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ON WEIGHTED FOURIER INEQUALITIES — SOME NEW SCALES OF EQUIVALENT CONDITIONS

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Abstract. For Lebesgue spaces on $\mathbb{R}^n$, we study two-weight $p \to q$-inequalities for Fourier transform. Some sufficient conditions on weights for such inequalities are known for special ranges of parameters p and q. In the same ranges of parameters we show, that in every case each of those conditions can be replaced by infinitely many conditions, even by continuous scales of conditions. We also derive some new such characterizations concerning the Fourier transform in weighted Lorentz spaces.

1. Introduction

The Fourier transform $\mathscr{F}$ of a complex-valued Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{C}$ in the Euclidean space $\mathbb{R}^n$ is defined as

$$(\mathscr{F}f)(\gamma) = \widehat{f}(\gamma) = \int\limits_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\gamma}dx,$$

where $\gamma \in \widehat{\mathbb{R}^n}(=\mathbb{R}^n)$ is a spectral variable and $x \in \mathbb{R}^n$ is a space variable.

In this paper we consider the weighted Fourier inequality

$$\|\hat{f}\|_{q,u} \leqslant C\|f\|_{p,v} \text{ for } p > 1,\ q > 1. \tag{1.1}$$

For the ranges of parameters $1 < p \leqslant q < \infty$, $1 \leqslant q < p \leqslant 2$ and $2 \leqslant q < p < \infty$ some sufficient weight conditions to guarantee that (1.1) holds are known. In the same ranges of parameters we show that in each case each of these conditions can be replaced by infinite many conditions, even by continuous scales of conditions. This gives new information concerning the continuity of the Fourier operator. We derive some similar new characterizations for inequalities of the type (1.1) with weighted Lorentz spaces. The proofs are based on some ideas from recent advances in the theory of Hardy-type inequalities, see [8, Chapter 7.3].

If f belongs to the Lebesgue space $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, then its L^p norm is denoted $\|f\|_p$. It is elementary to see that $\|\widehat{f}\|_\infty \leqslant \|f\|_1$, see e.g. [1, Section 2]. By

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using complex interpolation between this estimate and the Plancherel theorem $\|\widehat{f}\|_2 = \|f\|_2$, (see e.g. [2]) we obtain the Hausdorff-Young inequality

$$\|\widehat{f}\|_{p'} \leqslant \|f\|_p, \tag{1.2}$$

where $1 \leqslant p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\|f\|_r = \left(\int_{\mathbb{R}^n} |f(x)|^r dx \right)^{1/r}$, $1 \leqslant r \leqslant \infty$ (with the usual supremum interpretation for $r = \infty$). It is well known that the inequality (1.2) does not hold in general for any $p > 2$. Hence, the following result by J. Benedetto and H. Heinig [1, Theorem 1] complemented by J. Rastegari and G. Sinnamon [14], may be surprising (without restrictions $1 \leqslant p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$):

THEOREM A. *Let u and v be weight functions on $\mathbb{R}^n$, suppose $1 < p, q < \infty$. Then there is a constant $C > 0$ such that, for all $f \in L^p_v(\mathbb{R}^n)$ and $f \in L^1 + L^2$, the Fourier inequality*

$$\left(\int_{\mathbb{R}^n} |\widehat{f}(\gamma)|^q u(\gamma) d\gamma \right)^{1/q} \leqslant C \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p} \tag{1.3}$$

holds in the following ranges and with the following hypotheses on u and v:

(i) $1 < p \leqslant q < \infty$ and

$$A := \sup_{x>0} \left(\int_0^{1/x} u^*(t) dt \right)^{1/q} \left(\int_0^x \left[\left(\frac{1}{v} \right)^* (t) \right]^{p'-1} dt \right)^{1/p'} < \infty \tag{1.4}$$

(ii) $1 < q < p \leqslant 2$ or $2 \leqslant q < p < \infty$ and

$$B := \left(\int_0^\infty \left(\int_0^{1/x} u^*(t) dt \right)^{r/q} \left(\int_0^x \left[\left(\frac{1}{v} \right)^* (t) \right]^{p'-1} dt \right)^{r/q'} \left[\left(\frac{1}{v} \right)^* (x) \right]^{p'-1} dx \right)^{1/r} < \infty, \tag{1.5}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Moreover, for the best constant C in (1.3) it yields that $C \lesssim A$ and $C \lesssim B$ in (i) and (ii), respectively.

REMARK 1.1. In the original paper [1] part (ii) was claimed to hold for all $1 < q < p < \infty$ but it was later on pointed out in [14] that this claim indeed does not hold in the range $1 < q < 2 < p \infty$.

Here, as usual, by a weight function we mean a non-negative and measurable function, $C \lesssim A$ means that $C \leqslant cA$ for some $c > 0$, $u^*(t)$ denotes the non-increasing rearrangement of $u(x)$ (see Definition 2.2) and $L^p_v = L^p_v(\mathbb{R}^n)$, $p \geqslant 1$, denotes the weighted Lebesgue space defined by the norm

$$\|f\|_{p,v} := \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}.$$

In this paper we show that the condition (1.4) is not unique and can in fact be replaced by infinite many equivalent conditions, even by scales of conditions, see Theorem 4.8. Also the condition (1.5) can in a similar way be replaced even by scales of (equivalent) conditions, see Theorem 5.13. The results above can be interpreted as sufficient conditions for the continuity of the Fourier operator. The ideas of the results above are based on some recent results in Hardy-type inequalities, see e.g. Section 7.3 of the recent book [8] by A. Kufner et. al. Some of these ideas from Hardy-type inequalities, which are important for this paper, are presented in Section 3. In order not to disturb our discussions later on some necessary preliminaries are collected in Section 2. Especially the equivalence theorem (Theorem 2.6) seems to be of general interest, at least it was crucial for part of this new development of Hardy-type inequalities but also crucial in our proof of Theorem 4.8. Finally, in Section 6 we discuss shortly the fact that some of our results can be given also when weighted Lebesgue spaces are replaced by weighted Lorentz spaces. See Theorems 6.16 and 6.17.

2. Preliminaries

Let us first give the following important definitions.

DEFINITION 2.2. Let $(\mathbb{X},\mu)$ be a measure space, where $\mathbb{X}\subseteq\mathbb{R}^n$, and let f be a complex-valued μ-measurable function on $\mathbb{X}$. The distribution function $D_f:[0,\infty)\to[0,\infty)$ of f is defined as

$$D_f(\theta)=\mu\{x\in\mathbb{X}:|f(x)|>\theta\}. \tag{2.6}$$

Two measurable functions f and g on measurable spaces $(\mathbb{X},\mu)$ and $(\mathbb{Y},\nu)$, respectively, are called equimeasurable if $D_f=D_g$ on $[0,\infty)$. The non-increasing rearrangement of f on $(\mathbb{X},\mu)$ is the function $f^*:[0,\infty)\to[0,\infty)$ defined by

$$f^*(t)=\inf\{\theta\geqslant 0;D_f(\theta)\leqslant t\}. \tag{2.7}$$

We use the convention $\inf\theta=\infty$, so that if $D_f(s)>t$ for all $s\in[0,\infty)$, then $f^*(t)=\infty$.

