Essays on exploitation of natural resources

Optimal control theory applied on multispecies fisheries and fossil fuel extraction

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Chapter 1

Synthesis

1.1 Introduction

The main scope of this thesis is modeling and optimizing motivated by real-world problems in fishery and environmental economics. The work is interdisciplinary and the emphasis is on combining economics with biology, mathematics and numerics. Key words are *the art of modeling* and *optimal control theory*.

Optimal control theory is a well-established concept for the solution of management problems where one has to maximize/minimize some objective or function subject to the state of a dynamic system dependent on the level we choose for the free variables, the *control variables*. Control theory formulations are used in a variety of areas, including process control, robotics, bioengineering, economics, finance and management science. In resource economics such formulations are very popular, e.g. in deciding maximum sustainable yield of specific fish resources. More curious applications can also be found which show that the concept of optimal control is suitable for a wide range of decision problems. An example is Metric and Weitzman (1998) who uses optimal control theory to decide which crane bird species one should put most resources into preserving in order to maintain maximum biological diversity. Optimal control theory formulations have limitations, however, to which we will return later in this section.

In the essays of this thesis the focus is somewhat phenomenological, but adaptations that rest on econometrics and employ multivariate regression analysis and parameter estimations are also made. Since solving the models demands extensive use of numerical methods, some attention is also paid to operational analysis and mathematical technicalities. All in all the thesis is a cross between bioeconomics and applied mathematical modeling.

There are several attributes of a good model. Perhaps the most important are realism/accuracy and usefulness. A famous quote by the industrial statistician George Box (1979) is straight to the point. "All models are wrong; some are useful". Between accuracy and usefulness, usefulness should come first, since only a model that brings light to some shade is worthwhile. For optimizing problems that need to be simplified in a model, incorporating the full complexity of the real world is rarely possible, but the model should be a window on reality and present a clear and concise view of that reality. In this sense one could say that the art of modeling is the art of simplifying. First, important characteristics must be identified. Next, one or more of these characteristics should be incorporated in the problem formulation. Which characteristics to choose is mainly a question about their contribution to the realism of the formulation, but if they add much complexity one must find out if they are absolutely necessary. Solvability is essential since a problem formulation that cannot be solved loses much of its purpose regardless of how realistic it might be.

When considering using optimal control theory formulations for the purpose of managing complex systems, one has to weight up the importance of an accurate system description against the need to find the exact, one and only, optimal solution to a formulated problem. Control theory formulations usually have an exact solution, but the concept of usefulness is not fulfilled if the system description has no realism. Even with the best computer technology available, system complexity very often is too overwhelming for a model to allow optimizing when acceptable accuracy and realism are to be maintained. In many cases it is impossible to tell which strategy is the best; either to relax model complexity so that one can use e.g. optimal control theory to find the optimal solution, or to raise complexity to gain more realism and search for an acceptable strategy rather than an optimal one.

1.2 Models

The problem formulations of the essays in this thesis consist of one or more ordinary differential equations expressing the current state of the system, $x(t)$, and of an objective function that is to be maximized over time. The system is controlled by the control variable(s), $u(t)$. Typically, in the case of a fishery $x(t)$ could be a vector of biomasses and $u(t)$ could be a vector of catches. If we assume current profit given by $\Pi(x(t), u(t))$, discount rate δ and an infinite time horizon, the problem is given by

$$\max_{u} \int_{0}^{\infty} \Pi(x(t), u(t))\, e^{-r\,t}\, dt \tag{1.1}$$

subject to

$$x'(t) = f(x(t), u(t)). \tag{1.2}$$

In other words, we have to find optimal controls that maximizes the sum of all future discounted profit on condition that the state vector changes according to eq. (1.2). A related problem also explored in this thesis (*essay I*) is when the time horizon is finite but unknown. Instead of maximizing eq. (1.1), one maximizes

$$\max_{u} \int_{0}^{T} \Pi(x(t), u(t))\, e^{-r\,t}\, dt + \psi(x(T), T), \tag{1.3}$$

where $\psi(x(T),T)$ is some kind of salvage value remaining when the period $[0,T]$ of optimizing is ended. Since T is unknown, however, it is part of the problem to find its size.

The rest of the model formulations are specific to each essay and are therefore not described here.

An important common characteristic of all the models in this thesis is that the objective functions (eq. (1.1)) are autonomous (independent of time) when one disregards the discount factor. In addition, the state equations (1.2) are not explicitly time-dependent. These properties are very important since they allow us to find an autonomous "feedback solution" to the problems. The autonomous feedback $u = u(x)$ is a class of solutions that defines the optimal control, u, for every admissible state, x. The autonomity simplifies the solution. It makes it possible to find the optimal control without worrying about the optimal time-path, and is essential in all the solution procedures of this thesis. Without it, the feedback would have had the form $u = u(x,t)$, which would have increased the complexity considerably.

1.3 Background and perspectives

Two of the papers in this thesis concern management of capelin and cod in the Barents Sea (*essay III* and *essay IV*) and a third *essay (II)* regards the techniques used to solve the models in these essays. The first essay is, however, somewhat different. Its theme is carbon taxes and management of the resources of fossil fuel. This is currently a much hotter topic than it was when the essay was written. The climate question rose to the top of the world media's agenda when the Intergovernmental Panel of Climate Change and Al Gore shared the Nobel Peace Price of 2007. In the UN climate conference on Bali the participants finally reached an agreement on negotiations for a new agreement in Copenhagen in 2009 to replace the existing Kyoto agreement. There is now a consensus that emissions must be reduced. The battle now is about how and how much.

Basically, there are two leading approaches to how emissions can be reduced. Carbon markets and carbon taxes (see *Essay I*). In January 2005 the European Union Greenhouse Gas Emission Trading Scheme (EU ETS) emerged, but it is not working very well because the supply of emission permits is too large. Most economists also favor carbon tax over other approaches (54$\%$ according to a Wall Street Journal Survey in February 2007). The advantage with carbon taxes is that the extra costs are passed downstream to all the consumers in a way that does not unreasonably favor particular companies or countries. Although they do not set an actual limit on emissions, they turn the market in a more environment-friendly direction.

The British government addressed the time scale and volume of an optimal reduction in emissions with their publication of the Stern Review (Nordhaus 2007 a). The Stern Review claims that the current situation demands an immediate response. If we do not act, the report estimates all risks and impacts will cost 20% of GDP (gross domestic product) or more. Moreover, it suggests global emission reductions of between 30% and 70% over the next two decades. The Stern Review is, however, strongly criticized by Nordhaus (2007 a), who points out that the results rely heavily on an unrealistic model parameterization. In his opinion, answers in the Stern report are too bombastic and too dramatic: "How much, how fast, and how costly - remain open" (Nordhaus 2007 b).

In many of the world's fisheries the need for immediate management and control is even more urgent than in the fossil fuel market. Typically, knowledge about the situation is lacking just as much as the willingness to reach agreements. Fortunately, in the case of the Barents Sea fisheries the situation is much better. Norwegian and Russian scientists know much about the present state of the ecosystem, and assessment data are available for the most important stocks. Although unreported unlicensed fishery has been a considerable problem, the two countries more or less agree on the management.

The Norwegian-Russian cooperation was prepared by the introduction of *Exclusive Economic Zones* (EEZ) in 1977, where Norway extended its geographical area by approximately six times its land territory (Hoel 2005). Since then, *The Mixed Norwegian-Russian Fisheries Commission* (established in 1976) has played a decisive role in the management of cod, haddock and capelin in the area. Their conclusions are largely based on statements from *The International Council for the Exploration of the Sea (ICES)*, which provides a "quality control" on the advices from Norwegian and Russian scientists.

In spite of the well-established management of the fish resources in the Barents Sea, improvements are still possible. Much more could be done in combining biologicals and economic aspects in a multi-species setting. The Barents Sea fisheries are of great economic importance, yet the management of the most important species, capelin and cod, is not founded on scientific economic advice. The total allowable catch (TAC) of capelin determines a fishery from January to February and is decided on the basis of the expected size of the spawning stock when predation by cod is taken into account. Similarly, the catch of cod is based on the stock size, but in a single-species context with an aim of not changing the TAC too much from year to year. The cod stock is shared evenly between Norway and Russia with a small allocation to third countries (Hannesson 2006), and the division key for capelin is 60-40 in favor of Norway.

Failure of previous management made it clear in the early 1980s that traditional single species models for many species were incapable of predicting future stock sizes adequately for management purposes (Mehl 1986). Analysis of stomach content gave a better understanding of the dynamics in the fisheries (Mehl 1991), and the collapse of the capelin stock in the middle of the 1980s was understood and explained by the dynamics between capelin, cod, herring (*Clupea harengus*) and man (Hopkins and Nilssen 1991).

Hjermann, Ottersen and Stenseth (2004) conclude that capelin, cod and herring should be managed at a multi-species/community level, and that the "one-way" multi-species approach currently used should be extended to jointly decide harvest levels for all three stocks. The need of a multi-species perspective, however, was already addressed in *Models for multi-species management* (Rødseth 1998).

In both *essay III* and *essay IV* the management has a multi-species perspective. Only capelin and cod are considered in the former, but in *essay IV* the perspective is extended to include juvenile herring. Such an extension is rather challenging from a computational perspective, and the methods used are the theme of *essay II*, which establishes a technical basis for the next two essays.

The economic model used in both *essay III* and *essay IV* acknowledges the fact that the Barents Sea is jointly managed by Norway and Russia. This model assumes that the TACs of both Norway and Russia can be controlled by a social planner, and the objective of the planner is to maximize Norwegian surplus. In spite of the promising cooperation between Norwegian and Russian scientists today, there have been several disputes between the two countries during the last thirty years. Owing to pressure from the Russian side, TACs have often been set far above the level recommended by scientists. Moreover, in the beginning of the 1990s overfishing by the Russian TACs was a threat to the joint fishery, but after 1993 cooperation between Norwegian and Russian coast guards has eased the problem. There are, however, still problems with unreported fishing on both sides. Nevertheless, p.t. (in 2009) the stock situation in the Barents Sea is very encouraging.

The common basis of all the essays is the optimizing problem stated in eq. (1.1) and (1.2). Moreover, all the essays solve it using different variants of optimal control theory. These variants are Pontryagin's maximum principle (see Kamien and Schwarz 1991), which we make use of in *essay I*, and dynamic programming (see Bertsekas 2005) that we employ in the other works. Unfortunately, these methods impose some limits on the degree of complexity one can choose for the system to be optimized. In e.g. the case of multi-species fishery management one has to moderate the number of interacting species in the growth model describing the state of the system. The reason for this moderation is that the numerical cost (throughput) grows very fast with the dimensionality of the system. This also limits the possibility of working with year classes within each species. Although a lot of work has been done in the field of operational analysis, there is still a long way to go before optimal control theory is well suited for general problems in more than four dimensions, such as a three-species management model with two year classes per species, which results in a six-dimensional state space.

