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What should have happened if Hardy had discovered this?

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Abstract

First we present and discuss an important proof of Hardy's inequality via Jensen's inequality which Hardy and his collaborators did not discover during the 10 years of research until Hardy finally proved his famous inequality in 1925. If Hardy had discovered this proof, it obviously would have changed this prehistory, and in this article the authors argue that this discovery would probably also have changed the dramatic development of Hardy type inequalities in an essential way. In particular, in this article some results concerning power-weight cases in the finite interval case are proved and discussed in this historical perspective. Moreover, a new Hardy type inequality for piecewise constant $p = p(x)$ is proved with this technique, limiting cases are pointed out and put into this frame.

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1 Introduction

Hardy's inequality in its original continuous form reads: If f is non-negative and p-integrable on $(0,\infty)$,then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p > 1. \tag{1.1}$$

The dramatic more than 10 years period of research until Hardy finally proved (1.1) in 1925, [1] (he stated it already in 1920 [2]), was recently described by Kufner et al. [3]. In this development many other mathematicians (e.g., Landau, Shur, Pólya and Riesz) contributed in various ways.

This article may be regarded as a follow-up and complement of [3]. We claim that this prehistory described in [3] could have changed dramatically if Hardy and his collaborators had discovered what we present in this article.

What Hardy and others did not discover in this period was that (1.1) in fact is equivalent to the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le 1.\int_0^\infty g^p(x)\frac{dx}{x}, \tag{1.2}$$

where $f(x) = g(x^{1-1/p})x^{-1/p}$. Obviously, (1.2) holds with equality also when $p = 1$, while (1.1) has no meaning in this case. Moreover, (1.2) can easily be proved by using Jensen's inequality and reverse the order of integration:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)dy\right)\frac{dx}{x}$$
$$= \int_0^\infty g^p(y)dy \int_y^\infty \frac{dx}{x^2}dy = \int_0^\infty g^p(y)\frac{dy}{y}.$$

Jensen's inequality is from 1905 (see [4,5]) and Hardy used this inequality in many other situations.

The further development of what is nowadays called Hardy-type inequalities has been very intensive, see, e.g., the books [6-9] and the references given there. We mean that also this further development could have been influenced very much if Hardy had discovered this proof. For example, his first generalization of (1.1) to the power weighted case (see (2.1)) is in fact also equivalent to (1.2) via another simple substitution (see (2.2)) and of course he had then discovered himself that both (1.1) and (1.2) also hold for $p < 0$ (at least when $f(x) > 0$).

The proof above (via convexity) of (1.1) was first presented in 1965 by Godunova [10], but this wonderful discovery seems not to have been observed very much until it was rediscovered and complemented in 2002 by Kaijser et al. [11]. After that a great number of results based on this idea have been presented and applied. See, e.g., the PhD thesis [12] by Krulic from 2010 and the recent review articles [13,14] and the references given there.

It is also known that (1.1) holds for finite intervals, e.g., that

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\ell f^p(x)dx, \quad p > 1. \tag{1.3}$$

holds for any ℓ, $0 < \ell \leq \infty$, but the constant $\left(\frac{p}{p-1}\right)^p$ is sharp only for $\ell = \infty$.

In Section 2 of this article we prove a very general variant of (1.3) with sharp constant, (see Theorem 2.4). The key is to prove a variant of (1.2) yielding for $(0, \ell)$, $0 < \ell \leq \infty$, and prove that this inequality is in fact sharp and that a number of weighted Hardy type inequalities are in fact equivalent, all constants are sharp and the inequalities hold also for $p < 0$. Then our proof of Theorem 2.4 shows that all constants in this inequality must also be sharp. These facts have not been pointed out before in this generality, but for sure it should have been done early if Hardy had discovered the proof above.

In Section 3 we prove another new result namely when the parameter p is not the same for $x > 0$, i.e., $p = p(x)$ is piecewise constant. Also in this case the inequality is in a sense sharp (see Theorems 3.1 and 3.6).

Section 4 is reserved for some further remarks and examples including some limiting cases of our inequalities and suggestions of further research with this idea in mind. Also all inequalities in this Section including those by Carleman and Pólya-Knopp can easily be derived from our basic inequality ((1.2) when $\ell = \infty$) and also all these inequalities are sharp.

