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Wachpress Conjecture Restricted To Arrangements Of Three Conics

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1 Introduction

In this thesis we are going to focus on Wachpress conjecture for arrangements of three conics. We are going to introduce what Wachpress coordinates are, how they are useful and present the current state of our knowledge on these coordinates in polycons that arise from arrangements of three conics.

Wachpress coordinates are a generalization of Barycentric coordinates from simplexes to planar polytopes. Wachpress conjectured the existence of this set of rational barycentric coordinates, named Wachpress coordinates, on a generalization of polytopes called polycons. For polycons we allow quadratic boundaries where only linear ones were allowed for polytopes.

Barycentric coordinates are extremely important because one can define a finite element method of approximation [Flo15], which has linear precision. It is a way to approximate a function on a domain given only its evaluation in a finite amount of points. Moreover, if the function is linear, the approximation is identical to the function itself. This approximation sees many uses in numerous different fields, from modeling to statistics and far beyond.

Wachpress coordinates are not proven yet to be well defined for generic polycons and here lies the conjecture. It states that the coordinates are well defined for all polycons. Wachpress coordinates are rational and the polynomial at the denominator takes the name of adjoint polynomial. Its algebraic set is called adjoint curve, sometimes we refer to it by adjoint. If the adjoint has no intersection with the sides and interior of a polycon, Wachpress coordinates are well defined for this specific polycon. Proving that the adjoint does not intersect the sides and interior of a polycon, is proving the conjecture for the polycon. It may seem an easy problem to solve, since we know the adjoint will not intersect the sides of the polycon. But it is not. When a real algebraic curve has degree greater or equal three, it may have more than one connected component. For example, a real curve of degree three has one or two connected components. If an adjoint curve is of the latter case, with a oval and a pseudoline, one must prove that the oval does not lie completely in the interior of the polycon. The approach taken on this problem is partially a personalized one for different topological cases.

The history of Wachpress coordinates is a conjoint work of many mathematicians, first and foremost Wachpress himself [Fix78], who invented the coordinates as a generalization of barycentric coordinates to embrace all generic planar polytopes. Warren [War96] generalized the conjecture to higher dimensional polytopes. Many more mathematicians wrote about these coordinates, some of their names are Meyer, Barr, Lee and Desbrun, Schaefer and Hirani. Now the conjecture is stated for all polycons and there are means to compute the adjoint for every specific case. Wachpress grandson recently wrote his bachelor thesis on the subject [Wac20], in which he spent time on the arrangement of three conics and under some assumptions he proved some results. This is the same type of arrangement on which this thesis focuses. Recently, more mathematicians worked on the same topic, writing the article [Koh+21], which is the foundation of this thesis. In the aforementioned article, they viewed polycons as special cases of positive geometries and proved the conjecture for many topological cases of arrangements of three conics. The same authors proved the conjecture also for special three dimensional polytopes.

In this thesis we report the current state of our knowledge of Wachpress conjecture on arrangements of three conics. We explain the efforts on a catalog of the topological cases of the arrangements of three ellipses given a combinatorial description.

The true main objective of this thesis is to educate myself on real algebraic geometry, to spend some time on a practical problem and to learn how to use different tools.

2 Positive Geometries

Wachpress conjecture refers to polycons, which are geometric and algebraic objects. A good environment for their study does not forget about their topological or algebraic properties. There are many different ways to approach the conjecture on polycons. One that has shown good returns is the working environment of positive geometries. Positive geometries and their canonical rational forms, capture really well both the algebraic and the manifold nature of polycons. In this section we will introduce positive geometries. The content we will present is based on the article [ABL17].

2.1 Introduction To Positive Geometries

Any intuitive approach to the concept of positive geometries should start by pointing out that we are dealing with geometric objects, namely compact manifolds which are similar to CW-complexes. For more information on CW-complexes refer to [Hat05]. The algebraic nature of these objects is encoded by a canonical form. The canonical form of a positive geometry must be *unique, non-zero and rational*. The boundaries of the positive geometry are the poles of the canonical form and the residue on these poles will be the canonical form of the boundaries seen as positive geometries. The key observation is that the boundaries of a positive geometry is again a positive geometry. A requirement on the boundary will imply that, by passing recursively to the residue we will eventually reach a 0-dimensional positive geometry. Here the canonical form is particularly simple: either $+1$ or -1 , given by the chosen orientation. In regard to orientation refer to [Lee13].

Preliminary Definitions Positive geometries find themselves in the middle ground between differential geometry and algebraic geometry. Thus, there are many different notions which are needed to comprehend the definition of positive geometries. In regard to the algebraic definitions refer to [Ful08].

Definition 2.1: Standard Complex-Projective Space.

Consider the vector space \mathbb{C}^{N+1} . Let $x, y \in \mathbb{C}^{N+1}$. Set \sim as the equivalence relation

$$x \sim y \iff y = \alpha x$$

for some $\alpha \in \mathbb{C}^*$. Call the quotient

$$\pi : \mathbb{C}^{N+1} \setminus \{0\} \rightarrow \frac{\mathbb{C}^{N+1} \setminus \{0\}}{\sim}$$

the *standard projection*. Call the target space of π , namely $\frac{\mathbb{C}^{N+1}}{\sim}$, the *complex-projective-space* $\mathbb{C}\mathbb{P}^N$.

Remark 2.2. We view \mathbb{R}^N in \mathbb{C}^N by the embedding

$$i : \mathbb{R}^N \rightarrow \mathbb{C}^N, x_1, \dots, x_N \mapsto x_1 + 0 \cdot i, \dots, x_N + 0 \cdot i.$$

We can now view the real-projective space $\mathbb{R}\mathbb{P}^{N-1}$ as a subset of $\mathbb{C}\mathbb{P}^{N-1}$ via

$$\mathbb{R}\mathbb{P}^{N-1} := \pi (i (\mathbb{R}^N \setminus \{0\})) \subseteq \mathbb{C}\mathbb{P}^{N-1}.$$

Definition 2.3: Complex Algebraic Variety.

Let p_1, \dots, p_s be polynomials in $\mathbb{C}[X_1, \dots, X_{N+1}]$. Set $I = \langle p_1, \dots, p_s \rangle$ the ideal generated by p_1, \dots, p_s . We call the set

$$X = \{t \in \mathbb{C}^{N+1} \mid p_1(t) = 0, p_2(t) = 0, \dots, p_n(t) = 0\}$$

a *complex algebraic variety*. Its *real part*, denoted as $X(\mathbb{R})$ is by definition the following:

$$X(\mathbb{R}) := \{x \in \mathbb{R}^{N+1} \mid i(x) \in X\},$$

here i is the embedding defined in the remark 2.2. Its *real projective part*, denoted as $X(\mathbb{P}\mathbb{R})$ is $\pi(X(\mathbb{R}))$, where π is the projection defined in 2.1.

From now on we consider the dimension of X as a manifold to be D

Notation 2.4.

A *rational form* of degree D on X is a differential form which can be written as

$$\omega = f dx_1 \wedge \dots \wedge dx_D$$

where f is a *rational function*

$$f : X \rightarrow \mathbb{C}.$$

We call a D -rational smooth form on X a rational D -form on it's smooth interior. If f is *holomorphic* instead of being rational, the form ω is an *holomorphic form*.

As the last preliminary definition, we introduce the semi-algebraic sets in a real affine environment and then we extend it to a projective one.

Definition 2.5: Semi-Algebraic Sets.

Let $(p_i)_{i \in \{1, \dots, t\}}$ be a finite family of polynomials in $\mathbb{R}[X_1, \dots, X_N]$. Set

$$S_i := \{x \in \mathbb{R}^N \mid p_i(x) = 0 \text{ or } p_i(x) < 0\}.$$

Any

$$S = \bigcup_{i=r}^t \bigcap_{i=1}^{r-1} S_i$$

for $1 \leq r \leq t$ is a *semi-algebraic-set*.

Now follow two examples, respectively of semi-algebraic sets on \mathbb{R} and a non semi-algebraic set on \mathbb{R} .

Example 2.6. Consider \mathbb{R} . Every open interval $(-\infty, b) \subset \mathbb{R}$ is a semi-algebraic set. Every closed interval $(-\infty, b] \subset \mathbb{R}$ is a semi-algebraic set. Every complement, finite union or intersection of the former is a semi-algebraic set.

Example 2.7 (Non Semi-Algebraic Set). Consider \mathbb{R} . Let \mathbb{Z} be the set of integers. Let $i : \mathbb{Z} \rightarrow \mathbb{R}, x \mapsto x$ be the natural embedding of \mathbb{Z} in \mathbb{R} . The set $i(\mathbb{Z}) \subset \mathbb{R}$ is *not a semi-algebraic set*.

Definition 2.8: Projective Semi-Algebraic Sets.

Let S be a semi-algebraic set in \mathbb{R}^{n+1} . Let π and i be defined respectively as in 2.1 and 2.2. We view $\mathbb{R}\mathbb{P}^n \subseteq \mathbb{C}\mathbb{P}^n$ as in 2.2.

Then we define the *projective semi-algebraic set* as

$$\pi(i(S)) \subseteq \mathbb{R}\mathbb{P}^n \subseteq \mathbb{C}\mathbb{P}^n.$$

Given these preliminary definitions, we can define positive geometries.

2.1.1 Definition Of Positive Geometries

Positive geometries are defined recursively on the boundaries. We need to start defining the boundary components.

Definition 2.9: Boundary Components.

Let X be a complex variety and $X_{\geq 0}$ be a real semi-algebraic set such that $i(X_{\geq 0}) \subset X$ for i as in 2.2. Call $X_{>0}$ the euclidean interior of the semi-algebraic set $X_{\geq 0}$.

Let

$$\begin{aligned} \partial X_{\geq 0} &:= X_{\geq 0} \setminus X_{>0} \\ \partial X &:= \overline{\partial X_{\geq 0}} \end{aligned}$$

here the bar of the second line refers to the Zariski closure in the complex projective space.

Thus, we can write

$$\partial X = \{x \in X \mid p(x) = 0 \ \forall p : p(y) = 0 \ \forall y \in \partial X_{\geq 0}\}$$

where p is a homogeneous polynomial.

We see that ∂X is a closed algebraic subset of X . Call its irreducible components C^1, \dots, C^r .

Set $C_{\geq 0}^i$ to be the euclidean closure of the interior of $C^i \cap \partial X_{\geq 0}$.

The *Boundary Components* of $(X, X_{\geq 0})$ are the pairs

$$(C^i, C_{\geq 0}^i).$$

We see that ∂X is the largest set such that if an homogeneous polynomial vanishes on $\partial X_{\geq 0}$ then it vanishes on ∂X .

We also see that all the C_i are a subset of the real algebraic variety $C_i(\mathbb{R})$ and that the $C_{i,\geq 0}$ are a semi-algebraic set.

In our case we will see that all $C_{i,\geq 0}$ are co-dimension one with respect to $X_{\geq 0}$.

Since we will act recursively on the boundaries and we are working in a orientable setting, it follows the definition of a natural induced orientation on the boundary components.

Definition 2.10: Natural Orientation Of The Boundary Components.

Let $X_{\geq 0}$ be a real closed orientable set of dimension D with non empty interior. Let $U \subset \mathbb{R}^{D-1} \times \mathbb{R}$ be open. We denote by (x, y) the coordinates on \mathbb{R}^{D-1} and \mathbb{R} respectively. We find a local chart ϕ into $(x, y) \in U \cap (\mathbb{R}^{D-1} \times \mathbb{R}_{\geq 0})$ such that the boundary is mapped to $y = 0$. We can restrict ϕ to produce charts of the boundary preserving orientation from $X_{\geq 0}$ to its boundaries components.

We assume $\mathbb{R}^{D-1} \times \mathbb{R}$ to be positively oriented with respect to the standard Euclidean orientation. Let us approach the situation recursively as follows.

- For $D = 1$: If ϕ preserves orientation, then the orientation on $C_{\geq 0}^i$ becomes $+1$, otherwise it becomes -1 .
- For $D > 1$: We induce a orientation on $C_{\geq 0}^i$ which will have dimension $D - 1$ via the restriction of ϕ . We proceed to its boundaries until we reach the former case.

Now we are ready to produce the definition of a positive geometry.

Definition 2.11.

A D -dimensional *positive geometry* is a pair $(X, X_{\geq 0})$ of a *irreducible complex projective variety* of dimension D and a *semi-algebraic set of real dimension D* .

For X and $X_{\geq 0}$ we respectively ask for:

1. The complex variety X must have a singular locus of co-dimension two or more. If $C \subset X$ is a co-dimension one sub-variety, then $\overset{\circ}{X} \cap C$ is open and dense in C . Given a top rational form ω on X , then $Res_C \omega$ makes sense as a rational top form on C .
2. The semi-algebraic set $X_{\geq 0}$, which is a subset of $X(\mathbb{R})$, must *non-empty, oriented and closed*.

Let $X_{> 0}$ be the interior of $X_{\geq 0}$. Then $X_{> 0}$ is also a semi-algebraic set.

We assume $X_{> 0}$ as a *real, open, oriented sub-manifold of dimension D of $X(\mathbb{R})$* such that *its closure is $X_{\geq 0}$* .

In particular $X(\mathbb{R})$ must be a dimension D real algebraic variety.

Note that if $X_{> 0}$ has multiple connected components, one may have different choices of orientations.