For natural systems, like the marine ecosystem in the Barents Sea, it is impossible to model all attributes. One has to set a limit for the complexity. Where one sets this limit determines whether it is possible or impossible to find the optimal solution to the model.

If the model gets a much better assessment of the real system when more complexity is added to it, such an extension could also be evaluated when this means that it is impossible to find an exact solution to the model. In such cases scenario-modeling (see Schweder 2006), which we use in *essay III*, is a good method for evaluating different management strategies. One cannot expect to find the best strategy, but by evaluating a number of strategies one can choose the best strategy tested.

There is always room for fundamental stylized models that investigate new principles in management, but there is also much to be done in the field of applied research where real data analysis is given priority. The optimal control theory methods we make use of in *essay II*, *essay III* and *essay IV* can be employed in a wide range of applied studies as long as the dimension of the state space (fish species, year classes, geographical areas, etc.) is fairly restricted. A modest rise in dimensions is possible, but introduction of stochastics and applications in models with marine protected areas (MPAs) are more striking extensions of the work in this thesis.

1.4 Overview of the thesis

The thesis consists of four essays, all based on optimal control theory. The subject of *essay I* is optimal exploration of fossil fuel when negative externalities regarding CO_2 emissions are accounted for. In this essay, a Pontryagin approach is used to solve the optimal control problem described by eq. (1.1) and (1.2). The objective is to study how a tax on fossil fuel may shift total world exploitation of fossil fuel to a level that is optimal for society when negative externalities from exploitation and consumption are accounted for.

Essay II focuses on dynamic programming approaches to solving similar problems. This is most of all an essay about technicalities demonstrating the techniques employed in *essays III* and *IV*, but a few examples of concrete problems are also solved.

Essay III is a scenario-model. Optimal TACs of capelin and cod are found based on a top-down biological growth model from Agnarsson et. al. (2008).Consequences of this catch strategy are compared with regard to a bottom-up growth model from the Institutes of Marine Research in Bergen (IMR) as the real growth. Priority is largely given to stock scenarios resulting from the different strategies tested.

In *essay IV* a three-species top-down growth model for capelin, cod and herring in the Barents Sea is established on the basis of historical stock data. In accordance with this growth model optimal TACs for capelin and cod are found, and long-term effects of following this policy are investigated.

1.5 Summary of essays

1.5.1 Essay I

In this essay a principal model for optimal management of the world's resources of fossil fuel is constructed. The profit function is defined as the sum of consumer surplus and producer surplus, and the objective is to maximize discounted profit for the future. Negative externalities associated with emissions resulting from production and use of fossil fuel are, however, also accounted for. Atmospheric level of CO_2 and the remaining reserves of fossil fuel represent the environmental state of the system as a two-dimensional state vector, and a convex damage function damps down the profit.

Accumulation of CO_2 in the atmosphere is modeled to grow linearly with the world production of fossil fuel and decline non-linearly with a purification function $f(a)$ that has a skew normal distributed form and represents the self-cleaning ability of the ecosystem. Similar models have been made previously, but with a linear function representing this ability. With a linear $f(a)$ it is not possible to model scenarios with irreversible pollution. The coefficients of the function we use are calibrated from real historical data for emission and atmospheric CO_2 levels.

Since fossil fuel is a non-renewable resource in all practical interpretations of the term, there comes a time $t = T$ when either all fossil fuel is exhausted or a new technology replaces fossil fuel. Therefore the maximizing problem is given above by eq. (1.3), where the time of the technology shift T is decided endogenously.

We study the pure market solution and compare it with the optimal solution adjusted for externalities. The unregulated solution results in irreversible pollution since the CO_2-levels becomes so high that the modeled self-cleaning ability of the ecosystem collapses. When $f(a)$ is replaced by a linear purification function the optimal tax level is highly reduced compared with the non linear case. The reason is that linear purification over estimates the cleaning ability of the ecosystem for high levels of CO_2 in the atmosphere.

The influence of the discount rate factor is considerable in this model. Low discount rate gives higher tax and earlier shift in technology and, in the first years especially, tax is higher.

1.5.2 Essay II

This essay concentrates on a specific dynamic programming approach well suited for a lot of optimal control problems in fishery management. Since the term "curse of dimensionality" is frequently used in connection with dynamic programming, the approach was developed with the intention of solving optimal control problems with three- or four-dimensional state spaces within reasonable time.

Standard discretization of the state space and well-known fixed-point iterative methods are employed, but the main idea is to avoid discretization of the control space. Instead our method finds the optimal controls from first-order conditions on an approximated discrete version of the Hamilton-Jacobi-Bellman equation. This method can only be effectively used when the optimal value function is at least piece-wise differentiable. Other characteristics are that we shift between policy and value iteration and use interpolation to find approximated values of the value function to state values between the nodes of the state space grid. Such values are used in the iterative process when we calculate the optimal value function.

Interpolation is also used in the process of solving the first-order conditions. For many problems the first-order conditions cannot be solved algebraically, and then solving of the first order conditions is in itself a task deserving the attention of operational analysis experts. In stead of solving the exact first-order condition problem, we interpolate one or both of the functions in the equation to be able to find an algebraic solution to a related problem. This way an approximately optimal control can be found computationally cheaply.

At the end of the essay, three examples are solved with our methods. These are one investment problem and two problems for optimal management of fishery resources.

1.5.3 Essay III

The issue in this essay is optimal management of Barents Sea capelin (*Mallotus villosus*) and North East Arctic cod *Gadus morhua*). A two-species management model with pure economical objectives is presented, and its biological consequences are explored through growth simulations with an independent simulator, the Bifrost simulator. A growth model is also integrated in the management model, but the approach in the two growth models is very different. Although the growth of the management model is multi-species, it has a top-down approach, whereas the Bifrost simulator is much more bottom-up. The simulations are therefore a powerful validity test for the management model.

In order to find out how the management model performs compared with existing strategies, many different simulations are done. Four of them are based on the management model and two are inspired by existing management.

A simulation period of 565 years with 2006 as a starting year is chosen. The results show that the strategies revealed from the management model perform surprisingly well. First of all, none of the strategies leads to extinction of any of the stock. Second, with regard to long-term average of spawning stock biomasses and TACs of cod, the two best strategies are picked from the management model. It also emerges that when the herring assessment model Seastar runs concurrently with Bifrost to simulate a dynamical herring influence, the management model performs relatively better compared with existing strategies than it does when herring influence is constant.

The positive results from the simulations of this work indicate that further studies with both biologically and economically-based multi-species management should be performed. It is not a given that the biologically-based management is more fish-stock preserving and aware than economically-based management, and the multi-species approaches may contribute much.

1.5.4 Essay IV

In this essay we develop a three dimensional Barents Sea growth model for the key species capelin, cod and herring. This growth constitutes the state equations in an optimal control problem of the form defined in eq. (1.1) and (1.2). The profit function is derived from a Nordic ministry report (Agnarsson et. al. 2008), and represents the Norwegian fraction of profit from the Russian and Norwegian fishery of capelin and cod in the Barents Sea.

Logistic growth modified by predation is assumed as the overall structure of the growth model. Herring enters the model only for age groups one to three, since herring aged three to four years leave the Barents Sea area and join the mature part of the population further south-west (Huse 2002) and the zero group of herring is of little importance as far as predation relationships are concerned (Hjermann 2004). The inflow of herring is exogenous to the model, but the growth also depends on the size of the Barents Sea fraction.

Coefficients of the growth model are found through regression analysis based on data from 1973 to 2004. The statistical results are quite good, and both the equilibrium levels resulting from fishery closure and the equilibrium levels resulting from optimal harvest make sense. One should be careful, however, to extrapolating growth when stock sizes are very far from historical ones.

Optimal control problems are not usually considered suitable when the state space is three-dimensional because of high computational expenses, but with the methods presented in *essay II* this is no problem. As far as the concrete results are concerned, we find that according to this model optimal TACs of capelin are very dependent on the size of the capelin stock itself, the cod stock and also the herring stock. Cod is the top-predator in the model and therefore optimal TACs of cod are less dependent on the size of the capelin and herring stock. The most important finding in this work is that it indicates that historic TACs have been much too high for both capelin and cod. Starting with stock data from 1973 as initial condition, the biologic growth model and the feedback-catch solution would gives a cod stock on average more than 100 % bigger than actual historic average between the years 1973 to 2005. In spite of the huge cod average, the capelin stock given by the feedback solution is more also than 10 % bigger than historic average.

The feedback-solution implies that capelin to a greater extent should be left as food for cod, which is more valuable on the market of fish. That gives better growth conditions for cod, and a higher cod stock increase the cost effectivity for each unit of cod fished. The total harvested biomass of cod with the feedback solution is slightly lower than actual harvest, but the profit is much bigger due to lower costs. The conclusion is that from an economical point of view, the capelin and god stocks must be managed jointly.

From initial conditions with sensible combinations of stock sizes, optimal management leads to a single stable equilibrium with rather high stocks of all the species in the model. Another interesting finding is that a moderate presence of herring has a positive influence on the sum of yield from the capelin and cod fishery of the Barents Sea. According to the biological growth model a moderate herring stock stimulates cod growth without harming the capelin stock critically much. An obvious conclusion of this finding is that a focus on only the capelin-predator role of herring is too narrow.

1.6 The background of the essays

The four essays of this work are fully written by the main writer. However, parts of them are also read through and commented by one or both of the supervisors. As far as the models are concerned, there have been more contributions from the supervisors. A detailed description follows below. With regard to the amount of contribution by these essays to existing knowledge, the reader is referred to the introduction of each essay for details.

Essay I

The model was originally given by the second supervisor, Professor Leif K. Sandal, but some modeling details are provided by the main writer, who also is responsible for all the numerics and the writing. The mathematical principles behind the solution procedures are the end product of discussion between Sandal, Professor Emeritus Gerhard Berge and the undersigned. Whereas Sandal was the main contributor in the discussions, Berge was responsible for much of the proof readings.

Essay II

The essay is fully produced by the main author. However, the principals behind the solution procedure are a result of discussions with Sandal, who i.a. introduced the article by Grüne and Semmler (2004). All the examples are found by the main author, who also is responsible for the numerics.