2 The case with power weights

In 1928 Hardy himself (see [15]) proved the first weighted version of his inequality (1.1) namely the following:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \le \left(\frac{p}{p-1-a}\right)\int_0^\infty f^p(x)x^a dx \tag{2.1}$$

holds for all measurable and non-negative functions f on $(0, \infty)$ whenever $a < p - 1$, $p \ge 1$.

We remark that (2.1) in fact also is equivalent to (1.2) and, thus, to (1.1). This means in particular that this first generalization by Hardy was not a genuine generalization. In fact, the statement above follows by just using the substitution

$$f(t) = g\left(t^{\frac{p-1-\alpha}{p}}\right)t^{-\frac{1+\alpha}{p}}, \tag{2.2}$$

and making some variable transformations. The details can be found as a special case of our proof of Theorem 2.4 below, where we handle also the more general case, namely when the interval $(0, \infty)$ is replaced by an interval $(0, \ell)$, $0 < \ell \le \infty$. The basic result for the proof of our main result Theorem 2.4 is the following variant of inequality (1.2):

Theorem 2.1. *Let g be a non-negative and measurable function on* $(0, \ell)$,$0 < \ell \le \infty$.

(a) If $p < 0$ *or* $p \ge 1$, *then*

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le 1 \cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}. \tag{2.3}$$

(in the case $p < 0$ *we assume that* $g(x) > 0, 0 < x \le \ell$*)*

(b) If $0 < p \le 1$, *then* (2.3) *holds in the reversed direction*
(c) The constant $C = 1$ *is sharp in both (a) and (b).*

Proof. By using Jensen's inequality with the convex function $\Psi(u) = u^p$, $p \ge 1$, $p < 0$, and reversing the order of integration, we find that

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le \int_0^\ell \frac{1}{x}\int_0^x g^p(y)dy\frac{dx}{x} = \int_0^\ell g^p(y)\left(\int_y^\ell \frac{1}{x^2}dx\right)dy$$
$$= \int_0^\ell g^p(y)\left(\frac{1}{y}-\frac{1}{\ell}\right)dy = \int_0^\ell g^p(y)\left(1-\frac{y}{\ell}\right)\frac{dy}{y}.$$

The only inequality in this proof holds in the reversed direction when $0 < p \le 1$ so the proof of (b) follows in the same way.

Concerning the sharpness of the inequality (2.3) we first let $\ell < \infty$ and assume that

$$\int_0^\ell \left(\frac{1}{x}\int_0^\infty g(y)dy\right)^p \frac{dx}{x} \leq C\cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x} \tag{2.4}$$

for all non-negative and measurable functions g on $(0,\ell)$ with some constant C, $0 < C < 1$. Let $p \geq 1$ and $\varepsilon > 0$ and consider $g_\varepsilon(x) = x^\varepsilon$ (for the case $p < 0$ we assume that $\varepsilon < 0$). By inserting this function into (2.4) we obtain that

$$C \geq (\varepsilon p + 1)^{1-p},$$

so that, by letting $\varepsilon \to 0_+$ we have that $C \geq 1$. This contradiction shows that the best constant in (2.3) is $C = 1$. In the same way we can prove that the constant $C = 1$ is sharp also in the case (b). For the case $\ell = \infty$ the sharpness follows by just making a limit procedure with the result above in mind. The proof is complete.

Remark 2.2. For the case $\ell = \infty$ (2.3) coincides with the inequality (1.2) and, thus, the constant $C = 1$ is sharp, which in its turn, implies the well-known fact that the constant $C = \left(\frac{p}{p-1}\right)^p$ in (1.1) is sharp for $p > 1$ and as we see above also for $p < 0$.

The inequality (2.1) for the interval $(0, \ell)$, $0 < \ell \leq \infty$, $p \geq 1$, reads

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \leq \left(\frac{p}{p-1-a}\right)^p \int_0^\ell f^p(x)x^a\left[1-\left(\frac{x}{\ell}\right)^{\frac{p-a-1}{p}}\right]dx, \tag{2.5}$$

where $a < p - 1$, $p \geq 1$.

Remark 2.3. Inequality (2.5) was proved in [16] (see also [17]) and independently in [18] (see also [19,20]). However, in these articles it was not observed that (2.5) holds also for $p < 0$ and that the inequality is in fact equivalent to (2.2) (see our Theorem 2.4 below).

