Symmetry classification of viscid flows on space curves

Anna Duyunova,
Institute of Control Sciences of RAS, Bauman Moscow State Technical University,
anna.duyunova@yahoo.com
Valentin Lychagin,
Institute of Control Sciences of RAS, University of Tromsø,
valentin.lychagin@uit.no,
Sergey Tychkov,
Institute of Control Sciences of RAS,
sergey.lab06@ya.ru

July 8, 2020

Abstract

Symmetries and differential invariants of viscid flows with viscosity depending on temperature on a space curve are given. Their dependence on thermodynamic states of media is studied, and a classification of thermodynamic states is given.

1 Introduction

In this paper, we continue studies of viscid flows on space curves started in [4], [5]. Here, we consider media with viscosity ζ being a function of temperature T. The main goal of the paper is to provide classification from the admitted symmetries standpoint.

Recall that motion of viscid flows on an oriented Riemannian manifold (M, g) in the field of constant gravitational force satisfies the Navier–Stokes equations (see [2], [6] for details):

$$\begin{cases} \rho(\mathbf{u}_t + \nabla_{\mathbf{u}}\mathbf{u}) - \operatorname{div}\sigma - \mathbf{g}\rho = 0, \\ \rho_t + \mathbf{u}(\rho) + \rho \operatorname{div}\mathbf{u} = 0, \\ \rho T(s_t + \mathbf{u}(s)) - \operatorname{Tr}(\sigma_v^*(d_\nabla \mathbf{u})) + k\,\Delta_g T = 0, \end{cases} \tag{1}$$

where $\mathbf{u}$ is the flow velocity, p, ρ, s, T, $\sigma = -p\delta_{ij} + \sigma_v$ are the pressure, density, specific entropy, temperature, stress tensor of the fluid respectively, k is a constant thermal conductivity and $\mathbf{g}$ is the gravitational acceleration field.

We consider a flow on a naturally-parameterized curve

$$M = \{x = f(a),\, y = g(a),\, z = h(a)\}$$

in the three-dimensional Euclidean space. In this case vector $\mathbf{g}$ is the restriction of the vector field $(0, 0, \mathrm{g})$ on M, i.e.

$$\mathbf{g} = \mathrm{g}h'\partial_a, \quad \mathrm{g} < 0.$$

Because the system (1) is underdetermined, we use methods described in [3] to get two additional relations between thermodynamic quantities. The idea of this method is based on interpretation of media thermodynamic states as Legendrian, or Lagrangian, manifolds in contact, or symplectic, space correspondingly.

By the Navier–Stokes system $\mathcal{E}$, we mean the system (1) restricted to the curve M along with Lagrangian manifold L in a four-dimensional symplectic space endowed with structure form

$$\Omega = ds \wedge dT + \rho^{-2} d\rho \wedge dp.$$

Also we require [8] the restriction $\kappa|_L$ of the quadratic form

$$\kappa = d(T^{-1}) \cdot d\epsilon - \rho^{-2} d(pT^{-1}) \cdot d\rho$$

be negative definite, where ϵ is the specific internal energy.

The paper is organized as follows.

In Section 2 we consider two Lie algebras, namely, a symmetry algebra $\mathfrak{g}$ of the system (1) restricted to the curve M and a symmetry algebra $\mathfrak{g}_{\mathfrak{sym}}$ of the system $\mathcal{E}$. Then we give a classification of symmetry algebras depending on the functions ζ and h. It turns out that, from the admissible symmetry algebra standpoint, there are three different models of viscosity as a function of temperature and seven types of curves including arbitrary ones.

The following functions ζ have distinct symmetry algebras: (a) $\zeta(T) = \alpha T$, (b) $\zeta(T) = \alpha T^{\beta}$, $\beta \neq 1$, and (c) all others. It is worth to note that two well-known models of viscosity, namely, the elastic hard-ball model and power-law model fit into the case (b), which possesses more symmetries. While other models, including exponential law, are less symmetrical.

Though the case of constant viscosity may be considered as a part of case (b) given $\beta = 0$, sometimes it needs special consideration and additional computations concerning differential invariants and thermodynamic states, which can be found in [5].

Since the system $\mathcal{E}$ and, therefore, its symmetry algebra depend only on the z-component of the curve, that is the function h, we may give geometrical interpretation for the seven special cases of h. If a space curve is represented as a pair of plane curve $(x(\tau), y(\tau))$ and a function $z(\tau)$ considered as a way of lifting the plain curve, we can extend our classification of different forms of h to the 'lifting' function z. It appears that the distinct cases of h in the present classification match ones found in [4] (though corresponding algebras are different), therefore, we only recollect connection between the functions $h(a)$ and $z(\tau)$ in Appendix.

In Section 3 we consider thermodynamic states that admit a one-dimensional symmetry algebra. For each case studied in Section 2 corresponding thermodynamic states are found in the form of two relations on p, ρ, T and s.

In Section 4 we recall the notion of a differential invariant of the Navier–Stokes system and introduce two classes of differential invariants, namely, kinematic and Navier–Stokes invariants. Fields of differential invariants corresponding to different cases of the function h and ζ are described.

All computations for this paper are made with the DifferentialGeometry package [1] in Maple (see the Maple files *http://d-omega.org*).

2 Symmetry Lie algebra

Evidently, the symmetry Lie algebra $\mathfrak{g}_{\mathfrak{sym}}$ of the system $\mathcal{E}$ depends on the functions ζ and h. To describe this Lie algebra, we introduce a Lie algebra $\mathfrak{g}$, which is the algebra of point symmetries of the PDE system (1).

Depending on how a symmetry acts on thermodynamic quantities we distinct two kinds of point symmetries of the PDE system $\mathcal{E}$. Namely, if a symmetry acts on thermodynamic phase space trivially, we call it a *geometric symmetry.* The second kind may be specified as follows. Consider a Lie algebras homomorphism $\vartheta\colon \mathfrak{g} \to \mathfrak{h}$,

$$\vartheta\colon X \mapsto X(\rho)\partial_\rho + X(s)\partial_s + X(p)\partial_p + X(T)\partial_T,$$

where $\mathfrak{h}$ is a Lie algebra of vector fields on the thermodynamic space (p, ρ, s, T).

The algebra of geometric symmetries $\mathfrak{g}_{\mathfrak{m}}$ coincides with $\ker \vartheta$.

Let $\mathfrak{h}_{\mathfrak{t}}$ be the Lie subalgebra of the algebra $\mathfrak{h}$ that preserves thermodynamic state L.

Theorem 1 ([3])**.** *The Lie algebra $\mathfrak{g}_{\mathfrak{sym}}$ of symmetries of the Navier–Stokes system $\mathcal{E}$ coincides with*

$$\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}}).$$

2.1 $\zeta(T)$ is an arbitrary function

First of all, we consider the case of an arbitrary function $h(a)$. Then the Lie algebra $\mathfrak{g}$ is generated by the vector fields

$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s.$$

The corresponding pure thermodynamic part $\mathfrak{h}$ of the symmetry algebra is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s.$$

We may conclude that the system $\mathcal{E}$ admits the smallest Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}})$, when the function $h(a)$ is arbitrary.

It is natural to expect that the algebra $\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}})$ will be larger for some special forms of h. These special cases are listed below.