The boundary components of $X_{\geq 0}$ are the $C_{\geq 0}^i$. The positive geometry must satisfy the recursive axioms:

- For $D = 0$,

$$X_{\geq 0} = X = \{\text{single point}\}.$$

The canonical 0-form associated to the positive geometry is

$$\Omega(X, X_{\geq 0}) = \pm 1$$

Where the former value is either $+1$ or -1 depending on the orientation of $X_{\geq 0}$.

- For $D > 0$ we must have:
 - Every boundary component $(C^i, C_{\geq 0}^i)$ is a positive geometry of dimension $D - 1$.
 - There exists a *unique non-zero rational D -form $\Omega(X, X_{\geq 0})$* on X such that:

$$Res_{C^i} \Omega(X, X_{\geq 0}) = \Omega(C^i, C_{\geq 0}^i)$$

along every boundary component, with no singularities elsewhere.

Remark 2.12 (Residue Operator). In the environment we are considering, the singularities of the canonical form on the boundaries are logarithmic. Let $\nu = \omega \wedge d(\log C)$ be a form with logarithmic singularities on C . The residue of ν on C is meant as follows:

$$Res_C(\nu) = \omega|_C$$

From the definition of positive geometries 2.11 follows this lemma.

Lemma 2.13. *Given the Axiom 2.11, then, the D -form is unique if and only if there are no non-zero holomorphic D -forms on X .*

Proof. The statement must be proven in both directions:

1. Let Ω_0 , a non-zero holomorphic D -form on X , Let Ω_1 be a D -form that satisfies the Axiom 2.11.
Then the form $\Omega_1 + \Omega_0$ which is different from Ω_1 , satisfies 2.11 too.
2. If there are no non-zero holomorphic D -forms on X and Ω_1, Ω_2 satisfy 2.11, then $\Omega_1 - \Omega_2$ has a zero-residue on the boundary, thus $\Omega_1 - \Omega_2$ is an holomorphic D -form, vanishing thus everywhere.

□

Remark 2.14. Let S be a semi-algebraic set. It is not trivial to say if there are any non-zero holomorphic top forms on S .

As an example we write the case of one-dimensional positive geometries.

Example 2.15 (0-Dimensional Positive Geometries). Let $(X, X_{\geq 0})$ be a 0-dimensional positive geometry. Thus $X = \text{point} = X_{\geq 0}$. Its canonical form $\Omega(X, X_{\geq 0})$ will be either 1 or -1 depending on the orientation given to the point.

One-Dimensional Positive Geometries

Proposition 2.16 (One-Dimensional Positive Geometries). *Let $(X, X_{\geq 0})$ be a 1-dimensional positive geometry. Then X is isomorphic to \mathbb{CP}^1 . Set $X = \mathbb{CP}^1$, then $(X, X_{\geq 0})$ becomes*

$$(\mathbb{CP}^1, S), S = \bigcup_{i \in \{1, \dots, t\}} [a_i, b_i].$$

Moreover, the canonical form of $(X, X_{\geq 0})$ is

$$\Omega(\mathbb{CP}^1, S) = \sum_{i \in \{1, \dots, t\}} \Omega([a_i, b_i])$$

Given

$$\Omega([a_i, b_i]) = \frac{dx}{x - a_i} - \frac{dx}{x - b_i} = \frac{(b_i - a_i)dx}{(b_i - x)(x - a_i)}.$$

Proof. In 2.11 we assume that X is projective and normal with no non-zero holomorphic forms. Remember that a projective smooth curve of genus g has g independent holomorphic differentials. it follows that X must be isomorphic to the projective line \mathbb{CP}^1 .

From the requirements on $X_{\geq 0}$ in 2.11, $X_{\geq 0} \subset \mathbb{RP}^1 \equiv S^1$ must be:

- Non-empty
- Closed

- Of dimension $\dim_{\mathbb{R}}(X_{\geq 0}) = 1$
- The topological closure of its interior
- With finitely many connected components
- A proper subset for the condition on the boundaries

Thus it follows $X_{\geq 0}$ is a finite union of closed bounded intervals. Write

$$X_{\geq 0} = \bigcup_{i \in \{1, \dots, t\}} [a_i, b_i]$$

We anticipate the result of 2.2.1. It states that the canonical form of a positive geometry is the sum of the canonical forms of its triangulations. Thus

$$\Omega(X_{\geq 0}) = \sum_{i \in \{1, \dots, t\}} \Omega([a_i, b_i]).$$

Now fix a $[a_i, b_i]$, call $a := a_i$ and $b := b_i$.

Compute $\Omega([a, b])$ using the technique described in 2.4.1, which constructs the positive geometry recursively from the boundaries. Choose the orientation on the segment $[a, b] := \{t \in \mathbb{R}P^1 \mid t = (1-r)a + rb \text{ for } r \in [0, 1]\}$ in accordance with the growth of $r \in [0, 1]$.

Then

$$\Omega([a, b]) = \frac{dt}{t-a} - \frac{dt}{t-b} = \frac{(b-a)dt}{(b-t)(t-a)}.$$

□

Remark 2.17. In 2.16, if we were to allow $X_{\geq 0} = \mathbb{R}P^1$ or $X_{\geq 0} = \emptyset$ then $\Omega(X, X_{\geq 0}) = 0$. In this case, $(X, X_{\geq 0})$ is a pseudo-positive geometry, as defined in 2.20.

Example 2.18 (Simplexes). Consider the real projective space $\mathbb{R}P^m$ with the set of coordinates X_0, X_1, \dots, X_m . Consider the standard simplex $\Delta^m := \mathbb{R}P_{\geq 0}^m$ to be the convex hull of positive coordinate vectors, $\text{conv}_{\mathbb{R}}([1 : 0 : \dots : 0], \dots, [0 : \dots : 0 : 1])$. We claim that the pair $(\mathbb{R}P^m, \Delta^m)$ is a positive geometry whose canonical form is

$$\Omega(\Delta^m) = \prod_{i=1}^m \frac{dx_i}{x_i} = \prod_{i=1}^m d\log_i$$

where we are on a chart where $X_0 = 1$ and $x_i = \frac{X_i}{X_0}$. On this claim we work inductively on the dimension.

- Let $m = 1$. Then the simplex Δ^m is a segment from $e_0 = [1 : 0]$ to $e_1 = [0 : 1]$. The claim is $\Omega(\Delta^m) = \frac{dx_1}{x_1}$. The residues are

1. $\text{Res}_{X_1=0} \frac{dx_1}{x_1} = 1$
2. For the residue on $X_0 = 0$ we have to change charts, consider $X_1 = 1$. Let $y_0 = \frac{X_0}{X_1}$, in the current chart the claimed canonical form becomes

$$\Omega(\Delta^m) = \frac{d(1/y_0)}{1/y_0} = -\frac{dy_0}{y_0}$$

$$\text{and } \text{Res}_{X_0=0} -\frac{dy_0}{y_0} = -1.$$

- Let m be generic.

1. For the facets $X_i = 0$ for $i = 1, \dots, m$ note the sign alternates.

$$\text{Res}_{X_i=0} \prod_{j=1}^m \frac{dx_j}{x_j} = (-1)^{i+1} \prod_{j=1, j \neq i}^m \frac{dx_j}{x_j}$$

2. For the facet $X_0 = 0$, work in the chart $X_1 = 1$. Let $y_i = X_i/X_1$ for $i \neq 1$. Then $x_1 = 1/y_0$ and $x_i = y_i/y_0$ for $i \in \{2, \dots, m\}$.

$$\begin{aligned} \Omega(\Delta^m) &= \frac{d(1/y_0)}{1/y_0} \wedge \frac{d(y_2/y_0)}{y_2/y_0} \wedge \dots \wedge \frac{d(y_m/y_0)}{y_m/y_0} \\ &= -\frac{dy_0}{y_0} \wedge \frac{d(y_2/y_0)}{y_2/y_0} \wedge \dots \wedge \frac{d(y_m/y_0)}{y_m/y_0} \\ &= -\frac{dy_0}{y_0} \wedge \frac{dy_2}{y_2} \wedge \dots \wedge \frac{dy_m}{y_m} \end{aligned}$$

We can view any simplex as diffeomorphic to a standard simplex. Through this diffeomorphism, any simplex can be proven to be a positive geometry.

Example 2.19 (Polytopes). Polytopes are examples of positive geometry. Simplexes are positive geometries from 2.18 and for a polytope P we can give a triangulation in simplexes $\tau(P)$. We will show in 2.2.1 that we can write

$$\Omega(P) = \sum_{\mathcal{V} \in \tau(P)} \Omega(\tau(P)).$$

Which for a polytope will not be a null-geometry.

2.1.2 Pseudo-Positive Geometries

A half-circle is a positive geometry, but a circle is not. We want to introduce a generalization of positive geometries to *pseudo-positive geometries* which will be of aid in regard to triangulation. Since the positive geometries and the pseudo-positive geometries are very similar, we define the latter from their differences of the former.

Definition 2.20: Pseudo-Positive Geometries.

A D -dimensional *pseudo-positive geometry* is a pair $(X, X_{\geq 0})$ which differs from a positive geometry in the following way.

- The set $X_{\geq 0}$ is allowed to be empty.
- If $X_{\geq 0} = \emptyset$ then we set $\Omega(X, X_{\geq 0}) = 0$
- The recursive Axioms ask for a pseudo-positive geometry instead of a positive one.

Remark 2.21. Note that there are pseudo-positive geometries with $\Omega(X, X_{\geq 0}) \neq 0$. The geometries such that $\Omega(X, X_{\geq 0}) = 0$ are called *null-geometries*.

Example 2.22 (0-Dimensional Pseudo-Positive Geometries). The zero dimensional pseudo-positive geometries are as follows:

- Let $(X, X_{\geq 0})$ be a 0-dimensional positive geometry. Then $(X, X_{\geq 0})$ is also a 0-dimensional pseudo-positive geometry.
- Let $X = \text{point}$ and $X_{\geq 0} = \emptyset$, then $(X, X_{\geq 0})$ is a null-geometry, a pseudo-positive geometry with the canonical form $\Omega(X, X_{\geq 0}) = 0$.

2.2 Reduction Of Positive Geometries

There are methods to reduce complicated positive geometries to simpler ones. In this subsection, we are going to present in more detail the two methods quickly introduced in the following bullet point list.

- **Triangulation:** If a positive geometry $(X, X_{\geq 0})$ can be tiled by a collection of positive geometries $(X^i, X_{\geq 0}^i)_i$ with mutually non-overlapping interiors, then the canonical form for $(X, X_{\geq 0})$, namely $\Omega_{X_{\geq 0}}$ is given by the sum of the canonical forms on the tiles.

$$\Omega(X_{\geq 0}) = \sum_i \Omega(X_{\geq 0}^i).$$

Thus the canonical form is said to be *triangulation independent*.

- **Push-forward:** Given $(X, X_{\geq 0})$ positive geometry, and a *morphism of positive geometries* $\phi : X_{\geq 0} \rightarrow Y_{\geq 0}$, then $\Omega(Y_{\geq 0}) = \phi^* \Omega(X_{\geq 0})$

2.2.1 Triangulation Of Pseudo-Positive Geometries

Let X an irreducible algebraic variety and $X_{\geq 0} \subset X$ a closed semi-algebraic set as in 2.11. Let $(X, X_{\geq 0}^i)_i$ be a finite collection of pseudo-geometries that live on X .

We say that $\{(X, X_{\geq 0}^i)\}$ triangulates $(X, X_{\geq 0})$ if the following are satisfied.

- Each $X_{\geq 0}^i$ is contained in $X_{> 0}$ and the orientations agree.
- The interiors of $X_{\geq 0}^i, X_{\geq 0}^j$ are mutually disjoint.
- The union $\bigcup_i X_{\geq 0}^i = X_{\geq 0}$.

Naively, a triangulation of $X_{\geq 0}$ as a positive geometry is a tiling of positive geometries of $X_{\geq 0}$.

Remark 2.23. The aim of this section is to prove the following properties:

If $\{X_{\geq 0}^i\}_i$ triangulates $X_{\geq 0}$ then $X_{\geq 0}$ is a pseudo-positive geometry such that

$$\Omega(X, X_{\geq 0}) = \sum_i \Omega(X^i, X_{\geq 0}^i) \tag{2.1}$$

Notice that even if the family $\{X_{\geq 0}^i\}$ is made by only positive geometries, $X_{\geq 0}$ may not be a positive geometry. The half circle is a clear example of this.

The next paragraph will offer a generalization of the concept of triangulation for positive geometries. It will turn out to guide us into the use of triangulations to reduce the canonical form of positive geometries.

Signed Triangulations In a pseudo positive geometries environment there can be different types of triangulations. We use the term *signed triangulations* to embrace any between triangulation of interior, boundary and canonical form which will be defined in this paragraph.

Definition 2.24: Interior Triangulation.

Let $(X, X_{\geq 0}^i)_{i=1, \dots, t}$ be a collection of pseudo positive geometries. Let C be the union of their boundary components. Set $x \in \bigcup_i X_{\geq 0}^i \setminus C$.

Given that

$$\begin{aligned} & |\{i|x \in X_{>0}^i \text{ and } X_{>0}^i \text{ is positively oriented in } x\}| \\ & = \\ & |\{i|x \in X_{>0}^i \text{ and } X_{>0}^i \text{ is negatively oriented in } x\}| \end{aligned}$$

holds for any possible choice of x , We say that $\{X_{\geq 0}^i\}$ *triangulates the empty set*.

