UiT The Arctic University of Norway

Department of Mathematics and Statistics

Killing Tensors in Koutras—McIntosh Spacetimes

Wijnand Steneker

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Contents

Introduction

In general relativity we often want to solve the geodesic equations

$$\frac{d^2x^k}{d\lambda^2} + \Gamma^k_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0 \tag{1}$$

in a given spacetime M^n with metric $\mathrm{g} = \mathrm{g}_{ij}dx^i dx^j$. These equations describe the motion of free-falling particles. It turns out that the dynamics of geodesics can be encoded as a Hamiltonian system by passing to the cotangent bundle T^*M. Given canonical coordinates (x, p), we define the Hamiltonian

$$H(x,p) = \frac{1}{2}g^{ij}(x)p_ip_j \quad \text{where} \quad g^{ij} = (g_{ij})^{-1}. \tag{2}$$

Hamilton's equations of motion read

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = \mathrm{g}^{ij}(x)p_j \text{ and } \dot{p}_i = -\frac{\partial H}{\partial x^i} = -\frac{1}{2}\partial_i(\mathrm{g}^{jk})(x)p_jp_k. \tag{3}$$

It is well-known that the solutions of Hamilton's equations project onto geodesics in the spacetime. In order to integrate the geodesic equations it is beneficial to have integrals of motion, quantities which are conserved along the geodesic motion. A function $I \in C^\infty(T^*M)$ is an integral of motion if it satisfies

$$\{H, I\} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial I}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial I}{\partial p_i}\right) = 0. \tag{4}$$

We restrict our attention to integrals of motion that are polynomial in momenta, i.e.

$$I = \sum_{i_1,\ldots,i_n=1}^{n} a^{i_1\cdots i_d}(x)\; p_{i_1}\cdots p_{i_d}. \tag{5}$$

This assumption is not very restrictive in principle: if there is an integral analytic in momenta, then there is a polynomial integral. Also integrals in well-known examples are polynomial in momenta. Polynomial integrals of degree d in momenta are in one-to-one correspondence with Killing d-tensors. A Killing tensor of degree d is a symmetric d-tensor T whose symmetrized covariant-derivative vanishes:

$$\nabla_{(i}T_{j_1\ldots j_d)} = 0. \tag{6}$$

At the order $d = 1$ this correspondence is essentially Noether's theorem: symmetries (Killing vectors) are equivalent to conserved quantities (linear integrals). The Schouten–Nijenhuis bracket for symmetric tensors extends this to a correspondence between Killing tensors and polynomial integrals.

The key observation is that equation (4) defines an overdetermined first order linear partial differential equation on the coefficient functions $a^{i_1\cdots i_d}(x)$. We shall study this PDE using the geometric theory of PDEs. In the paper *Nonexistence of an integral of the 6th degree in momenta for the Zipoy–Voorhees metric* [KM12], Boris Kruglikov and Vladimir Matveev demonstrated for the first time nonexistence of integrals using Cartan's prolongation method. Cartan's prolongation method is an algorithm that allows us to compute the number of linearly independent integrals (equivalently Killing tensors) in a spacetime.

In this thesis we shall implement Cartan's prolongation method in the computer algebra software Maple and apply it to the Koutras–McIntosh metrics. Subcases of the Koutras–McIntosh metric include a *conformally flat pp-wave*,

$$\mathrm{g} = 2dx^3dx^4 + 2f(x^3)\ ((x^1)^2 + (x^2)^2)\ (dx^3)^2 - (dx^1)^2 - (dx^2)^2 \tag{7}$$

and the *Wils metric*,

$$\mathrm{g}_{\mathrm{Wils}} = 2\,x^1dx^3dx^4 - 2x^4dx^1dx^3 + \{2f(x^3)x^1((x^1)^2 + (x^2)^2) - (x^4)^2\}(dx^3)^2 - (dx^1)^2 - (dx^2)^2. \tag{8}$$

Cartan's prolongation method turns the problem into a linear algebraic one. In this way we are able to (rigorously!) prove the (non)-existence of Killing tensors up to degree 4.

The relevance of this result is that Cartan's prolongation method is a *feasible* method to prove the (non-)existence of higher order Killing tensors. There is a connection between existence of Killing 2-tensors and separation of the Hamilton–Jacobi method, which has been demonstrated by [Car68] for the Kerr metric. However, such methods cannot be applied to find Killing tensors of degree $d \geq 3$. The hope is that Cartan's prolongation method can advance the study of higher order Killing tensors.

Structure of thesis

- In Chapter 1 we discuss the required background from the theory of Hamiltonian dynamics and pseudo-Riemannian geometry. The main theorem here is the correspondence between polynomial integrals of the geodesic flow and Killing tensors.
- Chapter 2 is the core of the thesis, here we discuss the geometric theory of PDEs and develop the algorithms (coming from Cartan's prolongation method) that we shall apply in Chapters 4 and 5.
- Chapter 3 is a short detour to general relativity. We discuss the relevance of the Koutras–McIntosh metric, several approaches to classifying spacetimes and Hamilton–Jacobi method for Kerr metric following Carter.
- Chapter 4: Results for conformally flat pp-waves.
- Chapter 5: Results for Wils metrics.

New results

We prove using Cartan's prolongation method that there are no irreducible Killing tensors of degree 3 and 4 in several conformally flat pp-waves, in particular those with constant wave profile ($f(x^3) = \text{constant}$). We also deduce the existence of an irreducible Killing 2-tensor, but this Killing tensor was already obtained by Keane and Tupper [KT10] using the Koutras algorithm.

In Chapter 5 we prove for several Wils metrics the nonexistence of Killing tensors of degree 3 and 4. Moreover, we obtain explicitly the form of the function $f(x^3)$ for which the Wils metric has a Killing vector. We also show that the only Killing 2-tensors in the Wils metric are the metric and the symmetric product of this Killing vector (when it exists). A corollary of these results is that the Wils metric admits no irreducible Killing 2-tensors (apart from the metric).

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Chapter 1

Integrability of Hamiltonian Systems and Geodesic Flow

1.1 Hamiltonian Formalism

We discuss the Hamiltonian formalism of mechanics with an emphasis on symmetries. This section has benefited greatly from the references [Dui04], [Hec13], [Aud04], [Mei00], [Sil08].

Symplectic Manifolds

Definition 1.1.1. A symplectic manifold is a pair (M, ω) consisting of a smooth manifold M together with a closed, nondegenerate two-form $\omega \in \Omega^2(M)$. We refer to ω as the symplectic form on (M, ω). Thus, a symplectic form on M assigns to each point p in M a nondegenerate skew-symmetric bilinear map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ whose exterior derivative vanishes ($d\omega = 0$).

Let (M, ω) and (N, σ) be two symplectic manifolds. A symplectomorphism is a diffeomorphism $\varphi : M \to N$ which pulls back the symplectic form $\sigma \in \Omega^2(N)$ to the symplectic form $\omega \in \Omega^2(M)$, that is, $F^*\sigma = \omega$. In this case (M, ω) and (N, ρ) are said to be symplectomorphic. We denote the group of symplectomorphisms from a symplectic manifold (M, ω) to itself by

$$\mathrm{Symp}(M, \omega) := \{\varphi \in \mathrm{Diff}(M) : \varphi^*\omega = \omega\}.$$

By linear algebraic considerations it follows that any symplectic manifold (M, ω) is necessarily even-dimensional, say $2n$. Since the symplectic form ω is nondegenerate, it follows that the n-fold wedge product $\omega^{\wedge n}$ is a volume form giving M an orientation.

We now describe a symplectic structure on the cotangent bundle of any manifold. Cotangent bundles are important examples of symplectic manifolds, because they represent the phase spaces of mechanical systems. A phase space consists of all possible values for the positions and momenta of the dynamical system in consideration. (For example, if we consider a particle which is constrained to move along a manifold, then its phase space will be the cotangent bundle of that manifold.)

Example 1.1.2 (Cotangent Bundle as a Symplectic Manifold). Let Q be a smooth manifold with cotangent bundle $\pi : T^*Q \to Q$. A point $\xi \in T^*Q$ is a linear map $\xi : T_{\pi(\xi)}Q \to \mathbb{R}$. Note that differentiating the projection π at the point ξ gives a linear map $d\pi_\xi : T_\xi(T^*Q) \to T_{\pi(\xi)}$. Composing these maps allows us to define a one-form $\lambda \in \Omega^1(T^*Q)$ by the formula

$$\lambda : T^*Q \to T^*(T^*Q),\ \xi \mapsto \xi \circ d\pi_\xi = d\pi_\xi^* \xi.$$

Define $\omega \in \Omega^2(T^*Q)$ by $\omega = -d\lambda$. Clearly, ω is a closed as the exterior derivative squares to zero. We now show that ω is nondegenerate. To this end, let $(U, q^1, \ldots, q^n)$ be local coordinates on Q and consider the natural chart on $\pi^{-1}(U) \subseteq T^*Q$ given by

$$\pi^{-1}(U) \to \mathbb{R}^{2n},\ \xi = \sum_{i=1}^n p_i\ dq^i \mapsto (q^1, \ldots, q^n, p_1, \ldots, p_n). \tag{1.1}$$

In these coordinates we have

$$\lambda = \sum_{i=1}^n p_i\ dq^i \tag{1.2}$$

and so

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i. \tag{1.3}$$

In view of this coordinate expression it is readily checked that ω is nondegenerate. Thus the cotangent bundle $(T^*Q, \omega = -d\lambda)$ is an exact symplectic manifold. The forms λ and ω defined above are called the canonical forms on T^*Q. The one-form λ is called the tautological one-form, because $\lambda \in \Omega^1(T^*Q)$ is uniquely characterized by the following property: given any one-form $\alpha \in \Omega^1(Q)$, viewed as a section $\alpha : Q \to T^*Q$, we have

$$\alpha^* \lambda = \alpha. \tag{1.4}$$

We now describe a natural group homomorphism $\mathrm{Lift} : \mathrm{Diff}(Q) \to \mathrm{Symp}(T^*Q, \omega)$. If $f : Q \to Q$ is a diffeomorphism, then its differential $df : TQ \to TQ$ is a vector bundle isomorphism. Dualizing gives a bundle isomorphism $F =: \mathrm{Lift}(f)$, called the cotangent lift of f, given by

$$F := d(f^{-1})^* : T^*Q \to T^*Q,\ \xi \mapsto \xi \circ d(f^{-1})_{f(\pi(\xi))} = d(f^{-1})^*_{f(\pi(\xi))}\xi \tag{1.5}$$

By checking that $F^*\lambda$ satisfies the unique property Equation (1.4), we find that F preserves the canonical one-form and thus also the symplectic form.

It is a natural question to ask whether a symplectic manifold has local invariants. The answer to this question is negative, Gaston Darboux has shown at the end of the nineteenth century that all symplectic manifolds of the same dimension are symplectomorphic.

Theorem 1.1.3 (Darboux). *Let (M, ω) be a $2n$-dimensional symplectic manifold. For every point $p \in M$ there exists a chart $(U, q^1, \ldots, q^n, p_1, \ldots, p_n)$ centered at p on which we have*

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i. \tag{1.6}$$

Local coordinates as in Darboux's Theorem are called Darboux coordinates. We have seen that local coordinates on a manifold Q induce Darboux coordinates on the cotangent bundle (T^*Q, ω).

Poisson Brackets

Suppose (M, ω) is a symplectic manifold. The flow ϕ_V^t of a vector field $V \in \mathfrak{X}(M)$ is said to preserve the symplectic form if $(\phi_V^t)^*\omega = \omega$ is satisfied. Pulling back the symplectic form along the flow and differentiating yields

$$\frac{d}{dt}(\phi_V^t)^*\omega = (\phi_V^t)^*\mathcal{L}_V\omega = (\phi_V^t)^*(di_V\omega + i_V d\omega) = (\phi_V^t)^*(di_V\omega). \tag{1.7}$$

Here we have used the link between flows and Lie derivatives in the first equality, Cartan's homotopy formula in the second equality and closedness of the symplectic form in the final equality. Since the flow ϕ_V^0 at time $t = 0$ equals the identity map the above computation shows that the flow of V preserves the symplectic form if and only if the one-form $i_V\omega$ is closed. A vector field V such that $di_V\omega = 0$ is called a symplectic vector field. By virtue of the Poincaré lemma we have that a closed form is locally exact. Thus, if V is a symplectic vector field we obtain around each point on M a locally defined function $f : U \to \mathbb{R}$ such that

$$i_V\omega = df.$$

A special class of symplectic vector fields are those coming from a *globally* defined function.

Definition 1.1.4. Let (M, ω) be a symplectic manifold and $H : M \to \mathbb{R}$ a smooth function on M. The unique vector field $X_H \in \mathfrak{X}(M)$ satisfying

$$i_{X_H}\omega = dH \tag{1.8}$$

is called the Hamiltonian vector field associated to the Hamiltonian H. (Note that X_H exists by nondegeneracy of ω.) We call (M, ω, X_H) the Hamiltonian system defined by H.

The Lie bracket of symplectic vector fields is Hamiltonian.

Proposition 1.1.5. Let V, W be symplectic vector fields on (M, ω). Then the Lie bracket $[V, W] \in \mathfrak{X}(M)$ is a Hamiltonian vector field with Hamiltonian function $\omega(W, V)$.

Let (M^{2n}, ω, X_H) be a Hamiltonian system. The Hamiltonian formulation of classical mechanics can aid our intuition for Hamiltonian systems as follows. Suppose that (M, ω) represents the phase space of a physical particle with the Hamiltonian H describing the total energy of the particle in question. Hamilton's principle can then be formulated by the assertion that the traversed trajectory of the particle in (momentum) phase space is an integral curve of the Hamiltonian vector field X_H. The Hamiltonian vector field X_H can be written in Darboux coordinates (q, p) as

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{1.9}$$

The condition that a curve $\gamma(t) = (q(t), p(t))$ is an integral curve for X_H now reads as

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \ \dot{p}_i = -\frac{\partial H}{\partial q^i}, \ i = 1, \ldots, n, \tag{1.10}$$

which we recognize as Hamilton's equations of motion.

The notion of a Hamiltonian vector field leads to an important binary operation on the algebra of smooth functions.

Definition 1.1.6. Let (M, ω) be a symplectic manifold. We define the *Poisson bracket* $\{\cdot, \cdot\}$ on $C^\infty(M)$ by the following skew-symmetric bilinear map

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M), \ \{f, g\} := \omega(X_f, X_g) \tag{1.11}$$

It is readily seen that

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = X_g f = -X_f g. \tag{1.12}$$

Thus we can think of the Poisson bracket $\{f, g\}$ as measuring the infinitesimal change of f under the flow of the Hamiltonian vector field X_g.

In Darboux coordinates the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) \tag{1.13}$$

Note that the Darboux coordinates satisfy

$$\{q^i, q^j\} = 0 = \{p_i, p_j\}, \ \{p_i, q^j\} = \delta_i^j, \text{ for all } i, j = 1, \dots, n. \tag{1.14}$$

We summarize the main properties of the Poisson bracket in the following proposition.

Proposition 1.1.7. The algebra of smooth functions $(C^\infty(M), \{\cdot, \cdot\})$ together with the Poisson bracket is a Lie algebra. The map $C^\infty(M) \to \mathfrak{X}(M)$ sending a function to its Hamiltonian vector field is a Lie algebra anti-homomorphism. Moreover, the Poisson bracket satisfies the Leibniz identity.

Suppose that (M, ω, X_H) is a Hamiltonian system and consider the flow ϕ_H^t of X_H. By application of the link between flows and Lie derivatives, we obtain

$$\frac{d}{dt} f \circ \phi_H^t = (\phi_H^t)^* \mathcal{L}_{X_H} f = (\phi_H^t)^* \{f, H\}. \tag{1.15}$$

Thus, f is constant along the integral curves of X_H if and only if $\{f, H\} = 0$ (if and only if H is constant along the integral curves of X_f). This motivates the following definition.

Definition 1.1.8. Let (M, ω, X_H) be a Hamiltonian system. A function $f \in C^\infty(M)$ is an *integral of motion* if f and H Poisson commute, that is, $\{f, H\} = 0$.

Let (M, ω, X_H) be a Hamiltonian system. Skew-symmetry of the Poisson bracket implies that H is an integral of motion and so H is constant on the integral curves of X_H, which we can think of as conservation of total energy.

More generally, let f be an integral of motion and consider the flow ϕ_f^t of its Hamiltonian vector field X_f. For simplicity, we assume that the flow is complete. By the above, we have that

$$H \circ \phi_f^t = H \text{ for all } t \in \mathbb{R}. \tag{1.16}$$

The flow $t \mapsto \phi_f^t$ defines a Lie group action of the additive group $(\mathbb{R}, +)$ on the symplectic manifold (M, ω). Since $\mathcal{L}_{X_f} \omega = 0$ we have that ϕ_f^t is a symplectomorphism for all $t \in \mathbb{R}$, so the action can be encoded as a group homomorphism

$$\mathbb{R} \to \mathrm{Symp}(M, \omega), \ t \mapsto \phi_f^t. \tag{1.17}$$

Since this action preserves both the symplectic structure and the Hamiltonian H, we call this action a "symmetry" of the Hamiltonian system (M, ω, X_H). A crucial feature is that the vector field generated by this $\mathbb{R}$-action is a Hamiltonian vector field, namely the one defined by f. This is an example of *Noether's principle*, which (informally) states that there is a correspondence between symmetries and conserved quantities. In this case we started with a conserved quantity (the integral f) and deduced a symmetry (the $\mathbb{R}$-action).

Symmetries of Hamiltonian Systems

We would like to answer the following question:

$$\boxed{\text{How do we find integrals of a Hamiltonian system?}}$$

We take Noether's principle as our guiding philosophy and look for integrals by considering actions of Lie groups.

Suppose that $\psi : G \to \text{Diff}(M)$ is a smooth action of a Lie group G on a manifold M. The infinitesimal generator assigns to each element X in the Lie algebra $\mathfrak{g}$ of G its fundamental vector field X_M on M, which is defined pointwise by $(X_M) : x \mapsto \frac{d}{dt}\Big|_{t=0} \psi_{\exp(tX)}(x)$. The map $X \mapsto X_M$ defines a Lie algebra anti-homomorphism from $\mathfrak{g}$ to $\mathfrak{X}(M)$, that is, we have $[X, Y]_M = -[X_M, Y_M]$ for all $X, Y \in \mathfrak{X}(M)$.

Let (M, ω, X_H) be a Hamiltonian system, and suppose that $\psi : G \to \text{Symp}(M, \omega)$ is a symplectic action of a Lie group G on which preserves the Hamiltonian H. Let X be an element of the Lie algebra $\mathfrak{g}$ and consider the curve $t \mapsto \exp(tX)$ in G. By invariance of H under the action, we have that

$$H \circ \psi_{\exp(tX)} = H.$$

Differentiating this expression with respect to t gives

$$\mathcal{L}_{X_M} H = dH(X_M) = 0.$$

Supposing that the fundamental vector field X_M is Hamiltonian, then there exists by definition a function, say μ_X, such that

$$i_{X_M}\omega = d\mu_X.$$

It follows that μ_X is an integral of motion, as $\{H, \mu_X\} = \mathcal{L}_{X_M} H = 0$. Ideally, we can find a Hamiltonian function for all $X \in \mathfrak{g}$, in which case we obtain a linear map

$$\mu : \mathfrak{g} \to C^\infty(M), \ X \mapsto \mu_X.$$

By Proposition 1.1.5 we have that $d(\mu_{[X,Y]} - \{\mu_X, \mu_Y\}) = 0$. We shall require that $\mu : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism. For actions of connected Lie groups G, this condition is equivalent to the condition that $\mu : M \to \mathfrak{g}^*$ is equivariant with respect to the coadjoint action on the dual $\mathfrak{g}^*$. Here we identified $\text{Hom}_{\mathbb{R}}(\mathfrak{g}, C^\infty(M)) \cong C^\infty(M, \mathfrak{g}^*)$ to view μ as a map $M \to \mathfrak{g}^*$. This discussion leads to the following definition.

Definition 1.1.9. A symplectic action $\psi : G \to \mathrm{Symp}(M, \omega)$ is a Hamiltonian action if there exists a smooth map $\mu : M \to \mathfrak{g}^*$ satisfying the following properties:

- For every X in the Lie algebra $\mathfrak{g}$ the smooth function $\mu_X = \langle \mu, X \rangle$ is a Hamiltonian for the fundamental vector field $\underline{X}$ on M:

$$i_{\underline{X}}\omega = d\mu_X. \tag{1.18}$$

- The map $\mu : M \to \mathfrak{g}^*$ is equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$:

$$\mu(g \cdot x) = \mathrm{Ad}_g^* \cdot \mu(x), \text{ for all } g \in G, x \in M. \tag{1.19}$$

The map $\mu : M \to \mathfrak{g}^*$ is said to be a *momentum map* for the Hamiltonian action.

We can now make Noether's principle precise.

Theorem 1.1.10 (**Noether**). *Let (M, ω, X_H) be a Hamiltonian system. Suppose that $\psi : G \to Symp(M, \omega)$ is a Hamiltonian action with momentum map $\mu : M \to \mathfrak{g}^*$. Then the Hamiltonian H is invariant under the action of G if and only if the momentum mapping μ is an integral of motion, that is, $\{\mu_X, H\} = 0$ for all $X \in \mathfrak{g}$.*

Example 1.1.11. Let Q be a manifold equipped with a Lie group action $\psi : G \to \mathrm{Diff}(Q)$. The cotangent lift map induces a symplectic action $G \to \mathrm{Symp}(T^*Q, \omega = -d\lambda)$ which preserves the canonical one-form λ:

$$g \cdot \xi := \xi \circ d(\psi_{g^{-1}})_{\psi_g(\pi(\xi))}, \qquad (g \in G, \xi \in T^*Q).$$

This action on $(T^*Q,\ \omega = -d\lambda)$ is Hamiltonian and

$$\mu_X = i_{X_{T^*Q}}\lambda \quad (X \in \mathfrak{g}). \tag{1.20}$$

defines a momentum map. Equivalently, we can write the momentum map as

$$\mu_X(\xi) = \langle \xi, X_Q \rangle = \xi(X_Q)\ (\xi \in T^*Q), \tag{1.21}$$

which in practice is better suited for explicit computations. In cotangent coordinates this momentum map is linear in momenta.

Theorem 1.1.12 (**Symplectic Reduction**). *Let (M, ω, μ) be a Hamiltonian G-space and let $\xi \in \mathfrak{g}^*$. Suppose that the stabilizer G_ξ of the coadjoint action acts freely and properly on the level set $\mu^{-1}(\xi)$. Then the reduced space $M_\xi := \mu^{-1}(\xi)/G_\xi$ is a smooth manifold and there exists a symplectic form ω_ξ on M_ξ satisfying*

$$\pi_\xi^* \omega_\xi = i_\xi^* \omega,$$

where $\iota_\xi : \mu^{-1}(\xi) \hookrightarrow M$ and $\pi_\xi : \mu^{-1}(\xi) \to M_\xi$ denote the inclusion and projection map, respectively.

We can use symplectic reduction to reduce the number of variables in a Hamiltonian system, as follows.

Theorem 1.1.13 (**Reduced Dynamics**). *Let (M, ω, X_H) be a Hamiltonian system and suppose that the flow ϕ_H^t of X_H is complete. Suppose that a Lie group G acts on (M, ω) in a Hamiltonian way with momentum map $\mu : M \to \mathfrak{g}^*$ and preserves the Hamiltonian H. In addition, suppose that $\xi \in \mathfrak{g}^*$ is a regular value of μ and that its stabilizer G_ξ acts freely on the level set $\mu^{-1}(\xi)$. Then H descends to a unique function H_ξ on the symplectic reduced space (M_ξ, ω_ξ) satisfying*

$$H_\xi \circ \pi_\xi = H \circ i_\xi.$$

The vector field $(X_H)|_{\mu^{-1}(\xi)} \in \mathfrak{X}(\mu^{-1}(\xi))$ is well-defined and is π_ξ-related to the Hamiltonian vector field $X_{H_\xi} \in \mathfrak{X}(M_\xi)$.

Under the assumptions of the Reduced Dynamics Theorem, we have for all $t \in \mathbb{R}$, a commutative diagram

$$\begin{tikzcd} \mu^{-1}(\xi) \arrow[r, "\phi_H^t"] \arrow[d, "\pi_\xi"'] & \mu^{-1}(\xi) \arrow[d, "\pi_\xi"] \\ M_\xi \arrow[r, "\phi_{H_\xi^t}"'] & M_\xi. \end{tikzcd}$$

Here $\phi_{H_\xi}^t$ denotes the flow of the Hamiltonian vector field $X_{H_\xi} \in \mathfrak{X}(M_\xi)$. We would like to solve the dynamics on the reduced space (M_ξ, ω_ξ), which entails computing the flow $\phi_{H_\xi}^t$ explicitly. If we know the action of G_ξ explicitly, we can construct the flow ϕ_H^t on $\mu^{-1}(\xi)$ using the flow of the reduced Hamiltonian (See [AM08, Section 4.3]). Following [Mei00], [AKN07], [GPS02] we outline such a procedure for the Kepler problem.

Example 1.1.14 (**Kepler Problem**). Let $Q = \mathbb{R}^2(q^1, q^2)$ be the configuration space and consider its cotangent bundle $T^*\mathbb{R}^2 = \mathbb{R}^2(q^1, q^2) \times \mathbb{R}^2(p_1, p_2)$ with the standard symplectic form $\omega = dq^1 \wedge dp_1 + dq^2 \wedge dp_2 \in \Omega^2(T^*Q)$. The Hamiltonian of the *Kepler problem* is given by

$$H(q^1, q^2, p_1, p_2) = \frac{p_1^2 + p_2^2}{2} + V(q), \tag{1.22}$$

with potential $V(q) = \frac{-1}{\|q\|}$. This Hamiltonian describes the two-body problem where the force between the two bodies is an inverse square law. A physical example of a Kepler problem is the motion of the Earth with respect to the Sun under the effect of gravity. We derive the Kepler orbits with the point of view of symplectic reduction.