1. $h(a) = const.$

The Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3, \quad X_4 = \partial_a, \quad X_5 = t\,\partial_a + \partial_u, \quad X_6 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g} = \langle X_1, X_2, \ldots, X_6\rangle \supset \langle X_1, X_2, X_4\rangle \supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1 = \partial_p, \quad Y_2 = \partial_s, \quad Y_3 = p\,\partial_p + \rho\,\partial_\rho.$$

2. $h(a) = \lambda a$, $\lambda \neq 0$

The Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3, \quad X_4 = \partial_a, \quad X_5 = t\,\partial_a + \partial_u, \quad X_6 = t\,\partial_t + \left(\frac{\lambda g t^2}{2} + a\right)\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g} = \langle X_1, X_2, \ldots, X_6\rangle \supset \langle X_2, X_4, X_1 + \lambda g X_5\rangle \supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p + \rho\,\partial_\rho.$$

3. $h(a) = \lambda a^2$, $\lambda \neq 0$
If $\lambda < 0$, the Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3,$$
$$X_4 = \sin(\sqrt{2\lambda \mathrm{g}}\,t)\,\partial_a + \sqrt{2\lambda \mathrm{g}}\cos(\sqrt{2\lambda \mathrm{g}}\,t)\,\partial_u, \quad X_5 = \cos(\sqrt{2\lambda \mathrm{g}}\,t)\,\partial_a - \sqrt{2\lambda \mathrm{g}}\sin(\sqrt{2\lambda \mathrm{g}}\,t)\,\partial_u,$$

and, if $\lambda > 0$, by

$$X_1, \quad X_2, \quad X_3,$$
$$X_4 = e^{\sqrt{-2\lambda \mathrm{g}}\,t}\partial_a + \sqrt{-2\lambda \mathrm{g}}\,e^{\sqrt{-2\lambda \mathrm{g}}\,t}\partial_u, \quad X_5 = e^{-\sqrt{-2\lambda \mathrm{g}}\,t}\partial_a - \sqrt{-2\lambda \mathrm{g}}\,e^{-\sqrt{-2\lambda \mathrm{g}}\,t}\partial_u.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g} = \langle X_1, X_2, \ldots, X_5 \rangle \supset \langle X_4, X_5 \rangle \supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s.$$

4. $h(a) = \ln a$
The Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3, \quad X_4 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g} = \langle X_1, X_2, X_3, X_4 \rangle \supset \langle X_1, X_2 \rangle \supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p + \rho\,\partial_\rho.$$

The table below sums up the results of this subsection.

$h(a)$ is arbitrary	$\partial_t, \quad \partial_p, \quad \partial_s$
$h(a) = const$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad \partial_a, \quad t\,\partial_a + \partial_u, \quad t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho$
$h(a) = \lambda a$, $\lambda \neq 0$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad \partial_a, \quad t\,\partial_a + \partial_u,$ $t\,\partial_t + \left(\frac{\lambda g t^2}{2} + a\right)\,\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho$
$h(a) = \lambda a^2$, $\lambda \neq 0$	$\partial_t, \quad \partial_p, \quad \partial_s,$ $e^{\sqrt{2\lambda g}\,t}\partial_a + \sqrt{2\lambda g}\,e^{\sqrt{2\lambda g}\,t}\partial_u, \quad e^{-\sqrt{2\lambda g}\,t}\partial_a - \sqrt{2\lambda g}\,^{-\sqrt{2\lambda g}\,t}\partial_u$
$h(a) = \ln a$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho$

2.2 $\zeta(T) = \alpha T$

If the medium viscosity is proportional to the temperature, the symmetry algebra differs from the previous case only in one additional symmetry $p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s + T\,\partial_T$. See table below.

$h(a)$ is arbitrary	$\partial_t, \quad \partial_p, \quad \partial_s, \quad p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s + T\,\partial_T$
$h(a) = const$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad \partial_a, \quad t\,\partial_a + \partial_u,$ $p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s + T\,\partial_T, \quad t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho$
$h(a) = \lambda a$, $\lambda \neq 0$	$\partial_t, \quad \partial_a, \quad \partial_p, \quad \partial_s, \quad p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s + T\,\partial_T,$ $t\,\partial_a + \partial_u, \quad t\,\partial_t + \left(\frac{\lambda g t^2}{2} + a\right)\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho$
$h(a) = \lambda a^2$, $\lambda \neq 0$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s + T\,\partial_T,$ $e^{\sqrt{2\lambda g}\,t}\partial_a + \sqrt{2\lambda g}\,e^{\sqrt{2\lambda g}\,t}\partial_u, \quad e^{-\sqrt{2\lambda g}\,t}\partial_a - \sqrt{2\lambda g}\,e^{-\sqrt{2\lambda g}\,t}\partial_u$
$h(a) = \ln a$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s + T\,\partial_T,$ $t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho$

2.3 $\zeta(T) = \alpha T^\beta$, $\beta \neq 1$

First of all, we consider the case of an arbitrary function h. Then the Lie algebra $\mathfrak{g}$ is generated by the vector fields

$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s.$$

The corresponding pure thermodynamic part $\mathfrak{h}$ of the symmetry algebra is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s.$$

As before, we see that an arbitrary function h corresponds to the smallest Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}_t)$.

Below, the special cases of the function h are listed.

1. $h(a) = const$

The Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3, \quad X_4 = \partial_a, \quad X_5 = t\,\partial_a + \partial_u, \quad X_6 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho,$$
$$X_7 = a\,\partial_a + u\,\partial_u - \frac{2\beta}{\beta-1}p\,\partial_p - \frac{4\beta-2}{\beta-1}\rho\,\partial_\rho + \frac{2\beta}{\beta-1}s\,\partial_s - \frac{2}{\beta-1}T\,\partial_T.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g} = \langle X_1, X_2, \ldots, X_7\rangle \supset \langle X_1, X_2, X_3, X_4, X_5\rangle \supset \langle X_4\rangle \supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p + \rho\,\partial_\rho, \qquad Y_4 = (\beta-1)\rho\,\partial_\rho - \beta s\,\partial_s + T\,\partial_T.$$

2. $h(a) = \lambda a$, $\lambda \neq 0$

The Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3, \quad X_4=\partial_a, \quad X_5=t\,\partial_a+\partial_u,$$
$$X_6=t\,\partial_t+2a\,\partial_a+u\,\partial_u-\frac{3\beta-1}{\beta-1}p\,\partial_p-\frac{5\beta-3}{\beta-1}\rho\,\partial_\rho+\frac{2\beta}{\beta-1}s\,\partial_s-\frac{2}{\beta-1}T\partial_T,$$
$$X_7=t\,\partial_t+\left(\frac{\lambda g t^2}{2}+a\right)\partial_a+\lambda g t\,\partial_u-p\,\partial_p-\rho\,\partial_\rho.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g}=\langle X_1,X_2,\dots,X_7\rangle\supset\langle X_1,X_2,X_3,X_4,X_5\rangle\supset\langle X_4\rangle\supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1=\partial_p, \qquad Y_2=\partial_s, \qquad Y_3=p\,\partial_p+\rho\,\partial_\rho, \qquad Y_4=(\beta-1)\rho\,\partial_\rho-\beta s\,\partial_s+T\,\partial_T.$$

3. $h(a)=\lambda a^2$, $\lambda\neq 0$
If $\lambda<0$, the Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3,$$
$$X_4=\sin(\sqrt{2\lambda \mathrm{g}}\,t)\,\partial_a+\sqrt{2\lambda \mathrm{g}}\cos(\sqrt{2\lambda \mathrm{g}}\,t)\,\partial_u, \quad X_5=\cos(\sqrt{2\lambda \mathrm{g}}\,t)\,\partial_a-\sqrt{2\lambda \mathrm{g}}\sin(\sqrt{2\lambda \mathrm{g}}\,t)\,\partial_u,$$
$$X_6=a\,\partial_a+u\,\partial_u-\frac{2\beta}{\beta-1}p\,\partial_p-\frac{4\beta-2}{\beta-1}\rho\,\partial_\rho+\frac{2\beta}{\beta-1}s\,\partial_s-\frac{2}{\beta-1}T\partial_T$$

and, if $\lambda>0$, by

$$X_1, \quad X_2, \quad ,X_3,$$
$$X_4=e^{\sqrt{-2\lambda \mathrm{g}}\,t}\partial_a+\sqrt{-2\lambda \mathrm{g}}\,e^{\sqrt{-2\lambda \mathrm{g}}\,t}\partial_u, \quad X_5=e^{-\sqrt{-2\lambda \mathrm{g}}\,t}\partial_a-\sqrt{-2\lambda \mathrm{g}}\,e^{-\sqrt{-2\lambda \mathrm{g}}\,t}\partial_u,$$
$$X_6=a\,\partial_a+u\,\partial_u-\frac{2\beta}{\beta-1}p\,\partial_p-\frac{4\beta-2}{\beta-1}\rho\,\partial_\rho+\frac{2\beta}{\beta-1}s\,\partial_s-\frac{2}{\beta-1}T\partial_T.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g}=\langle X_1,X_2,\dots,X_6\rangle\supset\langle X_2,X_3,X_4,X_5\rangle\supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1=\partial_p, \qquad Y_2=\partial_s, \qquad Y_3=\beta p\,\partial_p+(2\beta-1)\rho\,\partial_\rho-\beta s\,\partial_s+T\,\partial_T.$$