Essay III

The idea of working with a capelin and cod model was suggested from the main supervisor, Arne Eide. The starting point was, however, a bottum-up biological model, which turned out to be a poor basis for economical optimizing performed by dynamic programming. Discussions with Geir Huse at the Institute of Marine Resource in Bergen did for a while lead the work in the direction of so-called individual based modelling. This work was left when Sandal in a meeting introduced the undersigned with some of the work behind Agnarsson et.al (2008), where a two dimensional growth model is presented along with a utility/profit model and a feedback-solution for optimal catch. We agreed that the undersigned should start working to improve the numerical algorithms in order to form the basis for solutions of similar models in more than two dimensions. E.g models with a three- or four-dimensional state space.

In a meeting between Eide and the undersigned, the improved numerical method was introduced to Eide, and in the proceeding discussions we agreed on the final model structure for *Essay III*, which came up as an approach that combined the starting point of bottum-up biological modelling with the two-species top-down management model. A scenario model was to be made to test the flexibility of the feedback solution. The flexibility should be tested by simulating possible stock scenarios when an independent assessment model was used as input for the biomass development of capelin and cod. In the coming weeks investigations were done concerning the availability of several assessment models in use at the Institute of Marine Research. Finally, it turned out that researcher Sigurd Tjelmeland had two assessment models that fitted the project well, namely Bifrost and SeaStar. The feedback-solution was integrated into the assessment models, and Tjelmeland provided figures that showed simulated scenarios.

Essay IV

The model in *Essay IV*, with a three-dimensional state space (capelin, cod and herring), is a natural extension of the two-dimensional model presented in Agnarsson et. al. (2008). Several discussions with the working environment at the Institute of Marine Research convinced the undersigned that a three-dimensional growth model is a significant improvement to existing two-dimensional models, as juvenile herring demonstrable impacts the capelin-cod system considerably.

On the way, several growth models were made, tested and discussed. Early in the process Eide had some very useful critical comments and Sandal had some good advices concerning both software to use and statistical methods. However, the final growth model is fully made and discussed by the undersigned. Both Sandal and Eide has done a very good job in proof-reading parts of the text. In addition Sandal has provided some assistance concerning interpretation of the feedback solution and choice of figures to present.

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Chapter 2

Essay I

Abstract

I denne artikkelen presenteres og analyseres en modell for velferdsoptimerende uttak av fossilt brensel i global målestokk. Modellen tar hensyn til forringelsen av miljøet ved bruk og utvinning. Forøvrig er velferdsmålet knyttet til produsent og konsumentoverskuddet i markedet for fossilt brensel, hvor tilbud/etterspørsel påvirkes av både pris og miljø. Den sosiale planleggeren bruker avgift til å justere omsetningen av fossilt brensel til et velferdsmaksimerende nivå. Med velferdsmaksierende mener vi at det tas hensyn til eksternalitetene i modellen. Ellers er tidsperspektivet uendelig, og vi antar at "ren" teknologi etterhvert blir mer lønnsom enn tradisjonell energiutvinning. Denne rene teknologien gir en alternativ profitt som øker med tiden. Tidspunktet for teknologiskiftet inngår endogent i modellen. I lignende modeller er det vanlig å bruke en lineær funksjon som representant for naturens egen rensning av CO_2 *fra atmosfæren. Dette arbeidet har ikke denne begrensningen. Lineær rensefunksjon overvurderer naturens egen evne til å rense opp. Det viser vi ved å demonstrere at en ikke-lineær og ikke-monoton rensefunksjon gir en langt mer restriktiv miljøpolitikk. Modellen løser vi numerisk.*

2.1 Innledning

I de senere tiår har verdenssamfunnet blitt mer og mer bekymret for global oppvarming forårsaket av den såkalte drivhuseffekten. Kyoto-konvensjonen, som trådde i kraft fra 1994, forpliktet de underskrivende land å sette øvre grenser for sine utslipp av CO_2. Imidlertid har flere land, deriblant Norge, vegret seg for å ratifisere avtalen. USA under George Bush erklærte i 2001 sogar å avvise den grunnet konflikten med økonomisk vekst. I lys av dette kan det være interessant å undersøke nærmere hva som er det optimale utslippsnivået av CO_2 i global målestokk sett fra en sosial planleggers side.

Ettersom bruk og utvinning av fossilt brensel utgjør den betydeligste delen av våre utslipp av CO_2, har vi valgt å lage en modell der dette representerer hele forurensningen. Vi undersøker virkningen av avgift på forurensende produksjon. Finnes det et avgiftsnivå som gir en optimal løsning for samfunnet som helhet når en veier fordeler og ulemper mot hverandre? I vår modell er dette avgiften som maksimerer summen av velferdsoverskudd når det tas høyde for negative velferdsvirkninger av flyt- og oppsamlingseksternaliter forbundet med utslipp av CO_2. Den sosialt optimale løsningen sammenlignes med den rene markedsløsningen. På denne måten illustreres effekten av en aktiv miljøbevisst politikk.

En viktig egenskap med modellen er at forurensningen ikke bare påvirker gjennom eksternalitetene. Vi åpner i tillegg for at tilbud og etterspørsel er direkte påvirket av dens akkumulerte nivå. Dermed kan vi ta høyde for kundepreferanser som miljøbevissthet i markedet. Likeledes tenker vi oss at kostnader knyttet til produksjon kan endres av forurensningen. Dette kan for eksempel være kostnader grunnet strengere krav til rensning.

I eksisterende økonomisk litteratur antas stort sett en lineær rensefunksjon. (Vellinga og Withagen 1996 og Tahvonen 1997). Denne antagelsen reduserer de matematiske utfordringene når en skal finne et optimalt skattenivå, men gjør også modellene mindre realistiske. Dersom rensefunksjonen virkelig var lineær, ville uendelig stor forurensning også gitt uendelig renseevne. Virkeligheten er mindre rosenrød. Mye tyder på at rensefunksjonen ikke bare flater ut, men også avtar og går helt i null for høye forurensningsnivå. Dette betyr at scenarier med irreversibel forurensning ikke kan utelukkes.

Farzin og Tahvonen (1996) modellerer rensefunksjonen ved å anta to typer karbonbeholdninger i atmosfæren. En type med lineær rensning og en type som naturen ikke klarer å rense. På denne måten mener de å fange opp forventede ikke-lineære egenskaper med den virkelige rensefunksjonen. Dessuten har Tahvonen og Salo (1996) gitt ut en artikkel der forurensningen er skapt ved uttak av en fornybar ressurs. Her er rensefunksjonen gjort mer realistisk ved at den er strengt økende og konkav for lave forurensningsnivåer og avtagende og konveks for høye nivåer. Denne rensefunksjonen er ikke definert eksplisitt. Istedenfor er den gitt som en generell funksjon med de ønskede egenskaper.

Foruten de allerede nevnte arbeidene er Sandal og Steinshamn (1998, 2000, 2001) og Sandal, Steinshamn og Grafton (2003) blant dem som har jobbet med modeller der det åpnes for ikke-lineære tilnærminger til rensefunksjonen.

Vi bruker i dette arbeidet en ikke-lineær rensefunksjon. Den øker med forurensningen for lave akkumulerte verdier, men når en topp og avtar deretter mot null for høy verdier. Etter å ha antatt en slik form kalibrerer vi rensefunksjonen basert på historiske data for utslipp av CO_2 og for aggregert mengde i atmosfæren. I tillegg sammenligner vi med en lineær estimering basert på de samme data.

Flere har påpekt at like viktig som avgiftsstørrelsen er utviklingen over tid - om avgiften er avtagende eller økende. (jfr. Ulph og Ulph 1994 og Sinclair 1994). I denne sammenheng er en realistisk rensefunksjon avgjørende. Dessuten er det svært viktig at man tar hensyn til at fossilt brensel er en ikke-fornybar ressurs som bare kan "høstes" i et endelig tidsperspektiv. Mange tar ikke dette hensynet. Da kan de anta at CO_2-nivået utvikler seg mot en likevekt der tilførselen av ny forurensning er lik renseevnen fra naturen. (Nordhaus 1982 og Stollery 1998).

Selv om optimeringsproblemet i vår modell har et uendelig tidsperspektiv, utvinnes der bare fossilt brensel i et begrenset tidsrom. Vi tenker oss at en alternativ ren energikilde etterhvert overtar. Profitten ved uttak av denne alternative energikilden er ikke konstant. Istedenfor åpnes det for at teknologiske fremskritt gir rom for høyere og høyere profitt ved bruk av ren teknologi. Vi kjenner ikke til at tidsavhengig alternativ teknologi har blitt behandlet i litteraturen tidligere.

Tidspunktet for teknologiskiftet, $t = T$, finnes endogent i modellen vår. I T er der to muligheter. Enten har optimal politikk vært å bruke opp hele ressursen av fossilt brensel, eller så er der fremdeles mer igjen. Matematisk vil vi behandle dette som to adskilte tilfeller.

I tilsvarende publiserte modeller antar man typisk T som en eksogen størrelse fordi det gjør modellene betydelig enklere. (Se f.eks. Ulph og Ulph (1994)). I forhold til våre resultater kan avgiftsprofil og uttaksnivå av fossilt brensel endres vesentlig som følge av en slik forenklingen. Relativt små endringer i modellstørrelser, f.eks. funksjonen for etterspørsel etter fossilt brensel, kan gi svært stor endring i T når denne størrelsen finnes endogent. Dette er hovedårsaken til at vi behandler T som en endogen størrelse.

2.2 Modellen

2.2.1 Generelle betraktninger

Nytten i modellen er knyttet til velferdsoverskuddet i markedet for fossilt brensel. For å maksimere dette overskuddet korrigeres markedet for alle flyt- og oppsamlingseksternaliteter. Med flyteksternaliteter mener vi i denne sammenheng de umiddelbare miljøkostnader forbundet med bruk og utvinning av fossilt brensel. Et eksempel kan være forringet luftkvalitet i nærheten av industriell virksomhet der det nyttes fossilt brensel. Oppsamlingseksternalitetene representerer de samfunnsmessige kostnader ved at utslipp i dag lagres i atmosfæren og får virkninger også for fremtiden.

Vi lar $y(t)$ være uttak av fossilt brensel ved tidspunkt t, mens $a(t)$ betegner akkumulert CO_2-nivå i atmosfæren og $s(t)$ er den gjenværende mengden fossilt brensel i naturen[1]. Videre lar vi den inverse etterspørselskurven $P(a, y)$ representere samfunnets etterspørsel etter fossilt brensel. Dette tenker vi oss som en ikke-separabel funksjon på formen $P(a, y) = p_0(a) - p_1(a)y$. (Se Fig. 2.1). Legg merke til at denne kurven har generell a-avhengighet. Dermed åpner vi for at miljøhensyn kan påvirke verdensmarkedets etterspørsel etter fossilt brensel.