For a given μ-measurable f on $(\mathbb{X},\mu)$, f^* is a non-negative, non-increasing, right continuous function on $[0,\infty)$ and f and f^* are equimeasurable, where f^* is considered as a Lebesgue measurable function on $[0,\infty)$.

DEFINITION 2.3. Let v be a weight function on $[0,\infty)$ and let $1<p<\infty$.

The weighted Lorentz space $\Lambda_p(v)$ is the set of Lebesgue measurable functions $f:\mathbb{R}^n\to\mathbb{C}$ with the property that

$$\rho(f):=\left(\int_0^\infty f^*(t)^p v(t)dt\right)^{1/p}<\infty. \tag{2.8}$$

Moreover, we say that $v \in \mathbf{B}_p$ if there is a constant $b_p > 0$ such that for all $x > 0$

$$\int_x^\infty \frac{v(t)}{t^p} dt \leqslant b_p \frac{1}{x^p} \int_0^x v(t) dt. \tag{2.9}$$

It is not difficult to see that if v is non-increasing, then $v \in \mathbf{B}_p$.

REMARK 2.4. G. G. Lorentz defined $\Lambda_p(v)$ in [9, Section 2] and proved that $\Lambda_p(v)$ is a normed linear space with $\|f\|_{\Lambda_p(v)} \equiv \rho(f)$ if and only if v is non-increasing on $(0,\infty)$ and proved also that if v is non-increasing and $\int_0^\infty v(t)dt = \infty$, then $\Lambda_p(v)$ is a Banach space with the norm $\|f\|_{\Lambda_p(v)} \equiv \rho(f)$.

Moreover, to define a Banach space associated with $\Lambda_p(v)$, in the case that v is not necessary non-increasing, we need the following condition: there exists a norm $\|\cdot\|$ on $\Lambda_p(v)$ so that $\|\cdot\| \approx \rho$ i.e. there are constants $0 < C_1 < C_2 < \infty$ such that, for all $f \in \Lambda_p(v)$,

$$C_1\|f\| \leqslant \rho(f) \leqslant C_2\|f\|. \tag{2.10}$$

An important result of E. Sawyer [15, Theorem 4] reads:

THEOREM 2.5. *Let v be a weight function on $(0,\infty)$ and let $1 < p < \infty$. The following conditions are equivalent:*

(i) $(\Lambda_p(v), \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is a norm on $\Lambda_p(v)$ satisfying (2.10).

(ii) $v \in \mathbf{B}_p$ with constant b_p.

(iii) There exists a constant $K > 0$ such that, for all $x > 0$,

$$\left(\int_0^x \left(\frac{1}{t} \int_0^t v(\tau) d\tau \right)^{1-p'} dt \right)^{1/p'} \leqslant Kx \left(\int_0^x v(t)dt \right)^{-1/p}.$$

We will need the following equivalence theorem in the proofs of our main results.

THEOREM 2.6. *For $-\infty \leqslant a < b \leqslant \infty$, α, β and s positive numbers and f, g, h measurable functions positive a.e. in (a,b), let*

$$F(x) := \int_x^b f(t)dt, \quad G(x) := \int_a^x g(t)dt \tag{2.11}$$

and denote $B_1(x;\alpha,\beta)$ - $B_{15}(x;\alpha,\beta,s;h)$ as follows:

$$B_1(x;\alpha,\beta) := F^\alpha(x) G^\beta(x);$$

$$B_2(x;\alpha,\beta,s) := \left(\int_x^b f(t) G^{\frac{\beta-s}{\alpha}}(t) dt \right)^\alpha G^s(x);$$

$$B_3(x;\alpha,\beta,s) := \left(\int_a^x g(t)F^{\frac{\alpha-s}{\beta}}(t)dt\right)^{\beta} F^s(x);$$

$$B_4(x;\alpha,\beta,s) := \left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\alpha} G^{-s}(x);$$

$$B_5(x;\alpha,\beta,s) := \left(\int_x^b g(t)F^{\frac{\alpha+s}{\beta}}(t)dt\right)^{\beta} F^{-s}(x);$$

$$B_6(x;\alpha,\beta,s) := \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x);$$

$$B_7(x;\alpha,\beta,s) := \left(\int_a^x g(t)F^{\frac{\alpha}{\beta+s}}(t)dt\right)^{\beta+s} G^{-s}(x);$$

$$B_8(x;\alpha,\beta,s) := \left(\int_a^x f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^{s}(x), \quad \alpha > s;$$

$$B_9(x;\alpha,\beta,s) := \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^{s}(x), \quad \alpha < s;$$

$$B_{10}(x;\alpha,\beta,s) := \left(\int_x^b g(t)F^{\frac{\alpha}{\beta-s}}(t)dt\right)^{\beta-s} G^{s}(x), \quad \beta > s;$$

$$B_{11}(x;\alpha,\beta,s) := \left(\int_a^x g(t)F^{\frac{\alpha}{\beta-s}}(t)dt\right)^{\beta-s} G^{s}(x), \quad \beta < s;$$

$$B_{12}(x;\alpha,\beta,s;h) := \left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\alpha} (h(x)+G(x))^{s}, \quad \beta < s;$$

$$B_{13}(x;\alpha,\beta,s;h) := \left(\int_a^x g(t)h^{\frac{\alpha-s}{\beta}}(t)dt\right)^{\beta} (h(x)+F(x))^{s}, \quad \alpha < s;$$

$$B_{14}(x;\alpha,\beta,s;h) := \left(\int_a^x f(t)(h(t)+G(t))^{\frac{\beta+s}{\alpha}}dt\right)^{\alpha} h^{-s}(x);$$

$$B_{15}(x;\alpha,\beta,s;h) := \left(\int_x^b g(t)(h(t)+F(t))^{\frac{\alpha+s}{\beta}}dt\right)^{\beta} h^{-s}(x). \tag{2.12}$$

Then the numbers

$$B_1 := \sup_{a<x<b} B_1(x;\alpha,\beta) \text{ and } B_i := \sup_{a<x<b} B_i(x;\alpha,\beta,s), \quad i = 2,3,\dots,11$$

and

$$B_i = \inf_{h\geqslant 0} \sup_{a<x<b} B_i(x;\alpha,\beta,s;h), \; i = 12,13,14,15$$

are mutually equivalent. The constants in the equivalence relations can depend on α,β *and* s.