4. $h(a)=\lambda_1 a^{\lambda_2}$, $\lambda_2\neq 0,1,2$
The Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3,$$
$$X_4=t\,\partial_t-\frac{2a}{\lambda_2-2}\partial_a-\frac{\lambda_2 u}{\lambda_2-2}\partial_u+\frac{\lambda_2(\beta+1)+2(\beta-1)}{(\beta-1)(\lambda_2-2)}p\,\partial_p+$$
$$\frac{\lambda_2(3\beta-1)+2(\beta-1)}{(\beta-1)(\lambda_2-2)}\rho\,\partial_\rho-\frac{2\beta\lambda_2}{(\beta-1)(\lambda_2-2)}s\,\partial_s+\frac{2\lambda_2}{(\beta-1)(\lambda_2-2)}T\,\partial_T.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g}=\langle X_1,X_2,X_3,X_4\rangle\supset\langle X_1,X_2,X_3\rangle\supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$\begin{aligned}&Y_1 = \partial_p, \qquad Y_2 = \partial_s,\\ &Y_3 = (\lambda_2(\beta+1) + 2(\beta-1))\, p\,\partial_p + (\lambda_2(3\beta-1) + 2(\beta-1))\rho\,\partial_\rho - 2\beta\lambda_2 s\,\partial_s + 2\lambda_2 T\,\partial_T.\end{aligned}$$

5. $h(a) = \lambda_1 e^{\lambda_2 a}$, $\lambda_2 \neq 0$
The Lie algebra $\mathfrak{g}$ is generated by

$$\begin{aligned}&X_1, \quad X_2, \quad X_3,\\ &X_4 = t\,\partial_t - \frac{2}{\lambda_2}\,\partial_a - u\,\partial_u + \frac{\beta+1}{\beta-1} p\,\partial_p + \frac{3\beta-1}{\beta-1}\rho\,\partial_\rho - \frac{2\beta}{\beta-1} s\,\partial_s + \frac{2}{\beta-1} T\partial_T.\end{aligned}$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g} = \langle X_1, X_2, X_3, X_4\rangle \supset \langle X_1, X_2, X_3\rangle \supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = (\beta+1)p\,\partial_p + (3\beta-1)\rho\,\partial_\rho - 2\beta s\,\partial_s + 2T\,\partial_T.$$

6. $h(a) = \ln a$
The Lie algebra $\mathfrak{g}$ is generated by

$$X_1, \quad X_2, \quad X_3, \quad X_4 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho.$$

This Lie algebra is solvable and its sequence of derived algebras is

$$\mathfrak{g} = \langle X_1, X_2, X_3, X_4\rangle \supset \langle X_1, X_2\rangle \supset 0.$$

The pure thermodynamic part $\mathfrak{h}$ is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p + \rho\,\partial_\rho.$$

The following table summarizes the results of this subsection.

$h(a)$ is arbitrary	$\partial_t, \quad \partial_p, \quad \partial_s$
$h(a)=const$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad \partial_a, \quad t\,\partial_a+\partial_u, \quad t\,\partial_t+a\,\partial_a-p\,\partial_p-\rho\,\partial_\rho,$ $a\,\partial_a+u\,\partial_u-\frac{2\beta}{\beta-1}p\,\partial_p-\frac{4\beta-2}{\beta-1}\rho\,\partial_\rho+\frac{2\beta}{\beta-1}s\,\partial_s-\frac{2}{\beta-1}T\partial_T$
$h(a)=\lambda a$, $\lambda\neq 0$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad \partial_a, \quad t\,\partial_a+\partial_u,$ $t\,\partial_t+2a\,\partial_a+u\,\partial_u-\frac{3\beta-1}{\beta-1}p\,\partial_p-\frac{5\beta-3}{\beta-1}\rho\,\partial_\rho+\frac{2\beta s}{\beta-1}\partial_s-\frac{2T}{\beta-1}\partial_T,$ $t\,\partial_t+\left(\frac{\lambda g t^2}{2}+a\right)\partial_a+\lambda g t\,\partial_u-p\,\partial_p-\rho\,\partial_\rho$
$h(a)=\lambda a^2$, $\lambda\neq 0$	$\partial_t, \quad \partial_p, \quad \partial_s,$ $e^{\sqrt{2\lambda g}\,t}\partial_a+\sqrt{2\lambda g}\,e^{\sqrt{2\lambda g}\,t}\partial_u, \quad e^{-\sqrt{2\lambda g}\,t}\partial_a-\sqrt{2\lambda g}\,e^{-\sqrt{2\lambda g}\,t}\partial_u,$ $a\,\partial_a+u\,\partial_u-\frac{2\beta}{\beta-1}p\,\partial_p-\frac{4\beta-2}{\beta-1}\rho\,\partial_\rho+\frac{2\beta}{\beta-1}s\,\partial_s-\frac{2}{\beta-1}T\partial_T$
$h(a)=\lambda_1 a^{\lambda_2}$, $\lambda_2\neq 0,1,2$	$\partial_t, \quad \partial_p, \quad \partial_s,$ $t\,\partial_t-\frac{2a}{\lambda_2-2}\partial_a-\frac{\lambda_2 u}{\lambda_2-2}\partial_u+\frac{\lambda_2(\beta+1)+2(\beta-1)}{(\beta-1)(\lambda_2-2)}p\,\partial_p+$ $\frac{\lambda_2(3\beta-1)+2(\beta-1)}{(\beta-1)(\lambda_2-2)}\rho\,\partial_\rho-\frac{2\beta\lambda_2 s}{(\beta-1)(\lambda_2-2)}\partial_s+\frac{2\lambda_2 T}{(\beta-1)(\lambda_2-2)}\partial_T$
$h(a)=\lambda_1 e^{\lambda_2 a}$, $\lambda_2\neq 0$	$\partial_t, \quad \partial_p, \quad \partial_s,$ $t\,\partial_t-\frac{2}{\lambda_2}\partial_a-u\,\partial_u+\frac{\beta+1}{\beta-1}p\,\partial_p+\frac{3\beta-1}{\beta-1}\rho\,\partial_\rho-\frac{2\beta s}{\beta-1}\partial_s+\frac{2T}{\beta-1}\partial_T$
$h(a)=\ln a$	$\partial_t, \quad \partial_p, \quad \partial_s, \quad t\,\partial_t+a\,\partial_a-p\,\partial_p-\rho\,\partial_\rho$

3 Thermodynamic states with a one-dimensional symmetry algebra

Recall that our approach to thermodynamics is based on geometric interpretation of thermodynamic states as Lagrangian manifolds [3].

In this section we study thermodynamic states that admit a one-dimensional symmetry algebra

$$Z=\gamma_1 Y_1+\gamma_2 Y_2+\ldots+\gamma_k Y_k,$$

where Y_i are pure thermodynamic symmetries, that is, elements of the algebra $\mathfrak{h}$. The cases of thermodynamic states admitting a two-dimensional symmetry algebra can be studied in the similar manner.

Following our approach, we come to a system on the Lagrangian manifold L

$$\begin{cases}\Omega|_L=0,\\ (\iota_Z\Omega)|_L=0\end{cases}$$

with the condition $\kappa|_L < 0$.

The latter is equivalent to an overdetermined PDE system on the specific energy $\epsilon = \epsilon(\rho, s)$. We can obtain two state equations from its solution with relations

$$T = \epsilon_s, \quad p = \rho^2 \epsilon_\rho.$$

3.1 $\zeta(T)$ is an arbitrary function

First of all, since the quadratic form κ is degenerated everywhere, there are no valid thermodynamic states admitting a one-dimensional symmetry algebra for an arbitrary function $h(a)$ and $h(a) = \lambda a^2$.

Note that for $h(a) = const$, $h(a) = \lambda a$ and $h(a) = \ln a$ the pure thermodynamic symmetries $\mathfrak{h}$ are same.

If a one-dimensional symmetry subalgebra is generated by

$$\gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3 = \gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 (p\,\partial_p + \rho\,\partial_\rho),$$

then the corresponding thermodynamic state (or Lagrangian manifold) is defined by the equations

$$p = \frac{-(\gamma_2 F' + C)\rho}{\gamma_3} - \frac{\gamma_1}{\gamma_3}, \quad T = F', \quad F = F\left(s - \frac{\gamma_2}{\gamma_3} \ln \rho\right)$$

where C is a constant and F is an arbitrary function. The admissibility condition $\kappa|_L < 0$ leads to the relations

$$F' > 0, \quad F'' > 0, \quad \frac{\gamma_2 F' + C}{\gamma_3} < 0$$

for all $s \in (-\infty, s_0]$.