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[1] I fortsettelsen vil det være underforstått at $a = a(t)$, $y = y(t)$ og $s = s(t)$.

Når det gjelder kostnadskurven/tilbudskurven har vi valgt å skille mellom privat kostnad, $C_p(a, y)$, og sosial kostnad, $C_s(a, y)$. Den private kostnadskurven omfatter bare kostnader som direkte vedrører produsenten/ressursuttakeren i markedet, mens den sosiale kostnadskurven også inkluderer flyteksternalitetene i modellen.

Oppsamlingseksternalitetene representeres av skadefunksjonen $D(a)$. ($D(0) = 0$, $D' > 0$ og $D'' > 0$ for $a > 0$). Uttakerne av ressursen er bare opptatte av å maksimere produsentoverskuddet (PS), og er derfor blinde for de velferdsdempende faktorene (utslipp/forurensning) knyttet til uttaket y. Vi kan heller ikke forvente at samfunnets etterspørsel skal gjenspeile alle de negative sider ved bruken av ressursen. Derfor tenker vi oss at en sosial planlegger justerer markedslikevekten slik at den utgjør et sosialt optimum når det tas hensyn til flyt- og mengdeeksternalitetene.

Den dynamiske utviklingen av tilstandsvariablene a og s, styres av y. Vi tenker oss at utslippene til atmosfæren er proporsjonale med y. Dessuten påvirkes dynamikken av rensefunksjonen, $f(a)$. Denne funksjonen kan vi se på som naturens egen nettorensning av CO_2 fra atmosfæren. Den skal være økende for lave a-verdier, men når en topp ved et visst nivå. For veldig høye a-verdier antar vi at rensningen avtar konvekst mot null. Dette åpner for scenarier med irreversibel forurensning.

Tilstandsligningene er gitt ved [2]

$$\begin{aligned} \dot{a} &= \mu y - f(a), \qquad \mu > 0 \\ \dot{s} &= -y. \end{aligned} \tag{2.1}$$

Vi antar at ressursen med fossilt brensel før eller siden brukes helt opp eller blir ulønnsom, og at en ny ren teknologi overtar. I denne siste perioden blir tilstandsligningene våre på formen

$$\begin{aligned} \dot{a} &= -f(a) \\ \dot{s} &= 0, \end{aligned} \tag{2.2}$$

mens vi tenker oss velferden som differansen mellom en kjent "profittfunksjon" $\widehat{\pi}(t)$ og skadefunksjonen $D(a)$.

2.2.2 Uregulert markedsløsning

I den uregulerte markedsløsningen blir markedsoverskuddet fordelt i sin helhet mellom produsenter og konsumenter, og vi kan se bort ifra både flyt- og oppsamlingseksternaliteter. Det vil si at uttaket y vil være bestemt av a i henhold til markedslikevekten $P(a, y) = C_p(a, y)$. Fig. 2.1(a) illustrerer denne likevekten, som er dannet av den private kostnadskurven og etterspørselskurven.

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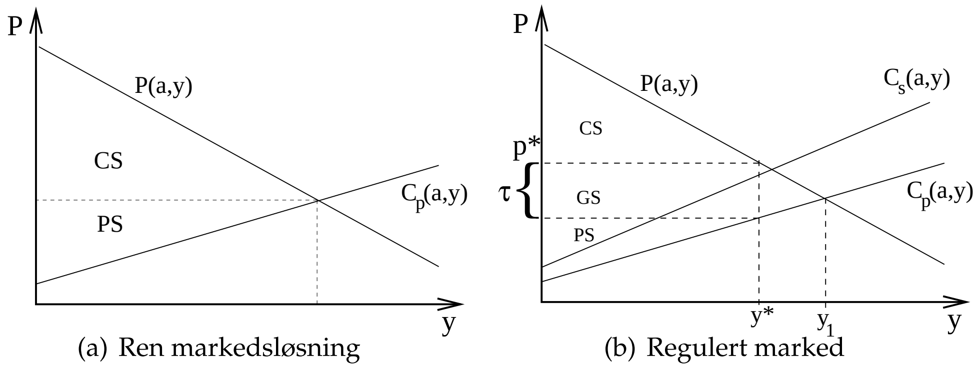
[2] Prikk over a og s markerer her tidsderiverte.

Denne løsningen av modellen er ikke sosialt optimerende. Oppsamlingseksternaliteten, $D(a)$, og flyteksternalitetene, som inkluderes i den sosiale kostnadskurven, $C_s(a, y)$, berører ikke markedsaktørene.

Uten en aktiv skattepolitikk er eneste miljøhensyn i modellen a-avhengigheten i kostnads- og etterspørselsfunksjoner. Teknologiskiftet kommer enten fordi de samlede ressurser av fossilt brensel er oppbrukt, eller fordi markedsaktørenes profitt ved ren teknologi blir større enn ved tradisjonell teknologi.



(a) Ren markedsløsning (b) Regulert marked

Figure 2.1: Sosialt optimum, $(y^\star, P^\star)$. Skatten τ forskyver likevekten fra $(y_1, P(a, y_1))$ til $(y^\star, P^\star)$.

2.2.3 Regulert markedsløsning

Ved hjelp av skatt kan den sosiale planleggeren korrigere markedslikevekten slik at den tar hensyn til både flyt- og oppsamlingseksternaliteter og blir sosialt optimerende. Dersom markedslikevekten i utgangspunktet ligger i skjæringspunktet mellom $P(a, y)$ og $C_p(a, y)$, flytter Pigou-skatten

$$\tau(a, y) = P(a, y) - C_p(a, y) \tag{2.3}$$

likevekten mot venstre til det sosialt optimale nivået $(y^\star, P^\star)$. Vi ser av Fig. 2.1(b) at dette punktet ligger noe til venstre for skjæringspunktet mellom den sosiale kostnadskurven og etterspørselskurven. Årsaken er at vi har valgt å ikke ta med oppsamlingseksternalitetene i den sosiale kostnadskurven. Disse representeres istedenfor av skadefunksjonen, $D(a)$. Den sosiale kostnadskurven skiller seg bare fra den private kostnadskurven ved at den inkluderer flyteksternalitetene. På grunn av manglende viten om hvordan disse størrelsene oppfører seg i virkeligheten, nøyer vi oss med å tilnærme dem lineært. Vi setter

$$C_p(a,y) = c_{0p}(a) + c_{1p}(a)y$$
$$C_s(a,y) = c_{0s}(a) + c_{1s}(a)y.$$

Uten skadefunksjonen ville markedslikevekten vært gitt ved skjæringspunktet mellom den inverse etterspørselskurven og den sosiale kostnadskurven. (Se Fig. 2.1(b)). Arealet mellom $P(a,y)$ og $C_s(a,y)$ ville da utgjort flyten av velferdsoverskudd i markedet for fossilt brensel. Dette arealet representerer vi som

$$\pi(a,y) = \int_0^y \big[P(a,x) - C_s(a,x)\big]\,dx = \beta(a)y - \gamma(a)y^2, \quad \beta,\gamma > 0, \tag{2.4}$$

hvor

$$\beta(a) = p_0(a) - c_{0s}(a) \quad \text{og} \quad \gamma(a) = \frac{1}{2}\big[p_1(a) + c_{1s}(a)\big]. \tag{2.5}$$

Den sosiale planleggeren tar hensyn til oppsamlingseksternaliteter representert ved skadefunksjonen $D(a)$. Målet er å maksimere summen av $PS + CS + GS$. [3] I appendiks A.1 viser vi at summen av $PS + CS + GS = \pi(a,y^\star) - D(a)$.

Dersom vi lar $\Phi(a(T),T) \equiv \int_T^\infty e^{-rt}[\widehat{\pi}(t) - D(a)]dt$, kan vi formulere vårt optimeringsproblem som et en-periode-problem med "skrapverdi". Neddiskontert velferdsmaksimum er da gitt ved

$$\max_{y,T} \left\{ \int_0^T e^{-rt}\big[\pi(a,y) - D(a)\big]\,dt + \Phi(a(T),T) \right\} \tag{2.6}$$

under bibetingelse

$$\begin{bmatrix} \dot{a} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} y - f(a) \\ -y \end{bmatrix}, \text{når t < T.} \tag{2.7}$$

I forhold til ligning (2.1) har vi her skalert både y og s slik at parameteren μ får verdien 1, og kan ignoreres. Dette betyr at fossilt brensel måles i hvor mye CO_2 det skaper. Skattenivået som korresponderer med det optimale uttaksnivået $y^\star$ blir da $\tau(a,y^\star) = P(a,y^\star) - C_p(a,y^\star)$, mens tilhørende ad-valorem skatt blir:

$$\theta(a,y^\star) = \frac{P(a,y^\star) - C_p(a,y^\star)}{C_p(a,y^\star)}. \tag{2.8}$$

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[3] GS (Government Surplus) er noe misvisende i denne sammenheng ettersom vi ser på uttak av fossilt brensel i global - ikke nasjonal - målestokk. PS og CS representerer henholdsvis produsent- og konsumentoverskudd.

Maksimeringsproblemet i (2.6) og (2.7) kan formuleres to-periodisk med y som enslig kontrollvariabel. Nåverdi-hamiltonfunksjon blir da [4]

$$\mathcal{H}(a, y, t) = \begin{cases} \pi(a, y) - D(a) + m(y - f(a)) - ny, & t \leq T \\ \widehat{\pi}(t) - D(a) - mf(a), & t > T, \end{cases} \tag{2.9}$$

med tilhørende første ordens betingelser:

$$\dot{m} = rm - \mathcal{H}_a \tag{2.10}$$

$$\dot{n} = rn - \mathcal{H}_s \tag{2.11}$$

$$\mathcal{H}_y = 0, \tag{2.12}$$

der $m(t)$ er skyggepris (kofaktor) for a og $n(t)$ er skyggepris for s. Vi refererer til (2.10) og (2.11) som henholdsvis kofaktor 1 og kofaktor 2, mens (2.12) er kravet for indre optimum.