The circle group $\mathrm{SO}(2) = \mathbb{S}^1$ acts on the configuration space Q by rotations. By using the cotangent lift we obtain a symplectic action of SO(2) on the cotangent bundle T^*Q which preserves the Hamiltonian. This action is Hamiltonian with momentum map

$$\mu : T^*Q \to \mathbb{R} \cong \mathfrak{so}(2)^*, \; \mu(q^1, q^2, p_1, p_2) = p_2 q^1 - q^2 p_1 = (q^1, q^2, 0) \times (p_1, p_2, 0), \tag{1.23}$$

where $(\cdot \times \cdot)$ denotes the cross-product. We recognize this momentum map as the *angular momentum.* In view of the rotational symmetry, we change to polar coordinates on Q:

$$q^1 = r\cos(\varphi), \; q^2 = r\sin(\varphi). \tag{1.24}$$

Using the transformation law for one-forms, we find the following relation for the momenta p_r, p_φ conjugate to r, φ:

$$p_1 = p_r \cos\varphi - \frac{p_\varphi}{r}\sin\varphi, \; p_2 = p_r \sin\varphi + \frac{p_\varphi}{r}\cos\varphi. \tag{1.25}$$

The symplectic form in polar coordinates equals

$$\omega = dr \wedge dp_r + d\varphi \wedge dp_\varphi \in \Omega^2(T^*Q). \tag{1.26}$$

and the Hamiltonian reads

$$H = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} - \frac{1}{r} \in C^\infty(T^*Q). \tag{1.27}$$

The momentum map is given by $\mu = p_\varphi = r^2\dot{\varphi}$, which is indeed a conserved quantity of the system because the Hamiltonian does not depend on φ. Let $\xi \neq 0$ be a nonzero value. It is readily seen that SO(2) acts freely on the level set $\mu^{-1}(\xi) = \{p_\varphi = \xi\} \subseteq T^*Q$. Let $i_\xi : \mu^{-1}(\xi) \hookrightarrow T^*Q$ denote the inclusion map. Since the angular momentum p_φ is conserved (on $\mu^{-1}(\xi)$), we have $dp_\varphi = 0$ and it follows that

$$i_\xi^*\omega = dr \wedge dp_r \in \Omega^2(\mu^{-1}(\xi)). \tag{1.28}$$

The two-form $i_\xi^*\omega$ is of constant rank and its kernel is the tangent bundle of the SO(2)-orbit. It follows that the reduced phase space $M_\xi = \mu^{-1}(\xi))/\mathrm{SO}(2)$ is diffeomorphic to $\mathbb{R}_{>0}(r) \times \mathbb{R}(p_r)$ and under this identification the reduced symplectic form equals $\omega_\xi = dr \wedge dp_r \in \Omega^2(M_\xi)$. The reduced Hamiltonian is given by

$$H_\xi(r, p_r) = \frac{p_r^2}{2} + \left(\frac{\xi^2}{2r^2} - \frac{1}{r}\right) = \frac{p_r^2}{2} + V_\xi(r) \in C^\infty(M_\xi). \tag{1.29}$$

Here we think of the second term $V_\xi(r) = \left(\frac{\xi^2}{2r^2} - \frac{1}{r}\right)$ as a *reduced potential.*
Hamilton's equations of motion on the reduced phase space (M_ξ, ω_ξ) imply

$$\dot{r} = \frac{\partial H_\xi}{\partial p_r} = p_r \tag{1.30}$$

On a hypersurface $H_\xi^{-1}(E) \subseteq M_\xi$ of constant energy E we obtain

$$\dot{r} = \sqrt{2(E - V_\xi(r))} \tag{1.31}$$

Lifting this up to $\mu^{-1}(\xi)$ gives

$$\frac{dr}{d\varphi} = \dot{r}\frac{dt}{d\varphi} = \dot{r}\frac{r^2}{\xi} = \frac{r^2\sqrt{2(E - V_\xi(r))}}{\xi}. \tag{1.32}$$

Here we have used that $\xi = r^2\dot{\varphi}$ on $\mu^{-1}(\xi)$. Applying separation of variables to the above differential equation yields

$$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{\xi\ dr}{r^2\sqrt{2(E - V_\xi(r))}}. \tag{1.33}$$

This integral can be solved by means of the substitution $u = \frac{1}{r}$ and after some algebraic manipulations we get the Kepler orbits:

$$r(\varphi) = \frac{\xi^2}{1 + e\cos(\varphi - \varphi_0)} \tag{1.34}$$

where $e := \sqrt{1 + 2E\xi^2}$ is the eccentricity of the orbit.

1.2 Integrable Systems and the Liouville–Mineur–Arnold Theorem

In this section we consider integrable systems. Informally, these are Hamiltonian systems with "enough" symmetries. To an integrable system one can associate action-angle coordinate in which Hamilton's equations can be readily integrated. The exposition follows Zung's paper *A conceptual approach to action-angle variables* [Zun18] and Duistermaat's *On global action-angle variables* [Dui80]. The philosophy is that there is a correspondence between integrable systems with compact fibers and Hamiltonian torus spaces. On the one hand, a Hamiltonian torus space gives an integrable system by its momentum map. On the other hand, an integrable system has a semilocal (i.e. in a neighborhood of a fiber) torus action which is Hamiltonian. The benefit of this approach is that we can construct the action-angle coordinates as a straightforward consequence from this Hamiltonian torus action.

Definition 1.2.1. Let (M^{2n}, ω) be a symplectic manifold. An *integrable system* is a collection of n Poisson commuting smooth functions $(f_1, \ldots, f_n)$ on (M^{2n}, ω) whose differentials $df_1, \ldots, df_n \in \Omega^1(M)$ are linearly independent on a dense open subset. A Hamiltonian system (M, ω, X_H) is called *integrable* if there is an integrable system $F = (f_1, \ldots, f_n)$ on M with $f_1 = H$.

Let $F = (f_1, \ldots, f_n) : (M^{2n}, \omega) \to \mathbb{R}^n$ be an integrable system. By Sard's theorem, any integrable system has a regular value in its image. Let $c \in F(M)$ be a regular value of F. We show that the level set

$$F^{-1}(c) = f_1^{-1}(c_1) \cap \cdots \cap f_n^{-1}(c_n) \tag{1.35}$$

is a Lagrangian submanifold of M. By the regular value theorem we have that $F^{-1}(c)$ is a submanifold of M of dimension n with tangent bundle given by $T(F^{-1}(c)) = \ker dF|_{F^{-1}(c)}$. We denote by ϕ_i^t the flow of the Hamiltonian vector field X_{f_i}. By assumption, we have that

$$\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0, \qquad \text{for all } i, j = 1, \ldots, n, \tag{1.36}$$

so the flow ϕ_i^t preserves the level set $F^{-1}(c)$. In particular, the Hamiltonian vector fields X_{f_i} are tangent to the level set $F^{-1}(c)$. By linear independence of the differentials $df_1, \ldots, df_n$ on $F^{-1}(c)$, it follows that the tangent bundle $F^{-1}(c)$ is spanned by the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$. This proves together with (1.36) that $F^{-1}(c)$ is Lagrangian.

Now suppose in addition that $F^{-1}(c)$ is compact and connected. The flows ϕ_i^t are complete on $F^{-1}(c)$ by compactness. Since

$$[X_{f_i}, X_{f_j}] = -X_{\{f_i, f_j\}} = 0 \tag{1.37}$$

the flows $\phi_i^{t_i}, \phi_j^{t_j}$ commute. Consequently, the *joint flow* defined by

$$\mathbb{R}^n \times F^{-1}(c) \to F^{-1}(c), \; ((t_1, \ldots, t_n), x) \mapsto \phi_1^{t_1} \circ \cdots \circ \phi_n^{t_n}(x). \tag{1.38}$$

defines a smooth action of $(\mathbb{R}^n, +)$ on the fiber $F^{-1}(c)$. By linear independence of $X_{f_1}, \ldots, X_{f_n}$ the orbit map $\Phi_x : \mathbb{R}^n \to F^{-1}(c)$ is a local diffeomorphism. This means that the isotropy subgroup $\mathbb{R}^n_x$ of the joint flow through a point $x \in F^{-1}(c)$ is discrete, and so the orbit is open in $F^{-1}(c)$. In turn the connectedness of the fiber implies that the action is transitive. Hence we obtain a diffeomorphism from $\mathbb{R}^n/\mathbb{R}^n_x \cong F^{-1}(c)$. The fiber is compact, so the isotropy subgroup is isomorphic to the lattice $\mathbb{Z}^n$. We conclude that the fiber is diffeomorphic to the n-torus $\mathbb{T}^n$ and is equipped with a *free* $\mathbb{T}^n$-action induced by the joint flow. Liouville's theorem extends this action to a tubular neighborhood of the fiber ([Zun18], [Aud04, p.91]).

Theorem 1.2.2 (Liouville). *Let $F = (f_1, \dots, f_n)$ be an integrable system on a symplectic manifold (M^{2n}, ω). Suppose that $c \in F(M)$ is a regular value and that the level set $T := F^{-1}(c)$ is compact and connected. There exists a tubular neighborhood $U(T)$ of T on which the joint flow induces a free torus action*

$$\Phi : \mathbb{T}^n \times U(T) \to U(T), \tag{1.39}$$

satisfying the following properties:

- *The orbits of the torus action are regular level sets of F.*
- *There exists a diffeomorphism*

$$\chi : U(T) \to \mathbb{D}^n \times \mathbb{T}^n \tag{1.40}$$

which maps each orbit in $U(T)$ diffeomorphically onto a corresponding torus $\{p\} \times \mathbb{T}^n$ in the image.

Accordingly, we call the level set T a Liouville torus, the action Φ the Liouville torus action and the diffeomorphism χ Liouville coordinates.

It is a nice exercise in de Rham cohomology to show that the Liouville torus action on $U(T)$ is Hamiltonian [Aud04, p.93]. Using the momentum map one can readily obtain the Liouville–Mineur–Arnold theorem cf [Zun18], [Aud04, p.96]. The classical proofs can be found in [Arn13], [Min37], [Lio55].

Theorem 1.2.3 (Liouville–Mineur–Arnold). *Let $F = (f_1, \dots, f_n) : M \to \mathbb{R}^n$ be an integrable system on a symplectic manifold (M^{2n}, ω). Suppose that $c \in F(M)$ is a regular value of F and that the level set $T := F^{-1}(c)$ is compact and connected. There exists a neighborhood $U(T)$ of the Liouville torus T and a symplectomorphism*

$$\Psi : (U(T),\ \omega) \to (\mathbb{D}^n \times \mathbb{T}^n,\ \sum_{i=1}^{n} d\theta^i \wedge dx^i) \qquad (x^i \in \mathbb{D}^n, \theta^i \in \mathbb{T}^n)$$

such that the coordinate representation $F \circ \Psi^{-1} : \mathbb{T}^n \times \mathbb{D}^n \to \mathbb{R}^n$ only depends on $x \in \mathbb{D}^n$.

Definition 1.2.4. The Darboux coordinates $\Psi = (I_1, \dots, I_n, \varphi_1, \dots, \varphi_n)$ constructed in the Liouville–Mineur–Arnold Theorem are called *action-angle variables*.

Action-angle coordinates allow us to explicitly integrate the solutions of a Hamiltonian system in a neighborhood of a Liouville torus. Indeed, let $F = (f_1, \dots, f_n)$ be an integrable system. In action-angle coordinates the Hamiltonian f_1 only depends on the action coordinates $(I_1, \dots, I_n)$, so Hamilton's equations read

$$\dot{I}_i = -\frac{\partial f_1}{\partial \varphi_i} = 0 \text{ and } \dot{\varphi}_i = \frac{\partial f_1}{\partial I_i} = c_i(I_1, \dots, I_n). \tag{1.41}$$

Integrating these equations yields

$$I_i(t) = I_i(0) \text{ and } \varphi_i(t) = c_i(I_1(0), \dots, I_n(0)) \cdot t + \varphi_i(0), \tag{1.42}$$

which shows that the angle variables are linear in time.

1.3 Geodesic Flow

This section serves to introduce our main object of study, which are geodesic flows of pseudo-Riemannian manifolds. As we shall see, the geodesic flow allows us to study geodesic motion from a Hamiltonian perspective. By applying tools from Hamiltonian dynamics we can study the behavior of a geodesic flow. We can pose the question if a geodesic flow is integrable or admits chaotic behavior.

1.3.1 Basics of Pseudo-Riemannian Geometry

Definition 1.3.1. Let M be a smooth manifold. A *metric* on M is a nondegenerate symmetric $(0,2)$-tensor field g on M of constant signature. A *pseudo-Riemannian manifold* is a pair (M, g) consisting of a smooth manifold M together with a metric $\mathrm{g} \in \mathcal{T}_2^0(M)$. A pseudo-Riemannian manifold whose metric has signature $(-,+,\ldots,+)$ is called a *Lorentzian manifold.*

Let (M, g_M) and (N, g_N) be pseudo-Riemannian manifolds. An *isometry* from M to N is a diffeomorphism $\varphi : M \to N$ which pulls back the metric g_N to the metric g_M, that is, $\varphi^*\mathrm{g}_N = \mathrm{g}_M$. In this case, the pseudo-Riemannian manifolds (M, g_M) and (N, g_N) are said to be *isometric.*

We denote the group of isometries from a pseudo-Riemannian manifold (M, g) to itself by

$$\mathrm{Iso}(M, \mathrm{g}) := \{\varphi \in \mathrm{Diff}(M) : \varphi^*\mathrm{g} = \mathrm{g}\}. \tag{1.43}$$

The isometry group describes the symmetries of the pseudo-Riemannian manifold.

On a pseudo-Riemannian manifold (M, g) we can use a local frame $(E_1, \ldots, E_n)$ of the tangent bundle TM with dual coframe $(\sigma^1, \ldots, \sigma^n)$ to write the metric locally as

$$g = g_{ij}\ \sigma^i \otimes \sigma^j.$$

Typically, we use the coordinate vector fields of some smooth chart as a local frame or construct an orthonormal frame using the Gram-Schmidt algorithm. As is customary, we employ the Einstein summation convention which means that summation over indices that appear both up and down is implied. In view of the symmetry $g_{ij} = g_{ji}$, we obtain $\frac{n(n+1)}{2}$ functions g_{ij}. On the domain of the frame the metric acts on tangent vectors $v = v^i E_i, w = w^j E_j$ by the formula

$$g(v, w) = g_{ij}\sigma^i(v)\sigma^j(w) = g_{ij}v^i w^j. \tag{1.44}$$

Similarly, a tensor (r, s)-tensor $T \in \mathcal{T}_s^r(M)$ is written as

$$T = T^{i_1\cdots i_r}_{j_1\cdots j_s} E_{i_1} \otimes \cdots \otimes E_{i_r} \otimes \sigma^{j_1} \otimes \cdots \otimes \sigma^{j_s}. \tag{1.45}$$

so that its component functions equal

$$T^{i_1\cdots i_r}_{j_1\cdots j_s} = T(\sigma^{i_1}, \ldots, \sigma^{i_r}, E_{j_1}, \ldots, E_{j_s}). \tag{1.46}$$

Since the metric is nondegenerate the assignment $X \mapsto \mathrm{g}(X, \cdot)$ defines an isomorphism from the space of vector fields $\mathfrak{X}(M)$ to the space of one-forms $\Omega^1(M)$. This isomorphism is explicitly given by declaring

$$(\cdot)^\flat : E_i \mapsto \mathrm{g}_{ij}\sigma^j \tag{1.47}$$

and requiring it to be $C^\infty(M)$-linear. If we denote by g^{ij} the entries of the inverse matrix $(g_{ij})^{-1}$ the inverse of the above isomorphism equals

$$(\cdot)^\sharp : \sigma^j \mapsto \mathrm{g}^{ij} E_i. \tag{1.48}$$

Using Equation (1.47) and Equation (1.48) we can change the tensor (r,s)-tensor T into a $(r-k, s+k)$-tensor $(r-k, s+k \geq 0)$, as follows. For simplicitly, we apply this process of *type change* to turn T into a $(r-1, s+1)$-tensor. Given $(r-1)$ one-forms $\omega^1, \ldots, \omega^{r-1}$ and $(s+1)$ vector fields $X_1, \ldots, X_{s+1}$, we convert the vector field X_1 into a one-form using the map $(\cdot)^\flat$ and subsequently apply T. In formula, we have

$$(\omega^1, \ldots, \omega^{r-1}, X_1, \ldots, X_{s+1}) \mapsto T(\omega^1, \ldots, \omega^{r-1}, X_1^\flat, X_2, \ldots, X_{s+1}). \tag{1.49}$$

We also denote this type-changed tensor by the symbol T. The components of T after type change are related to the original components by

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_s} = g_{j_1 k} T^{i_1 \cdots i_{r-1} k}_{j_2 \cdots j_{s+1}}. \tag{1.50}$$

Given a $(0,n)$-tensor T, we can use the symmetric group $\mathcal{S}_n$ to symmetrize and skew-symmetrize T. In tensor notation, *symmetrizatio*n is denoted by round brackets

$$T_{(j_1 \cdots j_n)} := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} T_{\sigma(i_1) \cdots \sigma(i_n)} \tag{1.51}$$

and *skew-symmetrization* is denoted by square brackets

$$T_{[j_1 \cdots j_n]} := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \operatorname{sign}(\sigma)\ T_{\sigma(i_1) \cdots \sigma(i_n)}. \tag{1.52}$$

We call $T_{(j_1 \cdots j_n)}$ and $T_{[j_1 \cdots j_n]}$ the *symmetric part* and *skew-symmetric part* of T, respectively. Note that in this notation a differential n-form is a $(0,n)$-tensor T which satisfies $T_{[j_1 \cdots j_n]} = T_{j_1 \cdots j_n}$.

In order to define the acceleration of a curve in a manifold we would like to differentiate the velocity vector field defined by the curve, just as in Euclidean space. However, a problem with this approach is that we cannot subtract two tangent vectors which live in different tangent spaces. Introducing a *connection* in the tangent bundle allows us to resolve this problem. A connection amounts to a method of taking derivatives of vector fields (or more generally, derivatives of sections of vector bundles). We need the following theorem, which asserts that on a pseudo-Riemannian manifold there is a canonical way of choosing a connection after requiring that the connection is compatible with the metric and Lie bracket.

Theorem 1.3.2 (**Fundamental Theorem of Pseudo-Riemannian Geometry**). *Let (M,g) be a pseudo-Riemannian manifold. There exists a unique $\mathbb{R}$-bilinear map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ (X,Y) \mapsto \nabla_X Y \tag{1.53}$$

satisfying the following properties:

- *The assignment $X \mapsto \nabla_X Y$ is tensorial:*

$$\nabla_{fX} Y = f \nabla_X Y, \tag{1.54}$$

  for all $f \in C^\infty(M)$.

- *The assignment $Y \mapsto \nabla_X Y$ satisfies the Leibniz rule:*

$$\nabla_X(fY) = f \cdot \nabla_X Y + X(f) \cdot Y, \tag{1.55}$$

for all $f \in C^\infty(M)$.

- *The map ∇ is torsion-free:*

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{1.56}$$

- *The map ∇ is compatible with the metric:*

$$\mathcal{L}_Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \tag{1.57}$$

*We call ∇ the **Levi-Civita connection** associated to the metric g and $\nabla_X Y$ is said to be the **covariant derivative** of Y in the direction X.*

Let (M, g) be a pseudo-Riemannian manifold, and consider the (Levi-Civita) connection ∇ on M. Since the connection is tensorial, the covariant derivative $\nabla_v Y$ of $Y \in \mathfrak{X}(M)$ in the direction of a tangent vector $v \in TM$ is well-defined. Given a local frame $(E_1, \ldots, E_n)$ of the tangent bundle TM, the functions Γ^k_{ij} defined by the formula

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k. \tag{1.58}$$

are called the Christoffel symbols of the connection with respect to the chosen frame. If the vector fields of the local frame commute, the vanishing of the torsion of ∇ is equivalent to the Christoffel symbols Γ^k_{ij} being symmetric in the lower two indices. The covariant derivative $\nabla_X Y$ can be computed in the local frame as

$$\nabla_X Y = (X^i \mathcal{L}_{E_i}(Y^k) + \Gamma^k_{ij} X^i Y^j)\ E_k. \tag{1.59}$$

Here we have used properties Equation (1.54), Equation (1.55) of the connection together with the definition of the Christoffel symbols. The local expression of $\nabla_X Y$ shows that the value of $\nabla_X Y$ at a point only depends on the values of Y along a curve passing through that point. This motivates the following definition.

Definition 1.3.3. Let (M^n, g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇. A curve γ in M is called a *geodesic* if the covariant derivative of the velocity vector field $\dot{\gamma}$ along itself vanishes, that is,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0. \tag{1.60}$$

If $(x^1, \ldots, x^n)$ are coordinates, then the condition for $\gamma : t \mapsto (x^1(t), \ldots, x^n(t))$ to be a geodesic becomes

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0\ (1 \leq i, j, k \leq n). \tag{1.61}$$

We call either of the above equations the *geodesic equations*.

We now define the covariant derivative of any tensor on M along a vector field X. For $f \in C^\infty(M)$ we let $\nabla_X f := \mathcal{L}_X f$ be the Lie derivative. There exists a unique tensor derivation ∇_X on M which restricts to the Lie derivative on functions and the usual covariant derivative on vector fields (cf [O'N83, p.45]). Indeed, the covariant derivative $\nabla_X T$ of a tensor T must necessarily be given by the formula

$$\begin{aligned}(\nabla_X T)(\omega^1,\dots,\omega^r,Y_1,\dots,Y_s) = \nabla_X(T(\omega^1,\dots,\omega^r,Y_1,\dots,Y_s)))\\ -\sum_{i=1}^{n} T(\omega^1,\dots,\omega^r,Y_1,\dots,\nabla_X Y_i,\dots,Y_r)\\ -\sum_{j=1}^{n} T(\omega^1,\dots,\nabla_X\omega^j,\dots,\omega^r,Y_1,\dots,Y_s).\end{aligned} \tag{1.62}$$

We emphasize that the first term on the right hand side is a Lie derivative.

Let $(E_1,\dots,E_n)$ be a local frame of (M,g) with dual coframe $(\sigma^1,\dots,\sigma^n)$. We write the covariant derivative of a tensor $T\in\mathcal{T}^r_s(M^n)$ as

$$\nabla T = \nabla_j T^{i_1\cdots i_r}_{j_1\cdots j_s}\ \sigma^j\otimes E_{i_1}\otimes\cdots\otimes E_{i_r}\otimes\sigma^{j_1}\otimes\cdots\otimes\sigma^{j_s} \tag{1.63}$$

where the component functions are defined by

$$\nabla_j T^{i_1\cdots i_r}_{j_1\cdots j_s} := (\nabla_{E_j}T)(\sigma^{i_1},\dots,\sigma^{i_r},E_{j_1},\dots,E_{j_s}). \tag{1.64}$$

1.3.2 Geodesic Flow and Noether's Theorem

Definition 1.3.4 (Geodesic Flow). Let (M,g) be a pseudo-Riemannian manifold. The Hamiltonian system on the cotangent bundle (T^*M,ω) defined by

$$H(q,p)=\frac{1}{2}g^{ij}(q)\ p_ip_j\in C^\infty(T^*M) \tag{1.65}$$

is called the *geodesic flow* of the metric g. Note that H can be written in a coordinate-independent way as $H:\xi\mapsto\frac{1}{2}\mathrm{g}(\xi^\sharp,\xi^\sharp)$.

The following proposition explains why it is called the geodesic flow.

Proposition 1.3.5. The geodesic flow of a pseudo-Riemannian manifold (M^n,g) projects onto geodesics. This means that if γ is an integral curve of the Hamiltonian vector field X_H defined by $H=\frac{1}{2}\mathrm{g}^{ij}p_ip_j$ and $\pi:T^*M\to M$ is the canonical projection, then $\pi\circ\gamma$ is a geodesic.

We wish to determine conserved quantities of the geodesic flow in a systematic manner. The isometries represent symmetries in the dynamics of geodesics. Noether's theorem serves as a computational tool to find integrals of motion of the geodesic flow. We start by looking at the infinitesimal notion of an isometry.

Definition 1.3.6 (Killing Vector Field). Let (M,g) be a pseudo-Riemannnian manifold with Levi-Civita connection ∇. A vector field $X\in\mathfrak{X}(M)$ is said to be a *Killing field* if its corresponding flow preserves the metric, that is,

$$\mathcal{L}_X\mathrm{g}=0. \tag{1.66}$$

Since the Levi-Civita connection is compatible with the metric, the condition for a vector field X to be a Killing field becomes the *Killing equation*

$$\nabla_{(i}X_{j)}=\frac{1}{2}(\nabla_iX_j+\nabla_jX_i)=0. \tag{1.67}$$

We denote by $\mathfrak{iso}(M,\mathrm{g})$ the Lie algebra of Killing fields.

It turns out that the isometry group $\mathrm{Iso}(M, \mathrm{g})$ is actually a finite-dimensional Lie group [Pet16, Theorem 8.1.6] and that its Lie algebra can be identified with $\mathfrak{iso}(M, \mathrm{g})$. The exponential map of the isometry group is given by the "flow at $t = 1$" map, $\exp : X \mapsto \phi^1_X$ whenever defined.

Clearly, we have a Lie group action of the isometry group on the pseudo-Riemannian manifold (M, g)

$$\mathrm{Iso}(M, \mathrm{g}) \times M \to M, \ (\varphi, p) \mapsto \varphi(p). \tag{1.68}$$

Following example 1.1.11 , this action lifts to a symplectic action of the isometry group on the cotangent bundle T^*M which leaves the Hamiltonian $H = \mathrm{g}^{ij} p_i p_j$ invariant. This action is Hamiltonian with momentum map given by

$$\mu : \mathfrak{iso}(M, \mathrm{g}) \to C^\infty(T^*M), \ \mu_X(\xi) = \xi(X). \tag{1.69}$$

By Noether's theorem, the momentum map is a conserved quantity of the geodesic flow. In cotangent coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ the momentum map equals $\mu_X = X^i p_i$.

Definition 1.3.7. Let (T^*M, ω, X_H) be a Hamiltonian system on the cotangent bundle defined by a Hamiltonian H, and let $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ denote cotangent coordinates. A *polynomial integral* of degree d is an integral $I : T^*M \to \mathbb{R}$ whose expression is a homogeneous polynomial of degree d in momenta:

$$I = \sum_{i_1, \ldots, i_n = 1}^{n} a^{i_1 \cdots i_d}(q^1, \ldots, q^n) \ p_{i_1} \cdots p_{i_d} \tag{1.70}$$

The smooth functions $a_{i_1 \cdots i_n}$ may only depend on the position coordinates $(q^1, \ldots, q^n)$. Polynomial integrals of degree 1,2,3,4,. . . are called linear, quadratic, cubic and quartic integrals, etc.

For example, the Hamiltonian of the geodesic flow is a quadratic integral.

Proposition 1.3.8. Consider the geodesic flow of a pseudo-Riemannian manifold (M, g). Then the map

$$\mathfrak{iso}(M, \mathrm{g}) \to (\{\text{linear integrals}\}, \{\cdot, \cdot\}), \ X \mapsto X^i p_i \tag{1.71}$$

is a Lie algebra isomorphism.