Remark 2.25 (Orientation). In 2.24 we arbitrary choose an orientation of $X(\mathbb{R})$ near x . Since all the $X_{>0}^i$ are open subsets, it suffices to check the property 2.24 on a dense subset of $\bigcup_i X_{\geq 0}^i \setminus C$.

Notation 2.26.

If $\{X_{\geq 0}^1, \dots, X_{\geq 0}^t\}$ interior triangulates the empty set, we may say that the interior of $\{X_{\geq 0}^2, \dots, X_{\geq 0}^t\}$ triangulates $X_{\geq 0}^{1-}$ (that is $X_{\geq 0}^1$ with inverted orientation).

We can see that an interior triangulation is a (genuine) triangulation of $X_{\geq 0} = \bigcup_i X_{\geq 0}^i$ exactly when each point $x \in X_{\geq 0}$ is contained in exactly one of the $X_{\geq 0}^i$.

Definition 2.27: Boundary triangulation.

Let $(X, X_{\geq 0}^i)_{i=1, \dots, t}$ be a collection of pseudo positive geometries. Suppose X has dimension D . The following definition will be recursive with respect to the passage to the boundary. We say that $\{X_{\geq 0}^i\}$ is a *boundary triangulation of the empty set* if:

- If $D = 0$:
we have $\sum_i \Omega(X, X_{\geq 0}^i) = 0$.
- If $D > 0$:
Let C be an irreducible sub variety of X of dimension $D - 1$. Let $(C, C_{\geq 0}^i)$ be the boundary component of $(X, X_{\geq 0}^i)$ along C . If such a boundary component does not exist set $C_{\geq 0}^i = \emptyset$. We require that for every C , the collection of $\{C_{\geq 0}^i\}$ forms a boundary triangulation of the empty set.

Notation 2.28.

If $\{X_{\geq 0}^1, \dots, X_{\geq 0}^t\}$ boundary triangulates the empty set, we may say that the collection $\{X_{\geq 0}^2, \dots, X_{\geq 0}^t\}$ boundary triangulates $X_{\geq 0}^{1-}$ (that is $X_{\geq 0}^1$ with inverted orientation).

Definition 2.29: Canonical Form Triangulation.

Let $(X, X_{\geq 0}^i)_{i=1, \dots, t}$ be a collection of pseudo positive geometries.

We say that $(X, X_{\geq 0}^i)_i$ is a *canonical form triangulation of the empty set*, if

$$\sum_{i=1}^t \Omega(X, X_{\geq 0}^i) = 0.$$

Notation 2.30.

If $X_{\geq 0}^1, \dots, X_{\geq 0}^t$ canonical form triangulates the empty set, we may say that $X_{\geq 0}^2, \dots, X_{\geq 0}^t$ canonical form triangulates $X_{\geq 0}^{1-}$ (that is $X_{\geq 0}^1$ with inverted orientation).

Relations Of Triangulations The three signed triangulations are related by

$$\text{interior triangulation} \Rightarrow \text{boundary triangulation} \Leftrightarrow \text{canonical form triangulation}.$$

We see by the following example that the viceversa of the first implication does not work.

Example 2.31 (Counter Example). Consider the null geometry $(\mathbb{CP}^1, \mathbb{R}^1)$. Its boundary triangulates the empty set, but it does not interior triangulate it.

Remark 2.32. If $\{X_{\geq 0}^i\}$ boundary triangulates the empty set and all $X_{\geq 0}^i$, except $X_{1, \geq 0}$ are known to be pseudo positive geometries, then $X_{\geq 0}^i$ is a positive geometry and its canonical form is:

$$\Omega(X^i, X_{\geq 0}^i) = - \sum_{i=2}^t \Omega(X^i, X_{\geq 0}^i).$$

The canonical form is triangulation independent. 2.3

Grothendieck group of pseudo-positive geometries

Definition 2.33: Grothendieck group of pseudo-positive geometries on X .

The free abelian group generated by all pseudo positive geometries on X is denoted by $P(X)$. Its elements are $\sum_{i=1}^t X_{\geq 0}^i$ whenever the collection boundary triangulates the empty set.

Remark 2.34. In $P(X)$ we have $X_{\geq 0} = -X_{> 0}^-$. Also, if $X_{\geq 0}^i$ forms an interior triangulation of $X_{\geq 0}$. We may extend Ω to an additive homomorphism from $P(X)$ to the space of meromorphic top forms on X via:

$$\Omega\left(\sum_{i=1}^t X_{\geq 0}^i\right) := \sum_{i=1}^t \Omega(X_{\geq 0}^i)$$

Notice that the homomorphism is injective because the boundary and canonical form triangulations are equivalent.

2.2.2 Morphisms of positive geometries

A function which preserves the structure of a positive geometry must keep in check both the differential nature and the algebraic nature of the object. The canonical form of the positive geometries capture both these aspects of positive geometries and thus any nice definition of morphism, will have to guarantee a respect of the canonical form.

Definition 2.35: Morphism Of Pseudo-Positive Geometries.

Let $(X, X_{\geq 0})$ and $(Y, Y_{\geq 0})$ be two pseudo-positive geometries. A *rational map*

$$\phi : X \longrightarrow Y$$

such that the restriction

$$\phi|_{\geq 0} : X_{\geq 0} \longrightarrow Y_{\geq 0}$$

is an *orientation preserving diffeomorphism*.

A morphism where $(X, X_{\geq 0}) = (\mathbb{P}^D, \Delta^D)$ is called *rational parametrization*.

Definition 2.36: Isomorphism Of Pseudo-Positive Geometries.

Given a morphism of pseudo-positive geometries

$$\phi : X \longrightarrow Y$$

that is also a *isomorphism of varieties*, then we call

$$\phi : (X, X_{\geq 0}) \longrightarrow (Y, Y_{\geq 0})$$

an *isomorphism of pseudo-positive geometries*.

Two positive geometries are said to be isomorphic if there is an isomorphism between them.

2.3 Physical And Spurious Poles

One may wonder what kind of poles we can have in the canonical forms of a triangulation of a positive geometry.

Let $\{X_{\geq 0}^i\}$ be a signed triangulation (any triangulation between: internal, boundary and canonical form triangulation) of $X_{\geq 0}$.

Definition 2.37: Physical and Spurious boundaries.

The boundary components of $X_{\geq 0}^i$, that are also boundary components of $X_{\geq 0}$ are called physical boundaries. Otherwise they are called spurious boundaries.

Definition 2.38: Physical and Sporadic poles.

Poles of $\Omega(X, X_{\geq 0}^i)$ at physical boundaries are called physical poles. Poles of $\Omega(X, X_{\geq 0}^i)$ at spurious boundaries are called spurious poles.

Sometimes we refer to the triangulation independence of the canonical form as cancellation of spurious poles, since spurious poles don't appear in the sum $\sum_i \Omega(X, X_{\geq 0}^i)$.

Remark 2.39. Spurious poles don't generally disappear in pairs. Spurious poles cancel among collections of boundary components that boundary triangulate the empty set

2.4 Computing Canonical Forms

This section regards the computational aspects of canonical forms, the main method used are given as follows:

1. *Direct construction from poles and zeros*: viewing the canonical form as a rational function and imposing the appropriate constraints from poles and zeros.
2. *Triangulations*: we triangulate a generalized polytope by generalized simplices and sum the canonical form of each piece.

These methods do not exhaust all the techniques that can be used to compute a canonical form of a positive geometry. Two notable approaches explained in [ABL17] are the *push-forwards* of forms and the *integral representation* technique.

- *Push-forwards*: Let $\phi : (X, X_{\geq 0}) \rightarrow (Y, Y_{\geq 0})$ be a morphism of two positive geometries. It is possible to compute $\Omega(Y_{\geq 0})$ as a push-forward of the canonical form on $X_{\geq 0}$ via ϕ ,

$$\Omega(Y) = \phi^* (\Omega(X)).$$

- *Integral representations*: We view the canonical forms as volume integral over a "dual geometry" or as a contour integral of a related geometry.

2.4.1 Direct Construction From Poles And Zeros

There is a recursive approach for the construction of the canonical form of a positive geometry in a standard space based on the recursive nature of its definition.

We can consider a positive geometry (\mathbb{CP}^m, A) in a projective space.

Since A is a bounded semi-algebraic set, we can write it as

$$A = \pi(\{y \in \mathbb{R}^{m+1} \mid q_1(y) \geq 0, \dots, q_t(y) \geq 0\}).$$

Where q_1, \dots, q_t are homogeneous polynomials. Then the canonical form of (\mathbb{CP}^m, A) is the following:

$$\Omega(A) = \frac{q}{\prod_{i=1}^t q_i} dx_1 \wedge \dots \wedge dx_m.$$

Where q is an homogeneous polynomial which must satisfy the following restriction on the degree:

$$\deg q = \sum_i^t \deg q_i - m - 1.$$

The canonical form is invariant under a *local* $GL(1)$ -action, namely

$$Y \rightarrow \alpha(Y)Y.$$

This method operates under the assumption that such a numerator exists for the canonical form, which is not a trivial fact for most cases. The procedure is based on the idea that the numerator can be found by imposing constraints that arise from the residue operator.

An example will clarify the process described above for the recursive construction of the canonical form of (\mathbb{CP}^m, A) .

Example 2.40. Consider the quadrilateral $A := A(Z_1, Z_2, Z_3, Z_4)$ in \mathbb{RP}^2 with facets given by the four following inequalities.

$$\begin{aligned} q_1 &= x \geq 0 \\ q_2 &= 2y - x \geq 0 \\ q_3 &= 3 - x - y \geq 0 \\ q_4 &= 2 - y \geq 0 \end{aligned}$$

The vertices of A are these four.

$$Z_1^I = (1, 0, 0), Z_2^I = (1, 2, 1), Z_3^I = (1, 1, 2), Z_4^I = (1, 0, 2)$$

considering the coordinates as $(1, x, y) \in \mathbb{CP}^2$.

Computing the canonical form for this positive geometry is finding the unknown coefficients A, B, C of the following expression.

$$\Omega(A) = \frac{(A + Bx + Cy) dx dy}{x(2y - x)(3 - x - y)(2 - y)}.$$

Notice that the numerator has to be linear for the restriction on the degree of the polynomial previously named as q .

For each element of the boundary of the positive geometry there is a residue, we can pair them to go down a dimension further. In this case there are thus four residues and

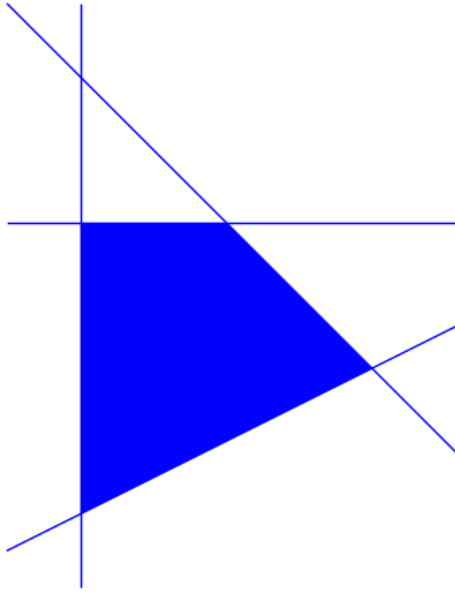


Figure 1: Quadrilateral A .

$\binom{4}{2} = 6$ double residues. The latter will give the canonical form of the zero-dimensional positive geometries in regards to the canonical form of the full-dimensional initial positive geometry. As such, those corresponding to vertices of the quadrilateral must have residue either one or minus one, where the sign is chosen with regards of the orientation. Those double residues corresponding to two opposite edges must have residue zero.

Each double residue can be written as the following expression.

$$Res_{ji} := Res_{q_j=0} Res_{q_i=0}.$$

The list of all double residues for the current example is the one that follows.

$$\begin{aligned} Res_{12} &= \frac{A}{12} = 1 \\ Res_{23} &= \frac{A + 2B + C}{6} = 1 \\ Res_{34} &= \frac{A + B + 2C}{6} = 1 \\ Res_{41} &= \frac{A + 2C}{4} = 1 \\ Res_{13} &= \frac{A + 3C}{6} = 0 \\ Res_{24} &= \frac{A + 4B + 2C}{12} = 0 \end{aligned}.$$

These constraints determine the unique solution to be the following.

$$(A, B, C) = (12, -1, -4).$$

We observe that there are many more equations than undetermined coefficients, thus the existence of a solution is non-obvious. Moreover, since to consider the positive geometry of the co-dimension one skeleton is necessary to take the residue on all the boundaries, the computational cost of this method increases quickly.

2.4.2 Triangulations

Another way to obtain the canonical form of a positive geometry from the canonical forms of easier positive geometries is to make use of triangulation. Recall that if a positive geometry is interior triangulated by a collection of positive geometries, its canonical form is given by the sum of the canonical forms of the collection that triangulates the initial positive geometry.

We can use an interior triangulation of *Projective polytopes* to obtain their canonical form from the simplexes that triangulate them. Let A be the projective polytope triangulated by the simplexes $\Delta_1, \dots, \Delta_r$. Consider the positive geometries (\mathbb{CP}^2, A) , $(\mathbb{CP}^2, \Delta_i)$ for all $i \in \{1, \dots, r\}$.

Call their canonical forms $\Omega(A)$ and $\Omega(\Delta_i)$ for each $i \in \{1, \dots, r\}$ respectively.