In our next theorem we will give another proof of (2.5) based on the fact that (2.5) in fact is equivalent to (2.3) and it directly follows that the constant $\left(\frac{p}{p-1-a}\right)^p$ in (2.5) is sharp. More generally, we will prove that all the inequalities in our next Theorem 2.4 are equivalent to the basic inequality (2.3):

Theorem 2.4. *Let* $0 \leq \ell \leq \infty$, *let* $p \in \mathbb{R}_+ \setminus \{0\}$ *and let f be a non-negative function.*

(a) Let f be a measurable function on $(0, \ell]$. *Then* (2.5), *i.e.,*

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \leq \left(\frac{p}{p-1-a}\right)^p \int_0^\ell f^p(x)x^a\left[1-\left(\frac{x}{\ell}\right)^{\frac{p-a-1}{p}}\right]dx$$

holds for the following cases:

$$(a_1)\quad p \geq 1,\quad a < p-1,$$
$$(a_2)\quad p < 0,\quad a > p-1.$$

(b) For the case $0 < p < 1$, $a < p - 1$ *inequality* (2.5) *holds in the reversed direction.*

(c) Let f be a measurable function on $[\ell, \infty)$. *Then*

$$\int_{\ell}^{\infty}\left(\frac{1}{x}\int_{x}^{\infty} f(y)dy\right)^{p} x^{a}dx \le \left(\frac{p}{a+1-p}\right)^{p}\int_{\ell}^{\infty} f^{p}(x)x^{a}\left[1-\left(\frac{\ell}{x}\right)^{\frac{a+1-p}{p}}\right]dx \tag{2.6}$$

holds for the following cases:

(c_1) $p \ge 1, \quad a > p-1,$
(c_2) $p < 0, \quad a < p-1.$

(d) For the case $0 < p \le$, *inequality* (2.6) *holds in the reversed direction.*
(e) All inequalities above are sharp.
(f) All inequalities above are equivalent to the basic inequality (2.3) *(and thus equivalent to each other) via suitable substitutions.*

Proof. First we prove that (2.5) in the case (a_1) in fact is equivalent to (2.3) via the relation

$$f(x) = g\left(x^{\dfrac{p-a-1}{p}}\right)x^{-\dfrac{a+1}{p}}.$$

In fact, with $f(x) = g\left(x^{\dfrac{p-a-1}{p}}\right)x^{-\dfrac{a+1}{p}}$ and $\ell_0 = \ell^{\dfrac{p}{p-a-1}}$, in (2.5) we get that

$$\begin{aligned}
\text{RHS} &= \left(\frac{p}{p-1-a}\right)^{p}\int_{0}^{\ell_0} g^{p}\left(x^{\dfrac{p-a-1}{p}}\right)\left[1-\left(\frac{x}{\ell_0}\right)^{\frac{p-1-a}{p}}\right]\frac{dx}{x} \\
&= \left(\frac{p}{p-1-a}\right)^{p+1}\int_{0}^{\ell_0^{\frac{p-a-1}{p}}} g^{p}(y)\left[1-\frac{y}{\ell_0^{\frac{p-1-a}{p}}}\right]\frac{dy}{y} \\
&= \left(\frac{p}{p-1-a}\right)^{p+1}\int_{0}^{\ell} g^{p}(y)\left[1-\frac{y}{\ell}\right]\frac{dy}{y},
\end{aligned}$$

where $y = x^{\frac{p-a-1}{p}}$, $dy = x^{-\frac{a+1}{p}}\left(\dfrac{p-1-a}{p}\right)dx$, and

$$\begin{aligned}
\text{LHS} &= \int_{0}^{\ell_0}\left(\frac{1}{x}\int_{0}^{x} g\left(y^{\frac{p-a-1}{p}}\right)y^{-\frac{a+1}{p}}dy\right)^{p} x^{a}dx \\
&= \left(\frac{p}{p-1-a}\right)^{p}\int_{0}^{\ell_0}\left(\frac{1}{x^{\frac{p-a-1}{p}}}\int_{0}^{x^{\frac{p-a-1}{p}}} g(s)ds\right)^{p}\frac{dx}{x} \\
&= \left(\frac{p}{p-1-a}\right)^{p+1}\int_{0}^{\ell}\left(\frac{1}{y}\int_{0}^{y} g(s)ds\right)^{p}\frac{dy}{y}.
\end{aligned}$$

Since we have only equalities in the calculations above we conclude that (2.3) and (2.5) are equivalent and, thus, by Theorem 2.1, (a) is proved for the case (a_1).