Proof. This map is a Lie algebra homomorphism being the corestriction of the momentum map (which is a Lie algebra homomorphism). Clearly, the map in consideration is injective. To prove surjectivity, let $I := a^i(q^1, \ldots, q^n) \ p_i$ be a linear integral of the geodesic flow. We must show that $Z := a^i(q^1, \ldots, q^n)\partial_{q^i}$ is a Killing field. The condition that Z is a Killing field is equivalent to the assertion that the Lie derivative of the inverse metric vanishes, in formula,

$$\mathcal{L}_Z(\mathrm{g}^{ij} \ \partial_i \otimes \partial_j) = \left(\mathcal{L}_Z(\mathrm{g}^{ij}) - \mathrm{g}^{jk}\partial_k(a^i) - \mathrm{g}^{ik}\partial_k(a^j)\right) \ \partial_i \otimes \partial_j = 0. \tag{1.72}$$

By assumption, we have that $\{H, I\} = 0$. By applying the Leibniz rule to $H = \mathrm{g}^{ij} p_i p_j$ and $I = a^i p_i$ repeatedly, we obtain

$$\begin{aligned} 0 = \{H, I\} &= \{\mathrm{g}^{ij} \ p_i p_j, a^k p_k\} \\ &= -\mathrm{g}^{ij} p_i a^k \{p_j, p_k\} + \mathrm{g}^{ij} p_i p_k \{p_j, a^k\} \\ &- \mathrm{g}^{ij} p_j a^k \{p_i, p_k\} + \mathrm{g}^{ij} p_j p_k \{p_i, a_k\} \\ &- p_i p_j p_k \{\mathrm{g}^{ij}, a^k\} + p_i p_j a^k \{\mathrm{g}^{ij}, p_k\}. \end{aligned} \tag{1.73}$$

Using that the momenta variables Poisson commute together with $\{g^{ij}, a^k\} = 0$, Equation (1.73) reduces to

$$0 = -\mathrm{g}^{ij}\partial_j(a^k)p_i p_k - \mathrm{g}^{ij}\partial_i(a^k)p_j p_k + a^k\partial_k(\mathrm{g}^{ij})p_i p_j. \tag{1.74}$$

Relabeling the indices and recognizing that $a^k\partial_k(\mathrm{g}^{ij}) = \mathcal{L}_Z(\mathrm{g}^{ij})$ leads to

$$0 = \left(\mathcal{L}_Z(\mathrm{g}^{ij}) - \mathrm{g}^{jk}\partial_k(a^i) - \mathrm{g}^{ik}\partial_k(a^j)\right) p_i p_j \tag{1.75}$$

It follows that Z satisfies Equation (1.72) and so is a Killing field. This proves that the assignment $X \mapsto X^i p_i$ is surjective. □

Thus Noether's theorem can be restated in this setting as:

$\boxed{\text{The symmetries of the geodesic flow coming from isometries generate linear integrals.}}$

Ideally we can prove integrability of a geodesic flow using only linear integrals. This is not always possible, a necessary condition is that $\mathfrak{iso}(M, \mathrm{g})$ contains a commutative subalgebra of dimension $n-1$. So it is of importance to have alternative methods of finding integrals. We can use methods from Lie theory to find additional integrals, as the following example demonstrates.

Example 1.3.9 (Integrability of the Geodesic Flow of the Schwarzschild Metric). Consider the *Schwarzschild metric* of a spherically symmetric massive body with Schwarzschild radius r_0.

$$\mathrm{g}_{\mathrm{Schw}} = -\left(1 - \frac{r_0}{r}\right) dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)\ d\varphi^2). \tag{1.76}$$

This spacetime has four Killing fields (with corresponding linear integrals) given by

$$\begin{aligned}
X_t &= \partial_t & I_t &= p_t \\
X_1 &= \sin(\varphi)\ \partial_\theta + \frac{\cos(\theta)\cos(\varphi)}{\sin(\theta)}\ \partial_\varphi & I_1 &= \sin(\varphi)\ p_\theta + \frac{\cos(\theta)\cos(\varphi)}{\sin(\theta)}\ p_\varphi \\
X_2 &= \cos(\varphi)\ \partial_\theta - \frac{\cos(\theta)\sin(\varphi)}{\sin(\theta)}\ \partial_\varphi & I_2 &= \cos(\varphi)\ p_\theta - \frac{\cos(\theta)\sin(\varphi)}{\sin(\theta)}\ p_\varphi \\
X_3 &= \partial_\varphi & I_3 &= p_\varphi.
\end{aligned} \tag{1.77}$$

It is readily verified that the nonzero Lie brackets of $\mathfrak{iso}(M, \mathrm{g})$ are given by

$$\{I_1, I_2\} = I_3,\ \{I_1, I_3\} = -I_2,\ \{I_2, I_3\} = I_1. \tag{1.78}$$

Hence, the Lie algebra decomposes as $\mathfrak{iso}(M, \mathrm{g}) = \mathbb{R} \oplus \mathfrak{so}(3)$, where the center is given by $\mathbb{R} = \langle I_t \rangle$ and $\mathfrak{so}(3) = \langle I_1, I_2, I_3 \rangle$.
In view of the Lie brackets and the Leibniz rule, it is readily seen that the function $I_{\mathrm{Cas}} := I_1^2 + I_2^2 + I_3^2$ is a *Casimir function* in $\mathbb{R}[I_1, I_2, I_3] \subseteq (C^\infty(T^*M), \{\cdot, \cdot\})$, that is, the integral I_{Cas} Poisson commutes with I_1, I_2 and I_3. The functions $H, I_1, I_3, I_{\mathrm{Cas}}$ are four Poisson-commuting independent integrals. We conclude that the geodesic flow of the Schwarzschild metric is integrable.

1.3.3 Killing Tensors

In the previous section we have seen that the symmetries of the geodesic flow coming from the isometry group are in a bijective correspondence with linear integrals of motion. So-called hidden symmetries of the system are responsible for higher order polynomial integrals. These hidden symmetries are most conveniently encoded in an algebraic object which satisfy a generalization of the Killing equation.

Definition 1.3.10 (Killing Tensor). Let (M^n, g) be a pseudo-Riemannian manifold. A symmetric d-tensor $K \in \mathcal{T}_d^0(M)$ is a *Killing tensor* of rank d if the symmetric part of its covariant derivative vanishes:

$$\nabla_{(i} K_{j_1 \ldots j_d)} = 0 \tag{1.79}$$

Let K_d denote the vector space of Killing d-tensors.

Note that the metric tensor is trivially a Killing 2-tensor.

Theorem 1.3.11 (Correspondence Killing Tensors and Polynomial Integrals). *Consider the geodesic flow of a pseudo-Riemannian manifold (M, g). There is a one-to-one correspondence*

$$K_d \to \{\textit{Polynomial integrals of degree } d\} \tag{1.80}$$

which assigns to a Killing d-tensor K the function $K^{i_1 \cdots i_d} p_{i_1} \cdots p_{i_d} \in C^\infty(T^*M)$.

Proof. Following Woodhouse' article [Woo75], we sketch a proof.
Let $S^d(M) := \Gamma(S^d TM)$ be the symmetric $(0, p)$-tensors on M. We have a linear isomorphism

$$\widehat{(\cdot)} : S^d(M) \to \{\text{Homogeneous polynomial in momenta of degree } d\},\ T \mapsto (\xi \mapsto T(\xi, \ldots, \xi)) \tag{1.81}$$

The Lie bracket $[\cdot, \cdot]$ on $S^1(M) = \mathfrak{X}(M)$ extends to a Lie bracket on $S(M)$ called the *Schouten–Nijenhuis bracket*. Given $A \in S^d(M), B \in S^e(M)$ we have that $[A, B] \in S^{d+e-1}(M)$. We have

$$\widehat{[A, B]} = \{\widehat{A}, \widehat{B}\}. \tag{1.82}$$

Taking $B = \mathrm{g}$, we find that $\widehat{A} \in C^\infty(T^*M)$ is an integral if and only if $[A, \mathrm{g}] = 0$. The condition $[A, \mathrm{g}] = 0$ is equivalent to $\nabla^{(i} K^{j_1 \ldots j_d)} = 0$. Since a connection commutes with type change, this proves the claim. $\square$

We mention that polynomial integrals can be reconstructed from its values on an arbitrarily small neighborhood of the cotangent bundle (cf. [KM16]). The argument uses the fact that any two points can be joined by a broken geodesic. Since integrals are constant along geodesics, this provides a way of determining the values of the integral outside the given neighborhood. As polynomials are determined by a *finite* number of values, this approach is constructive.

Graded structure of Killing Tensors
By the above theorem the Poisson bracket on functions corresponds to the Lie bracket for symmetric tensors. The pointwise product on functions corresponds to the symmetric product for tensors, which gives rise to a grading on the space of Killing tensors: $K^* = \bigoplus_{k\in\mathbb{N}}$

$$K_{d_1} \otimes K_{d_2} \mapsto K_{d_1+d_2}, \; c_{l,m}\ \alpha^l \otimes \beta^m \mapsto c_{l,m}\alpha^l \cdot \beta^m. \tag{1.83}$$

Definition 1.3.12 (**Relation among Killing Tensors**.)**.** A *relation (syzygy) among Killing tensors* of rank d_1 and d_2 with $d_1 \neq d_2$ is an element of the kernel of the map

$$K_{d_1} \otimes K_{d_2} \to K_{d_1+d_2}. \tag{1.84}$$

If $d_1 = d_2 =: d$, a relation is given by an element in the kernel of the map $S^2K_d \to K_{2d}$.

Definition 1.3.13 (**Irreducible Killing Tensor**)**.** A Killing d-tensor ($d \geq 2$) is *irreducible* if it cannot be written as the symmetric product of lower rank Killing tensors.

The space of irreducible Killing 2-tensors can be identified with the cokernel of map $\iota_2 : S^2K_1 \to K_2$, it fits into a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\ \iota_2 \longrightarrow S^2K_1 \to K_2 \longrightarrow \mathrm{Coker}\ \iota_2 \longrightarrow 0. \tag{1.85}$$

The space of irreducible Killing 3-tensors can be identified with the cokernel of the map $\iota_3 : K_1 \otimes K_2 \to K_3$, etc.

Chapter 2

Geometric Theory of Partial Differential Equations

The main goal of this thesis is to determine the dimension of the space K_d of Killing d-tensors. In the previous chapter we have seen that this problem is equivalent to completely integrating a linear partial differential equation obtained from the Poisson bracket. To this end, we shall study partial differential equations from the geometric point of view. This approach amounts to solving a PDE up to a given order k and determining whether this potential solution can be extended to a solution up to order $k+1$, etc. If we can continue this process indefinitely we have constructed a *formal solution* of the PDE, which essentially is the Taylor series of a potential solution. In order to formalize this approach, we dive into the language of jet theory. The advantage of jet theory is that we can apply methods of differential geometry and cohomology to study PDE's.

2.1 Basics of Jet Theory

In this section we discuss the required background from the theory of jets and PDEs, it has profited a lot from the references [Sau89], [Olv95], [Boc+99], [KL08], [Sei10], [Yud16].

2.1.1 Jet Bundles

Definition 2.1.1. Let E, B and F be manifolds. A smooth surjection $\pi : E \to B$ is a *fiber bundle* with fiber F over B if there exists an open covering $\mathcal{U} = (U_i)_{i \in I}$ of B together with diffeomorphisms $\varphi_{U_i} : \pi^{-1}(U_i) \to U_i \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\varphi_{U_i}"] \arrow[dr, "\pi"'] & & U_i \times F \arrow[dl, "\mathrm{pr}_{U_i}"] \\ & U_i. & \end{tikzcd}$$

Here $\mathrm{pr}_{U_i} : U_i \times F \to U_i$ is the projection onto the first factor. We say that φ_{U_i} are *local trivializations* of the fiber bundle $\pi : E \to B$. The map π is called the projection, the manifold E is called the *total space*, and B is called the *base space*.

Given a fiber bundle $\pi : E \to B$, the existence of local trivializations imply that the projection π is a submersion. Another consequence of the local triviality property $\mathrm{pr}_{U_i} \circ \varphi_{U_i} = \pi$ is that each fiber of π is diffeomorphic to F. A local trivialization allows us to construct coordinate charts on the total space E compatible with the fiber bundle structure, as follows. Let $(x^1, \dots, x^n)$ and $(u^1, \dots, u^m)$ denote coordinates on the base B and fiber F, respectively. By restricting their coordinate domains (if necessary) and composing with the local trivialization, we obtain coordinates $(x^1, \dots, x^n, u^1, \dots, u^m)$ on the total space. We call such coordinates on the total space *adapted to* π.

Let $\pi : E \to M$ be a fiber bundle. A *section* of π is a smooth map $\sigma : B \to E$ satisfying $\pi \circ \sigma = \mathrm{Id}_B$. Thus, a section σ maps a point $x \in E$ to an element of the corresponding fiber $E_p := \pi^{-1}(p)$ above x. We shall also consider local sections, i.e., a smooth map $\sigma : U \to E$ defined on an open $U =: \mathrm{Dom}(\sigma) \subseteq B$ satisfying $\pi \circ \sigma = \mathrm{Id}_U$. We denote the space of (global) sections and local sections by $\Gamma(E)$ and $\Gamma_{\mathrm{loc}}(E)$, respectively. Moreover, let $\Gamma_{\mathrm{loc}}(E)_p$ denote the space of local sections whose domain contains the point $p \in B$. In contrast to vector bundles, a fiber bundle does not always admit a global section. Using a local trivialization of π, it is readily seen that for any $e \in E$ and $x \in B$ there exists a local section σ such that $\sigma(x) = e$.

If $\sigma \in \Gamma_{\mathrm{loc}}(E)$ is a local section and $(x, u) = (x^1, \dots, x^n, u^1, \dots, u^m)$ are adapted coordinates, then locally we have $\sigma(x) = (x, \sigma^1(x), \dots, \sigma^m(x))$ where $\sigma^j := u^j \circ \sigma$ are called the component functions of σ. Hence, we think of the x-coordinates on the base as the *independent variables* and the u-coordinates in the fibers as the *dependent variables*. We now want to define the k-jet of a local section $\sigma : x \mapsto (x, \sigma^1(x), \dots, \sigma^m(x))$. It should encode all the partial derivatives of σ up to order k, or equivalently its k'th order Taylor polynomial. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ of length $|\alpha| := \alpha_1 + \dots + \alpha_n$, we define the operator

$$\partial_p^\alpha := \frac{\partial^{|\alpha|}}{\partial x^\alpha}\Big|_{x=p} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\Big|_{x=p}. \tag{2.1}$$

We define an equivalence relation $\sim_p^k$ on $\Gamma_{\mathrm{loc}}(E, p)$ as follows. Two local sections $\sigma_1, \sigma_2 \in$ are said to be *k-equivalent* at p if

$$\partial_p^\alpha(\sigma_1^j) = \partial_p^\alpha(\sigma_2^j) \tag{2.2}$$

for all $1 \leq j \leq m$ and all multi-indices α with $|\alpha| \leq k$. One should verify that this construction is independent of the choice of coordinate chart (see [Sau89, Lemma 6.2.1.]). The resulting equivalence class of a section $\sigma \in \Gamma_{\mathrm{loc}}(E)_p$ is called the *k-jet at p* of σ and is denoted by $j_p^k\sigma$. In other words, two sections have the same k-jet if and only if their k-th order Taylor polynomials in one (and hence any) trivialization coincide. By taking the quotient of $\Gamma_{\mathrm{loc}}(E)_p$ under this equivalence relation, we obtain the *k-th jet space at p*:

$$J^k(E)_p := \Gamma_{\mathrm{loc}}(E)_p / \sim_p^k = \{ j_p^k \sigma : \sigma \in \Gamma_{\mathrm{loc}}(E)_p \}. \tag{2.3}$$

Next, we construct the *k-th jet bundle* of $\pi : E \to B$ by setting

$$J^k E := \bigsqcup_{p \in B} J^k(E)_p. \tag{2.4}$$

The space $J^k E$ is naturally endowed with a smooth structure turning $J^k E$ into a manifold of dimension $\dim J^k E = n + m\binom{n+k}{k}$. If (x, u) are adapted coordinates for the fiber bundle

π, then we obtain coordinates on J^kE which assigns to a k-jet $j^k_p\sigma$ the coordinates of the basepoint p together with all its partials up to order k. In formula, this assignment equals

$$j^k_p\sigma \mapsto (x(p), \partial_p^\alpha(\sigma^j))_{1\leq j\leq m,\ 0\leq|\alpha|\leq k}. \tag{2.5}$$

We denote these coordinates on J^kE by (x, u^α), which emphasizes that these coordinates are induced by some adapted coordinates on the total space E.

There are several natural projections associated to the jet bundle J^kE. Firstly, we have the projection to the base,

$$\pi_k : J^kE \to M,\ j^k_p\sigma \mapsto p \tag{2.6}$$

Moreover, we obtain a projection $\pi_{k,l}$ of J^kE onto a jet bundle J^lE of lower order l by "forgetting" all partial derivatives of order greater than l, that is,

$$\pi_{k,l} : J^kE \to J^lE,\ j^k_p\sigma \mapsto j^l_p\sigma\ (k \geq l). \tag{2.7}$$

Under the identification $J^0E = E$ through the map $j^0_p\sigma \mapsto \sigma(p)$ we have that $\pi_{k,0} : J^kE \to E$ is the projection to the total space. The maps $\pi_{k,l}$ and π_k are smooth fiber bundles, which justifies naming J^kE a jet *bundle*. Even more is true, if additionally π is a vector bundle then the maps π_k are vector bundles as well. Local sections of $\pi : E \to B$ give rise to local sections of $\pi_k : J^kE \to B$. We define the k-jet of a local section $\sigma \in \Gamma(E)_{\mathrm{loc}}$ to be the section $j^k\sigma : p \mapsto j^k_p\sigma$ of $J^kE \to B$. The converse is not true, a local section of J^kE is generally not a k-jet of a local section of E.

These maps fit into a sequence of fiber bundles

$$\cdots \longrightarrow J^kE \xrightarrow{\pi_{k,k-1}} J^{k-1}E \xrightarrow{\pi_{k-1,k-2}} \ldots \xrightarrow{\pi_{2,1}} J^1E \xrightarrow{\pi_{1,0}} E \xrightarrow{\pi} M, \tag{2.8}$$

which in its inverse limit yields the *infinite jet bundle* $J^\infty E := \varprojlim J^kE$. It turns out that the infinite jet bundle is actually an infinite-dimensional manifold modelled on a Frechét space. An element of $J^\infty E$ can be thought of as a *formal* infinite Taylor series, as this series does not converge in general (give a citation). However, by Borel's theorem (citation), we do have that for any $z \in J^\infty E$ there exists a local section $\sigma \in \Gamma_{\mathrm{loc}}(E)$ such that its infinite prolongation $j^\infty\sigma$ equals z.

2.1.2 Partial Differential Equations

Definition 2.1.2 (**Geometric PDE**)**.** A *(geometric) partial differential equation of pure order* k on a fiber bundle $\pi : E \to B$ is a submanifold $\mathcal{E} \subseteq J^kE$ with the property that the restriction $\pi|_{\mathcal{E}} : \mathcal{E} \to B$ is a fiber bundle. A *solution* of the PDE is defined to be a local section σ of π such that its k-jet $j^k\sigma$ takes values in $\mathcal{E}$, that is,

$$j^k_p\sigma \in \mathcal{E} \text{ for all } p \in \mathrm{Dom}(\sigma). \tag{2.9}$$

We denote by $\mathrm{Sol}(\mathcal{E})$ the space of all solutions of the PDE.

Let us look into this definition with more detail. Suppose that $\mathcal{E} \subseteq J^kE$ is a PDE. Since $\mathcal{E}$ is a submanifold, we can find around any point in the PDE an open $U \subseteq J^kE$ and a submersion $F : U \to \mathbb{R}^{\mathrm{codim}(\mathcal{E})}$ such that locally $\mathcal{E}$ is the level set of F:

$$U \cap \mathcal{E} = F^{-1}(0). \tag{2.10}$$

This means that a point $j^k_p\tau \in U$ is an element of the PDE if and only

$$F(j^k_p\tau) = 0. \tag{2.11}$$

If $\sigma \in \mathrm{Sol}(\mathcal{E})$ is a solution of the PDE, then clearly we have

$$F(j^k_p\sigma) = 0. \tag{2.12}$$

Comparing Equation (2.11) and Equation (2.12) leads to an interpretation of $\mathcal{E}$ in terms of its solutions:

"Elements of a k'th order geometric PDE $\mathcal{E} \subseteq J^k E$ are solutions *up to order k*."

Example 2.1.3. We can identify a k-jet $j^k_p\sigma$ with the Taylor polynomial of σ about p in adapted coordinates (x, u) through the map

$$j^k_p\sigma \mapsto (\mathrm{Taylor}(\sigma^1, p, k), \ldots, \mathrm{Taylor}(\sigma^m, p, k)) \tag{2.13}$$

where m is the number of dependent variables.

Of course, we would then like to find the solutions of the PDE $\mathcal{E}$ up to order $k+1$ (then up to order $k+2$, and so on). This is done by differentiating the system Equation (2.11) with respect to the base variables and imposing the resulting constraints. To this end, suppose that $\sigma(x) = (x, u(x))$ is a solution of the PDE $\mathcal{E}$. We can write its k-jet as $(j^k\sigma)(x) = (x, u^\alpha(x))$. Now differentiating and applying the chain rule yields

$$\frac{\partial}{\partial x^i}(F(x, u^\alpha(x)) = \frac{\partial F}{\partial x^i}(x, u^\alpha(x)) + \sum_{0\le|\alpha|\le k} \frac{\partial F}{\partial u^\alpha}(x, u^\alpha(x))\frac{\partial u^\alpha}{\partial x^i}(x) = 0. \tag{2.14}$$

We introduce the notation $\alpha + 1_i$ to be the multi-index obtained by adding 1 to the i'th entry of α. In view of the expression (2.14) we define the i'th *total derivative* of F at a point (x, u^α) by

$$(D_iF)(x, u^\alpha) := \frac{\partial F}{\partial x^i}(x, u^\alpha) + \sum_{0\le|\alpha|\le k} \frac{\partial F}{\partial u^\alpha}(x, u^\alpha)\ u^{\alpha+1_i} \tag{2.15}$$

Now, a point $(x, u^\alpha) \in J^{k+1}E$ is said to be a solution of $\mathcal{E}$ up to order $k+1$ if it satisfies the following system of equations:

$$\begin{cases} F(x, u^\alpha) = 0 \\ (D_iF)(x, u^\alpha) = 0 \text{ for all } i = 1, \ldots, n. \end{cases} \tag{2.16}$$

The resulting system of equations is called the *first prolongation* of $\mathcal{E}$. By construction, a solution of $\mathcal{E}$ is still a solution of the prolongation $\mathcal{E}^{(1)}$. In the language of jet theory, we can phrase this prolongation procedure as follows.

Definition 2.1.4 (Prolongation of a PDE). Let $\mathcal{E}$ be a PDE of order k on a fiber bundle $\pi : E \to B$. The *(first) prolongation* of $\mathcal{E}$ is defined to be

$$\mathcal{E}^{(1)} := J^1(\mathcal{E}) \cap J^{k+1}E. \tag{2.17}$$

(Note that since $\mathcal{E} \subseteq J^kE$ we have $J^1(\mathcal{E}) \subseteq J^1(J^kE)$. However, $J^1(J^kE)$ consists of 1-jets of sections of $\pi_k : J^kE \to B$, so we intersect $J^1(\mathcal{E})$ with the jet bundle $J^{k+1}E$.)

It is instructive to check that the definition Equation (2.17) written in jet coordinates (x, u^α) is equivalent to Equation (2.16).

Example 2.1.5. We work out the procedure of prolongation for a simple PDE. Consider on the bundle $\mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, u) \mapsto (x, y)$ the first-order PDE defined by

$$\mathcal{E} = \begin{cases} u_x = yu \\ u_y = u. \end{cases} \tag{2.18}$$

(Here u_x denotes the partial derivative of a section $u = u(x, y)$ with respect to x, etc.) Differentiating the equations of $\mathcal{E}$ with respect to x, y and applying the product rule once yields the first prolongation:

$$\mathcal{E}^{(1)} = \begin{cases} u_{xx} = yu_{xx}, & u_x = yu \\ u_{xy} = u + yu_y & \\ u_{xy} = u_x, & u_y = u \\ u_{yy} = u_y. & \end{cases} \tag{2.19}$$

Note that we can rewrite the equation from the second row as

$$u_{xy} = u + yu = (1 + y)u. \tag{2.20}$$

On the other hand, combining the equation from the third row shows and the second equation from the third row shows that

$$u_{xy} = u_x = yu. \tag{2.21}$$

Comparing Equation (2.20) and Equation (2.21), we obtain that $u = 0$. Thus, by prolonging the PDE we were able to deduce that the only solution of the PDE $\mathcal{E}$ is given by the trivial solution $u = 0$.

In order to simplify the exposition, we make the following assumption.

Regularity Assumption I. Throughout, we shall assume that the prolongation of a PDE is again a PDE.

Under this assumption, we can inductively define the l'th prolongation $\mathcal{E}^{(l)}$ of a k'th order PDE $\mathcal{E} \subseteq J^k E$ by

$$\mathcal{E}^{(l)} := (\mathcal{E}^{(l-1)})^{(1)} \subseteq J^{k+l+1}E. \tag{2.22}$$

Note that the PDE $\mathcal{E}^{(l)}$ is of order $k + l$. It is readily seen that the l'th prolongation consists of Equation (2.16) together with the equations obtained by differentiating this system up to order k with respect to the base variables. Since a solution of a prolongation is necessarily a solution of the original PDE, the projection $\pi_{k+1,k}$ restricts to a well-defined map $\pi : \mathcal{E}^{(1)} \to \mathcal{E}$. We say that a solution up to order k, denoted $\tau \in \mathcal{E}$, can be *extended* to a solution up to order $k+1$ if there exists $\tilde{\tau} \in \mathcal{E}^{(1)}$ such that $\pi(\tilde{\tau}) = \tau$. An element of the inverse limit of the sequence

$$\ldots \xrightarrow{\pi} \mathcal{E}^{(l)} \xrightarrow{\pi} \mathcal{E}^{(l-1)} \xrightarrow{\pi} \ldots \xrightarrow{\pi} \mathcal{E}^{(1)} \xrightarrow{\pi} \mathcal{E} \tag{2.23}$$

is called a *formal solution* of $\mathcal{E}$. The ∞-jet of a solution always defines a formal solution, but the converse is not necessarily true.