Then the following property

$$\Omega(A) = \sum_i \Omega(\Delta_i)$$

holds and we can use it to reduce the problem of finding the canonical form of A into finding the canonical form of the simplexes Δ_i .

Instead of remaining on polytopes and their triangulation in simplexes, in the following example we show this method on a triangulation of a positive geometry defined by some conic inequalities and some linear ones.

Example 2.41. Let

$$\begin{aligned} p_0 &= x^2 + y^2 - 1 \\ p_1 &= -(1/2)x + 1 \\ p_2 &= -3x + 3 \\ p_3 &= 3x - 3 \\ p_4 &= (1/2)x - 1 \end{aligned}$$

Set the semi-algebraic set T as follows

$$T := \{(x, y) \in \mathbb{R}^2 \mid p_0 \leq 0, p_1 \leq 0, p_2 \leq 0, p_3 \geq 0, p_4 \geq 0\}$$

and consider the positive geometry (\mathbb{CP}^2, T) .

By slicing T with the vertical line

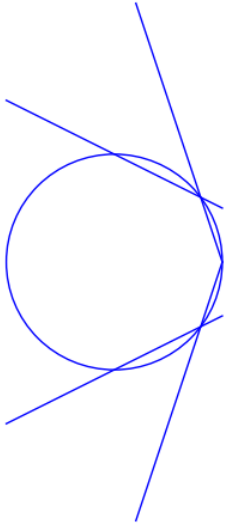
$$s : x = 0$$

we produce an interior triangulation of T , namely $\mathcal{T}(T) := \{T_1, T_2\}$.

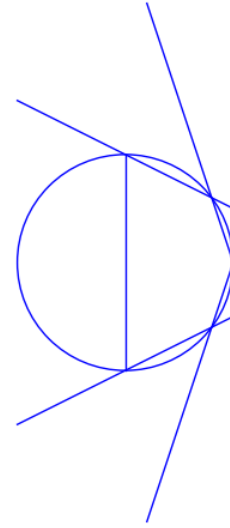
Set $\Omega(T)$ the canonical form of T . Set $\Omega(T_1), \Omega(T_2)$ the canonical forms of T_1, T_2 respectively. Now we can reduce the computation of $\Omega(T)$ to the computation of $\Omega(T_1)$ and $\Omega(T_2)$, because

$$\Omega(T) = \Omega(T_1) + \Omega(T_2).$$

The rational form $\Omega(T_2)$ with regards to T will have spurious poles on s and physical poles on p_0 .



(a) Semi-Algebraic Set T



(b) Triangulation $\mathcal{T}(T)$

3 Wachpress Coordinates

Any discussion of Wachpress coordinates needs to start from barycentric coordinates on simplexes, because Wachpress coordinates aim to be a generalization of the latter coordinates to more complicated geometrical objects. Barycentric coordinates are very useful because they allow a very natural approximation scheme with linear precision named finite element method. It has so many practical applications that writing a complete list would be tedious if not utterly impossible.

3.1 Barycentric Coordinates

The barycentric coordinates defined on a simplex are a finite partition of unity. Every point in the simplex can be written as a finite linear combination of these coordinates and thus these coordinates really simplify the treatment of every linear function defined on this domain. Since most used functions admit a linear approximation, at least locally, we can use barycentric coordinates as a mean to approximate any good enough function. This is known as a finite element method with linear precision and it is used in a great amount of practical circumstances [Flo15].

The moment has come for a more precise, formal and through explanations of these coordinates.

Definition 3.1.

Given a simplex Δ^n , for every vertex $v \in V(\Delta^n)$ an associated function B_v can be defined such that the following three properties are satisfied.

1. *Non-Negativity*, $B_v(x) \geq 0 \forall v \in V(\Delta^n)$
2. *Linear precision*, given a linear function $L(x)$, the approximations given by

$$L(x) = \sum_{v \in V(\Delta^n)} L(v) B_v(x).$$

3. *Minimal degree*, over every vertex $v \in V(\Delta^n)$ the barycentric coordinate B_v is a linear polynomial.

The set $\{B_v\}_{v \in V(\Delta^n)}$ is the set of barycentric coordinates of the simplex Δ^n .

We can easily see that the barycentric coordinates are a partition of unity. We can consider a constant linear function, then the second property of the list, namely the linear precision, implies that the barycentric coordinates form a partition of unity.

3.2 Polycons

Wachpress coordinates are conjectured to be well defined for a generalization of simplexes, called polycons. Polycons are not the first natural generalization of simplexes, which are polytopes. They are the second step up in the generalization ladder. Polycons are the generalization of polytopes too. We need to introduce what these objects are, but to do so we must ask for a little patience, since a convenient way of defining polycons is to define them as a special case of its own generalization, the polypol.

3.2.1 Polypols

Now the question of what is a polycon has been modified to what is a polypol and what property should we ask to a polypol to be a polycon.

Remark 3.2. For the purpose of this thesis is sufficient to introduce rational polypols.

Definition 3.3: Rational Polypols.

Let C be an algebraic planar curve in \mathbb{CP}^2 with $k \geq 2$ rational irreducible components C^1, \dots, C^k . Assume there are k points $v_{12} \in C^1 \cap C^2, \dots, v_{k1} \in C^k \cap C^1$ such that v_{ij} is a non-singular point for both irreducible components C^i and C^j and such that these two components intersect transversely at v_{ij} .

Then we say that the irreducible curves C^i and the points v_{ij} form a *polypol* P . The set of points $V(P) := \{v_{ij}\}$ is called the *vertices* of P , and the complement $R(P) := \text{Sing}(C) \setminus V(P)$ of the vertices in the singular locus of C is called the set of *residual points* of C . We call $d = \prod_i \text{deg}(C^i)$ the *degree* of P .

Notation 3.4.

Every time we will refer to a polypol, we assume it to be rational.

Definition 3.5: Real and Quasi-Regular Polypols.

Let P be a polypol. We call P a *real polypol* when

- It has real boundaries C^i .
- It has real vertices v_{ij} .
- It has a given choice of *sides*, segments of C^i connecting circularly the vertices.
- There is a semi-algebraic set $P_{\geq 0}$ whose interior is a union of simply connected sets and whose boundary is exactly the union of the sides of P .

We call a real polypol *quasi-regular* when it has non-singular sides.

Given that the polypol has been defined, it is now time to introduce a proper definition of a real polycon.

Definition 3.6: Real Polycon.

A real polycon is a quasi-regular polypol with either linear or quadratic boundaries.

Notation 3.7.

Every time we will refer to a polycon, we assume it to be real.

In this thesis we describe an approach to face Wachpress conjecture which uses positive geometries for the reasons described in the section 2. Thus, we need to relate polycons and positive geometries. The next theorem shows that one can view quasi-regular polypols as positive geometries.

Theorem 3.8. *Let P be a quasi-regular polypol in \mathbb{RP}^2 . Consider P in the affine chart which contains $P_{\geq 0}$ defined in 3.5. Then P is a positive geometry with canonical form:*

$$\Omega(P) := \frac{\alpha_P}{f_1 \dots f_k} dx \wedge dy$$

Where the different symbols are defined as the following:

- All the f_i are real polynomials defining the curves C^i which are the boundaries of the quasi-regular polypol.
- α_P is the real polynomial defining A_P , which is the adjoint of the quasi-regular polypol.

Remark 3.9 (Orientability). We want to stress the importance of considering affine charts of the projective space to view the polypols as positive geometries. The real projective plane is not orientable, and thus we must be careful on considering positive geometries, which require orientation, as purely defined on this space. We rest assured being aware that a choice of charts will guarantee orientability of the polypol.

3.3 The Adjoint Curve

The adjoint curve to the polycon is our main concern throughout the thesis, because proving that there are no points in the intersection of interior of the polycon and the adjoint curve means that the coordinates of Wachpress will be well defined.

The adjoint, as defined by Wachpress in [Fix78] is the following:

Definition 3.10: Adjoint.

Let P be a planar polycon. Let $V(P)$ be its vertices. Let $R(P)$ be the set of residual points as in 3.3.

Set A_P the curve of minimal degree passing through all the $R(P)$. We call A_P the *adjoint curve* to the polycon. We call the *real adjoint curve* the real part of the adjoint curve.

Remark 3.11 (Generalization). In case of rational polypols with boundaries with complicated singularities, or which intersect non-transversally, Wachpress required the adjoint curve to have appropriate multiplicities at the resulting residual points.

The adjoint of a polycon can be defined differently, as in [Koh+21]. I decided to opt for Wachpress definition for the intuition and simplicity of this definition.

Adjoint Of Polygons Consider the linear case polygons, which are polycons with only linear boundaries. Polygons arise from arrangements of lines, a generalization of this would be arrangements of pseudolines for which refer to [FG17]

With the following two remark we aim to count the residual points of an arrangement which produces a polygon.

1. In the remark 3.12 we move the focus from a polygon P to the arrangement of lines which generates it, namely L . Moreover, it gives some intersection properties of its irreducible components. We shift from an affine environment to a real projective one.
2. In the remark 3.13 we count the residue points for the arrangement L with respect to the polygon P .

Remark 3.12. Let P be a polygon of k vertices \mathbb{R}^2 . We can see P as the intersection of k half spaces. Call the lines that produce these half spaces l_1, \dots, l_k . Call the arrangement of these lines $L = \bigcup_{i=1}^k l_i$. Embed P and L in $\mathbb{R}P^2$. Here every pair of distinct lines (l_i, l_j) , $i < j$ has a transversal intersection. By construction of L , there is no intersection for a triple of distinct lines (l_i, l_j, l_s) , $i < j < s$.

Remark 3.13. Let L as stated above in 3.12. We have $\binom{k}{2} = \frac{k \cdot (k-1)}{2}$ intersections in pairs of irreducible components of L . These are the only singular points of L . Of these singular points, k are vertices of P . By definition, we conclude that there are $\frac{k(k-1)}{2} - k$ residue points.

Now we know how many residue points we have for a polygon P and its arrangement L .

Now focus on the family of curves of a fixed degree d passing through a certain amount of distinct points t .

1. In 3.14 it is given a projective real space equivalent to \mathcal{V}_d , defined as the space of planar curves of a fixed degree d .
2. In 3.15 it is specified what entails for the curves in the space \mathcal{V}_d the imposition of passing through a chosen point.

Remark 3.14. Let $\phi \in \mathcal{V}_d$. By definition, ϕ is a variety generated by the principal ideal $I = \langle P \rangle$, where $P = P(X, Y)$ is a polynomial of degree d unique up to scalar multiplication. The polynomial P has $(d+1) + \binom{d(d+1)}{2}$ coefficients. It follows that ϕ can be seen as a point of

$$\mathbb{R}P^{(d+1) + \binom{d(d+1)}{2}}.$$

For a generic ϕ we have no restrictions on the choice of coefficients. Thus, \mathcal{V}_d can be seen as $\mathbb{R}P^{(d+1) + \binom{d(d+1)}{2}}$.

Remark 3.15. The elements of \mathcal{V}_d passing through a point p in $\mathbb{R}P^2$ must satisfy a linear constraint on their coefficients. Thus the family of curves V_d^p of degree d passing through p form a co-dimension one linear subspace of \mathcal{V}_d .

In the case of polygons we have the following proposition about the degree of the adjoint.

Proposition 3.16 (Degree Of The Adjoint). *Given a polygon P of k vertices, its adjoint polynomial α_P will have degree $k - 3$.*

Degree Of The Adjoint. As we discussed earlier in 3.13 and 3.12, for a polygon of k vertices, there are $\frac{k \cdot (k-1)}{2} - k$ residue points.

Set m the degree of the adjoint. From 3.14 and 3.15, we know that the adjoint will lay in a linear sub-space of \mathcal{V}_m of co-dimension $\frac{k \cdot (k-1)}{2} - k$. For the adjoint to exist, m has to satisfy

$$m(m + 3) - k(k - 3) \geq 0.$$

By definition, the adjoint has minimal degree, thus $m = k - 3$. This implies that the adjoint polynomial α_P which defines the adjoint curve will be of degree $k - 3$. \square

We can generalize the result obtained in the case of polygons to polycons.

Proposition 3.17 (Degree Of The Adjoint Of A Polycon). *Given a polycon P of degree k then its adjoint polynomial α_P will be of degree $k - 3$.*

Remark 3.18 (Positive Geometries). Remember 3.8, which states that any rational polycon P is a positive geometry with the canonical form

$$\Omega(\mathbb{P}^2, P) = \frac{\alpha_P}{\prod_i b_i} dx \wedge dy.$$

The numerator in the canonical form, α_P is the adjoint polynomial to P . The zero locus of the b_i are the boundaries C^i of P .

Useful Topological Definitions The concepts of *oval* and *pseudoline* are repeating continuously in the discussion of the topology of the adjoint curve A_P . In the next paragraph we present their definitions.

We give a definition of Oval and Pseudoline for a projective setting as in [Ore21]. Let us consider the projective real plane \mathbb{RP}^2 .

Definition 3.19: Oval And Pseudoline.

Let η be a closed curve in \mathbb{RP}^2 . Call the complement of η in the real projective space $\mathbb{RP}^2 \setminus \eta$.

1. If $\mathbb{RP}^2 \setminus \eta$ has *Two Maximal Connected Components*, η is an *Oval*.
2. If $\mathbb{RP}^2 \setminus \eta$ has *One Maximal Connected Component*, η is a *Pseudoline*.