For the case (a_2) all calculations above are still valid and, according to Theorem 2.1, (2.3) holds also in this case and (a) is proved also for the case (a_2).

For the case $0 < p \le 1$, $a < p - 1$, all calculations above are still true and (2.3) holds in the reversed direction according to Theorem 2.1. Hence also (b) is proved.

For the proof of (c) we consider (2.5) with $f(x)$ replaced by $f(1/x)$, with a replaced by a_0 and with ℓ replaced by $\ell_0 = 1/\ell$:

$$\int_0^{\ell_0}\left(\frac{1}{x}\int_0^x f(1/y)dy\right)^p x^{a_0}dx$$
$$\le \left(\frac{p}{p-1-a_0}\right)^p \int_0^{\ell_0} f^p(1/x)x^{a_0}\left[1-\left(\frac{x}{\ell_0}\right)^{\frac{p-a_0-1}{p}}\right]dx.$$

Moreover, by making the variable substitution $y = 1/s$, we find that

$$LHS = \int_0^{\ell_0}\left(\frac{1}{x}\int_{1/x}^{\infty}\frac{f(s)}{s^2}ds\right)^p x^{a_0}dx = \int_\ell^\infty\left(y\int_y^\infty\frac{f(s)}{s^2}ds\right)^p y^{-a_0-2}dy$$
$$= \int_\ell^\infty\left(\frac{1}{y}\int_y^\infty\frac{f(s)}{s^2}ds\right)^p y^{-a_0-2+2p}dy$$
$$\left[\mathrm{Put}\frac{f(s)}{s^2} = g(s)\right]$$
$$= \int_\ell^\infty\left(\frac{1}{y}\int_y^\infty g^p(y)\right)^p y^{2p-a_0-2}dy.$$

and

$$\mathrm{RHS} = \left(\frac{p}{p-1-a_0}\right)^p \int_\ell^\infty f^p(y)y^{-a_0}\left[1-\left(\frac{\ell}{y}\right)^{\frac{p-a_0-1}{p}}\right]y^{-2}dy$$
$$= \left(\frac{p}{p-1-a_0}\right)^p \int_\ell^\infty g^p(y)y^{2p-a_0-2}\left[1-\left(\frac{\ell}{y}\right)^{\frac{p-a_0-1}{p}}\right]dy.$$

Now replace $2p - a_0 - 2$ by a and g by f and we have $a_0 = 2p - a - 2$, $p - 1 - a_0 = a + 1 - p$. Hence, it yields that

$$\int_0^\ell\left(\frac{1}{x}\int_x^\infty f(s)ds\right)^p x^a dx \le \left(\frac{p}{a+1-p}\right)^p \int_\ell^\infty f^p(x)x^a\left[1-\left(\frac{\ell}{x}\right)^{\frac{a+1-p}{p}}\right]dx$$

and, moreover,

$$a_0 < p-1 \Leftrightarrow 2p-a-2 < p-1 \Leftrightarrow a > p-1$$

We conclude that (c) with the conditions (c_1) and (c_2) are in fact equivalent to (a) with the conditions (a_1) and (a_2), respectively, and also (c) is proved.

The calculations above hold also in the case (d) and the only inequality holds in the reversed direction in this case so also (d) is proved.

Next we note that the proof above only consists of suitable substitutions and equalities to reduce all inequalities to the sharp inequality (2.3) and we obtain a proof also of the statements (e) and (f) according to Theorem 2.1. The proof is complete.

3 The case with piecewise constant *p* = *p*(*x*)

For simplicity we state our first result only for the case

$$p(x) = \begin{cases} p_0, \, 0 \le x \le 1, \\ p_1, \, x > 1, \end{cases} \tag{3.1}$$

where p_0, $p_1 \in \mathbb{R}\backslash\{0\}$.