The condition that any solution up to order k can be extended to a solution up to order $k+1$ is equivalent to the assertion that $\pi : \mathcal{E}^{(1)} \to \mathcal{E}$ is surjective. This line of thought leads to the following definition.

Definition 2.1.6 (**Formal Integrability**). A PDE $\mathcal{E}$ is *formally integrable* if $\pi : \mathcal{E}^{(l)} \to \mathcal{E}^{(l-1)}$ is surjective for all $l \geq 1$.

Put differently, formal integrability means that any solution up to order k can be extended to a formal solution. There exist examples of formally integrable PDEs that do not allow smooth solutions, see Lewy example [Lew57].

As a mathematical curiosity, we mention that criteria for formal integrability can be described in terms of a cohomology theory called Spencer cohomology due to the works by Bott, Goldschmidt, Guillemin, Quillen, Spencer, Sternberg ([Gol67]).

Cartan Distribution.
We now discuss an important geometric structure associated to PDEs called the Cartan distribution. The integral submanifolds of this distribution are solutions of the PDE. Frobenius' theorem on the integrability of distributions will allow us to deduce some local solvability results of PDEs later.

Definition 2.1.7. Let M^n be a manifold. A *distribution* of rank d is a subbundle D of the tangent bundle TM.

A distribution D of rank d can be specified in several ways [Lee13]:

- There exist d locally defined vector fields $X_1, \ldots, X_d$ that span the distribution:
$$D = \langle X_1, \ldots, X_d \rangle := \operatorname{span}\{X_1, \ldots, X_d\}. \tag{2.24}$$
- There exist $n-d$ locally defined one-forms $\omega_1, \ldots, \omega_{n-d}$ such that for each point p on this neighborhood we have
$$D_p = \ker \omega_1|_p \cap \cdots \cap \ker \omega_{n-d}|_p. \tag{2.25}$$

Definition 2.1.8. Let D be a distribution of rank d on a manifold M. An *integral submanifold* of D is a connected, immersed d-dimensional submanifold $S \subseteq M$ such that
$$T_pS = D_p \text{ for all } p \in S. \tag{2.26}$$

A distribution is said to be *involutive* if $\Gamma(D)$ is closed under the Lie bracket, that is, $[X, Y] \in \Gamma(D)$ for vector fields $X, Y \in \Gamma(D)$.

Theorem 2.1.9 (**Frobenius' Theorem**). *An involutive distribution is completely integrable, that is, the manifold can be foliated by integral submanifolds.*

Let $\pi : E \to B^n$ be a fiber bundle, and let $x_{k+1} = j_p^{k+1}\sigma \in J^{k+1}E$ be a point in its $(k+1)$'th jet bundle for some representative local section σ. Denote $x_k = \pi_{k+1,k}(x_{k+1}) = j_p^k\sigma$. We define $L(x_{k+1})$ to be the n-dimensional tangent plane at x_k by the image of the k-jet of σ:

$$L(x_{k+1}) := T_{x_k}(\mathrm{Im}(j^k\sigma)) \subseteq T_{x_k}J^kE. \tag{2.27}$$

This is well-defined, because the plane $L_{x_{k+1}}$ is independent of the local section σ representing the equivalence class x_{k+1}. Next, we define the subspace $\mathcal{C}(x_k)$ at x_k by

$$\mathcal{C}_{x_k} := \mathrm{span}\{L(x_{k+1}) : x_{k+1} \in \pi_{k+1,k}^{-1}(x_k))\} = (d\pi_{k,k-1})^{-1}(L(x_k)) \subseteq T_{x_k}(J^kE). \tag{2.28}$$

Definition 2.1.10 (Cartan Distribution). Let $\pi : E \to B^n$ be a fiber bundle. The *Cartan distribution* $\mathcal{C}$ in J^kE is the distribution given by

$$\mathcal{C} : x_k \mapsto \mathcal{C}(x_k). \tag{2.29}$$

If $\mathcal{E} \subseteq J^kE$ is a PDE of order k on π, then

$$\mathcal{C}(\mathcal{E}) := \mathcal{C} \cap T\mathcal{E} \subseteq T\mathcal{E}. \tag{2.30}$$

is called the *Cartan distribution of the PDE* $\mathcal{E}$.

There might be some singular behavior of the Cartan distribution (in the sense that the rank is not constant), so we assume the following.

Regularity Asssumption II. The Cartan distribution $\mathcal{C}(\mathcal{E})$ of a PDE $\mathcal{E}$ is a smooth subdistribution of $\mathcal{C}$.

We can locally describe the Cartan distribution on J^kE, as follows.

- The Cartan distribution $\mathcal{C}$ is locally spanned by vector fields

$$X_i := \frac{\partial}{\partial x^i} + \sum_{0\le|\alpha|\le k-1}\sum_{j=1}^{m} u_j^{\alpha+1_i}\frac{\partial}{\partial u_j^\alpha} \tag{2.31}$$

  for all $1 \le i \le n$ and the coordinate vector fields $\frac{\partial}{\partial u_j^\alpha}$ for all $1 \le j \le m$ and multi-indices α satisfying $|\alpha| = k$. In particular, the rank of the Cartan distribution is given by $\mathrm{rank}(\mathcal{C}) = n + m\binom{n+k-1}{k}$.

- Dually, the Cartan distribution can be described by the common kernel of the *Cartan forms*, which are defined by

$$\omega_j^\alpha := du_j^\alpha - \sum_{i=1}^{n} u_j^{\alpha+1_i}dx^i \tag{2.32}$$

  for all $1 \le j \le m$ and $0 \le |\alpha| \le k-1$.

The Cartan distribution $\mathcal{C}$ on a jet bundle J^kE is not involutive. For example, using the coordinate formula for Lie derivative, we have

$$\left[\frac{\partial}{\partial u_j^{\alpha+1_i}}, X_i\right] = \frac{\partial}{\partial u_j^\alpha} \tag{2.33}$$

where α is a multi-index of length $|\alpha| = k - 1$. Thus, we cannot guarantee the existence of integral manifolds for the Cartan distribution using Frobenius' theorem. However, we have the following. The Cartan distribution distinguishes (graphs of) solutions from arbitrary submanifolds of a jet bundle in the following way.

Proposition 2.1.11. Let $\pi : E \to B^n$ be a fiber bundle, and let $\mathcal{E} \subseteq J^k E$ be a PDE. The image of a section of $\pi_k : J^k E \to B$ is an integral submanifold of the Cartan distribution if and only if it comes from the k-jet of a section of $\pi : E \to B$. Moreover, the image of a section of $\pi : \mathcal{E} \to B$ is an integral submanifold of the Cartan distribution $\mathcal{C}(\mathcal{E})$ if and only if it comes from the k-jet of a solution of $\mathcal{E}$.

Proof. Let $j^k\sigma$ be the k-jet of a section σ of $\pi : E \to B$, and write $(j^k\sigma)(x) = (x, u_j^\alpha(x))$ in adapted coordinates. On the image of $j^k\sigma$ we have $du_j^\alpha = \sum_{i=1}^n (\frac{\partial u_j^\alpha}{\partial x^i}) dx^i$, which implies that the Cartan forms vanish. We conclude that the image of $j^k\sigma$ is an integral submanifold for the Cartan distribution.

Conversely, suppose that $\tau : B \to J^k E$ is a section such that the image $S := \mathrm{Im}\tau$ is an integral submanifold. We write $\tau(x) = (x, \tau_j(x), \tau_j^\alpha(x))$ where $1 \leq |\alpha| \leq k$. As S is integral, the Cartan forms are identically zero on S. It follows that

$$\frac{\partial \tau_j^\alpha}{\partial x_i} = \tau_j^{\alpha+1_i} \tag{2.34}$$

for all $1 \leq j \leq m$ and $1 \leq |\alpha| \leq k$. This shows that the higher order jets of τ are the derivatives of the 0-jets τ_j. In other words, τ is the k-jet of the section $B \to E$, $x \mapsto (x, \tau_j(x))$. ☐

Motivated by this proposition, one can also define generalized solutions as arbitrary integral manifolds of the Cartan distribution. [Boc+99].

Note that the integral manifolds S of $\mathcal{C}(\mathcal{E})$ that come from k-jets of solutions are *horizontal* with respect to the projection $\pi : \mathcal{E} \to B$ (i.e. : $S \to TB$ is an isomorphism). Conversely, π-horizontal integral manifolds are locally of this form [Boc+99, p.81]. For a generalized solution the integral submanifold is not assumed to be horizontal, so it may be multi-valued.

The above proposition also suggests a method to find solutions of a PDE. First, we look for horizontal integral submanifolds of the Cartan distribution. (These are guaranteed if the horizontal part of the distribution is involutive.) If the integral submanifolds are horizontal with respect to the projection $\pi : \mathcal{E} \to B$, then locally it is the graph of a k-jet of a solution!

2.2 Cartan's Prolongation–Projection Method

Cartan's prolongation–projection method is an algorithm which can be applied to turn PDEs from a special class into a formally integrable PDE. The goal of this section is to explain this method, as well as find such a special class of PDEs on which the method is applicable. This method involves the interaction between two geometric operations: *prolongation* and *projection*, which we now briefly sketch. Suppose we have a PDE $\mathcal{E} \subseteq J^k E$ of order k. As mentioned before, we think of an element in $\mathcal{E}$ as a solution up to order k. Then by *prolonging* the PDE to $\mathcal{E}^{(1)}$ we obtain all solutions up to order $k+1$. Consider the map $\pi : \mathcal{E}^{(1)} \to \mathcal{E}$. A natural question we can ask is whether the map π is surjective. If it is surjective, then every solution up to order k can be extended. Intuitively, this means that the (local) equations defining $\mathcal{E}^{(1)}$ do not impose any additional constraints on solutions up to order k. However, if π is not surjective, then we have found new equations, satisfied by solutions, which shrink the subspace of $\mathcal{E}$ consisting of elements that potentially come from formal solutions. In this case, *projecting* $\mathcal{E}^{(1)}$ gives a strict inclusion $\pi(\mathcal{E}^{(1)}) \subset \mathcal{E}$. So we shall use the procedure of projection to test if we obtain (essentially) new equations after prolongation. We will now discuss these principles in more detail.

Prolongation.

We have seen that prolonging a PDE $\mathcal{E}$ raises the order by one; these higher order equations being obtained from differentiating the equations defining $\mathcal{E}$ with respect to the independent variables. In practice, the prolongation of a PDE is easily computed starting from a local representation of the PDE. Ideally, we can then extract useful information about the PDE from its prolongations. One class of PDEs for which this works are called PDEs of finite type.

Definition 2.2.1 (**Finite Type**). Let $\mathcal{E} \subseteq J^k E$ be a PDE. A PDE is called of *finite type* l if after l prolongations all the highest order derivatives of the dependent variables can be expressed algebraically in terms of the lower order derivatives. A PDE is called of *Frobenius type* if it is of finite type 0.

Given a PDE $\mathcal{E}$ of finite type, it is readily seen that the space of formal solutions $\mathcal{E}^{(\infty)}$ is necessarily finite-dimensional (under some regularity conditions, see [Kru11]). Since any solution is a formal solution, we find in particular that the solution space $\mathrm{Sol}(\mathcal{E})$ is finite-dimensional.

We examine some examples of PDE and check whether they are of finite type or not.

Example 2.2.2. Consider the bundle $\mathbb{R}^4 \to \mathbb{R}^2,\ (x, y, a, b) \mapsto (x, y)$. The first-order PDE $\mathcal{E}$ given by

$$\mathcal{E} = \begin{cases} a_x = 0 \\ a_y + b_x = 0 \\ b_y = 0 \end{cases} \tag{2.35}$$

is of finite type 1. Indeed, prolonging the PDE yields

$$\mathcal{E}^{(1)} = \begin{cases} a_{xx} = 0, & a_x = 0 \\ a_{xy} = 0 & \\ a_{xy} + b_{xx} = 0, & a_y + b_x = 0 \\ a_{yy} + b_{xy} = 0 & \\ b_{xy} = 0, & b_y = 0 \\ b_{yy} = 0. & \end{cases} \tag{2.36}$$

We can rewrite the first equation from the third row to $b_{xx} = 0$ using the equation from the second row. Similarly, the fourth equation reduces to $a_{yy} = 0$. Thus, all highest order derivatives are expressed.

The condition that a PDE is Frobenius type (or finite type) can also be interpreted geometrically in terms of its symbol, as we now discuss.

Definition 2.2.3. Let $\mathcal{E} \subseteq J^k E$ be a PDE of order k. The *symbol* g is the subbundle of the Cartan distribution given by

$$g := \ker(d\pi_{k,k-1}|_{T\mathcal{E}}) \subseteq \mathcal{C}(\mathcal{E}) \tag{2.37}$$

Thus, the symbol is the vertical bundle of $\pi_{k,k-1}|_{\mathcal{E}}$.

If $\mathcal{E}$ is defined by the vanishing of a function F, then $T\mathcal{E} = \ker dF$. Thus for a tangent vector $v \in T\mathcal{E}$ we have that

$$dF(v) = \frac{\partial F}{\partial x^i} dx^i(v) + \sum_{|\alpha| \leq k} \sum_{j=1}^{m} \frac{\partial F}{\partial u_j^\alpha} d(u_j^\alpha)(v) = 0. \tag{2.38}$$

In view of definition (2.37) it is readily seen that

$$g = \{v \in T\mathcal{E} : \sum_{|\alpha| = k} \sum_{j=1}^{m} \frac{\partial F}{\partial u_j^\alpha} d(u_j^\alpha)(v) = 0\}. \tag{2.39}$$

Thus the symbol only captures the highest degree part of the differential equation. Under some regularity conditions a PDE is of Frobenius type if and only if its symbol vanishes.

Definition 2.2.4 (Cartan–Ehresmann Connection). A *Cartan–Ehresmann connection* on a PDE $\mathcal{E}$ is a subbundle H of the Cartan distribution $\mathcal{C}(\mathcal{E})$ such that $d\pi_k : H \to TB$ is an isomorphism. In this way we obtain a direct sum decomposition of the Cartan distribution

$$\mathcal{C}(\mathcal{E}) \cong H \oplus g \tag{2.40}$$

into its horizontal and vertical part (cf. [Boc+99, Theorem 2.1]).

Theorem 2.2.5 (**Frobenius Theorem**). *Let $\mathcal{E} \subseteq J^kE$ be a PDE of finite type l. The solutions of $\mathcal{E}$ are determined uniquely by their $(k+l-1)$-jets. If in addition $\mathcal{E}$ is formally integrable, then for every $x_{k+l} \in \mathcal{E}^{(l)}$ there exists a solution σ of $\mathcal{E}$ satisfying $j_x^{k+l}\sigma = x_{k+l}$.*

Proof. The first assertion holds by definition of finite type and the Picard-Lindelöf fundamental theorem for ODEs. Now suppose that $\mathcal{E}$ is also formally integrable. Since $\mathcal{E}$ is of finite type l and formally integrable, it follows that $\pi : \mathcal{E}^{(l)} \to \mathcal{E}^{(l-1)}$ is an isomorphism. The unique section $\xi : \mathcal{E}^{(l-1)} \to \mathcal{E}^{(l)}$ of π gives rise to a Cartan–Ehresmann connection H on $\mathcal{E}^{(l-1)}$. Explicitly, it is given by the assignment

$$H : \mathcal{E}^{(l-1)} \ni x_{k+l-1} \mapsto L(\xi(x_{k+l-1})) \subset \mathcal{C}(\mathcal{E}^{(l-1)}), \tag{2.41}$$

where $L(\xi(x_{k+l-1}))$ is the horizontal jet plane (2.27) at x_{k+l-1}. The symbol $g^{(l)} = \ker(d\pi|_{T\mathcal{E}^{(l)}})$ of $\mathcal{E}^{(l)}$ vanishes by the finite type assumption. This implies that $d\pi$ maps the Cartan distribution $\mathcal{C}(\mathcal{E}^{(l)})$ bijectively onto the horizontal subbundle $H \subset \mathcal{C}(\mathcal{E}^{(l-1)})$. We conclude that $\mathcal{C}(\mathcal{E}^{(l)})$ is horizontal, and so $\mathcal{C}(\mathcal{E}^{(l)})$ defines a Cartan–Ehresmann connection on $\mathcal{E}^{(l)}$.

Furthermore, this connection is flat as there is no vertical part in $\mathcal{C}(\mathcal{E}^{(l)})$. By virtue of Frobenius' theorem (2.1.9) there exists through every point x_{k+l} an integral manifold S of the Cartan distribution $\mathcal{C}(\mathcal{E}^{(l)})$. The integral submanifold S is horizontal, so locally it can be described by the graph of a section σ of the bundle $\mathcal{E}^{(l)} \to B^n$ ([Boc+99, p.81]). Application of Proposition (2.1.11) guarantees that σ is a solution of $\mathcal{E}^{(l)}$. Since any solution of $\mathcal{E}^{(l)}$ is a solution of $\mathcal{E}$, this proves the second assertion. □

Note that if a PDE is of finite type, then all of its prolongations are finite type as well. Thus, if we can turn a PDE of finite type into a formally integrable one by prolonging enough times, then we have local existence and uniqueness of solutions.

Projection.

Let $\mathcal{E} \subseteq J^kE$ be a PDE. Consider its l'th prolongation $\mathcal{E}^{(l)} \subseteq J^{k+l}$ and the map $\pi : \mathcal{E}^{(l)} \to \mathcal{E}$. The *projection* of $\mathcal{E}^{(l)}$ is defined to be the subspace $\pi(\mathcal{E}^{(l)}) \subseteq \mathcal{E}$. So prolongation raises the order of a PDE, whereas projection lowers the order of a PDE. In general these operations are not inverses of each other.

If the projection $\pi(\mathcal{E}^{(l)})$ is strictly contained in $\mathcal{E}$, then we say that we have found compatibility conditions.

Definition 2.2.6 (**Compatibility Condition.**). Let $\mathcal{E} \subseteq J^kE$ be a PDE of order k. Suppose that $\mathcal{E} = \{F(x, u^\alpha) = 0\}$ is a local representation of the PDE. A *compatibility condition* of $\mathcal{E}$ is an equation which is algebraically independent of F and which is satisfied by all formal solutions.

Compatibility conditions can be interpreted as the obstructions of a PDE to being formally integrable.

One can obtain a local representation of a projection, as follows. Suppose we have a PDE of order k with local representation $\mathcal{E} = \{F(x, u^\alpha) = 0\}$. Then the local representation of its prolongation consists of equations which are affine linear in $(k+1)$-jets. By using solely Gauss manipulation, we now try to rewrite the equations of order $k+1$ to equations of lower order. If we succeed in rewriting and if in addition the resulting equation is algebraically independent from the equations defining the PDE, then we have obtained an compatibility condition. The local representation of the projection is given by the equations $F(x, u_\alpha) = 0$ together with all the compatibility conditions. On the other hand, if none of the highest order equations can be rewritten in such a way, then the projection is given by $\pi(\mathcal{E}^{(1)}) = \mathcal{E}$. We demonstrate this technique with the following example.

Example 2.2.7 ([Sei10])**.** Consider the bundle $\mathbb{R}^4 \to \mathbb{R}^3$, $(x, y, z, u) \mapsto (x, y, z)$ together with the PDE $\mathcal{E}$ defined by

$$\mathcal{E} = \begin{cases} u_x + zu_y = 0 \\ u_z = 0 \end{cases} \tag{2.42}$$

The prolongation is easily computed:

$$\mathcal{E}^{(1)} = \begin{cases} u_{xx} + zu_{xy} = 0, & u_x + zu_y = 0 \\ u_{xy} + zu_{yy} = 0 & \\ u_{xz} + u_y + zu_{yz} = 0 & \\ u_{xz} = 0, & u_z = 0 \\ u_{yz} = 0 & \\ u_{zz} = 0. & \end{cases} \tag{2.43}$$

Note that this system is linear in the dependent variables (u, u_x, u_y, u_z, u_{xx}, etc). We can rewrite the equation in the third row to $u_y = 0$ using the equations from the fourth and fifth row. The compatibility condition $u_y = 0$ is responsible for the projection $\pi(\mathcal{E}^1)$ being strictly contained in $\mathcal{E}$. We thus obtain the following coordinate representation of the projection:

$$\pi(\mathcal{E}^{(1)}) = \begin{cases} u_x = 0 \\ u_y = 0 \\ u_z = 0, \end{cases} \tag{2.44}$$

from which we read off that the only solutions of $\mathcal{E}$ are the constants.

As we see in the above example, the process of projecting after prolongation is essentially a linear algebraic problem.

Cartan's Prolongation–Projection Method.

Algorithm 1. (Cartan's Prolongation–Projection Method).
(**Input**: A PDE $\mathcal{E}$ of finite type l.)

- Step 1.) Prolong the PDE $\mathcal{E}$ to $\mathcal{E}^{(l)}$.

  Put $\mathcal{F}_0 := \mathcal{E}^{(l)}$ and inductively define $\mathcal{F}_N := (\mathcal{F}_{N-1})^{(1)}$.
  Now set $N := 0$.

- Step 2.) Compute the projection of the prolonged PDE $\mathcal{F}_{N+1}$ in $\mathcal{F}_N$.

  If we have obtained compatibility conditions from this projection, increase N by 1 and repeat Step 2.

- Step 3.) Return the PDE $\mathcal{F}_N$.

(**Output**: A formally integrable PDE of Frobenius type with the same solutions as $\mathcal{E}$.)

From the description of Step 2, it is clear that the algorithm terminates after a finite number of steps. In order to show that the output is actually a formally integrable PDE, it remains to show that this algorithm finds *all* compatibility conditions in Step 2. This is guaranteed by Cartan's prolongation theorem. By Frobenius' theorem the PDE obtained from Cartan's method enjoys local existence and uniqueness of solutions.

Theorem 2.2.8 (**Cartan's Prolongation Theorem**). *An overdetermined PDE of finite type l can be turned into a formally integrable one using Cartan's prolongation-projection method. Moreover, we have the following criterion to test if the N'th prolongation is formally integrable: if $\pi : \mathcal{E}^{(N)} \to \mathcal{E}^{(N-1)}$ is surjective for some $N > l$, then $\pi : \mathcal{E}^{(p)} \to \mathcal{E}^{(p-1)}$ is surjective for all $p \geq N$.*

Proof. By prolonging if necessary, we may assume without loss of generality that $\mathcal{E}$ is of Frobenius type and additionally that $\pi : \mathcal{E}^{(1)} \to \mathcal{E}$ is surjective. Furthermore, we can reduce the order of the PDE through the replacing of derivatives by new dependent variables (see ([Sei10, p.519]) for more details on why this reduction respects compatibility conditions). Hence, we may also assume that the PDE $\mathcal{E}$ is of first order.

Thus, let $\mathcal{E} \subseteq J^1E$ be a first order PDE of Frobenius type such that $\pi : \mathcal{E}^{(1)} \to \mathcal{E}$ is surjective. The theorem follows by induction if we can prove that the projection $\pi : \mathcal{E}^{(2)} \to \mathcal{E}$ is surjective. Let $x = (x_1, \dots, x_n)$ denote the independent variables. Since the system is of Frobenius type, we can write $u_x = \varphi(x, u)$, that is, one equation for each first derivative. Explicitly, a local representation of $\mathcal{E}$ is given by

$$\mathcal{E} = \begin{cases} u_{x_1} = \varphi_1(x, u) \\ \cdots \\ u_{x_n} = \varphi_n(x, u). \end{cases} \tag{2.45}$$

From this local representation, we see that the only compatibility conditions can arise from cross-derivatives. A cross-derivative is of the form

$$\begin{aligned}
u_{x_i x_j} - u_{x_j x_i} &= \chi_{ij}(x, u, u_x)) \\
&= \chi_{ij}(x, u, \varphi(x, u)) \\
&= \chi_{ij}(x, u) = 0
\end{aligned} \tag{2.46}$$

for some function χ_{ij}. (Here $u_{x_i x_j}$ is notation for the expression obtained from differentiating u_{x_j} with respect to the base variable x_i.) By the assumption that $\pi(\mathcal{E}^1) = \mathcal{E}$, it follows that the functions χ_{ij} vanish identically on $\mathcal{E}$ for all $1 \leq i, j \leq n$. The compatibility conditions of the projection $\mathcal{E}^{(2)} \to \mathcal{E}$ can only come from the following cross-derivatives:

$$u_{x_k x_i x_j} - u_{x_k x_j x_i} = \nu_{kij}(x, u, u_x, u_{xx}) = \nu_{kij}(x, u) = 0. \tag{2.47}$$

Here we have used the Frobenius type condition to express the higher order jets u_x, u_{xx} in terms of (x, u). In order to show that $\pi(\mathcal{E}^{(2)}) = \mathcal{E}$, it suffices to show that the equation $\nu_{kij} = 0$ is automatically satisfied on $\mathcal{E}$ for all $1 \leq i, j, k \leq n$. By using that χ_{ij} vanishes on $\mathcal{E}$, it follows that

$$\nu_{kij}(x, u) = D_k(u_{x_i x_j} - u_{x_j x_i}) = D_k(\chi_{ij}) = 0. \tag{2.48}$$

In other words, the equations $\nu_{kij} = 0$ are not compatibility conditions. We conclude that $\pi : \mathcal{E}^{(2)} \to \mathcal{E}$ is surjective. $\square$

The criterion in Cartan's theorem can be rephrased as follows. Under the assumptions of Cartan's theorem, if after a prolongation (of a Frobenius type PDE) we obtain no new compatibility conditions, then no more compatibility conditions can be found by subsequent prolongations. This allows for an effective implementation of the algorithm in computer algebra systems, which we shall discuss in the next section.