Another concept which will be heavily used from now is the hyperbolicity of a real curve. On this property is based the most effective argument for solving the conjecture for some polycons in arrangements of three ellipses.

Definition 3.20: Hyperbolic Curve.

Let C be a real curve of $\deg(C) = n$. Then we say C is *Hyperbolic* when

1. If n is even, C consists of $\frac{n}{2}$ *nested ovals*.
2. If n is odd, C consists of $\lfloor \frac{n}{2} \rfloor$ *nested ovals and one pseudoline*.

3.4 Definition Of Wachpress Coordinates

Wachpress coordinates are rational barycentric coordinates defined on polycons and are defined as:

Definition 3.21: Wachpress coordinates.

Given a polycon P . For each vertex of the polycon $v \in V(P)$ we define

$$\phi_v(x) = \frac{k_v p^v(x)}{\alpha(x)}$$

as the *Wachpress coordinate* of P in v . In the previous formula, p^v is the polynomial which defines the boundaries of P which do not pass through v . The polynomial α defines the adjoint of the polycon. While

$$k_v = \frac{\alpha(v)}{p^v(v)}$$

is the *normalizing factor*, which guarantees that for every v vertex of P

$$\phi_v(v) = 1.$$

Now will follow a couple of examples of Wachpress coordinates, the first on a polygon and a second on a proper planar polycon.

Example 3.22 (Pentagon). Consider the arrangement of the five lines:

$$\begin{aligned} l_1 : y &= 1 \\ l_2 : y &= -x + 3 \\ l_3 : y &= \frac{2}{3}x - 2 \\ l_4 : y &= -\frac{2}{3}x - 2 \\ l_5 : y &= x + 3 \end{aligned}$$

Which intersects in $\binom{5}{2} = 10$ points. Five of these points along with segments of the lines of the arrangement form a pentagon.

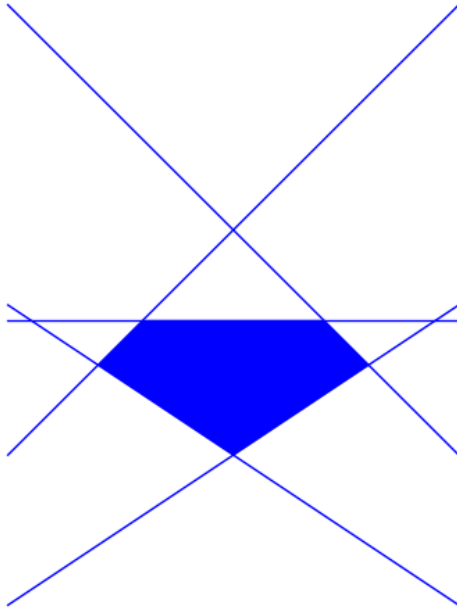


Figure 3: Pentagon

The five vertices of the pentagon are the following:

$$v_1 = (-2, 1)$$

$$v_2 = (2, 1)$$

$$v_3 = (3, 0)$$

$$v_4 = (0, -2)$$

$$v_5 = (-3, 0)$$

The five residue points are the following:

$$r_1 = \left(-\frac{9}{2}, 1\right)$$

$$r_2 = (0, 3)$$

$$r_3 = \left(\frac{9}{2}, 1\right)$$

$$r_4 = (15, -12)$$

$$r_5 = (-15, -12)$$

Let us find the adjoint curve to the pentagon. From the theory we know that the adjoint will be a conic.

$$A_P = a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{10}x + a_{01}y + a_{00}$$

To obtain an explicit adjoint polynomial A_P , we must solve the following linear system. Each row is a linear constraint given by the passing of the curve through a residual

point and every column is associated to a coefficient of the curve.

$$\begin{bmatrix} \frac{81}{4} & 1 & \frac{-9}{2} & \frac{-9}{2} & 1 & 1 \\ 0 & 9 & 0 & 0 & 3 & 1 \\ \frac{81}{4} & 1 & \frac{9}{2} & \frac{9}{2} & 1 & 1 \\ (15)^2 & (12)^2 & (15) \cdot (-12) & 15 & -12 & 1 \\ (15)^2 & (12)^2 & (-15) \cdot (12) & -15 & 12 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{20} \\ a_{02} \\ a_{11} \\ a_{10} \\ a_{01} \\ a_{00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution space of the system above is the one dimensional space:

$$\text{span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 339/88 \\ 93/22 \\ -93/22 \\ -465/88 \\ -207/11 \end{bmatrix} \right)$$

They are the coefficients of the adjoint up to scalar multiplication.

$$A_P = x^2 + 339/88y^2 + 93/22xy + -93/22x - 465/88y + -207/11$$

The Wachpress coordinates of the pentagon will be the five rational functions:

$$\begin{aligned} \phi_{v_1} &= \frac{k_{v_1}(l_2 \cdot l_3 \cdot l_4)}{A_P} \\ \phi_{v_2} &= \frac{k_{v_2}(l_3 \cdot l_4 \cdot l_5)}{A_P} \\ \phi_{v_3} &= \frac{k_{v_3}(l_4 \cdot l_5 \cdot l_1)}{A_P} \\ \phi_{v_4} &= \frac{k_{v_4}(l_5 \cdot l_1 \cdot l_2)}{A_P} \\ \phi_{v_5} &= \frac{k_{v_5}(l_1 \cdot l_2 \cdot l_3)}{A_P} \end{aligned}$$

Where all the polynomials l_1, \dots, l_5, A_P have been determined.

Example 3.23 (Quarter Of A Circle). Consider the arrangement of the following curves:

$$\begin{aligned} l_1 : y &= 0 \\ l_2 : x &= 0 \\ c : x^2 + y^2 - 1 &= 0 \end{aligned}$$

From which arises a polycon P as the upper right quarter of the circle. Since the curves meet pairwise in a transversal way. We count the total number of intersections as the sum of the products of the degrees of the curves taken pairwise. Thus the curves of the arrangement will meet in five points, three of which are vertices of the polycon.

The vertices of the polycon are :

$$v_1 = (0, 0)$$

$$v_2 = (1, 0)$$

$$v_3 = (0, 1)$$

The residue points are :

$$r_1 = (-1, 0)$$

$$r_2 = (0, -1)$$

Since we have defined the adjoint as the curve of minimal degree passing through all the residue points, in this setting, the adjoint will be a line passing through r_1 and r_2 . The polynomial that defines it will be

$$A_P = x + y + 1$$

The Wachpress coordinates of P are:

$$\begin{aligned}\phi_{v_1} &= \frac{x^2 + y^2 + 1}{x + y + 1} \\ \phi_{v_2} &= \frac{x(x^2 + y^2 + 1)}{x + y + 1} \\ \phi_{v_3} &= \frac{y(x^2 + y^2 + 1)}{x + y + 1}\end{aligned}$$

3.5 Wachpress on Polypols of Low Degree

On all planar polypols of degree at most five on the real projective plane, Wachpress conjecture has already been proven in the following proposition.

Proposition 3.24 (Polypols of Total Degree at most Five). *Wachpress conjecture holds for polypols of total degree at most five.*

Proof. Let P be a polypol of degree four. Then the degree of the adjoint curve must be one, so it is a real line in the projective real plane. The inner part of the polypol $P_{>0}$ cannot contain a real line that does not intersect the bounding sides.

If the degree of P is five, the adjoint is a real conic, thus the curve is a single connected component. The conjecture follows once it is shown that there is always a real residual point, which is outside of the polycon by regularity.

The possible degrees of the bounding curves are the following:

$$(1, 4), (2, 3), (1, 1, 3), (1, 2, 2), (1, 1, 1, 2), (1, 1, 1, 1, 1).$$

In the first, second and third case, the rational real quartic and respectively cubic curve has at least one real singularity, which is a residual point. In the two following cases, one of the lines intersects the conic in a vertex, which must be a real point, this means that the other intersection point is a real residual point. The last case is a convex pentagon with five real residual points.

Finding a residual point for every possible configuration, concludes the proof. \square

3.6 Wachpress On Convex Polygons

In the case of convex polygons on the real plane, Wachpress [Fix78] proved that his barycentric coordinates are well defined.

Moreover in this article [Koh+21], the authors write a complete topological description of the adjoint curve of convex polygons.

The following theorem gathers the topological information of the adjoint curve. From its hyperbolicity to the list of connected components and how they are arranged in respect to the others and the polygon itself. This theorem gives a sufficient result to prove Wachpress conjecture for planar polygons.

Theorem 3.25 (Topology Of The Adjoint Curve). *The adjoint curve A_P of a convex k -gon named $P_{\geq 0}$ is hyperbolic with respect to every point $e \in P_{\geq 0}$ of the polygon. Moreover, the curve A_P is strictly hyperbolic, which means it does not have any real singularities.*

A more precise description can be written as A_P has $\lfloor \frac{k-3}{2} \rfloor$ disjoint nested ovals. If the total degree k is even, there is additionally a pseudoline contained in the region in the complement of the ovals that is not simply connected. In this case, the residual intersection point C^i and $C^{i+k/2}$ lies on the pseudoline component (reading the index modulo k). In general, for k even or odd, the residual intersection point of C^i and C^{i+1+m} for a positive integer $m < \frac{k}{2} - 1$ lies on the m -th oval counting from the inside, from the convex polygon outwards.

It is interesting to see the sign of the adjoint polynomial on the Zarinski closure of the boundaries, namely on all the C^i . The adjoint curve separates the plane in areas where the real evaluation of the adjoint polynomial assumes different signs. If the adjoint curve has a pseudoline component, the sign of the real evaluation of the adjoint polynomial on each C^i changes at infinity if the curve intersects the line at infinity.

4 Three conics boundary

4.1 Introduction

In this section we will look at the easiest case of polycons of degree six which is yet to be solved. The case of three irreducible conics that intersect transversely whenever they meet. This special arrangement type has been analyzed by both J. Wachpress in [Wac20] and in the article [Koh+21]. Most of the possible topological cases of the arrangement have been now solved, but not all of them. The study of the case of three irreducible conics is complicated by the fact that the adjoint might not be hyperbolic. This is in contrast to the case of polygons, where the hyperbolicity of the adjoint can be exploited.

A small summary of the approach which has been taken in the article [Koh+21] for these three irreducible conics arrangements can be written as this bullet points list:

- Classify all the possible arrangements of three ellipses that meet transversely up to real diffeomorphisms.
- For each case of the previous point, produce a complete list of the distinct polycons that arise from the configuration.
- Resolve the statement for the polycons when it is possible to prove that the adjoint is hyperbolic and the oval component of the adjoint lies outside of the polycon.
- Look for more complicated arguments for the remaining cases.
- For the ones that have not been solved, find an example and compute the adjoint.

Remark 4.1. Differently from the case of polytopes, in arrangements of three ellipses there are polycons whose adjoint is not hyperbolic.

The following theorem from [Koh+21] is the most important result of the section. It states the current situation of the Wachpress Conjecture on arrangement of three conics.

Theorem 4.2 (3.13). *33 out of 44*

There are exactly 44 topologically non-equivalent configurations of 3 ellipses in \mathbb{RP}^2 such that each pair of ellipses intersect each other transversally in at least two real points, and all 3 of them do not intersect at a common real point. In 33 of these configurations, the adjoint curve A_P of any regular polycon P in the configuration does not intersect $P_{>0}$, the interior of the polycon.

Remark 4.3. The proof of the previous theorem is divided in the following main two parts:

- For 28 of these configurations, Wachpress conjecture is proven via hyperbolicity of A_P . This means that the conjecture is proven by ensuring that A_P has two connected components and that the oval must be outside of the arrangement of the conics. This argument solves most of the polycons of all configurations, for each arrangement not completely solved, just one polycon causes problems.
- The argument to prove five of the remaining arrangements is divided in many different cases.

The details of the proof of the theorem will be explained carefully later in the section.

In what follows, we report the algorithm used in [Koh+21] to produce the catalog of topological configurations of all arrangements of three ellipses.

After the catalog, we bring the *approach using hyperbolicity*. It is effective enough to prove Wachpress conjecture for 28 of these configurations and all but one polycons for each of the other configurations. This approach using hyperbolicity consists of showing that the adjoint curve of degree three has to be hyperbolic, which means it has an oval component lying strictly outside the polycon.

Moreover, a convoluted argument will be presented. It is reasoned on cases distinctions that will solve the conjecture for five more cases. This leaves 11 problematic configurations, which are yet to be solved.

4.2 The Catalog

We want to classify all the *topologically stable* distinct configurations of three ellipses in \mathbb{R}^2 . A topologically stable configuration keeps its topological properties under small enough perturbations of its curves. It is easy to see that each ellipse must intersect only transversely the others and the intersection of all three ellipses must be empty. To have a proper degree six polycon, we finally ask that each pair of ellipses will intersect at least twice in the plane \mathbb{R}^2 .

Two configurations are equivalent if there exist a planar diffeomorphism $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that restricts to a diffeomorphism of the configurations. If the conjecture is proven for a configuration, it is proven for all equivalent configurations too. Thus, we want to consider only one representative for each equivalence class of topologically equivalent configurations for the catalog.

This work has been done in the article that motivated the whole thesis, namely [Koh+21].

The algorithm follows the idea of selecting one property to distinguish non-equivalent topological cases. In this way we obtain a partition of the possible cases. We consider a representative for each element of the partition. We consider a sub-arrangement of two ellipses. We build an excessive catalog from the possible intersections of an oval with the two ellipses partitioned by the former property. By identifying new properties that must distinguish between non-equivalent cases, the partition becomes finer. We eliminate the impossible arrangements of two ellipses and an oval. Repeating this process until we cannot distinguish any further between obtainable cases leads to the complete catalog.