Theorem 3.1. *Let* $0 < \ell \le \infty$, *and let* $p(x)$ *be defind by* (3.1). *Then, for every non-negative and measurable function f,*

$$\begin{aligned} \int_0^\ell \left(\frac{1}{x} \int_0^x f(t)dt \right)^{p(x)} \frac{dx}{x} &\le 1 \cdot \int_0^\ell (f(x))^{p(x)} \left(1 - \frac{x}{\ell}\right) \frac{dx}{x} \\ &+ \max\left\{0, 1 - \frac{1}{\ell}\right\} \int_0^1 \left[(f(x))^{p_1} - (f(x))^{p_0} \right] dx, \end{aligned} \tag{3.2}$$

whenever $p(x) \ge 1$ *or* $p(x) < 0$ *(for the case* $p(x) < 0$ *we also assume that* $f(x) > 0$).

For the case $0 < p(x) < 1$ *inequality* (3.2) *holds in the reversed direction.*

The constant $C = 1$ *in front of the first integral is sharp.*

Remark 3.2. For the case $p_0 = p_1 = p$ (3.2) will coincide with (1.2). Hence, our inequality (3.2) is a genuine generalization not only of (1.2) but also of all Hardy type inequalities we have derived from (1.2) in this article (see, e.g., Theorem 2.1). Also all inequalities discussed in Section 4 follows from (3.2).

Remark 3.3. For the "breaking point" 1 in the definition we could have used any other number *a*. Then (3.2) just reads:

$$\begin{aligned} \int_0^\ell \left(\frac{1}{x} \int_0^x f(t)dt \right)^{p(x)} \frac{dx}{x} &\le 1 \cdot \int_0^\ell (f(x))^{p(x)} \left(1 - \frac{x}{\ell}\right) \frac{dx}{x} \\ &+ \max\left\{0, \frac{1}{a} - \frac{1}{\ell}\right\} \int_0^1 \left[(f(x))^{p_1} - (f(x))^{p_0} \right] dx. \end{aligned}$$

This fact is easily seen by analyzing the proof of Theorem 3.1 we present now.

Proof. For $0 < \ell \le 1$, $p_0 \ge 1$ or $p_0 < 0$ we have, by Jensen's inequality,

$$\begin{aligned} \int_0^\ell \left(\frac{1}{x} \int_0^x f(t)dt \right)^{p(x)} \frac{dx}{x} &\le \int_0^\ell \frac{1}{x^2} \int_0^x (f(t))^{p_0} dtdx = \int_0^\ell (f(t))^{p_0} \int_t^\ell \frac{1}{x^2} dxdt \\ &= \int_0^\ell (f(t))^{p(t)} \left(1 - \frac{t}{\ell}\right) \frac{dt}{t} \end{aligned}$$

The reversed inequality holds if $0 < p_0 < 1$.

For $0 < \ell \le 1$, $p_0 \ge 1$ or $p_0 < 0$, and $p_1 \ge 1$ or $p_1 < 0$, we have, by Jensen's inequality and the Fubini theorem,

$$\begin{aligned}
\int_0^\ell \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p(x)} \frac{dx}{x} &= \int_0^1 \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p_0} \frac{dx}{x} + \int_1^\ell \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p_1} \frac{dx}{x} \\
&\le \int_0^1 \int_0^x (f(t))^{p_0} dt \frac{dx}{x^2} + \int_1^\ell \int_0^1 (f(t))^{p_0} dt \frac{dx}{x^2} + \int_1^\ell \int_1^x (f(t))^{p_1} dt \frac{dx}{x^2} \\
&= \int_0^1 (f(t))^{p_0} \int_t^1 \frac{dx}{x^2} dt + \left(1 - \frac{1}{\ell}\right) \int_0^1 (f(t))^{p_1} dt + \int_1^\ell (f(t))^{p_1} \int_t^\ell \frac{dx}{x^2} dt \\
&= \int_0^1 (f(t))^{p_0} \left(\frac{1}{t} - 1\right) dt + \left(1 - \frac{1}{\ell}\right) \int_0^1 (f(t))^{p_1} dt + \int_1^\ell (f(t))^{p_1} \left(1 - \frac{t}{\ell}\right) \frac{dt}{t} \\
&= \int_0^\ell (f(t))^{p(t)} \left(1 - \frac{t}{\ell}\right) \frac{dt}{t} + \left(1 - \frac{1}{\ell}\right) \int_0^1 \left[(f(t))^{p_1} - (f(t))^{p_0}\right] dt.
\end{aligned}$$

The proof of the case $\ell = \infty$ only consists of small modifications.

For the case $0 < p(x) < 1 (0 < p_0, p_1 < 1)$ the only inequality in the proof above holds in the reversed direction and so also the final inequality holds in the reversed direction.