Når det gjelder transversalitetskravene til maksimeringsproblemet (2.6) og (2.7), blir transversalitetskravet svarende til at ressursen er endelig[5]

$$n(T)s(T) = 0, \quad s \geq 0, \quad n(T) \geq 0. \tag{2.13}$$

Filipov-Cesaris eksistensteorem [6], garanterer eksistens av en optimal løsning for problemet vårt. (Se appendiks A.2). En formulering av hvordan vi har løst optimeringsproblemet vårt følger i appendiks A.3.

2.3 Numeriske eksempler

Før vi presenterer numeriske løsninger av modellen, skal vi gjøre noen spesifikasjoner over aktuelle modellstørrelser. Forurensningen, a, måles i gigatonn CO_2, og vi har valgt å gjøre skaleringen $a \to a - \tilde{a}$, der $\tilde{a} = 2187$ er pre-industrielt a-nivå. Dette er fornuftig fordi CO_2-nivået i atmosfæren var relativt stabilt før den industrielle revolusjon, iallfall mellom istidene. Dessuten er skaleringen hensiktsmessig i definisjonen av rensefunksjonen. Denne er valgt på en form som hovedsakelig representerer en forskjøvet normalfordeling. Vi har satt

$$f(a) = k_1 \max\{0, \mathrm{e}^{-\mathrm{k}_2(\frac{\mathrm{a}}{\hat{\mathrm{a}}}-1)^2} - \mathrm{e}^{-\mathrm{k}_2}\}, \tag{2.14}$$

der $\hat{a}$ er topp-punktet for $f(a)$. Dette er gitt verdien $\hat{a} = 625$.

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[4] Prinsippene for optimal kontrollteori kan f.eks finnes i Kamien og Schwarz (1991).

[5] En mer generell diskusjon av transversalitetskrav finner man i Seierstad og Sydsæter (1987) og Kamien og Schwarz (1991).

[6] Se Seierstad og Sydsæter (1987).

Legg merke til at $f(0) = 0$. Dette oppnådde vi ved skaleringen av parameteren a. Konstantene k_1 og k_2 er funnet ved kalibrering av rensefunksjonen med data for atmosfærisk CO_2-nivå og tilsvarende utslipp.[7] Resultatet ble $k_1 = 10.885$ og $k_2 = 2.610$. Her måles k_1 og $f(a)$ i gigatonn CO_2 pr. år, mens k_2 er dimensjonsløs. (Se Tab. 2.2 for en oversikt over dimensjonene til alle størrelsene i modellen). Verdien $\hat{a} = 625$ kan kanskje kritiseres for å være noe pessimistisk ettersom det følger at dagens forurensning, $a_0 = 625$, utgjør maksimum for rensefunksjonen. De nyeste dataene som er tilgjengelige underbygger likevel vår modellering. De viser at oppsamling av CO_2 i atmosfæren har aksellerert de siste årene samtidig som utslippene har flatet ut. Dette tyder på at naturens egen rensning er svekket og passer veldig godt med vår antagelse om en rensefunksjon som er avtagende for forurensningsnivåer høyere enn vår a_0-verdi. Rensefunksjonen vi har brukt gir mindre og mindre rensning dess mer a stiger når $a > a_0$. Når a passerer $2\hat{a} = 1250$ vil den opphøre helt. Da får vi irreversibel forurensning. Et slikt verste utfall-scenario kan ikke avvises fra et biologisk ståsted.

Virkninger av den ikke-lineære rensefunksjonen skal vi i avsnitt 2.3.2 sammenligne med virkningene av en lineær tilnærming som er kalibrert med de samme data. I de andre avsnittene undersøker vi bare den ikke-lineære tilnærmingen.

Vi fant følgende lineære tilnærming:

$$f_{lin}(a) = 0.017a. \tag{2.15}$$

Skadefunksjonen er i likhet med rensefunksjonen en veldig usikker størrelse. Denne skal gi et økonomisk mål (NOK) på de negative ringvirkninger av oppsamling av CO_2 i atmosfæren. Vi har valgt å følge Sandal og Steinshamn (2000), som brukte

$$D(a) = \frac{a^2}{100000}. \tag{2.16}$$

Denne funksjonen har de ønskelige egenskapene $D(0) = 0$, $D' \geq 0$ når $a > 0$ og $D'' > 0$.

For den tidsavhengige alternative profitten benytter vi

$$\widehat{\pi}(t) = 140 - 70 \cdot e^{-0.07t}. \tag{2.17}$$

Vi tenker oss dermed en alternativ teknologi som utvikler seg veldig fort i begynnelsen, men etterhvert tar ut hele sitt potensiale og stabiliserer seg mot $\lim_{t\to\infty} \widehat{\pi}(t) = 140$. Det må understrekes at både $\widehat{\pi}(t)$ og $D(a)$ er høyst usikre størrelser. Det gjenstår mye økonometri og dataanalyse før vi får full klarhet i disse. Det ligger imidlertid utenfor

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[7] Data for atmosfærisk CO_2-nivå og data for historiske utslipp er tilgjengelige på henholdsvis http://cdiac.esd.ornl.gov/ftp/maunaloa-co2/maunaloa.co2 og http://cdiac.esd.ornl.gov/ftp/ndp030/global00.ems.

dette arbeidet, hvor hovedfokus er å studere mulige effekter av en ikke-lineær rensefunksjon. Man skal derfor ikke overdrive vektleggingen av parameterstørrelser og enheter. At $D(a)$ og $\widehat{\pi}(t)$ måles i NOK pr. tidsenhet, mens $t = 1$ tilsvarer ett år, er av mindre betydning enn modellens prinsipielle elementer.

Vi har i modellen vår antatt generell a-avhengighet i markedet for fossilt brensel. Dermed gis det rom for at miljøbevissthet blant forbrukerne kan påvirke etterspørselen. Et tilfelle uten a-avhengighet og et tilfelle med a-avhengighet undersøkes. Disse sammenlignes i seksjon 2.3.1. I de andre seksjonene undersøker vi bare a-avhengighet.

I tilfellet med a-avhengighet satte vi $p_0(a) = 16 - 0.0012 \cdot a$. Da er $p_o(a_0) = 15.25$, som er nesten lik verdien i tilfellet med p_o konstant lik 15.3. Dette er ingen sterk avhengighet for den inverse etterspørselen $P(a, y) = p_0(a) - p_1(a)y$, men vi skal se at den likevel får en del å si for a og T.

Det kan være verdt å legge merke til at vi har valgt C_{0p} som en egen enhet (jfr. Tab. 2.2) og satt $C_{0p} = C_{0s} = 1$ i Tab. 2.1. Uten skaleringen $\mu = 1$, (jfr. ligning (2.1) og (2.7)), ville enheten C_{0p} ha tilsvart prisen på den billigste oljen verdensmarkedet er i stand til å produsere. Vi nøyer oss med at C_{0p} er proporsjonal med denne prisen.

Vi har satt diskontering $r = 0.05$ dersom annet ikke er spesifisert. De andre parametrene for etterspørsels- og kostnadskurver finner vi i Tab. 2.1. Disse er usikre ettersom mye økonometri og dataanalyse mangler.

Markedslikevekt	Parameter	verdi	parameter	verdi
avhengig av a	p_0	$16 - 0.0012 \cdot a$	p_1	0.6
uavhengig	p_0	15.3	p_1	0.6
av a	c_{0s}	1	c_{1s}	0.09
	c_{0p}	1	c_{1p}	0.02

Table 2.1: Benyttede verdier for parameterne i etterspørsels- og kostnadsfunksjonen.

Table 2.2: Enheter for parametere og funksjoner.

Parameter	enhet	funksjon	enhet
$[a]$, $[s]$	Gt. CO_2	$f(a)$	Gt. CO_2/pr. år
$[y]$, $[k_1]$	Gt. CO_2 pr./år	D(a)	NOK/pr. år
$[c_{0p}(a)]$, $[c_{0s}(a)]$, $[p_0(a)]$	$C_{0p}(a)$	$[\pi(a, y)]$	NOK/pr. år
$[c_{1p}(a)]$, $[c_{1s}(a)]$, $[p_1(a)]$	$C_{0p}\cdot$ år/Gt. CO_2	$[\widehat{\pi}(t)]$	NOK/pr. år
$[\theta]$, $[k_2]$	dimensjonløs	$[\phi(a(T), y(T))]$	NOK

2.3.1 Regulert versus uregulert markedsløsning

I denne seksjonen skal vi fokusere på virkningen av eksternalitetene i modellen. Det vil si at vi skal sammenligne den uregulerte markedsløsningen med den regulerte. Samtidig ser vi også på virkningen av at etterspørselen gjøres negativt korrelert med akkumulert forurensning. Dette kan vi kalle idealistisk marked.

La oss først konsentrere oss om virkningen av å regulere markedet med skatt eller avgift. Skatten, som viser seg å ligge mellom 130 og 160 $\%$ over uregulert markedspris, får store konsekvenser i dette numeriske eksempelet. I Fig. 2.2(a) og 2.2(b) gir et uregulert marked irreversibel forurensning. Dette ser vi fordi den akkumulerte forurensningen ikke avtar etter teknologiskiftet. Istedenfor stabiliseres den på et veldig høyt nivå. Denne egenskapen ville modellen vår ikke fanget opp dersom rensefunksjonen var lineær. Det at vi har antatt dagens nivå for rensning som det maksimale, gjør at virkeligheten kan komme til å se noe lysere ut enn i vår numeriske beregning, men det er ikke gitt at vårt anslag for rensefunksjonen er for pessimistisk.

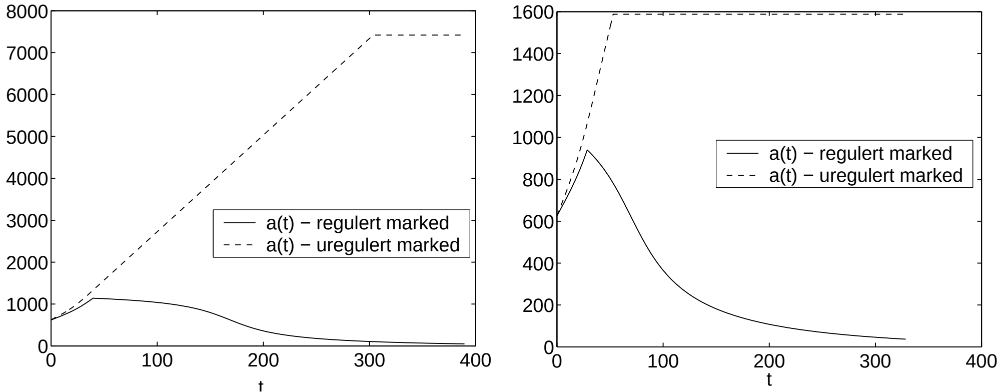
Tab. 2.3 viser at tiden frem til teknologiskiftet forlenges kraftig når markedet er uregulert. Særlig i tilfellet der etterspørselskurven ikke er a-avhengig. Da brukes ressursen av fossilt brensel opp, og som en konsekvens av dette blir ikke teknologien ren før etter over 300 år. Med a-avhengighet kommer teknologiskiftet allerede etter 53 år. Dermed ser vi at forbrukernes miljøbevissthet kan ha veldig mye å si. Vi registrerer også at uten a-avhengighet så er forurensningen ved teknologiskiftet nesten 5 ganger høyere for det uregulerte tilfellet enn for det regulerte. Dessuten legger vi merke til at $y(0) = y(T)$ for tilfellet uten a-avhengighet i den rene markedsløsningen.

