $\mathbf{Pro}(\mathbf{K})$: for a double diagram $(\mathbf{X}_{i,j})_{i\in\mathbf{I}, j\in\mathbf{J}}$ in $\mathbf{Pro}(\mathbf{K})$, where $\mathbf{I}$ is cofiltered, and $\mathbf{J}$ is finite,*

$$\varprojlim_{i\in\mathbf{I}} \varinjlim_{j\in\mathbf{J}} \mathbf{X}_{i,j} \simeq \varinjlim_{j\in\mathbf{J}} \varprojlim_{i\in\mathbf{I}} \mathbf{X}_{i,j},$$
$$\varprojlim_{i\in\mathbf{I}} \varprojlim_{j\in\mathbf{J}} \mathbf{X}_{i,j} \simeq \varprojlim_{j\in\mathbf{J}} \varprojlim_{i\in\mathbf{I}} \mathbf{X}_{i,j},$$

Proof. (**1**) The statement is dual to [Kashiwara and Schapira, 2006, Theorem 6.1.8].
(**2**) See [Kashiwara and Schapira, 2006, dual to Corollary 6.1.17].
(**3**) See [Kashiwara and Schapira, 2006, dual to Proposition 6.1.18].
(**4**) The statement is dual to [Kashiwara and Schapira, 2006, Proposition 6.1.19]. ∎

2.2. (Pre)cosheaves. Throughout this paper, we will consider (pre)cosheaves with values in $\mathbf{Pro}(\mathbf{K})$ ($\mathbf{K}$ is a cocomplete category), or $\mathbf{Pro}(k)$, and (pre)sheaves with values in $\mathbf{L}$ ($\mathbf{L}$ is a complete category) or $\mathbf{Mod}(k)$. Pre(co)sheaves can be defined on small sites (in particular) or on small categories (in general). Most of our constructions and statements are also valid for those generalized pre(co)sheaves.

Moreover, we will constantly use the pairings

$$\langle \bullet, \bullet \rangle : \mathbf{Pro}(k)^{op} \times \mathbf{Mod}(k) \longrightarrow \mathbf{Mod}(k),$$
$$\langle \bullet, \bullet \rangle : \mathbf{pCS}(X, \mathbf{Pro}(k))^{op} \times \mathbf{Mod}(k) \longrightarrow \mathbf{pS}(X, \mathbf{Mod}(k))$$

from Definition A.1.1(1, 2) where $\mathbf{pCS}$ denotes the category of precosheaves, while $\mathbf{pS}$ denotes the category of presheaves.

Let $X = (\mathbf{C}_X, Cov(X))$ be a small site (Definition B.1.3), and let $\mathbf{K}$ be a category. Assume that $\mathbf{K}$ is cocomplete. Remind Definition 2.0.11 for $\mathbf{C}_U$ and Definition 2.0.12 for $\mathbf{C}_R$.

2.2.1. DEFINITION.

1. *A* ***precosheaf*** $\mathcal{A}$ *on* X *with values in* $\mathbf{K}$ *is a functor* $\mathcal{A}:\mathbf{C}_X\to\mathbf{K}$.

2. *For any* $U\in\mathbf{C}_X$ *and a covering sieve (Definition B.1.3)* R *over* U *there is a natural morphism*

$$\varphi(U,R):\varinjlim_{(V\to U)\in\mathbf{C}_R}\mathcal{A}(V)\longrightarrow\mathcal{A}(U)$$

where $\mathbf{C}_R$ *is the comma-category (Definition 2.0.11).*

   (a) *A precosheaf* $\mathcal{A}$ *on* X *is* ***coseparated*** *provided* $\varphi(U,R)$ *is an* ***epimorphism*** *for any* $U\in\mathbf{C}_X$ *and for any covering sieve .*

   (b) *A precosheaf* $\mathcal{A}$ *on* X *is a* ***cosheaf*** *provided* $\varphi(U,R)$ *is an* ***isomorphism*** *for any* $U\in\mathbf{C}_X$ *and for any covering sieve* R *over* U.

2.2.2. REMARK. *The morphism in the above definition is isomorphic to the following:*

$$\psi(U,R):\mathcal{A}\otimes_{\mathbf{Set}^{\mathbf{C}_X}}R\longrightarrow\mathcal{A}\otimes_{\mathbf{Set}^{\mathbf{C}_X}}h_U.$$

The pairing $\otimes_{\mathbf{Set}^{\mathbf{C}_X}}$ *is introduced in Definition A.1.1(5). The isomorphisms*

$$\mathcal{A}\otimes_{\mathbf{Set}^{\mathbf{C}_X}}R\simeq\varinjlim_{(V\to U)\in\mathbf{C}_R}\mathcal{A}(V)$$

and

$$\mathcal{A}\otimes_{\mathbf{Set}^{\mathbf{C}_X}}h_U\simeq\mathcal{A}(U)$$

follow from Proposition B.1.8, because the comma-category $\mathbf{C}_U\simeq\mathbf{C}_{h_U}$ *(Remark 2.0.13) has a terminal object* $(U,\mathbf{1}_U)$.

2.2.3. NOTATION. *Denote by* $\mathbf{CS}(X,\mathbf{K})$ *the category of cosheaves, and by* $\mathbf{pCS}(X,\mathbf{K})$ *the category of precosheaves on* X *with values in* $\mathbf{K}$.

2.2.4. REMARK. *Compare to Definition B.1.14 and Notation B.1.16 for (pre)sheaves.*

2.2.5. DEFINITION.

1. *Assume that* $\mathbf{K}$ *is cocomplete. Given a precosheaf* $\mathcal{A}\in\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K}))$, *let*

$$\mathcal{A}_+(U):=\left[U\longmapsto\check{H}_0(U,\mathcal{A})\right]$$

(see Definition B.2.1 (2)). $\mathcal{A}_+$ *is clearly a precosheaf, and we have natural morphisms*

$$\begin{aligned}\lambda_+(\mathcal{A})&:\mathcal{A}_+\longrightarrow\mathcal{A},\\ \lambda_{++}(\mathcal{A})&=\lambda_+(\mathcal{A})\circ\lambda_+(\mathcal{A}_+):\mathcal{A}_{++}\longrightarrow\mathcal{A}.\end{aligned}$$

2. *Assume that* $\mathbf{K}$ *is complete. Given a presheaf* $\mathcal{B}\in\mathbf{pS}(X,\mathbf{K})$*, let*

$$\mathcal{B}^{+}(U):=\left[U\longmapsto\check{H}^{0}(U,\mathcal{A})\right]$$

(see Definition B.2.1 (2). $\mathcal{B}^{+}$ *is clearly a presheaf, and we have natural morphisms*

$$\begin{array}{rcl}\lambda^{+}(\mathcal{B}) & : & \mathcal{B}\longrightarrow\mathcal{B}^{+},\\ \lambda^{++}(\mathcal{B}) & = & \lambda^{+}\left(\mathcal{B}^{+}\right)\circ\lambda^{+}(\mathcal{B}):\mathcal{B}\longrightarrow\mathcal{B}^{++}.\end{array}$$

It is well-known that $\mathcal{B}^{++}$ *is a sheaf. Apply, e.g., [Prasolov, 2016, Theorem 3.1(3)] to* $\mathbf{K}^{op}$.

The following theorem has been partially proved in [Prasolov, 2016]:

2.2.6. Theorem. *Assume that* $\mathbf{K}$ *is cocomplete. In (3-4) below assume in addition that* $\mathbf{K}$ *admits finite limits. Let*

$$\begin{array}{rcl}\mathcal{A} & \in & \mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K})),\\ \mathcal{B} & \in & \mathbf{pCS}(X,\mathbf{K})\subseteq\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K})),\\ \mathcal{C} & \in & \mathbf{pCS}(X,\mathbf{Pro}(k)).\end{array}$$

Then:

1. $\mathcal{B}$ *is coseparated (a cosheaf) iff it is coseparated (a cosheaf) when considered as a precosheaf with values in* $\mathbf{Pro}(\mathbf{K})$.

2. *The full subcategory of cosheaves*

$$\mathbf{CS}(X,\mathbf{Pro}(\mathbf{K}))\subseteq\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K}))$$

is coreflective (Definition 2.0.3), and the coreflection

$$\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K}))\longrightarrow\mathbf{CS}(X,\mathbf{Pro}(\mathbf{K}))$$

is given by

$$\mathcal{A}\longmapsto\mathcal{A}_{\#}:=\mathcal{A}_{++}.$$

3. *The functor*

$$()_{+}:\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K}))\longrightarrow\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K}))$$

is ***right exact*** *(Remark 2.0.2 (3)).*

4. *The functor*

$$()_{\#}=()_{++}:\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K}))\longrightarrow\mathbf{CS}(X,\mathbf{Pro}(\mathbf{K}))$$

is ***exact*** *(Remark 2.0.2 (3).*

5. *$\mathcal{C}$ is **co**separated iff the presheaf $\langle \mathcal{C}, T\rangle$ (see Definition A.1.1(2)) is **separated** (Definition B.1.14) for any injective $T \in \mathbf{Mod}(k)$.*

6. *$\mathcal{C}$ is a **co**sheaf iff the presheaf $\langle \mathcal{C}, T\rangle$ is a **sheaf** (Definition B.1.14) for any injective $T \in \mathbf{Mod}(k)$.*

7.
$$\langle \mathcal{C}_+, T\rangle \simeq \langle \mathcal{C}, T\rangle^+,$$
$$\langle \mathcal{C}_\#, T\rangle \simeq \langle \mathcal{C}, T\rangle^\#,$$

*naturally in $\mathcal{C}$ and T, for **any** (not necessarily injective) $T \in \mathbf{Mod}(k)$.*

PROOF. (**1**, **2**) See [Prasolov, 2016, Theorem 3.1(4)].

(**3**) Let $U \in \mathbf{C}_X$, and let $R \subseteq h_U$ be a sieve. Then the functor

$$\left[\mathcal{A} \longmapsto H_0(R, \mathcal{A}) = \varinjlim_{(V\to U)\in \mathbf{C}_R} \mathcal{A}(V)\right] : \mathbf{pCS}(\mathbf{C}_X, \mathbf{Pro}(\mathbf{K})) \longrightarrow \mathbf{Pro}(\mathbf{K})$$

preserves arbitrary colimits (not necessarily finite!) because colimits commute with colimits. Therefore, the above functor is right exact. Since cofiltered limits are exact in the category $\mathbf{Pro}(\mathbf{K})$ (Proposition 2.1.13(4)), the functor

$$\left[\mathcal{A} \longmapsto \mathcal{A}_+(U) = \varprojlim_{R\in Cov(U)} \varinjlim_{(V\to U)\in \mathbf{C}_R} \mathcal{A}(V)\right] : \mathbf{pCS}(\mathbf{C}_X, \mathbf{Pro}(\mathbf{K})) \longrightarrow \mathbf{Pro}(\mathbf{K})$$

is right exact as the composition of two right exact functors. Let $U \in \mathbf{C}_X$ vary. It follows that the corresponding functor

$$()_+ : \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K})) \longrightarrow \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K}))$$

is right exact.

(**4**) Consider the composition

$$()_{++} = \iota \circ ()_\# : \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K})) \longrightarrow \mathbf{CS}(X, \mathbf{Pro}(\mathbf{K})) \longrightarrow \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K})),$$

which is right exact, due to (3). Since ι is fully faithful, the functor

$$()_\# : \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K})) \longrightarrow \mathbf{CS}(X, \mathbf{Pro}(\mathbf{K}))$$

is **right** exact as well. However, $()_\#$, being a right adjoint, preserves **arbitrary** (e.g., finite) limits, therefore it is **left** exact.

(**5**) If $\mathcal{C}$ is coseparated, then it follows from [Prasolov, 2016, Proposition 2.10(1)] that $\langle \mathcal{C}, T\rangle$ is separated for **any** (not necessarily injective) $T \in \mathbf{Mod}(k)$.

Assume now that $\langle \mathcal{C}, T\rangle$ is separated for any injective $T \in \mathbf{Mod}(k)$. Let $R \in Cov(U)$ be a sieve. It follows that

$$\begin{aligned}&[\langle \mathcal{C} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R, T\rangle \longleftarrow \langle \mathcal{C} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} h_U, T\rangle \simeq \langle \mathcal{C}, T\rangle (U) =: \langle \mathcal{C}(U), T\rangle] \simeq \\ &\simeq [Hom_{\mathbf{Set}^{\mathbf{C}_X}}(R, \langle \mathcal{C}, T\rangle) \longleftarrow Hom_{\mathbf{Set}^{\mathbf{C}_X}}(h_U, \langle \mathcal{C}, T\rangle)]\end{aligned}$$

is a monomorphism, and, due to Proposition A.2.8(8)

$$\mathcal{C} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R \longrightarrow \mathcal{C} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} h_U \simeq \mathcal{A}(U)$$

is an epimorphism.

(**6**) Proved analogously, using [Prasolov, 2016, Proposition 2.10(2)] and Proposition A.2.8(8).

(**7**) See [Prasolov, 2016, Proposition 2.11]. ∎

2.3. Quasi-projective (pre)cosheaves.

2.3.1. Definition. *Let X be a small site.*

1. *Assume that $\mathcal{A}$ is a precosheaf:*

$$\mathcal{A} \in \mathbf{pCS}(X, \mathbf{Pro}(k)).$$

*$\mathcal{A}$ is called **quasi-projective** iff for any injective $T \in \mathbf{Mod}(k)$, the presheaf*

$$\langle \mathcal{A}, T\rangle \in \mathbf{pS}(X, \mathbf{Mod}(k))$$

*is **injective**.*

2. *A cosheaf*

$$\mathcal{B} \in \mathbf{CS}(X, \mathbf{Pro}(k))$$

*is called **quasi-projective** iff for any injective $T \in \mathbf{Mod}(k)$, the sheaf*

$$\langle \mathcal{B}, T\rangle \in \mathbf{S}(X, \mathbf{Mod}(k))$$

*is **injective**.*

2.3.2. Notation. *Denote by*

$$\mathbf{Q}(\mathbf{pCS}(X, \mathbf{Pro}(k))) \subseteq \mathbf{pCS}(X, \mathbf{Pro}(k))$$

the full subcategory of quasi-projective precosheaves, and by

$$\mathbf{Q}(\mathbf{CS}(X, \mathbf{Pro}(k))) \subseteq \mathbf{CS}(X, \mathbf{Pro}(k))$$

the full subcategory of quasi-projective cosheaves.

2.3.3. Definition.

1. *A small category* $\mathbf{C}$ *is called* ***discrete*** *iff its only morphisms are identities* $(\mathbf{1}_U)_{U\in\mathbf{C}}$.
2. *A site* $X=(\mathbf{C}_X, Cov(X))$ *is called* ***discrete*** *iff* $\mathbf{C}_X$ *is a discrete category and all sieves are covering sieves.*

2.3.4. Example. *Let* $\mathbf{D}$ *be a discrete category, and assume that* $\mathcal{A}(U)$ *is a quasi-projective pro-module (Definition A.2.1) for any* $U\in\mathbf{D}$. *Then the precosheaf* $\mathcal{A}$ *is quasi-projective. Indeed, for any injective* $T\in\mathbf{Mod}(k)$, *the* k*-modules* $\langle\mathcal{A}(U),T\rangle$ *are injective (remember that* k *is quasi-noetherian!). Since the functor*

$$Hom_{\mathbf{pS}(\mathbf{D},\mathbf{Mod}(k))}(\bullet,\langle\mathcal{A},T\rangle)\simeq\prod_{U\in\mathbf{D}}Hom_{\mathbf{Mod}(k)}(\bullet(U),\langle\mathcal{A}(U),T\rangle)$$

is exact, the presheaf $\langle\mathcal{A},T\rangle$ *is injective, and the precosheaf* $\mathcal{A}$ *is quasi-projective.*

Below are necessary definitions, notations and properties of **left** and **right Kan extensions** used in Proposition 2.3.7.

2.3.5. Definition. *Let* $\mathbf{I}$ *and* $\mathbf{J}$ *be small categories and let* $\mathbf{C}$ *be an arbitrary category. For*

$$\varphi:\mathbf{J}\longrightarrow\mathbf{I}$$

denote by φ_* *the following functor:*

$$\varphi_*:\mathbf{C}^{\mathbf{I}}\longrightarrow\mathbf{C}^{\mathbf{J}}\ \ (\varphi_*(f):=f\circ\varphi),$$

where $f:\mathbf{I}\longrightarrow\mathbf{C}$ *is an arbitrary diagram. Then the following* ***left*** *adjoint* ($\varphi^\dagger\dashv\varphi_*$)

$$\varphi^\dagger:\mathbf{C}^{\mathbf{J}}\longrightarrow\mathbf{C}^{\mathbf{I}}$$

to φ_* *(if exists!) is called the* ***left*** *Kan extension of* φ. *The following* ***right*** *adjoint* ($\varphi_*\dashv\varphi^\ddagger$)

$$\varphi^\ddagger:\mathbf{C}^{\mathbf{J}}\longrightarrow\mathbf{C}^{\mathbf{I}}$$

to φ_* *(if exists!) is called the* ***right*** *Kan extension of* φ. *See [Kashiwara and Schapira, 2006, Definition 2.3.1].*

2.3.6. Proposition. *Let* $\varphi:\mathbf{J}\longrightarrow\mathbf{I}$ *be a functor and* $\beta\in\mathbf{C}^{\mathbf{J}}$.

1. *Assume that*

$$\varinjlim_{(\varphi(j)\to i)\in\varphi\downarrow i}\beta(j)$$

exists in $\mathbf{C}$ *for any* $i\in\mathbf{I}$. *Then* $\varphi^\dagger\beta$ *exists, and we have*

$$\varphi^\dagger\beta(i)=\varinjlim_{(\varphi(j)\to i)\in\varphi\downarrow i}\beta(j)$$

for $i\in\mathbf{I}$.

2. Assume that

$$\varprojlim_{(i\to\varphi(j))\in i\downarrow\varphi} \beta(j)$$

exists in $\mathbf{C}$ *for any* $i\in\mathbf{I}$*. Then* $\varphi^{\ddagger}\beta$ *exists, and we have*

$$\varphi^{\ddagger}\beta(i) = \varprojlim_{(i\to\varphi(j))\in i\downarrow\varphi} \beta(j)$$

for $i\in\mathbf{I}$*.*

3. Assume that $\mathbf{C}$ *is abelian, and that* $\varphi^{\dagger}$ *exists. Then* $\varphi^{\dagger}$ *converts projective objects of* $\mathbf{C}^{\mathbf{J}}$ *into projective objects of* $\mathbf{C}^{\mathbf{I}}$*.*

4. Assume that $\mathbf{C}$ *is abelian, and that* $\varphi^{\ddagger}$ *exists. Then* $\varphi^{\ddagger}$ *converts injective objects of* $\mathbf{C}^{\mathbf{J}}$ *into injective objects of* $\mathbf{C}^{\mathbf{I}}$*.*

Proof. For (1) and (2) see [Kashiwara and Schapira, 2006, Theorem 2.3.3].

(3) φ_* is clearly exact. If $\mathcal{A}\in\mathbf{C}^{\mathbf{J}}$ is projective, then the functor

$$Hom_{\mathbf{C}^{\mathbf{I}}}\left(\varphi^{\dagger}\mathcal{A},\bullet\right)\simeq Hom_{\mathbf{C}^{\mathbf{J}}}\left(\mathcal{A},\varphi_*(\bullet)\right):\mathbf{C}^{\mathbf{I}}\longrightarrow\mathbf{Ab}$$

is exact, therefore $\varphi^{\dagger}\mathcal{A}$ is projective.

(4) If $\mathcal{A}\in\mathbf{C}^{\mathbf{J}}$ is injective, then the functor

$$Hom_{\mathbf{C}^{\mathbf{I}}}\left(\bullet,\varphi^{\ddagger}\mathcal{A}\right)\simeq Hom_{\mathbf{C}^{\mathbf{J}}}\left(\varphi_*(\bullet),\mathcal{A}\right):\mathbf{C}^{\mathbf{I}}\longrightarrow\mathbf{Ab}$$

is exact, therefore $f^{\ddagger}\mathcal{A}$ is injective. ■

2.3.7. Proposition. *Let* $\mathbf{D}$ *and* $\mathbf{E}$ *be small categories, and let*

$$f:\mathbf{E}\longrightarrow\mathbf{D}$$

be a functor. Then

$$f^{\dagger}:\mathbf{pCS}(\mathbf{E},\mathbf{Pro}(k))\longrightarrow\mathbf{pCS}(\mathbf{D},\mathbf{Pro}(k)),$$

where $f^{\dagger}$ *is the left Kan extension of* f *(Definition 2.3.5) converts quasi-projectives into quasi-projectives.*

Proof. Let $\mathcal{A}\in\mathbf{pCS}(\mathbf{E},\mathbf{Pro}(k))$ be quasi-projective, and $T\in\mathbf{Mod}(k)$ be injective. It follows from Proposition A.2.8(5) that

$$\left\langle f^{\dagger},T\right\rangle\simeq\langle f,T\rangle^{\ddagger}.$$

Since $\langle f,T\rangle^{\ddagger}$ converts injectives into injectives (Proposition 2.3.6(4)), the presheaf $\left\langle f^{\dagger}\mathcal{A},T\right\rangle$ is injective for any injective T, and the precosheaf $f^{\dagger}\mathcal{A}$ is quasi-projective. ■

2.3.8. Definition.

1. *A cosheaf $\mathcal{A} \in \mathbf{CS}(X, \mathbf{Pro}(k))$ on a **topological space** X is called **flabby** iff $\mathcal{A}(V \to U)$ is a monomorphism for any $(V \to U) \in \mathbf{C}_X$.*

2. *A cosheaf $\mathcal{A} \in \mathbf{CS}(X, \mathbf{Pro}(k))$ on a **small site** X is called **flasque** iff*

$$H_s(R, \mathcal{A}) = 0$$

(see Definition B.2.5 (4, 5)) for any $s > 0$, and any covering sieve $R \subseteq h_U$.

2.3.9. Remark.

1. *A cosheaf $\mathcal{A}$ on a topological space is flabby iff $\langle \mathcal{A}, T \rangle$ is a flabby sheaf [Bredon, 1997, Definition II.5.1] for all injective $T \in \mathbf{Mod}(k)$. Indeed, $\langle \mathcal{A}, T \rangle$ is flabby iff*

$$\langle \mathcal{A}, T \rangle (V \to U) \simeq \langle \mathcal{A}(V \to U), T \rangle$$

is an epimorphism for any $(V \to U) \in \mathbf{C}_X$. The latter is equivalent, since T varies through all injective modules, to the fact that $\mathcal{A}(V \to U)$ is a monomorphism for any $(V \to U) \in \mathbf{C}_X$.