2.3 Prolongation-Projection for the Geodesic Flow

We now wish to apply the prolongation-projection method to find Killing tensors in an n-dimensional (M^n, g) be an n-dimensional spacetime (for most purposes $n = 4$). In Chapter 1 we have seen that a Killing d-tensor corresponds to a degree d polynomial integral of the geodesic flow. Thus, we search for a nontrivial degree d polynomial

$$I_d := a^{i_1 \cdots i_d}(x)\ p_{i_1} \cdots p_{i_d}. \tag{2.49}$$

which Poisson commutes with the Hamiltonian $H = g^{ij} p_i p_j$ of the geodesic flow, that is,

$$\{H, I_d\} = 0. \tag{2.50}$$

Since the Hamiltonian is quadratic in momenta, the Poisson bracket $\{H, I_d\}$ will be of degree $d+1$ in momenta. Therefore, we can instead equate the coefficients of $\{H, I_d\}$ with respect to the momentum variables to zero. On the vector bundle

$$\pi : \mathbb{R}^{n+\binom{n+d-1}{d}} \to \mathbb{R}^n,\ (x, a^{i_1 \cdots i_d}) \mapsto x. \tag{2.51}$$

this leads to a first order linear PDE given by

$$\mathcal{E} := \{F = 0 : F \in \mathrm{coeffs}_p(\{H, I_d\})\} \subseteq J^1. \tag{2.52}$$

Here we use the notation

$$\mathrm{coeffs}_p(P) = \{\text{the coefficients of } P \text{ with respect to the momentum variables } p_1, \ldots, p_n\}). \tag{2.53}$$

Linearity of the PDE eq. (2.52) means that the resulting system of equations is linear in the dependent variables $a^{i_1 \cdots i_d}$ and its first order derivatives $a^{i_1 \cdots i_d}_\alpha$ ($|\alpha| = 1$). We introduce the notation

$$\mathcal{V}_{k,d} = \{a^{i_1 \cdots i_d}_\alpha : |\alpha| \leq k\}. \tag{2.54}$$

It is readily seen that the k'th prolongation $\mathcal{E}^k \subseteq J^{k+1}$ is also a linear PDE, i.e., it is linear in the variables $\mathcal{V}_{k+1,d}$. The following theorem is well-known, cf [Tho86] and [Wol98].

Theorem 2.3.1 (**Killing is Finite Type**). *The PDE $\mathcal{E}$ (eq. (2.52)) defining a Killing d-tensor of the geodesic flow is a first order linear PDE of finite type d. Hence, it can be turned into a formally integrable PDE by Cartan's prolongation–projection method. Moreover, no compatibility conditions can be found before achieving Frobenius type.*

Example 2.3.2. Consider on $\mathbb{R}^2$ the flat metric

$$\mathrm{g} = (dx^1)^2 + (dx^2)^2 \qquad (H = p_1^2 + p_2^2). \tag{2.55}$$

The equation for $I = a(x)\ p_1 + b(x)\ p_2$ to be a linear integral (Killing 1-tensor) is given by the linear first order PDE

$$\mathcal{E} = \begin{cases} a_x = 0 \\ a_y + b_x = 0 \\ b_y = 0. \end{cases} \tag{2.56}$$

In Example 2.2.2 we saw that $\mathcal{E}$ is indeed a PDE of finite type 1. There are no compatibility conditions for this metric.

In the previous section we have made regularity assumptions on the PDE. In practice, this assumption can be circumvented by restricting to an open submanifold of the PDE which does not contain any singularities.

In the next section we shall apply Cartan's prolongation method to the PDEs defining Killing tensors on a spacetime (M, g). This PDE $\mathcal{E}$ is linear so each prolongation $\mathcal{E}^{(k)}$ will be linear in the $(k+1)$-jets of the dependent variables. We shall convert this linear system of equations to a matrix-valued function $M_k(x)$ on the spacetime. In order to make use of computer algebra software, we would like to insert a point to obtain a matrix with *rational* coefficients. The first thing to do is to find the points that work nicely with Cartan's prolongation method. We call a point $z \in (M, \mathrm{g})$ a *regular point* if the function $x \mapsto \mathrm{rank}(M_k(x))$ attains its maximum at z for all $k \geq 0$, that is, at each step we find the maximal number of compatibility conditions. Note that a regular point is generic, i.e., the regular points form an open, dense subset of M. A point is *singular* if it is not regular.

Example 2.3.3 (Toy Example of a Singular Point.). Consider the two functions $f_1(x) = x,\ f_2(x) = x^2$. The Wronskian of f_1 and f_2 is given by $W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$. Since the Wronskian does not vanish identically, we know that f_1 and f_2 are linearly independent on $\mathbb{R}$. However, the Wronskian vanishes at the (singular) point $x = 0$, so the 0- and 1-jets of f_1, f_2 are not enough to detect linear independence. For the singular point, we need to include 2-jets to see the linear independence. Indeed, the 1-jet at 0 of the equation $ax + bx^2 = 0$ yields $a = 0$ and the 2-jet at 0 yields $2b = 0$.

This example builds our intuition that for singular points more prolongations are needed to find all the compatibility conditions. Since the number of equations and jet variables grow drastically while prolonging it is ideal (necessary) to use regular points.

Example 2.3.4 (Singular Points of Geodesic Flow on a Surface of Revolution). A *surface of revolution* is an embedded submanifold $S \subseteq \mathbb{R}^3$ of Euclidean space obtained by a parametrization $(u, \theta) \mapsto (r(u)\cos(\theta), r(u)\sin(\theta), z(u))$ for some functions $r(u), z(u)$ with $r(u) > 0$ for all u. In these coordinates the induced metric on S reads

$$\mathrm{g} = du^2 + r^2 d\theta^2 \qquad H = p_u^2 + \frac{p_\theta^2}{(r(u))^2} \tag{2.57}$$

By Noether's theorem, the circle action generates a linear integral given by $\mu = p_\theta$. By Clairaut's theorem this linear integral equals $r(u)\cos(\theta)$ along geodesics. The points on S with angular coordinate $\theta = \frac{\pi}{2}$ are singular.

We will not comment further on regular or singular points.

2.3.1 Computer Implementation of Cartan's Method

We now discuss several algorithms based on Cartan's prolongation method, which can be implemented in computer algebra programs. We state the algorithms and then prove correctness and termination.

Algorithm 2. (Cartan's Method for Geodesic Flow).
(**Input**: A nonnegative integer d, a local representation of an n-dimensional spacetime (M^n, g), a regular point $z \in M$.)

- Step 1.) Compute the Poisson bracket $\{H, I_d\}$ of the function I_d with the Hamiltonian H of the geodesic flow. (Their Poisson bracket is a polynomial in momenta of degree $d+1$.)
- Step 2.) Collect the coefficients of $\{H, I_d\}$ with respect to the momentum variables. Define the first order linear PDE $\mathcal{E} := \{F = 0 : F \in \mathrm{coeffs}_p(\{H, I_d\})\}$.
- Step 3.) Set $k := 0$.
  - Write $\mathcal{E}^{(k)} = \{F_i^{(k)} = 0 : i \in I\}$ and $\mathcal{V}_{k+1,d} = \{v_j : j \in J\}$ where I and J are indexing sets.
  - Convert the linear system of equations $\mathcal{E}^{(k)}$ into the matrix M_k with entries defined by
  $$(i, j) \mapsto \mathrm{coeff}(F_i^{(k)}, v_j) \tag{2.58}$$
  - Set $\delta_k := \mathrm{columns}(M_k) - \mathrm{rank}(M_k)$.

  If $(k < d)$ or $(k > 0$ and $\delta_k \neq \delta_{k-1})$, increase k by 1 and repeat Step 3.
- Step 4.) Return (δ_k, k)

(**Output**: The dimension $\dim K_d = \delta_k$ of the space of Killing d-tensors. The integer k indicates the number of prolongations we need to achieve formal integrability.)

Proposition 2.3.5. Algorithm 2 is correct and it terminates.

Proof. Termination of algorithm 2 is guaranteed by Cartan's prolongation theorem. We now verify correctness. In view of algorithm 1, it remains to check that all the compatibility conditions of the projection $\pi : \mathcal{E}^{(k)} \to \mathcal{E}$ can be detected using the value $\delta_k = \mathrm{columns}(M_k) - \mathrm{rank}(M_k)$. Since the PDE $\mathcal{E}$ defined by the Poisson bracket is linear, the equations defining the prolongation $\mathcal{E}^{(k)}$ are linear in the dependent variables $\mathcal{V}_{k+1,d}$. Consequently, we can compute compatibility conditions using only linear algebra. $\square$

Example 2.3.6. Consider on the manifold $\mathbb{R}_{>0}(x^1) \times \mathbb{R}(x^2)$ the metric

$$\mathrm{g} = (dx^1)^2 + x^1 \cdot (dx^2)^2. \tag{2.59}$$

The Hamiltonian of the geodesic flow is $H = p_1^2 + \frac{1}{x_1} p_2^2$. We apply algorithm 2 to find the number of Killing 1-tensors.

Step 1.) Let $I_1 = a^1(x)p_1 + a^2(x)p_2$ be an arbitrary polynomial (in momenta) of degree 1. Using the Leibniz rule we compute

$$\begin{aligned}\{H, I_1\} &= \{p_1^2 + \frac{1}{x^1}p_2^2, a^1(x)p_1 + a^2(x)p_2\} \\ &= -2a^1_{[1,0]}p_1^2 - 2\left(a^2_{[1,0]} + \frac{a^1_{[0,1]}}{x^1}\right)p_1p_2 - \left(\frac{a^1}{x_1^2} + 2\frac{a^2_{[0,1]}}{x^1}\right)p_2^2.\end{aligned} \tag{2.60}$$

Step 2.) Collecting the coeffients in *Equation* (2.60) with respect to momentum variables, we obtain the following PDE:

$$\mathcal{E} = \{2a^1_{[1,0]} = 0,\ 2a^2_{[1,0]} + \frac{a^1_{[0,1]}}{x^1} = 0,\ \frac{a^1}{(x^1)^2} + 2\frac{a^2_{[0,1]}}{x^1} = 0\} \tag{2.61}$$

In view of the second equation, we see that $\mathcal{E}$ is not of Frobenius type yet. Theorem 2.3.1 guarantees that the next prolongation is of Frobenius type. We now set $k := 0$.

Step 3.) We write $\mathcal{E}$ as above and the variables as $\mathcal{V}_{1,1} = \{a^1, a^1_{[1,0]}, a^1_{[0,1]}, a^2, a^2_{[1,0]}, a^2_{[0,1]}\}$. By differentiating the left hand sides of the equations from the PDE $\mathcal{E}$ with respect to the variables in $\mathcal{V}_{1,1}$ we obtain the matrix M_0:

$$M_0 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{x^1} & 0 & 2 & 0 \\ \frac{1}{(x^1)^2} & 0 & 0 & 0 & 0 & \frac{2}{x^1}, \end{pmatrix} \tag{2.62}$$

which clearly has rank 3 for all $x^1 > 0$. Thus, $\delta_0 = 6 - 3 = 3$.

We continue the loop, summarizing the results in the following table:

	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$
rows(M_k)	3	9	18	30	45
columns(M_k)	6	12	20	30	42
rank(M_k)	3	9	18	29	41
δ_k	3	3	2	1	1

Step 4. We conclude that g has one Killing field (which is ∂_{x^2}).

The number of rows and columns of the matrices M_k show that doing this algorithm by hand quickly gets unfeasible, even for this low-dimensional example.

Remark 2.3.7 (**Pseudo-Stabilization**). Algorithm 2 requires the use of a regular point. If a singular point is chosen, the algorithm can have two consecutive values in the sequence $(\delta_k)_k$ that agree before additional compatibility conditions are found. We call this phenomenon *pseudo-stabilization.* After pseudo-stabilization the sequence $(\delta_k)_k$ will continue to decrease until it stabilizes at the correct value. It is thus important to run algorithm 2 several times with different points to check if a point is possibly singular.

Finding relations amongst Killing tensors using Cartan's prolongation method

Recall that a *relation* among Killing tensors is an element of the kernel of the map $K_{d_1} \otimes K_{d_2} \to K_{d_1+d_2}$. We can use Cartan's prolongation method to find relations.

Notation. We introduce some more notation. Denote by

$$I_{k,d} := \text{Taylor}(a^{i_1\cdots i_d}, z, k) \cdot p_{i_1} \cdots p_{i_d} \tag{2.63}$$

an arbitrary polynomial of degree d whose coefficient functions are Taylor approximated in a point z up to order k. For example, with 4 independent coordinates we have

$$I_{1,1} = \sum_{i=1}^{4} (a^i_{[1,0,0,0]}(z)(x_1 - z_1) + a^i_{[0,1,0,0]}(z)(x_2 - z_2) + a^i_{[0,0,1,0]}(z)(x_3 - z_3) + a^i_{[0,0,0,1]}(z)(x_4 - z_4) + a^i_{[0,0,0,0]}(z))p_i. \tag{2.64}$$

Algorithm 3. (Finding Relations among Killing Tensors).
(**Input**: Nonnegative integers d_1, d_2, a local representation of an n-dimensional spacetime (M^n, g) and a regular point $z \in M$.)

- Step 1.) For $i = 1, 2$: run algorithm 2 to obtain $\dim K_{d_i}$ and the number of prolongation k_i needed to achieve formal integrability.

  Consider the polynomial I_{k_i+1,d_i}.

- Step 2.) Consider the linear algebraic system of equations $\{\text{Taylor}(c, z, k) = 0 : c \in \text{coeffs}_p(\{H, I_{k_i+1,d_i}\})\}$ on the variables $\mathcal{V}_{k_i+1,d_i}(z) := \{a_\alpha^{i_1\cdots i_{d_i+1}}(z) : |\alpha| \leq k_i + 1\}$, for $i = 1, 2$. Solve the equation and find a basis $v_1^i, \cdots, v^i_{\dim K_{d_i}}$ for its solution set.

  Set $I^j_{k_i+1,d_i} = \text{eval}(I_{k_i+1,d_i}, v^i_j)$ for $1 \leq j \leq \dim K_{d_i}$.

- Step 3.) Set

$$T := \sum_{l=1}^{\dim K_{d_1}} \sum_{m=1}^{\dim K_{d_2}} c_{l,m}\ I^l_{k_1+1,d_1} \cdot I^m_{k_2+1,d_2}. \tag{2.65}$$

  Define $S := \{\text{Taylor}(c, z, d_1 + d_2) : c \in \text{coeffs}_p(T)\}$).

- Step 4.) Solve the linear algebraic system of equations $\{F = 0 : F \in \text{coeffs}_x(S)\}$ in terms of the coefficients $c_{l,m}$, and call the resulting solution space R.

- Step 5.) Return R and $\dim R$.

(**Output**: The relations among Killing tensors of rank d_1 and d_2. The dimension of R is the number of (linearly independent) relations.

Proposition 2.3.8. Algorithm 3 is correct and it terminates.

Proof. For $d \geq 1$, consider the PDE

$$\mathcal{E} = \{F = 0 : F \in \text{coeffs}_p(\{H, I_d\})\} \tag{2.66}$$

on the bundle

$$\pi : \mathbb{R}^{n+\binom{n+d-1}{d}} \to \mathbb{R}^n,\ (x, a^{i_1\cdots i_d}) \mapsto x. \tag{2.67}$$

An arbitrary $(k+1)$-jet $j_z^{k+1}\sigma$ of a section $\sigma = (a^{i_1\cdots i_d}(x))$ can be identified with the Taylor polynomial $I_{k+1,d} = \text{Taylor}(a^{i_1\cdots i_d}, z, k+1)p_{i_1} \cdots p_{i_d}$. Under this correspondence, we have that $j_z^{k+1}\sigma \in \mathcal{E}^{(k)}$ if and only if $\{H, I_{k+1,d}\}$ vanishes up to order k. This observation explains steps 1 and 2.

By Theorem 2.3.1 K_d is determined by d-jets in the sense that we can compute all jets of a Killing d-tensor at a point if we know the d-jets (or equivalently the Taylor polynomial up to order $d+1$). Thus, in step 3 we must include the jets up to order $d_1 + d_2$ in order to determine uniquely the corresponding $(d_1 + d_2)$-tensor. (Remark W: mtaylor command in maple has other idea of up to order k) $\square$

Example 2.3.9. Consider on the manifold $\mathbb{R}^3$ the flat metric (with Hamiltonian)

$$\mathrm{g} = dx^2 + dy^2 + dz^2 \qquad (H = p_x^2 + p_y^2 + p_z^2). \tag{2.68}$$

We show using algorithm 3 that there exists one relation among the Killing 1-tensors.

Step 1. Applying algorithm 2 shows that we need one prolongation to achieve formal integrability and that $\dim K_1 = 6$.

Step 2. The linear integrals up to order 2 (actually up to any order) are given by

$$I_1 = p_x,\ I_2 = p_y, I_3 = p_z,\ I_4 = xp_y - yp_x,\ I_5 = yp_z - zp_y,\ I_6 = xp_z - zp_x. \tag{2.69}$$

Step 3. We set

$$T := \sum_{l=1}^{6}\sum_{m=1}^{6} c_{l,m} I_l \cdot I_m. \tag{2.70}$$

The coefficients of T (up to order 2) with respect to the momentum variables are given by

$$\begin{aligned}
p_x^2 &: c_{1,1} - yc_{1,4} - zc_{1,6} + y^2c_{4,4} + yzc_{4,6} + z^2c_{6,6}\\
p_xp_y &: c_{1,2} + xc_{1,4} - z(c_{1,5} + c_{2,6}) - yc_{2,4} - 2xyc_{4,4}\\
p_xp_z &: c_{1,3} + y(c_{1,5} - c_{3,4}) + xc_{1,6} - zc_{3,6} - 2xzc_{6,6}\\
p_y^2 &: c_{2,2} + xc_{2,4} - zc_{2,5} + x^2c_{4,4} - xzc_{4,5} + z^2c_{5,5}\\
p_yp_z &: c_{2,3} + yc_{2,5} + x(c_{2,6} + c_{3,4}) - zc_{3,5} - 2yzc_{5,5}\\
p_z^2 &: c_{3,3} + yc_{3,5} + xc_{3,6} + y^2c_{5,5} + xyc_{5,6} + x^2c_{6,6}
\end{aligned} \tag{2.71}$$

Step 4. We equate the polynomials in Equation (2.71) to zero and solve. The solution is given by

$$c_{1,5} = -c_{2,6} = c_{3,4}. \tag{2.72}$$

with all the other coefficients being zero.

Step 5. We conclude that we have found one relation among the Killing 1-tensors. It is given by

$$p_x(yp_z - zp_y) + p_z(xp_y - yp_x) = p_y(xp_z - zp_x). \tag{2.73}$$

We discuss an application, which we shall use in later chapters.

Application. We can use algorithms 2 and 3 to prove the existence or nonexistence of irreducible Killing tensors, as we now briefly describe. Suppose that we have a local representation of a spacetime (M, g) and consider the short exact sequence

$$0 \longrightarrow \text{Ker } \iota_2 \longrightarrow S^2K_1 \xrightarrow{\iota_2} K_2 \longrightarrow \text{Coker } \iota_2 \longrightarrow 0. \tag{2.74}$$

Recall that Ker ι_2 consists of the relations among Killing 1-tensors and that Coker ι_2 can be identified with the irreducible Killing 2-tensors. First, using algorithm 2 we compute the dimensions of S^2K_1 and K_2. Then we use algorithm 3 to determine the dimension of the kernel Ker ι_2. Finally, the number of (linearly independent) irreducible Killing 2-tensors is given by

$$\dim \text{Coker } \iota_2 = \dim K_2 - \dim S^2K_1 + \text{Ker } \iota_2. \tag{2.75}$$

Here we used the exact sequence property to relate the dimensions. This method can be readily generalized to higher order Killing tensors.

Cartan's Prolongation Method for a regular Metric

We can also use Cartan's method for a family of metrics g_f parametrized by a function f. Since computer algebra programs assume regularity (i.e., the function f is not assumed to satisfy nontrivial equations.) In order to circumvent this problem, we need to modify algorithm 2 to keep track of possible nontrivial equations on the function f which alter the results of the prolongation method. For simplicity, we shall assume that f is a function of just one variable.

Consider the PDE $\mathcal{E}$ defining a Killing tensor. In Cartan's prolongation method we can consider two sets of variables, those that are determined and those that are free. The determined jet variables are written in terms of the free variables. The Killing d-tensor PDE is determined by the d-jets, so all jets of order greater than d are determined. After reaching Frobenius type, we prolong to find compatibility conditions. These compatibility conditions increase the number of determined variables, and in turn decrease the number of free variables. We keep prolonging until we acquire stabilization (i.e., no more compatibility conditions can be found). The next algorithm is based on this idea, with the key feature that we control the assumptions made on f.

Algorithm 4. (Cartan's Method for a General Metric).
(**Input**: A nonnegative integer d, a family of local representations of an n-dimensional spacetime (M^n, g_f) depending on a function f of one variable.)

- Step 1.) Compute the Poisson bracket $\{H, I_d\}$ of the function I_d with the Hamiltonian H of the geodesic flow. Consider the PDE $\mathcal{E} := \{F = 0 : F \in \mathrm{coeffs}_p(\{H, I_d\})\}$.

- Step 2.) For k from 0 to $d-1$, do the following. Consider the prolongation $\mathcal{E}^{(k)}$. Express, using the equations defining $\mathcal{E}^{(k)}$, the maximal number of $(k+1)$-jets in terms of the remaining *free variables*. We set

  $\mathrm{sub}_{k+1} :=$ {expressions of $(k+1)$-jets in terms of the free variables $\subseteq \mathcal{V}_{k+1,d}$}

  (Note that the number of free variables increase as we prolong.)
  Denote by $\mathrm{sub}_{1\cdots d} = \cup_{j=1}^{d} \mathrm{sub}_j$ the expressed jets up to order d.

- Step 3.) For the Frobenius type PDE $\mathcal{E}^{(d)}$, express *all* $(d+1)$-jets in terms of the free variables ($\subseteq \mathcal{V}_{d,d}$) by using the expressions obtained in $\mathrm{sub}_{1\cdots d}$. Set

  $\mathrm{sub}_{d+1} :=$ {expressions of all $(d+1)$-jets in terms of the free variables $\subseteq \mathcal{V}_{d,d}$}.

- Step 4.) Set $k := 1$. For $\mathcal{E}^{(d+k)}$, do the following.

  – Express all $(d + k + 1)$-jets in terms of the free variables ($\subseteq \mathcal{V}_{d,d}$) and denote the set of these expressions by sub_{d+k+1}.

  – After expressing, we collect the unused equations into

  $$\mathrm{int}_{(d+k)} := \{\text{the remaining equations defining } \mathcal{E}^{(d+k)}\}$$

  The set $\mathrm{int}_{(d+k)}$ consists of potential compatibility conditions.

  – Simplify $\mathrm{sub}_{1\cdots d}$ using these compatibility conditions, which reduces the number of free variables. Here, potential conditions on f can arise (as a consequence of division by differential expressions in f). Keep track off this "branching".

  Increase k by 1 and repeat step 4 until no additional compatibility conditions can be found.

- Step 5.) Return the number of remaining free variables in $\mathrm{sub}_{1\cdots d}$ for each branch (nontrivial equations satisfied by f).

(**Output**: The dimension $\dim K_d$ of Killing d-tensors as a function of f.)

Chapter 3

Metrics in General Relativity

3.1 Classifying Spacetimes

We discuss several possible approaches to classifying Lorentzian spacetimes.

Petrov Classification

The Petrov classification is an algebraic classification of the Weyl tensor. We briefly recall the situation in dimension 4. The Riemann curvature tensor is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \; R(X,Y,Z,W) = \mathrm{g}(R(X,Y)Z,W) \tag{3.1}$$

By contraction of the curvature tensor we can construct the *Ricci curvature* and *scalar curvature*, respectively given by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y) \qquad \mathrm{Scal} := \mathrm{tr}(\mathrm{Ric}). \tag{3.2}$$

The Weyl tensor is the completely trace-free $(0,4)$-tensor defined by

$$W := R - \frac{1}{2}\left(\mathrm{Ric} - \frac{\mathrm{Scal}}{4}\mathrm{g}\right) \owedge \mathrm{g} + \frac{\mathrm{Scal}}{24}\mathrm{g} \owedge \mathrm{g}. \tag{3.3}$$

Here $\cdot \owedge \cdot$ is the Kulkarni–Nomizu product for symmetric $(0,2)$-tensors. By the Ricci decomposition [Lee16, p.216], we can write the curvature tensor in terms of the Weyl tensor and two additional tensors coming from the Ricci curvature. The Weyl tensor enjoys the same algebraic symmetries as the curvature tensor. Hence, we can view the Weyl tensor at an event q in spacetime as a curvature operator taking bivectors to bivectors,

$$W : \wedge^2 T_q M \to \wedge^2 T_q M. \tag{3.4}$$

One then proceeds by converting the (6×6)-matrix associated to this linear map to a complex symmetric trace-free (3×3)-matrix [Ste+09, p.49]. The Petrov classification is then obtained by classifying the possible sets of eigenvalues (with multiplicites). This leads to six different Petrov types (I, II, D, III, N, O). For example, an event is said to be of Petrov type O if its Weyl tensor vanishes. By the Weyl–Schouten theorem [Ste+09] a 4-dimensional spacetime is conformally flat if and only if its Weyl tensor vanishes everywhere.

Note that the curvature tensors of *vacuum* spacetime are distinguished by the Petrov classification, since vacuum spacetimes are Ricci-flat and so $R = W$. For physical interpretations of the different Petrov types in terms of tidal forces see [d'I92].

Scalar Curvature Invariants

Ideally, we can locally classify spacetimes (up to isometry) by finding some simple criteria to decide if there exists an isometry between two spacetimes. The method of scalar curvature invariants provides some progress in this direction.

Consider a local representation of two Lorentzian metrics

$$\mathrm{g} = \mathrm{g}_{ij}(x)\ dx^i dx^j \qquad \tilde{\mathrm{g}} = \tilde{\mathrm{g}}_{ij}(\tilde{x})\ d\tilde{x}^i d\tilde{x}^j \tag{3.5}$$

Given a coordinate transformation $\varphi : x \mapsto \tilde{x}(x)$ the isometry condition $\varphi^*\tilde{\mathrm{g}} = \mathrm{g}$ reads

$$\tilde{\mathrm{g}}_{ij}(\tilde{x}(x))\ d\tilde{x}^i d\tilde{x}^j = \mathrm{g}_{ij}(x)\ dx^i dx^j \tag{3.6}$$

This equation defines a nonlinear first order PDE on the functions $\tilde{x}^i(x)$, which in practice is virtually impossible to solve explicitly. Instead we construct simple scalar invariants to compare both metrics. Here a scalar invariant means a scalar function that is invariant under the pseudogroup of coordinate transformations, that is, we know how the object transforms under coordinate changes. We observe that if g and $\tilde{\mathrm{g}}$ are isometric, then their invariants will be the same. A difficulty that arises is that the invariants are computed with respect to different coordinate systems, so it might be hard to compare them.