For en sosial planlegger kan det på Tab. 2.3 og Fig. 2.2 se ut som betydningen av miljøbevissthet i markedet er mindre enn for det uregulerte markedet. Vi kan forstå dette som at planleggerens skatter gjør markedets bevissthet mindre nødvendig, og de negative konsekvensene av forurensningen blir i dette tilfellet ikke så store fordi forurensningen begrenses av skatten som blir pålagt. Studerer vi imidlertid Fig. 2.2(a) og 2.2(b) nøye, ser vi at miljøbevisstheten likevel har stor betydning. I det "idealistiske" tilfellet har a-kurven en spiss topp - det vil si at den avtar fort etter teknologiskiftet. Årsaken er at forurensningen på sitt høyeste nivå ikke er høyere enn at naturens rensning fremdeles bidrar vesentlig. Uten a-avhengighet blir imidlertid a-kurven mye flatere. Dette fordi stor forurensning har svekket naturens egen renseevne så mye at teknologiskiftet i første omgang bare utløser en beskjeden reduksjon i akkumulert forurensning.

I de neste seksjonene vil vi la etterspørselen være a-avhengig.

Table 2.3: Randstørrelser i regulert/uregulert marked. Konstant/a-avhengig etterspørsel.

Marked	etterspørsel	y(0)	y(T)	a(T)	s(T)	T	$\theta(0)$	$\theta(T)$
Uregulert	konstant	23.065	23.065	7419.04	0	303.497	0	0
	a-avhengig	22.984	21.121	1587.6	5826.19	53.002	0	0
Regulert	konstant	20.006	19.825	1140.31	6216.26	39.505	1.3542	1.4383
	a-avhengig	19.459	19.451	940.03	6447.26	28.569	1.5733	1.3049



(a) Ikke-idealistisk marked. (b) Idealistisk marked.

Figure 2.2: Akkumulert forurensning. (Gt. CO_2). Regulert versus uregulert marked.

2.3.2 Lineær versus ikke-lineær rensefunksjon

I denne seksjonen skal vi se hvilken betydning formen på rensefunksjonen har for optimeringsproblemet vårt. Vi tester den ikke-lineære formen (2.14) mot den lineære (2.15).

Forskjellen i virkninger av lineær og ikke-lineær rensefunksjon illustreres kanskje best i det uregulerte tilfellet. Dersom vi betrakter a-kurvene for dette tilfellet i Fig. 2.3, ser vi at det tar relativt lang tid før teknologiskiftet finner sted slik at forurensningen er svært høy. I tilfellet med lineær rensning (Fid. 2.3(a)) avløses skiftet i teknologi umiddelbart av en sterk nedgang i akkumulert forurensning. Dette fordi antagelsen om lineær rensning gir svært optimistiske renseverdier for høye forurensningsnivå. I motsetning ser vi at langt lavere rensning for høye a-nivå i det ikke-lineære tilfellet gir irreversibel forurensning.

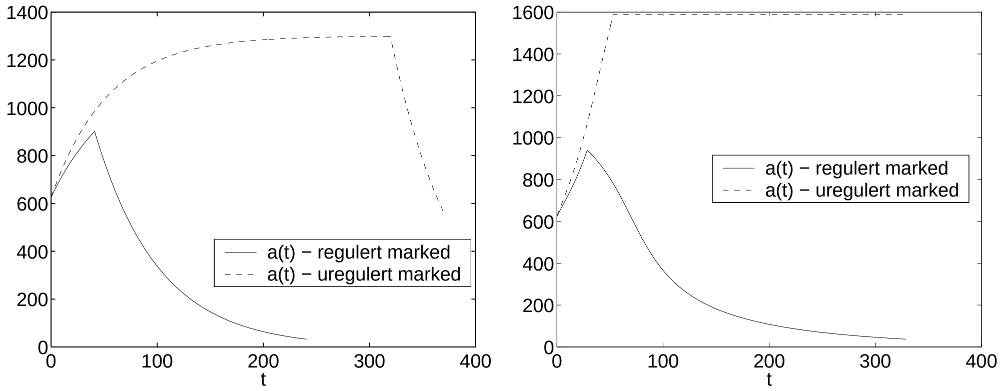
En interessant egenskap ved linearitet, som er særlig tydelig i det uregulerte markedet, er at etterhvert som tiden går flater den akkumulerte forurensningen ut. Årsaken er at rensningen øker lineært med forurensningen. Dermed er vi i en slags "likevekt" med svært store utslipp og svært stor forurensning. (Se Fig. 2.3(a)). I vårt numeriske eksempel kommer teknologiskiftet som et resultat av at den alternative rene teknologien blir den mest lønnsomme, men dersom vi hadde endret litt på modellparameterne kunne skiftet kommet på grunn av knapphet på fossilt brensel.

I det regulerte markedet ser vi en mer moderat virkning av ikke-linearitet. Tendensene er likevel de samme. Ikke-lineariteten fremskynder teknologiskiftet kraftig, og de fremtidige skadevirkninger av forurensningen er større fordi anslagene for naturens renseevne er mer pessimistiske.

Når det gjelder skattetrykket ser vi i Fig. 2.4 at dette blir noe større i det ikke-lineære tilfellet. Forskjellene er imidlertid ikke dramatiske. Initielt er skatten henholdsvis 138 % og 157 %, mens den for begge tilfellene er konkav frem til teknologiskiftet og da har gått betydelig ned.

Marked	Rensefunksjon	y(0)	y(T)	a(T)	s(T)	T	$\theta(0)$	$\theta(T)$
Uregulert	Lineær	22.984	21.680	1298.88	1357.23	319.700	0	0
	Ikke-lineær	22.984	21.121	1587.6	5826.2	53.002	0	0
Regulert	Lineær	19.857	19.858	901.26	6190.60	41.070	1.3876	1.1497
	Ikke-lineær	19.459	19.451	940.03	6447.26	28.569	1.5733	1.3049

Table 2.4: Randstørrelser. Lineær versus ikke-lineær rensefunksjon.

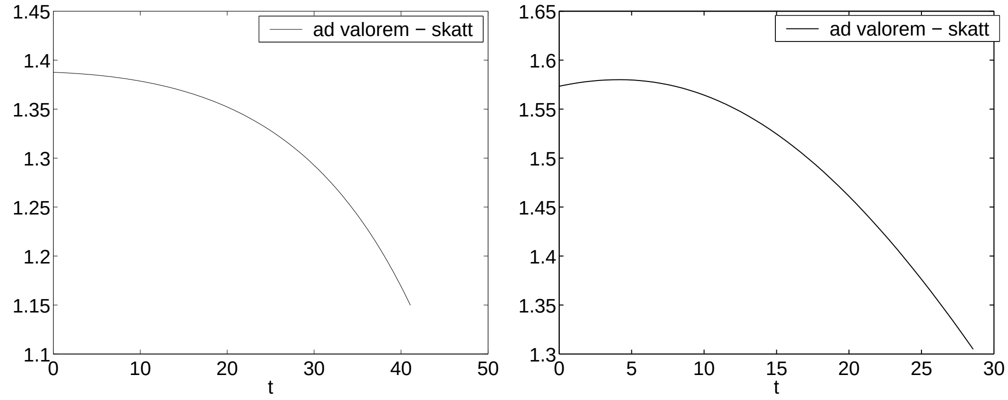


(a) Lineær rensefunksjon $f_{lin}(a)$. (b) Ikke-lineær rensefunksjon $f(a)$.

Figure 2.3: Akkumulert forurensning. (Gt. CO_2).

2.3.3 Virkning av diskontering

Vi skal nå se hvilken innvirkning diskonteringen, r, har for modellen vår. Dette tester vi bare ut for den regulerte markedsløsningen ettersom diskonteringen ikke har noe



(a) Lineær rensefunksjon $f_{lin}(a)$. (b) Ikke-lineær rensefunksjon $f(a)$.

Figure 2.4: Ad-valorem skatt, $\theta(t)$. (Skatteprosent= $\theta \cdot 100\%$)

å si når markedet er uregulert. (Se seksjon 2.2.2). Særlig skal vi konsentrere oss om sammenhengen mellom diskontering og skatt.

Vi ser av Tab. 2.5 at forurensning og uttaksperiode stiger klart med økende diskonteringer. I tillegg ser vi at initielt uttak øker tydelig, mens uttaket like før teknologiskiftet er noe mindre følsomt. Dette er slik vi på forhånd kunne forvente ettersom diskonteringen øker betydningen av umiddelbar profitt.

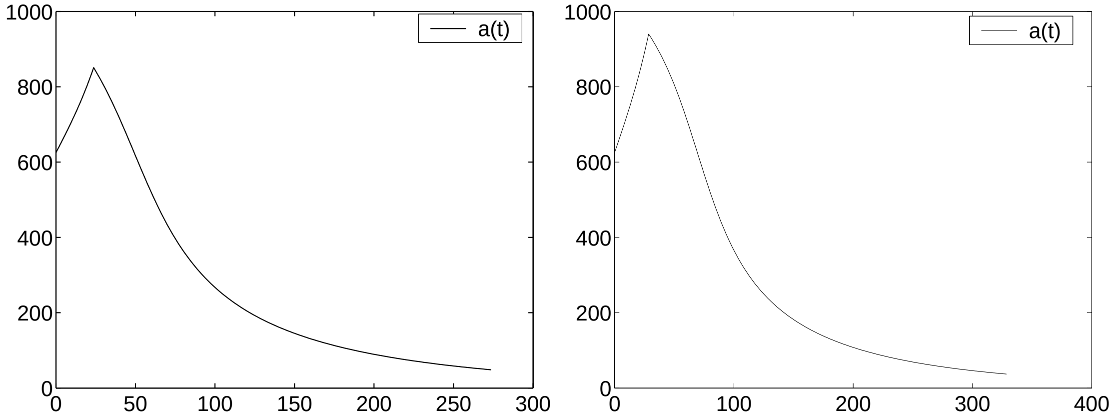
Table 2.5: Randstørrelser. Diskonteringens virkning i to typer marked.