2. *A cosheaf $\mathcal{A}$ on a general site is flasque iff $\langle \mathcal{A}, T \rangle$ is a flasque sheaf ([Tamme, 1994, Definition 3.5.1] or [Artin, 1962, Definition 2.4.1]) for all injective $T \in \mathbf{Mod}(k)$. Indeed, $\langle \mathcal{A}, T \rangle$ is flasque iff*

$$0 = H^s(R, \langle \mathcal{A}, T \rangle) \simeq \langle H_s(R, \mathcal{A}), T \rangle$$

for all $s > 0$ and all covering sieves R. The latter is equivalent, since T varies through all injective modules, to the fact that $H_s(R, \mathcal{A})$ is zero for all $s > 0$ and all covering sieves R.

3. *On a topological space, any flabby cosheaf is flasque, because it follows from [Bredon, 1997, Theorem II.5.5], that $\langle \mathcal{A}, T \rangle$ is a flasque sheaf whenever it is flabby.*

2.3.10. Definition. *Let $\mathbf{E}$ be a small category and $V \in \mathbf{E}$.*

1. *Let $A \in \mathbf{Pro}(k)$, considered as a precosheaf on the one-object category $\{V\}$. Denote by A^V and A_V the following precosheaves on $\mathbf{E}$:*

$$A^V := (\{V\} \longrightarrow \mathbf{E})^{\ddagger}(A),$$
$$A_V := (\{V\} \longrightarrow \mathbf{E})^{\dagger}(A),$$

If A is a quasi-projective pro-module, then, due to Example 2.3.4 and Proposition 2.3.7, A_V is a quasi-projective cosheaf on $\mathbf{E}$.

2. *Let $A \in \mathbf{Mod}(k)$, considered as a presheaf on the one-object category $\{V\}$. Denote by A^V and A_V the following presheaves on $\mathbf{E}$:*

$$A^V := (\{V\} \longrightarrow \mathbf{E})^{\ddagger}(A),$$
$$A_V := (\{V\} \longrightarrow \mathbf{E})^{\dagger}(A),$$

If A is an injective module, then, A^V is an injective presheaf on $\mathbf{E}$ (compare to Example 2.3.4 and Proposition 2.3.7).

2.3.11. Remark.

1. *The presheaves* $\{k_V \mid V \in \mathbf{E}\}$ *form a set of generators for the category of presheaves* $\mathbf{pS}(\mathbf{E}, \mathbf{Mod}(k))$. *Indeed,*

$$Hom_{\mathbf{pS}(\mathbf{E},\mathbf{Mod}(k))}(k_V, \mathcal{A}) \simeq (\{V\} \longrightarrow \mathbf{E})_* \mathcal{A} \simeq Hom_{\mathbf{Pro}(k)}(k, \mathcal{A}(V)) \simeq \mathcal{A}(V)$$

for any $\mathcal{A} \in \mathbf{pS}(\mathbf{E}, \mathbf{Mod}(k))$. *Therefore, for any* ***proper*** *subpresheaf* $\mathcal{B} \subsetneq \mathcal{A}$, *there exist a* $V \in \mathbf{E}$, *and an* $a \in \mathcal{A}(V)$, $a \notin \mathcal{B}(V)$. *The morphism* $k_V \to \mathcal{A}$, *corresponding to* a, ***does not factor*** *through* $\mathcal{B}$.

2. *The sheaves* $\left\{(k_V)^\# \mid V \in \mathbf{E}\right\}$ *form a set of generators for the category of sheaves* $\mathbf{S}(X, \mathbf{Mod}(k))$. *Indeed,*

$$Hom_{\mathbf{S}(X,\mathbf{Mod}(k))}\left((k_V)^\#, \mathcal{A}\right) \simeq Hom_{\mathbf{pS}(X,\mathbf{Mod}(k))}(k_V, \mathcal{A}) \simeq \mathcal{A}(V)$$

for any $\mathcal{A} \in \mathbf{S}(X, \mathbf{Mod}(k))$. *Therefore, for any* ***proper*** *subsheaf* $\mathcal{B} \subsetneq \mathcal{A}$, *there exist a* $V \in \mathbf{E}$, *and an* $a \in \mathcal{A}(V)$, $a \notin \mathcal{B}(V)$. *The morphism* $(k_V)^\# \to \mathcal{A}$, *corresponding to* a, ***does not factor*** *through* $\mathcal{B}$.

3. *We cannot build a* ***set*** *of cogenerators for* $\mathbf{pCS}(X, \mathbf{Pro}(k))$ *or* $\mathbf{CS}(X, \mathbf{Pro}(k))$. *However, it is possible to build a* ***class*** $\mathfrak{G}$ *of cogenerators, see Theorem 3.1.1 (12) and Theorem 3.3.1 (10).*

3. Main results

3.1. Category of precosheaves.

3.1.1. Theorem. *Let* $\mathbf{E}$ *be a small category.*

1. *The category* $\mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$ *of precosheaves is abelian, complete and cocomplete, and satisfies both the AB3 and AB3* axioms ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]).*

2. *For any diagram*

$$\mathcal{X} : \mathbf{I} \longrightarrow \mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$$

and any $T \in \mathbf{Mod}(k)$ *(not necessarily injective!)*

$$\left\langle \varinjlim_{i\in\mathbf{I}} \mathcal{X}_i, T \right\rangle \simeq \varprojlim_{i\in\mathbf{I}} \langle \mathcal{X}_i, T \rangle$$

in $\mathbf{pS}(\mathbf{E}, \mathbf{Mod}(k))$.

3. *For any diagram*

$$\mathcal{X}:\mathbf{I}\longrightarrow\mathbf{pCS}\left(\mathbf{E},\mathbf{Pro}\left(k\right)\right)$$

and any $T\in\mathbf{Mod}\left(k\right)$

$$\left\langle \varprojlim_{i\in\mathbf{I}}\mathcal{X}_{i},T\right\rangle \simeq\varinjlim_{i\in\mathbf{I}}\left\langle \mathcal{X}_{i},T\right\rangle$$

in $\mathbf{pS}\left(\mathbf{E},\mathbf{Mod}\left(k\right)\right)$ *if either* $\mathcal{X}$ *is cofiltered or* T *is injective.*

4. *For any family* $\left(\mathcal{X}_{i}\right)_{i\in I}$ *in* $\mathbf{pCS}\left(\mathbf{E},\mathbf{Pro}\left(k\right)\right)$ *and any* $T\in\mathbf{Mod}\left(k\right)$ *(not necessarily injective!)*

$$\left\langle \prod_{i\in I}\mathcal{X}_{i},T\right\rangle \simeq\bigoplus_{i\in I}\left\langle \mathcal{X}_{i},T\right\rangle$$

in $\mathbf{pS}\left(\mathbf{E},\mathbf{Mod}\left(k\right)\right)$.

5. *Let* $\mathbf{D}$ *and* $\mathbf{E}$ *be small categories, let*

$$f:\mathbf{E}\longrightarrow\mathbf{D}$$

be a functor, and let $T\in\mathbf{Mod}\left(k\right)$*. Then*

$$\left\langle f^{\dagger}\left(\bullet\right),T\right\rangle =\left(f^{op}\right)^{\ddagger}\left\langle \bullet,T\right\rangle :\mathbf{pS}\left(\mathbf{E},\mathbf{Mod}\left(k\right)\right)\longrightarrow\mathbf{pS}\left(\mathbf{D},\mathbf{Mod}\left(k\right)\right),$$

where $f^{\dagger}$ *and* $g^{\ddagger}$ *are the left and the right Kan extensions (Definition 2.3.5).*

6. *Let* $\mathcal{M}\in\mathbf{pCS}\left(\mathbf{E},\mathbf{Pro}\left(k\right)\right)$*. Then* $\mathcal{M}\simeq0$ *iff* $\left\langle \mathcal{M},T\right\rangle \simeq0$ *for any injective* $T\in\mathbf{Mod}\left(k\right)$.

7. *Let*

$$\mathcal{E}=\left(\mathcal{M}\overset{\alpha}{\longleftarrow}\mathcal{N}\overset{\beta}{\longleftarrow}\mathcal{K}\right)$$

be a sequence of morphisms in $\mathbf{pCS}\left(\mathbf{E},\mathbf{Pro}\left(k\right)\right)$ *with* $\beta\circ\alpha=0$*, and let* $T\in\mathbf{Mod}\left(k\right)$ *be injective. Then*

$$H\left(\mathcal{E}\right):=\frac{\ker\left(\alpha\right)}{im\left(\beta\right)}$$

satisfies

$$\left\langle H\left(\mathcal{E}\right),T\right\rangle \simeq H\left(\left\langle \mathcal{E},T\right\rangle \right):=\frac{\ker\left(\left\langle \beta,T\right\rangle \right)}{im\left(\left\langle \alpha,T\right\rangle \right)}.$$

8. *Let*

$$\mathcal{E}=\left(\mathcal{M}\overset{\alpha}{\longleftarrow}\mathcal{N}\overset{\beta}{\longleftarrow}\mathcal{K}\right)$$

be a sequence of morphisms in $\mathbf{pCS}\left(\mathbf{E},\mathbf{Pro}\left(k\right)\right)$ *with* $\beta\circ\alpha=0$*. Then* $\mathcal{E}$ *is exact iff the sequence*

$$\left\langle \mathcal{M},T\right\rangle \overset{\left\langle \alpha,T\right\rangle }{\longrightarrow}\left\langle \mathcal{N},T\right\rangle \overset{\left\langle \beta,T\right\rangle }{\longrightarrow}\left\langle \mathcal{K},T\right\rangle$$

is exact in $\mathbf{pS}\left(\mathbf{E},\mathbf{Mod}\left(k\right)\right)$ *for all injective* $T\in\mathbf{Mod}\left(k\right)$.

9. *The category* $\mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$ *satisfies the AB*4 *axiom ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]).*

10. *The category* $\mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$ *satisfies the AB*4^* *axiom ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]).*

11. *The category* $\mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$ *satisfies the AB*5^* *axiom ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]): cofiltered limits are exact in* $\mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$.

12. *The* ***class*** *(**not** a set)* $\left\{A^V \mid V \in \mathbf{E},\ A \in \mathfrak{G} \subseteq \mathbf{Pro}(k)\right\}$ *where* $\mathfrak{G}$ *is the class from Proposition A.2.8(13) forms a class of cogenerators ([Grothendieck, 1957, 1.9], [Bucur and Deleanu, 1968, Ch. 5.9]) of the category* $\mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$.

Proof. The category $\mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$ inherits most properties from the category $\mathbf{Pro}(k)$, therefore we can apply Proposition A.2.8.

(**1-4**) Follow from Proposition A.2.8(1-4).

(**5**) Let $\mathcal{A} \in \mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$, and $U \in \mathbf{E}$. It follows from Proposition 2.3.6 that

$$\left[f^{\dagger}\mathcal{A}\right](U) \simeq \varinjlim_{(f(V)\to U)\in f\downarrow U} \mathcal{A}(V),$$

$$\left[(f^{op})^{\ddagger}\mathcal{B}\right](U) \simeq \varprojlim_{(f(V)\to U)\in f\downarrow U} \mathcal{B}(V).$$

Therefore

$$\left\langle f^{\dagger}\mathcal{A}, T\right\rangle(U) = \left\langle \left[f^{\dagger}\mathcal{A}\right](U), T\right\rangle = \left\langle \varinjlim_{(f(V)\to U)\in f\downarrow U} \mathcal{A}(V), T\right\rangle \simeq$$

$$\simeq \varprojlim_{(f(V)\to U)\in f\downarrow U} \langle \mathcal{A}(V), T\rangle \simeq \left[(f^{op})^{\ddagger}\langle \mathcal{A}, T\rangle\right](U).$$

(**6-8**) Follow from Proposition A.2.8(6-8).

(**9-11**) Follow from Proposition A.2.8(10-12).

(**12**) Let

$$(\varphi : \mathcal{C} \twoheadrightarrow \mathcal{D}) \in \mathbf{pCS}(\mathbf{E}, \mathbf{Pro}(k))$$

be a non-trivial epimorphism. It follows that

$$\varphi(U) : \mathcal{C}(U) \longrightarrow \mathcal{D}(U)$$

is an epimorphism in $\mathbf{Pro}(k)$ for any $U \in \mathbf{E}$, and that there exists a $V \in \mathbf{E}$, such that

$$\varphi(V) : \mathcal{C}(V) \longrightarrow \mathcal{D}(V)$$

is **non-trivial** epimorphism. Due to Proposition A.2.8(13), there exist an $A \in \mathfrak{G} \subseteq \mathbf{Pro}(k)$, and a morphism

$$(\psi : \mathcal{C}(V) \longrightarrow A) \in \mathbf{Pro}(k),$$

which does **not** factor through $\mathcal{D}(V)$. The morphism

$$\xi : \mathcal{C} \longrightarrow A^V,$$

which corresponds to ψ under the adjunction

$$Hom_{\mathbf{pCS}(\mathbf{E},\mathbf{Pro}(k))}\left(\mathcal{C}, A^V\right) \simeq Hom_{\mathbf{Pro}(k)}\left(\mathcal{C}(V), A\right)$$

does **not** factor through $\mathcal{D}$. ■

3.2. Precosheaf homology.

3.2.1. Theorem. *Let $X = (\mathbf{C}_X, Cov(X))$ be a small site. Let also*

$$\mathcal{A} \in \mathbf{pCS}(X, \mathbf{Pro}(k)).$$

Remind that $\check{H}$ and ${}^{Roos}\check{H}$ are (isomorphic when the topology is generated by a pre-topology!) Čech homologies from Definition B.2.5.

1. *There exists a functorial epimorphism*

$$\pi : \mathcal{P}(\mathcal{A}) \twoheadrightarrow \mathcal{A},$$

where $\mathcal{P}(\mathcal{A})$ is quasi-projective (Definition 2.3.1(1)).

2.

   (a) *The full subcategory*

$$\mathbf{Q}(\mathbf{pCS}(X, \mathbf{Pro}(k))) \subseteq \mathbf{pCS}(X, \mathbf{Pro}(k))$$

   (Notation 2.3.2) is F-projective (Definition A.3.1) with respect to the functors

$$F(\bullet) = H_0(R, \bullet),$$

   where $R \subseteq h_U$ runs through the sieves (Definition B.1.1) on X;

   (b) *The full subcategory*

$$\mathbf{Q}(\mathbf{pCS}(X, \mathbf{Pro}(k))) \subseteq \mathbf{pCS}(X, \mathbf{Pro}(k))$$

   is F-projective with respect to the functors

$$F(\bullet) = {}^{Roos}\check{H}_0(U, \bullet) \simeq \check{H}_0(U, \bullet), \ U \in \mathbf{C}_X.$$

3.

(a) If the sieve R is generated by a base-changeable (Definition B.2.2) family $\{U_i \to U\}$, then the left satellites (Definition A.3.4) $L_n H_0(R, \mathcal{A})$ satisfy

$$L_n H_0(R, \mathcal{A}) \simeq H_n(\{U_i \to U\}, \mathcal{A}),$$

naturally in $\mathcal{A}$ and R.

(b) The left satellites $L_n \check{H}_0(U, \mathcal{A})$ are naturally, in U and $\mathcal{A}$, isomorphic to

$$\check{H}_n(U, \mathcal{A}) \simeq {}^{Roos}\check{H}_n(U, \mathcal{A}).$$

4. There are isomorphisms, natural in $\mathcal{A}$, R, and T,

(a)

$$\langle H_n(R, \mathcal{A}), T\rangle \simeq H^n(R, \langle \mathcal{A}, T\rangle)$$

for any injective $T \in \mathbf{Mod}(k)$.

(b)

$$\left\langle {}^{Roos}\check{H}_n(U, \mathcal{A}), T\right\rangle \simeq \left\langle \check{H}_n(U, \mathcal{A}), T\right\rangle \simeq \check{H}^n(U, \langle \mathcal{A}, T\rangle) \simeq {}^{Roos}\check{H}^n(U, \langle \mathcal{A}, T\rangle)$$

for any injective $T \in \mathbf{Mod}(k)$ (see Notation 3.2.2).

Proof. (**1**) Let $\mathbf{D}^\delta$ be the discrete category with the same set of objects as $\mathbf{D}$:

$$Ob\left(\mathbf{D}^\delta\right) = Ob(\mathbf{D}),$$

and let $f : \mathbf{D}^\delta \longrightarrow \mathbf{D}$ be the evident functor, identical on objects. Define the precosheaf $\mathcal{P}(\mathcal{A})$ by the following:

$$\mathcal{P}(\mathcal{A}) := f^\dagger \mathcal{G}(f_*(\mathcal{A})),$$

where $\mathcal{F}$ is the functor from Proposition A.2.8(5), and

$$\mathcal{G}(U) := \mathbf{F}([f_*(\mathcal{A})](U)) = \mathbf{F}(\mathcal{A}(U))$$

for $U \in \mathbf{C}_X$. It follows from Proposition 2.3.6 that

$$\mathcal{P}(\mathcal{A})(U) = \bigoplus_{V \to U} \mathbf{F}(\mathcal{A}(V)).$$

The morphism

$$\mathcal{G}(f_*(\mathcal{A})) \longrightarrow f_*(\mathcal{A})$$

induces, by adjunction, the desired homomorphism

$$\pi : \mathcal{P}(\mathcal{A}) = f^\dagger \mathcal{G}(f_*(\mathcal{A})) \longrightarrow \mathcal{A}.$$

Indeed, $\mathcal{G}(f_*(\mathcal{A}))$ is quasi-projective due to Example 2.3.4, and $\mathcal{P}(\mathcal{A})$ is quasi-projective due to Proposition 2.3.7. For any $U \in \mathbf{C}_X$, the composition

$$\mathbf{F}(\mathcal{A}(U)) \hookrightarrow \mathcal{P}(\mathcal{A})(U) = \bigoplus_{V \to U} \mathbf{F}(\mathcal{A}(V)) \xrightarrow{\pi(U)} \mathcal{A}(U)$$

is the epimorphism from Proposition A.2.8(5), therefore π is an epimorphism as well.

(**2**) We have just proved the condition (1) of Definition A.3.1. It remains to check the other two conditions.

Given a short exact sequence

$$0 \longrightarrow \mathcal{B}' \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}'' \longrightarrow 0$$

of precosheaves, assume that

$$\mathcal{B}, \mathcal{B}'' \in Q(\mathbf{pCS}(X, \mathbf{Pro}(k))).$$

Therefore, for any injective $T \in \mathbf{Mod}(k)$, the sequence

$$0 \longrightarrow \langle \mathcal{B}'', T \rangle \longrightarrow \langle \mathcal{B}, T \rangle \longrightarrow \langle \mathcal{B}', T \rangle \longrightarrow 0$$

is exact. Since $\langle \mathcal{B}'', T \rangle$ and $\langle \mathcal{B}, T \rangle$ are injective presheaves, it follows that the sequence above is **split** exact, and

$$\langle \mathcal{B}, T \rangle \simeq \langle \mathcal{B}', T \rangle \times \langle \mathcal{B}'', T \rangle.$$

The presheaf $\langle \mathcal{B}', T \rangle$, being a direct summand of the injective presheaf $\langle \mathcal{B}, T \rangle$, is injective (for any injective T), therefore $\mathcal{B}'$ is quasi-projective. The condition (2) of Definition A.3.1 is proved!

Let now $R \subseteq h_U$ be a sieve. Since both $H^0(R, \bullet)$ and $\check{H}^0$ are **additive** functors

$$\mathbf{pS}(X, \mathbf{Pro}(k)) \longrightarrow \mathbf{Mod}(k),$$

the sequences of k-modules

$$\begin{array}{rcl} 0 & \longrightarrow & H^0(R, \langle \mathcal{B}'', T \rangle) \longrightarrow H^0(R, \langle \mathcal{B}, T \rangle) \longrightarrow H^0(R, \langle \mathcal{B}', T \rangle) \longrightarrow 0, \\ 0 & \longrightarrow & \check{H}^0(U, \langle \mathcal{B}'', T \rangle) \longrightarrow \check{H}^0(U, \langle \mathcal{B}, T \rangle) \longrightarrow \check{H}^0(U, \langle \mathcal{B}', T \rangle) \longrightarrow 0, \end{array}$$

are exact (in fact, split exact). It follows from Proposition A.2.8(8) that the corresponding sequences of pro-modules

$$\begin{array}{rcl} 0 & \longrightarrow & H_0(R, \mathcal{B}') \longrightarrow H_0(R, \mathcal{B}) \longrightarrow H_0(R, \mathcal{B}'') \longrightarrow 0, \\ 0 & \longrightarrow & \check{H}_0(U, \mathcal{B}') \longrightarrow \check{H}_0(U, \mathcal{B}) \longrightarrow \check{H}_0(U, \mathcal{B}'') \longrightarrow 0, \end{array}$$

are exact, because

$$\begin{array}{c} \langle H_0(R, \mathcal{E}), T \rangle \simeq H^0(R, \langle \mathcal{E}, T \rangle) \\ \langle \check{H}_0(U, \mathcal{E}), T \rangle \simeq \check{H}^0(U, \langle \mathcal{E}, T \rangle) \end{array}$$

for any precosheaf $\mathcal{E}$ (see the statement (4) of our theorem).