Concerning the sharpness statement we first let $1 < \ell < \infty$ and assume that (3.2) holds with some constant $C < 1$ instead of $C = 1$. We test this with the function $f_\varepsilon(x) = x^\varepsilon$, $\varepsilon > 0$ and we find that, for $p_0, p_1 > 1$

$$\begin{aligned}
&\frac{1}{(1+\varepsilon p_0)^{p_0} \varepsilon p_0} + \frac{1}{(1+\varepsilon p_1)^{p_1} \varepsilon p_1} (\ell^{\varepsilon p_1} - 1) \\
&C\left(\frac{1}{\varepsilon p_0(\varepsilon p_0 + 1)} + \frac{\ell^{(\varepsilon p_1)}}{\varepsilon p_1(\varepsilon p_1 + 1)} + \left(1 + \frac{1}{\ell}\right) \frac{\varepsilon(p_0 - p_1)}{(\varepsilon p_0 + 1)(\varepsilon p_1 + 1)}\right).
\end{aligned}$$

By letting $\varepsilon \to 0$ we find that $C \ge 1$ and this contradiction shows that $C = 1$ is the sharp constant. For the case $p_0 < 0$ and/or $p_1 < 1$ the same arguments can be used but here in the definition of $f_\varepsilon(x)$ we must have $\varepsilon < 0$ on $0 \le x \le 1$ if $p_0 \le 0$ and $\varepsilon < 0$ on $1 \le x < \ell$ if $p_1 < 0$.

For the case $\ell = \infty$ we can just use the test functions above and a limiting procedure when $\ell \to \infty$. For the case $\ell \le 1$ Theorem 3.1 is a special case of Theorem 2.1 and the sharpness is proved already there. The proof is complete.

Remark 3.4. It is obvious from the proof above that Theorem 3.1 can be generalized to the situation when $p(x) = p_i$, $a_i \le x \le a_{i+1}$, $a_0 = 0$, $a_{N+1} \le \infty$, $i = 0,1,..., N$, $N \in \mathbb{Z}_+$. The only difference is that the second term on the right hand side in (3.2) will be more complicated.

Remark 3.5. For the case when $\int_0^1 (f(t))^{p_1} dx \le \int_0^1 (f(x))^{p_0} dx$ we can directly delete the correction term in (3.2) and we see that (3.2) is more general than (1.2).

Theorem 3.1 can be generalized in various directions. For example, by using the same arguments as in the proof of Theorem 3.1 we can prove the following more general version when the weight $\frac{1}{x}$ is replaced by the weight x^{α}, $\alpha < 1$, and with a general "breaking point" $x = a$. (c.f. also Remark 3.3):

Theorem 3.6. *Let* $a > 0$ *and*

$$p(x) = \begin{cases} p_0, & 0 \le x \le a, \\ p_1, & x > a, \end{cases}$$

where $p_0, p_1 \in \mathbb{R}\backslash\{0\}$. *Moreover, let* $\alpha < 1$, $0 < a \le \ell \le \infty$. *Then*

$$\begin{aligned} \int_0^{\ell} \left(\frac{1}{x} \int_0^x f(t)dt \right)^{p(x)} x^{\alpha} \frac{dx}{x} &\le \frac{1}{1-\alpha} \int_0^{\ell} (f(x))^{p(x)} x^{\alpha} \left(1 - \left(\frac{x}{\ell}\right)^{1-\alpha} \right) \frac{dx}{x} \\ &+ \max\left\{0, \frac{a^{\alpha-1} - \ell^{\alpha-1}}{1-\alpha}\right\} \int_0^{\ell} \left[(f(x))^{p_1} - (f(x))^{p_0} \right] dx, \end{aligned} \tag{3.3}$$

whenever $p(x) \ge 1$ *or* $p(x) < 0$.

For the case $0 < p(x) < 1$ (3.3) *holds in the reversed direction. The inequality* (3.3) *is sharp in the sense that the constant* $C = \frac{1}{1-\alpha}$ *in front of the first integral on the right hand side of* (3.3) *does not hold in general with any* $C < \frac{1}{1-\alpha}$.