Marked	Diskontering	y(0)	y(T)	a(T)	s(T)	T	$\theta(0)$	$\theta(T)$
reg- ulert	$r = 0.01$	18.2161	19.1623	850.75	6558.92	23.71	2.1666	1.5171
	$r = 0.03$	18.9675	19.3353	898.60	6497.02	26.41	1.8053	1.3945
	$r = 0.05$	19.4586	19.4507	940.03	6447.26	28.57	1.5733	1.3049

Akkumulert forurensning viser seg å bli en del større ved 5 $\%$ diskontering enn ved 1 $\%$ diskontering. (Se Fig. 2.5). Både høyere uttaksnivå og senere teknologiskifte må ta skylden, men forskjellene er ikke dramatiske.

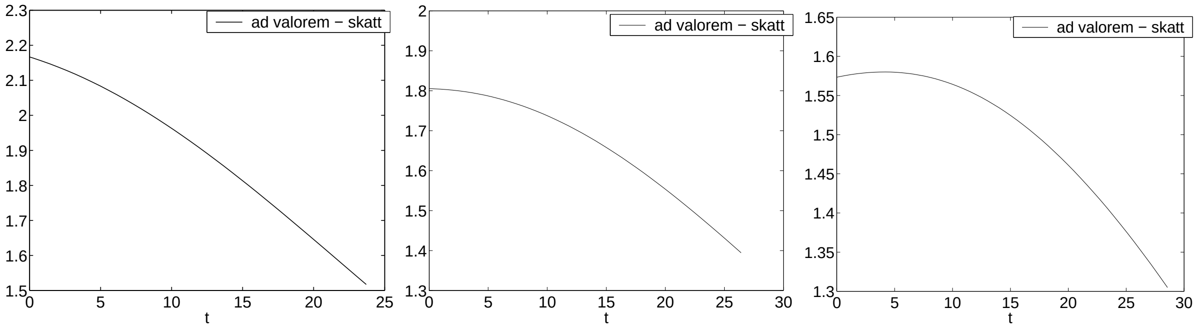
Når det gjelder skattetrykket er ikke virkningene av diskonteringen så veldig store. Vi ser av Fig. 2.6 at skatten faller etterhvert som vi nærmer oss teknologiskiftet. Ved lav diskontering er der mye høyere skatt initielt, men ved teknologiskiftet er forskjellene mindre. Dette er også hva vi kunne forvente ettersom vi så at initielt uttak var følsomt for diskontering. Ellers ser vi at skatten er konkav i tiden. Fig. 2.7 viser også at skatten avtar med økte mengder av akkumulert forurensning. Når a-nivåene ikke er høyere enn i disse eksemplene, ser vi også at skatten er konkav i a.

Det at skatten avtar når akkumulert forurensning øker er kanskje ikke helt intuitivt. Mange vil kanskje forvente at høy forurensning også vil gi høy skatt ettersom skaden av utslippene da er større. Men vi må også huske at når forurensningen allerede er veldig stor, så vil en liten økning være enda mindre relativt sett. En økning i skattenivå som reduserer uttaket kan da gi et større tap for markedsaktørene enn den inntekt det gir for planleggeren. I tillegg kommer at flyteksternalitetenes innvirkning i noen grad kan sies å ta over skattens uttaksdempende funksjon i tilfellet med a-avhengig etterspørsel. Numeriske eksempler viser at skattens konkavitetsegenskaper forsvinner for store a-verdier i tilfellet uten idealistiske markedsaktører.



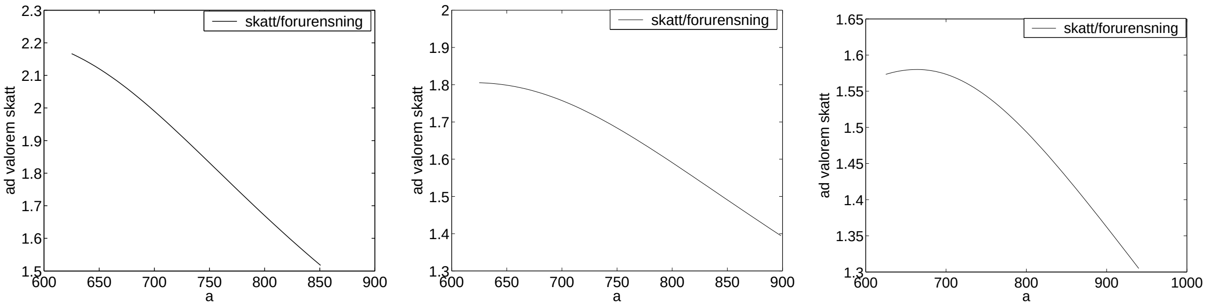
(a) $r = 0.01$. (b) $r = 0.05$.

Figure 2.5: Akkumulert forurensning (Gt. CO_2) for ulike diskonteringsverdier.



(a) Diskontering $r = 0.01$. (b) Diskontering $r = 0.03$. (c) Diskontering $r = 0.05$.

Figure 2.6: Ad-valorem skatt, $\theta(t)$, for ulike diskonteringsverdier.



(a) Diskontering $r = 0.01$.

(b) Diskontering $r = 0.03$.

(c) Diskontering $r = 0.05$.

Figure 2.7: Skatt som funksjon av forurensning ($\theta(a)$) når $r = 0.05$.

2.4 Oppsummering og konklusjoner

I dette arbeidet sammenlignet vi først en uregulert markedsløsning med en regulert markedsløsning. Vi såg at skatt som justerte markedslikevekten til å ta hensyn til oppsamlingseksternaliteter fikk store konsekvenser for akkumulert forurensningsnivå og teknologiskiftetidspunkt. Uten skatt gav modellen en tilstand med irreversibel forurensning ettersom det akkumulerte CO_2-nivå ble så høyt at naturens egen rensning kollapset.

Modellformuleringen med etterspørsel som avtok med økt forurensning virket sterkt inn på resultatene. Dette gav også irreversibel forurensning i den rene markedsløsningen, men hadde vi valgt større negativ forurensningskorrelasjon for etterspørselen kunne dette gitt et annet resultat.

Egenskapen med irreversibel forurensning er kun et mulig scenario når der brukes en ikke-lineær rensefunksjon. Dersom rensefunksjonen velges lineær, vil naturen uavhengig av forurensningsnivå rense seg selv relativt kort tid etter et teknologiskifte. Dette fordi rensningen stiger lineært med akkumulert forurensning. Uendelig stor forurensning gir dermed uendelig renseevne.

Det viser seg at antagelsen om lineær rensefunksjon resulterer i langt lavere avgifter på uttak av fossilt brensel enn ikke-lineære funksjon av den typen vi har brukt. Da vi betraktet tidshorisonten frem til teknologiskiftet for de to tilfellene, såg vi at forskjellene kunne bli svært store. Lineær rensning utsatte teknologiskiftet betydelig. Sammen med lavere avgifter gav dette et mye høyere samlet uttak over tid, og dermed ble langt mer av ressursen brukt opp. I det uregulerte tilfellet ble nesten hele ressursen brukt opp. Alt i alt er det nærliggende å konkludere med at det å bruke en lineær tilnærming til rensefunksjonen er urealistisk og misvisende når en skal si noe om den virkelige verden.

I den siste seksjonen undersøkte vi hvordan diskonteringsnivået påvirker den regulerte markedsløsningen. Resultatene var svært følsomme for diskontering slik vi normalt kan forvente. Høy diskontering gav sent teknologiskifte og lav skatt initielt. Resultatet var høyere forurensning. Skattekurvene vi observerte var konkave og fallende med tiden. Dessuten såg vi at de falt med forurensningsnivået og var svakt konkave også som funksjon av dette. Denne konkaviteten såg ut til å forsvinne for store forurensningsverdier.

I modellen vår fant vi tidspunktet frem til teknologiskiftet endogent. Den store forskjellen i T for de ulike variantene av optimeringsproblemet vi testet ut, viser hvor avgjørende det kan være å bestemme T endogent. En eksogen tilnærming gir rom for store feil.

Ellers brukte vi en tidsavhengig alternativ profitt, og viste at det lot seg gjøre å bestemme den optimale løsning med a-avhengig etterspørsel ("idealistisk marked"). Begge deler kan være med og gjøre modellen mer realistisk.

Som en avsluttende konklusjon mener vi at dette arbeidet demonstrerer viktigheten av en ytterligere økonometrisk kartlegging av modellens størrelser. Rensefunksjonens betydning viser at å få denne type basale elementer riktigere er bedre enn å spekulere i ulike typer stokastiske utvidelser av problemet.

2.5 Referanser

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S. Aanestad – *"Damping down the curse of dimensionality in dynamic programming,"*

Chapter 3

Essay II

Abstract

This work focuses on lightening the curse of dimensionality in a range of to multi-dimensional dynamic programming problems. Discretization in state space and iterative methods are used to find the optimal value function. The differentiability of the optimal value function is utilized to solve the two-stage discrete Hamilton-Jacobi-Bellman equation.

The procedure suits for a wide range of optimal control problems in resource economics. Its efficiency is exemplified by the solving of a few simple problems.

3.1 Introduction

Dynamic programming (DP) is a popular technique for the solving of optimal control problems and stems from the early contribution of Bellman (1957). The technique, which is built upon the Hamilton-Jacobi-Bellman equation, provides a mathematical formalization of the trade-off between current and future profit. Unlike trajectorywise approaches such as direct discretizations or methods based on the Pontryargin's maximum principle, it gives a global solution to the problem stated. It is well known, however, that for many problems the computational requirements are so overwhelming that DP is considered unsuitable.

Nevertheless, the concept of dynamic programming is more powerful than many scientists seem to realize. A wide range of problems in higher state and control space dimensions are solvable with DP, and extensive work in the field of operational research has been undertaken to overcome the curse of dimensionality (see Rust (1996) for an overview of computational complexity). Methods like neuro-dynamic programming (Bertsekas and Tsitsiklis 1995), higher-order approximations, randomization and adaptive space discretization have been proposed (Grüne and Semmler 2004, Grüne 2004). The techniques presented in this work may be categorized as discretization and approximation methods. A discretization of the state space is done, but discretization of the policy space is avoided in order to keep down numerical costs. Instead the optimal controls are found from using first-order conditions in an approximated discrete version of the Hamilton-Jacobi-Bellman equation. This method relies heavily on the differentiability of the optimal value function, which is satisfied when the object function is concave (see Cotter and Park (2005), and Benveniste and Scheinkman (1979)).