3.2 $\zeta(T) = \alpha T$

1. $h(a)$ is arbitrary, $h(a) = \lambda a^2$

If a one-dimensional symmetry subalgebra is generated by

$$\gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3 = \gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 (p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s + T\,\partial_T),$$

then the corresponding thermodynamic state is defined by the equations

$$p = \rho^2 \left(s - \frac{\gamma_2}{\gamma_3}\right) F' + C\rho - \frac{\gamma_1}{\gamma_3}, \quad T = \rho F', \quad F = F\left(\left(s - \frac{\gamma_2}{\gamma_3}\right)\rho\right),$$

where C is a constant and F is an arbitrary function. The admissibility condition $\kappa|_L < 0$ leads to the relations

$$F' > 0, \quad F'' > 0, \quad CF'' - (F')^2 > 0.$$

2. $h(a) = const$, $h(a) = \lambda a$, $h(a) = \ln a$

If a one-dimensional symmetry subalgebra is generated by

$$\gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3 + \gamma_4 Y_4 = \gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 (p\,\partial_p + \rho\,\partial_\rho) + \gamma_4 (s\,\partial_s - T\,\partial_T),$$

then the corresponding thermodynamic state is defined by the equations

$$p = C\rho - \frac{F'(\gamma_4 s + \gamma_2)\rho^{-\frac{\gamma_4}{\gamma_3}+1} + \gamma_1}{\gamma_3}, \quad T = \rho^{-\frac{\gamma_4}{\gamma_3}} F', \quad F = F\left(\left(s + \frac{\gamma_2}{\gamma_4}\right)\rho^{-\frac{\gamma_4}{\gamma_3}}\right),$$

where C is a constant and F is an arbitrary function. The admissibility condition $\kappa|_L < 0$ leads to the relations

$$F' > 0, \quad F'' > 0, \quad F'F''(\gamma_3 + \gamma_4)(\gamma_4 s + \gamma_2)\rho^{-\frac{\gamma_4}{\gamma_3}} - C\gamma_3^2 F'' + \gamma_4^2 (F')^2 < 0$$

for all $s \in (-\infty, s_0]$.

3.3 $\zeta(T) = \alpha T^\beta$, $\beta \neq 1$

Since the quadratic form κ is degenerated everywhere, there are no valid thermodynamic states admitting a one-dimensional symmetry algebra for an arbitrary function $h(a)$.

1. $h(a) = const$, $h(a) = \lambda a$, $\lambda \neq 0$

If a one-dimensional symmetry subalgebra is generated by

$$\gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3 + \gamma_4 Y_4 = \gamma_1\,\partial_p + \gamma_2\,\partial_s + \gamma_3(p\,\partial_p + \rho\,\partial_\rho) + \gamma_4\left((\beta-1)\rho\,\partial_\rho - \beta s\,\partial_s + T\,\partial_T\right),$$

then the corresponding thermodynamic state is defined by the equations

$$p = \frac{F'(\beta\gamma_4 s - \gamma_2)\rho^{\frac{\beta\gamma_4+\gamma_3}{\gamma_3+(\beta-1)\gamma_4}} - F\gamma_4(\beta-1)\rho^{\frac{\gamma_3}{\gamma_3+(\beta-1)\gamma_4}}}{\gamma_3 + (\beta-1)\gamma_4} - \frac{\gamma_1}{\gamma_3}, \quad T = F'\rho^{\frac{\gamma_4}{\gamma_3+(\beta-1)\gamma_4}},$$

$$F = F\left(\left(s - \frac{\gamma_2}{\beta\gamma_4}\right)\rho^{\frac{\beta\gamma_4}{\gamma_3+(\beta-1)\gamma_4}}\right)$$

where F is an arbitrary function. The admissibility condition $\kappa|_L < 0$ leads to the relations

$$F' > 0, \quad F'' > 0, \quad F'F''(\gamma_3 - \gamma_4)(\beta\gamma_4 s - \gamma_2)\rho^{\frac{\beta\gamma_4}{\gamma_3+(\beta-1)\gamma_4}} - \gamma_4(\gamma_3(\beta-1)FF'' + \gamma_4 F'^2) > 0$$

for all $s \in (-\infty, s_0]$.

2. $h(a) = \lambda a^2$, $\lambda \neq 0$

If a one-dimensional symmetry subalgebra is generated by

$$\gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3 = \gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3\left(\beta p\,\partial_p + (2\beta-1)\rho\,\partial_\rho - \beta s\,\partial_s + T\,\partial_T\right),$$

then the corresponding thermodynamic state is defined by the equations

$$p = \frac{F'(\beta\gamma_3 s - \gamma_2)\rho^{\frac{2\beta}{2\beta-1}} - F\gamma_3(\beta-1)\rho^{\frac{\beta}{2\beta-1}}}{\gamma_3(2\beta-1)} - \frac{\gamma_1}{\beta\gamma_3}, \quad T = F'\rho^{\frac{1}{2\beta-1}}, \quad F = F\left(\left(s - \frac{\gamma_2}{\beta\gamma_3}\right)\rho^{\frac{\beta}{2\beta-1}}\right),$$

where F is an arbitrary function. The admissibility condition $\kappa|_L < 0$ leads to the relations

$$F' > 0, \quad F'' > 0, \quad F'F''(\beta-1)(\beta s - \frac{\gamma_2}{\gamma_3})\rho^{\frac{\beta}{2\beta-1}} - \beta(\beta-1)FF'' - F'^2 > 0$$

for all $s \in (-\infty, s_0]$.

If $\beta = 1/2$ the thermodynamic state is defined by the equations

$$p = \frac{\rho^2 F'(\rho) + 2\gamma_1 s}{2\gamma_2 - \gamma_3 s}, \quad T = \frac{\gamma_3\rho F(\rho) - 4\gamma_2(C\rho + \gamma_1)}{\rho(2\gamma_2 - \gamma_3 s)^2},$$

where C is a constant and F is an arbitrary function. The admissibility condition gives

$$\frac{2\gamma_2 - \gamma_3 s}{\gamma_3} > 0, \quad (p^2 + 4p\rho s T)\gamma_3^2 - ((2\rho^3 F'' + 8(\gamma_2 p - \gamma_1 s))\rho T - 4\gamma_1 p)\gamma_3 + 4\gamma_1^2 < 0.$$

3. $h(a) = \lambda_1 a^{\lambda_2}$, $\lambda_2 \neq 0, 1, 2$
If a one-dimensional symmetry subalgebra is generated by

$$\gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3 = \gamma_1 \partial_p + \gamma_2 \partial_s + \\ \gamma_3 \left((\lambda_2(\beta+1) + 2(\beta-1))\, p\, \partial_p + (\lambda_2(3\beta-1) + 2(\beta-1))\rho\, \partial_\rho - 2\beta\lambda_2 s\, \partial_s + 2\lambda_2 T\, \partial_T \right),$$

then the corresponding thermodynamic state is defined by the equations

$$p = \frac{(2\beta\lambda_2\gamma_3 s - \gamma_2)F'\rho^{\frac{2\lambda_2}{A}+1} - 2(\beta-1)\lambda_2\gamma_3 F \rho^{\frac{2\lambda_2(1-\beta)}{A}+1}}{A\gamma_3} - \frac{\gamma_1}{\gamma_3(\lambda_2(\beta+1) + 2(\beta-1))},$$

$$T = \rho^{\frac{2\lambda_2}{A}} F', \quad F = F\left(\rho^{\frac{2\beta\lambda_2}{A}}\left(s - \frac{\gamma_2}{2\beta\lambda_2\gamma_3}\right)\right),$$

where $A = \lambda_2(3\beta - 1) + 2(\beta - 1)$ and the admissibility condition $\kappa|_L < 0$ gives

$$F' > 0, \quad F'' > 0,$$

$$\left(2\beta\lambda_2 s - \frac{\gamma_2}{\gamma_3}\right)(\beta-1)(\lambda_2+2)F'F''\rho^{\frac{2\beta\lambda_2}{A}} - 2\lambda_2\left((\lambda_2(\beta+1) + 2(\beta-1))(\beta-1)FF'' + 2\lambda_2 F'^2\right) > 0.$$