Remark 3.7. By using Theorem 3.6 with $p_0 = p_1 = p$ we obtain the following weighted generalization of our basic inequality (2.3):

$$\int_0^{\ell} \left(\frac{1}{x} \int_0^x g(t)dt \right)^{p} x^{\alpha} \frac{dx}{x} \le \frac{1}{1-\alpha} \int_0^{\ell} (f(x))^{p} x^{\alpha} \left(1 - \left(\frac{x}{\ell}\right)^{1-\alpha} \right) \frac{dx}{x}$$

for any $\alpha < 1$. The inequality is sharp. (For the case when $p(x) < 0$ we also assume that $f(x) > 0$ a.e.)

4 Concluding remarks and examples

First we discuss the discrete version of Hardy's inequality (1.1).

Remark 4.1. We remark that Hardy was in fact originally more interested of the following discrete form of (1.1): If $\{a_n\}_1^{\infty}$ is a sequence of non-negative numbers, then

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right)^p \le \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \quad p > 1. \tag{4.1}$$

Moreover, (1.1) ⇒ (4.1). In fact, by applying (1.1) with step functions we obtain (4.1). This fact was pointed out to Hardy in a private letter from Landau already in 1921 and here Landau even included a proof of (4.1) so it should maybe be more correct to call (4.1) the Hardy-Landau inequality.

We shall now point out some limit cases of inequalities (1.1) and (4.1).

Example 4.2. (Carleman's inequality): If $\{a_n\}_1^{\infty}$ is a sequence of positive numbers, then

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \dots a_n} \quad \le \quad e \sum_{n=1}^{\infty} a_n. \tag{4.2}$$

The constant $C = e$ is sharp. This inequality may be regarded as a limit inequality of (4.1) when $p \to \infty$ (see Remark 4.3).

Remark 4.3. This inequality was proved by Carleman in 1922 (see [21]) in connection to his important work on quasianalytical functions. Carleman's proof was based on the Lagrange multiplier method but most easily it follows as a limiting case of the Hardy inequalities (4.1) as $p \to \infty$ according to the following:

Replace a_i with $a_i^{1/p}$ in (4.1) and we obtain that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{1/p}\right)^p \le \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n, \quad p > 1.$$

Moreover, when $p \to \infty$ we have that

$$\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{1/p}\right)^p \to \left(\prod_{i=1}^{n}a_i\right)^{1/n} \quad \text{and} \quad \left(\frac{p}{p-1}\right)^p \to e.$$

Example 4.4. (Pólya-Knopp's inequality): If f is a positive and integrable function on $(0,\infty)$, then

$$\int_0^{\infty}\exp\left(\frac{1}{x}\int_0^x \ln f(y)dy\right)dx \le e\int_0^{\infty}f(x)dx. \tag{4.3}$$

The constant $C = e$ is sharp. This inequality can be regarded as a limit inequality of (1.1) as $p \to \infty$.

Remark 4.5. Sometimes (4.3) is referred to as the Knopp inequality with reference to his 1928 article [22]. But it is clear that it was known before and in his 1925 article [1] Hardy informed that Pólya had pointed out the fact that (4.3) is in fact a limit inequality (as $p \to \infty$) of the inequality (1.1) for $p > 1$ and the proof is literally the same as that above that (4.1) implies (4.2). Accordingly, nowadays (4.3) is many times referred to as Pólya-Knopp's inequality.

Remark 4.6. By using the ideas in this article we can obviously also prove some Hardy-type inequalities with more general weights and also in multi-dimensional cases. In addition to this information in this article for what is known in this journey so far we just refer to the recent PhD thesis by Krulic [12] and the review articles [13,14] and the references therein.

Remark 4.7. It is nowadays a popular subject to study Lebesgue spaces with varying exponent $p = p(x)$ instead of just a constant p. Also some interesting result concerning Hardy-type inequalities are obtained in this case, see, e.g., Diening-Samko [23] and the references given there. However, our result in Section 3 can never be derived from such general results because we both permit cases when $p(x) < 1$ and we have sharp constants.

Remark 4.8. We remark that most inequalities obtained in this article (including the original inequalities of Hardy, Carleman, Pólya-Knopp, power weight extensions of these inequalities, e.g., Theorem 3.1, etc.) are easy consequences of the basic inequality (2.3). Our main question in this article is the following: How would the development of Hardy type inequalities be if Hardy himself had discovered this proof? The development after 2002 when this technique was (re)discovered shows that many new and old results easily follow in this way. The present authors strongly believe that these ideas will clearly influence on the further development of Hardy type inequalities and that these ideas and proofs will be used in analysis books in the future (in almost all such books even the classical Hardy inequality (1.1) is usually proved by using essentially his original proof via partial integration and Pólya's simplification).