The first property we use to distinguish non-equivalent topological cases is the *intersection type*. The intersection type is defined as a count of the number of intersections of the ellipses pairwise.

Since symmetric cases are equivalent, for our cases the possible values of the intersection type are the following:

1. The value (222) is used when all three pairs intersect exactly in two different points of the real plane \mathbb{R}^2 .
2. The value (224) is used when only one pair of ellipses intersects in four distinct real points, the other pairs will intersect in two.
3. The value (244) is used when only one pair of ellipses intersects in two distinct real points, the other pairs will meet in four.

4. The value (444) is used when all pairs of ellipses intersect in four real points, this case that realizes the maximum amount of real intersections possible it is referred to as the M -case.

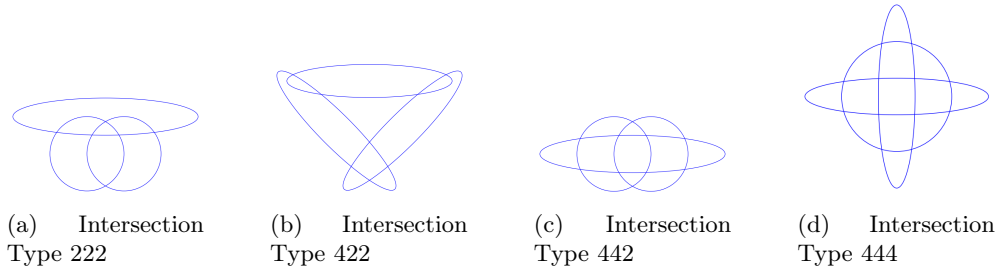


Figure 4: Intersection Types

The latter case, of full real intersections, is referred to as the M -case, [And].

Remark 4.4. The article [Kha96] contains statistical information about all possible topological configurations of three real non-singular conics transversely intersecting in $\mathbb{R}P^2$. From [Kha96] we know there are (105) configurations of the M -case in this setting.

Notice that there are more projective non-equivalent configurations of three conics than affine configurations of three ellipses. In particular, a configuration of two ellipses intersecting each other in four real points cannot be obtained from a configuration consisting of an ellipse and a hyperbola intersecting each other in four real points even from a projective diffeomorphism.

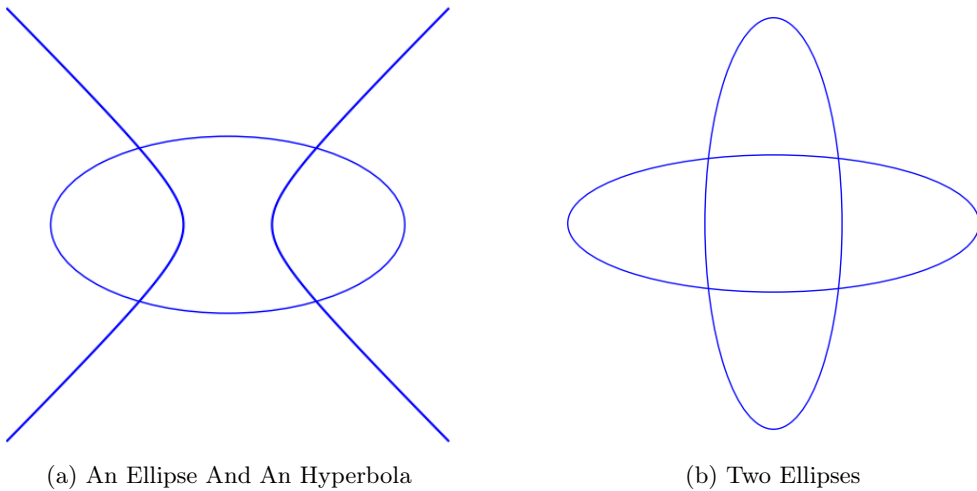


Figure 5: Projective Inequivalent Conics Arrangements

The algorithm in the article [Koh+21] to find all possible not equivalent configurations of three ellipses in the real plane consists of four steps. This method is sufficient for

this case of intersection of three ellipses, but it might need some rework to function in a more general problem, for example if we work with more than three ellipses.

1. *Subdivision according to intersection types*

Start by dividing the cases per intersection type. The intersection type must be one of the following.

$$(222), (224), (244), (444)$$

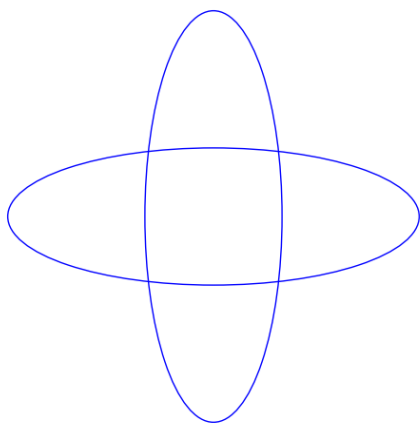
2. *Obtaining a preliminary excessive catalog*

During this step, consisting of three sub-steps, we create all possible topological configurations of two ellipses and an oval (that is not necessarily convex) of the intersection type chosen in the previous step.

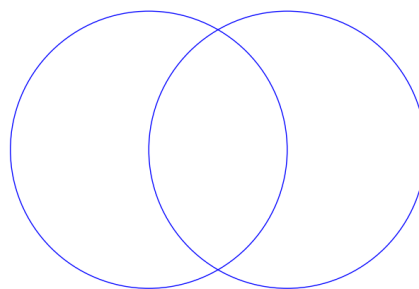
Remember that by oval we refer to a simple closed curve when we work on the real plane \mathbb{R}^2 . But in a projective setting an oval is defined as a C^1 -curve such that its projective complement will be of two connected components.

Notice that during this step we enumerate configurations of two ellipses and an oval, identifying those which can be obtained from another by a continuous deformation or a global symmetry.

- Draw two ellipses, one vertical and one horizontal, that intersect each other in a way consistent with the chosen intersection type. The intersection type allows the horizontal ellipse to intersect the vertical one either two or four points. This will produce a splitting of the two ellipses in respectively two or four arcs for each ellipse.



(a) Four Real Intersections



(b) Two Real Intersections

Figure 6: Two ellipses

Observe that the intersection type determines whether the third ovals can intersect the vertical ellipse in two or four real points. We need to list all the topologically distinct cases of how this oval can intersect the vertical ellipse locally.

Which means we must choose two or, respectively four short segments that meet the vertical ellipse in all topologically distinct ways (up to symmetries).

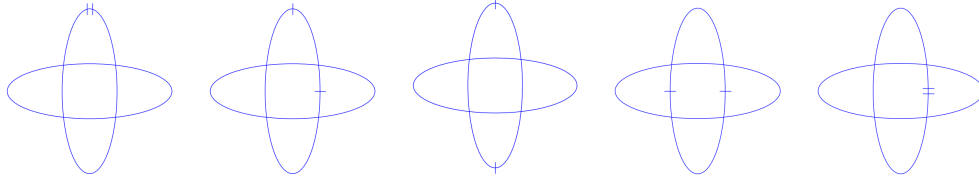


Figure 7: Vertical cuts for intersection type (422)

Let the intersection type be (422). In 7 we see the five possible topologically distinct cases of how the third ellipse can intersect the vertical ellipse locally.

Remark 4.5. Keep in mind that an arrangement of three ellipses with an intersection type and with a specific position of the cuts of the oval onto the vertical ellipse is not unique. There are possibly many non-diffeomorphic arrangements of three ellipses with the same intersection type and cuts-position on the vertical ellipse. We see an example in 8.

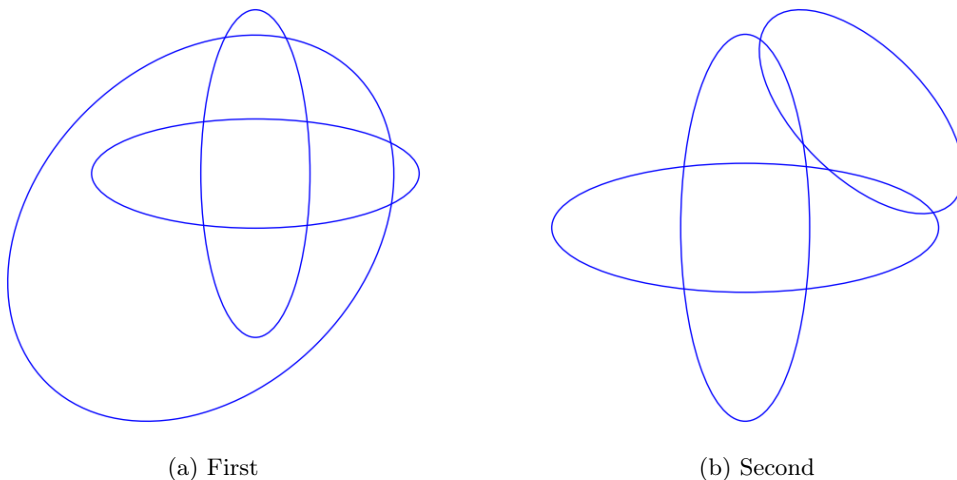


Figure 8: Example Of Inequivalent Arrangements Which Are Projectively Equivalent

- Subdivide each case obtained in the prior step further by connecting the short segments in all admissible ways consistent with the chosen intersection type, inside the union of the two ellipses. This determines how the third oval intersects the horizontal ellipse in the interior of the vertical ellipse.
- Finally, for each sub-case obtained in the previous step, complete the curve in all possible ways to get topologically distinct ovals that intersects the initial two ellipses according to the chosen intersection type.

3. Reduction

In this step we determine which configurations cannot be realized by three convex ovals using the following arguments:

- The intersection of two ellipse interiors must be convex.
- A line intersects a convex oval in at most two points.

4. Identification

During this step, we decide which configurations of three ellipses that are remaining are topologically equivalent and which are not.

Consider I the connected union of the filled ellipses in an arrangement and call its boundary ∂I . Call any connected subset of an ellipse ϵ an arc of ϵ .

Now,

$$\partial I = \cup_{j_1 \in J_1} arc_{j_1} \cup_{j_2 \in J_2} arc_{j_2} \cup_{j_3 \in J_3} arc_{j_3}$$

for J_1, J_2 and J_3 finite and for arc_{j_1}, arc_{j_2} and arc_{j_3} arcs of the different ellipses in the configuration. Any two equivalent arrangements will have the same triple $(|J_1|, |J_2|, |J_3|)$ up to permutation.

For each configuration, we call the latter triple ordered decreasingly the *outer-arc type*.

Note that two configurations with distinct outer-arc types cannot be equivalent.

Now all is left is to decide which configurations of the same outer-arc type are topologically equivalent.

If the number of polycons inside two configurations differ, then the configurations are different. Similarly, we can count regions with more than three sides to distinguish between distinct configurations. Finally, we identify that two configurations are equivalent by considering all permutations of the three ovals in one of the configurations.

5. Realization

Visually represent the topological configurations of three ovals remaining after the latest step or show they are not realizable by arrangements of ellipses using a method suggested by S. Orevkov in Ore99.

Of the configurations left in the latest step, they were able to realize all configurations with three ellipses, except the five configurations of M -type intersection shown underneath in 9.

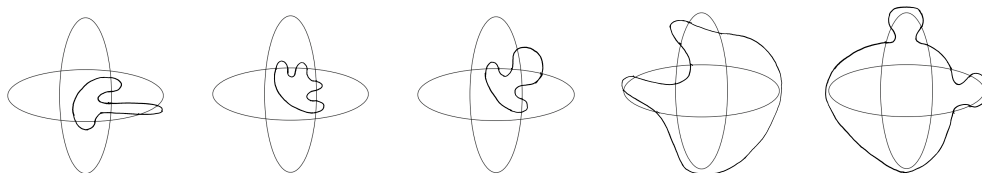


Figure 9: Arrangements Removed In The Step Realization.

Proposition 4.6. *None of these five configurations of two ellipses and an oval represented above can be realized by using three ellipses.*

For the proof of the proposition 4.6 we refer directly to the article [Koh+21].

4.2.1 A Combinatorial Approach To The Topological Catalog

It would be interesting to produce the catalog of stable topological cases of arrangements of three ellipses in a combinatorial fashion with a minimal set of satisfied properties.

One may think to construct the catalog recursively on the sub arrangements, up to diffeomorphism.

Sub Arrangement Of Two Ellipses Let $\mathcal{E}_1, \mathcal{E}_2$ be two ellipses with transversal and non-empty intersection. Then

$$|\mathcal{E}_1 \cap \mathcal{E}_2| = \begin{cases} 4 \\ 2 \end{cases}$$

for Bezout's theorem. In the first case the two ellipses would be divided in four arcs each, half internal, half external. In the second, we have a division in two arcs of each ellipse.

$$\mathcal{E}_j = \bigcup_{i=1}^t \text{arc}_i(\mathcal{E}_j) \text{ for } j = 1 \text{ or } 2,$$

for $t = 2$ or 4 depending on $|\mathcal{E}_1 \cap \mathcal{E}_2|$. This concludes the discussion for the case of two ellipses $\mathcal{E}_1, \mathcal{E}_2$.

Extension To Three Ellipses If we were to add a third ellipse, namely \mathcal{E}_3 , we know the cases for $|\mathcal{E}_3 \cap \mathcal{E}_*|$ ($* = 1$ or 2) are analogous to 4.2.1.