Proof. The proof is carried out by deriving explicit positive constants c_i and d_i so that, for $i = 2,\dots,15$,

$$c_i \sup_{a<x<b} B_i(x;\alpha,\beta,s) \leqslant \sup_{a<x<b} B_1(x;\alpha,\beta) \leqslant d_i \sup_{a<x<b} B_i(x;\alpha,\beta,s). \tag{2.13}$$

The details of the proofs of equivalences between B_1, B_2, B_3, B_4 and B_5 can be found in the book [8] (see Theorem 7.29). Concerning the proof of the equivalence of also the other constants see [5, Theorem 1] and [4, Theorem 1]. For the readers convenience we include the proof of (2.13) for $i = 2$, i.e. that

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \approx \sup_{a<x<b} B_2(x;\alpha,\beta,s). \tag{2.14}$$

(i) Let $s \leqslant \beta$. Then $\frac{\beta-s}{\alpha} \geqslant 0$, and since $G(x)$ is increasing, we have that for $t \geqslant x$

$$G^{\frac{\beta-s}{\alpha}}(t) \geqslant G^{\frac{\beta-s}{\alpha}}(x). \tag{2.15}$$

Consequently,

$$\begin{aligned} B_2(x;\alpha,\beta,s) &= \left(\int_x^b f(t) G^{\frac{\beta-s}{\alpha}}(t)dt \right)^{\alpha} G^s(x) \\ &\geqslant \left(\int_x^b f(t)dt \right)^{\alpha} \left(G^{\frac{\beta-s}{\alpha}}(x) \right)^{\alpha} G^s(x) \\ &= F^{\alpha}(x) G^{\beta}(x). \end{aligned}$$

(ii) Let $s > \beta$ and denote

$$W(x) := \int_x^b f(t) G^{\frac{\beta-s}{\alpha}}(t)dt,$$

i.e. $-dW(x) = f(x)G^{\frac{\beta-s}{\alpha}}(x)dx$. Then

$$\begin{aligned} F^\alpha(x)G^\beta(x) &= G^\beta(x)\left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)G^{\frac{s-\beta}{\alpha}}(t)W^{\frac{s-\beta}{s}}(t)W^{\frac{\beta-s}{s}}(t)dt\right)^\alpha \\ &\leqslant \left(\sup_{x<t<b} G^{s-\beta}(t)W^{\frac{(s-\beta)\alpha}{s}}(t)\right)G^\beta(x)\left(-\int_x^b W^{\frac{\beta-s}{s}}(t)dW(t)\right)^\alpha \\ &= \left(\sup_{x<t<b} G^s(t)W^\alpha(t)\right)^{\frac{s-\beta}{s}}\left(\frac{s}{\beta}\right)^\alpha G^\beta(x)W^{\frac{\beta}{s}\alpha}(x) \\ &\leqslant \left(\frac{s}{\beta}\right)^\alpha\left(\sup_{x<t<b} G^s(t)W^\alpha(t)\right)^{1-\frac{\beta}{s}}\left(\sup_{x<t<b} G^s(t)W^\alpha(t)\right)^{\frac{\beta}{s}} \\ &= \left(\frac{s}{\beta}\right)^\alpha \sup_{x<t<b} B_2(t;\alpha,\beta,s). \end{aligned} \tag{2.16}$$

Hence, for every $s>0$ it follows from (2.15) and (2.16) that

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \leqslant \left(\max\left\{1,\frac{s}{\beta}\right\}\right)^\alpha \sup_{a<x<b} B_2(x;\alpha,\beta,s). \tag{2.17}$$

Also for the proof of the opposite estimate we need to consider two cases.

(iii) Now, let $s>\beta$. Then we have an inequality opposite to (2.15) and hence

$$\begin{aligned} B_2(x;\alpha,\beta,s) &= G^s(x)\left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)dt\right)^\alpha \\ &\leqslant G^s(x)\left(\int_x^b f(t)dt\right)^\alpha G^{\beta-s}(x) = F^\alpha(x)G^\beta(x). \end{aligned} \tag{2.18}$$

(iv) For $s<\beta$ we have

$$\begin{aligned} G^s(x)W^\alpha(x) &= G^s(x)\left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)F^{\frac{\beta-s}{\beta}}(t)F^{\frac{s-\beta}{\beta}}dt\right)^\alpha \\ &\leqslant \left(\sup_{x<t<b} G^{\frac{\beta-s}{\alpha}}(t)F^{\frac{\beta-s}{\beta}}(t)\right)^\alpha G^s(x)\left(\int_x^b F^{\frac{s}{\beta}-1}(t)\left(-dF(t)\right)\right)^\alpha \\ &= \left(\sup_{x<t<b} G^\beta(t)F^\alpha(t)\right)^{\frac{\beta-s}{\beta}}\left(\frac{\beta}{s}\right)^\alpha G^s(x)F^{\frac{\alpha s}{\beta}}(x) \\ &\leqslant \left(\sup_{x<t<b} G^\beta(t)F^\alpha(t)\right)^{\frac{\beta-s}{\beta}}\left(\frac{\beta}{s}\right)^\alpha\left(\sup_{a<x<b} G^\beta(x)F^\alpha(x)\right)^{\frac{s}{\beta}} \\ &\leqslant \left(\frac{\beta}{s}\right)^\alpha \sup_{a<x<b} B_1(x;\alpha,\beta). \end{aligned}$$

Therefore, by combining this estimate with (2.18), for every $s>0$ it follows that

$$\sup_{a<x<b} B_2(x;\alpha,\beta,s) \leqslant \left(\max\left\{1,\frac{\beta}{s}\right\}\right)^{\alpha} \sup_{a<x<b} B_1(x;\alpha,\beta). \tag{2.19}$$

The proof of (2.14) follows by combining (2.17) and (2.19). $\square$

REMARK 2.7. This information is useful e.g. for obtaining good estimates of the best constant in Hardy-type inequalities (see the book [8]) and correspondingly for some of the Fourier inequalities in this paper.

3. A comparison with Hardy-type inequalities

The inequality (1.3) remind us about the following well-known Hardy inequality

$$\|Hf\|_{q,u} \leqslant C_{p,q}\|f\|_{p,v},\ f\geqslant 0, \tag{3.20}$$

with parameters a,b,p,q such that $-\infty\leqslant a<b\leqslant\infty$, $0<q\leqslant\infty$, $1\leqslant p\leqslant\infty$ and with $u\geqslant 0$, $v\geqslant 0$ given weight functions. Here $(Hf)(x)=\int\limits_a^x f(t)dt$ is the Hardy operator for all measurable functions $f\geqslant 0$ on (a,b) and the norms (quasi-norms when $q<1$) in (3.20) are considered in $L_u^q(a,b)$ and $L_v^p(a,b)$, respectively.

For the case $1<p\leqslant q<\infty$ a necessary and sufficient condition on the weights $u\geqslant 0$, $v\geqslant 0$ for (3.20) to hold for all $f\geqslant 0$ is either the well-known Muckenhoupt-Bradley condition (e.g., see [11] for $q=p$ and the generalization in [3])

$$A_{MB}(x) := \sup_{a<x<b}\left(\int\limits_x^b u(t)dt\right)^{1/q}\left(\int\limits_a^x v^{1-p'}(t)dt\right)^{1/p'} \tag{3.21}$$

or the following two alternatives, which can be found in [12, Theorem 1] (see also [8, Theorem 1.1]):

$$A_{PS}^{(1)} = \sup_{a<x<b}\left(\int\limits_a^x\left(\int\limits_a^t v^{1-p'}(\tau)d\tau\right)^q u(t)dt\right)^{1/q}\left(\int\limits_a^x v^{1-p'}(t)dt\right)^{-1/p} < \infty$$

$$A_{PS}^{(2)} = \sup_{a<x<b}\left(\int\limits_x^b\left(\int\limits_t^b u(\tau)d\tau\right)^{p'} v^{1-p'}(t)dt\right)^{1/p\prime}\left(\int\limits_x^b u(t)dt\right)^{-1/q'} < \infty,$$

where $p'=\frac{p}{p-1}$. Moreover, for the best constant $C_{p,q}$ in (3.20) it yields that $C_{p,q}\approx A_{MB}\approx A_{PS}^{(1)}\approx A_{PS}^{(2)}$. Besides these conditions there exists also other (equivalent) conditions in the literature. The latest in this development is that it is nowadays known that there are infinite many such characterizing conditions, even a number of scales of conditions. The historical development and these scales of conditions are described in the review article [7] and Section 7.3 of the new book [8].