4. $h(a) = \lambda_1 e^{\lambda_2 a}$, $\lambda_2 \neq 0$
If a one-dimensional symmetry subalgebra is generated by

$$\gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3 = \gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 \left((\beta+1)p\, \partial_p + (3\beta-1)\rho\, \partial_\rho - 2\beta s\, \partial_s + 2T\, \partial_T\right),$$

then the corresponding thermodynamic state is defined by the equations

$$p = \frac{(2\beta\gamma_3 s - \gamma_2)\rho^{\frac{3\beta+1}{3\beta-1}} F' - 2\gamma_3(\beta-1)\rho^{\frac{\beta+1}{3\beta-1}} F}{\gamma_3(3\beta-1)} - \frac{\gamma_1}{\gamma_3(\beta+1)}, \quad T = \rho^{\frac{2}{3\beta-1}} F'$$

$$F = F\left(\rho^{\frac{2\beta}{3\beta-1}}\left(s - \frac{\gamma_2}{2\beta\gamma_3}\right)\right)$$

and

$$F' > 0, \quad F'' > 0, \quad (\beta-1)\left(2\beta s - \frac{\gamma_2}{\gamma_3}\right) F'F''\rho^{\frac{2\beta}{3\beta-1}} - 2(\beta^2-1)FF'' - 4F'^2 > 0.$$

If $\beta = 1/3$ then the thermodynamic state is defined by the equations

$$p = \frac{\rho^2 F'(\rho) - 3\gamma_1 s(\gamma_3 s - 3\gamma_2)}{(2\gamma_3 s - 3\gamma_2)^2}, \quad T = \frac{27\gamma_2^2(C\rho + \gamma_1) - 4\gamma_3\rho F(\rho)}{\rho(2\gamma_3 s - 3\gamma_2)^3},$$

and the admissibility condition $\kappa|_L < 0$ gives

$$\frac{6\gamma_3}{2\gamma_3 s - 3\gamma_2} < 0, \qquad \frac{6\gamma_3(2F' + \rho F'')}{2\gamma_3 s - 3\gamma_2} + \frac{\left(4\gamma_3\rho^2 F' + 27\gamma_1\gamma_2^2\right)^2}{\rho^3 T\left(2\gamma_3 s - 3\gamma_2\right)^4} < 0$$

for all $s \in (-\infty, s_0]$.

If $\beta = -1$ then the thermodynamic state is defined by the equations

$$p = \rho^{\frac{1}{2}} F'\left(\frac{s}{2} + \frac{\gamma_2}{4\gamma_3}\right) - \frac{\gamma_1}{4\gamma_3}(\ln \rho - 1) - F, \quad T = \rho^{-\frac{1}{2}} F', \quad F = F\left(\rho^{\frac{1}{2}}\left(s + \frac{\gamma_2}{2\gamma_3}\right)\right),$$

where

$$F' > 0, \quad F'' > 0, \quad \left(s + \frac{\gamma_2}{2\gamma_3}\right)\rho^{\frac{1}{2}} F' F'' - F'^2 - \frac{\gamma_1}{\gamma_3} F'' > 0$$

for all $s \in (-\infty, s_0]$.

5. $h(a) = \ln a$

In this case the pure thermodynamic part of the symmetry algebra coincides with the thermodynamic part of the case when $\zeta(T)$ is an arbitrary function.

4 Differential invariants

In this section we recollect the notions of kinematic and Navier–Stokes differential invariants.

As in [3], we consider the prolonged group actions generated by the Lie algebras $\mathfrak{g}_{\mathfrak{m}}$ and $\mathfrak{g}_{\mathfrak{s}\mathfrak{y}\mathfrak{m}}$ on the Navier–Stokes system $\mathcal{E}$.

We call [3] a function J on the manifold $\mathcal{E}_k$ a *kinematic differential invariant of order* $\leq k$ if

1. J is a rational function along fibers of the projection $\pi_{k,0} : \mathcal{E}_k \to \mathcal{E}_0$,
2. J is invariant with respect to the prolonged action of the Lie algebra $\mathfrak{g}_{\mathfrak{m}}$, i.e., for all $X \in \mathfrak{g}_{\mathfrak{m}}$,

$$X^{(k)}(J) = 0, \tag{2}$$

where $\mathcal{E}_k$ is the prolongation of the system $\mathcal{E}$ to k-jets, and $X^{(k)}$ is the k-th prolongation of a vector field $X \in \mathfrak{g}_{\mathfrak{m}}$.

Note that fibers of the projection $\mathcal{E}_k \to \mathcal{E}_0$ are irreducible algebraic manifolds.

A kinematic invariant is *a Navier–Stokes invariant* if condition (2) holds for all $X \in \mathfrak{g}_{\mathfrak{s}\mathfrak{y}\mathfrak{m}}$.

We say that a point $x_k \in \mathcal{E}_k$ and its $\mathfrak{g}_{\mathfrak{m}}$-orbit $\mathcal{O}(x_k)$ (or $\mathfrak{g}_{\mathfrak{s}\mathfrak{y}\mathfrak{m}}$-orbit) are *regular*, if there are exactly $\operatorname{codim} \mathcal{O}(x_k)$ independent kinematic (or Navier–Stokes) invariants in a neighborhood of this orbit. Otherwise, the point and the corresponding orbit are *singular*.

Since the Navier–Stokes system and the symmetry algebras $\mathfrak{g}_{\mathfrak{m}}$ and $\mathfrak{g}_{\mathfrak{s}\mathfrak{y}\mathfrak{m}}$ satisfy the Lie–Tresse theorem (see [7]), and, therefore, the kinematic and Navier–Stokes differential invariants separate regular $\mathfrak{g}_{\mathfrak{m}}$- and $\mathfrak{g}_{\mathfrak{s}\mathfrak{y}\mathfrak{m}}$-orbits on the Navier–Stokes system $\mathcal{E}$ correspondingly.

We call a total derivative

$$A\frac{\mathrm{d}}{\mathrm{d}t} + B\frac{\mathrm{d}}{\mathrm{d}a}$$

$\mathfrak{g}_{\mathfrak{m}}$- or $\mathfrak{g}_{\mathfrak{s}\mathfrak{y}\mathfrak{m}}$-invariant, if it commutes with the prolonged action of algebra $\mathfrak{g}_{\mathfrak{m}}$ or $\mathfrak{g}_{\mathfrak{s}\mathfrak{y}\mathfrak{m}}$, and A, B are rational functions along fibers of the projection $\pi_{k,0} : \mathcal{E}_k \to \mathcal{E}_0$ for some $k \geq 0$.

4.1 Kinematic invariants

Theorem 2. *1. The kinematic invariants field is generated by first-order basis differential invariants and by basis invariant derivatives. This field separates regular orbits.*

2. *For the cases of arbitrary $h(a)$, as well as for $h(a) = \lambda_1 a^{\lambda_2}$, $h(a) = \lambda_1 e^{\lambda_2 a}$ and $h(a) = \ln a$, the basis differential invariants are*

$$a, \quad u, \quad \rho, \quad s, \quad u_t, \quad u_a, \quad \rho_a, \quad s_t, \quad s_a,$$

and the basis invariant derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}t}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

3. *For the cases $h(a) = const$, $h(a) = \lambda a$ the basis differential invariants are*

$$\rho, \quad s, \quad u_a, \quad u_t + uu_a, \quad \rho_a, \quad s_a, \quad s_t + us_a,$$

and basis invariant derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

4. *For the case $h(a) = \lambda a^2$ the basis differential invariants are*

$$\rho, \quad s, \quad u_a, \quad u_t + uu_a - 2\lambda g a, \quad \rho_a, \quad s_a, \quad s_t + us_a,$$

and basis invariant derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

5. *The number of independent invariants of pure order k is equal to 5 for $k \geq 1$.*

4.2 Navier–Stokes invariants

Let the thermodynamic state admit a one-dimensional symmetry algebra generated by the vector field A.

We get basis first-order Navier–Stokes differential invariants finding first integrals of an action of the vector A on the field of kinematic invariants.

Below we list basis invariants depending on the form of functions $\zeta(T)$ and $h(a)$.