If we order decreasingly the number of intersections of pairs of ellipses in the arrangement, we obtain the triple we called *intersection type*. It is enough to consider the triples ordered decreasingly because they are well defined up to ordering.

Remark 4.7. The intersection type of an arrangement is an excellent combinatorial invariant for topological cases up to diffeomorphism. Obviously the intersection type is not a sufficient combinatorial description of an arrangement up to diffeomorphism.

One may think that if we kept track of the arcs of \mathcal{E}_1 and \mathcal{E}_2 which \mathcal{E}_3 intersects, we might have a sufficient combinatorial description of the arrangement.

Remark 4.8. It turns out that if we require the cases to be expressed up to *real diffeomorphisms*, the description is not sufficient. We can see it in the counter-example 4.9. It might hold for purely projective diffeomorphisms, but we have not thoroughly checked this claim yet.

Example 4.9. The example of two arrangements diffeomorphic with purely projective diffeomorphisms, but not diffeomorphic with real diffeomorphisms in picture 8 shows that the combinatorial description is not enough for real diffeomorphisms.

Notation 4.10.

From now on we are only going to differentiate between cases with different intersections of \mathcal{E}_3 with the arcs of \mathcal{E}_1 and \mathcal{E}_2 .

Let (e_1, e_2, e_3) be an intersection type. Call $e_1 := |\mathcal{E}_1 \cap \mathcal{E}_2|$, let $e_j = |\mathcal{E}_3 \cap \mathcal{E}_{j-1}|$ for $j = 2$ or 3 .

Recall that we have a division of $\mathcal{E}_1, \mathcal{E}_2$ in e_1 number of arcs. For each triple (e_1, e_2, e_3) we produce a family of functions $\{\phi_i\}_{i \in I}$, by all the ϕ_i such

$$\phi_i : \{\text{arcs of } \mathcal{E}_1, \mathcal{E}_2\} \rightarrow \{1, \dots, 4\}$$

Such That:

$$\sum_{j=1}^{e_1} \phi_i(\text{arcs of } \mathcal{E}_1) = e_2$$

$$\sum_{j=1}^{e_1} \phi_i(\text{arcs of } \mathcal{E}_2) = e_3.$$

Each ϕ_i counts the number of intersection points of arcs in the sub-arrangement $\{\mathcal{E}_1, \mathcal{E}_2\}$ with \mathcal{E}_3 . Notice that $I = I((e_1, e_2, e_3))$ since for every intersection type we obtain a different family $\{\phi_i\}_{i \in I}$. Let $\bigcup I := \bigcup_{\text{all intersection types}} I(\text{intersection type})$. The set

$$A := \{\phi_j\}_{j \in \bigcup I} \quad (4.1)$$

gives an excessive catalog of cases, because we work with respect to 4.10.

Reduction Of Cases Let

$$f : A \rightarrow F, \phi \mapsto \tau$$

such that A is as in 4.1. The target set F is the set of all possible

$$\tau : \{\text{faces of sub arrangement } \{\mathcal{E}_1, \mathcal{E}_2\}\} \rightarrow \{1, \dots, 5\}.$$

Let $\tau := f(\phi)$ be defined as

$$\tau : \{\text{faces of sub arrangement } \{\mathcal{E}_1, \mathcal{E}_2\}\} \rightarrow \{1, \dots, 5\},$$

$$\text{face} \mapsto \left(\sum_{\text{arc}_j \in \partial \text{face}} \frac{\phi(\text{arc}_j)}{2} \right) + 1$$

The function ϕ counts for each arc of the sub arrangement $\{\mathcal{E}_1, \mathcal{E}_2\}$ the number of intersections with the third ellipse \mathcal{E}_3 . The function τ counts the number of faces of the complete arrangement contained in each face of the sub arrangement $\{\mathcal{E}_1, \mathcal{E}_2\}$.

Let \tilde{A} be the restriction of A on which f is well defined. All the $y \in A - \tilde{A}$ are associated to impossible configurations.

We can further restrict \tilde{A} to \tilde{A}^2 by removing all the ϕ that satisfy these two properties:

1. Exists two arcs of the sub-arrangement $\mathcal{E}_1, \mathcal{E}_2$, arc_1 internal and arc_2 external for which

$$\begin{aligned} \phi(\text{arc}_1) &> 0 \\ \phi(\text{arc}_2) &> 0. \end{aligned} \quad (4.2)$$

2. Do not exist two arcs of the sub-arrangement $\mathcal{E}_1, \mathcal{E}_2$, arc_a internal and arc_b external which bound a common face of $\mathcal{E}_1, \mathcal{E}_2$ which

$$\begin{aligned} \phi(\text{arc}_a) &> 0 \\ \phi(\text{arc}_b) &> 0. \end{aligned} \quad (4.3)$$

If one found more properties that ϕ must satisfy to describe an existing arrangement, proceed from \tilde{A}^n to \tilde{A}^{n+1} until the sequence stabilizes.

Quotient Since the sequence \tilde{A}^n must stabilize, let $\tilde{A}^N := \lim_{n \rightarrow \infty} \tilde{A}^n$. The set \tilde{A}^N can be seen as a catalog of existing topological cases, not unique up to diffeomorphism. Let $\phi \in \tilde{A}^N$. Let $\tau = f(\phi)$. Let

$$T_\phi = \{\tau(\text{most inner face}), \tau(\text{most external face})\}.$$

Let $P(\tilde{A}^N)$ be a partition of \tilde{A}^N such that for all $P \in P(\tilde{A}^N)$,

$$\phi, \psi \in P \iff T_\phi = T_\psi.$$

Let $D(4)$ be the four Dihedral group. There is a natural action of $D(4)$ onto

$$\tilde{f}(P) := \{\tau|_{\text{middle faces}} \mid \tau = f(\phi) \text{ for } \phi \in P\}$$

for each $P \in P(\tilde{A}^N)$.

Partition each $P \in P(\tilde{A}^N)$ via the induced orbits from the former action. Take a representative for each set of this latter partition. This latest quotient eliminates multiple copies of diffeomorphic configurations. At this point the algorithm does not guarantee a complete reduction in a catalog of not equivalent cases.

Remark 4.11. As we pointed out at the beginning of 4.2.1, the combinatorial description we use during this algorithm is not well defined up to real diffeomorphisms. It might be well defined up to projective diffeomorphisms. If we were to implement such a technique now, we would expand the cases recursively in a more local fashion, eliminating problematic configurations and multiple symmetric cases as they arise from the expansion. We would define the added ellipse in the arrangement through a circular path in the facets of the sub arrangement, to encode more topological information into the combinatorial description and produce less impossible configurations from the expansions.

4.2.2 An Approach Using Hyperbolicity

The adjoint curve of a polycon defined by three irreducible conics is a cubic curve. Thus, there are only two possible topologically inequivalent cases for the adjoint curve, if it is not singular.

The topology of a curve of degree three will be one of the following two possibilities:

1. The real curve of the adjoint is connected. In this case, the connected component is a pseudoline and its complement in the real projective plane \mathbb{RP}^2 is *connected*.
2. The real curve of the adjoint has two connected components. In this case it is a hyperbolic cubic, the two connected components are a pseudoline and an oval, while the pseudoline does not divide the projective plane into two connected components, the oval does.

Usage Of Hyperbolicity If we show that the adjoint of a polycon P is hyperbolic and that its oval component has at least one point outside the polycon, the existence of Wachpress coordinates is proven for P . The only possible problem would be to have the oval completely contained inside the interior of the polycon, since the adjoint cannot cross the polycon's boundaries in any point.

The adjoint curve of an arrangement of three conics is a cubic curve, passing through the residual points of the arrangement. Since every polynomial of all the boundaries C_i of the arrangement is an irreducible conic, the residue of the adjoint polynomial over all the C_i will be a linear polynomial. This linear polynomial has simple roots in all the residual points and nowhere else on all the boundaries C_i , thus determining all the sign change of the evaluation of the adjoint polynomial over all the C_i . The adjoint curve has to pass through every residual point and separate the arcs of C_i with mismatching signs. Every small enough real circle around each residual point intersects the adjoint curve in two real points.

Proposition 4.12 (Provable Hyperbolicity). *In 28 of the configurations of three ellipses, the real part of the adjoint curve of any regular polycon P in the configuration is hyperbolic and does not intersect the interior $P_{\geq 0}$.*

Proof. In 28 different topological cases the previous local information of intersection of the adjoint curve with small circles around residual points of the configuration is enough to determine the existence and the position of the oval component of the adjoint. In these cases a triangle formed by arcs of the C_i which has all three sides of the same sign with respect to the adjoint polynomial, lies in the simply connected region of the complement of all the arcs of the conics with the adjoint that assumes the opposite sign. The adjoint curve cannot cross the boundary of this simply connected region, thus it is forced to have the oval component here. □

4.2.3 Problematic Configurations

For 16 topological cases of the arrangement of three ellipses the previous argument does not suffice. It proves, however, the conjecture for all but one polycon up to symmetry of each case.

Remark 4.13 (Five Solved Configurations). Five of the problematic configurations have been solved. These five configurations share the property of having three triangles of sides of sign opposite to the sign of the sides of the polycon.

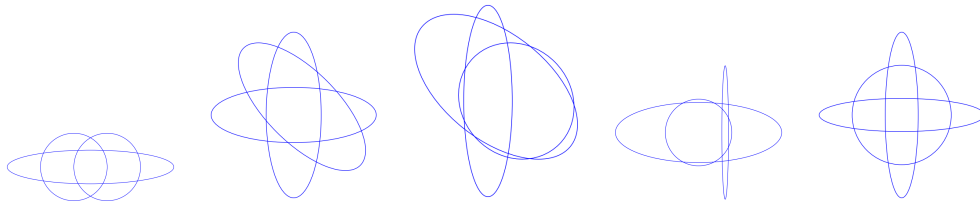


Figure 10: Five Problematic Configurations Solved

The following *key-lemma* is used heavily throughout the whole subsection.

Lemma 4.14 (Key Lemma). *Given P a regular polycon in \mathbb{R}^2 defined by three ellipses. If for every point p of the interior of $P_{\geq 0}$ there is a line passing through p that intersects the adjoint curve A_P outside of $P_{\geq 0}$ at least twice, then the adjoint curve A_P does not intersect the interior of the polycon $P_{\geq 0}$.*

Key Lemma. This is a count of intersection points because the adjoint curve does not intersect the sides of the polycon. So a connected component of the adjoint curve

$A_P(\mathbb{R})$ inside the interior of the polycon $P_{\geq 0}$ would have to be an oval (possibly singular). In the case of a non-singular oval, we choose a point p in its interior and see from the statement that exists a line through p that meets A_P in at least four points. This is not possible since the adjoint is a cubic.

In the singular case, we choose p to be the singular point of A_P inside $P_{\geq 0}$. Then the line from the statement passes through p with multiplicity two and arrive at the same contradiction.

□

Since in the arrangement of three conics the adjoint curve is a cubic, the real topologies, as discussed earlier, are either a pseudoline or an oval and a pseudoline, with possible singularities. If the real adjoint curve A_P has an oval or a singularity outside the polycon, then there cannot be a connected component of A_P strictly contained inside the polycon, implying Wachpress conjecture. Hence, in the following we assume that all the real residual points lie on one connected non-singular pseudoline of the real adjoint curve.

Knowing how the adjoint passes through the residual points, separating elements of the boundaries on which the real evaluation of the adjoint assumes mismatching signs, we have useful restriction on the regions in which we can connect the adjoint. For each region bounded by four arcs of alternating residual sign of the adjoint, we can only connect the real curve of the adjoint along side the arcs of one chosen side.

Lemma 4.15 (Helping Lemma). *If in one of the five configurations above two adjoint curve segments connect along sides of the same sign as the sides of the problematic polycon, the adjoint curve does not intersect $P_{\geq 0}$.*

Helping Lemma. We start the proof by observing that if there is one of the three ellipses that satisfies the following two properties, then, by the 4.14 the proof terminates.

1. The polycon is the interior of the ellipse
2. The pseudoline of the adjoint separates the ellipse into disjoint regions such that one residual point p' on the ellipse lies in a different region than the polycon.

We claim that the statement of 4.14 is satisfied given that these two properties are satisfied by one of the ellipses in the arrangement because for every point p in the polycon, the line segment from p to p' is contained inside the ellipse and must intersect the adjoint at least twice, once at p' and once where the pseudoline separates the ellipse.

To conclude the proof, we must check that for each problematic polycon of the five configurations in 10, when the adjoint connects along side arcs of the same sign as the arcs of the polycon, then the two previously stated properties are satisfied. This is easy to check given the drawings of the considered arrangements and a simple two color coloring of the arcs that arises from the sign of the real evaluation of the adjoint polynomial on the arcs.

□

Following these results, we see that for each of the problematic polycons considered, there are six branches of the pseudoline of the adjoint that leave the arrangement. We call these *tentacles*.

We see that the initial remark does not have a complete proof for now.

Remember that a non-singular pseudoline is the only topological possibility for the adjoint curve outside the polycon which does not prove the conjecture. To conclude the topological picture of the pseudoline in the five cases above, all is left is to distinguish how the six tentacles connect outside the considered arrangements.