For the case $1 < q < p < \infty$ the inequality (3.20) is usually characterized by the Maz'ya-Rozin or by the Persson-Stepanov conditions. In this case it is known, see [10], [8], [16, Theorem 2.4] and [12, Theorem 3], that the inequality (3.20) holds for some finite constant $C_{p,q} > 0$, if and only if one of the following quantities is finite, the Maz'ya-Rozin condition:

$$B_{MR} := \left(\int_a^b \left(\int_x^b u(t)dt \right)^{r/p} \left(\int_a^x v^{1-p'}(t)dt \right)^{r/p'} u(t)dt \right)^{1/r} < \infty \tag{3.22}$$

or the Persson-Stepanov condition:

$$B_{PS} := \left(\int_a^b \left(\int_a^x \left(\int_a^t v^{1-p'}(\tau)d\tau \right)^q u(t)dt \right)^{r/p} \left(\int_a^x v^{1-p'}(t)dt \right)^{q-r/p} \right)^{1/r} < \infty, \tag{3.23}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, and for the best constant $C_{p,q}$ in (3.20) it holds that $C_{p,q} \approx B_{MR} \approx B_{PS}$. Moreover, these conditions are not unique and can be replaced by scales of conditions depending on a continuous parameter $s > 0$. See [13] and again for historical remarks and newest developments [7] and [8]. The descriptions above indicate that the conditions (1.4) and (1.5) to guarantee the Benedetto-Heinig inequality (1.3) are not unique and maybe can be replaced by alternative conditions which can be easier to verify and give new possibilities to estimate the best constants in (1.3). The crucial fact that confirms that indeed it is so, is the following exact relation between the conditions (1.4) and (1.5) and the Muckenhoupt-Bradley and Maz'ya-Rozin conditions, respectively.

Crucial observation 3.1 By making the substitution $s = 1/t$, we have

$$\int_0^{1/x} u^*(t)dt = \int_x^\infty \frac{u^*(1/t)}{t^2}dt.$$

Hence, the condition (1.4) can be rewritten as

$$\sup_{x>0} \left(\int_x^\infty \frac{u^*(1/t)}{t^2}dt \right)^{1/q} \left(\int_0^x \left[\left(\frac{1}{v} \right)^* (t) \right]^{p'-1} dt \right)^{1/p'} < \infty,$$

which is analogous to the Muckenhoupt-Bradley condition (3.21) ($A_{MB} < \infty$) with the weights $u^*\left(\frac{1}{t}\right)t^{-2}$ and $\left[\left(\frac{1}{v}\right)^*(t) \right]^{-1}$ instead of $u(t)$ and $v(t)$, respectively. Similarly, condition (1.5) can be written as

$$\left(\int_0^\infty \left(\int_x^\infty \frac{u^*(1/t)}{t^2}dt \right)^{r/q} \left(\int_0^x \left[\left(\frac{1}{v} \right)^* (t) \right]^{p'-1} dt \right)^{r/q'} \left[\left(\frac{1}{v} \right)^* (x) \right]^{p'-1} dx \right)^{1/r} < \infty,$$

which is analogous to the Maz'ya-Rozin condition (3.22) ($B_{MR} < \infty$), again with the same relation between the involved weights.

4. Continuity of the Fourier operator – the case $1 < p \leqslant q < \infty$

Our main result in this case reads:

THEOREM 4.8. *Let $1 < p \leqslant q < \infty$, $0 < s < \infty$ and u, v, h be weight functions on $\mathbb{R}^n$. Denote*

$$U(x) = \int_x^\infty \frac{u^*(1/t)}{t^2} dt, \; V(x) = \int_0^x \left[\left(\frac{1}{v} \right)^* (t) \right]^{p'-1} dt$$

where u^* and $(1/v)^*$ are the decreasing rearrangements of u and $\frac{1}{v}$, respectively.

Define

$$A_1(x,s) := \left(\int_x^\infty \frac{u^*(1/t)}{t^2} V^{q(\frac{1}{p'}-s)}(t)dt \right)^{1/q} V^s(x);$$

$$A_2(x,s) := \left(\int_0^x \left[\left(\frac{1}{v} \right)^* (t) \right]^{p'-1} U^{p'(\frac{1}{q}-s)}(t)dt \right)^{1/p'} U^s(x);$$

$$A_3(x,s) := \left(\int_0^x \frac{u^*(1/t)}{t^2} V^{q(\frac{1}{p'}+s)}(t)dt \right)^{1/q} V^{-s}(x);$$

$$A_4(x,s) := \left(\int_x^\infty \left[\left(\frac{1}{v} \right)^* (t) \right]^{p'-1} U^{p'(\frac{1}{q}+s)}(t)dt \right)^{1/p'} U^{-s}(x);$$

$$A_5(x,s) := \left(\int_x^\infty \frac{u^*(1/t)}{t^2} V^{\frac{q}{p'(1+sq)}}(t)dt \right)^{\frac{1+sq}{q}} U^{-s}(x);$$

$$A_6(x,s) := \left(\int_0^x \left[\left(\frac{1}{v} \right)^* (t) \right]^{p'-1} U^{\frac{p'}{q(1+sp')}}(t)dt \right)^{\frac{1+sp'}{p'}} V^{-s}(x);$$

$$A_7(x,s) := \left(\int_0^x \frac{u^*(1/t)}{t^2} V^{\frac{q}{p'(1-sq)}}(t)dt \right)^{\frac{1-sq}{q}} U^s(x), \quad qs < 1;$$

$$A_8(x,s) := \left(\int_x^\infty \frac{u^*(1/t)}{t^2} V^{\frac{q}{p'(1-sq)}}(t)dt \right)^{\frac{1-sq}{q}} U^s(x), \quad qs > 1;$$

$$A_9(x,s) := \left(\int_x^\infty \left[\left(\frac{1}{v}\right)^*(t)\right]^{p'-1} U^{\frac{p'}{q(1-sp')}}(t)dt\right)^{\frac{1-sp'}{p'}} V^s(x), \quad p's<1;$$

$$A_{10}(x,s) := \left(\int_0^x \left[\left(\frac{1}{v}\right)^*(t)\right]^{p'-1} U^{\frac{p'}{q(1-sp')}}(t)dt\right)^{\frac{1-sp'}{p'}} V^s(x), \quad p's>1;$$

$$A_{11}(x,h,s) := \left(\int_x^\infty \frac{u^*(1/t)}{t^2} h(t)^{q(\frac{1}{p'}-s)}dt\right)^{1/q} (h(x)+V(x))^s, \quad p's>1;$$

$$A_{12}(x,h,s) := \left(\int_0^x \left[\left(\frac{1}{v}\right)^*(t)\right]^{p'-1} h(t)^{p'(\frac{1}{q}-s)}dt\right)^{1/p'} (h(x)+U(x))^s, \quad qs>1;$$

$$A_{13}(x,h,s) := \left(\int_0^x \frac{u^*(1/t)}{t^2} (h(t)+V(t))^{q(\frac{1}{p'}+s)}dt\right)^{1/q} h^{-s}(x);$$

$$A_{14}(x,h,s) := \left(\int_x^\infty \left[\left(\frac{1}{v}\right)^*(t)\right]^{p'-1} (h(t)+U(t))^{p'(\frac{1}{q}+s)}(t)\right)^{1/p'} h^{-s}(x). \tag{4.24}$$