4.2.1 $\zeta(T)$ is an arbitrary function

1. $h(a) = const$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_6 = \xi_1 \partial_p + \xi_2 \partial_s + \xi_3 (t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$s + \frac{\xi_2}{\xi_3}\ln\rho, \quad \frac{u_a}{\rho}, \quad \frac{u_t + uu_a}{\rho}, \quad \frac{\rho_a}{\rho^2}, \quad \frac{s_a}{\rho}, \quad \frac{s_t + us_a}{\rho}$$

and by the invariant derivatives

$$\rho^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}\right), \quad \rho^{-1}\frac{\mathrm{d}}{\mathrm{d}a}.$$

2. $h(a)=\lambda a,\ \lambda\neq 0$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_6=\xi_1\partial_p+\xi_2\partial_s+\xi_3\left(t\,\partial_t+\left(\frac{\lambda g t^2}{2}+a\right)\partial_a+\lambda g t\,\partial_u-p\,\partial_p-\rho\,\partial_\rho\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$s+\frac{\xi_2}{\xi_3}\ln\rho,\quad \frac{u_a}{\rho},\quad \frac{u_t+uu_a-\lambda g}{\rho},\quad \frac{\rho_a}{\rho^2},\quad \frac{s_a}{\rho},\quad \frac{s_t+us_a}{\rho}$$

and by the invariant derivatives

$$\rho^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}+u\frac{\mathrm{d}}{\mathrm{d}a}\right),\quad \rho^{-1}\frac{\mathrm{d}}{\mathrm{d}a}.$$

4. $h(a)=\ln a$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4=\xi_1\partial_p+\xi_2\partial_s+\xi_3\left(t\,\partial_t+a\,\partial_a-p\,\partial_p-\rho\,\partial_\rho\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$s-\frac{\xi_2}{\xi_3}\ln a,\quad u,\quad a\rho,\quad au_t,\quad au_a,\quad a^2\rho_a,\quad as_t,\quad as_a$$

and by the invariant derivatives

$$\rho^{-1}\frac{\mathrm{d}}{\mathrm{d}t},\quad \rho^{-1}\frac{\mathrm{d}}{\mathrm{d}a}.$$

4.2.2 $\zeta(T)=\alpha T$

First of all, if $h(a)$ is an arbitrary function and if the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4=\xi_1\partial_p+\xi_2\partial_s+\xi_3(p\,\partial_p+\rho\,\partial_\rho-s\,\partial_s+T\,\partial_T),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$a,\quad u,\quad \left(s-\frac{\xi_2}{\xi_3}\right)\rho,\quad u_t,\quad u_a,\quad \frac{\rho_a}{\rho},\quad \rho s_t,\quad \rho s_a$$

and by the invariant derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t},\quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

1. $h(a)=const$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4+\xi_4X_7=\xi_1\partial_p+\xi_2\partial_s+\xi_3(p\,\partial_p+\rho\,\partial_\rho-s\,\partial_s+T\,\partial_T)+\xi_4(t\,\partial_t+a\,\partial_a-p\,\partial_p-\rho\,\partial_\rho),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$\frac{\rho}{u_a}\left(s-\frac{\xi_2}{\xi_3}\right),\quad u_a\rho^{\frac{\xi_4}{\xi_3-\xi_4}},\quad \frac{u_t+uu_a}{u_a},\quad \frac{\rho_a}{\rho u_a},\quad \frac{\rho s_a}{u_a^2},\quad \frac{\rho(s_t+us_a)}{u_a^2}$$

and by the invariant derivatives

$$\rho^{\frac{\xi_4}{\xi_3-\xi_4}}\left(\frac{\mathrm{d}}{\mathrm{d}t}+u\frac{\mathrm{d}}{\mathrm{d}a}\right),\quad \rho^{\frac{\xi_4}{\xi_3-\xi_4}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

2. $h(a)=\lambda a$, $\lambda\neq 0$
If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4+\xi_4X_7=\xi_1\partial_p+\xi_2\partial_s+ \\ \xi_3(p\,\partial_p+\rho\,\partial_\rho-s\,\partial_s+T\,\partial_T)+\xi_4\left(t\,\partial_t+\left(\frac{\lambda g t^2}{2}+a\right)\partial_a+\lambda g t\,\partial_u-p\,\partial_p-\rho\,\partial_\rho\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$\frac{\rho}{u_a}\left(s-\frac{\xi_2}{\xi_3}\right),\quad u_a\rho^{\frac{\xi_4}{\xi_3-\xi_4}},\quad \frac{u_t+uu_a-\lambda g}{\rho},\quad \frac{\rho_a}{\rho u_a},\quad \frac{\rho s_a}{u_a^2},\quad \frac{\rho(s_t+us_a)}{u_a^2}$$

and by the invariant derivatives

$$\rho^{\frac{\xi_4}{\xi_3-\xi_4}}\left(\frac{\mathrm{d}}{\mathrm{d}t}+u\frac{\mathrm{d}}{\mathrm{d}a}\right),\quad \rho^{\frac{\xi_4}{\xi_3-\xi_4}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

3. $h(a)=\lambda a^2$, $\lambda\neq 0$
If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4=\xi_1\partial_p+\xi_2\partial_s+\xi_3(p\,\partial_p+\rho\,\partial_\rho-s\,\partial_s+T\,\partial_T),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$\left(s-\frac{\xi_2}{\xi_3}\right)\rho,\quad u_t+uu_a-2\lambda g a,\quad u_a,\quad \frac{\rho_a}{\rho},\quad \rho\left(s_t+us_a\right),\quad \rho s_a$$

and by the invariant derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}+u\frac{\mathrm{d}}{\mathrm{d}a},\quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

4. $h(a)=\ln a$
If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4+\xi_4X_5=\xi_1\partial_p+\xi_2\partial_s+\xi_3(p\,\partial_p+\rho\,\partial_\rho-s\,\partial_s+T\,\partial_T)+\xi_4\left(t\,\partial_t+a\,\partial_a-p\,\partial_p-\rho\,\partial_\rho\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$u,\quad \rho a^{1-\frac{\xi_3}{\xi_4}},\quad a\rho\left(s-\frac{\xi_2}{\xi_3}\right),\quad au_t,\quad au_a,\quad \frac{a\rho_a}{\rho},\quad a^2\rho s_t,\quad a^2\rho s_a$$

and by the invariant derivatives

$$\rho^{\frac{\xi_4}{\xi_3-\xi_4}}\frac{\mathrm{d}}{\mathrm{d}t},\quad \rho^{\frac{\xi_4}{\xi_3-\xi_4}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

4.2.3 $\zeta(T) = \alpha T^\beta$, $\beta \neq 1$

1. $h(a) = const$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_6 + \xi_4 X_7 = \xi_1 \partial_p + \xi_2 \partial_s + \xi_3 \left(t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho\right) + \xi_4 \left(a\,\partial_a + u\,\partial_u - \frac{2\beta}{\beta-1} p\,\partial_p - \frac{4\beta-2}{\beta-1}\rho\,\partial_\rho + \frac{2\beta}{\beta-1} s\,\partial_s - \frac{2}{\beta-1} T\partial_T\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$\frac{\rho^3 u_a}{\rho_a^2}\left(s + \frac{\xi_2(\beta-1)}{2\xi_4\beta}\right), \quad u_a \rho^{\frac{-\xi_3(\beta-1)}{\xi_3(\beta-1)+2\xi_4(2\beta-1)}}, \quad (u_t + u u_a)\rho^{\frac{-(\xi_3+\xi_4)(\beta-1)}{\xi_3(\beta-1)+2\xi_4(2\beta-1)}}, \quad \frac{\rho_a(u_t+uu_a)}{\rho u_a^2},$$

$$\frac{\rho^4 u_a s_a}{\rho_a^3}, \quad \frac{\rho^3(s_t + u s_a)}{\rho_a^2}$$

and by the invariant derivatives

$$\rho^{\frac{-\xi_3(\beta-1)}{\xi_3(\beta-1)+2\xi_4(2\beta-1)}}\left(\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}\right), \quad \rho^{\frac{-(\xi_3+\xi_4)(\beta-1)}{\xi_3(\beta-1)+2\xi_4(2\beta-1)}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