(**3**) Choose a quasi-projective resolution

$$0 \longleftarrow \mathcal{A} \longleftarrow \mathcal{P}_0 \longleftarrow \mathcal{P}_1 \longleftarrow \mathcal{P}_2 \longleftarrow \dots \longleftarrow \mathcal{P}_n \longleftarrow \dots$$

and construct a bicomplex

$$X_{s,t} = \check{C}_s \left(\{U_i \to U\}, \mathcal{P}_t \right).$$

Due to Theorem A.4.3, one gets two spectral sequences

$${}^{ver}E^r_{s,t},\ {}^{hor}E^r_{s,t} \implies H_{s+t}\left(Tot_\bullet (X)\right).$$

Apply $\langle \bullet, T\rangle$ where $T \in \mathbf{Mod}(k)$ is an arbitrary injective module. It follows that

$$\langle \mathcal{P}_t, T\rangle \in \mathbf{pS}\left(X, \mathbf{Mod}(k)\right)$$

are injective **presheaves** for all t. Due to [Artin, 1962, Corollary 1.4.2] or [Tamme, 1994, Theorem 2.2.3], and the fact that

$$\left\langle \check{C}_s \left(\{U_i \to U\}, \mathcal{P}_t \right), T \right\rangle \simeq \check{C}^s \left(\{U_i \to U\}, \langle \mathcal{P}_t, T\rangle \right)$$

the sequence

$$\left\langle {}^{hor}E^0_{0,t}, T \right\rangle \longrightarrow \left\langle {}^{hor}E^0_{1,t}, T \right\rangle \longrightarrow \left\langle {}^{hor}E^0_{2,t}, T \right\rangle \longrightarrow \dots \longrightarrow \left\langle {}^{hor}E^0_{s,t}, T \right\rangle \longrightarrow \dots$$

is exact for all $s > 0$ (and all T!), therefore the sequence

$${}^{hor}E^0_{0,t} \longleftarrow {}^{hor}E^0_{1,t} \longleftarrow {}^{hor}E^0_{2,t} \longleftarrow \dots \longleftarrow {}^{hor}E^0_{s,t} \longleftarrow \dots$$

is exact for all $s > 0$, and

$${}^{hor}E^1_{s,t} = 0$$

if $s > 0$. The spectral sequence ${}^{hor}E^r$ degenerates from E^2 on, implying

$$H_n\left(Tot_\bullet (X)\right) \simeq {}^{hor}E^2_{0,n} \simeq L_n H_0 \left(\{U_i \to U\}, \mathcal{A} \right).$$

On the other hand, since products are exact in $\mathbf{Mod}(k)$, one gets exact (for $t > 0$) sequences

$$\left\langle {}^{ver}E^0_{s,0}, T \right\rangle \longrightarrow \left\langle {}^{ver}E^0_{s,1}, T \right\rangle \longrightarrow \left\langle {}^{ver}E^0_{s,2}, T \right\rangle \longrightarrow \dots \longrightarrow \left\langle {}^{ver}E^0_{s,t}, T \right\rangle \longrightarrow \dots$$

in $\mathbf{Mod}(k)$, and exact (for $t > 0$) sequences

$${}^{ver}E^0_{s,0} \longleftarrow {}^{ver}E^0_{s,1} \longleftarrow {}^{ver}E^0_{s,2} \longleftarrow \dots \longleftarrow {}^{ver}E^0_{s,t} \longleftarrow \dots$$

It follows that ${}^{ver}E^1_{s,t} = 0$ for $t > 0$, and the sequence ${}^{ver}E^r$ degenerates from E^2 on, therefore

$$L_n H_0 \left(\{U_i \to U\}, \mathcal{A} \right) \simeq H_n\left(Tot_\bullet (X)\right) \simeq {}^{ver}E^2_{n,0} \simeq H_n \left(\{U_i \to U\}, \mathcal{A} \right).$$

Apply $\varprojlim_{R \in Cov(U)}$ to the bicomplexes $X_{\bullet,\bullet}$ to get the bicomplex $\check{X}_{\bullet,\bullet}$. The two spectral sequences for $\check{X}_{\bullet,\bullet}$ degenerate from E^2 on, giving the desired isomorphisms.

(**4**) See the proof of (3). It remains only to remind (Proposition 2.1.13 (1)) that $\langle \bullet, T\rangle$ converts cofiltered limits $\varprojlim$ into filtered colimits $\varinjlim$. ■

3.2.2. NOTATION. *For a sieve $R \subseteq h_U$, the left satellites $L_n H_0(R, \bullet)$ are denoted by $H_n(R, \bullet)$.*

3.3. CATEGORY OF COSHEAVES.

3.3.1. THEOREM. *Let $X = (\mathbf{C}_X, Cov(X))$ be a site.*

1. *The category $\mathbf{CS}(X, \mathbf{Pro}(k))$ is abelian, complete and cocomplete, and satisfies both the AB3 and AB3* axioms ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]).*
2. *For any diagram*
$$\mathcal{X} : \mathbf{I} \longrightarrow \mathbf{CS}(X, \mathbf{Pro}(k))$$
and any $T \in \mathbf{Mod}(k)$ (not necessarily injective!)
$$\left\langle \varinjlim_{i \in \mathbf{I}} \mathcal{X}_i, T \right\rangle \simeq \varprojlim_{i \in \mathbf{I}} \langle \mathcal{X}_i, T \rangle$$
in $\mathbf{S}(X, \mathbf{Mod}(k))$.
3. *For any diagram*
$$\mathcal{X} : \mathbf{I} \longrightarrow \mathbf{CS}(X, \mathbf{Pro}(k))$$
and any $T \in \mathbf{Mod}(k)$
$$\left\langle \varprojlim_{i \in \mathbf{I}} \mathcal{X}_i, T \right\rangle \simeq \varinjlim_{i \in \mathbf{I}} \langle \mathcal{X}_i, T \rangle$$
in $\mathbf{S}(X, \mathbf{Mod}(k))$ if either $\mathcal{X}$ is cofiltered or T is injective.
4. *For any family $(\mathcal{X}_i)_{i \in I}$ in $\mathbf{CS}(X, \mathbf{Pro}(k))$ and any $T \in \mathbf{Mod}(k)$ (not necessarily injective!)*
$$\left\langle \prod_{i \in I} \mathcal{X}_i, T \right\rangle \simeq \bigoplus_{i \in I} \langle \mathcal{X}_i, T \rangle$$
in $\mathbf{S}(X, \mathbf{Mod}(k))$.
5. *Let $\mathcal{M} \in \mathbf{CS}(X, \mathbf{Pro}(k))$. Then $\mathcal{M} \simeq 0$ iff $\langle \mathcal{M}, T \rangle = 0$ for any injective $T \in \mathbf{Mod}(k)$.*
6. *Let*
$$\mathcal{E} = \left(\mathcal{M} \xleftarrow{\alpha} \mathcal{N} \xleftarrow{\beta} \mathcal{K} \right)$$
be a sequence of morphisms in $\mathbf{CS}(X, \mathbf{Pro}(k))$ with $\beta \circ \alpha = 0$, and let $T \in \mathbf{Mod}(k)$ be injective. Then
$$H(\mathcal{E}) := \frac{\ker(\alpha)}{im(\beta)}$$
satisfies
$$\langle H(\mathcal{E}), T \rangle \simeq H(\langle \mathcal{E}, T \rangle) := \frac{\ker(\langle \beta, T \rangle)}{im(\langle \alpha, T \rangle)}.$$

7. *Let*

$$\mathcal{E}=\left(\mathcal{M}\xleftarrow{\alpha}\mathcal{N}\xleftarrow{\beta}\mathcal{K}\right)$$

be a sequence of morphisms in $\mathbf{CS}(X,\mathbf{Pro}(k))$ *with* $\beta\circ\alpha=0$. *Then* $\mathcal{E}$ *is exact iff the sequence*

$$\langle\mathcal{M},T\rangle\xrightarrow{\langle\alpha,T\rangle}\langle\mathcal{N},T\rangle\xrightarrow{\langle\beta,T\rangle}\langle\mathcal{K},T\rangle$$

is exact in $\mathbf{S}(X,\mathbf{Mod}(k))$ *for all injective* $T\in\mathbf{Mod}(k)$.

8. *The category* $\mathbf{CS}(X,\mathbf{Pro}(k))$ *satisfies the* $AB4^*$ *axiom ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]).*

9. *The category* $\mathbf{CS}(X,\mathbf{Pro}(k))$ *satisfies the* $AB5^*$ *axiom ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]): cofiltered limits are exact in* $\mathbf{CS}(X,\mathbf{Pro}(k))$.

10. *The **class** (**not** a set)* $\left\{\left(A^V\right)_\#\mid V\in\mathbf{E},\ A\in\mathfrak{G}\subseteq\mathbf{Pro}(k)\right\}$ *where* $\mathfrak{G}$ *is the class from Proposition A.2.8(13) forms a class of cogenerators ([Grothendieck, 1957, 1.9], [Bucur and Deleanu, 1968, Ch. 5.9]) of the category* $\mathbf{CS}(X,\mathbf{Pro}(k))$.

PROOF. (**1**)

- *Kernels.* Given a morphism f of cosheaves

$$f:\mathcal{A}\longrightarrow\mathcal{B},$$

let

$$\mathcal{K}=\ker(\iota f:\iota\mathcal{A}\longrightarrow\iota\mathcal{B})$$

in $\mathbf{pCS}(X,\mathbf{Pro}(k))$. Then, for any $\mathcal{C}\in\mathbf{CS}(X,\mathbf{Pro}(k))$,

$$\begin{aligned}&Hom_{\mathbf{CS}(X,\mathbf{Pro}(k))}(\mathcal{C},\mathcal{K}_\#)\simeq Hom_{\mathbf{pCS}(X,\mathbf{Pro}(k))}(\mathcal{C},\mathcal{K})\simeq\\&\simeq\ker\left(Hom_{\mathbf{CS}(X,\mathbf{Pro}(k))}(\mathcal{C},\mathcal{A})\longrightarrow Hom_{\mathbf{CS}(X,\mathbf{Pro}(k))}(\mathcal{C},\mathcal{B})\right),\end{aligned}$$

therefore $\mathcal{K}_\#$ is the kernel of f in $\mathbf{CS}(X,\mathbf{Pro}(k))$.

- *Cokernels.* The cokernel of ιf is clearly a cosheaf, therefore

$$\operatorname{coker} f:=\operatorname{coker}\iota f$$

is the desired cokernel.

- *Products.* Let

$$(\mathcal{A}_i)_{i\in I}$$

be a family of cosheaves, and let

$$\mathcal{B}:=\prod_{i\in I}\iota(\mathcal{A}_i)$$

in $\mathbf{pCS}(X, \mathbf{Pro}(k))$. Then, for any $\mathcal{C} \in \mathbf{CS}(X, \mathbf{Pro}(k))$,

$$
\begin{aligned}
&Hom_{\mathbf{CS}(X,\mathbf{Pro}(k))}(\mathcal{C}, \mathcal{B}_{\#}) \simeq Hom_{\mathbf{pCS}(X,\mathbf{Pro}(k))}(\mathcal{C}, \mathcal{B}) \simeq \\
&\simeq \prod_{i \in I} Hom_{\mathbf{CS}(X,\mathbf{Pro}(k))}(\mathcal{C}, \mathcal{A}_i),
\end{aligned}
$$

therefore $\mathcal{B}_{\#}$ is the product of $\mathcal{A}_i$ in $\mathbf{CS}(X, \mathbf{Pro}(k))$.

- *Coproducts.* The coproduct

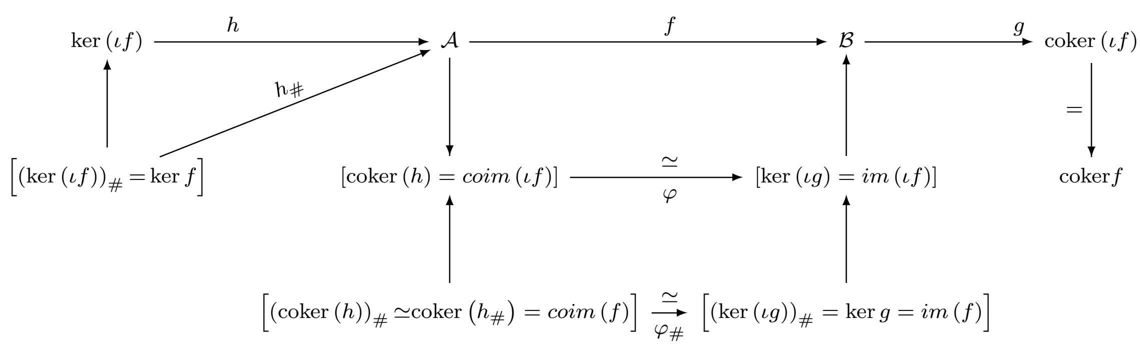
$$\bigoplus_{i \in I} \iota(\mathcal{A}_i)$$

is clearly a cosheaf, and can therefore serve as a coproduct in $\mathbf{CS}(X, \mathbf{Pro}(k))$.

- *Limits* $\varprojlim$ are built as combinations of products and kernels. The category $\mathbf{CS}(X, \mathbf{Pro}(k))$ is *complete.*
- *Colimits* $\varinjlim$ are built as combinations of coproducts and cokernels. The category $\mathbf{CS}(X, \mathbf{Pro}(k))$ is *cocomplete.*
- *Images and coimages.* Let

$$(f : \mathcal{A} \longrightarrow \mathcal{B}) \in \mathbf{CS}(X, \mathbf{Pro}(k)).$$

Consider the diagram of (pre)cosheaves



The cosheafification functor $()_{\#}$ is exact, due to Theorem 2.2.6 (4), therefore

$$(\mathrm{coker}(h))_{\#} \simeq \mathrm{coker}(h_{\#}).$$

Since the category of precosheaves $\mathbf{pCS}(X, \mathbf{Pro}(k))$ is abelian,

$$\varphi : coim(\iota f) \longrightarrow im(\iota f)$$

is an isomorphism. It follows that

$$\varphi_{\#} : coim(f) = (coim(\iota f))_{\#} \longrightarrow (im(\iota f))_{\#} = im(f)$$

is an isomorphism as well, and the category of cosheaves $\mathbf{CS}(X, \mathbf{Pro}(k))$ is abelian.

(**2**) Follows from Theorem 3.1.1(2), because the inclusion functor

$$\iota : \mathbf{CS}\left(X, \mathbf{Pro}\left(k\right)\right) \longrightarrow \mathbf{pCS}\left(X, \mathbf{Pro}\left(k\right)\right),$$

being a left adjoint to $()_{\#}$, preserves **colimits**, while

$$\iota : \mathbf{S}\left(X, \mathbf{Mod}\left(k\right)\right) \longrightarrow \mathbf{pS}\left(X, \mathbf{Mod}\left(k\right)\right),$$

being a right adjoint to $()^{\#}$, preserves **limits**.

(**3**) If $\mathcal{X}$ is cofiltered, then

$$\begin{aligned} &\left\langle \varprojlim_{i\in\mathbf{I}} \mathcal{X}_i, T\right\rangle \simeq \left(\iota\left\langle \varprojlim_{i\in\mathbf{I}} \mathcal{X}_i, T\right\rangle\right)^{\#} \simeq \left\langle \left(\iota \circ \varprojlim_{i\in\mathbf{I}} \mathcal{X}_i\right)_{\#}, T\right\rangle \simeq \\ &\simeq \left\langle \left(\varprojlim_{i\in\mathbf{I}} \left(\iota \circ \mathcal{X}_i\right)\right)_{\#}, T\right\rangle \simeq \left\langle \varprojlim_{i\in\mathbf{I}} \left(\iota \circ \mathcal{X}_i\right), T\right\rangle \simeq \\ &\simeq \varinjlim_{i\in\mathbf{I}} \left\langle \iota \circ \mathcal{X}_i, T\right\rangle \simeq \varinjlim_{i\in\mathbf{I}} \left\langle \left(\iota \circ \mathcal{X}_i\right)_{\#}, T\right\rangle \simeq \varinjlim_{i\in\mathbf{I}} \left\langle \mathcal{X}_i, T\right\rangle, \end{aligned}$$

since $\mathcal{X}_i$ are cosheaves. If T is injective, then it is enough to prove:

1. $\langle \bullet, T\rangle$ converts products into coproducts: done in (4);
2. $\langle \bullet, T\rangle$ converts kernels into cokernels: done in (6).

(**4**) If $A \subseteq I$ is finite, then the isomorphism

$$\left\langle \prod_{i\in A} \mathcal{X}_i, T\right\rangle \simeq \bigoplus_{i\in A} \left\langle \mathcal{X}_i, T\right\rangle$$

follows from the additivity of $\langle \bullet, T\rangle$. Let now $\mathbf{J}$ be the poset of finite subsets of I. $\mathbf{J}$ is clearly filtered, and $\mathbf{J}^{op}$ is cofiltered. Due to (3),

$$\begin{aligned} &\left\langle \prod_{i\in I} \mathcal{X}_i, T\right\rangle \simeq \left\langle \varprojlim_{A\in\mathbf{J}^{op}} \left(\prod_{i\in A} \mathcal{X}_i\right), T\right\rangle \simeq \varinjlim_{A\in\mathbf{J}} \left\langle \prod_{i\in A} \mathcal{X}_i, T\right\rangle \simeq \\ &\simeq \varinjlim_{A\in\mathbf{J}} \left(\bigoplus_{i\in A} \left\langle \mathcal{X}_i, T\right\rangle\right) \simeq \bigoplus_{i\in I} \left\langle \mathcal{X}_i, T\right\rangle. \end{aligned}$$

(**5**) Follows from Theorem 3.1.1(6).

(**6**)

$$H\left(\mathcal{E}\right) = \frac{\ker\left(\alpha\right)}{im\left(\beta\right)} = \operatorname{coker}\left(\mathcal{K} \longrightarrow \ker\left(\alpha\right) = \left(\ker\left(\iota\alpha\right)\right)_{\#}\right).$$

It follows from Theorem 3.1.1 (7) that

$$\langle H(\mathcal{E}), T\rangle \simeq \left\langle \operatorname{coker}\left(\iota\mathcal{K} \longrightarrow (\ker(\iota\alpha))_{\#}\right), T\right\rangle \simeq \ker\left(\left\langle (\ker(\iota\alpha))_{\#} \longrightarrow \iota\mathcal{K}, T\right\rangle\right) \simeq$$
$$\simeq \ker\left(\left\langle (\ker(\iota\alpha))_{\#}, T\right\rangle \longrightarrow \langle\iota\mathcal{K}, T\rangle\right) \simeq \ker\left(\langle\ker(\iota\alpha), T\rangle^{\#} \longrightarrow \langle\iota\mathcal{K}, T\rangle\right) \simeq$$
$$\simeq \ker\left((\operatorname{coker}\langle\iota\alpha, T\rangle)^{\#} \longrightarrow \langle\iota\mathcal{K}, T\rangle\right) \simeq \ker(\operatorname{coker}\langle\alpha, T\rangle \longrightarrow \langle\mathcal{K}, T\rangle) \simeq \frac{\ker(\langle\beta, T\rangle)}{im(\langle\alpha, T\rangle)}.$$

(**7**) Follows from (6) and (5).
(**8**) Follows, since $\mathbf{S}(X, \mathbf{Mod}(k))$ satisfies AB4, from (4).
(**9**) Follows, since $\mathbf{S}(X, \mathbf{Mod}(k))$ satisfies AB5, from (3).
(**10**) Let

$$(\varphi : \mathcal{C} \twoheadrightarrow \mathcal{D}) \in \mathbf{CS}(X, \mathbf{Pro}(k))$$

be a non-trivial epimorphism. It means that

$$\ker(\varphi) \neq 0.$$

Since

$$\operatorname{coker}(\varphi) \simeq \operatorname{coker}(\iota\varphi),$$

$\iota\varphi$ is an epimorphism in $\mathbf{pCS}(X, \mathbf{Pro}(k))$ as well. It is non-trivial ($\ker(\iota\varphi) \neq 0$), because if it were trivial, then

$$0 \neq \ker(\varphi) = (\ker(\iota\varphi))_{\#} = 0_{\#} = 0.$$

It follows from Theorem 3.1.1 (12) that there exists an $A \in \mathfrak{G}$, $V \in \mathbf{C}_X$, and a morphism

$$\psi : \mathcal{C} \longrightarrow A^V,$$

that cannot be factored through $\mathcal{D}$. In other words,

$$Hom_{\mathbf{pCS}(X,\mathbf{Pro}(k))}\left(\mathcal{D}, \left(A^V\right)_{\#}\right) \simeq Hom_{\mathbf{pCS}(X,\mathbf{Pro}(k))}\left(\mathcal{D}, A^V\right)$$
$$\longrightarrow Hom_{\mathbf{pCS}(X,\mathbf{Pro}(k))}\left(\mathcal{C}, A^V\right) \simeq Hom_{\mathbf{pCS}(X,\mathbf{Pro}(k))}\left(\mathcal{C}, \left(A^V\right)_{\#}\right)$$

is **not** an epimorphism. It follows that the corresponding morphism

$$\psi_{\#} : \mathcal{C} \longrightarrow \left(A^V\right)_{\#}$$

cannot be factored through $\mathcal{D}$. ∎

3.4. Cosheaf homology.

3.4.1. Theorem. *Let X be a small site. Let also $\check{H}$ and ${}^{Roos}\check{H}$ be (isomorphic when the topology is generated by a pre-topology!) Čech homologies from Definition B.2.5.*

1. *For an arbitrary cosheaf $\mathcal{A} \in \mathbf{CS}(X, \mathbf{Pro}(k))$, there exists a functorial epimorphism*
$$\sigma(\mathcal{A}) : \mathcal{R}(\mathcal{A}) \twoheadrightarrow \mathcal{A},$$
where $\mathcal{R}(\mathcal{A})$ is quasi-projective.

2. *The full subcategory*
$$\mathbf{Q}(\mathbf{CS}(X, \mathbf{Pro}(k))) \subseteq \mathbf{CS}(X, \mathbf{Pro}(k))$$
is F-projective (Definition A.3.1) with respect to the functors:
   (a)
$$F(\bullet) = \Gamma(U, \bullet) := \bullet(U);$$
   (b)
$$F = \iota : \mathbf{CS}(X, \mathbf{Pro}(k)) \hookrightarrow \mathbf{pCS}(X, \mathbf{Pro}(k)).$$

3. *The left satellites $L_n\Gamma(U, \bullet)$ satisfy*
$$\langle L_n\Gamma(U, \bullet), T\rangle \simeq H^n(U, \langle \bullet, T\rangle)$$
for any injective $T \in \mathbf{Mod}(k)$.

4. *The left satellites $L_n\iota$ satisfy*
   (a)
$$\langle (L_n\iota)\bullet, T\rangle \simeq \mathcal{H}^n(\langle \bullet, T\rangle),$$
   for any injective $T \in \mathbf{Mod}(k)$ (see Notation 3.4.2 for $\mathcal{H}^n$),
   (b)
$$[(L_n\iota)\mathcal{A}](U) \simeq H_n(U, \mathcal{A}).$$

5. $$(\mathcal{H}_t\mathcal{A})_+ = 0$$
for all $t > 0$.

6.
   (a) *For any $U \in \mathbf{C}_X$ and any covering sieve R on U there exists a natural spectral sequence*
$$E^2_{s,t} = H_s(R, \mathcal{H}_t(\mathcal{A})) \implies H_{s+t}(U, \mathcal{A}),$$
   converging to the homology of $\mathcal{A}$ (see Notation 3.4.2 for $\mathcal{H}_t$).

(b) *For any $U \in \mathbf{C}_X$ there exists a natural spectral sequence*

$$E^2_{s,t} = {}^{Roos}\check{H}_s\left(U, \mathcal{H}_t\left(\mathcal{A}\right)\right) \implies H_{s+t}\left(U, \mathcal{A}\right),$$

converging to the homology of $\mathcal{A}$.