Then the Fourier inequality (1.3) *holds for all measurable functions* $f \geqslant 0$ *if any of the quantities* $A_i(s) = \sup_{x>0} A_i(x,s),\ i=1,2,\ldots,10,$ *and* $A_i(s) = \inf_{h>0}\sup_{x>0} A_i(x;h;s),\ i=11,\ldots,14,$ *is finite for any fixed* $s>0$. *Moreover, for the best constant* C *in* (1.4) *we have* $C \lesssim \min_{i,s} A_i(s),\ i=1,2,\ldots,14.$

REMARK 4.9. Since $\int_0^{1/x} u^*(t)dt = \int_x^\infty \frac{u^*(1/t)}{t^2}dt$, we see that the condition $A_1(x,1/p')<\infty$ coincides with the condition (1.4) in Theorem A. Hence, Theorem 4.8 is a generalization of this result.

Proof of Theorem 4.8. The basic ideas are just to combine Remark 4.9, crucial observation 3.1, (the equivalence) Theorem 2.6 and Theorem A. More exactly, in (2.11) and (2.12) we put $a=0,\ b=\infty,\ f(x)=\frac{u^*(1/x)}{x^2},\ g(x)=\left[\left(\frac{1}{v}\right)^*(x)\right]^{p'-1}$, so that $F(x)=U(x),\ G(x)=V(x)$, and choose $\alpha=1/q,\ \beta=1/p'$. Then the assertion follows from the fact that the following expressions

$$A_0 := \sup_{x>0} B_1(x;1/q,1/p');$$

$$A_1(s) := \sup_{x>0} B_2(x;1/q,1/p',s),\ s>0;$$

$$A_2(s) := \sup_{x>0} B_3(x;1/q,1/p',s),\ s>0;$$

$$\begin{aligned}
A_3(s) &:= \sup_{x>0} B_4(x;1/q,1/p',s), \ s>0;\\
A_4(s) &:= \sup_{x>0} B_5(x;1/q,1/p',s), \ s>0;\\
A_5(s) &:= \sup_{x>0} B_6(x;1/q,1/p',s), \ s>0;\\
A_6(s) &:= \sup_{x>0} B_7(x;1/q,1/p',s), \ s>0;\\
A_7(s) &:= \sup_{x>0} B_8(x;1/q,1/p',s), \ qs<1;\\
A_8(s) &:= \sup_{x>0} B_9(x;1/q,1/p',s), \ qs>1;\\
A_9(s) &:= \sup_{x>0} B_{10}(x;1/q,1/p',s), \ p'\,s<1;\\
A_{10}(s) &:= \sup_{x>0} B_{11}(x;1/q,1/p',s), \ p'\,s>1;\\
A_{11}(s,h) &:= \inf_{h>0}\sup_{x>0} B_{12}(x;1/q,1/p',s), \ p's>1;\\
A_{12}(s,h) &:= \inf_{h>0}\sup_{x>0} B_{13}(x;1/q,1/p',s), \ qs>1;\\
A_{13}(s,h) &:= \inf_{h>0}\sup_{x>0} B_{14}(x;1/q,1/p',s), \ s>0;\\
A_{14}(s,h) &:= \inf_{h>0}\sup_{x>0} B_{15}(x;1/q,1/p',s), \ s>0;
\end{aligned}$$

are all equivalent to A from (1.4) according to Theorem 2.6 and the finiteness of A is sufficient for the inequality (1.3) to hold. Moreover, since for the least constant C in (1.3) we have

$$C \lesssim \sup_{x>0}\left(\int_0^{1/x} u^*(t)dt\right)^{1/q}\left(\int_0^x \left[\left(\frac{1}{v}\right)^*(t)\right]^{p'-1} dt\right)^{1/p'} =: A,$$

it is clear that $C \lesssim \min_{i,s} A_i(s), \ i=1,2,\dots,14$. The proof is complete. □

REMARK 4.10. Theorem 4.8 gives us a great variety of possibilities to verify the Fourier inequality (1.3). By also using known estimates from (2.13) we can like in the case with the Hardy operator (see [7] and the book [8]) obtain better estimates of the best constant C in (1.3). Here we just give one example of alternative estimate of the best constant C in (1.3).

EXAMPLE 4.11. Let $1<p\leqslant q<\infty$, and $s\in(0,1/p]$. Then the inequality (1.3) in Theorem A holds for all measurable $f\geqslant 0$ if $A_3(s)<\infty$, where $A_3(s)$ is defined in (4.24). Moreover, if C is the best constant in (1.3). Then

$$C \leqslant p' \min_{s\in(0,\frac{1}{p}]} A_3(s).$$

Compare also with [5, Corollary 1] and c.f. also the book [8, Example 7.28:(7.46)].

REMARK 4.12. The result in Theorem 4.8 may be regarded as a statement concerning the continuity of the Fourier operator $\mathscr{F}: f \to \widetilde{f}$ between the weighted Lebesgue spaces $L_v^p(\mathbb{R}^n) \to L_u^q(\mathbb{R}^n), 1 < p \leqslant q < \infty$, namely that the operator $\mathscr{F}$ maps continuously between these spaces whenever $A_i(s) < \infty$ for any $i,\ i = 1,2,\ldots,14$ and for any fixed $s > 0$. Moreover, for the operator norm we have the estimate $\lesssim \min\limits_{i,s} A_i(s),\ i = 1,2,\ldots,14$ and $s > 0$.