2. $h(a) = \lambda a$, $\lambda \neq 0$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_6 + \xi_4 X_7 = \xi_1 \partial_p + \xi_2 \partial_s + \xi_4 \left(t\,\partial_t + \left(\frac{\lambda g t^2}{2} + a\right)\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho\right) + \xi_3 \left(t\,\partial_t + 2a\,\partial_a + u\,\partial_u - \frac{3\beta-1}{\beta-1} p\,\partial_p - \frac{5\beta-3}{\beta-1}\rho\,\partial_\rho + \frac{2\beta}{\beta-1} s\,\partial_s - \frac{2}{\beta-1} T\partial_T\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$\frac{\rho^3 u_a}{\rho_a^2}\left(s + \frac{\xi_2(\beta-1)}{2\xi_3\beta}\right), \quad u_a \rho^{\frac{-(\xi_3+\xi_4)(\beta-1)}{\xi_3(5\beta-3)+\xi_4(\beta-1)}}, \quad (u_t + u u_a - \lambda g)\rho^{\frac{-\xi_4(\beta-1)}{\xi_3(5\beta-3)+\xi_4(\beta-1)}},$$

$$\frac{\rho_a(u_t+uu_a-\lambda g)}{\rho u_a^2}, \quad \frac{\rho^4 u_a s_a}{\rho_a^3}, \quad \frac{\rho^3(s_t + u s_a)}{\rho_a^2}$$

and by the invariant derivatives

$$\rho^{\frac{-(\xi_3+\xi_4)(\beta-1)}{\xi_3(5\beta-3)+\xi_4(\beta-1)}}\left(\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}\right), \quad \rho^{\frac{-(2\xi_3+\xi_4)(\beta-1)}{\xi_3(5\beta-3)+\xi_4(\beta-1)}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

3. $h(a) = \lambda a^2$, $\lambda \neq 0$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_6 = \xi_1 \partial_p + \xi_2 \partial_s + \xi_3 \left(a\,\partial_a + u\,\partial_u - \frac{2\beta}{\beta-1} p\,\partial_p - \frac{4\beta-2}{\beta-1}\rho\,\partial_\rho + \frac{2\beta}{\beta-1} s\,\partial_s - \frac{2}{\beta-1} T\partial_T\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$\frac{\rho^3 u_a}{\rho_a^2}\left(s+\frac{\xi_2(\beta-1)}{2\xi_3\beta}\right),\quad u_a,\quad (u_t+uu_a-2\lambda ga)\rho^{\frac{\beta-1}{2(2\beta-1)}},\quad \frac{\rho_a(u_t+uu_a-2\lambda ga)}{\rho u_a^2},$$

$$\frac{\rho^4 u_a s_a}{\rho_a^3},\quad \frac{\rho^3(s_t+us_a)}{\rho_a^2}$$

and by the invariant derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}+u\frac{\mathrm{d}}{\mathrm{d}a},\quad \rho^{-\frac{\beta-1}{2(2\beta-1)}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

If $\beta=1/2$ then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$\rho,\quad \frac{2\xi_3 s-\xi_2}{\rho_a^2},\quad u_a,\quad \rho_a(u_t+uu_a-2\lambda ga),\quad \frac{s_a}{\rho_a^3},\quad \frac{s_t+us_a}{\rho_a^2}$$

and by the invariant derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}+u\frac{\mathrm{d}}{\mathrm{d}a},\quad \frac{1}{\sqrt{\xi_2-2\xi_3 s}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

4. $h(a)=\lambda_1 a^{\lambda_2}$, $\lambda\neq 0,1,2$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4=\xi_1\partial_p+\xi_2\partial_s+\xi_3\left(t\,\partial_t-\frac{2a}{\lambda_2-2}\partial_a-\frac{\lambda_2 u}{\lambda_2-2}\partial_u-\frac{2\beta\lambda_2}{(\beta-1)(\lambda_2-2)}s\,\partial_s+\right.$$
$$\left.\frac{\lambda_2(\beta+1)+2(\beta-1)}{(\beta-1)(\lambda_2-2)}p\,\partial_p+\frac{\lambda_2(3\beta-1)+2(\beta-1)}{(\beta-1)(\lambda_2-2)}\rho\,\partial_\rho+\frac{2\lambda_2}{(\beta-1)(\lambda_2-2)}T\,\partial_T\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$u^2a^{-\lambda_2},\quad \rho u a^{1+\frac{\lambda_2\beta}{\beta-1}},\quad au\rho\left(s-\frac{\xi_2(\beta-1)(\lambda_2-2)}{2\lambda_2\beta\xi_3}\right),\quad \frac{au_t}{u^2},\quad \frac{au_a}{u},\quad \frac{a\rho_a}{\rho},\quad a^2\rho s_t,\quad a^2u\rho s_a$$

and by the invariant derivatives

$$\rho^{\frac{(\lambda_2-2)(\beta-1)}{\lambda_2(3\beta-1)+2(\beta-1)}}\frac{\mathrm{d}}{\mathrm{d}t},\quad \rho^{\frac{-2(\beta-1)}{\lambda_2(3\beta-1)+2(\beta-1)}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

5. $h(a)=\lambda_1 e^{\lambda_2 a}$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4=\xi_1\partial_p+\xi_2\partial_s+$$
$$\xi_3\left(t\,\partial_t-\frac{2}{\lambda_2}\,\partial_a-u\,\partial_u+\frac{\beta+1}{\beta-1}p\,\partial_p+\frac{3\beta-1}{\beta-1}\rho\,\partial_\rho-\frac{2\beta}{\beta-1}s\,\partial_s+\frac{2}{\beta-1}T\partial_T\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$u^2e^{-\lambda_2 a},\quad u\rho e^{\frac{\lambda_2\beta a}{\beta-1}},\quad au\rho\left(s-\frac{\xi_2(\beta-1)}{2\beta\xi_3}\right),\quad \frac{u_t}{u^2},\quad \frac{u_a}{u},\quad \frac{\rho_a}{\rho},\quad \rho s_t,\quad u\rho s_a$$

and by the invariant derivatives

$$\rho^{\frac{\beta-1}{3\beta-1}}\frac{\mathrm{d}}{\mathrm{d}t},\qquad \frac{\mathrm{d}}{\mathrm{d}a}.$$

If $\beta = 1/3$ then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$u^2e^{-\lambda_2 a},\quad \rho,\quad u\left(s+\frac{\xi_2}{\xi_3}\right),\quad \frac{u_t}{u^2},\quad \frac{u_a}{u},\quad \rho_a,\quad s_t,\quad us_a$$

and by the invariant derivatives

$$(\xi_3 s+\xi_2)\frac{\mathrm{d}}{\mathrm{d}t},\qquad \frac{\mathrm{d}}{\mathrm{d}a}.$$

6. $h(a) = \ln a$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by

$$\xi_1X_2+\xi_2X_3+\xi_3X_4 = \xi_1\partial_p+\xi_2\partial_s+\xi_3\left(t\,\partial_t+a\,\partial_a-p\,\partial_p-\rho\,\partial_\rho\right),$$

then the field of Navier–Stokes differential invariants is generated by the first-order differential invariants

$$s-\frac{\xi_2}{\xi_3}\ln a,\quad u,\quad a\rho,\quad au_t,\quad au_a,\quad a^2\rho_a,\quad as_t,\quad as_a$$

and by the invariant derivatives

$$\rho^{-1}\frac{\mathrm{d}}{\mathrm{d}t},\qquad \rho^{-1}\frac{\mathrm{d}}{\mathrm{d}a}.$$

Appendix

Let us define a space curve as a pair of plane curve $(x(\tau), y(\tau))$ and function $z(\tau)$ that serves as a way of lifting the plain curve. We denote length of the plane curve $\int_0^\tau \sqrt{x_\theta^2+y_\theta^2}\,d\theta$ by $l(\tau)$.

Depending on the particular form of the function $h(a)$, we get different forms of the 'lifting' function. Below, we enumerate all six particular cases of the function h arising in classification of Lie algebras (see Section 2). In each case, we find a relation between the 'lifting' function z and the length of the plane curve $l(\tau)$.

1. $h(a) = const$

The first way of lifting a plane curve is to translate the whole curve along z-axis, i.e., if $h(a) = const$ then $z(\tau) = const$.