(c) *There are natural (in U and $\mathcal{A}$) isomorphisms*

$$H_0\left(U, \mathcal{A}\right) \simeq \check{H}_0\left(U, \mathcal{A}\right),$$
$$H_1\left(U, \mathcal{A}\right) \simeq \check{H}_1\left(U, \mathcal{A}\right),$$

and a natural (in U and $\mathcal{A}$) epimorphism

$$H_2\left(U, \mathcal{A}\right) \twoheadrightarrow \check{H}_2\left(U, \mathcal{A}\right).$$

7. *Assume that the topology on X is generated by a pretopology (Definition B.1.6). Then:*

(a) *The spectral sequence from (6a) becomes*

$$E^2_{s,t} = H_s\left(\left\{U_i \to U\right\}, \mathcal{H}_t\left(\mathcal{A}\right)\right) \implies H_{s+t}\left(U, \mathcal{A}\right).$$

(b) *The spectral sequence from (6b) becomes*

$$E^2_{s,t} = \check{H}_s\left(U, \mathcal{H}_t\left(\mathcal{A}\right)\right) \implies H_{s+t}\left(U, \mathcal{A}\right).$$

Proof. (**1**) Define the following functor

$$\begin{array}{rcl} \mathcal{Q} & : & \mathbf{CS}\left(X, \mathbf{Pro}\left(k\right)\right) \longrightarrow \mathbf{CS}\left(X, \mathbf{Pro}\left(k\right)\right): \\ \mathcal{Q}\left(\mathcal{A}\right) & = & \left[\mathcal{P}\left(\mathcal{A}\right)\right]_{\#}, \end{array}$$

where $\mathcal{P}$ is from Theorem 3.2.1(1) One has the following natural epimorphism

$$\rho\left(\mathcal{A}\right): \mathcal{Q}\left(\mathcal{A}\right) = \mathcal{P}\left(\mathcal{A}\right)_{\#} \longrightarrow \mathcal{P}\left(\mathcal{A}\right) \longrightarrow \mathcal{A}.$$

For ordinals α, define $\mathcal{Q}_\alpha$ using transfinite induction:

$$\mathcal{Q}_\alpha\left(\mathcal{A}\right) := \mathcal{Q}\left(\mathcal{Q}_\beta\left(\mathcal{A}\right)\right)$$

if $\alpha = \beta + 1$, and

$$\mathcal{Q}_\alpha := \varprojlim_{\beta < \alpha} \mathcal{Q}_\beta\left(\mathcal{A}\right)$$

if α is a limit ordinal. The sheaves $\left(\left(k_V\right)^{\#}\right)_{V \in Ob(\mathbf{C}_X)}$ form a set of generators of $\mathbf{S}(X,$ $\mathbf{Mod}(k)$ (Remark 2.3.11). Consider the coproduct

$$\mathcal{G} := \bigoplus_{V \in Ob(\mathbf{C}_X)} \left(k_V\right)^{\#} = \left(\bigoplus_{V \in Ob(\mathbf{C}_X)} k_V\right)^{\#}$$

in the category of sheaves. Let W be the set of representatives of all subsheaves of $\mathcal{G}$. Let further, for $\mathcal{E} \in W$,

$$S(\mathcal{E}) = \left(\coprod_{U \in Ob(\mathbf{C}_X)} \mathcal{E}(U) \right) \in \mathbf{Set},$$

be the coproduct in the category **Set**, and let β be any cardinal of cofinality larger than

$$\sup \left(card \left(S \left(\mathcal{E} \right) \right) \right)_{\mathcal{E} \in W}.$$

We claim that the epimorphism

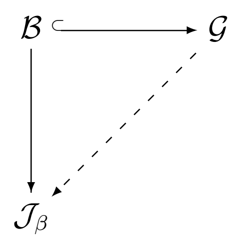
$$\mathcal{R}(\mathcal{A}) := \mathcal{Q}_\beta(\mathcal{A}) \longrightarrow \mathcal{A}$$

is as desired. Indeed, it is enough to prove that $\mathcal{Q}_\beta(\mathcal{A})$ is quasi-projective.

Let T be any injective k-module, and let

$$\mathcal{J}_\alpha := \langle \mathcal{Q}_\alpha(\mathcal{A}), T \rangle, \alpha \leq \beta.$$

We have to prove that $\mathcal{J}_\beta$ is an injective sheaf. Since $\mathcal{G}$ is a generator for $\mathbf{S}(X, \mathbf{Mod}(k))$, it is enough [Grothendieck, 1957, Lemme 1.10.1] to prove the existence of the dashed arrow in any diagram of the form

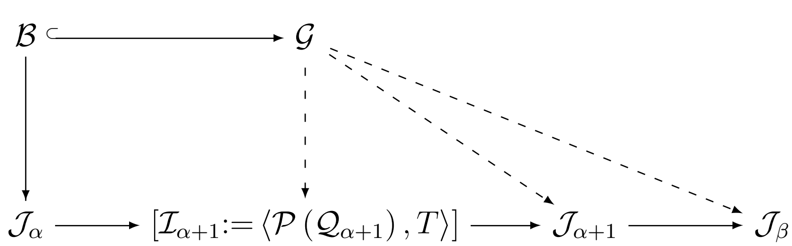


where $\mathcal{B}$ is a subsheaf of $\mathcal{G}$. Since

$$card(S(\mathcal{B})) = card \left(\coprod_{U \in Ob(\mathbf{C}_X)} \mathcal{E}(U) \right) < \beta,$$

there exists an $\alpha < \beta$, such that $\mathcal{B} \to \mathcal{J}_\beta$ factors through $\mathcal{J}_\alpha$.

Consider the commutative diagram



The second vertical arrow exists, because $\mathcal{I}_{\alpha+1}$ is an injective **pre**sheaf, and the morphism $\mathcal{B} \hookrightarrow \mathcal{G}$, being a **mono**morphism of sheaves, is a **mono**morphism of **pre**sheaves, as well.

(**2**) The first condition in Definition A.3.1 follows from (1). Let now

$$0 \longrightarrow \mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}'' \longrightarrow 0$$

be an exact sequence with $\mathcal{P}$, $\mathcal{P}'' \in Q\left(\mathbf{CS}\left(X, \mathbf{Pro}\left(k\right)\right)\right)$. For any injective $T \in \mathbf{Mod}\left(k\right)$, the sequence of **sheaves**

$$0 \longrightarrow \left\langle \mathcal{P}'', T\right\rangle \longrightarrow \left\langle \mathcal{P}, T\right\rangle \longrightarrow \left\langle \mathcal{P}', T\right\rangle \longrightarrow 0$$

is exact in $\mathbf{S}\left(X, \mathbf{Mod}\left(k\right)\right)$, while $\left\langle \mathcal{P}'', T\right\rangle$ and $\left\langle \mathcal{P}, T\right\rangle$ are injective. Therefore the above sequence splits, and

$$\left\langle \mathcal{P}, T\right\rangle \simeq \left\langle \mathcal{P}'', T\right\rangle \oplus \left\langle \mathcal{P}', T\right\rangle.$$

The sheaf $\left\langle \mathcal{P}', T\right\rangle$, being a direct summand of an injective sheaf, is injective, therefore the cosheaf $\mathcal{P}'$ is quasi-projective,

$$\mathcal{P}' \in Q\left(\mathbf{CS}\left(X, \mathbf{Pro}\left(k\right)\right)\right).$$

The second condition in Definition A.3.1 is proved!

Apply the functor $\mathcal{A} \mapsto \mathcal{A}\left(U\right)$ to the split exact sequence above, and get the following split exact sequences in $\mathbf{Mod}\left(k\right)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left\langle \mathcal{P}'', T\right\rangle\left(U\right) & \longrightarrow & \left\langle \mathcal{P}, T\right\rangle\left(U\right) & \longrightarrow & \left\langle \mathcal{P}', T\right\rangle\left(U\right) & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
0 & \longrightarrow & \left\langle \mathcal{P}''\left(U\right), T\right\rangle & \longrightarrow & \left\langle \mathcal{P}\left(U\right), T\right\rangle & \longrightarrow & \left\langle \mathcal{P}'\left(U\right), T\right\rangle & \longrightarrow & 0
\end{array}$$

It follows that the sequence

$$0 \longrightarrow \mathcal{P}'\left(U\right) \longrightarrow \mathcal{P}\left(U\right) \longrightarrow \mathcal{P}''\left(U\right) \longrightarrow 0$$

is exact in $\mathbf{Pro}\left(k\right)$, and the third condition for the F-projectivity is proved for the functor

$$F\left(\bullet\right) = \Gamma\left(U, \bullet\right) = \bullet\left(U\right).$$

Consider now the following split exact sequences of presheaves

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \iota\left\langle \mathcal{P}'', T\right\rangle & \longrightarrow & \iota\left\langle \mathcal{P}, T\right\rangle\left(U\right) & \longrightarrow & \iota\left\langle \mathcal{P}', T\right\rangle & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
0 & \longrightarrow & \left\langle \iota\mathcal{P}'', T\right\rangle & \longrightarrow & \left\langle \iota\mathcal{P}, T\right\rangle & \longrightarrow & \left\langle \iota\mathcal{P}', T\right\rangle & \longrightarrow & 0
\end{array}$$

It follows that the sequence

$$0 \longrightarrow \iota\mathcal{P}' \longrightarrow \iota\mathcal{P} \longrightarrow \iota\mathcal{P}'' \longrightarrow 0$$

is exact in $\mathbf{pCS}(X, \mathbf{Pro}(k))$, and the third condition for the F-projectivity is proved for the inclusion functor

$$\iota : \mathbf{CS}(X, \mathbf{Pro}(k)) \longrightarrow \mathbf{pCS}(X, \mathbf{Pro}(k)).$$

(**3**) Let

$$0 \longleftarrow \mathcal{A} \longleftarrow \mathcal{P}_0 \longleftarrow \mathcal{P}_1 \longleftarrow \mathcal{P}_2 \longleftarrow ... \longleftarrow \mathcal{P}_n \longleftarrow ...$$

be a quasi-projective resolution. Apply the functor $\Gamma(U, \bullet) = \bullet(U)$, and get a chain complex of pro-modules:

$$0 \longleftarrow \mathcal{P}_0(U) \longleftarrow \mathcal{P}_1(U) \longleftarrow \mathcal{P}_2(U) \longleftarrow ... \longleftarrow \mathcal{P}_n(U) \longleftarrow ...$$

Let $T \in \mathbf{Mod}(k)$ be injective. Apply $\langle \bullet, T\rangle$, and get an injective resolution of $\langle \mathcal{A}, T\rangle$:

$$0 \longrightarrow \langle \mathcal{A}, T\rangle \longrightarrow \langle \mathcal{P}_0, T\rangle \longrightarrow \langle \mathcal{P}_1, T\rangle \longrightarrow \langle \mathcal{P}_2, T\rangle \longrightarrow ... \longrightarrow \langle \mathcal{P}_n, T\rangle \longrightarrow ...$$

It follows that

$$\langle L_n\Gamma(U, \mathcal{A}), T\rangle \simeq \langle H_n(\mathcal{P}_\bullet(U)), T\rangle \simeq H^n \langle \mathcal{P}_\bullet(U), T\rangle \simeq H^n(U, \langle \mathcal{A}, T\rangle).$$

(**4**) Apply the inclusion functor ι to the quasi-projective resolution above, and get a chain complex of precosheaves:

$$0 \longleftarrow \iota\mathcal{P}_0 \longleftarrow \iota\mathcal{P}_1 \longleftarrow \iota\mathcal{P}_2 \longleftarrow ... \longleftarrow \iota\mathcal{P}_n \longleftarrow ...$$

The precosheaf $L_n\iota$ is defined by

$$(L_n\iota)\mathcal{A} := H_n(\iota\mathcal{P}_\bullet)$$

The functor

$$\mathcal{B} \longmapsto \mathcal{B}(U) : \mathbf{pCS}(X, \mathbf{Pro}(k)) \longrightarrow \mathbf{Pro}(k)$$

is exact, therefore

$$[(L_n\iota)\mathcal{A}](U) \simeq H_n(\iota\mathcal{P}_\bullet(U)) \simeq H_n(\mathcal{P}_\bullet(U)) \simeq H_n(U, \mathcal{A}),$$

proving (b). Moreover,

$$\langle (L_n\iota)\mathcal{A}, T\rangle \simeq \langle H_n(\iota\mathcal{P}_\bullet), T\rangle \simeq H^n \langle \iota\mathcal{P}_\bullet, T\rangle \simeq \mathcal{H}^n \langle \mathcal{A}, T\rangle,$$

proving (a).

(**5**) It follows from [Prasolov, 2016, Theorem 2.12(2, 3)] that

$$(\mathcal{H}_t\mathcal{A})_\# \longrightarrow (\mathcal{H}_t\mathcal{A})_+$$

is an epimorphism. Therefore, it is enough to prove that $(\mathcal{H}_t\mathcal{A})_\# = 0$ for $t > 0$. Apply the **exact** (due to Theorem 2.2.6 (4)) functor $()_\#$ to the chain complex

$$0 \longleftarrow \iota\mathcal{P}_0 \longleftarrow \iota\mathcal{P}_1 \longleftarrow \iota\mathcal{P}_2 \longleftarrow ... \longleftarrow \iota\mathcal{P}_n \longleftarrow ...$$

Since $()_{\#} \circ \iota = 1_{\mathbf{CS}(X,\mathbf{Pro}(k))}$, one gets an acyclic complex

$$(\iota\mathcal{P}_\bullet)_{\#} \simeq (\mathcal{P}_\bullet).$$

Therefore,

$$0 = H_t\mathcal{P}_\bullet \simeq H_t\left[(\iota\mathcal{P}_\bullet)_{\#}\right] \simeq \left[H_t(\iota\mathcal{P}_\bullet)\right]_{\#} \simeq (\mathcal{H}_t\mathcal{A})_{\#}$$

for $t > 0$.

(**6**) Let $X_{\bullet,\bullet}$ be the following bicomplex in $\mathbf{Pro}(k)$:

$$(X_{s,t}, d, \delta) := \left(\bigoplus_{(U_0 \to U_1 \to \ldots \to U_s \to U) \in \mathbf{C}_R} \mathcal{P}_t(U_0), d, \delta\right),$$

where δ is inherited from the above quasi-projective resolution, and d is as in Definition B.2.4. Consider the two spectral sequences

$$\begin{array}{rcl} {}^{ver}E^2_{s,t} & \Longrightarrow & H_{s+t}(Tot_\bullet(X)), \\ {}^{hor}E^2_{s,t} & \Longrightarrow & H_{s+t}(Tot_\bullet(X)). \end{array}$$

Since $\mathcal{P}_t$ are quasi-projective cosheaves, thus quasi-projective **pre**cosheaves, it follows that

$$\begin{array}{rcl} {}^{hor}E^1_{s,t} & = & {}^{hor}H_s(X_{\bullet,\bullet}) = H_s(R, \mathcal{P}_t) = \begin{cases} H_0(R,\mathcal{P}_t) \simeq \mathcal{P}_t(U) & \text{if} \quad s = 0, \\ 0 & \text{if} \quad s \neq 0. \end{cases} \\ {}^{hor}E^2_{s,t} & = & \begin{cases} H_t(U,\mathcal{A}) & \text{if} \quad s = 0, \\ 0 & \text{if} \quad s \neq 0. \end{cases} \end{array}$$

The spectral sequence degenerates from E_2 on, implying

$$H_n(Tot_\bullet(X)) \simeq H_n(U,\mathcal{A}),$$

Furthermore,

$$\begin{array}{rcl} {}^{ver}E^1_{s,t} & = & {}^{ver}H_t(X_{\bullet,\bullet}) = \displaystyle\bigoplus_{(U_0 \to U_1 \to \ldots \to U_s \to U) \in \mathbf{C}_R} \mathcal{H}_t\mathcal{A}(U_0), \\ {}^{ver}E^2_{s,t} & = & H_s(R, \mathcal{H}_t\mathcal{A}) \Longrightarrow H_{s+t}(Tot_\bullet(X)) \simeq H_{s+t}(U,\mathcal{A}), \end{array}$$

proving (a).

Apply $\varprojlim$ over all covering sieves, to the above spectral sequence, and get the desired spectral sequence

$$E^2_{s,t} = \check{H}_s(U, \mathcal{H}_t\mathcal{A}) \Longrightarrow H_{s+t}(U,\mathcal{A}),$$

proving (b).

To prove (c), notice that

$$\begin{array}{rcl} & & \mathcal{H}_0\mathcal{A} \simeq \mathcal{A}, \\ & & E^2_{s,0} \simeq \check{H}_s(U,\mathcal{A}), \\ E^2_{0,t} & = & 0, \ t > 0. \end{array}$$

It follows that

$$\check{H}_0(U,\mathcal{A}) \simeq E^2_{0,0} \simeq E^\infty_{0,0} \simeq H_0(U,\mathcal{A}).$$

Moreover, there is a short exact sequence

$$0 \longrightarrow \left[E^\infty_{0,1} = 0\right] \longrightarrow H_1(U,\mathcal{A}) \longrightarrow \left[E^\infty_{1,0} = \check{H}_1(U,\mathcal{A})\right] \longrightarrow 0,$$

implying

$$H_1(U,\mathcal{A}) \simeq \check{H}_1(U,\mathcal{A}).$$

Finally,

$$E^\infty_{2,0} \simeq E^3_{2,0} \simeq \ker\left(E^2_{2,0} \longrightarrow \left[E^2_{0,1} = 0\right]\right) \simeq E^2_{2,0} \simeq \check{H}_2(U,\mathcal{A}),$$

and there is, since $E^\infty_{0,2} = 0$, a short exact sequence

$$0 \longrightarrow E^\infty_{1,1} \longrightarrow H_2(U,\mathcal{A}) \longrightarrow \left[E^2_{2,0} \simeq \check{H}_2(U,\mathcal{A})\right] \longrightarrow 0,$$

implying

$$H_2(U,\mathcal{A}) \twoheadrightarrow \check{H}_2(U,\mathcal{A}).$$

(**7**) Follows from Proposition B.2.7. ∎

3.4.2. Notation.

1. *Denote by $\mathcal{H}_n$ the left satellites of the embedding*

$$\iota : \mathbf{CS}(X, \mathbf{Pro}(k)) \longrightarrow \mathbf{pCS}(X, \mathbf{Pro}(k));$$

2. *Denote by $\mathcal{H}^n$ the right satellites of the embedding*

$$\iota : \mathbf{S}(X, \mathbf{Pro}(k)) \longrightarrow \mathbf{pS}(X, \mathbf{Pro}(k));$$

4. Examples

4.0.1. Example. *Let X be the convergent sequence from [Prasolov, 2016, Example 4.8]:*

$$X = \{x_0\} \cup \{x_1, x_2, x_3, ...\} = \{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\} \subseteq \mathbb{R},$$

let $G \in \mathbf{Ab}$, $G \neq \{0\}$, and let $\mathcal{A} = G_\#$ be the constant cosheaf. Then

$$H_n(X,\mathcal{A}) = \check{H}_n(X,\mathcal{A}) = \textit{pro-}H_n(X,G) = \begin{cases} \mathbf{B} & \textit{if} \quad n = 0, \\ 0 & \textit{if} \quad n \neq 0, \end{cases}$$

*where $\mathbf{B}$ is an abelian pro-group which is **not** rudimentary (Remark 2.1.6), i.e.*

$$\mathbf{B} \notin \mathbf{Ab} \subseteq \mathbf{Pro}(\mathbf{Ab}).$$

PROOF. Let $T \in \mathbf{Ab}$ be injective. It is easy to check that the cosheaf $\mathcal{A}$ is flabby, therefore the sheaf $\langle \mathcal{A}, T\rangle$ is flabby, thus acyclic. Due to Theorem 3.4.1(3), the cosheaf $\mathcal{A}$ is acyclic, too:

$$H_n(X,\mathcal{A}) = \begin{cases} \mathbf{0} & \text{if} \quad n>0; \\ \check{H}_0(X,\mathcal{A}) = \mathcal{A}(X) = \textit{pro-}H_0(X,G) & \text{if} \quad n=0. \end{cases}$$

It remains to calculate $\textit{pro-}H_0(X,G)$. Since

$$\mathbf{Pro}(\mathbf{Ab}) \longrightarrow \left(\mathbf{Set}^{\mathbf{Ab}}\right)^{op}$$

is a full embedding by Definition 2.1.4, it is enough to describe the functor

$$Hom_{\mathbf{Pro}(\mathbf{Ab})}(\textit{pro-}H_0(X,G), \bullet) : \mathbf{Ab} \longrightarrow \mathbf{Set}.$$

During the proof of [Prasolov, 2016, Proof of Theorem 3.11(3)], it is established a natural in X isomorphism

$$\textit{pro-}H_0(X,G) \simeq G \otimes_{\mathbf{Set}} \textit{pro-}\pi_0(X).$$

Moreover, in [Prasolov, 2016, Proposition 3.13] another natural (in X and $T \in \mathbf{Ab}$) isomorphism is proved:

$$Hom_{\mathbf{Pro}(\mathbf{Ab})}(G \otimes_{\mathbf{Set}} \textit{pro-}\pi_0(X), T) \simeq [Hom_{\mathbf{Ab}}(G,T)]^X,$$

where $[Hom_{\mathbf{Ab}}(G,T)]^X$ is the set of continuous mappings from X to $Hom_{\mathbf{Ab}}(G,T)$, where the latter space is supplied with the discrete topology. In fact,

$$[Hom_{\mathbf{Ab}}(G,T)]^X \simeq \check{H}^0(X, Hom_{\mathbf{Ab}}(G,T)),$$

where $\check{H}^0$ is the classical Čech **co**homology for topological spaces, but we **do not need** this fact. Continuous mappings to a discrete space are locally constant, and vice versa. Consider such a mapping

$$f : X \longrightarrow Hom_{\mathbf{Ab}}(G,T).$$

Since it is locally constant at $x = x_0$, there exists an $n \in \mathbb{Z}$ such that for all $i > n$

$$f(x_i) = f(x_0).$$

Therefore,

$$[Hom_{\mathbf{Ab}}(G,T)]^X \simeq \varinjlim \left(C_1 \xrightarrow{q_{1\to 2}} C_2 \xrightarrow{q_{2\to 3}} \dots \longrightarrow C_n \xrightarrow{q_{n\to n+1}} \dots \right),$$

where

$$C_n = [Hom_{\mathbf{Ab}}(G,T)]^{n+1} = [Hom_{\mathbf{Ab}}(G,T)]^{\{0,1,2,\dots,n\}},$$

and

$$\begin{array}{rcl} q_{n\to n+1}(\varphi_0,\varphi_1,\dots\varphi_{n-1},\varphi_n) & = & (\varphi_0,\varphi_1,\dots\varphi_{n-1},\varphi_n,\varphi_0), \\ \varphi_i & \in & Hom_{\mathbf{Ab}}(G,T). \end{array}$$

One gets a sequence of natural isomorphisms:

$$[Hom_{\mathbf{Ab}}(G,T)]^X \simeq \varinjlim \left(C_1 \xrightarrow{q_{1\to 2}} C_2 \xrightarrow{q_{2\to 3}} ... \longrightarrow C_n \xrightarrow{q_{n\to n+1}} ... \right) \simeq$$
$$\varinjlim \left(Hom_{\mathbf{Ab}}(B_1,T) \xrightarrow{Hom_{\mathbf{Ab}}(r_{1\leftarrow 2},T)} Hom_{\mathbf{Ab}}(B_1,T) \xrightarrow{Hom_{\mathbf{Ab}}(r_{2\leftarrow 3},T)} ... \longrightarrow Hom_{\mathbf{Ab}}(B_n,T) \longrightarrow ... \right)$$
$$\simeq Hom_{\mathbf{Pro(Ab)}}(\mathbf{B},T),$$

where

$$\begin{aligned} \mathbf{I} &= (1 \longleftarrow 2 \longleftarrow 3 \longleftarrow ... \longleftarrow n \longleftarrow ...), \\ \mathbf{B} &= (B_i)_{i\in \mathbf{I}} = \left(B_1 \xleftarrow{r_{1\leftarrow 2}} B_2 \xleftarrow{r_{2\leftarrow 3}} B_3 \longleftarrow ... \longleftarrow B_n \xleftarrow{r_{n\leftarrow n+1}} ... \right), \\ B_n &= G^{n+1} = G^{\{0,1,2,...,n\}}, \end{aligned}$$

and

$$\begin{aligned} r_{n\leftarrow n+1}(g_0,g_1,...g_n,g_{n+1}) &= (g_0+g_{n+1},g_1,...g_{n-1},g_n), \\ g_i &\in G. \end{aligned}$$

We have proved that

$$Hom_{\mathbf{Pro(Ab)}}(\mathbf{B},\bullet) \simeq Hom_{\mathbf{Pro(Ab)}}(pro\text{-}H_0(X,G),\bullet)$$

in $\mathbf{Set}^{\mathbf{Ab}}$, therefore $\mathbf{B} \simeq pro\text{-}H_0(X,G)$ in $\mathbf{Pro}(\mathbf{Ab})$.