5. Continuity of the Fourier operator – the cases $1 < q < p \leqslant 2$ and $2 \leqslant q < p < \infty$

The most important facts we used in the proof of Theorem 4.8 was Theorem A (i), equivalence Theorem 2.6 and the crucial observation 3.1. However, none of these arguments work directly in these cases but by using other recent results concerning Hardy-type inequalities we can work in a similar way. We introduce the following scales of conditions: $D_1(s) < \infty$, $D_2(s) < \infty$ and $D_3(s) < \infty$, where $s > 0$ and

$$D_1(s) = \left(\int_0^\infty \left[\int_x^\infty u(t) V^{q(\frac{1}{p'}-s)}(t)dt \right]^{r/p} V^{q(\frac{1}{p'}-s)+rs}(x)U(x)dx \right)^{1/r},$$

$$D_2(s) = \left(\int_0^\infty \left[\int_0^x u(t) V^{q(\frac{1}{p'}+s)}(t)dt \right]^{r/p} V^{q(\frac{1}{p'}+s)-rs}(x)U(x)dx \right)^{1/r},$$

$$D_3(s) = \left(\int_0^\infty \left[\int_x^\infty u(t) V^{q(\frac{1}{p'}-\frac{s}{r})}(t)dt \right]^{r/q} V^{s-1)}(x)v^{1-p'}(x)dx \right)^{q/r},$$

here $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, $u(t)$ and $v(t)$ are weight functions, and

$$U(x) := \int_x^\infty u(t)dt \text{ and } V(x) := \int_0^x v^{1-p'}(t)dt. \tag{5.25}$$

The main result of this section reads:

THEOREM 5.13. *Let $1 < q < p \leqslant 2$ or $2 \leqslant q < p < \infty$, $s > 0$, and let $D_1^*(s), D_2^*(s)$ and $D_3^*(s)$ denote $D_3(s), D_3(s)$ and $D_3(s)$, respectively, with $u(t)$ and $v(t)$ replaced by $u^*\left(\frac{1}{t}\right)t^{-2}$ and $\left[\left(\frac{1}{v}\right)^*(t)\right]^{-1}$, respectively. Then the Fourier inequality (1.3) holds* *for all measurable functions* $f \geqslant 0$ *if and only if any of the quantities* $D_i^*(s)$, $i = 1,2,3$, *is finite for any fixed* $s > 0$. *Moreover, for the best constant in* (1.4) *it yields that*

$$C \lesssim \min_{i,s} D_i^*(s), \; i = 1,2,3, \; s > 0. \tag{5.26}$$

REMARK 5.14. Some obvious substitutions show that the condition $D_3^*\left(\frac{r}{p'}\right) < \infty$ coincides with the condition (1.5). Hence, Theorem 5.13 is a generalization of Theorem A (ii) e.g. with more possibilities to estimate the best constant in the Fourier inequality (1.3).

Proof of Theorem 5.13. It is well known that any of the conditions $D_i(s) < \infty$, $i = 1,2,3$ and any fixed $s > 0$ are necessary and sufficient to characterize the Hardy inequality (3.20). Moreover, for the best constant C it yields that $C \approx A_i(s)$, $i = 1,2,3$ and all $s > 0$. See [13, Theorem 1], or the book [8], Theorem 7.25a) in the case $A_1(s)$ and $A_2(s)$ and [6, Theorem 2.1] concerning the case $A_3(s)$.

Next we note that the Maz'ya-Rozin constant B_{MR} is equivalent to the constant

$$D_{MR} := \int_0^\infty U^{r/q}(x) V^{r/q'}(x) dV(x),$$

with $U(x)$ and $V(x)$ defined by (5.25). More exactly, $B_{MR}^r = \frac{q}{p'} D_{MR}^r$ (see [16, Remark on p. 93]).

Moreover, we observe that the condition (1.5), via the substitution $x \to 1/x$ (see crucial observation 3.1), is equivalent to the condition $D_{MR} < \infty$ with $u(t)$ and $v(t)$ replaced by $u^*\left(\frac{1}{t}\right)t^{-2}$ and $\left[\left(\frac{1}{v}\right)^*(t)\right]^{-1}$, respectively. The proof follows by using Theorem A(ii) and the equivalences and the relations above. The estimate (5.26) is just a consequence of the corresponding estimates of C in Hardy inequalities. $\square$

REMARK 5.15. The results in Theorem 5.13 may be interpreted in terms of continuity of the Fourier operator in a similar way as in Remark 4.12 for the case $1 < p \leqslant q < \infty$.

6. Continuity of the Fourier operator in Lorentz spaces

The techniques developed in this paper can be used also for other function spaces. In this Section we shall illustrate this fact in the case of Lorentz spaces, thereby generalizing also some other results by J. Benedetto and H. Heinig [1] in a similar way.

In Lorentz spaces, sufficient and for the radial characteristic function $f(x) = \chi_{0,r}(|x|)$ even necessary conditions insuring the continuity of the Fourier operator $\widehat{f}$: $\Lambda_p(v) \to \Lambda_q(u)$ for the case $1 < p \leqslant q$, $q \geqslant 2$ are given in [1, Theorem 2]. According to this result the continuity of the Fourier operator $\widehat{f} : \Lambda_p(v) \to \Lambda_q(u)$ is connected with the following inequality

$$\|\hat{f}^*\|_{q,u} \leqslant C\|f^*\|_{p,v}, \tag{6.27}$$

where u is non-increasing and $v \in \mathbf{B}_p$. For the case $1 < p \leqslant q$, $q \geqslant 2$ the inequality (6.27) holds for all $f \in L^1 + L^2$ if the following condition is satisfied

$$E_1 := \sup_{x>0} x \left(\int_0^{1/x} u(t)dt \right)^{1/q} \left(\int_0^x v(t)dt \right)^{-1/p} < \infty. \tag{6.28}$$

For the case $2 < q \leqslant p < \infty$, it is shown in [1, Theorem 3] that the following condition is sufficient for (6.27) to hold for all $f \in L^1 + L^2$:

$$E_2 := \left(\int_0^\infty \left[\frac{1}{x} \left(\int_0^x u(t)dt \right)^{1/p} \left(\int_0^{1/x} v(t)dt \right)^{-1/p} \right]^r u(x)dx \right)^{1/r} < \infty. \tag{6.29}$$

In this Section we will use our technique from previous Sections and improve the Benedetto-Heinig result by proving that there are infinite many conditions, even scales of them, which can replace (6.28) and (6.29), to guarantee that (6.27) holds.

The case $1 < p \leqslant q, q \geqslant 2$. Let us define

$$\widetilde{U}(x) := \int_x^\infty \frac{u(1/t)}{t^2} dt, \; \widetilde{V}(x) := \int_0^x v^{p'-1}(t)dt \tag{6.30}$$

and

$$\widetilde{A}_1 := \sup_{x>0} \widetilde{U}^{1/q} \widetilde{V}^{1/p'}(x). \tag{6.31}$$

The constant $\widetilde{A}_1$ in (6.31) has been defined using the Muckenhoupt-Bradley quantity (see (3.21)). According to (the equivalence) Theorem 2.6, we can write infinitely many equivalent quantities to (6.31) for $s > 0$ as follows:

$$\widetilde{A}_2(s) := \sup_{x>0} \left(\int_x^\infty \frac{u(1/t)}{t^2} \widetilde{V}^{q(\frac{1}{p'}-s)}(t)dt \right)^{1/q} \widetilde{V}^s(x);$$

$$\widetilde{A}_3(s) := \sup_{x>0} \left(\int_0^x v^{1-p'} \widetilde{U}^{p'(\frac{1}{q}-s)}(t)dt \right)^{1/p'} \widetilde{U}^s(x);$$

$$\widetilde{A}_4(s) := \sup_{x>0} \left(\int_0^x \frac{u(1/t)}{t^2} \widetilde{V}^{q(\frac{1}{p'}+s)}(t)dt \right)^{1/q} \widetilde{V}^{-s}(x);$$

$$\widetilde{A}_5(s) := \sup_{x>0} \left(\int_x^\infty v^{1-p'} \widetilde{U}^{p'(\frac{1}{q}+s)}(t)dt \right)^{1/p'} \widetilde{U}^{-s}(x);$$