The scalar curvature is an invariant that we can use to compare metrics. For example, if the scalar curvature $\mathrm{Scal}(x)$ of the first metric is constant and the $\widetilde{\mathrm{Scal}}(\tilde{x})$ nonconstant, then we conclude that $\mathrm{g}, \tilde{\mathrm{g}}$ are not isometric. However, if the scalar curvatures are both (non)constant, then we cannot rule out that the spacetimes are isometric. In this case, we may look at other scalar invariants. A scalar curvature invariant [CHP09] is a scalar function obtained from the curvature R, Ricci curvature Ric, the Weyl tensor W and their covariant derivatives. Thus, some examples of scalar curvature invariants are the scalar curvature Scal, the Kretschmann scalar $R_{abcd}R^{abcd}$, the Weyl invariant $W_{abcd}W^{abcd}$ and $\nabla_i \mathrm{Ric}_{jk} \nabla^i \mathrm{Ric}^{jk}$. In order to compare both metrics, we can look at syzygies (relations) in the set of scalar curvature invariants. If the syzygies are different, the metrics are not isometric.

A natural question to ask whether metrics can be classified using their scalar curvature invariants. This is true in the Riemannian case (see the corollary after theorem 1 in note 19 on p.357 in [KN63] together with [Wey39]). However, there exist Lorentzian spacetimes whose scalar curvature invariants all vanish (so-called *VSI Spacetimes*). It is thus impossible to distinguish a VSI spacetime from Minkowski space using only scalar curvature invariants. It is well-known that all VSI spacetimes are of Kundt class ([Pra+02]). A Kundt metric can be written in coordinates (x^1, x^2, u, v) as [Col+09]

$$\mathrm{g} = 2dudv + 2h(x^1, x^2, u)(du^3)^2 + W_i(x^1, x^2, u, v)dx^i du + g_{ij}(x^1, x^2, u)dx^i dx^j \tag{3.7}$$

for some smooth functions h, W_i, g_{ij}. (It is vacuum if and only if $\partial^2_{x^1} h + \partial^2_{x^2} h = 0$.)

Cartan's Method of Moving Frames

Fortunately, it is still possible to classify Lorentzian spacetimes using Cartan's method. In Cartan's method of moving frames, one uses the orthonormal frame bundle to construct invariants from the components of the Riemann curvature tensor and its covariant derivatives. For 4-dimensional spacetimes, this procedure was carefully specified by Karlhede [Kar80] and so is nowadays called the Cartan–Karlhede algorithm. We briefly sketch an approach following [KMS19].

Let (M, g) be a Lorentzian spacetime. The orthonormal frame bundle $\mathcal{P} \to M$ is a principal bundle with structure group the Lorentz group $O(3,1)$. A fiber over $x \in M$ consists of all the orthonormal frames of g at x. The free and transitive action of $O(3,1)$ preserving the fibers is

$$O(3,1) \times \mathcal{P} \to \mathcal{P},\ E_i' := \Lambda \cdot E_i := \Lambda_i^j E_j \tag{3.8}$$

A key observation is that the components of a tensor can be viewed as a *function* on the orthonormal frame bundle $\mathcal{P}$. We define the *Cartan invariants* of order k to be rational combinations of components of the Riemann curvature tensors and its k'th covariant derivatives. In particular, Cartan invariants are functions on the total space $\mathcal{P}$. If a Cartan invariant is invariant under the action of the structure group (i.e. constant on the fibers), then it descends to a function on the base M. We call functions on M obtained in this way *invariants* on M. The advantage of this approach is the following: to establish the existence of an isometry $\varphi : (M, \mathrm{g}) \to (\tilde{M}, \tilde{\mathrm{g}})$ 'downstairs', we can instead compare their invariants that are obtained from the Cartan invariants 'living upstairs':

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \tilde{\mathcal{P}} \arrow[d] \\ M \arrow[r, "\varphi"] & \tilde{M}. \end{tikzcd}$$

We start by considering the Cartan invariants of order 0. If a Cartan is invariant under the structure group, we obtain an invariant on M. If it is not, we aim to set this Cartan invariant equal to a constant by fixing the group parameters. In turn this reduces the structure group from $O(3,1)$ to a lower-dimensional subgroup H_0. Both the structure group and total space are reduced in dimension. By doing this for all Cartan invariants of order 0, we obtain a principal H_0 subbundle $\mathcal{P}_0 \to M$ of the bundle $\mathcal{P} \to M$. Here the new total space $\mathcal{P}_0$ consists of all the *admissible* orthonormal frames.

Next, we look at the Cartan invariants of order 1. We collect the new invariants on M. The Cartan invariants which are not constant on the fibers allow us to reduce the structure group and total space. This reduction yields a principal H_1-subbundle $\mathcal{P}_1 \to M$ of $\mathcal{P}_0 \to M$ with $H_1 \subseteq H_0$. We continue this process for higher order Cartan invariants. One counts decrease of the dimension of the structure group and, simultaneously, the number of functionally independent invariants on the base M. These represent the vertical and horizontal freedom in the frame bundle (cf. [Bro+18]), respectively. The stabilization should occur for both numbers, hence the number of steps cannot exceed $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ where $n = \dim M$.

After doing the above procedure for two metrics $\mathrm{g}, \tilde{\mathrm{g}}$, one can now decide if the two are isometric or not. This is done by examining the syzygies among the invariants on the base (cf. [KMS19]).

Differential Invariants

An alternative approach to classification is the method of differential invariants. If a Lie (pseudo)group acts on the total space E of a bundle, one can prolong this action to its jet bundles [Olv95]. A differential invariant of order k is then just a function $I : J^kE \to \mathbb{R}$ which is invariant under this prolonged group action. In [KMS19] differential invariants for Kundt waves are computed and compared with the invariants obtained from the Cartan–Karlhede algorithm.

Koutras–McIntosh Metric

We now introduce the metric that we shall study in the next chapters using the techniques of chapter 2. We define the *Koutras–McIntosh metric* to be the 2-parameter family of metrics which is given in coordinates x^1, x^2, x^3, x^4 by

$$\mathrm{g} = 2(ax^1+b)dx^3dx^4 - 2adx^1dx^3 + \{2f(x^3)(ax^1+b)((x^1)^2+(x^2)^2) - a^2(x^4)^2\}(dx^3)^2 - (dx^1)^2 - (dx^2)^2 \tag{3.9}$$

Setting $(a, b) = (0, 1)$ we obtain a conformally flat pp-wave

$$\mathrm{g} = 2dx^3dx^4 + 2f(x^3)((x^1)^2 + (x^2)^2)(dx^3)^2 - (dx^1)^2 - (dx^2)^2 \tag{3.10}$$

and setting $(a, b) = (1, 0)$ we obtain the *Wils metric*

$$\mathrm{g}_{\mathrm{Wils}} = 2\,x^1dx^3dx^4 - 2x^4dx^1dx^3 + \{2f(x^3)x^1((x^1)^2 + (x^2)^2) - (x^4)^2\}(dx^3)^2 - (dx^1)^2 - (dx^2)^2 \tag{3.11}$$

These latter two spacetimes are pure radiation solutions, that is, they satisfy Einstein's equations with energy-momentum tensor given by $T = \phi\; l \otimes l$ for some scalar field ϕ and a null one-form l. A. Koutras and C. McIntosh showed in [KM96] that the metric (3.9) has no scalar curvature invariants.

Plane Waves and Singularities

We discuss the physical relevance of metrics (3.10), following the lecture notes *Plane Waves and Penrose Limits* by M. Blau [Bla11].

A special class among the pp-waves are the gravitational *plane waves* which in Brinkmann coordinates (x^1, x^2, u, v) take the form

$$\mathrm{g} = 2\;dudv + A_{ab}(u)\;x^ax^bdu^2 + (dx^1)^2 + (dx^2)^2 \tag{3.12}$$

for some symmetric matrix $A_{ab}(u)$ called the *wave profile*. A key feature of a plane wave is the existence of a parallel null vector field, namely the coordinate field ∂_v. In particular, we see that conformally flat pp-waves (3.10) are plane waves with $A_{ab}(u) = \delta_{ab}f(u)$.

Scalar curvature invariants are a useful tool in proving the existence of physical singularities (e.g. the Kretschmann scalar in Schwarzschild spacetime). For plane waves all scalar curvature invariants vanish, so one might wonder if physical singularities exist in plane waves. This question can be answered by examining the Jacobi equation for variations of geodesics, as we now briefly discuss. Let $\bar{c}(t, s)$ denote a variation of a geodesic c so that $s \mapsto \bar{c}(t, s)$

is a geodesic for all t. Then the vector field $J : s \mapsto \frac{\partial \bar{c}}{\partial t}(0, s)$ along c is a *Jacobi field*. This means that J satisfies the *Jacobi equation*

$$\ddot{J} + R(J, \dot{c})\dot{c} = 0. \tag{3.13}$$

(Here the second derivative of J is taken with respect to the connection.) Physically, the Jacobi equation describes the tidal forces on geodesic congruences. By choosing a Jacobi field with $J^u = 0$ and taking s to be the affine parameter along the geodesic variation, one can show (cf.[HE73, Section 4.1, 4.2], [Bla11, p.18], [Str12, p.68]) that the Jacobi equation implies

$$\frac{d^2}{ds^2} J^a = A_{ab}(u(s))J^b. \tag{3.14}$$

If the wave profile A_{ab} is singular for some u, then the tidal forces become infinite in which case we have a physical singularity.

R. Penrose shows in the paper *Any Space-Time has a Plane Wave as its Limit* [Pen76] that to any pair (g, γ) of a spacetime (M, g) together with a null geodesic γ one can associate a plane wave, the so-called *Penrose limit*. Determining the Penrose limit of a spacetime entails computing the wave profile $A_{ab}(u)$ from the null geodesic and the metric. In the original description the Penrose limit arises by writing the metric in terms of a 1-parameter family of coordinates and taking the limit of the parameter. In [Bla+04] it is shown that the wave profile A_{ab} can also be computed from

$$A_{ab}(u) = -R_{a+b+}|_{\gamma(u)} \tag{3.15}$$

with respect to a suitable *pseudo-orthonormal* frame (E^+, E^-, E^a). (On the left hand side u is a coordinate of the plane wave, on the right hand side u is the affine parameter of the null geodesic.)

For a large class of spacetimes with singularities, the Penrose limit, obtained from a null geodesic approaching a singularity, has waveprofile $A_{ab} \propto \frac{1}{u^2}$ ([Bla+04]). In Chapter 4, we apply Cartan's prolongation method to the plane wave (3.10) with $f(x^3) = \frac{1}{(x^3)^2}$.

3.2 Integrability in General Relativity

Kerr Metric

The *Kerr solution* in Boyer–Lindquist coordinates $q = (r, t, \theta, \varphi)$ is given by the two-parameter family of metrics

$$\begin{aligned} \mathrm{g}_{\mathrm{Kerr}} =& \frac{1}{\Xi}\left[-(\Delta - a^2\sin^2(\theta))\, dt^2 + 2a\sin^2(\theta)(\Delta - r^2 - a^2)\, dt\, d\varphi \right. \\ & \left. + \sin^2(\theta)((r^2 + a^2)^2 - \Delta a^2 \sin^2(\theta))\, d\varphi^2 \right] + \Xi\left(\frac{dr^2}{\Delta} + d\theta^2\right) \end{aligned} \tag{3.16}$$

where the two quantities Δ, Ξ are defined by

$$\Delta(r) = r^2 - 2mr + a^2, \ \Xi(r, \theta) = r^2 + a^2\cos^2(\theta). \tag{3.17}$$

The geometry of the Kerr metric describes a stationary rotating black hole with total mass m and angular momentum $J := am$. The metric coefficients are independent of the variables t, φ, it follows that $\partial_t, \partial_\varphi$ are Killing vector fields. In view of the Arnold–Liouville–Mineur theorem, an additional integral would be beneficial to integrate the geodesic equations. In order to obtain the geodesics, B. Carter used the Hamilton–Jacobi method. This essentially entails to solving the Hamilton–Jacobi equation

$$H\left(q^i, \frac{\partial S}{\partial q^i}\right) = \mathrm{g}^{ij}_{\mathrm{Kerr}}(q)\frac{\partial S}{\partial q^i}\frac{\partial S}{\partial q^j} = -\frac{1}{2}\mu^2 \qquad (\mu = \text{constant}) \tag{3.18}$$

for the generating function S of a symplectomorphism. Carter [Car68] solved this first order PDE by additive separation of variables. The separation constant $\mathcal{C}$ corresponding to the θ-dependent part, called *Carter's constant*,

$$\mathcal{C} := p_\theta^2 + \cos^2(\theta)\left(a^2(\mu^2 - p_t^2) + \frac{p_\varphi^2}{\sin^2(\theta)}\right). \tag{3.19}$$

is the fourth integral of motion. By means of this complete set of integrals, he solved the geodesics by quadrature.

The main takeaway of this story is that the additive separability of the Hamilton–Jacobi equation is closely related to the existence of *quadratic* integrals. Woodhouse' article [Woo75] explores this relation in detail.

The techniques from Chapter 2 provide another way of proving the existence of a quadratic integral. Application of algorithm 2 to the Kerr metric (with parameters $m = 50, a = \frac{1}{2}$) gives the following tables. In each table we denote the prolongation $\mathcal{E}^{(k)}$ together with its corresponding value δ_k.

Linear	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$
δ	10	10	2	**2**

Quadratic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$
δ	30	50	50	6	5	**5**

The first computation shows that p_t, p_φ are the only linear integrals for the Kerr metric. The second table shows that there are 5 quadratic integrals. The linear integrals and the Hamiltonian account for 4 quadratic integrals $(p_t^2, p_\varphi p_t, p_\varphi^2, H)$. It follows that there is an additional irreducible quadratic integral.

Summarizing, the Hamilton–Jacobi method provides an effective way of finding quadratic integrals. Cartan's prolongation method allows us to prove the (non-)existence of higher order Killing tensors. Even though Cartan's method does not give us an explicit form of the integrals, we can use algorithm 3 to obtain the integrals up to a given order.

Chapter 4

Killing Tensors of Conformally Flat pp-Waves

The metric of a conformally flat pp-wave is given by

$$\mathrm{g} = 2dx^3dx^4 + 2f(x^3)\ \left((x^1)^2 + (x^2)^2\right)\ (dx^3)^2 - (dx^1)^2 - (dx^2)^2 \tag{4.1}$$

where f is a function of the coordinate x^3. The Hamiltonian of the geodesic flow is given by

$$H := -p_1^2 - p_2^2 + 2p_3p_4 - 2f(x^3)((x^1)^2 + (x^2)^2)p_4^2 \tag{4.2}$$

The paper *Symmetry classes of pp-waves* by Sippel and Goenner [SG86] has classified pp-waves in terms of their isometry group. For *conformally flat* pp-waves there are three classes:

- the generic case has $\dim K_1 = 6$;
- the second class given by $f = k$ for some constant $k \neq 0$ has $\dim K_1 = 7$ and the seventh integral is given by p_3;
- the third class given by $f = \frac{k}{(x^3)^2}$ for some constant $k \neq 0$ has $\dim K_1 = 7$ and the seventh integral is given by $x^3p^3 - x^4p_4$.

In light of this, we will consider four metrics.

- Metric 1: $f(x^3) = 1$
- Metric 2: $f(x^3) = x^3$
- Metric 3: $f(x^3) = (x^3)^2$.
- Metric 4: $f(x^3) = \frac{1}{(x^3)^2}$.

Metric 1 belongs to the second class, metrics $2, 3$ belong to the generic case and metric 4 belongs to the third class. We will apply Cartan's prolongation method (algorithm 2) to the PDE

$$\mathcal{E} = \{F = 0 : F \in \mathrm{coeffs}_p(\{H, I_d\})\} \tag{4.3}$$

to compute the number of Killing d-tensors for $d = 1, 2, 3, 4$. Given Killing tensors of degree d_1 and d_2 their product will be a Killing tensor of degree $d_1 + d_2$. Thus, the existence of Killing 1-tensors imply the existence of higher order Killing tensors. However, relations between lower order Killing tensors will 'lower' the number of (independent) higher order Killing tensors. On the other hand, there might be additional irreducible Killing tensors. In order to compute the relations among Killing tensors we shall apply algorithm 3. In turn this will allow us to confirm the (non-)existence of irreducible Killing tensors up to order 4.

Note on the computational complexity.

We briefly discuss the computational difficulties associated with Cartan's method and how we deal with them. The dimension of the prolongation matrix M_k from algorithm 2 equals

$$\text{rows}(M_k) = \binom{n+d}{d+1}\left(\sum_{i=0}^{k}\binom{n+i-1}{n-1}\right),\ \text{columns}(M_k) = \binom{n+d-1}{d}\left(\sum_{i=0}^{k+1}\binom{n+i-1}{n-1}\right) \tag{4.4}$$

In particular, we see that the number of rows grows faster than the number of columns. We highlight several elements that have made the computer implementation more efficient:

- (**LinBox**). The LinBox package ([Dum+02]) in Sage allows for incredibly fast rank computations of large sparse integer matrices. For example, computing the rank of the quartic prolongation matrix M_{19} for metric 2 with size $(495880) \times (371910)$ took less than an hour. In comparison, rank computations of smaller matrices (say 50000 by 40000) would take several days in Maple or not give a result at all. Thanks to LinBox, the time to compute the ranks is negligible. The generating of a prolongation matrix takes by far the longest time of the steps in algorithm 2.
- (**Exploiting Sparsity**.) The prolongation matrices M_k that we encounter here are sparse (with density < 0.001). It is important that the generation of the matrix reflects this. We generate a matrix with only zeroes and then substitute the nonzero values.
- (**Combinatorial Description of Prolongations**.) For the quartic case, we used a combinatorial description of the prolongation equations. We demonstrate this for metric 2. Since I_4 is of degree 4, we have that $\{H, I_4\}$ is of degree 5 in momenta. Thus, we can write $\{H, I_4\} = \text{coeff}_\tau p^\tau$ where $p^\tau = p_1^{\tau_1} p_2^{\tau_2} p_3^{\tau_3} p_4^{\tau_4}$. Given a multi-index τ of length 5, we obtain the p^τ-coefficient in terms of the coefficients of I:

$$\begin{aligned}\text{coeff}_\tau(\{H, I_4\}) &= 2\partial_1(a^{\tau-1_1}) + 2\partial_2(a^{\tau-1_2}) - 2\partial_4(a^{\tau-1_3}) - 2\partial_3(a^{\tau-1_4})\\ &\quad + 4x^3((x^1)^2 + (x^2)^2)\partial_4(a^{\tau-1_4}) - 2((x^1)^2 + (x^2)^2)\frac{(\tau + 1_3 - 2\cdot 1_4)!}{(\tau - 2\cdot 1_4)}\\ &\quad - 4x^1x^3\frac{(\tau + 1_1 - 2\cdot 1_4)!}{(\tau - 2\cdot 1_4)!}a^{\tau+1_1-2\cdot 1_4} - 4x^2x^3\frac{(\tau + 1_2 - 2\cdot 1_4)!}{(\tau - 2\cdot 1_4)!}a^{\tau+1_2-2\cdot 1_4}\end{aligned} \tag{4.5}$$

  Using the product rule for multi-index notation, we can subsequently determine the general expression for $\partial^\alpha(\text{coeff}_\tau(\{H, I_4\}))$, where α is a multi-index. In this way we obtain the equations of the prolongation as a function of the multi-indices τ and α. This combinatorial description significantly reduces the time needed to generate the equations in Maple, especially as the order increases.

4.1 Dimensions of Space of Killing Tensors

Metric 1 ($f(x^3) = 1$)

We present the results of algorithm 2 applied to metric 1 for the degrees $d = 1, 2, 3, 4$. In the tables below we collect for a given degree d, the k'th prolongation $\mathcal{E}^{(k)}$ together with its value δ_k obtained from step 3. The generic point we used for algorithm 2 is $z := (1, 2, 3, 4) \in \mathbb{R}^4$. The used Maple worksheet "ConformallyFlatPPWave1_Algorithm2" is supplied separately in the attached zip-file.

Linear	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$
δ	10	10	7	**7**

Table 4.1: The values of δ obtained in algorithm 2 (step 3) for $d = 1$.

Quadratic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	30	50	50	37	29	28	**28**

Table 4.2: The values of δ obtained in algorithm 2 (step 3) for $d = 2$.

Cubic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$
δ	65	125	175	175	134	101	87	84	**84**

Table 4.3: The values of δ obtained in algorithm 2 (step 3) for $d = 3$.

Quartic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	119	...	...	490	490	...	299
cont'd	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	
δ	...	...	...	...	210	**210**	...

Table 4.4: The values of δ obtained in algorithm 2 (step 3) for $d = 4$.

Recall that the PDE $\mathcal{E}$ defined by the Poisson bracket $\{H, I_d\}$ is of finite type d. This manifests itself by the fact that the value δ_d of $\mathcal{E}^{(d)}$ is maximal (10 for the linear case, 50 for the quadratic case, 175 for the cubic case and 490 for the quartic case). By prolonging further (as in step 3) we obtain compatibility conditions. As a result, the rank of the matrix M_k (2.58) increases and so the value δ_k decreases. If two subsequent values δ_k, δ_{k+1} are equal (with $k \geq d$), the sequence of δ-values stabilize and we can read off the number of Killing d-tensors. These values are printed bold-faced in the tables above. We conclude that

$$\dim K_1 = 7, \ \dim K_2 = 28, \ \dim K_3 = 84, \ \dim K_4 = 210. \tag{4.6}$$

Metric 2 ($f(x^3) = x^3$)

We present the results of algorithm 2 applied to metric 2 for the degrees $d = 1, 2, 3, 4$. In the tables below we collect for a given degree d, the k'th prolongation $\mathcal{E}^{(k)}$ together with its value δ_k obtained from step 3. The generic point we used for algorithm 2 is $z := (1, 2, 3, 4) \in \mathbb{R}^4$. The used Maple worksheet "ConformallyFlatPPWave2_Algorithm2" is supplied separately in the attached zip-file.

Linear	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(3)}$
δ	10	10	7	6	**6**

Table 4.5: The values of δ obtained in algorithm 2 (step 3) for $d = 1$.

Quadratic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$
δ	30	50	50	35	28	24	22	**22**

Table 4.6: The values of δ obtained in algorithm 2 (step 3) for $d = 2$.

Cubic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	65	125	175	175	131	100	...
cont'd	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	$\mathcal{E}^{(13)}$
δ	...	68	...	64	63	62	**62**

Table 4.7: The values of δ obtained in algorithm 2 (step 3) for $d = 3$.

Quartic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$
δ	119	...	...	490	490	389	292	245	213	194
cont'd	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	$\mathcal{E}^{(13)}$	$\mathcal{E}^{(14)}$	$\mathcal{E}^{(15)}$	$\mathcal{E}^{(16)}$	$\mathcal{E}^{(17)}$	$\mathcal{E}^{(18)}$	$\mathcal{E}^{(19)}$
δ	181	170	163	158	156	154	152	150	148	**148**

Table 4.8: The values of δ obtained in algorithm 2 (step 3) for $d = 4$..

We conclude that the dimensions of the space K_d of Killing d-tensors are given by

$$\dim K_1 = 6,\ \dim K_2 = 22,\ \dim K_3 = 62,\ \dim K_4 = 148. \tag{4.7}$$

Metric 3 ($f(x^3) = (x^3)^2$)

We present the results of algorithm 2 applied to metric 3 for the degrees $d = 1, 2, 3, 4$. In the tables below we collect for a given degree d, the k'th prolongation $\mathcal{E}^{(k)}$ together with its value δ_k obtained from step 3. The generic point we used for algorithm 2 is $z := (1, 2, 3, 4) \in \mathbb{R}^4$. The used Maple worksheet "ConformallyFlatPPWave3_Algorithm2" is supplied separately in the attached zip-file.

Linear	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(3)}$
δ	10	10	7	6	**6**

Table 4.9: The values of δ obtained in algorithm 2 (step 3) for $d = 1$.

Quadratic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$
δ	30	50	50	35	28	24	22	**22**

Table 4.10: The values of δ obtained in algorithm 2 (step 3) for $d = 2$.

Cubic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	65	125	175	175	131	100	...
cont'd	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	$\mathcal{E}^{(13)}$
δ	...	68	...	64	63	62	**62**

Table 4.11: The values of δ obtained in algorithm 2 (step 3) for $d = 3$.

Quartic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$
δ	119	...	...	490	490	389	292	245	213	194
cont'd	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	$\mathcal{E}^{(13)}$	$\mathcal{E}^{(14)}$	$\mathcal{E}^{(15)}$	$\mathcal{E}^{(16)}$	$\mathcal{E}^{(17)}$	$\mathcal{E}^{(18)}$	$\mathcal{E}^{(19)}$
δ	181	170	163	158	156	154	152	150	148	**148**

Table 4.12: The values of δ obtained in algorithm 2 (step 3) for $d = 4$.

Note that the δ-values are exactly the same as for metric 2. We conclude that

$$\dim K_1 = 6,\ \dim K_2 = 22,\ \dim K_3 = 62,\ \dim K_4 = 148. \tag{4.8}$$

Metric 4 ($f(x^3) = \frac{1}{(x^3)^2}$)

We present the results of algorithm 2 applied to metric 4 for the degrees $d = 1, 2, 3, 4$. In the tables below we collect for a given degree d, the k'th prolongation $\mathcal{E}^{(k)}$ together with its value δ_k obtained from step 3. The generic point we used for algorithm 2 is $z := (1, 2, 1, 4) \in \mathbb{R}^4$. The used Maple worksheet "ConformallyFlatPPWave4_Algorithm2" is supplied separately in the attached zip-file.

Linear	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$
δ	10	10	7	**7**

Table 4.13: The values of δ obtained in algorithm 2 (step 3) for $d = 1$.

Quadratic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	30	50	50	35	29	28	**28**

Table 4.14: The values of δ obtained in algorithm 2 (step 3) for $d = 2$.

Cubic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$
δ	65	125	175	175	...	...	87	84	**84**

Table 4.15: The values of δ obtained in algorithm 2 (step 3) for $d = 3$.

Quartic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	119	...	...	490	490	389	294
cont'd	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	
δ	245	223	215	211	210	**210**	...

Table 4.16: The values of δ obtained in algorithm 2 (step 3) for $d = 4$.