It remains to show that $\mathbf{B}$ is **not a rudimentary** pro-object. Assume on the contrary that $\mathbf{B} \simeq Z$ where $Z \in \mathbf{Ab}$. I follows from Proposition 2.1.7 that there exists a homomorphism

$$\tau_0 : B_{i_0} \longrightarrow Z,$$

satisfying the property: for any morphism $s : i \to i_0$, there exist a morphism $\sigma : Z \to B_i$ and a morphism $t : j \to i$ satisfying

$$\begin{aligned} \tau_0 \circ B(s) \circ \sigma &= \mathbf{1}_Z, \\ \sigma \circ \tau_0 \circ B(s) \circ B(t) &= B(t). \end{aligned}$$

Take $s = (i_0 \leftarrow i_0+1)$. Choose a **nonzero** element $a \in \ker B(s)$, say

$$a = (g,0,...,0,-g), \ g \neq 0.$$

Since $B(t)$ is surjective, choose $b \in B_j$ with $[B(t)](b) = a$. Apply the second equation from above:

$$\begin{aligned} [\sigma \circ \tau_0 \circ B(s) \circ B(t)](b) &= [B(t)](b), \\ [\sigma \circ \tau_0 \circ B(s)](a) &= a, \\ 0 &= a \neq 0. \end{aligned}$$

Contraction. ■

4.0.2. EXAMPLE. *Let X be the **pseudocircle**, i.e. the 4-point topological space*

$$X = \{a, b, c, d\}$$

with the topology

$$\tau = \{\varnothing, \{a, b, c, d\}, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}.$$

*This space can be also described as the **non-Hausdorff suspension** [McCord, 1966, Section 8, p. 472] $\mathfrak{S}(Y)$, where*

$$X \supseteq Y = \{a, c\} \simeq S^0.$$

Let again $\mathcal{A} = G_{\#}$ be the constant cosheaf ($\{0\} \neq G \in \mathbf{Ab}$). Then

$$\check{H}_n(X, \mathcal{A}) \simeq H_n(X, \mathcal{A}) \simeq H_n^{sing}(X, G) = \begin{cases} G & if \quad n = 0, 1, \\ 0 & if \quad n \neq 0, 1. \end{cases}$$

where $H_\bullet^{sing}$ is the ordinary singular homology. Notice that the pro-homology pro-$H_1(X, G)$ is zero (Remark 1.0.6) and

$$\mathbf{0} = pro\text{-}H_1(X, G) \not\cong H_1(X, \mathcal{A}).$$

*The reason is that we **could not** apply Conjecture 1.0.3(1) because X is **not** Hausdorff.*

PROOF. Let

$$\mathcal{U} = \{U_{0,1,2,3} \longrightarrow X\} = \{\{a\}, \{c\}, \{a, b, c\}, \{a, c, d\}\}.$$

Define the bicomplex

$$C_{s,t} = \check{C}_s(\mathcal{U}, \mathcal{P}_t)$$

where $\mathcal{P}_\bullet \to \mathcal{A}$ is a **qis** (Notation A.3.2) from a quasi-projective complex to $\mathcal{A}$, considered as a complex concentrated in degree 0 (i.e. $\mathcal{P}_\bullet \to \mathcal{A}$ is a quasi-projective resolution of $\mathcal{A}$). Since $\mathbf{Pro}(\mathbf{Ab})$ is an abelian (Proposition A.2.6) category, we can apply Theorem A.4.3 in order to obtain two spectral sequences converging to the total complex $Tot_\bullet(C)$. Notice that $\mathcal{A}|_{U_{i_1} \times ... \times U_{i_s}}$ is flabby (Definition 2.3.8) for each s, therefore

$$H_t(U_{i_1} \times ... \times U_{i_s}, \mathcal{A}) = 0$$

if $t > 0$. Calculate the entries in the first spectral sequence:

$$\begin{aligned} {}^{ver}E^1_{s,t} &= \begin{cases} H_t(U_{i_1} \times ... \times U_{i_s}, \mathcal{A}) = 0 & \text{if} \quad t > 0 \\ \mathcal{A}(U_{i_1} \times ... \times U_{i_s}) & \text{if} \quad t = 0 \end{cases} \\ {}^{ver}E^2_{s,t} &= \begin{cases} 0 & \text{if} \quad t > 0 \\ H_s(\mathcal{U}, \mathcal{A}) & \text{if} \quad t = 0 \end{cases} \end{aligned}$$

It follows that

$$H_n(Tot_\bullet(C)) \simeq {}^{ver}E^2_{n,0} \simeq H_n(\mathcal{U}, \mathcal{A}).$$

The second spectral sequence gives

$$\begin{aligned} {}^{hor}E^1_{s,t} &= \begin{cases} 0 & \text{if} \quad s>0 \quad \text{since} \quad \mathcal{P}_t \text{ is quasi-projective as a (pre)cosheaf} \\ \mathcal{P}_t(X) & \text{if} \quad s=0 \quad \text{since} \quad \mathcal{P}_t \text{ is a cosheaf} \end{cases} \\ {}^{hor}E^2_{s,t} &= \begin{cases} 0 & \text{if} \quad s>0 \\ H_t(X,\mathcal{A}) & \text{if} \quad s=0 \end{cases} \end{aligned}$$

It follows that

$$H_n(Tot_\bullet(C)) \simeq {}^{hor}E^2_{0,n} \simeq H_n(X,\mathcal{A}).$$

Finally

$$H_n(X,\mathcal{A}) \simeq H_n(\mathcal{U},\mathcal{A}).$$

The latter pro-groups (in fact, *rudimentary* pro-groups, i.e. just ordinary groups) can be easily calculated. It remains to apply [McCord, 1966, Theorem 2 and the example in §5]:

$$H_n(\mathcal{U},\mathcal{A}) = \begin{cases} G & \text{if} \quad n=0,1, \\ 0 & \text{if} \quad n\neq 0,1; \end{cases} \simeq H^{sing}_n\left(S^1,G\right) \simeq H^{sing}_n(X,G).$$

■

A. Categories

A.1. Pairings.

A.1.1. Definition. *Let* $\mathbf{D}$ *be a small category. Various bifunctors are defined below:*

1.

$$\langle\bullet,\bullet\rangle : \mathbf{Pro}(k)^{op} \times \mathbf{Mod}(k) \longrightarrow \mathbf{Mod}(k).$$

If

$$\mathbf{A} = (A_i)_{i\in\mathbf{I}} \in \mathbf{Pro}(k)$$

is a pro-module, and $G \in \mathbf{Mod}(k)$*, let*

$$\langle\mathbf{A},G\rangle := Hom_{\mathbf{Pro}(k)}(\mathbf{A},G) = \varinjlim_{i\in\mathbf{I}} Hom_{\mathbf{Mod}(k)}(A_i,G) \in \mathbf{Mod}(k).$$

2.

$$\langle\bullet,\bullet\rangle : \mathbf{pCS}(\mathbf{D},\mathbf{Pro}(k))^{op} \times \mathbf{Mod}(k) \longrightarrow \mathbf{pS}(\mathbf{D},\mathbf{Mod}(k)).$$

If

$$\mathcal{A} : \mathbf{D} \longrightarrow \mathbf{Pro}(k)$$

is a functor, and $G \in \mathbf{Mod}(k)$*, let*

$$\begin{aligned} \langle\mathcal{A},G\rangle &= Hom_{\mathbf{Pro}(k)}(\mathcal{A},G) := \left[U \longmapsto Hom_{\mathbf{Pro}(k)}(\mathcal{A}(U),G)\right], \\ \langle\mathcal{A},G\rangle &: \mathbf{D}^{op} \longrightarrow \mathbf{Mod}(k). \end{aligned}$$

3.

$$\bullet \otimes_{\mathbf{Set}} \bullet : \mathbf{K} \times \mathbf{Set} \longrightarrow \mathbf{K}.$$

If $A \in \mathbf{K}$ (say, $\mathbf{K} = \mathbf{Mod}(k)$ or $\mathbf{K} = \mathbf{Pro}(k)$), and $B \in \mathbf{Set}$, let

$$A \otimes_{\mathbf{Set}} B = B \otimes_{\mathbf{Set}} A = \coprod_B A$$

be the coproduct in $\mathbf{K}$ of B copies of A.

4.

$$\bullet \otimes_{\mathbf{Set}} \bullet : \mathbf{K} \times \mathbf{Pro}(\mathbf{Set}) \longrightarrow \mathbf{Pro}(\mathbf{K}).$$

Let $\mathbf{Y} = (Y_i)_{i \in \mathbf{I}} \in \mathbf{Pro}(\mathbf{Set})$, and $X \in \mathbf{K}$. Define

$$X \otimes_{\mathbf{Set}} \mathbf{Y} = \mathbf{Y} \otimes_{\mathbf{Set}} X \in \mathbf{Pro}(\mathbf{K})$$

by

$$X \otimes_{\mathbf{Set}} \mathbf{Y} = (X \otimes_{\mathbf{Set}} Y_i)_{i \in \mathbf{I}}.$$

5.

$$\bullet \otimes_{\mathbf{Set}^{\mathbf{D}}} \bullet : \mathbf{pCS}(\mathbf{D}, \mathbf{Pro}(\mathbf{K})) \times \mathbf{pS}(\mathbf{D}, \mathbf{Set}) \longrightarrow \mathbf{Pro}(\mathbf{K}).$$

If

$$\begin{aligned} \mathcal{A} &: \mathbf{D} \longrightarrow \mathbf{Pro}(\mathbf{K}), \\ \mathcal{B} &: \mathbf{D}^{op} \longrightarrow \mathbf{Set}, \end{aligned}$$

are functors, let

$$\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{D}}} \mathcal{B} \in \mathbf{Pro}(\mathbf{K})$$

*be the **coend** [Mac Lane, 1998, Chapter IX.6] of the bifunctor $(U, V) \mapsto \mathcal{A}(U) \otimes_{\mathbf{Set}} \mathcal{B}(V)$, i.e.*

$$\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{D}}} \mathcal{B} := \operatorname{coker}\left(\coprod_{U \to V} \mathcal{A}(U) \otimes_{\mathbf{Set}} \mathcal{B}(V) \rightrightarrows \coprod_U \mathcal{A}(U) \otimes_{\mathbf{Set}} \mathcal{B}(U) \right).$$

6.

$$Hom_{\mathbf{Set}^{\mathbf{D}}}(\bullet, \bullet) : \mathbf{pS}(\mathbf{D}, \mathbf{Set})^{op} \times \mathbf{pS}(\mathbf{D}, \mathbf{K}) \longrightarrow \mathbf{Mod}(k)$$

If

$$\begin{aligned} \mathcal{A} &: \mathbf{D}^{op} \longrightarrow \mathbf{K}, \\ \mathcal{B} &: \mathbf{D}^{op} \longrightarrow \mathbf{Set}, \end{aligned}$$

are functors, let

$$Hom_{\mathbf{Set}^{\mathbf{D}}}(\mathcal{B}, \mathcal{A}) \in \mathbf{Mod}(k)$$

*be the **end** [Mac Lane, 1998, Chapter IX.6] of the bifunctor $(U, V) \mapsto Hom_{\mathbf{Set}}(\mathcal{B}(U), \mathcal{A}(V))$, i.e.*

$$Hom_{\mathbf{Set}^{\mathbf{D}}}(\mathcal{B}, \mathcal{A}) := \ker\left(\prod_U Hom_{\mathbf{Set}}(\mathcal{B}(U), \mathcal{A}(U)) \rightrightarrows \prod_{U \to V} Hom_{\mathbf{Set}}(\mathcal{B}(U), \mathcal{A}(V)) \right).$$

A.2. Quasi-projective pro-modules.

A.2.1. Definition. *A pro-module* $\mathbf{P}$ *is called* ***quasi-projective*** *iff the functor*

$$Hom_{\mathbf{Pro}(k)}(\mathbf{P},\bullet):\mathbf{Mod}(k)\longrightarrow\mathbf{Mod}(k)$$

is exact (see [Kashiwara and Schapira, 2006, dual to Definition 15.2.1]).

A.2.2. Proposition. *A pro-module* $\mathbf{P}$ *is quasi-projective iff it is isomorphic to a pro-module* $(Q_i)_{i\in\mathbf{I}}$ *where all modules* $Q_i\in\mathbf{Mod}(k)$ *are projective.*

Proof. The statement is dual to [Kashiwara and Schapira, 2006, Proposition 15.2.3]. ■

A.2.3. Remark. *The category* $\mathbf{Pro}(k)$ *does not have enough projectives (compare with [Kashiwara and Schapira, 2006, Corollary 15.1.3]). However, it has enough quasi-projectives (see Proposition A.2.8(5) below).*

A.2.4. Definition. *A commutative ring* k *is called* ***quasi-noetherian*** *iff*

$$\langle\mathbf{P},T\rangle=Hom_{\mathbf{Pro}(k)}(\mathbf{P},T)$$

is an injective k*-module for any quasi-projective pro-module* $\mathbf{P}$ *and an injective* k*-module* T*.*

A.2.5. Proposition. *A noetherian ring is quasi-noetherian.*

Proof. See [Prasolov, 2013, Proposition 2.28]. ■

A.2.6. Proposition. *If* $\mathbf{K}$ *is an abelian category, then* $\mathbf{Pro}(\mathbf{K})$ *is an abelian category as well.*

Proof. See [Kashiwara and Schapira, 2006, dual to Theorem 8.6.5(i)]. ■

A.2.7. Notation. *For a* k*-module* M*, denote by* M^* *the following* k*-module:*

$$M^*:=Hom_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}).$$

A.2.8. Proposition.

1. *The category* $\mathbf{Pro}(k)$ *is abelian, complete and cocomplete, and satisfies both the* $AB3$ *and* $AB3^*$ *axioms ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]).*
2. *For any diagram*

$$\mathbf{X}:\mathbf{I}\longrightarrow\mathbf{Pro}(k)$$

and any $T\in\mathbf{Mod}(k)$ *(not necessarily injective!)*

$$\left\langle\varinjlim_{i\in\mathbf{I}}\mathbf{X}_i,T\right\rangle\simeq\varprojlim_{i\in\mathbf{I}}\langle\mathbf{X}_i,T\rangle$$

in $\mathbf{Mod}(k)$*.*

3. *For any diagram*

$$\mathbf{X}:\mathbf{I}\longrightarrow\mathbf{Pro}\left(k\right)$$

and any $T\in\mathbf{Mod}\left(k\right)$

$$\left\langle \varprojlim_{i\in\mathbf{I}}\mathbf{X}_i,T\right\rangle \simeq \varinjlim_{i\in\mathbf{I}}\left\langle \mathbf{X}_i,T\right\rangle$$

in $\mathbf{Mod}\left(k\right)$ *if either* $\mathbf{I}$ *is cofiltered or* T *is injective.*

4. *For any family* $\left(\mathbf{X}_i\right)_{i\in I}$ *in* $\mathbf{Pro}\left(k\right)$ *and any* $T\in\mathbf{Mod}\left(k\right)$ *(not necessarily injective!)*

$$\left\langle \prod_{i\in I}\mathbf{X}_i,T\right\rangle \simeq \bigoplus_{i\in I}\left\langle \mathbf{X}_i,T\right\rangle$$

in $\mathbf{Mod}\left(k\right)$.

5. *For an arbitrary pro-module* $\mathbf{M}\in\mathbf{Pro}\left(k\right)$*, there exists a functorial surjection*

$$\mathbf{F}\left(\mathbf{M}\right)\twoheadrightarrow\mathbf{M},$$

where $\mathbf{F}\left(\mathbf{M}\right)$ *is quasi-projective.*

6. *Let* $\mathbf{M}\in\mathbf{Pro}\left(k\right)$*. Then* $\mathbf{M}\simeq\mathbf{0}$ *iff* $\left\langle \mathbf{M},T\right\rangle =0$ *for any injective* $T\in\mathbf{Mod}\left(k\right)$*.*

7. *Let*

$$\mathcal{E}=\left(\mathbf{M}\overset{\alpha}{\longleftarrow}\mathbf{N}\overset{\beta}{\longleftarrow}\mathbf{K}\right)$$

be a sequence of morphisms in $\mathbf{Pro}\left(k\right)$ *with* $\beta\circ\alpha=0$*, and let* $T\in\mathbf{Mod}\left(k\right)$ *be injective. Then*

$$H\left(\mathcal{E}\right):=\frac{\ker\left(\alpha\right)}{im\left(\beta\right)}$$

satisfies

$$\left\langle H\left(\mathcal{E}\right),T\right\rangle \simeq H\left(\left\langle \mathcal{E},T\right\rangle \right):=\frac{\ker\left(\left\langle \beta,T\right\rangle \right)}{im\left(\left\langle \alpha,T\right\rangle \right)}.$$

8. *Let*

$$\mathcal{E}=\left(\mathbf{M}\overset{\alpha}{\longleftarrow}\mathbf{N}\overset{\beta}{\longleftarrow}\mathbf{K}\right)$$

be a sequence of morphisms in $\mathbf{Pro}\left(k\right)$ *with* $\beta\circ\alpha=0$*. Then* $\mathcal{E}$ *is exact iff the sequence*

$$\left\langle \mathbf{M},T\right\rangle \overset{\left\langle \alpha,T\right\rangle }{\longrightarrow}\left\langle \mathbf{N},T\right\rangle \overset{\left\langle \beta,T\right\rangle }{\longrightarrow}\left\langle \mathbf{K},T\right\rangle$$

is exact in $\mathbf{Mod}\left(k\right)$ *for all injective* $T\in\mathbf{Mod}\left(k\right)$*.*

9. *Let* $T\in\mathbf{Mod}\left(k\right)$ *be an injective module. Then the corresponding rudimentary (Remark 2.1.6) pro-module* T *is an injective object of* $\mathbf{Pro}\left(k\right)$*.*

10. *The category* $\mathbf{Pro}(k)$ *satisfies the* $AB4$ *axiom ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]).*

11. *The category* $\mathbf{Pro}(k)$ *satisfies the* $AB4^*$ *axiom ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]).*

12. *The category* $\mathbf{Pro}(k)$ *satisfies the* $AB5^*$ *axiom ([Grothendieck, 1957, 1.5], [Bucur and Deleanu, 1968, Ch. 5.8]): cofiltered limits are exact in the category* $\mathbf{Pro}(k)$*.*

13. *The* ***class*** *(**not** a set)*

$$\mathfrak{G} = \{\mathbf{G}(S) \mid S \in \mathbf{Set}\} \subseteq \mathbf{Pro}(k),$$

where $\mathbf{G}(S)$ *is the rudimentary pro-module (Remark 2.1.6) corresponding to the* k*-module*

$$\prod_S k^* = \prod_S Hom_{\mathbf{Ab}}(k, \mathbb{Q}/\mathbb{Z})$$

forms a class of cogenerators ([Grothendieck, 1957, 1.9], [Bucur and Deleanu, 1968, Ch. 5.9]) of the category $\mathbf{Pro}(k)$*.*

Proof. (**1**) It follows from Proposition A.2.6 that $\mathbf{Pro}(k)$ is abelian. Due to Proposition 2.1.13 (2, 3), $\mathbf{Pro}(k)$ is complete and cocomplete. $AB3$ and $AB3^*$ follow immediately.

(**2**) Follows from the definition of a colimit.

(**3**) If $\mathbf{X}$ is cofiltered, then the statement follows from Proposition 2.1.13 (1). If not, then notice that limits in any category can be constructed as combinations of products and kernels. Let $T \in \mathbf{Mod}(k)$. It follows from (4) that the pairing $\langle \bullet, T\rangle$ converts products into coproducts. If T is injective, then $\langle \bullet, T\rangle$ converts kernels into cokernels. Finally, $\langle \bullet, T\rangle$ converts arbitrary limits into colimits.