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References

1. Hardy, GH: Notes on some points in the integral calculus, LX. An inequality between integrals. Messenger Math. **54**, 150–156 (1925)
2. Hardy, GH: Notes on a theorem of Hilbert. Math Z. **6**, 314–317 (1920). doi:10.1007/BF01199965
3. Kufner, A, Maligranda, L, Persson, LE: The prehistory of the Hardy inequality. Am Math Mon. **113**(8):715–732 (2006). doi:10.2307/27642033
4. Jensen, JLWV: Om konvexe funktioner og uligheter mellom middelvaerdier. Nyt Tidsskr Math. **16B**, 49–69 (1905)
5. Jensen, JLWV: Sur les fonctions convexes et les inegalites entre les valeurs moyennes. Acta Math. **30**, 175–193 (1906). doi:10.1007/BF02418571
6. Kokilashvili, V, Meshki, A, Persson, LE: Weighted Norm Inequalities for Integral Transforms with Product Weights. Nova Scientific Publishers, Inc., New York (2010)
7. Kufner, A, Maligranda, L, Persson, LE: The Hardy Inequality–About its History and Some Related Results. Vydavatelsky Servis Publishing House. Pilsen (2007)
8. Kufner, A, Persson, LE: Weighted inequalities of Hardy type. World Scientific Publishing Co. Inc., River Edge, NJ (2003)
9. Opic, B, Kufner, A, Hardy-type Inequalities: Pitman Research Notes in Mathematics, vol. 219. Longman Scientific & Technical, Harlow (1990)
10. Godunova, EK: Inequalities with convex functions. Am Math Soc Trans II Ser **88**, 57–66 (1970). Translation from Izv. Vyssh. Uchebn. Zaved. Mat. 4(47), 45-53 (1965)
11. Kaijser, S, Persson, LE, Öberg, A: On Carleman's and Knopp's inequalities. J Approx Theory. **117**, 140–151 (2002). doi:10.1006/jath.2002.3684
12. Krulic, K: Generalizations and refinements of Hardy's inequalities. PhD Thesis, Department of Mathematics, University of Zagreb (2010)
13. Oguntuase, JA, Persson, L-E: Hardy-type inequalities via convexity–the journey so far. Aust J Math Appl V. **7**(2):1–19 (2011)
14. Persson, L-E, Samko, N: Some remarks and new developments concerning Hardy-type inequalities. Rend Circ Mat Palermo Serie II. **82**, 1–29 (2010)
15. Hardy, GH: Notes on some points in the integral calculus, LXIV. Futher inequalities between integrals. Messenger Math. **57**, 12–16 (1928)
16. Yang, B, Zeng, Z, Debnath, L: On new generalizations of Hardy's integral inequality. J Math Anal Appl. **217**(1):321–327 (1998). doi:10.1006/jmaa.1998.5758
17. Yang, B, Debnath, L: Generalizations of Hardy's integral inequalities. Int J Math Sci. **22**(3):535–542 (1999). doi:10.1155/S0161171299225355
18. Cizmesija, A, Pecaric, J: Some new generalizations of inequalities of Hardy and Levin-Cochran-Lee. Bull Aust Math Soc. **63**, 105–113 (2001). doi:10.1017/S000497270001916X
19. Cizmesija, A, Pecaric, J: Multivariable mixed means and inequalities of Hardy and Levin-Cochran-Lee type. Math Inequal Appl. **5**(3):397–415 (2002)
20. Cizmesija, A, Pecaric, J, Persson, L-E: Strengthened of Hardy and Pólya-Knopp's inequalities. J Approx Theory. **125**, 74–84 (2003). doi:10.1016/j.jat.2003.09.007
21. Carleman, T: Sur les functions quasi-analytiques. Comptes Rendus du V Congres des Mathematiciens Scandinaves. pp. 181–196.Helsinki (1922)
22. Knopp, K: Über reihen mit positivern gliedern. J Lond Math Soc. **3**, 205–211 (1928). doi:10.1112/jlms/s1-3.3.205
23. Diening, L, Samko, S: Hardy inequality in variable exponent Lebesgue spaces. Frac Calc Appl Anal. **10**(1):1–18 (2007)

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