We conclude, for metric 4, that:

$$\dim K_1 = 7,\ \dim K_2 = 28,\ \dim K_3 = 84,\ \dim K_4 = 210. \tag{4.9}$$

4.2 Relations and Irreducible Killing Tensors

Relations and Irreducible Killing Tensors of Metric 1

In this section we determine the relations among Killing tensors using algorithm 3. The computations were done in Maple, see the Maple file "ConformallyFlatPPWave1_Algorithm3". The generic point we used is again $z = (1, 2, 3, 4)$. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\ \iota_2 \longrightarrow S^2K_1 \xrightarrow{\iota_2} K_2 \longrightarrow \mathrm{Coker}\ \iota_2 \longrightarrow 0. \tag{4.10}$$

Algorithm 3 shows that there is one relation, that is,

$$\dim \mathrm{Ker}\ \iota_2 = 1. \tag{4.11}$$

We have that $\dim K^2 = 28$ and $\dim S^2K_1 = \binom{8}{2} = 28$, so we compute that

$$\dim \mathrm{Coker}\ \iota_2 = 28 - 28 + 1 = 1. \tag{4.12}$$

Thus, there exists an irreducible Killing 2-tensor. We will show later that it is not the Hamiltonian. Actually, Maple is able to solve the Killing 2-tensors explicitly. The irreducible Killing 2-tensor is given by

$$I_q := x^3((dx^1)^2+(dx^2)^2)-x^1dx^1dx^3-x^2dx^2dx^3-x^3dx^3dx^4+(2x^4-2x^3((x^1)^2+(x^2)^2))\,(dx^3)^2 \tag{4.13}$$

or equivalently, by the quadratic integral

$$I_q = x^3p_1^2 + x^1p_1p_4 + x^2p_2p_4 - 2x^3p_3p_4 + 2x^3((x^1)^2 + (x^2)^2 + x^4)p_4^2. \tag{4.14}$$

Next, we look at the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\ \iota_3 \longrightarrow K_1 \otimes K_2 \xrightarrow{\iota_3} K_3 \longrightarrow \mathrm{Coker}\ \iota_3 \longrightarrow 0. \tag{4.15}$$

Algorithm 3 gives us 112 relations among the Killing tensors of rank 1 and 2:

$$\dim \mathrm{Ker}\ \iota_3 = 112. \tag{4.16}$$

It follows that

$$\dim \mathrm{Coker}\ \iota_3 = 84 - 196 + 112 = 0 \tag{4.17}$$

We conclude that all Killing 3-tensors are reducible. Since all Killing 3-tensors are reducible, we look at the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\ \iota_4 \longrightarrow S^2K_2 \xrightarrow{\iota_4} K_4 \longrightarrow \mathrm{Coker}\ \iota_4 \longrightarrow 0. \tag{4.18}$$

(If there was an irreducible Killing 3-tensor, the source space of ι_4 would be $K_1 \otimes K_3$ instead of S^2K_2.) Algorithm 3 gives us 196 relations:

$$\dim \mathrm{Ker}\ \iota_4 = 196. \tag{4.19}$$

It follows that

$$\dim \mathrm{Coker}\ \iota_4 = 210 - 406 + 196 = 0. \tag{4.20}$$

We conclude that all Killing 4-tensors are reducible. The following short exact sequence gives us a lower bound on the number of Killing 5-tensors:

$$0 \longrightarrow \mathrm{Ker}\ \iota_4 \longrightarrow K_1 \otimes K_4 \xrightarrow{\iota_5} K_5 \longrightarrow \mathrm{Coker}\ \iota_5 \longrightarrow 0. \tag{4.21}$$

Reducibility of the Hamiltonian

We prove that the Hamiltonian of metric 1 is reducible, i.e., it decomposes as a sum of products of Killing 1-tensors. In algorithm 3 (step 3), we replace T by

$$H - \sum_{l=1}^{7} \sum_{m=l}^{7} c_{l,m} I_{2,1}^{l} I_{2,1}^{m}, \tag{4.22}$$

where $I_{2,1}^{j}$ $(1 \leq j \leq 7)$ are the linear integrals up to order 2 (found in steps 1 and 2). Then proceeding with steps 4 and 5 in algorithm 3, we obtain a solution in terms of the coefficients $c_{l,m}$.

We summarize these results in the following theorem.

Theorem 4.2.1. *Consider a conformally flat pp-wave as in eq.* (4.1) *with $f(x^3) = k$ for some constant $k \neq 0$. The dimension of the space K_d of Killing d-tensors for $d = 1, 2, 3, 4$ is given by:*

$$\dim K_1 = 7,\ \dim K_2 = 28,\ \dim K_3 = 84,\ \dim K_4 = 210. \tag{4.23}$$

There exists one irreducible Killing 2*-tensor, which is not the Hamiltonian (metric). The Killing tensors of rank* 3 *and* 4 *are all reducible.*

Proof. We can rescale the coordinates x^3, x^4 such that $f = 1$. The theorem follows from the results for $f = 1$. $\square$

Relations and Irreducible Killing Tensors of Metrics 2 and 3

In this section we determine the relations among Killing tensors using algorithm 3 for metrics 2 and 3 (the results are identical). The computations were done in Maple, see the Maple file "ConformallyFlatPPWave2_Algorithm3" and "ConformallyFlatPPWave3_Algorithm3". The generic point we used is again $z = (1, 2, 3, 4)$. Consider the short exact sequence

$$0 \longrightarrow \text{Ker } \iota_2 \longrightarrow S^2 K_1 \xrightarrow{\iota_2} K_2 \longrightarrow \text{Coker } \iota_2 \longrightarrow 0 \tag{4.24}$$

We find one relation using algorithm 3:

$$\dim \text{Ker } \iota_2 = 1, \tag{4.25}$$

and so we compute

$$\dim \text{Coker } \iota_2 = 22 - 21 + 1 = 2. \tag{4.26}$$

We conclude that there are 2 irreducible Killing 2-tensors. We show later that the Hamiltonian is irreducible. Similar to metric 1, metric 2 and 3 have an irreducible Killing 2-tensor, which is not the Hamiltonian. Next, we look at the short exact sequence

$$0 \longrightarrow \text{Ker } \iota_3 \longrightarrow K_1 \otimes K_2 \xrightarrow{\iota_3} K_3 \longrightarrow \text{Coker } \iota_3 \longrightarrow 0. \tag{4.27}$$

Among the Killing tensors of rank 1 and 2 we find 70 relations:

$$\dim \text{Ker } \iota_3 = 70, \tag{4.28}$$

and it follows that

$$\dim \text{Coker } \iota_3 = 62 - 132 + 70 = 0. \tag{4.29}$$

We conclude that all Killing 3-tensors of metrics 2 and 3 are reducible. We consider the exact sequence

$$0 \longrightarrow \text{Ker } \iota_4 \longrightarrow S^2K_2 \xrightarrow{\iota_4} K_4 \longrightarrow \text{Coker } \iota_4 \longrightarrow 0. \tag{4.30}$$

Algorithm 3 shows that there are 105 relations:

$$\dim \text{Ker } \iota_4 = 105. \tag{4.31}$$

Thus, we find that

$$\dim \text{Coker } \iota_4 = 248 - 253 + 105 = 0. \tag{4.32}$$

We conclude that all Killing 4-tensors of metrics 2 and 3 are reducible.

Irreducibility of the Hamiltonian

We prove that the Hamiltonian of metrics 2 and 3 is reducible, i.e., it decomposes as a sum of products of Killing 1-tensors. In algorithm 3 (step 3), we replace T by

$$H - \sum_{l=1}^{6}\sum_{m=l}^{6} c_{l,m} I_{2,1}^{l} I_{2,1}^{m}, \tag{4.33}$$

where $I_{2,1}^{j}$ $(1 \leq j \leq 7)$ are the linear integrals up to order 2 (found in steps 1 and 2). Steps 4 and 5 return no solution, which means that the Hamiltonian is irreducible.

We summarize these results in the following theorem.

Theorem 4.2.2. *Consider a conformally flat pp-wave as in eq.* (4.1) *with* $f(x^3) = x^3$ *or* $f(x^3) = (x^3)^2$. *The dimension of the space* K_d *of Killing* d*-tensors for* $d = 1, 2, 3, 4$ *is given by:*

$$\dim K_1 = 6, \ \dim K_2 = 22, \ \dim K_3 = 62, \ \dim K_4 = 148. \tag{4.34}$$

There exist two irreducible Killing 2*-tensors, one of which is the Hamiltonian (metric). The Killing tensors of rank* 3 *and* 4 *are all reducible.*

Relations and Irreducible Killing Tensors in Metric 4

For $d = 1, 2, 3, 4$, we find that the number of Killing d-tensors and the number of relations (syzygies) among Killing d-tensors of metric 4 are identical to those of metric 1, see the attached Maple worksheet "ConformallyFlatPPWave4_Algorithm3". Thus, there exists one irreducible Killing 2-tensor and the Killing 3- and 4-tensors are all reducible. However, in constrast to metric 1, the irreducible Killing tensor of metric 4 is the Hamiltonian. We summarize:

Theorem 4.2.3. *Consider a conformally flat pp-wave as in eq.* (4.1) *with* $f(x^3) = \frac{1}{(x^3)^2}$. *The dimension of the space* K_d *of Killing* d*-tensors for* $d = 1, 2, 3, 4$ *is given by:*

$$\dim K_1 = 7, \ \dim K_2 = 28, \ \dim K_3 = 84, \ \dim K_4 = 210. \tag{4.35}$$

There exists one irreducible Killing 2*-tensor, which is the Hamiltonian (metric). The Killing tensors of rank* 3 *and* 4 *are all reducible.*

4.3 The Isometry Algebra of Conformally Flat pp-Waves

The previous sections deduced results on the number of Killing tensors without computing them explicitly. In this section we study the isometry algebra structure using the explicit form of the linear integrals.

For a generic metric of a conformally flat pp-wave, there exist 6 Killing fields ([SG86], [BO03]) given by

$$\begin{aligned} I_1 &= x^2 p_1 - x^1 p_2 \\ I_2 &= -\phi_1'(x^3) x^1 p_4 + \phi_1(x^3) p_1 \\ I_3 &= -\phi_2'(x^3) x^1 p_4 + \phi_2(x^3) p_1 \\ I_4 &= \phi_1'(x^3) x^2 p_4 - \phi_1(x^3) p_2 \\ I_5 &= \phi_2'(x^3) x^2 p_4 - \phi_2(x^3) p_2 \\ I_6 &= c\; p_4, \end{aligned} \tag{4.36}$$

where ϕ_1, ϕ_2 are solutions of the second order linear homogeneous ODE $\phi'' + 2f\,\phi = 0$. There are two special classes of conformally flat pp-waves which admit a seventh independent linear integral in addition to the six above:

- If $f = k$ for some constant $k \neq 0$, then p_3 is an integral.
- If $f = \frac{k}{(x^3)^2}$ for some constant $k \neq 0$, then $x^3 p_3 - x^4 p_4$ is an integral. (This integral comes from the boost $(x^1, x^2, x^3, x^4) \mapsto (x^1, x^2, ax^3, a^{-1}x^4)$).

The constant c in linear integral I_6 can be normalized such that the brackets of the generic case are given by

$$\begin{aligned} &\{I_1, I_2\} = I_4,\ \{I_1, I_3\} = I_5,\ \{I_1, I_4\} = -I_2,\ \{I_1, I_5\} = -I_3 \\ &\{I_2, I_3\} = I_6,\ \{I_4, I_5\} = I_6. \end{aligned} \tag{4.37}$$

We see that the center is given by $\mathcal{Z}(K_1) = \langle I_6 \rangle$. This Lie algebra is graded ($[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$):

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{4.38}$$

where $\mathfrak{g}_0 = \langle I_1 \rangle$, $\mathfrak{g}_1 = \langle I_2, I_3, I_4, I_5 \rangle$ and $\mathfrak{g}_2 = \langle I_6 \rangle$. Thus, the nilradical (maximal nilpotent ideal) of the Lie algebra is given by $\mathrm{nil}(\mathfrak{g}) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. It is a solvable Lie algebra, i.e. the derived series $\mathfrak{g}_{(j)}$ terminates:

$$\mathfrak{g}_{(1)} = \langle I_2, I_3, I_4, I_5, I_6 \rangle,\ \mathfrak{g}_{(2)} = \langle I_6 \rangle,\ \mathfrak{g}_{(3)} = 0. \tag{4.39}$$

Moreover, the nilradical $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{heis}(5)$ is the Heisenberg algebra and $\mathfrak{g}_0$ is generated by a complex structure on $\mathfrak{g}_1$. We now examine metrics 1,2,3,4.

Metric 1

There are 7 linear integrals in metric 1 given by:

$$
\begin{aligned}
&I_0 = \frac{p_3}{\sqrt{2}},\ I_1 = x^2 p_1 - x^1 p_2 \\
&I_2 = \sqrt{2}\ \cos(\sqrt{2}x^3) x^1 p_4 + \sin(\sqrt{2}x^3) p_1 \\
&I_3 = -\sqrt{2}\ \sin(\sqrt{2}x^3) x^1 p_4 + \cos(\sqrt{2}x^3) p_1 \\
&I_4 = -\sqrt{2}\ \cos(\sqrt{2}x^3) x^2 p_4 - \sin(\sqrt{2}x^3) p_2 \\
&I_5 = \sqrt{2}\ \sin(\sqrt{2}x^3) x^2 p_4 - \cos(\sqrt{2}x^3) p_2 \\
&I_6 = \sqrt{2}\ p_4
\end{aligned}
\tag{4.40}
$$

The Hamiltonian of metric 1 is reducible. We can decompose the Hamiltonian into the linear integrals, as follows:

$$
H = -I_1 I_6 - I_2^2 - I_3^2 - I_4^2 - I_5^2 + 2I_0 I_6 - I_3 I_4 + I_2 I_5. \tag{4.41}
$$

The (nonzero) structural relations of the isometry algebra are given by:

$$
\begin{aligned}
&\{I_1, I_2\} = I_4,\ \{I_1, I_3\} = I_5,\ \{I_1, I_4\} = -I_2,\ \{I_1, I_5\} = -I_3 \\
&\{I_2, I_3\} = I_6,\ \{I_2, I_0\} = I_3,\ \{I_4, I_0\} = -I_2 \\
&\{I_4, I_5\} = I_6,\ \{I_4, I_0\} = I_5,\ \{I_5, I_0\} = -I_4.
\end{aligned}
\tag{4.42}
$$

We see that the center is given by $\mathcal{Z}(K_1) = \langle I_1 \rangle$. This Lie algebra is still graded ($[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$):

$$
\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{4.43}
$$

where $\mathfrak{g}_0 = \langle I_0, I_1 \rangle$, $\mathfrak{g}_1 = \langle I_2, I_3, I_4, I_5 \rangle$ and $\mathfrak{g}_2 = \langle I_6 \rangle$. Moreover, the subalgebra $\mathfrak{g}_0$ is generated by *two* commuting complex structures on $\mathfrak{g}_1 \cong \mathbb{R}^2 \otimes \mathbb{R}^2$.

Metrics 2 and 3

We do the same for metric 2 (which is isomorphic to the isometry algebra of metric 3).

$$
\begin{aligned}
&I_1 = x^2 p_1 - x^1 p_2 \\
&I_2 = -2^{\frac{1}{3}} \mathrm{AiryAi}(1, -2^{\frac{1}{3}} x^3) x^1 p_4 + \mathrm{AiryAi}(-2^{\frac{1}{3}} x^3) p_1 \\
&I_3 = -2^{\frac{1}{3}} \mathrm{AiryBi}(1, -2^{\frac{1}{3}} x^3) x^1 p_4 + \mathrm{AiryBi}(-2^{\frac{1}{3}} x^3) p_1 \\
&I_4 = 2^{\frac{1}{3}} \mathrm{AiryAi}(1, -2^{\frac{1}{3}} x^3) x^2 p_4 - \mathrm{AiryAi}(-2^{\frac{1}{3}} x^3) p_2 \\
&I_5 = -2^{\frac{1}{3}} \mathrm{AiryBi}(1, -2^{\frac{1}{3}} x^3) x^2 p_4 - \mathrm{AiryBi}(-2^{\frac{1}{3}} x^3) p_2 \\
&I_6 = \frac{2^{\frac{1}{3}}}{\pi}\ p_4
\end{aligned}
\tag{4.44}
$$

The Airy functions AiryAi, AiryBi are solutions of the second order ODE $\phi'' + 2x^3\phi = 0$. In particular, we see that this Lie algebra can be viewed as a subalgebra of metric 1's isometry algebra.

Metric 4

The linear integrals of metric 4 are given by

$$
\begin{aligned}
&I_0 = x^3p_3 - x^4p_4,\ I_1 = x^2p_1 - x^1p_2\\
&I_2 = \frac{-1}{2\sqrt{x^3}}\left[\left\{\sqrt{7}\cos\left(\frac{\sqrt{7}\log(x^3)}{2}\right) + 2\sin\left(\frac{\sqrt{7}\log(x^3)}{2}\right)\right\}x^1p_4 + 2x^3\sin\left(\frac{\sqrt{7}\log(x^3)}{2}\right)p_1\right]\\
&I_3 = \frac{1}{2\sqrt{x^3}}\left[\left\{\sqrt{7}\sin\left(\frac{\sqrt{7}\log(x^3)}{2}\right) - 2\cos\left(\frac{\sqrt{7}\log(x^3)}{2}\right)\right\}x^1p_4 - 2x^3\cos\left(\frac{\sqrt{7}\log(x^3)}{2}\right)p_1\right]\\
&I_4 = \frac{1}{2\sqrt{x^3}}\left[\left\{\sqrt{7}\cos\left(\frac{\sqrt{7}\log(x^3)}{2}\right) + 2\sin\left(\frac{\sqrt{7}\log(x^3)}{2}\right)\right\}x^2p_4 + 2x^3\sin\left(\frac{\sqrt{7}\log(x^3)}{2}\right)p_2\right]\\
&I_5 = \frac{-1}{2\sqrt{x^3}}\left[\left\{\sqrt{7}\sin\left(\frac{\sqrt{7}\log(x^3)}{2}\right) - 2\cos\left(\frac{\sqrt{7}\log(x^3)}{2}\right)\right\}x^2p_4 - 2x^3\cos\left(\frac{\sqrt{7}\log(x^3)}{2}\right)p_2\right]\\
&I_6 = \frac{\sqrt{7}}{2}p_4
\end{aligned}
\tag{4.45}
$$

The (nonzero) structural relations of this isometry algebra are given by:

$$
\begin{aligned}
&\{I_1, I_2\} = I_4,\ \{I_1, I_3\} = I_5,\ \{I_1, I_4\} = -I_2,\ \{I_1, I_5\} = -I_3\\
&\{I_2, I_3\} = I_6,\ \{I_4, I_5\} = I_6,\ \{I_0, I_6\} = -I_6\\
&\{I_0, I_2\} = \frac{-1}{2}(I_2 + \sqrt{7}I_3),\ \{I_0, I_3\} = \frac{1}{2}(\sqrt{7}I_2 - I_3)\\
&\{I_0, I_4\} = \frac{-1}{2}(I_4 + \sqrt{7}I_5),\ \{I_0, I_5\} = \frac{1}{2}(\sqrt{7}I_4 - I_5),
\end{aligned}
\tag{4.46}
$$

We see that the element I_6 is no longer central, so the isometry algebra has trivial center. The element I_6 is a common eigenvector for the adjoint representation of $\mathfrak{g}$ (cf. Lie's Theorem on solvable Lie algebras). This Lie algebra is graded ($[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$):

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{4.47}$$

where $\mathfrak{g}_0 = \langle I_0, I_1\rangle$, $\mathfrak{g}_1 = \langle I_2, I_3, I_4, I_5\rangle$ and $\mathfrak{g}_2 = \langle I_6\rangle$ and $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{heis}(5)$. The subalgebra g_0 is generated by a complex structure on $\mathfrak{g}_1$ and an automorphism on $\mathfrak{g}_1$ that commutes with the complex structure.

Existence of irreducible Killing 2-tensor in generic case

Since metrics 1, 2 and 3 have an irreducible Killing 2-tensor which is not the Hamiltonian, it is natural to ask whether any generic conformally flat pp-wave has an irreducible Killing 2-tensor. We can look at the explicit expression of the quadratic integral of metric 1 and make an ansatz for the general case.

The following theorem is not a new result, it was already proven by [KT10, p. 5] using the Koutras algorithm. However, the methods we have employed in this chapter give an alternative view on these results. Moreover, these methods are applicable to all spacetimes in contrast to the Koutras algorithm (which requires a homothetic Killing field).

Theorem 4.3.1. *A conformally flat pp-wave* (M, g) *(eq.* (4.1)*) with either* $\dim K_1 = 6$ *(generic case) or* $f = k \neq 0$ *(second class) admits an irreducible Killing* 2*-tensor not being the Hamiltonian. It is given by the quadratic integral*

$$I_q := -x^3 H + x^1 p_1 p_4 + x^2 p_2 p_4 + 2x^4 p_4^2, \tag{4.48}$$

where H *is the Hamiltonian of the geodesic flow.*

Proof. It is readily computed that I_q is an integral. To see that I_q is irreducible, note that the linear integrals do not depend on x^4, whereas I_q does. □

Remark 4.3.2. The Koutras algorithm uses a homothetic Killing field to generate a Killing 2-tensor. The homothetic Killing field here is given by $L := x^1 p_1 + x^2 p_2 + 2x^4 p_4$. Indeed, we have that $\{H, L\} = \text{constant} \cdot H$

Module Structure of K_d over K_1

The space of Killing d-tensors is a module over K_1 with respect to the Poisson bracket: given $I_d \in K_d,\ I \in K_1$, their Poisson bracket $\{I_d, I\}$ is again a Killing d-tensor. Since the Killing tensors of rank 3 and 4 are all reducible, it suffices to compute the Poisson bracket of the quadratic integral I_q with the linear integrals. For the generic case (in particular metrics 2 and 3) we obtain the following brackets

$$\begin{aligned} &\{I_q, I_6\} = \frac{I_6^2}{c},\ \{I_q, I_1\} = 0,\ \{I_q, I_2\} = c\ I_6 I_2,\ \{I_q, I_3\} = c\ I_3 I_6 \\ &\{I_q, I_4\} = c\ I_4 I_6,\ \{I_q, I_5\} = c\ I_5 I_6. \end{aligned} \tag{4.49}$$

For metric 1 we get the above together with the additional bracket $\{I_q, I_0\} = cH$. These brackets show that the quadratic integral I_q is not a good candidate to prove integrability of the geodesic flow. Indeed, I_q only Poisson commutes with the linear integral I_1. Also, note that $\{I_q, K_1\} := \{\{I_q, I\} : I \in K_1\} \subseteq \iota_2(S^2 K_1)$. (This is also true for metric 1, because in that case the Hamiltonian is reducible.) For metrics $1, 2, 3, 4$, we have that the Hamiltonian is the only Killing tensor that commutes with all other Killing tensors (up to order 4).

Note that the Poisson algebra (4.49) is weighted. If we assign the weights $\mathrm{w}(I_0) = \mathrm{w}(I_1) = 0$, $\mathrm{w}(I_2) = \mathrm{w}(I_3) = \mathrm{w}(I_4) = \mathrm{w}(I_5) = 1$ and $\mathrm{w}(I_6) = \mathrm{w}(I_q) = \mathrm{w}(H) = 2$, then the Poisson bracket of two integral with weights w_1 and w_2 is an integral of weight $w_1 + w_2$.

4.4 Integrability of Conformally Flat pp-Waves

Theorem 4.4.1. *The geodesic flow of a conformally flat pp-wave* (M, g) *is integrable.*

Proof. Consider the function $F := (H, I_1, I_3, I_5) : T^*M \to \mathbb{R}^4$, where I_1, I_3, I_6 are the linear integrals as in eq. (4.36). In view of the brackets (4.37), we see that the component functions are pairwise Poisson-commuting. The component functions are linearly independent almost everywhere, because the Jacobian matrix

$$\left(\frac{\partial F^i}{\partial p_j}\right)_{1 \leq i,j \leq 4} = \begin{pmatrix} -2p_1 & -2p_2 & 2p_4 & 2p_3 - 4f(x^3)((x^1)^2 + (x^2)^2)p_4 \\ 0 & 0 & 0 & 1 \\ \phi_1(x^3) & 0 & 0 & -\phi_1'(x^3)\ x^1 \\ 0 & -\phi_2(x^3) & 0 & -\phi_2(x^3)\ x^2 \end{pmatrix} \tag{4.50}$$

has full rank almost everywhere. □

In contrast to the Kerr metric, the linear integrals imply integrability of the geodesic flow. By the Arnold–Mineur–Liouville theorem, the geodesic equations can be integrated and its geodesics exhibit regular dynamics (outside of singular points).

4.5 Applications

We mention some applications of the results in this chapter.

Penrose Limits. As discussed in Chapter 3, Penrose limits are plane waves associated to a null geodesic in spacetime. The number of Killing vectors is *hereditary*, which means that if the original spacetime has N Killing vectors, then the resulting Penrose plane wave also has atleast N Killing vectors. Thus, the spacetimes that have a conformally flat pp-wave as their Penrose limit, have no more than 7 Killing vectors. In Chapter 3 we mentioned that Penrose limits of null geodesics approaching a singularity have wave profile proportional to $\frac{1}{(x^3)^2}$. For example, the Penrose limit for a singular FLRW spacetime is in the same class as metric 4 ([Bla11, p.63]). Furthermore, the Penrose limit of the Schwarzschild metric for a null geodesic with constant r gives rise to a conformally flat pp-wave with $f(x^3) = \text{constant}$ ([Bla11, p.58]). To summarize, the results obtained in this chapter have implications for the (non)existence of Killing tensors in Penrose limits.

Superintegrability. Even though the quadratic integral I_q is not needed to prove integrability, it does restrict the possible motion of the geodesics. This phenomenon, having more integrals than necessary, is called 'superintegrability'. The main point is that in a 'superintegrable' system of dimension $2n$ the motion does not occur on an n-dimensional torus (or cylinder in the non-compact case) but on one with dimension smaller than n.

Extending Spacetimes. Integrals can also be used for extending spacetimes ([Ger69]). This is important for the global analysis of spacetimes.

Chapter 5

Absence of Killing Tensors in Wils Metric

The *Wils metric* is given by

$$g = 2x^1\ dx^3 dx^4 - 2x^4\ dx^1 dx^3 + \{2f(x^3)\ x^1((x^1)^2 + (x^2)^2) - (x^4)^2\}\ (dx^3)^2 - (dx^1)^2 - (dx^2)^2 \tag{5.1}$$

where f is a nonzero function of the coordinate x^3. For $f = 0$ this metric is of constant sectional curvature and thus admits the maximal number of Killing tensors, all of which are reducible (cite Thompson 88). The original motivation for studying this metric ([KM96]) was to find an example of a metric without symmetries that has vanishing scalar curvature invariants. This is in contrast to the flat metric which has maximal symmetry. In this paper A. Koutras and C. McIntosh show in particular that a Wils metric generically has no Killing fields. The purpose of this chapter is to give a precise characterization of f for which the Wils metric does admit Killing 1-tensors and Killing 2-tensors (apart from the metric).