(**4**) Let

$$\mathbf{Fin}(I) = \mathbf{Cat}(X(I))^{op}$$

(see Example 2.1.3) where $X(I)$ is the set of **finite** subsets of I, ordered by inclusion. Then $X(I)$ is a directed poset, and $\mathbf{Fin}(I)$ is a cofiltered category (see Example 2.1.3 again). It is easy to check that

$$\prod_{i\in I} \mathbf{X}_i \simeq \varprojlim_{A\in\mathbf{Fin}(I)} \left[\prod_{j\in A} \mathbf{X}_j\right].$$

It follows from the statement (3) of our theorem that

$$\left\langle \prod_{i\in I} \mathbf{X}_i \simeq, T\right\rangle \simeq \varinjlim_{A\in\mathbf{Fin}(I)^{op}} \left\langle \prod_{j\in A} \mathbf{X}_j, T\right\rangle \simeq$$
$$\simeq \varinjlim_{A\in\mathbf{Fin}(I)^{op}} \bigoplus_{j\in A} \langle \mathbf{X}_j, T\rangle \simeq \bigoplus_{j\in I} \langle \mathbf{X}_i, T\rangle.$$

(**5**) The statement is dual to the rather complicated Theorem 15.2.5 from [Kashiwara and Schapira, 2006]. However, the proof is much simpler in our case. Given $\mathbf{M} = (M_i)_{i \in \mathbf{I}}$, let

$$\mathbf{F}(\mathbf{M}) = (Q_i)_{i \in \mathbf{I}},$$

where $Q_i = F(M_i)$ is the free k-module generated by the set of symbols $([m])_{m \in M}$. A family of epimorphisms

$$f_i : Q_i \longrightarrow M_i \quad \left(f_i \left(\sum_j \alpha_j [m_j] \right) = \sum_j \alpha_j m_j, \alpha_j \in k, m_j \in M_i \right),$$

defines an epimorphism $f : \mathbf{F}(\mathbf{M}) \to \mathbf{M}$.

$$\mathbf{F}(\mathbf{M}) = (F(M_i))_{j \in \mathbf{I}}$$

is quasi-projective (Proposition A.2.2), and the epimorphism $F(\mathbf{M}) \twoheadrightarrow \mathbf{M}$ is as desired.

(**6**) The "only if" part is trivial. Assume now that $\mathbf{M}$ is **not** isomorphic to $\mathbf{0}$. Since

$$\mathbf{Pro}(k) \hookrightarrow \left(\mathbf{Set}^{\mathbf{Mod}(k)} \right)^{op}$$

is a full embedding (by definition!), there exists a $N \in \mathbf{Mod}(k)$ with

$$\langle \mathbf{M}, N \rangle = Hom_{\mathbf{Pro}(k)}(\mathbf{M}, N) \neq 0.$$

Choose an embedding $N \hookrightarrow T$ into an injective k-module. Then

$$\langle \mathbf{M}, N \rangle \longrightarrow \langle \mathbf{M}, T \rangle$$

is a monomorphism. It follows that $\langle \mathbf{M}, T \rangle \neq 0$ as well.

(**7**) Due to Proposition 2.1.11, one can assume that $\mathcal{E}$ is a level diagram:

$$(M_i)_{i \in \mathbf{I}} \xleftarrow{(\alpha_i)} (N_i)_{i \in \mathbf{I}} \xleftarrow{(\beta_i)} (K_i)_{i \in \mathbf{I}}.$$

Since T is injective, the sequences

$$\langle \mathcal{E}_i, T \rangle = \left[\langle M_i, T \rangle \xrightarrow{\langle \alpha_i, T \rangle} \langle N_i, T \rangle \xrightarrow{\langle \beta_i, T \rangle} \langle K_i, T \rangle \right]$$

satisfy

$$H \langle \mathcal{E}_i, T \rangle \simeq \langle H(\mathcal{E}_i), T \rangle.$$

The category $\mathbf{I}^{op}$ is filtered, and filtered colimits are exact in the category $\mathbf{Mod}(k)$, therefore

$$\langle H(\mathcal{E}), T \rangle \simeq \varinjlim_{i \in \mathbf{I}^{op}} \langle H(\mathcal{E}_i), T \rangle \simeq \varinjlim_{i \in \mathbf{I}^{op}} H \langle \mathcal{E}_i, T \rangle \simeq H(\langle \mathcal{E}, T \rangle).$$

(**8**) It follows from the statement (7) of our theorem that

$$\langle H(\mathcal{E}), T\rangle \simeq H(\langle \mathcal{E}, T\rangle).$$

Applying (6) of our theorem, one gets

$$H(\mathcal{E}) = \mathbf{0} \iff \forall(\text{injective } T)\,[H(\langle \mathcal{E}, T\rangle) = 0],$$

therefore $\mathcal{E}$ is exact iff $\langle \mathcal{E}, T\rangle$ is exact for all injective $T \in \mathbf{Mod}(k)$.

(**9**) Follows easily from (8).

(**10**) Let

$$(f_i : A_i \longrightarrow B_i)_{i\in I}$$

be a family of monomorphisms, and $T \in \mathbf{Mod}(k)$ be injective. Then all the homomorphisms

$$\langle f_i, T\rangle : \langle B_i, T\rangle \longrightarrow \langle A_i, T\rangle$$

are epimorphisms in $\mathbf{Mod}(k)$. Therefore, the homomorphism

$$\left\langle \bigoplus_{i\in I} f_i, T\right\rangle = \prod_{i\in I} \langle f_i, T\rangle : \prod_{i\in I} \langle B_i, T\rangle = \left\langle \bigoplus_{i\in I} B_i, T\right\rangle \longrightarrow \left\langle \bigoplus_{i\in I} A_i, T\right\rangle = \prod_{i\in I} \langle B_i, T\rangle$$

is an epimorphism in $\mathbf{Mod}(k)$ for any injective T. It follows that $\bigoplus_{i\in I} f_i$ is a monomorphism in $\mathbf{Pro}(k)$.

(**11**) Let

$$(f_i : A_i \longrightarrow B_i)_{i\in I}$$

be a family of epimorphisms, and $T \in \mathbf{Mod}(k)$ be injective. Then all the homomorphisms

$$\langle f_i, T\rangle : \langle B_i, T\rangle \longrightarrow \langle A_i, T\rangle$$

are monomorphisms in $\mathbf{Mod}(k)$. Therefore, the homomorphism

$$\left\langle \prod_{i\in I} f_i, T\right\rangle = \bigoplus_{i\in I} \langle f_i, T\rangle : \bigoplus_{i\in I} \langle B_i, T\rangle = \left\langle \prod_{i\in I} B_i, T\right\rangle \longrightarrow \left\langle \prod_{i\in I} A_i, T\right\rangle = \bigoplus_{i\in I} \langle B_i, T\rangle$$

is a monomorphism in $\mathbf{Mod}(k)$ for any injective T. It follows that $\prod_{i\in I} f_i$ is an epimorphism in $\mathbf{Pro}(k)$.

(**12**) Follows from Proposition 2.1.13 (4).

(**13**) Since

$$Hom_{\mathbf{Mod}(k)}(\bullet, k^*) \simeq Hom_{\mathbf{Ab}}(\bullet, \mathbb{Q}/\mathbb{Z}),$$

and $\mathbb{Q}/\mathbb{Z}$ is a cogenerator in the category $\mathbf{Ab}$, k^* is an **injective** cogenerator in $\mathbf{Mod}(k)$. In fact, k^* is **injective** in $\mathbf{Pro}(k)$ as well. Indeed, it is enough to apply part (9) of our theorem to $T = k^*$.

Let now

$$f : M \twoheadrightarrow N$$

be a non-trivial (not an isomorphism!) epimorphism in $\mathbf{Pro}(k)$. Let

$$K = \ker f \neq 0.$$

We can assume that f and $h : K \rightarrowtail M$ are level morphisms:

$$0 \longrightarrow (K_i {:=} \ker f_i)_{i\in\mathbf{I}} \xrightarrow{h=(h_i)} (M_i)_{i\in\mathbf{I}} \xrightarrow{f=(f_i)} (N_i)_{i\in\mathbf{I}}.$$

Due to Corollary 2.1.8, there exists an $i \in \mathbf{I}$, such that $K(t) \neq 0$ for any $t : j \to i$. It follows that $K_i \neq 0$. Let

$$S = \{(t : j \to i) \in \mathbf{I}\},$$

and let

$$\mathbf{G}(S) \in \mathbf{Pro}(k)$$

be the rudimentary pro-module corresponding to $\prod_{t\in S} k^*$. Due to (9), $\mathbf{G}(S)$ is an injective pro-module. Since k^* is an injective cogenerator for $\mathbf{Mod}(k)$, we can for each $(t : j \to i) \in S$, choose a homomorphism

$$\varphi_t : K_i \longrightarrow k^*,$$

such that the composition

$$\varphi_t \circ K(t)$$

is nonzero. Let

$$\varphi = \left(\prod_{t\in S} \varphi_t\right) : K_i \longrightarrow \prod_{t\in S} k^*.$$

The corresponding morphism

$$\Phi : \mathbf{K} \longrightarrow \mathbf{G}(S)$$

is **nonzero**. Indeed, if it is zero, then there exists a $t : j \to i$ with

$$\Phi \circ K(t) = 0.$$

However,

$$\pi_t \circ \Phi \circ K(t) = \varphi_t \circ K(t) \neq 0,$$

where

$$\pi_t : \prod_{t\in S} k^* \longrightarrow k^*$$

is the t-th projection. Denote by the same letter φ_i the corresponding morphism

$$(\varphi_i : K \longrightarrow k^*) \in \mathbf{Pro}(k).$$

The morphism Φ can be extended, due to injectivity of $\mathbf{G}(S)$, to a morphism

$$\Psi : M \longrightarrow \mathbf{G}(S).$$

Since the composition $\Phi = \Psi \circ h$ is nonzero, the morphism Ψ **cannot** be factored through N. ∎

A.3. Derived categories. We use here the "classical" definition of an F-projective category. The subcategories, which are called "F-projective" in [Kashiwara and Schapira, 2006, Definition 13.3.4], will be called **weak F-projective** in this paper.

A.3.1. Definition. *Let*

$$F : \mathbf{C} \longrightarrow \mathbf{E}$$

be a right exact additive functor of abelian categories, and let $\mathbf{P}$ *be a full additive subcategory of* $\mathbf{C}$*. Then:*

$\mathbf{P}$ *is called* ***weak*** F***-projective*** *if* $\mathbf{P}$ *satisfies the definition of an* F*-projective subcategory in [Kashiwara and Schapira, 2006, Definition 13.3.4].*

$\mathbf{P}$ *is called* F*-**projective** if it satisfies the following conditions:*

1. *The category* $\mathbf{P}$ *is* ***generating*** *in* $\mathbf{C}$ *(i.e. for any object* $X \in \mathbf{C}$ *there exists an epimorphism* $P \twoheadrightarrow X$ *with* $P \in \mathbf{P}$*);*
2. *For any exact sequence*

   $$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

   in $\mathbf{C}$ *with* $X,\ X'' \in \mathbf{P}$*, we have* $X' \in \mathbf{P}$*;*
3. *For any exact sequence*

   $$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

   in $\mathbf{C}$ *with* $X,\ X'' \in \mathbf{P}$*, the sequence*

   $$0 \longrightarrow F(X') \longrightarrow F(X) \longrightarrow F(X'') \longrightarrow 0$$

   is exact.

A.3.2. Notation. *For an abelian category* $\mathbf{E}$*, let:*

1. $C(\mathbf{E})$ *denote the category of* ***bounded below*** *chain complexes in* $\mathbf{E}$*;*
2. *a* ***qis*** *denote a* ***quasi-isomorphism*** *in* $C(\mathbf{E})$*, i.e. a homomorphism*

   $$X_\bullet \longrightarrow Y_\bullet$$

   inducing an isomorphism of the homologies;
3. *a complex* $X_\bullet$ *be* ***qis*** *to* $Y_\bullet$ *iff there is a qis* $X_\bullet \to Y_\bullet$*;*
4. $K(\mathbf{E})$ *denote the homotopy category of* $C(\mathbf{E})$*, i.e. morphisms*

   $$X_\bullet \longrightarrow Y_\bullet$$

   in $K(\mathbf{E})$ *are* ***classes*** *of homotopic maps* $X_\bullet \to Y_\bullet$*;*
5. $D(\mathbf{E})$ *denote the corresponding derived category of* $K(\mathbf{E})$*, i.e.*

   $$D(\mathbf{E}) = K(\mathbf{E})/N(\mathbf{E})$$

   where $N(\mathbf{E})$ *is the full subcategory of* $K(\mathbf{E})$ *consisting of complexes qis to* $\mathbf{0}$*.*

A.3.3. PROPOSITION. *Let $F : \mathbf{C} \to \mathbf{E}$ be an additive functor of abelian categories, and let $\mathbf{P}$ be a full additive subcategory of $\mathbf{C}$. Assume $\mathbf{P}$ is F-projective. Then:*

1. *$\mathbf{P}$ is weak F-projective.*
2. *The left satellite*
$$LF : D(\mathbf{C}) \longrightarrow D(\mathbf{E})$$
exists, and
$$LF(X_\bullet) \simeq F(Y_\bullet)$$
for any qis
$$Y_\bullet \longrightarrow X_\bullet$$
with $Y_\bullet \in K(\mathbf{P})$.

PROOF. Follows from [Kashiwara and Schapira, 2006, dual to Proposition 13.3.5 and Corollary 13.3.8]. ■

Using F-projective subcategories, one can define left **satellites** of the functor F.

A.3.4. DEFINITION. *In the conditions of Proposition A.3.3 let $X \in \mathbf{C}$. Considering X as a complex concentrated in degree 0, take a qis $P_\bullet \longrightarrow X$, i.e. a **resolution***

$$0 \longleftarrow X \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow ... \longleftarrow P_n \longleftarrow ...$$

with $P_\bullet \in K(\mathbf{P})$. Define
$$L_nF(X) := H_n(P_\bullet).$$
It is easy to check that L_nF, $n \geq 0$, are additive functors

$$L_nF : \mathbf{C} \longrightarrow \mathbf{E},$$

that $L_nF = \mathbf{0}$ if $n < 0$, and that $L_0F \simeq F$ if F is right exact.

*The functors L_nF are called the **left satellites** of F.*

A.4. BICOMPLEXES. In this section, $\mathbf{K}$ is assumed to be an abelian category. We consider only **first quadrant chain** bicomplexes.

A.4.1. DEFINITION. *A **bicomplex** in $\mathbf{K}$ is a collection*

$$X_{\bullet,\bullet} = (X_{s,t}, d_{s,t}, \delta_{s,t})_{s,t\in\mathbb{Z}}$$

of objects and morphisms

$$\begin{array}{rcl} X_{s,t} & \in & \mathbf{K}, \\ d_{s,t} & \in & Hom_{\mathbf{K}}(X_{s+1,t}, X_{s,t}), \\ \delta_{s,t} & \in & Hom_{\mathbf{K}}(X_{s,t+1}, X_{s,t}), \end{array}$$

such that for all $s, t \in \mathbb{Z}$

$$\begin{aligned} X_{s,t} &= 0 \text{ if } s < 0 \text{ or } t < 0, \\ d_{s-1,t} \circ d_{s,t} &= 0, \\ \delta_{s,t-1} \circ \delta_{s,t} &= 0, \\ d_{s-1,t-1} \circ \delta_{s,t-1} &= \delta_{s-1,t-1} \circ d_{s-1,t}. \end{aligned}$$

A.4.2. Definition. *If* $(X_{\bullet,\bullet}, d, \delta)$ *be a bicomplex, let* $Tot_{\bullet}(X)$ *be the following chain complex:*

$$Tot_n(X) = \bigoplus_{s+t=n} X_{s,t} = \bigoplus_{s+t=n} X_{s,t} \simeq \prod_{s+t=n} X_{s,t}$$

with the differential

$$\partial_n : Tot_{n+1}(X) \longrightarrow Tot_n(X),$$

given by

$$\partial_n \circ \iota_{s,t} = \iota_{s-1,t} \circ d + (-1)^s \iota_{s,t-1} \circ \delta,$$

where

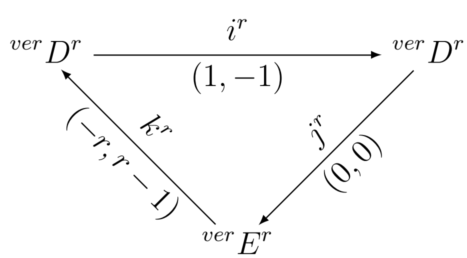
$$\iota_{s,t} : X_{s,t} \rightarrowtail Tot_n(X)$$

is the natural embedding into the coproduct.

A.4.3. Theorem. *Let* $(X_{\bullet,\bullet}, d, \delta)$ *be a first quadrant bicomplex in* $\mathbf{K}$. *All objects below depend* ***functorially*** *on* $X_{\bullet,\bullet}$, *and all morphisms are* ***natural*** *in* $X_{\bullet,\bullet}$.

1. *There exist two families* $(r \geq 1)$ *of (**vertical** and **horizontal**) bigraded derived exact couples, and two corresponding spectral sequences* (i^r, j^r, *and* k^r *have bidegrees indicated on the corresponding diagrams):*

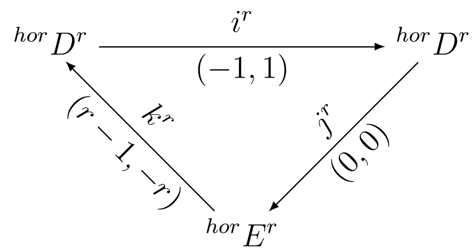
   (a)



   where

$$\begin{aligned} {}^{ver}D^r_{s,t} &\neq 0 \text{ only if } s, s+t \geq 0, \\ {}^{ver}E^r_{s,t} &\neq 0 \text{ only if } s, t \geq 0, \\ {}^{ver}E^r_{s,t} &= \left({}^{ver}E^r_{s,t},\ {}^{ver}d^r = j^r \circ k^r : {}^{ver}E^r_{s,t} \longrightarrow {}^{ver}E^r_{s-r,t+r-1}\right), \\ &\quad {}^{ver}E^{r+1}_{s,t} \simeq H\left({}^{ver}E^r_{s,t},\ {}^{ver}d^r\right). \end{aligned}$$

(b)



where

$$\begin{aligned} {}^{hor}D^r_{s,t} &\neq 0 \textit{ only if } t, s+t \geq 0, \\ {}^{hor}E^r_{s,t} &\neq 0 \textit{ only if } s, t \geq 0, \\ {}^{hor}E^r_{s,t} &= \left({}^{hor}E^r_{s,t},\ {}^{hor}d^r = j^r \circ k^r :\ {}^{hor}E^r_{s,t} \longrightarrow\ {}^{hor}E^r_{s+r-1,t-r} \right), \\ & {}^{hor}E^{r+1}_{s,t} \simeq H\left({}^{hor}E^r_{s,t},\ {}^{hor}d^r \right). \end{aligned}$$

2. *We introduce an extra entry E^0:*

   (a)
   $$ {}^{ver}E^0_{\bullet,\bullet} := \left(X_{\bullet,\bullet}, d^0 = \delta \right). $$

   (b)
   $$ {}^{hor}E^0_{\bullet,\bullet} := \left(X_{\bullet,\bullet}, d^0 = d \right). $$

3.

   (a)
   $$ {}^{ver}E^1_{\bullet,t} \simeq \left({}^{ver}H_t\left(X_{\bullet,\bullet} \right), d^1 = d|_{{}^{ver}H(X_{\bullet,\bullet})} \right). $$

   (b)
   $$ {}^{hor}E^1_{s,\bullet} \simeq \left({}^{hor}H_s\left(X_{\bullet,\bullet} \right), d^1 = \delta|_{{}^{hor}H(X_{\bullet,\bullet})} \right). $$

4.

   (a)
   $$ {}^{ver}E^2_{s,t} \simeq\ {}^{hor}H_s\left({}^{ver}H_t\left(X_{\bullet,\bullet} \right) \right). $$

   (b)
   $$ {}^{hor}E^2_{s,t} \simeq\ {}^{ver}H_t\left({}^{hor}H_s\left(X_{\bullet,\bullet} \right) \right). $$

5.

   (a) For each pair (s,t) the sequence ${}^{ver}D^r$ stabilizes:
   $$ {}^{ver}D^r_{s,t} \longrightarrow\ {}^{ver}D^{r+1}_{s,t} =:\ {}^{ver}D^\infty_{s,t} $$
   is an isomorphism whenever $r \gg 0$.

(b) *For each pair* (s,t) *the sequence* ${}^{hor}D^r$ *stabilizes:*

$$ {}^{hor}D^r_{s,t} \longrightarrow {}^{hor}D^{r+1}_{s,t} =: {}^{hor}D^\infty_{s,t} $$

is an isomorphism whenever $r \gg 0$.

6.

(a) *For each pair* (s,t) *the sequence* ${}^{ver}E^r$ *stabilizes:*

$$ {}^{ver}E^r_{s,t} \longrightarrow {}^{ver}E^{r+1}_{s,t} =: {}^{ver}E^\infty_{s,t} $$

is an isomorphism whenever $r \gg 0$.

(b) *For each pair* (s,t) *the sequence* ${}^{hor}E^r$ *stabilizes:*

$$ {}^{hor}E^r_{s,t} \longrightarrow {}^{hor}E^{r+1}_{s,t} =: {}^{hor}E^\infty_{s,t} $$

is an isomorphism whenever $r \gg 0$.

7. *The two spectral sequences converge to* $H_\bullet(Tot_\bullet(X))$ *in the following sense:*

(a) *For each* $n \geq 0$, *the sequence below consists of monomorphisms*

$$ \left[0 = {}^{ver}D^\infty_{-1,n+1}\right] \rightarrowtail {}^{ver}D^\infty_{0,n} \rightarrowtail {}^{ver}D^\infty_{1,n-1} \rightarrowtail \ldots \rightarrowtail {}^{ver}D^\infty_{n,0} \simeq H_n(Tot_\bullet(X)), $$

and for each s, t

$$ \operatorname{coker}\left({}^{ver}D^\infty_{s-1,t+1} \rightarrowtail {}^{ver}D^\infty_{s,t}\right) \simeq {}^{ver}E^\infty_{s,t}. $$

(b) *For each* $n \geq 0$, *the sequence below consists of monomorphisms*

$$ \left[0 = {}^{hor}D^\infty_{n+1,-1}\right] \rightarrowtail {}^{hor}D^\infty_{n,0} \rightarrowtail {}^{hor}D^\infty_{n-1,1} \rightarrowtail \ldots \rightarrowtail {}^{hor}D^\infty_{0,n} \simeq H_n(Tot_\bullet(X)), $$

and for each s, t

$$ \operatorname{coker}\left({}^{hor}D^\infty_{s+1,t-1} \rightarrowtail {}^{hor}D^\infty_{s,t}\right) \simeq {}^{hor}E^\infty_{s,t}. $$

8. *Let* $f_{\bullet,\bullet} : X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$ *be a morphism of bicomplexes, and let* $r \geq 1$.

(a) *If for some* r

$$ {}^{ver}E^r_{s,t}(f) : {}^{ver}E^r_{s,t}(X) \longrightarrow {}^{ver}E^r_{s,t}(Y) $$

is an isomorphism for all s, t, *then*

$$ H_n(Tot_\bullet(f)) : H_n(Tot_\bullet(X)) \longrightarrow H_n(Tot_\bullet(Y)) $$

is an isomorphism for all n.