2. $h(a) = \lambda a$, $\lambda \neq 0$

The second way to lift a plane curve is lifting proportional to its length, i.e., if $h(a) = \lambda a$ then the relation between the 'lifting' function $z(\tau)$ and the length $l(\tau)$ of plane projection of curve has the form

$$z(\tau) = \pm\frac{\lambda}{\sqrt{1-\lambda^2}}\,l(\tau)+C,$$

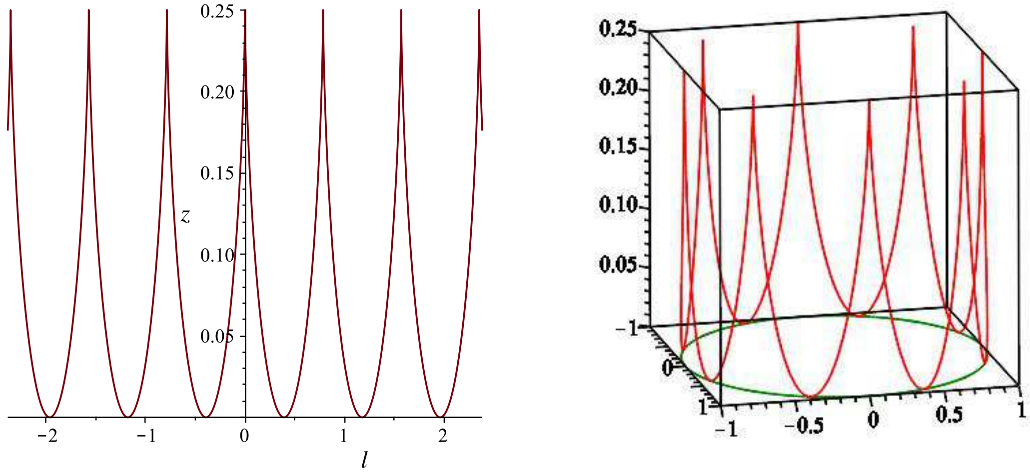
where $\lambda^2 < 1$ and C is a constant. Here, if $\lambda = \pm 1$ then $x(t) = y(t) = const$, and we have a vertical line.

3. $h(a) = \lambda a^2$, $\lambda \neq 0$

In this case the relation between the 'lifting' function $z(\tau)$ and the length of the plane curve is the following

$$\sqrt{4\lambda z(1-4\lambda z)} - \arccos(\sqrt{4\lambda z}) = \pm 4\lambda l(\tau).$$

This relation is presented in Figure 1a. An example of lifting of a unit circle with the function is given in Figure 1b.



(a) (b)

Figure 1

4. $h(a) = \lambda_1 a^{\lambda_2}$, $\lambda_2 \neq 0, 1, 2$

In this case, relation between the 'lifting' function $z(\tau)$ and the length of the plane curve is rather complex and involves hypergeometric functions. For example, for the case $\lambda_1 = 1$, $\lambda_2 = \frac{11}{3}$ we get

$$z^{\frac{3}{11}}\, {}_2F_1\left(-\frac{1}{2}, \frac{3}{16}; \frac{19}{16}; \frac{121}{9} z^{\frac{16}{11}}\right) = \pm l(\tau).$$

5. $h(a) = \lambda_1 e^{\lambda_2 a}$

The relation between the 'lifting' function $z(\tau)$ and the length of the plane curve is

$$\sqrt{1-\lambda_2^2 z^2} - \frac{1}{2}\ln\frac{1+\sqrt{1-\lambda_2^2 z^2}}{1-\sqrt{1-\lambda_2^2 z^2}} = \pm \lambda_2 l(\tau),$$

where $\lambda_2^2 z^2 < 1$.

This relation is demonstrated in Figure 2a. In Figure 2b, we show an example of lifting of a unit circle, given positive z and l. In fact, the space curve starts from the height of one unit above the circle and never intersects with it.

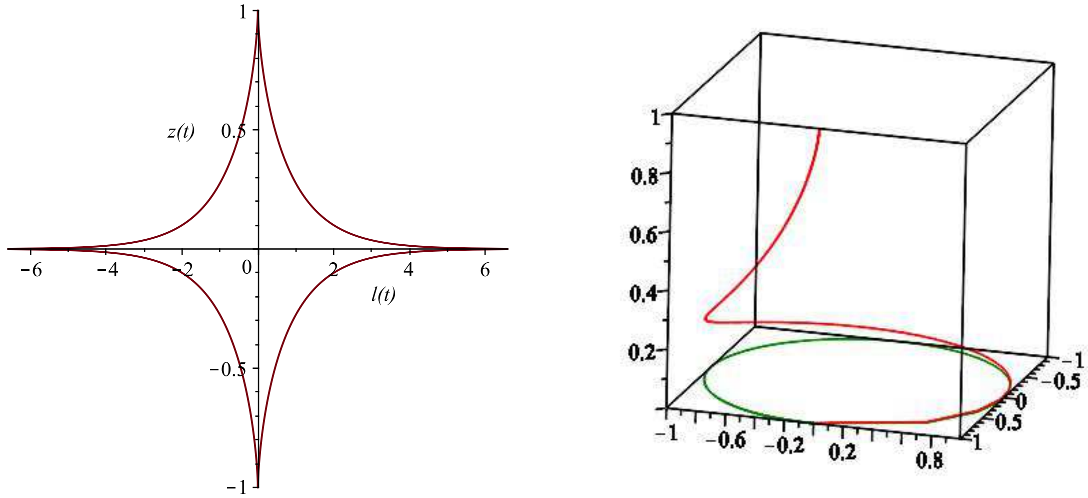
6. $h(a) = \ln a$

The relation between functions $l(\tau)$ and $z(\tau)$ is

$$\sqrt{e^{2z}-1} - \arctan\sqrt{e^{2z}-1} = \pm l(\tau).$$

In Figure 3a, we show this relation when l is positive. In Figure 3b the corresponding lifting of a unit circle is demonstrated.

Acknowledgments. The research was partially supported by RFBR Grant No 18-29-10013.



(a) (b)

Figure 2

References

[1] Anderson, Ian M. and Torre, Charles G., *The Differential Geometry Package* (2016). Downloads. Paper 4. http://digitalcommons.usu.edu/dg_downloads/4

[2] Batchelor G. K. *An introduction to fluid dynamics.* Cambridge university press, 2000.

[3] Duyunova A., Lychagin V., Tychkov S. *Differential invariants for flows of fluids and gases.* arXiv:2004.01567 [math-ph].

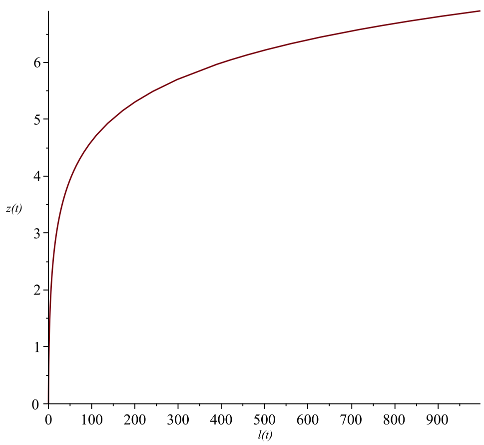
[4] Duyunova A., Lychagin V., Tychkov S. *Symmetries and differential invariants for inviscid flows on a curve.* Lobachevskii Journal of Mathematics, 2020, vol. 41, iss. 12.

[5] Duyunova A., Lychagin V., Tychkov S. *Symmetries and differential invariants for viscid flows on a curve*. Global and Stochastic Analysis, 2020, [in press].

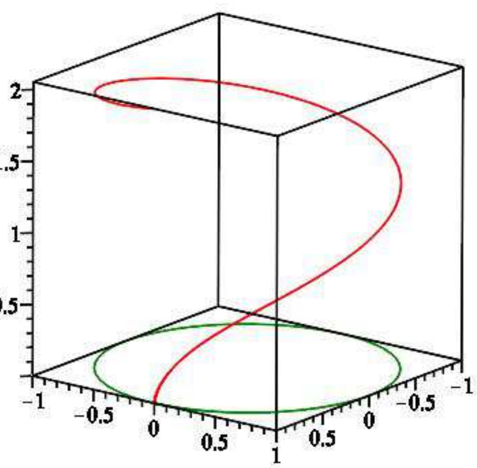
[6] Duyunova A., Lychagin V., Tychkov S. *Continuum mechanics of media with inner structures.* arXiv:2005.05840 [math-ph].

[7] Kruglikov B., Lychagin V., *Global Lie-Tresse theorem*. Selecta Math. 2016, 22, 1357-1411.

[8] V. Lychagin, *Contact Geometry, Measurement and Thermodynamics,* in: Nonlinear PDEs, Their Geometry and Applications. Proceedings of the Wisla 18 Summer School, Springer Nature, Switzerland, 3-54 (2019).


z(t)
l(t)
0
1
2
3
4
5
6
100
200
300
400
500
600
700
800
900

(a)


2
1.5
1
0.5
−1
−0.5
0
0.5
1

(b)

Figure 3