$$\widetilde{A}_6(s) := \sup_{x>0} \left(\int_x^\infty \frac{u(1/t)}{t^2} \widetilde{V}^{\frac{q}{p'(1+sq)}}(t)dt \right)^{\frac{1+sq}{q}} \widetilde{U}^{-s}(x);$$

$$\widetilde{A}_7(s) := \sup_{x>0} \left(\int_0^x v]^{1-p'} \widetilde{U}^{\frac{p'}{q(1+sp')}}(t)dt \right)^{\frac{1+sp'}{p'}} \widetilde{V}^{-s}(x);$$

$$\widetilde{A}_8(s) := \sup_{x>0} \left(\int_0^x \frac{u(1/t)}{t^2} \widetilde{V}^{\frac{q}{p'(1-sq)}}(t)dt \right)^{\frac{1-sq}{q}} \widetilde{U}^{s}(x), \quad qs < 1;$$

$$\widetilde{A}_9(s) := \sup_{x>0} \left(\int_x^\infty \frac{u(1/t)}{t^2} \widetilde{V}^{\frac{q}{p'(1-sq)}}(t)dt \right)^{\frac{1-sq}{q}} \widetilde{U}^{s}(x), \quad qs > 1;$$

$$\widetilde{A}_{10}(s) := \sup_{x>0} \left(\int_x^\infty v^{1-p'} \widetilde{U}^{\frac{p'}{q(1-sp')}}(t)dt \right)^{\frac{1-sp'}{p'}} \widetilde{V}^{s}(x), \quad p's < 1;$$

$$\widetilde{A}_{11}(s) := \sup_{x>0} \left(\int_0^x \left[\left(\frac{1}{v} \right)^* (t) \right]^{1-p'} \widetilde{U}^{\frac{p'}{q(1-sp')}}(t)dt \right)^{\frac{1-sp'}{p'}} \widetilde{V}^{s}(x), \quad p's > 1;$$

$$\widetilde{A}_{12}(h,s) := \inf_{h>0} \sup_{x>0} \left(\int_x^\infty \frac{u(1/t)}{t^2} h(t)^{q(\frac{1}{p'}-s)} dt \right)^{1/q} (h(x) + \widetilde{V}(x))^s, \quad p's > 1;$$

$$\widetilde{A}_{13}(h,s) := \inf_{h>0} \sup_{x>0} \left(\int_0^x v^{1-p'} h(t)^{p'(\frac{1}{q}-s)} dt \right)^{1/p'} (h(x) + \widetilde{U}(x))^s, \quad qs > 1;$$

$$\widetilde{A}_{14}(h,s) := \inf_{h>0} \sup_{x>0} \left(\int_0^x \frac{u(1/t)}{t^2} (h(t) + \widetilde{V}(t))^{q(\frac{1}{p'}+s)} dt \right)^{1/q} h^{-s}(x);$$

$$\widetilde{A}_{15}(h,s) := \inf_{h>0} \sup_{x>0} \left(\int_x^\infty v^{1-p'} (h(t) + \widetilde{U}(t))^{p'(\frac{1}{q}+s)}(t) \right)^{1/p'} h^{-s}(x). \tag{6.32}$$

Our main result in this case reads

THEOREM 6.16. *Let $1 < p \leqslant q$ and $q \geqslant 2$, let u and v be weight functions on $(0,\infty)$, let h be a measurable function positive a.e. on $(0,\infty)$ and let $\widetilde{U}$ and $\widetilde{V}$ be defined by* (6.30)*. If any of the quantities E_1, $\widetilde{A}_1$ or $\widetilde{A}_i(s)$ for any fixed $s > 0$, $i = 2,3,\ldots,15$, is finite then the Fourier inequality* (6.27) *holds for all $f \in L^1 + L^2$.*

Moreover, for the best constant in (6.27) *(the operator norm) it yields that*

$$C \lesssim \min_{i,s}\left(E_1,\widetilde{A}_1,\widetilde{A}_i(s)\right),$$

where $s>0$, $i=2,3,\dots,15$, *and further restricted as in* (6.32)

Proof. Suppose that $\widetilde{A}_1$ is finite. According to [1, Theorem 2] inequality (6.27) is true if the quantity D_1 in (6.28) is finite. Indeed it is so since by making the substitution $t=1/x$ and using Hölder's inequality we get that

$$\begin{aligned}
& x\left(\int_0^{1/x} u(t)dt\right)^{1/q}\left(\int_0^x v(t)dt\right)^{-1/p} \\
&= x\left(\int_x^\infty \frac{u(1/t)}{t^2}dt\right)^{1/q}\left(\int_0^x v(t)dt\right)^{-1/p} \\
&= \left(\int_0^x v^{-1/p}(t)v^{1/p}(t)dt\right)\left(\int_x^\infty \frac{u(1/t)}{t^2}dt\right)^{1/q}\left(\int_0^x v(t)dt\right)^{-1/p} \\
&\leqslant \left(\int_0^x v^{1-p'}(t)dt\right)^{1/p'}\left(\int_0^x v(t)dt\right)^{1/p}\left(\int_x^\infty \frac{u(1/t)}{t^2}dt\right)^{1/q}\left(\int_0^x v(t)dt\right)^{-1/p} \\
&= \left(\int_x^\infty \frac{u(1/t)}{t^2}dt\right)^{1/q}\left(\int_0^x v^{1-p'}(t)dt\right)^{1/p'} = \widetilde{A}_1.
\end{aligned}$$

Hence, $D_1 \leqslant \widetilde{A}_1$.

Moreover, according to Theorem 2.6, the finiteness of $\widetilde{A}_1$ is equivalent to the finiteness of any of the conditions $\widetilde{A}_i(s)<\infty$. The proof is complete. □

The case $2<q\leqslant p<\infty$. Now we introduce the following condition based on the Maz'ya-Rozin condition in (3.22) to characterize Hardy's inequality (3.20):

$$\widetilde{\mathbb{B}}_1 := \left(\int_0^\infty\left(\int_x^\infty \frac{u(1/t)}{t^2}dt\right)^{r/p}\left(\int_0^x v^{1-p'}(t)dt\right)^{r/p'}\frac{u(1/x)}{x^2}dx\right)^{1/r}<\infty.$$

Similarly, it is natural to consider the alternative condition based on the Persson-Stepanov condition (see (3.23)):

$$\widetilde{\mathbb{B}}_2 := \left(\int_0^\infty\left(\int_0^x \frac{u(1/t)}{t^2}\left(\int_0^t v^{1-p'}(\tau)d\tau\right)dt\right)^{r/p}\left(\int_0^x v^{1-p'}(t)dt\right)^{q-r/p}\frac{u(1/x)}{x^2}dx\right)^{1/r}<\infty,$$

where $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$. By using the notation (6.30) we have that

$$\widetilde{\mathbb{B}}_1 := \left(\int_0^\infty \widetilde{U}^{r/p}(x)\widetilde{V}^{r/p'}(x)\frac{u(1/x)}{x^2}dx\right)^{1/r}$$

and

$$\widetilde{\mathbb{B}}_2 := \left(\int_0^\infty \left(\int_0^x \frac{u(1/t)}{t^2}\widetilde{V}^q(t)dt\right)^{r/p} \frac{u(1/x)}{x^2}\widetilde{V}^{q-r/p}(x)dx\right)^{1/r}.$$