To start our study of the Wils metric, we shall consider three cases

- Metric 1: $f(x^3) = 1$
- Metric 2: $f(x^3) = x^3$
- Metric 3: $f(x^3) = (x^3)^2$.

We apply Cartan's prolongation method (algorithm 2) to the PDE

$$\mathcal{E} = \{F = 0 : F \in \text{coeffs}_p(\{H, I_d\})\} \tag{5.2}$$

to compute the number of Killing d-tensors for $d = 1, 2, 3, 4$. After these computations, we consider a Wils metric with arbitrary f. By applying algorithm 4, we deduce the precise form of f for which a Wils metric admits a Killing 1-tensor and a Killing 2-tensor (apart from the metric). This leads to theorems (5.2.1) and (5.2.3).

5.1 Dimensions of Killing Tensors

Metric 1 ($f(x^3) = 1$)

We present the results of algorithm 2 applied to metric 1 for the degrees $d = 1, 2, 3, 4$. In the tables below we collect for a given degree d, the k'th prolongation $\mathcal{E}^{(k)}$ together with its value δ_k obtained from step 3. The generic point we used for algorithm 2 is $z := (1, 2, 1, 4) \in \mathbb{R}^4$. See the attached Maple worksheet "Wils1_Algorithm2.mw".

Linear	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$
δ	10	10	7	4	1	**1**

Table 5.1: The values of δ obtained in algorithm 2 (step 3) for $d = 1$.

Quadratic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	30	50	50	31	14	2	**2**

Table 5.2: The values of δ obtained in algorithm 2 (step 3) for $d = 2$.

Cubic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$
δ	65	125	175	175	115	...	3	2	**2**

Table 5.3: The values of δ obtained in algorithm 2 (step 3) for $d = 3$.

Quartic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$
δ	119	245	385	490	490
cont'd	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$
δ	353	135	4	3	**3**

Table 5.4: The values of δ obtained in algorithm 2 (step 3) for $d = 4$.

The sequence of δ-values decreases quickly after reaching Frobenius type. This allows us to compute the number of Killing 4-tensors in a reasonable amount of time (2-3 hours). We conclude that

$$\dim K_1 = 1,\ \dim K_2 = 2,\ \dim K_3 = 1,\ \dim K_4 = 3. \tag{5.3}$$

The Killing vector is given by ∂_{x^3}, because the metric coefficients do not depend on the coordinate x^3. The three reducible Killing 4-tensors are given by $(\partial^\flat_{x^3})^4,\ \mathrm{g}(\partial^\flat_{x^3})^2, \mathrm{g}^2$.

Metric 2 ($f(x^3) = x^3$)

We present the results of algorithm 2 applied to metric 2 for the degrees $d = 1, 2, 3, 4$. In the tables below we collect for a given degree d, the k'th prolongation $\mathcal{E}^{(k)}$ together with its value δ_k obtained from step 3. The generic point we used for algorithm 2 is $z := (1, 2, 1, 4) \in \mathbb{R}^4$. See the attached Maple worksheet "Wils2_Algorithm2.mw".

Linear	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	10	10	7	4	1	0	**0**

Table 5.5: The values of δ obtained in algorithm 2 (step 3) for $d = 1$.

Quadratic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	30	50	50	31	14	1	**1**

Table 5.6: The values of δ obtained in algorithm 2 (step 3) for $d = 2$.

Cubic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$
δ	65	125	175	175	115	41	1	0	**0**

Table 5.7: The values of δ obtained in algorithm 2 (step 3) for $d = 3$.

Quartic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$
δ	119	245	385	490	490
cont'd	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$
δ	353	135	2	1	**1**

Table 5.8: The values of δ obtained in algorithm 2 (step 3) for $d = 4$.

We conclude that

$$\dim K_1 = 0,\ \dim K_2 = 1,\ \dim K_3 = 0,\ \dim K_4 = 1. \tag{5.4}$$

Metric 3 ($f(x^3) = (x^3)^2$)

We present the results of algorithm 2 applied to metric 3 for the degrees $d = 1, 2, 3, 4$. In the tables below we collect for a given degree d, the k'th prolongation $\mathcal{E}^{(k)}$ together with its value δ_k obtained from step 3. The generic point we used for algorithm 2 is $z := (1, 2, 1, 4) \in \mathbb{R}^4$. See the attached Maple worksheet "Wils3_Algorithm2.mw".

Linear	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	10	10	7	4	1	0	**0**

Table 5.9: The values of δ obtained in algorithm 2 (step 3) for $d = 1$.

Quadratic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	30	50	50	31	14	1	**1**

Table 5.10: The values of δ obtained in algorithm 2 (step 3) for $d = 2$.

Cubic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$
δ	65	125	175	175	115	41	1	0	**0**

Table 5.11: The values of δ obtained in algorithm 2 (step 3) for $d = 3$.

Quartic	$\mathcal{E}$	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(4)}$
δ	119	245	385	490	490
cont'd	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$
δ	353	135	2	1	**1**

Table 5.12: The values of δ obtained in algorithm 2 (step 3) for $d = 4$.

We conclude that

$$\dim K_1 = 0,\ \dim K_2 = 1,\ \dim K_3 = 0,\ \dim K_4 = 1. \tag{5.5}$$

5.2 Nonexistence of Killing Tensors in General Case

5.2.1 Linear Case

We know that the Wils metric generically has no Killing vector fields. However, the computations from metric 1 show that there exist special cases of the Wils metric for which we can find a Killing field. By application of algorithm 4, we determine the number of Killing fields in an arbitrary Wils spacetime exactly.

Theorem 5.2.1. *The Wils metric (5.1) generically has no Killing vector fields. It admits one Killing vector if and only if f is of the form*

$$f(x^3) = \frac{c_1}{(c_2 c_3^2 + 2c_2 c_3 x^3 + c_2 (x^3)^2 - 4)^2} \tag{5.6}$$

for some constants c_1, c_2, c_3 with c_1 being nonzero. If f is of this form, the Killing vector is

$$X := -(c_2 c_3^2 + 2c_2 c_3 x^3 + c_2 x_3^3)\ \partial_{x^3} + 2c_2 (c_3 x^4 + x^3 x^4 + x^1)\ \partial_{x^4}. \tag{5.7}$$

Proof. We prove this theorem by use of algorithm 4. See the attached worksheet "WilsGeneralCaseLinearIntegrals.mw" which is a Maple-implementation of algorithm 4. In order to simplify the calculations we evaluate $x^1 = 1, x^2 = 2, x^4 = 4$.

Step 1 and 2.) Using, the equations defining the PDE $\mathcal{E}$, we express the 1-jets $a^1_1, a^1_2, a^1_3, a^1_4, a^2_3, a^2_4, a^3_2, a^3_4, a^4_1, a^4_2$ in terms of the free variables $a^1, a^2, a^3, a^4, a^2_1, a^2_2, a^3_1, a^3_3, a^4_3, a^4_4$ and the function $f(x^3)$.

Step 3.) For the first prolongation $\mathcal{E}^{(1)}$, we can express all 2-jets in terms of lower order jets without making any assumptions on f.

Step 4.) Consider $\mathcal{E}^{(2)}$. If we assume that $f \neq 0$, we obtain the following compatibility conditions:

$$a^3_1 = 0,\ a^3_3 = -\frac{a^1 f + f' a^3}{2f},\ a^4_4 = 0. \tag{5.8}$$

We are left with 7 free variables. For $\mathcal{E}^{(3)}$, we obtain the additional compatibility conditions:

$$a^1 = 0,\ a^2_2 = a^4_3,\ a^2_1 = \frac{2a^2 f^2 - 4a^3 f f' + 2a^3 f f'' - 3a^3 (f')^2}{6f^2} \tag{5.9}$$

We are left with 4 free variables. The prolongation $\mathcal{E}^{(4)}$ gives three additional compatibility conditions: $a^4_3 = 0$ and two expressions for a^2 and a^4. We are left with 1 free variable, namely a^3. The prolongation $\mathcal{E}^{(5)}$ does not give an additional compatibility condition if and only if f is a solution of the ODE

$$f''' = \frac{3f'(6ff'' - 5f')^2)}{4f^2}. \tag{5.10}$$

The solutions of this ODE are given by

$$f(x^3) = \frac{c_1}{(c_2 c_3^2 + 2c_2 c_3 x^3 + c_2 (x^3)^2 - 4)^2} \tag{5.11}$$

for some constants c_1, c_2, c_3 with $c_1 \neq 0$. If this differential equation is not satisfied, then there are no Killing vector fields for the Wils metric. The Killing vector for Wils metric with f as in (5.11) is given by

$$X := -(c_2 c_3^2 + 2c_2 c_3 x^3 + c_2 x_3^3)\ \partial_{x^3} + 2c_2(c_3 x^4 + x^3 x^4 + x^1)\ \partial_{x^4}. \tag{5.12}$$

□

Remark 5.2.2. Note that the above computation is consistent with the results obtained from algorithm 2. We start with 10 free variables, then 7, then 4, then 1 and finally 1 or 0 depending on the function f.

5.2.2 Quadratic Case

We continue to play this game for the quadratic case.

Theorem 5.2.3. *The Wils metric (5.1) generically has no Killing 2-tensors apart from the metric. It admits one additional Killing 2-tensor if and only if f is of the form*

$$f(x^3) = \frac{c_1}{(c_2 c_3^2 + 2c_2 c_3 x^3 + c_2 (x^3)^2 - 4)^2} \tag{5.13}$$

for some constants c_1, c_2, c_3 with c_1 being nonzero.

Proof. We prove the theorem with help of algorithm 4. See the attached Maple worksheet "WilsGeneralCaseQuadraticIntegrals.mw". Note that a quadratic polynomial $I = a^{ij}(x) p_i p_j$ in four variables has 10 coefficient functions. It is readily seen that there are 10 0-jets, 40 1-jets and 100 2-jets.

Step 1 and 2. We express 20 1-jets in terms of 30 free-variables (10 0-jets and 20 1-jets). Collect these expressions into sub_1. We prolong to $\mathcal{E}^{(1)}$. Substitute the expressions sub_1 into the equations defining the first prolongation. Next, we express 80 2-jets in terms of 50 free variables, namely 10 0-jets, 20 1-jets and the 20 remaining 2-jets.

Step 3. Consider the prolongation $\mathcal{E}^{(2)}$, which is of Frobenius type. We can express all 3-jets in terms of the 50 free variables without making any assumptions on f.

Step 4. The prolongation $\mathcal{E}^{(3)}$ gives 19 compatibility conditions, and so we are left with 31 free variables. The prolongation $\mathcal{E}^{(4)}$ gives an additional 17 compatibility conditions, we are left with 14 free variables. The prolongation $\mathcal{E}^{(5)}$ gives 12 compatibility conditions and the potential compatibility condition

$$a^{33}\ (4ff''' - 18ff'f'' + 15(f')^3) = 0. \tag{5.14}$$

If the second factor is nonzero, we obtain a 13th compatibility condition $a^6 = 0$. In that case, we are left with 1 free variable. The corresponding Killing 2-tensor is the metric. If f satisfies the ODE (5.10) then the second factor is identically zero. Consequently, we cannot resolve for the 0-jet a^{33}, and so we have 2 free variables. The two corresponding Killing 2-tensors are given by the metric and the symmetric product of the vector field (5.12) with itself. □

Corollary 5.2.4. *The Wils metric (5.1) admits no irreducible Killing 2-tensors for any f.*

5.3 Applications

We mention some applications of Killing tensors in the Wils metric.

VSI Spacetimes. The Wils metric is an example of a spacetime with vanishing curvature invariants which generically has no nontrivial Killing 1- and 2-tensors. One could pose the question if '*hidden symmetries*' (i.e. higher order Killing tensors) are responsible for the vanishing of scalar curvature invariants. Corollary (5.2.4) show that the Wils metric admits no irreducible Killing 2-tensors. Moreover, our computations for metrics 1,2,3 show that in these cases there are no irreducible Killing 3- and 4-tensors as well.

Tensor Tomography. Suppose we have a pseudo-Riemannian submanifold (M, g) of a pseudo-Riemannian manifold (N, g_N) without boundary. Consider the unit cotangent bundle SM which is defined as $SM := H^{-1}(1)$ where H is the Hamiltonian of the geodesic flow. For a given $\xi \in SM$ define $\tau(\xi)$, the *travel time*, to be the time when the N-geodesic with initial data ξ 'leaves' (M, g) and 'enters' the ambient manifold N. Explicitly, one defines

$$\tau(\xi) = \inf\{t > 0 : \gamma_\xi(t) \in N \backslash M\} \tag{5.15}$$

where γ_ξ is the N-geodesic with initial data ξ. We assume that the travel time is finite for all $\xi \in SM$. Let ϕ_N^t denote the geodesic flow on the ambient manifold N. The *geodesic ray transform* If of a function $f \in C^\infty(SM)$ is defined by

$$If(\xi) := \int_0^{\tau(\xi)} f(\phi_N^t(\xi))\, dt \qquad (\xi \in S^+M).$$

(Here $S^+M := \{\xi \in SM : \mathrm{g}(\nu, \xi^\sharp) \leq 0\}$ with ν an outward-pointing normal to ∂M.)

The problem of *tensor tomography* is to recover the function f from its geodesic ray transform ([PSU13]). In applications, one would like that the geodesic ray transform is injective (in a suitable way). The absence of Killing tensors is useful in this setting, because the geodesic ray transform of an integral of motion reduces to the function $\xi \mapsto c\,\tau(\xi)$ for some constant c. Thus, we cannot recover an integral from its geodesic ray transform.

Linear Stability of Einstein Equations. Marsden and Arms [AM79] show that the absence of Killing fields implies linear stability of the Einstein equations with respect to some Cauchy hypersurface.

Outlook

The starting point of this thesis was to view the geodesic flow as a Hamiltonian system on the cotangent bundle of a pseudo-Riemannian manifold. By Noether's theorem we have a bijection between Killing vectors (symmetries) and conserved quantities (linear integrals). Generalizing the Lie bracket of vector fields to the Schouten–Nijenhuis bracket extends this duality to a one-to-one correspondence between Killing tensors and integrals which are polynomial in momenta. The Poisson bracket of the Hamiltonian with an arbitrary polynomial (in momenta) leads to a PDE on its coefficient functions.

Since the condition for a polynomial to be an integral is a PDE, we then moved on to the study of PDEs from the geometric perspective. By means of prolongation and projection one can obtain compatibility conditions, essentially new equations which must be satisfied by solutions of the PDE. For systems of linear PDEs of finite type (such as the Killing equations) Cartan's prolongation method computes the number of solutions through linear algebra. Subsequently we can also obtain the syzygies among Killing tensors using algorithm 3. Combining these results then proves the (non)existence of irreducible Killing tensors in the spacetime!

In Chapter 4 we applied these methods to several examples of conformally flat pp-waves. We reproved a result by Keane and Tupper [KT10] that there exists an irreducible Killing 2-tensor in generic conformally flat pp-waves. Moreover, we proved by Cartan's prolongation method that all Killing 3- and 4- tensors in these examples are reducible.

Finally, we studied the Wils metric. For several examples of a Wils metric, we showed that there are no Killing tensors up to degree 4 apart from the Hamiltonian (and possibly a Killing vector). Next we considered the entire family of the Wils metric, which has as parameter a function of one variable. By application of algorithm 4 we deduced the exact form of the function for which the Wils metric admits one Killing vector. This makes a statement of Koutras and McIntosh [KM96] about the Wils metric [Wil89] more precise. We also showed that the only additional Killing 2-tensor in a Wils metric comes from this Killing vector (if it exists). In particular, we obtained that any Wils metric admits no irreducible Killing 2-tensors.

To conclude, we hope that this thesis convinced the reader that Cartan's prolongation method is a *feasible* approach to finding higher rank Killing tensors (or disproving their existence). Further research would include applying these methods to other spacetimes with the hope that we can find new examples of spacetimes admitting Killing tensors of rank 3 or greater. Finding such examples would provide testing ground.

Bibliography

[AKN07] Vladimir I Arnold, Valery V Kozlov, and Anatoly I Neishtadt. *Mathematical aspects of classical and celestial mechanics*. Vol. 3. Springer Science & Business Media, 2007.

[AM08] Ralph Abraham and Jerrold E Marsden. *Foundations of mechanics*. 364. American Mathematical Soc., 2008.

[AM79] Judith M Arms and Jerrold E Marsden. "The absence of Killing fields is necessary for linearization stability of Einstein's equations". In: *Indiana University mathematics Journal* 28.1 (1979), pp. 119–125.

[Arn13] Vladimir Arnol'd. *Mathematical methods of classical mechanics*. Vol. 60. Springer Science & Business Media, 2013.

[Aud04] Michele Audin. *Torus actions on symplectic manifolds*. Vol. 93. Springer Science & Business Media, 2004.

[Bla+04] Matthias Blau, Monica Borunda, Martin O'Loughlin, and George Papadopoulos. "Penrose limits and spacetime singularities". In: *Classical and Quantum Gravity* 21.7 (2004), p. L43.

[Bla11] Matthias Blau. "Plane waves and Penrose limits". In: *Lecture Notes for the ICTP School on Mathematics in String and Field Theory (June 2-13 2003)* (2011).

[BO03] Matthias Blau and Martin O'Loughlin. "Homogeneous plane waves". In: *Nuclear Physics B* 654.1-2 (2003), pp. 135–176.

[Boc+99] A.V. Bocharov, Chetverikov, S.V. V.N. Duzhin, Krasil'shchik I.S. Khor'kova N.G., A.V. Samokhin, Torkhov, A.M. Yu. N. Verbovetsky, and A.M. Vinogradov. *Symmetries and conservation laws for differential equations of mathematical physics*. Vol. 182. American Mathematical Society Providence, RI, 1999.

[Bro+18] Dario Brooks, Paul-Christopher Chavy-Waddy, Alan Albert Coley, Adam Forget, Daniele Gregoris, Malcolm AH MacCallum, and David Duncan McNutt. "Cartan invariants and event horizon detection". In: *General Relativity and Gravitation* 50.4 (2018), pp. 1–18.

[Car68] Brandon Carter. "Global structure of the Kerr family of gravitational fields". In: *Physical Review* 174.5 (1968), p. 1559.

[CHP09] Alan Coley, Sigbjørn Hervik, and Nicos Pelavas. "Spacetimes characterized by their scalar curvature invariants". In: *Classical and Quantum Gravity* 26.2 (2009), p. 025013.

[Col+09] Alan Coley, Sigbjørn Hervik, G Papadopoulos, and N Pelavas. "Kundt Spacetimes". In: *Classical and Quantum Gravity* 26.10 (2009), p. 105016.

[d'I92] Ray A d'Inverno. "Introducing Einstein's relativity". In: *Introducing Einstein's relativity by Ray D'Inverno. New York: Oxford University Press* (1992).

[Dui04] Hans Duistermaat. *Symplectic Geometry*. 2004.

[Dui80] Hans Duistermaat. "On global action-angle coordinates". In: *Communications on pure and applied mathematics* 33.6 (1980), pp. 687–706.

[Dum+02] Jean-Guillaume Dumas, Thierry Gautier, Mark Giesbrecht, Pascal Giorgi, Bradford Hovinen, Erich Kaltofen, B David Saunders, Will J Turner, Gilles Villard, et al. "LinBox: A generic library for exact linear algebra". In: *Proceedings of the 2002 International Congress of Mathematical Software, Beijing, China*. 2002, pp. 40–50.

[Ger69] Robert Geroch. "Limits of Spacetimes". In: *Communications in Mathematical Physics* 13.3 (1969), pp. 180–193.

[Gol67] Hubert Goldschmidt. "Integrability criteria for systems of nonlinear partial differential equations". In: *Journal of Differential Geometry* 1.3-4 (1967), pp. 269–307.

[GPS02] Herbert Goldstein, Charles Poole, and John Safko. *Classical Mechanics*. American Association of Physics Teachers, 2002.

[HE73] Stephen W Hawking and George Francis Rayner Ellis. *The large scale structure of space-time*. Vol. 1. Cambridge University Press, 1973.

[Hec13] Gert Heckman. *Symplectic geometry*. 2013.

[Kar80] Anders Karlhede. "A review of the geometrical equivalence of metrics in general relativity". In: *General Relativity and Gravitation* 12.9 (1980), pp. 693–707.

[KL07] Boris Kruglikov and Valentin Lychagin. "Spencer δ-cohomology, restrictions, characteristics and involutive symbolic PDEs". In: *Acta Applicandae Mathematicae* 95.1 (2007), pp. 31–50.

[KL08] Boris Kruglikov and Valentin Lychagin. "Geometry of differential equations". In: *Handbook of Global Analysis, Editors D. Krupka and D. Saunders, Elsevier, Amsterdam* (2008), pp. 725–771.

[KM12] Boris S Kruglikov and Vladimir S Matveev. "Nonexistence of an integral of the 6th degree in momenta for the Zipoy-Voorhees metric". In: *Physical Review D* 85.12 (2012), p. 124057.

[KM16] Boris Kruglikov and Vladimir S Matveev. "The geodesic flow of a generic metric does not admit nontrivial integrals polynomial in momenta". In: *Nonlinearity* 29.6 (2016), p. 1755.

[KM96] Andreas Koutras and Colin McIntosh. "A metric with no symmetries or invariants". In: *Classical and Quantum Gravity* 13.5 (1996), p. L47.

[KMS19] Boris Kruglikov, David McNutt, and Eivind Schneider. "Differential Invariants of Kundt waves". In: *Classical and Quantum Gravity* 36.15 (2019), p. 155011.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*. Vol. 2. New York, London, 1963.

[Kru11] Boris Kruglikov. “Finite-dimensionality in Tanaka theory”. In: *Annales de l’IHP Analyse non linéaire*. Vol. 28. 1. 2011, pp. 75–90.

[KT10] Aidan J Keane and Brian OJ Tupper. “Killing tensors in pp-wave spacetimes”. In: *Classical and Quantum Gravity* 27.24 (2010), p. 245011.

[KVLG17] Boris S Kruglikov, Andreas Vollmer, and Georgios Lukes-Gerakopoulos. “On integrability of certain rank 2 sub-Riemannian structures”. In: *Regular and Chaotic Dynamics* 22.5 (2017), pp. 502–519.

[Lee13] John M Lee. “Smooth manifolds”. In: *Introduction to Smooth Manifolds*. Springer, 2013, pp. 1–31.

[Lee16] John M Lee. *Riemannian manifolds: an introduction to curvature*. Vol. 176. Springer Science & Business Media, 2016.

[Lew57] Hans Lewy. “An example of a smooth linear partial differential equation without solution”. In: *Annals of Mathematics* (1957), pp. 155–158.

[Lio55] Joseph Liouville. “Note sur l’intégration des équations différentielles de la Dynamique, présentée au Bureau des Longitudes le 29 juin 1853.” In: *Journal de Mathématiques pures et appliquées* (1855), pp. 137–138.

[Mei00] Eckhard Meinrenken. “Symplectic geometry lecture notes”. In: *University of Toronto* (2000).

[Min37] Henri Mineur. “Sur les systèmes mécaniques dans lesquels figurent des paramètres fonctions du temps. Étude des systèmes admettant n intégrales premières uniformes en involution. Extension à ces systèmes des conditions de quantification de Bohr-Sommerfeld”. In: *J. Ecole Polytechn* 3 (1937), pp. 173–191.

[MS10] Vladimir S Matveev and Vsevolod V Shevchishin. “Differential invariants for cubic integrals of geodesic flows on surfaces”. In: *Journal of Geometry and Physics* 60.6-8 (2010), pp. 833–856.

[Olv95] Peter J Olver. *Equivalence, invariants and symmetry*. Cambridge University Press, 1995.

[O’N83] Barrett O’Neill. *Semi-Riemannian geometry with applications to relativity*. Academic press, 1983.

[Pen76] Roger Penrose. “Any space-time has a plane wave as a limit”. In: *Differential geometry and relativity*. Springer, 1976, pp. 271–275.

[Pet16] Peter Petersen. *Riemannian Geometry*. Vol. 171. Springer, 2016.

[Pra+02] Vojtech Pravda, Alena Pravdová, A Coley, and Robert Milson. “All spacetimes with vanishing curvature invariants”. In: *Classical and Quantum Gravity* 19.23 (2002), p. 6213.

[PSU13] Gabriel P Paternain, Mikko Salo, and Gunther Uhlmann. “Tensor tomography: progress and challenges”. In: *arXiv preprint arXiv:1303.6114* (2013).

[Sau89] D. J. Saunders. *The geometry of jet bundles*. Vol. 142. Cambridge University Press, 1989.

[Sei10] Werner M Seiler. *Involution*. Vol. 24. Springer, 2010.

[SG86] R Sippel and H Goenner. “Symmetry classes ofpp-waves”. In: *General relativity and gravitation* 18.12 (1986), pp. 1229–1243.

[Sil08] A. Cannas da Silva. *Lectures on Symplectic Geometry*. Vol. 3575. Springer, 2008.

[Spe69] Donald C Spencer. "Overdetermined systems of linear partial differential equations". In: *Bulletin of the American Mathematical Society* 75.2 (1969), pp. 179–239.

[Ste+09] Hans Stephani, Dietrich Kramer, Malcolm MacCallum, Cornelius Hoenselaers, and Eduard Herlt. *Exact solutions of Einstein's field equations*. Cambridge university press, 2009.

[Str12] Norbert Straumann. *General relativity*. Springer Science & Business Media, 2012.

[Tho86] G Thompson. "Killing tensors in spaces of constant curvature". In: *Journal of mathematical physics* 27.11 (1986), pp. 2693–2699.

[Wey39] Hermann Weyl. "The Classical Groups". In: Princeton university press, 1939.

[Wil89] Patrick Wils. "Homogeneous and conformally Ricci flat pure radiation fields". In: *Classical and Quantum Gravity* 6.9 (1989), p. 1243.

[Wol98] Thomas Wolf. "Structural equations for Killing tensors of arbitrary rank". In: *Computer physics communications* 115.2-3 (1998), pp. 316–329.

[Woo75] Nick MJ Woodhouse. "Killing tensors and the separation of the Hamilton-Jacobi equation". In: *Communications in Mathematical Physics* 44.1 (1975), pp. 9–38.

[WP70] Martin Walker and Roger Penrose. "On quadratic first integrals of the geodesic equations for type {22} spacetimes". In: *Communications in Mathematical Physics* 18.4 (1970), pp. 265–274.

[Yud16] Ori Yudilevich. "Lie Pseudogroups à la Cartan from a modern perspective". PhD thesis. University Utrecht, 2016.

[Zun18] Nguyen Tien Zung. "A conceptual approach to the problem of action-angle variables". In: *Archive for Rational Mechanics and Analysis* 229.2 (2018), pp. 789–833.