(b) *If for some* r

$$ {}^{hor}E^r_{s,t}(f) : {}^{hor}E^r_{s,t}(X) \longrightarrow {}^{hor}E^r_{s,t}(Y) $$

is an isomorphism for all s, t, *then*

$$ H_n(Tot_\bullet(f)) : H_n(Tot_\bullet(X)) \longrightarrow H_n(Tot_\bullet(Y)) $$

is an isomorphism for all n.

9.

(a) *For all* r, $1 \le r \le \infty$, *and all* n,

$$ {}^{ver}D^r_{0,n} \simeq {}^{ver}E^r_{0,n}. $$

The composition

$$ {}^{ver}H_n(X_{0,\bullet}) = {}^{ver}E^1_{0,n} \twoheadrightarrow {}^{ver}E^\infty_{0,n} \simeq {}^{ver}D^\infty_{0,n} \rightarrowtail H_n(Tot_\bullet(X)) $$

is induced (up to sign) by the embedding of complexes $X_{0,\bullet} \hookrightarrow Tot_\bullet(X)$.
Let φ_n *be the composition*

$$ H_n(Tot_\bullet(X)) \simeq {}^{ver}D^\infty_{n,0} \twoheadrightarrow {}^{ver}E^\infty_{n,0} \rightarrowtail {}^{ver}E^2_{n,0}. $$

Then the following diagram commutes (up to sign):

$$ \begin{array}{ccccccc} Tot_{n+1}(X) & \xrightarrow{\partial} & Tot_n(X) & \longrightarrow & \text{coker } \partial & \hookleftarrow & H_n(Tot_\bullet(X)) \\ \downarrow & & \downarrow & & & & \\ X_{n+1,0} & & X_{n,0} & & \downarrow & & \downarrow \varphi_n \\ \downarrow & & \downarrow & & & & \\ \left[\text{coker}\delta_{n+1,0} = {}^{ver}E^1_{n+1,0}\right] & \xrightarrow{d|_{E^1_{n+1,0}}} & \left[\text{coker } \delta_{n,0} = {}^{ver}E^1_{n,0}\right] & \longrightarrow & \text{coker} d|_{E^1_{n+1,0}} & \hookleftarrow & {}^{ver}E^2_{n,0} \end{array} $$

(b) *For all* r, $1 \le r \le \infty$, *and all* n,

$$ {}^{hor}D^r_{n,0} \simeq {}^{hor}E^r_{n,0}. $$

The composition

$$ {}^{hor}H_n(X_{\bullet,0}) = {}^{hor}E^1_{n,0} \twoheadrightarrow {}^{hor}E^\infty_{n,0} \simeq {}^{hor}D^\infty_{n,0} \rightarrowtail H_n(Tot_\bullet(X)) $$

is induced (up to sign) by the inclusion of complexes $X_{\bullet,0} \hookrightarrow Tot_\bullet(X)$.
Let ψ_n *be the composition*

$$ H_n(Tot_\bullet(X)) \simeq {}^{hor}D^\infty_{0,n} \twoheadrightarrow {}^{hor}E^\infty_{0,n} \rightarrowtail {}^{hor}E^2_{0,n}. $$

Then the following diagram commutes (up to sign):

$$
\begin{array}{ccccccc}
Tot_{n+1}(X) & \xrightarrow{\partial} & Tot_n(X) & \longrightarrow & \text{coker } \partial & \hookleftarrow & H_n(Tot_\bullet(X)) \\
\downarrow & & \downarrow & & & & \\
X_{0,n+1} & & X_{0,n} & & \downarrow & & \downarrow \psi_n \\
\downarrow & & \downarrow & & & & \\
\left[\text{coker } d_{0,n+1} = {}^{hor}E^1_{0,n+1}\right] & \xrightarrow{\delta|_{E^1_{0,n+1}}} & \left[\text{coker} d_{0,n} = {}^{hor}E^1_{0,n}\right] & \longrightarrow & \text{coker} d|_{E^1_{0,n+1}} & \hookleftarrow & {}^{hor}E^2_{0,n}
\end{array}
$$

Proof. The proof of various forms of this theorem is scattered around several papers and books. See [Eckmann and Hilton, 1966], [Weibel, 1994, Chapter 5], [Gelfand and Manin, 2003, §III.7], and [Kashiwara and Schapira, 2006, Theorem 12.5.4 and Corollary 12.5.5(3)]. ■

B. Topologies

B.1. Grothendieck topologies.

B.1.1. Definition. *Let* $\mathbf{C}$ *be a category. A* ***sieve*** R *over* $U \in \mathbf{C}$ *is a subfunctor* $R \subseteq h_U$ *of*

$$h_U = Hom_{\mathbf{C}}(\bullet, U) : \mathbf{C}^{op} \longrightarrow \mathbf{Set}.$$

B.1.2. Remark. *Compare with [Kashiwara and Schapira, 2006, Definition 16.1.1].*

B.1.3. Definition. *A Grothendieck site (or simply a* ***site****)* X *is a pair* $(\mathbf{C}_X, Cov(X))$ *where* $\mathbf{C}_X$ *is a category, and*

$$Cov(X) = \bigcup_{U \in \mathbf{C}_X} Cov(U),$$

where $Cov(U)$ *are the sets of* ***covering sieves*** *over* U*, satisfying the axioms GT1-GT4 from [Kashiwara and Schapira, 2006, Definition 16.1.2], or, equivalently, the axioms T1-T3 from [Artin et al., 1972a, Definition II.1.1]:*

1. $h_U \in Cov(U)$.
2. *If* $R_1 \subseteq R_2 \subseteq h_U$ *and* $R_1 \in Cov(U)$*, then* $R_2 \in Cov(U)$.
3. *If* $\alpha : U \to V$ *is a morphism in* $\mathbf{C}_X$ *and* $R \in Cov(V)$*, then*

$$(h_\alpha)^{-1}(R) \in Cov(U).$$

4. *Let* R *and* $R' \in Cov(U)$ *be sieves over* U*. Assume that*

$$(h_\alpha)^{-1}(R) \in Cov(V)$$

for any

$$(\alpha : V \longrightarrow U) \in R'(V).$$

Then $R \in Cov(U)$.

The site is called ***small*** *iff* $\mathbf{C}_X$ *is a small category.*

B.1.4. Remark. *The class (or a set, if* X *is small)* $Cov(X)$ *is called the* ***topology*** *on* X*.*

B.1.5. Notation. *Given* $U \in \mathbf{C}_X$*, and* $R \in Cov(X)$*, denote simply*

$$\mathbf{C}_U := (\mathbf{C}_X)_U, \ \mathbf{C}_R := (\mathbf{C}_X)_R,$$

where $(\mathbf{C}_X)_U$ *and* $(\mathbf{C}_X)_R$ *are the comma-categories defined earlier in Definition 2.0.11 and Definition 2.0.12.*

B.1.6. Definition. *We say that the topology on a small site* X *is induced by a* ***pretopology*** *if each object* $U \in \mathbf{C}_X$ *is supplied with base-changeable (Definition B.2.2)* ***covers*** $\{U_i \to U\}_{i\in I}$*, satisfying [Artin et al., 1972a, Definition II.1.3] (compare to [Kashiwara and Schapira, 2006, Definition 16.1.5]), and the covering sieves* $R \in Cov(X)$ *are* ***generated*** *by covers:*

$$R = R_{\{U_i \to U\}} \subseteq h_U,$$

where $R_{\{U_i\to U\}}(V)$ *consists of morphisms* $(V \to U) \in h_U(V)$ *admitting a decomposition*

$$(V \to U) = (V \to U_i \to U).$$

B.1.7. Remark. *We use the word* ***covers*** *for general sites, and reserve the word* ***coverings*** *for open coverings of topological spaces.*

B.1.8. Proposition. *Let* $G \in \mathbf{Mod}(k)$*, let* $\mathcal{A} \in \mathbf{pCS}(X, \mathbf{Pro}(k))$*, and let* $R \subseteq h_U$ *be a sieve. Then:*

1.
$$Hom_{\mathbf{Pro}(k)}\left(\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R, G\right) \simeq Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}\left(R, Hom_{\mathbf{K}}(\mathcal{A}, G)\right) \simeq$$
$$\simeq \varprojlim_{(V\to U)\in \mathbf{C}_R} Hom_{\mathbf{K}}(\mathcal{A}(V), G) \simeq Hom_{\mathbf{K}}\left(\varinjlim_{(V\to U)\in \mathbf{C}_R} \mathcal{A}(V), G\right)$$

naturally in G*,* $\mathcal{A}$ *and* R*. The presheaf of* k*-**modules*** $Hom_{\mathbf{Pro}(k)}(\mathcal{A}, G)$ *is introduced in Definition A.1.1(2).*

2.
$$\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R \simeq \varinjlim_{(V\to U)\in \mathbf{C}_R} \mathcal{A}(V).$$

Proof. See [Prasolov, 2016, Proposition 2.3] ∎

B.1.9. Example. *Let X be a topological space. We will call the site $OPEN\left(X\right)$ below the **standard site** for X:*

$$OPEN\left(X\right)=\left(\mathbf{C}_{OPEN\left(X\right)},Cov\left(OPEN\left(X\right)\right)\right).$$

$\mathbf{C}_{OPEN(X)}$ has open subsets of X as objects and inclusions $U\subseteq V$ as morphisms. The pretopology on $OPEN\left(X\right)$ consists of open coverings

$$\left\{U_i\subseteq U\right\}_{i\in I}\in\mathbf{C}_{OPEN(X)}.$$

The corresponding topology consists of sieves $R_{\{U_i\subseteq U\}}\subseteq h_U$ where

$$\left(V\subseteq U\right)\in R_{\{U_i\subseteq U\}}\left(U\right)\iff\exists i\in I\ \left(V\subseteq U_i\right).$$

B.1.10. Remark. *We will always denote the standard site $OPEN\left(X\right)$ simply by X.*

B.1.11. Definition. *An open covering is called **normal** [Mardešić and Segal, 1982, §I.6.2], iff there is a partition of unity subordinated to it.*

B.1.12. Example. *Let again X be a topological space. Consider the site*

$$NORM\left(X\right)=\left(\mathbf{C}_{NORM\left(X\right)},Cov\left(NORM\left(X\right)\right)\right)$$

*where $\mathbf{C}_{NORM(X)}=\mathbf{C}_X$, while the pretopology on $NORM\left(X\right)$ consists of **normal** (Definition B.1.11) coverings $\{U_i\subseteq U\}$.*

See Conjecture 1.0.3.

B.1.13. Example. *Let X be a noetherian scheme, and define the site X^{et} by: $\mathbf{C}_{X^{et}}$ is the category of schemes Y/X étale, finite type, while the pretopology on X^{et} consists of finite surjective families of maps. See [Artin, 1962, Example 1.1.6], or [Tamme, 1994, II.1.2].*

Let $X=\left(\mathbf{C}_X,Cov\left(X\right)\right)$ be a small site (Definition B.1.3), and let $\mathbf{K}$ be a complete (Remark 2.0.2 (1)) category.

B.1.14. Definition.

1. *A **presheaf** $\mathcal{A}$ on X with values in $\mathbf{K}$ is a functor $\mathcal{A}:\left(\mathbf{C}_X\right)^{op}\to\mathbf{K}$.*
2. *A presheaf $\mathcal{A}$ on X is **separated** provided*

$$\mathcal{A}\left(U\right)\simeq Hom_{\mathbf{Set}^{\mathbf{C}_X}}\left(h_U,\mathcal{A}\right)\longrightarrow Hom_{\mathbf{Set}^{\mathbf{C}_X}}\left(R,\mathcal{A}\right)\simeq\varprojlim_{(V\to U)\in\mathbf{C}_R}\mathcal{A}\left(V\right)$$

is a monomorphism for any $U\in\mathbf{C}_X$ and for any covering sieve (Definition B.1.1 and B.1.3) R over U. The pairing $Hom_{\mathbf{Set}^{\mathbf{C}_X}}\left(\bullet,\bullet\right)$ is introduced in Definition A.1.1(6).

3. *A presheaf $\mathcal{A}$ on X is a **sheaf** provided*

$$\mathcal{A}\left(U\right)\simeq Hom_{\mathbf{Set}^{\mathbf{C}_X}}\left(h_U,\mathcal{A}\right)\longrightarrow Hom_{\mathbf{Set}^{\mathbf{C}_X}}\left(R,\mathcal{A}\right)\simeq\varprojlim_{(V\to U)\in\mathbf{C}_R}\mathcal{A}\left(V\right)$$

is an isomorphism for any $U\in\mathbf{C}_X$ and for any covering sieve R over U.

B.1.15. Remark. *The isomorphisms*

$$Hom_{\mathbf{Set}^{\mathbf{C}_X}}(R,\mathcal{A}) \simeq \varprojlim_{(V\to U)\in\mathbf{C}_R} \mathcal{A}(V)$$

and

$$\mathcal{A}(U) \simeq Hom_{\mathbf{Set}^{\mathbf{C}_X}}(h_U,\mathcal{A})$$

follow from [Prasolov, 2016, Proposition B.6], because the comma-category $\mathbf{C}_U \simeq \mathbf{C}_{h_U}$ *(Definition 2.0.11 and Remark 2.0.13) has a terminal object* $(U,\mathbf{1}_U)$*.*

B.1.16. Notation. *Denote by* $\mathbf{S}(X,\mathbf{K})$ *the category of sheaves, and by* $\mathbf{pS}(X,\mathbf{K})$ *the category of presheaves on* X *with values in* $\mathbf{K}$*.*

B.1.17. Remark. *Compare to Definition 2.2.1 and Notation 2.2.3.*

B.2. Čech (co)homology. In this section, we give different definitions of Čech (co)homology in two cases:

1. General Grothendieck topology.
2. A topology generated by a pretopology.

However, those definitions are equivalent, due to Proposition B.2.7.

Let us summarize this in the following

B.2.1. Definition. *Let* X *be a small site, let* $V\in\mathbf{C}_X$ *and let* R *be a covering sieve on* U*. Let also*

$$\mathcal{A}\in\mathbf{pCS}(\mathbf{C}_X,\mathbf{K}) \ \text{(respectively } \mathcal{B}\in\mathbf{pS}(\mathbf{C}_X,\mathbf{K})\text{)}$$

1. *In general,*

$$H_n(R,\mathcal{A}) := H_n\left({}^{Roos}C_n(R,\mathcal{A})\right),$$
$$H^n(R,\mathcal{A}) := H^n\left({}^{Roos}C^n(R,\mathcal{A})\right)$$

as in Definition B.2.5 (2). If R *is generated by a cover* $\{V_i\to V\}$*, then*

$$H_n(R,\mathcal{A}) := H_n(\{V_i\to V\},\mathcal{A}) := H_n\check{C}_\bullet(\{V_i\to V\},\mathcal{A}),$$
$$H^n(R,\mathcal{B}) := H^n(\{V_i\to V\},\mathcal{B}) := H^n\check{C}^\bullet(\{V_i\to V\},\mathcal{B})$$

as in Definition B.2.5 (3).

2. *In general,*

$$\check{H}_n(V,\mathcal{A}) := {}^{Roos}\check{H}_n(V,\mathcal{A}) := \varprojlim_{R\in Cov(V)} H_n(R,\mathcal{A}),$$

$$\check{H}^n(V,\mathcal{A}) := {}^{Roos}\check{H}^n(V,\mathcal{B}) := \varinjlim_{R\in Cov(V)} H^n(R,\mathcal{B})$$

as in Definition B.2.5 (4). If the topology on X is generated by a pretopology, then

$$\check{H}_n(V,\mathcal{A}) := \varprojlim_{\{V_i\to V\}\in Cov(V)} H_n(\{V_i \to V\},\mathcal{A}),$$
$$\check{H}^n(V,\mathcal{B}) := \varinjlim_{\{V_i\to V\}\in Cov(V)} H^n(\{V_i \to V\},\mathcal{B}).$$

as in Definition B.2.5 (5)

B.2.2. Definition. *A morphism $V \to U$ in a category $\mathbf{D}$ is called **base-changeable** ("quarrable" in ([Artin et al., 1972a, Def. II.1.3]), iff for every other morphism $U' \to U$ the fiber product $V \underset{U}{\times} U'$ exists.*

B.2.3. Definition. *Let $\mathbf{D}$ and $\mathbf{K}$ be categories. Assume that $\mathbf{D}$ is small and $\mathbf{K}$ is abelian. Let $\{U_i \to U\}$ be a family of base-changeable morphisms in $\mathbf{D}$. For a pre(co)sheaf*

$$\mathcal{A} \in \mathbf{pCS}(\mathbf{D},\mathbf{K}) \ \ \textit{(respectively } \mathcal{B} \in \mathbf{pS}(\mathbf{D},\mathbf{K})\textit{)}$$

*on $\mathbf{D}$ with values in $\mathbf{K}$, define the following **Čech chain complex** $\check{C}_\bullet$ and the **Čech cochain complex** $\check{C}^\bullet$. Assume that $\mathbf{K}$ is complete in the case of a presheaf, and cocomplete in the case of a precosheaf:*

$$\check{C}^\bullet(\{U_i \to U\},\ \mathcal{B}) := \left(\check{C}^n(\{U_i \to U\},\ \mathcal{B}), d^n\right)_{n\geq 0},$$
$$\check{C}_\bullet(\{U_i \to U\},\ \mathcal{A}) := \left(\check{C}_n(\{U_i \to U\},\ \mathcal{A}), d_n\right)_{n\geq 0},$$

where

$$\begin{aligned}
\check{C}^n(\{U_i \to U\},\ \mathcal{B}) &= \prod_{i_0,i_1,\dots,i_n\in I} \mathcal{B}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \dots \underset{U}{\times} U_{i_n}\right),\\
\check{C}_n(\{U_i \to U\},\ \mathcal{A}) &= \bigoplus_{i_0,i_1,\dots,i_n\in I} \mathcal{A}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \dots \underset{U}{\times} U_{i_n}\right),\\
d^n &= \sum_{k=0}^{n+1} (-1)^k d^n_{(k)},\\
d_n &= \sum_{k=0}^{n+1} (-1)^k d_n^{(k)},
\end{aligned}$$

$d^n_{(k)} : \check{C}^n \to \check{C}^{n+1}$ are defined by the compositions

$$\begin{aligned}
\left[\pi_{i_0,i_1,\dots,i_n,i_{n+1}}\right] \circ d^n_{(k)} := \Bigg[&\prod_{i_0,i_1,\dots,i_n\in I} \mathcal{B}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \dots \underset{U}{\times} U_{i_n}\right)\\
\xrightarrow{\pi_{i_0,\dots,\widehat{i_k},\dots,i_n}}\ & \mathcal{B}\left(U_{i_0} \underset{U}{\times} \dots \underset{U}{\times} \widehat{U_{i_k}} \underset{U}{\times} \dots \underset{U}{\times} U_{i_n} \underset{U}{\times} U_{i_{n+1}}\right)\\
\xrightarrow{\mathcal{B}\left(\sigma_{k,i_0,i_1,\dots,i_n,i_{n+1}}\right)}\ & \mathcal{B}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \dots \underset{U}{\times} U_{i_n} \underset{U}{\times} U_{i_{n+1}}\right)\Bigg],
\end{aligned}$$

and

$$\pi_{i_0,i_1,...,i_n,i_{n+1}} \;:\; \left[\prod_{i_0,i_1,...,i_{n+1}\in I} \mathcal{B}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_{n+1}}\right)\right] \longrightarrow \mathcal{B}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_{n+1}}\right),$$

$$\pi_{i_0,...,\widehat{i_k},...,i_n} \;:\; \left[\prod_{i_0,i_1,...,i_n\in I} \mathcal{B}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_n}\right)\right] \longrightarrow \mathcal{B}\left(U_{i_0} \underset{U}{\times} ... \underset{U}{\times} \widehat{U_{i_k}} \underset{U}{\times} ... \underset{U}{\times} U_{i_{n+1}}\right),$$

$$\sigma_{k,i_0,i_1,...,i_n,i_{n+1}} \;:\; U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_n} \underset{U}{\times} U_{i_{n+1}} \longrightarrow U_{i_0} \underset{U}{\times} ... \underset{U}{\times} \widehat{U_{i_k}} \underset{U}{\times} ... \underset{U}{\times} U_{i_{n+1}},$$

are the natural projections.

$d_n^{(k)} : \check{C}_{n+1} \to \check{C}_n$ *are defined dually to* $d^n_{(k)}$*, by the compositions*

$$\begin{aligned} d_n^{(k)} \circ \left[\rho_{i_0,i_1,...,i_{n+1}}\right] := \Big[& \mathcal{A}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_n} \underset{U}{\times} U_{i_{n+1}}\right) \\ & \xrightarrow{\mathcal{A}\left(\sigma_{k,i_0,i_1,...,i_n,i_{n+1}}\right)} \mathcal{A}\left(U_{i_0} \underset{U}{\times} ... \underset{U}{\times} \widehat{U_{i_k}} \underset{U}{\times} ... \underset{U}{\times} U_{i_{n+1}}\right) \\ & \xrightarrow{\rho_{i_0,...,\widehat{i_k},...,i_{n+1}}} \bigoplus_{i_0,i_1,...,i_n\in I} \mathcal{A}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_n}\right)\Big], \end{aligned}$$

where

$$\rho_{i_0,i_1,...,i_n,i_{n+1}} \;:\; \mathcal{A}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_{n+1}}\right) \longrightarrow \left[\bigoplus_{i_0,i_1,...,i_{n+1}\in I} \mathcal{A}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_{n+1}}\right)\right],$$

$$\rho_{i_0,...,\widehat{i_k},...,i_n} \;:\; \mathcal{A}\left(U_{i_0} \underset{U}{\times} ... \underset{U}{\times} \widehat{U_{i_k}} \underset{U}{\times} ... \underset{U}{\times} U_{i_{n+1}}\right) \longrightarrow \left[\bigoplus_{i_0,i_1,...,i_n\in I} \mathcal{A}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} ... \underset{U}{\times} U_{i_n}\right)\right],$$

are the natural embeddings.