The adjoint for the considered cases is a non-singular pseudoline which intersects the line at infinity in one or three points, as tentacles pair up to meet at the infinity line and if the intersections were even we would have an oval outside the polycon. So the number of connected components of the adjoint in the real plane is one or three. This implies that the number of connected components of the adjoint in the union of the three filled ellipses is either one, two or three.

Since in the five problematic configurations of 10 the arcs of the adjoint form a single connected non-singular pseudoline and cannot intersect the ellipses in non-residual points, only neighboring tentacles can be directly connected. By directly connected we mean that the tentacles meet in the real plane, without crossing first the line at infinity. In particular it must be noted that the two neighboring tentacles to the polycon are either directly connected or both meet the infinity line. They cannot be directly connected to their other neighbor, since this would form an oval.

Remark 4.16 (Cases Distinction). As a consequence of the comments above, for the five problematic polycons of 10, we divide the last part of the proof onto three different cases.

The three cases arise from how the six tentacles are connected with each other with particular emphasis to the tentacles nearest the polycon.

1. The two tentacles nearest the polycon are directly connected
2. All six tentacles meet the line at infinity before connecting to any other tentacle
3. The two tentacles nearest the polycon meet the line at infinity and the other tentacles are directly connected.

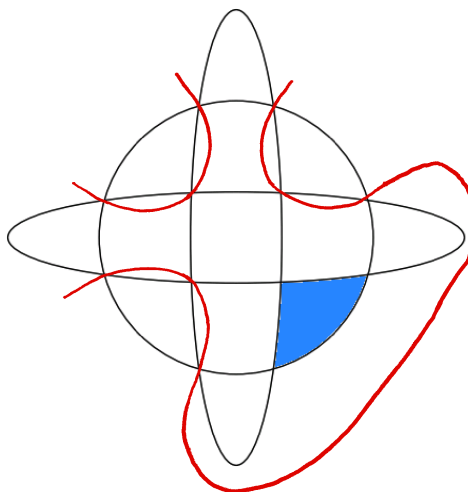


Figure 11: Two tentacles nearest the polycon are directly connected

The Two Tentacles Nearest The Polycon Are Directly Connected. One of these two similar arguments holds for each arrangement in 10:

1. Let p be any point in the interior of the problematic polycon. The polycon is outside the sub-arrangement of two ellipses $\mathcal{E}_1, \mathcal{E}_2$ and inside the third ellipse \mathcal{E}_3 . Let p' be any of the three residual points on the intersection of \mathcal{E}_1 and \mathcal{E}_2 . Consider the line spanned by p and p' and split it into three segments. One from the point at infinity to p' , another from p' to p and the last one from p to infinity. Since p' lies on the boundary of the ellipse \mathcal{E}_1 and \mathcal{E}_2 and p lies outside of the two ellipses, the line segment from p to infinity intersects neither of the two ellipses \mathcal{E}_1 and \mathcal{E}_2 . The line has to leave the polycon via its side of the ellipse \mathcal{E}_3 and then intersect the adjoint curve segment that connects the two tentacles closest to the polycon.

It is important to notice that this inference affirms that for each point p in the interior of the polycon, there is a line passing through it which will intersect the adjoint curve in two points, one in the connection between the two nearest tentacles to the polycon, and one in the residue point p' . By 4.14, the conjecture is proven for the two arrangements considered.

2. The problematic polycon is inside two ellipses, denoted by \mathcal{E}_1 and \mathcal{E}_2 and outside one, namely \mathcal{E}_3 . Let p' be the residual point on the intersection of \mathcal{E}_1 with \mathcal{E}_2 . We consider again the line through p and p' as above. Since both points p and p' are inside the two ellipses \mathcal{E}_1 and \mathcal{E}_2 , the line segment from p to infinity must intersect the boundary of one of the two ellipses exactly once, while the segment from p' to infinity must intersect the boundary of the other ellipse exactly once. In these considered configurations, the residual point p' lies inside the ellipse \mathcal{E}_3 . Since p lies outside of this ellipse, the line segment from p to infinity does not meet \mathcal{E}_3 . This line passing through p and p' leaves the configuration of the three ellipses via one of the two arcs that lie between the two residual points on the boundary of the configuration that are closest to the polycon, which is the same as saying that the line leaves the arrangement between the two tentacles that are nearest the polycon. We see that for each point p in the polycon there is a line that meets a residual point p' which lies on the adjoint curve and that the same line must meet the adjoint curve in the part that directly connects the tentacles nearest the polycon.

All Six Tentacles Meet The Line At Infinity Before Connecting To Any Other Tentacle. In this case, the pseudoline has three connected components in the real plane. Our objective is again to be able to use the 4.14. By contradiction we assume that there is a point p inside the polycon such that every line through p meets the pseudoline of the adjoint in exactly one point.

Consider the complement of the three lines L_1, L_2, L_3 spanned by the point p in the polycon and the three points at infinity where the paired six tentacles of the adjoint meet. This space will be the tessellation of the plane minus the three lines above defined into six pointed cones in p . Half of these cones can be obtained from the other three by a symmetry action around the point p . To avoid having a double intersection of one of the lines L_1, L_2, L_3 with the pseudoline of the adjoint, we must assume that each of the connected component of the adjoint will lay in a cone. Moreover, since the pseudoline has to approach infinity from opposite directions to avoid having singularities, we can assume that each connected component of the pseudoline will lay in

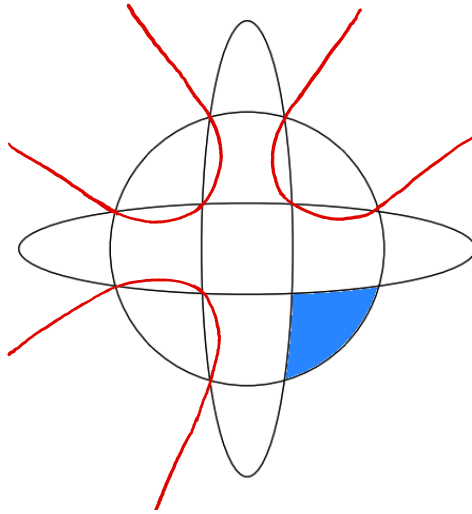


Figure 12: All Tentacles Intersect The Infinity Line Before Any Other Tentacle

every other cone of the tessellation.

For every arrangement which contains a problematic polycon that we are considering, there is an ellipse \mathcal{E} that satisfies these properties.

1. The problematic polycon lays outside of the ellipse \mathcal{E} .
2. Every region bounded by sides of opposite residual sign of the adjoint than the polycon has a side in the ellipse \mathcal{E} .

Remember that by 4.15 the pseudoline of the adjoint connects along side of opposite residual sign than the polycon. Thus, each of the three connected components of the pseudoline of the adjoint has to go around one of the three regions which sides' sign mismatch the polycon.

The ellipse \mathcal{E} must pass through every cone containing a branch of the pseudoline. Since the point p is outside the ellipse \mathcal{E} , the second intersection point of each of the lines L_1, L_2, L_3 with the ellipse \mathcal{E} lies on the same cone's boundary as the first one. Since the pseudoline components are in every other cone, we obtain a contradiction between p not being in \mathcal{E} and the convexity of the filled ellipse \mathcal{E} .

The Two Tentacles Nearest The Polycon Meet The Line At Infinity And Two Other Tentacles Are Directly Connected. Recall that only neighboring tentacles can be connected and that their connection cannot create an oval. For any of the five problematic polycons considered, this leaves only two pairs of tentacles that can be directly connected. To tackle this case, we need to further divide it into three sub-cases.

1. *The last pair of tentacles is also directly connected.*

We show that this assumption would give a contradiction on the number of real inflection points of the adjoint curve.

A real plane curve of degree three, which is non singular has exactly three real

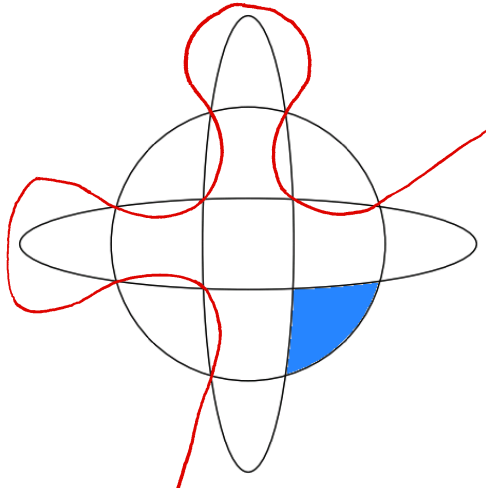


Figure 13: All Tentacles Are Connected Other Than The Nearest To The Polycon.

inflection points [CW03]. To give a lower bound on the inflection that the adjoint curve must have in this situation, we count the minimum number of inflections of the pseudoline component of the adjoint starting from one of the tentacles nearest the polycon. For the same connected component of the adjoint in the real plane, there are four real transitions between the union of the ellipses and its complement. This is enough to see that the lower bound of real inflection points in these configurations is four, which is impossible for a plane cubic.

2. The last pair of tentacles meet the line at infinity. In this case a subdivision follows:
 - *The two tentacles closest to the polycon meet the line at infinity in distinct points.*

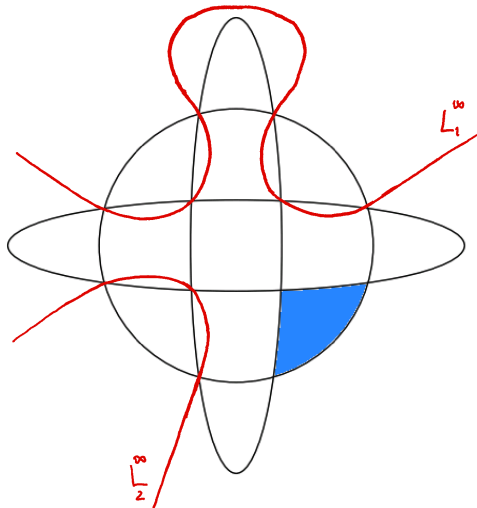


Figure 14: Distinct infinity points for the tentacles nearest the polycon

In this case, the line at infinity meets the adjoint curve in three real points. Since only four out of the six tentacles of the pseudoline segments intersect the line at infinity, there has to be another branch of the adjoint curve which goes to infinity in two points. Due to the non-singularity of the pseudoline, this new branch must be placed in between two of the four tentacles with points at infinity, which are not the nearest to the polycon, otherwise it would either create an oval, a singularity or contradict the assumptions made for this case. We can argue the solution of this case as we did when we supposed the two nearest tentacles to the polycon to be directly connected because the newly found branch of the adjoint works in the argument of this case exactly as the direct connection between the two nearest tentacles to the polycon did back in the other proof.

- *The two tentacles closest to the polycon meet the line at infinity in the same point.*

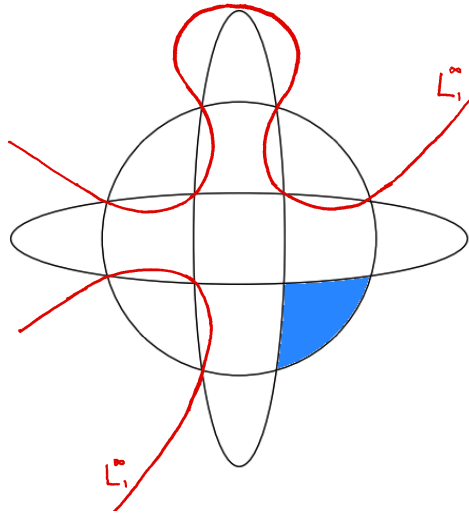


Figure 15: Distinct infinity points for the tentacles nearest the polycon

As in the previous case, there must be another branch of the adjoint that meets the line at infinity twice. This time the branch has to be located between the tentacles nearest the polycon. The argument based on counting the inflection points will work in this case too, considering the two tentacles not nearest the polycon that meet up at infinity to be connected by this newly found branch.

5 Conclusion

An objective of this thesis is to state the current knowledge we have on the veracity of Wachpress Conjecture on polycons which arise from arrangements of three ellipses. A second objective of this thesis is to use a practical question: “Can we define Wachpress coordinates of Polycons of degree six?” to motivate a dive into different sides of real algebraic geometry. In the thesis, we have discussed multiple generalizations of simplexes which begin with polytopes, continue to polycons, then to real rational polypols and terminate with positive geometries.

This thesis is structured in six sections. In the first section we introduce the thesis. In the second section, namely 2, we introduce many preliminary concepts which are necessary throughout the entirety of the thesis. We introduce positive geometries, which balance a good encoding of differential properties and of algebraic ones. These positive geometries are the chosen approach to deal with Wachpress conjecture for polycons that arise from arrangements of three ellipses. The chosen approach follows what has been described in the article [Koh+21]. In the third section 3 we look at what barycentric coordinates on simplexes are and how they are useful, how they can be generalized to polytopes via Wachpress coordinates. The usefulness of barycentric coordinates has always been the drive that motivates the existence of Wachpress coordinates and the study of Wachpress conjecture. Wachpress conjecture is about the existence of such these coordinates for polycons. In the fourth section 4 we move into the analysis of the polycons which arise from arrangements of three ellipses, which are the lowest degree polycons for which the conjecture has not been solved yet. To tackle the conjecture is necessary a catalog of all the topologically stable and distinct configurations. We bring the algorithm of [Koh+21]. Moreover, in this section, we explain the efforts we put into a different approach to the topological catalog. We faced the problem through a combinatorial description, we describe the approach and its problems. I hope to eventually spend more time on this topic and to improve my overall knowledge of real algebraic geometry.

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