Similarly as in Section 5 it is natural to introduce the following scales around these constants:

$$\widetilde{\mathbb{B}}_1(s) := \left(\int_0^\infty \left[\int_x^\infty \frac{u(1/t)}{t^2}\widetilde{V}^{q(\frac{1}{p'}-s)}(t)dt\right]^{r/p} \widetilde{V}^{q(\frac{1}{p'}-s)(x)+rs}(x)\frac{u(1/x)}{x^2}dx\right)^{1/r},$$

$$\widetilde{\mathbb{B}}_2(s) := \left(\int_0^\infty \left[\int_0^x \frac{u(1/t)}{t^2}\widetilde{V}^{q(\frac{1}{p'}+s)}(t)dt\right]^{r/p} \widetilde{V}^{q(\frac{1}{p'}+s)-rs}(x)\frac{u(1/x)}{x^2}dx\right)^{1/r},$$

and also the alternative Kufner-Kuliev scale (see[6]),

$$\widetilde{\mathbb{B}}_3(s) := \left(\int_0^\infty \left[\int_x^\infty \frac{u(1/t)}{t^2}\widetilde{V}^{q(\frac{1}{p'}-s/r)}(t)dt\right]^{r/p} \widetilde{V}^{s-1}(x)v^{1-p'}(x)dx\right)^{1/r}.$$

Here $s>0$, $\widetilde{\mathbb{B}}_1\left(\frac{1}{p'}\right)=\widetilde{\mathbb{B}}_1$ and $\widetilde{\mathbb{B}}_2\left(\frac{1}{p}\right)=\widetilde{\mathbb{B}}_2$.

The main theorem in this case reads:

THEOREM 6.17. *Let $2<q\leqslant p<\infty$, let u and v be weight functions on $(0,\infty)$, such that u is non-increasing and $v\in\mathbf{B}_p$. Moreover, let $\widetilde{U}$ and $\widetilde{V}$ be defined by* (6.30). *If any of E_2, $\widetilde{\mathbb{B}}_1(s)$, $\widetilde{\mathbb{B}}_2(s)$, $\widetilde{\mathbb{B}}_3(s)$ is finite for any $s>0$, then there is $C>0$ so that the Fourier inequality* (6.27) *holds for all $f\in L^1+L^2$.*

Proof. As before, the equivalence of the quantities $\widetilde{\mathbb{B}}_1(s)$, $\widetilde{\mathbb{B}}_2(s)$ and $\widetilde{\mathbb{B}}_3(s)$ for any $s>0$ is a direct consequence of [6, Theorem 2.1] and [13, Theorem 1].

Assume now that $\widetilde{\mathbb{B}}_1=\widetilde{\mathbb{B}}_1\left(\frac{1}{p'}\right)$ is finite. Then also the constant E_2 defined by (6.29) is finite since, by Hölder's inequality and the substitution $x\to 1/x$, $(t\to 1/t)$

we find that

$$
\begin{aligned}
E_2^r &:= \int_0^\infty \left[\frac{1}{x} \left(\int_0^x u(t)dt \right)^{1/p} \left(\int_0^{1/x} v(t)dt \right)^{-1/p} \right]^r u(x)dx \\
&= \int_0^\infty \left(\int_0^x v^{-1/p}(t) v^{1/p}(t)dt \right)^r \left(\int_0^x u(t)dt \right)^{r/p} \left(\int_0^{1/x} v(t)dt \right)^{-r/p} u(x)dx \\
&\leqslant \int_0^\infty \left(\int_0^{1/x} v^{1-p'}(t)dt \right)^{r/p'} \left(\int_0^x u(t)dt \right)^{r/p} u(x)dx \\
&= \int_0^\infty \left(\int_0^{1/x} u(t)dt \right)^{r/p} \left(\int_0^x v^{1-p'}(t)dt \right)^{r/p'} \frac{u(1/x)}{x^2}dx \\
&= \int_0^\infty \left(\int_x^\infty \frac{u(1/t)}{t^2}dt \right)^{r/p} \left(\int_0^x v^{1-p'}(t)dt \right)^{r/p'} \frac{u(1/x)}{x^2}dx = \widetilde{\mathbb{B}}_1^r.
\end{aligned}
$$

The proof follows by using these two facts together with [1, Theorem 3]. □

REFERENCES

[1] J. J BENEDETTO AND H. P. HEINIG, *Weighted Fourier inequalities: new proofs and generalizations*, J. Fourier Anal. Appl., 9 (1): 1–37, 2003.

[2] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces – An introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[3] J. S. BRADLEY, *Hardy inequalities with mixed norms*, Canad. Math. Bull., 21 (4): 405–408, 1978.

[4] A. GOGATISHVILI, A. KUFNER, AND L. E. PERSSON, *Some new scales of characterizations of Hardy's inequality*, Proc. Est. Acad. Sci., 59 (1): 7–18, 2010.

[5] A. GOGATISHVILI, A. KUFNER, L. E. PERSSON, AND A. WEDESTIG, *An equivalence theorem for integral conditions related to Hardy's inequality*, Real Anal. Exchange, 29 (2): 867–880, 2003/04.

[6] A. KUFNER AND K. KULIEV, *Some characterizing conditions for the Hardy inequality*, Eurasian Math. J., 1 (2): 86–98, 2010.

[7] A. KUFNER, L. E. PERSSON, AND N. SAMKO, *Some new scales of weight characterizations of Hardy-type inequalities*, Operator theory, pseudo-differential equations, and mathematical physics, Oper. Theory Adv. Appl., 228: 261–274, 2013.

[8] A. KUFNER, L. E. PERSSON, AND N. SAMKO, *Weighted Inequalities of Hardy Type*, World Scientific, New Jersey, 2017, second edition.

[9] G. G. LORENTZ, *On the theory of spaces* Λ, Pacific J. Math., 1: 411–429, 1951.

[10] V. MAZ'JA, *Sobolev spaces*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985. Translated from Russian by T. O. Shaposhnikova.

[11] B. MUCKENHOUPT, *Hardy's inequality with weights*, Studia Math., 44: 31–38, 1972. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I.

[12] L. E. PERSSON AND V. STEPANOV, *Weighted integral inequalities with the geometric mean operator*, J. Inequal. Appl., 7 (5): 727–746, 2002.

[13] L. E. PERSSON AND V. STEPANOV, *Some scales of equivalent weight characterizations of Hardy's inequality: the case* $q < p$, Math. Inequal. Appl., 10 (2): 267–279, 2007.

[14] J. RASTEGARI AND G. SINNAMON, *Weighted Fourier inequalities via rearrangement*, J. Fourier Anal. Appl., 24: 1225–1248, 2018.

[15] E. SAWYER, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math., 96 (2): 145–158, 1990.

[16] G. SINNAMON AND V. STEPANOV, *The weighted Hardy inequality: new proofs and the case* $p = 1$, J. London Math. Soc. (2), 54 (1): 89–101, 1996.

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