B.2.4. Definition. *Let* $\mathbf{D}$ *and* $\mathbf{K}$ *be categories. Assume that* $\mathbf{D}$ *is small and* $\mathbf{K}$ *is abelian. Let* $R \subseteq h_U$ *be a sieve on* $\mathbf{D}$*. For a pre(co)sheaf*

$$\mathcal{A} \in \mathbf{pCS}\left(\mathbf{D},\mathbf{K}\right) \;\; \textit{(respectively } \mathcal{B} \in \mathbf{pS}\left(\mathbf{D},\mathbf{K}\right) \textit{)}$$

on $\mathbf{D}$ *with values in* $\mathbf{K}$*, define the following* ***Roos chain complex*** ${}^{Roos}C_\bullet$ *and the* ***Roos cochain complex*** ${}^{Roos}C^\bullet$ *(see [Roos, 1961] and [Noebeling, 1962]). Assume that* $\mathbf{K}$ *is complete in the case of a presheaf, and cocomplete in the case of a precosheaf:*

$${}^{Roos}C^\bullet\left(R,\ \mathcal{B}\right) := \left({}^{Roos}C^n\left(R,\ \mathcal{B}\right), d^n\right)_{n\geq 0},$$
$${}^{Roos}C_\bullet\left(R,\ \mathcal{A}\right) := \left({}^{Roos}C_n\left(R,\ \mathcal{A}\right), d_n\right)_{n\geq 0},$$

where

$$\langle i_0, i_1, ..., i_n\rangle := \left[U_0 \xrightarrow{i_0} U_1 \xrightarrow{i_1} ... \xrightarrow{i_{n-1}} U_n \xrightarrow{i_n} U\right] \in \mathbf{C}_R,$$

$$\begin{aligned} {}^{Roos}C^n(R,\ \mathcal{B}) &= \prod_{\langle i_0,i_1,\dots,i_n\rangle\in\mathbf{C}_R} \mathcal{B}(U_0), \\ {}^{Roos}C_n(R,\ \mathcal{A}) &= \bigoplus_{\langle i_0,i_1,\dots,i_n\rangle\in\mathbf{C}_R} \mathcal{A}(U_0), \\ d^n &= \sum_{k=0}^{n+1}(-1)^k\, d^n_{(k)}, \\ d_n &= \sum_{k=0}^{n+1}(-1)^k\, d^{(k)}_n, \end{aligned}$$

$d^n_{(k)}:\ {}^{Roos}C^n \to\ {}^{Roos}C^{n+1}$ are defined by the compositions

$$\left[\pi_{\langle i_0,i_1,\dots,i_{n+1}\rangle}\right]\circ d^n_{(k)} := \left[\left(\prod_{\langle i_0,i_1,\dots,i_n\rangle}\mathcal{B}(U_0)\right)\xrightarrow{\pi_{\langle i_0,\dots,i_k\circ i_{k-1},\dots,i_{n+1}\rangle}}\mathcal{B}(U_0)\right],$$

if $k\neq 0$,

$$\left[\pi_{\langle i_0,i_1,\dots,i_{n+1}\rangle}\right]\circ d^n_{(0)} := \left[\left(\prod_{\langle i_0,i_1,\dots,i_n\rangle}\mathcal{B}(U_0)\right)\xrightarrow{\pi_{\langle i_1,i_2,\dots,i_{n+1}\rangle}}\mathcal{B}(U_1)\xrightarrow{\mathcal{B}(i_0)}\mathcal{B}(U_0)\right],$$

if $k=0$, and

$$\begin{aligned} \pi_{\langle i_0,i_1,\dots,i_{n+1}\rangle} &: \left[\prod_{\langle i_0,i_1,\dots,i_{n+1}\rangle}\mathcal{B}(U_0)\right]\longrightarrow\mathcal{B}(U_0), \\ \pi_{\langle i_0,i_1,\dots,i_n\rangle} &: \left[\prod_{\langle i_0,i_1,\dots,i_n\rangle}\mathcal{B}(U_0)\right]\longrightarrow\mathcal{B}(U_0), \end{aligned}$$

are the natural projections.

$d^{(k)}_n:\ {}^{Roos}C_{n+1}\to\ {}^{Roos}C_n$ are defined dually to $d^n_{(k)}$, by the compositions

$$d^{(k)}_n\circ\left[\rho_{\langle i_0,i_1,\dots,i_{n+1}\rangle}\right] := \left[\mathcal{A}(U_0)\xrightarrow{\rho_{\langle i_0,\dots,i_{k+1}\circ i_k,\dots,i_{n+1}\rangle}}\left(\bigoplus_{\langle i_0,i_1,\dots,i_n\rangle}\mathcal{A}(U_0)\right)\right],$$

if $k \neq 0$,

$$d_n^{(k)} \circ \left[\rho_{\langle i_0,i_1,\dots,i_{n+1}\rangle}\right] := \left[\mathcal{A}(U_0) \xrightarrow{\mathcal{A}(i_0)} \mathcal{A}(U_1) \xrightarrow{\rho_{\langle i_1,i_2,\dots,i_{n+1}\rangle}} \left(\bigoplus_{\langle i_0,i_1,\dots,i_n\rangle} \mathcal{A}(U_0)\right)\right],$$

if $k = 0$, *where*

$$\begin{aligned} \rho_{\langle i_0,i_1,\dots,i_{n+1}\rangle} &: \mathcal{A}(U_0) \longrightarrow \left[\bigoplus_{\langle i_0,i_1,\dots,i_{n+1}\rangle} \mathcal{A}(U_0)\right], \\ \rho_{\langle i_0,i_1,\dots,i_n\rangle} &: \mathcal{A}(U_0) \longrightarrow \left[\bigoplus_{\langle i_0,i_1,\dots,i_n\rangle} \mathcal{A}(U_0)\right], \end{aligned}$$

are the natural embeddings.

B.2.5. Definition. *Let* $X = (\mathbf{C}_X, Cov(X))$ *be a small site,* $\mathcal{A}$ *a precosheaf, and* $\mathcal{B}$ *a presheaf on* X*:*

$$\begin{aligned} \mathcal{A} &\in \mathbf{pCS}(X, \mathbf{Pro}(k)), \\ \mathcal{B} &\in \mathbf{pS}(X, \mathbf{Mod}(k)). \end{aligned}$$

Let also R *be a sieve on* X*, and* $\{V_i \to V\}$ *be a family of base-changeable morphisms in* $\mathbf{C}_X$*.*

1.

$$\begin{aligned} H_0(R, \mathcal{A}) &:= \mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R \simeq \varinjlim_{(V\to U)\in \mathbf{C}_R} \mathcal{A}(V), \\ H^0(R, \mathcal{B}) &:= Hom_{\mathbf{Set}^{\mathbf{C}_X}}(R, \mathcal{B}) \simeq \varprojlim_{(V\to U)\in \mathbf{C}_R} \mathcal{B}(V), \end{aligned}$$

see Definition A.1.1(5,6), Notation B.1.5, Proposition B.1.8(2) and Remark B.1.15;

2.

$$\begin{aligned} H_n(R, \mathcal{A}) &:= H_n\left({}^{Roos}C_\bullet(R, \mathcal{A})\right), \\ H^n(R, \mathcal{B}) &:= H^n\left({}^{Roos}C^\bullet(R, \mathcal{B})\right); \end{aligned}$$

3.

$$\begin{aligned} H_n(\{V_i \to V\}, \mathcal{A}) &:= H_n \check{C}_\bullet(\{V_i \to V\}, \mathcal{A}), \\ H^n(\{V_i \to V\}, \mathcal{B}) &:= H^n \check{C}^\bullet(\{V_i \to V\}, \mathcal{B}); \end{aligned}$$

4.

$$ {}^{Roos}\check{H}_n(U,\mathcal{A}) := \varprojlim_{R\in Cov(U)} H_n(R,\mathcal{A}), $$
$$ {}^{Roos}\check{H}^n(U,\mathcal{B}) := \varinjlim_{R\in Cov(U)} H^n(R,\mathcal{B}), $$

see Notation 3.2.2.

5. *Assume that the topology on X is generated by a pretopology (Definition B.1.6). Then define:*

$$ \check{H}_n(U,\mathcal{A}) := \varprojlim_{\{U_i\to U\}\in Cov(U)} H_n(\{U_i\to U\},\mathcal{A}), $$
$$ \check{H}^n(U,\mathcal{B}) := \varinjlim_{\{U_i\to U\}\in Cov(U)} H^n(\{U_i\to U\},\mathcal{B}). $$

6. *Let $\mathcal{A}_+$ and $\mathcal{A}_\#$ be the following precosheaves:*

$$ \mathcal{A}_+ := \left(U \longmapsto \check{H}_0(U,\mathcal{A})\right), $$
$$ \mathcal{A}_\# := \mathcal{A}_{++}, $$

and let $\mathcal{B}^+$ and $\mathcal{B}^\#$ be the following presheaves:

$$ \mathcal{B}^+ := \left(U \longmapsto \check{H}^0(U,\mathcal{B})\right), $$
$$ \mathcal{B}^\# := \mathcal{B}^{++}. $$

There are natural morphisms of functors:

$$ \begin{aligned} \lambda_+ &: (\bullet)_+ \longrightarrow 1_{\mathbf{pCS}(X,\mathbf{Pro}(k))} : \lambda_+(\mathcal{A}) : \mathcal{A}_+ \longrightarrow \mathcal{A}, \\ \lambda^+ &: 1_{\mathbf{pS}(X,\mathbf{Mod}(k))} \longrightarrow (\bullet)^+ : \lambda^+(\mathcal{B}) : \mathcal{B} \longrightarrow \mathcal{B}^+, \\ \lambda_{++} &: (\bullet)_\# = (\bullet)_{++} \longrightarrow 1_{\mathbf{pCS}(X,\mathbf{Pro}(k))} : \lambda_{++}(\mathcal{A}) = \lambda_+(\mathcal{A}) \circ \lambda_+(\mathcal{A}_+) : \mathcal{A}^{++} \longrightarrow \mathcal{A}, \\ \lambda^{++} &: 1_{\mathbf{pS}(X,\mathbf{Mod}(k))} \longrightarrow (\bullet)^{++} = (\bullet)^\# : \lambda^{++}(\mathcal{B}) = \lambda^+\left(\mathcal{B}^+\right) \circ \lambda_+(\mathcal{B}) : \mathcal{B} \longrightarrow \mathcal{B}^{++}. \end{aligned} $$

B.2.6. Remark. *Compare to Definition 2.2.5.*

B.2.7. Proposition. *Assume that the topology on X is generated by a pretopology.*

1. *If a sieve R is generated by a cover $\{U_i \to U\}$, then the groups $H_n(R,\mathcal{A})$, $H^n(R,\mathcal{B})$ from Definition B.2.5(2) are naturally isomorphic to the groups $H_n(\{U_i\to U\},\mathcal{A})$, $H^n(\{U_i\to U\},\mathcal{B})$ from Definition B.2.5(3).*

2. *The groups ${}^{Roos}\check{H}_n(U,\mathcal{A})$ and ${}^{Roos}\check{H}^n(U,\mathcal{B})$ from Definition B.2.5(4) are naturally isomorphic to the groups $\check{H}_n(U,\mathcal{A})$ and $\check{H}^n(U,\mathcal{B})$ from Definition B.2.5(5).*

Proof. The reasoning below is similar to [Artin et al., 1972b, Proposition V.2.3.4 and Exercise V.2.3.6]. Let us prove the statement for the pre**co**sheaf $\mathcal{A}$. The proof for the presheaf $\mathcal{B}$ is similar. Assume that the sieve R is generated by a family $\{U_i \to U\}$. We construct first natural isomorphisms

$$H_n(R, \mathcal{A}) \simeq H_n(\{U_i \to U\}, \mathcal{A}).$$

Applying $\varprojlim$ will give us the desired natural isomorphisms

$$^{Roos}\check{H}_n(U, \mathcal{A}) = \varprojlim_{R \in Cov(U)} H_n(R, \mathcal{A}) \simeq \varprojlim_{\{U_i \to U\} \in Cov(U)} H_n(\{U_i \to U\}, \mathcal{A}) = \check{H}_n(U, \mathcal{A}).$$

Let $X_{\bullet,\bullet}$ be the following bicomplex:

$$(X_{s,t}, d, \delta) := \left(\bigoplus_{U_0 \to U_1 \to \ldots \to U_s \to U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t}} \mathcal{A}(U_0), d, \delta \right)$$

where the **horizontal** differentials $d_{\bullet,\bullet}$ are like in Definition B.2.4, while the **vertical** differentials $\delta_{\bullet,\bullet}$ are like in Definition B.2.3. Consider the two spectral sequences converging to the total homology:

$$\begin{array}{rcl} ^{ver}E^2_{s,t} & = & ^{hor}H_s\ ^{ver}H_t(X_{\bullet,\bullet}) \Longrightarrow H_{s+t}(Tot_\bullet(X)), \\ ^{hor}E^2_{s,t} & = & ^{ver}H_t\ ^{hor}H_s(X_{\bullet,\bullet}) \Longrightarrow H_{s+t}(Tot_\bullet(X)). \end{array}$$

Since the comma category

$$\mathbf{C}_X \downarrow \left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t} \right)$$

has a terminal object

$$U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t} \xrightarrow{1_{U_{i_0} \times \ldots \times U_{i_t}}} U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t},$$

it follows that

$$^{hor}E^1_{s,t} = {}^{hor}H_s(X_{\bullet,t}) = \varprojlim^s_{\mathbf{C}_X \downarrow \left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t} \right)} \mathcal{A} = \begin{cases} \mathcal{A}\left(U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t} \right) & \text{if} \quad s = 0, \\ 0 & \text{if} \quad s > 0, \end{cases}$$

Therefore

$$^{hor}E^2_{s,t} = {}^{ver}H_t\ ^{hor}H_s(X_{\bullet,\bullet}) = \begin{cases} H_n(\{U_i \to U\}, \mathcal{A}) & \text{if} \quad s = 0, \\ 0 & \text{if} \quad s > 0, \end{cases}$$

the spectral sequence degenerates from E^2 on, and

$$H_n\left(Tot_\bullet\left(X\right)\right) \simeq H_n\left(\left\{U_i \to U\right\}, \mathcal{A}\right).$$

The vertical spectral sequence is as follows:

$${}^{ver}E^1_{s,t} = {}^{ver}H_t\left(X_{s,\bullet}\right),$$

where $X_{s,\bullet}$ allows the following description:

$$X_{s,t} = \bigoplus_{\substack{U_0 \to U_1 \to \ldots \to U_s \to U \\ \varphi \in T\left(U_s \to U, U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t}\right)}} \mathcal{A}\left(U_0\right),$$

where

$$T\left(U_s \to U, U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t}\right) := \coprod_{i_0, i_1, \ldots, i_t} Hom_U\left(U_s \to U, U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t}\right),$$

and the coproduct (disjoint union) is taken in the category of sets. Denote temporarily $T\left(U_s \to U, U_{i_0} \underset{U}{\times} U_{i_1} \underset{U}{\times} \ldots \underset{U}{\times} U_{i_t}\right)$ by S. It follows that

$${}^{ver}H_t\left(X_{s\bullet}\right) = H_t\left[\bigoplus_X \mathcal{D} \longleftarrow \bigoplus_{X \times X} \mathcal{D} \longleftarrow \ldots \longleftarrow \bigoplus_{X^n} \mathcal{D} \longleftarrow \ldots\right] = \begin{cases} \mathcal{D} & \text{if} \quad t = 0 \ \& \ S \neq \varnothing \\ 0 & \text{if} \quad t \neq 0 \ \& \ S \neq \varnothing \\ 0 & \text{if} \quad S = \varnothing \end{cases},$$

where

$$\mathcal{D} = \bigoplus_{U_0 \to U_1 \to \ldots \to U_s \to U} \mathcal{A}\left(U_0\right).$$

The set S is non-empty iff $\left(U_s \to U\right) \in \mathbf{C}_R$. Finally,

$$\begin{aligned} {}^{ver}E^1_{s,t} &= {}^{ver}H_t\left(X_{s\bullet}\right) = \begin{cases} \bigoplus_{\left(U_0 \to U_1 \to \ldots \to U_s \to U\right) \in \mathbf{C}_R} \mathcal{A}\left(U_0\right) & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases}, \\ {}^{ver}E^2_{s,t} &= {}^{hor}H_s \, {}^{ver}H_t\left(X_{s\bullet}\right) = \begin{cases} H_s\left(R, \mathcal{A}\right) & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases}, \end{aligned}$$

the spectral sequence degenerates from E^2 on, and

$$H_n\left(\left\{U_i \to U\right\}, \mathcal{A}\right) \simeq {}^{ver}E^2_{0,n} \simeq Tot_n\left(X\right) \simeq {}^{hor}E^2_{n,0} \simeq H_n\left(R, \mathcal{A}\right).$$

■

B.3. Pro-homotopy and pro-homology. Let **Top** be the category of topological spaces and continuous mappings. The following categories are closely related to **Top**: the category $H(\mathbf{Top})$ of homotopy types, the category $\mathbf{Pro}(H(\mathbf{Top}))$ of pro-homotopy types, and the category $H(\mathbf{Pro}(\mathbf{Top}))$ of homotopy types of pro-spaces. The latter category is used in *strong shape theory*. It is finer than the former which is used in *shape theory*. The pointed versions $\mathbf{Pro}(H(\mathbf{Top}_*))$ and $H(\mathbf{Pro}(\mathbf{Top}_*))$ are defined similarly.

One of the most important tools in strong shape theory is a *strong expansion* (see [Mardešić, 2000], conditions (S1) and (S2) on p. 129). In this paper, it is sufficient to use a weaker notion: an $H(\mathbf{Top})$-*expansion* ([Mardešić and Segal, 1982, §I.4.1], conditions (E1) and (E2)). Those two conditions are equivalent to the following

B.3.1. Definition. *Let X be a topological space. A morphism $X \to (Y_j)_{j\in\mathbf{I}}$ in $\mathbf{Pro}(H(\mathbf{Top}))$ is called an $H(\mathbf{Top})$-expansion (or simply **expansion**) if for any polyhedron P the following mapping*

$$\varinjlim_j [Y_j, P] = \varinjlim_j Hom_{H(\mathbf{Top})}(Y_j, P) \longrightarrow Hom_{H(\mathbf{Top})}(X, P) = [X, P]$$

is bijective where $[Z, P]$ is the set of homotopy classes of continuous mappings from Z to P.

*An expansion is called **polyhedral** (or an $H(\mathbf{Pol})$-expansion) if all Y_j are polyhedra.*

B.3.2. Remark.

1. *The pointed version of this notion (an $H(\mathbf{Pol}_*)$-expansion) is defined similarly.*
2. *For any (pointed) topological space X there exists an $H(\mathbf{Pol})$-expansion (an $H(\mathbf{Pol}_*)$-expansion), see [Mardešić and Segal, 1982, Theorem I.4.7 and I.4.10].*
3. *Any two $H(\mathbf{Pol})$-expansions ($H(\mathbf{Pol}_*)$-expansions) of a (pointed) topological space X are isomorphic in the category $\mathbf{Pro}(H(\mathbf{Pol}))$ ($\mathbf{Pro}(H(\mathbf{Pol}_*))$), see [Mardešić and Segal, 1982, Theorem I.2.6].*

B.3.3. Remark. *Theorem 8 from [Mardešić and Segal, 1982, App.1, §3.2], shows that an $H(\mathbf{Pol})$- or an $H(\mathbf{Pol}_*)$-expansion for X can be constructed using nerves of normal (see Definition B.1.11) open coverings of X.*

Pro-homotopy is defined in [Mardešić and Segal, 1982, p. 121]:

B.3.4. Definition. *For a (pointed) topological space X, define its pro-homotopy pro-sets*

$$pro\text{-}\pi_n(X) := (\pi_n(Y_j))_{j\in\mathbf{J}}$$

where $X \to (Y_j)_{j\in\mathbf{J}}$ is an $H(\mathbf{Pol})$-expansion if $n = 0$, and an $H(\mathbf{Pol}_*)$-expansion if $n \geq 1$.

Similar to the "usual" algebraic topology, pro-π_0 is a pro-set (an object of $\mathbf{Pro}(\mathbf{Set})$), pro-$\pi_1$ is a pro-group (an object of $\mathbf{Pro}(\mathbf{Gr})$), and pro-$\pi_n$ are abelian pro-groups (objects of $\mathbf{Pro}(\mathbf{Ab})$) for $n \geq 2$.

Pro-homology groups are defined in [Mardešić and Segal, 1982, §II.3.2], as follows:

B.3.5. Definition. *For a topological space X, and an abelian group G, define its pro-homology groups as*

$$\text{pro-}H_n(X,G) := (H_n(Y_j,G))_{j\in\mathbf{J}}$$

where $X \to (Y_j)_{j\in\mathbf{J}}$ is a polyhedral expansion.

References

Adámek, J. and Rosický, J. (1994). *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge.

Artin, M. (1962). *Grothendieck topologies. Notes on a Seminar held in Spring 1962.* Harvard University, Cambridge, MA.

Artin, M., Grothendieck, A., and Verdier, J.-L. (1972a). *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

Artin, M., Grothendieck, A., and Verdier, J.-L. (1972b). *Théorie des topos et cohomologie étale des schémas. Tome 2.* Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

Artin, M. and Mazur, B. (1986). *Etale homotopy*, volume 100 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin. Reprint of the 1969 original.

Bredon, G. E. (1997). *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition.

Bucur, I. and Deleanu, A. (1968). *Introduction to the theory of categories and functors.* With the collaboration of Peter J. Hilton and Nicolae Popescu. Pure and Applied Mathematics, Vol. XIX. Interscience Publication John Wiley & Sons, Ltd., London-New York-Sydney.

Eckmann, B. and Hilton, P. J. (1966). Exact couples in an abelian category. *J. Algebra*, 3:38–87.

Gelfand, S. I. and Manin, Y. I. (2003). *Methods of homological algebra.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition.

Grothendieck, A. (1957). Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221.

Kashiwara, M. and Schapira, P. (2006). *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.

Mac Lane, S. (1998). *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.

Mardešić, S. (2000). *Strong shape and homology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin.

Mardešić, S. and Segal, J. (1982). *Shape theory*, volume 26 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam.

McCord, M. C. (1966). Singular homology groups and homotopy groups of finite topological spaces. *Duke Math. J.*, 33:465–474.

Noebeling, G. (1962). Über die Derivierten des Inversen und des direkten Limes einer Modulfamilie. *Topology*, 1:47–61.

Prasolov, A. V. (2013). On the universal coefficients formula for shape homology. *Topology Appl.*, 160(14):1918–1956.

Prasolov, A. V. (2016). Cosheafification. *Theory Appl. Categ.*, 31:1134–1175.

Roos, J.-E. (1961). Sur les foncteurs dérivés de $\varprojlim$. Applications. *C. R. Acad. Sci. Paris*, 252:3702–3704.

Schneiders, J.-P. (1987). Cosheaves homology. *Bull. Soc. Math. Belg. Sér. B*, 39(1):1–31.

Sugiki, Y. (2001). *The category of cosheaves and Laplace transforms*, volume 2001-33 of *UTMS Preprint Series*. University of Tokyo, Tokyo.

Tamme, G. (1994). *Introduction to étale cohomology*. Universitext. Springer-Verlag, Berlin. Translated from the German by Manfred Kolster.

Weibel, C. A. (1994). *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

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