The efficiency of the method depends on the nature of the problem explored. It is, however, well adapted for many problems in resource economics and economics in general, where the object function is concave. Although it does not overcome the dimensionality problem of higher dimensions, it may ease the problem, and a state space of at least four dimensions works fine.

At the end of this work some simple examples with a two-dimensional state space are given, and the relevance of the proposed method is demonstrated and discussed.

3.2 Problem formulation

The general discounted optimal control problem to be solved can be formulated

$$V(x) = \max_{u \in \mathcal{U}} \int_0^\infty e^{-\delta t} g\big(x(t), u(t)\big)\, dt, \tag{3.1}$$

with continuous state equation

$$\frac{d}{dt} x(t) = f\big(x(t), u(t)\big), \qquad x(0) = x_0, \tag{3.2}$$

where $x(t) \in \Omega$ and both $\Omega \subset \mathcal{R}^n$ and $\mathcal{U} \subset \mathcal{R}^m$ are compact sets. Further on, $g(x(t), u(t))$ is continuous and concave with respect to the control variable, $u(t)$.

The solving of this problem requires in many cases numerical procedures. Since numerics is by nature discrete, we may as well give the discrete formulation straight away. By replacing eq. (3.2) with the discrete first order approximation

$$x_h(0) = x, \quad x_h(i+1) = \varphi(x, u) \equiv x_h(i) + f\big(x_h(i), u_h(i)\big)\, h, \tag{3.3}$$

where h is the discrete time-step ($h << 1$), it is shown in Grüne and Semmler (2004) that the corresponding discrete optimal value function is given by

$$V_h(x) \approx \max_{u \in \mathcal{U}} h \sum_{i=0}^{\infty} \beta^i\, g(x_h(i), u_i), \tag{3.4}$$

where $\beta \equiv 1 - \delta h$. Now, if we put eq. (3.3) into eq. (3.4) and take the first period out of the summation sign, we arrive at the discrete Hamilton-Jacobi-Bellman equation

$$V_h(x) \approx \max_u \Big\{ h\, g(x, u) + \beta\, V_h\big(\varphi(x, u)\big) \Big\}, \tag{3.5}$$

which is a sufficient condition for optimum.

Solving equation (3.3) repeatedly based on an estimate of $V_h(x)$ is the core of the numerical procedure that will be described and demonstrated in this work. A guess-estimate $V_h^0(x)$ is assumed as a starting value for a fixed-point iteration. Based on this estimate an approximation to $V_h(\varphi(x,u))$ is calculated, and $u^0(x)$ is found as the value that maximizes the right side of the equation based on these estimates. This maximum value, defined as $V_h^1(x)$, is a new and improved estimate of $V_h(x)$, and it leads to a new optimal policy, $u^1(x)$. A sequence of further fixed-point iterations continues until the sequence $\{V_h^1,\, V_h^2,\, \ldots\, V_h^n\}$ convergences ($V_h^{n+1}(x) = V_h^n(x)$).

This procedure will in the following be explained mathematically. For notation purposes we start by defining the linear operator

$$L(u)(V_h) \equiv h\, g(x,u) + \beta\, V_h\big(\varphi(x,u)\big). \tag{3.6}$$

The dynamic programming operator

$$T_h(V_h)(x) = \max_{u \in \mathcal{U}} \Big\{ L(u)(V_h) \Big\} \tag{3.7}$$

can be used to successively solve the Hamilton-Jacobi-Bellman equation with fixed-point iterations

$$V_h^{i+1}(x) = T_h(V_h^i)(x), \tag{3.8}$$

and through this a mathematical procedure to solve the original problem stated in eq. (3.1) and (3.2) is established. Since the procedure solves equation (3.5) directly, an optimal solution with many equilibriums is no threat to the procedure. Local versus global extreme points are handled properly as long is this equations is solved. There are, however, many short cuts to make use of, and traps to fall into, when the first steps are taken in concrete numerics.

3.2.1 Numerical procedures

Using first-order conditions in selection of control

To solve the problem defined in eq. (3.1) and (3.2) by numerical methods, we find the optimal value function on a discrete state space grid, $G_x \in \hat{\Omega} \subset \Omega$. This is done by making an initial guess at the value function on G_x and employing the fixed-point iteration in eq. (3.8).

Most literature, e.g. Bertsekas (2001), (2005) and Grüne (2004) deals with working in a discrete "control-space", $\hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_m)$ being a subspace of the continuous value-space $\mathcal{U}$. This should be avoided if possible. First of all the discrete sub-space is sub-optimal unless the problem is in itself discrete with a limited number of nodes or grid points. Second, depending on the size of the vector $\hat{u}$ and on the dimension of the problem, n, the discretization may be very computer-demanding. Working in $\mathcal{R}^1$ this is no issue, but in $\mathcal{R}^3$ it is very difficult to work with a discrete control vector of some size without struggling with the curse of dimensionality.

One way to cope better with the dimensionality problem is to avoid discretization in control space. Unfortunately, when one is working with a continuous control space, a new problem rises. Assume $x_h(i) = x_i \in \Omega$. Then according to eq. (3.3) admissible controls in a continuous control-space may give a new state variable, $x_h(i+1)$ outside of $\hat{\Omega}$ but inside Ω. The updating of $V_h^{i+1}(x)$ in accordance with (3.8) is therefore not straight-forward. Interpolation in state space, however, is an effective way, which is well documented in literature (see Judd and Solnick (1994)), to find approximate values of the optimal value function for state-values outside the initial selected grid, G_x. In the examples below we use linear interpolation to approximate the value-function outside G_x.

In the selection of optimal control, u, in each fixed-point iteration step that updates the optimal value function according to (3.8), we may take advantage of the first-order Taylor approximation of V_h with respect to x. Inserting

$$V_h\big(\varphi(x,u)\big) \approx V_h(x) + (\nabla V_h)^T(x) f(x,u)h. \tag{3.9}$$

into eq. (3.5), we get

$$V_h(x) \approx \frac{h}{1-\beta} \max_{\underline{u}} \left\{ g(x,u) + \beta(\nabla V_h)^T(x)\, f(x,u) \right\}. \tag{3.10}$$

Only a few values need to be tested to find optimal control, u, that maximize this equation. These are the lower bound $\underline{u} = \underline{0}$, the upper bound and the interior solution(s) solving

$$\frac{\partial g(x,u)}{\partial u} + \beta(\nabla V_h)^T(x)\, \frac{\partial f(x,u)}{\partial u} = 0 \tag{3.11}$$

with respect to u on all the nodes or grid-points of x.

In this equation the only unknown, except from u which is to be found, is $\nabla V_h(x)$. For every fixed-point iteration, however, an estimate for this size can be found. Typically, we start the fixed-point iteration with $V_h^0 = 0$ and use the nodes or grid points of x to estimate $\nabla V_h^k(x)$ from

$$\nabla V_h^k(x) \approx \frac{V_h^{k-1}(x+\Delta x) - V_h^{k-1}(x)}{\Delta x}. \tag{3.12}$$

Then $V_h^k(x)$ will, for each fixed-point iteration, approach its true value, $V_h(x)$, and $u^k(x)$ will approach $u_h(x)$.

3.2.2 Policy and value iterations

The number of heavy approximations should be reduced to a minimum in order to improve time of convergence. One way to do that is to mix policy iterations and what we call "value iterations" (see Grüne and Semmler (2004)).

In the solving procedure of the Hamilton-Jacobi-Bellman equation (3.5), each step of using the dynamic programming operator (3.7) could be called a policy iteration. To speed up convergence switching between this policy-iteration and less numerically expensive value iterations can be done. In the "value-iterations" the policy is fixed, $u = u^f$, and therefore also the profit, $g(x, u^f)$, is fixed in the linear value iteration operator (3.6). This means that the value iterations

$$V_h^{n+1} = L(u^f)V_h^n \tag{3.13}$$

stabilize rather quick. When they do, we shift to policy-iterations in accordance with equation (3.7), before we again return to value-iterations. The alternation between value and policy iterations continues until convergence.

3.2.3 Interpolating first-order conditions

The usefulness of utilizing first order conditions in accordance with the procedure described in sec. 3.2.1 depends on the numeric costs of solving equation (3.11) with respect to u. When exact roots can be found algebraically the method is very favorable, but it might be preferable also in cases where the numerics are more challenging.

For many cases an exact root of (3.11) is not required. When approximate solutions are sufficient, a passable and effective procedure is to find unique interpolated functions on each node of G_x representing the u-dependence. Say, for node i on G_x we find $h_1(i, u)$ and/or $h_2(i, u)$ that satisfies $h_1(i, u) \approx \frac{\partial g(x,u)}{\partial u}$ and/or $h_2(i, u) \approx \frac{\partial f(x,u)}{\partial u}$. If these interpolations are sufficiently good approximations and have forms that make it possible algebraically to find roots of

$$h_1(i, u) + \beta(\nabla V_h)^T \, h_2(i, u) = 0, \tag{3.14}$$

the method can be used. (Eq. (3.14) corresponds to eq. (3.11).

Note that interpolating $f(x, u)$ and $g(x, u)$, and taking the partial derivative of these functions instead of interpolating the partial derivatives directly, may lead to large errors.

3.3 Examples

In this section we will look at some examples from two classes of problems where the first-order conditions in eq. (3.11) are effective in the finding of optimal control. The first class of problems consists of cases where an algebraic solution is reachable (examples 3.3.1 and 3.3.2), and the second class is problems where an algebraic solution is reachable only when $\frac{\partial f(x,u)}{\partial u}$ and/or $\frac{\partial g(x,u)}{\partial u}$ is replaced by simpler functions on G_x (see example 3.3.3). Such a replacement should only be done when the interpolated functions are sufficiently close to the original ones.

3.3.1 Example 1: A 2d investment model with relative adjustment cost

First we will look at a problem where eq. 3.11 is easily solved algebraically without any interpolation. The problem from Haunschmied and colleagues (2003) is also presented in Grüne and Semmler (2004), where it is solved by adaptive grid schemes. The inputs to equation (3.1) and (3.2) are respectively given by

$$g(x,u) = k_1\sqrt{x_1} - \frac{x_1}{1+k_2x_1^4} - c_1x_2 - \frac{c_2x_2^2}{2} - \frac{\alpha u^2}{2}, \tag{3.15}$$

and

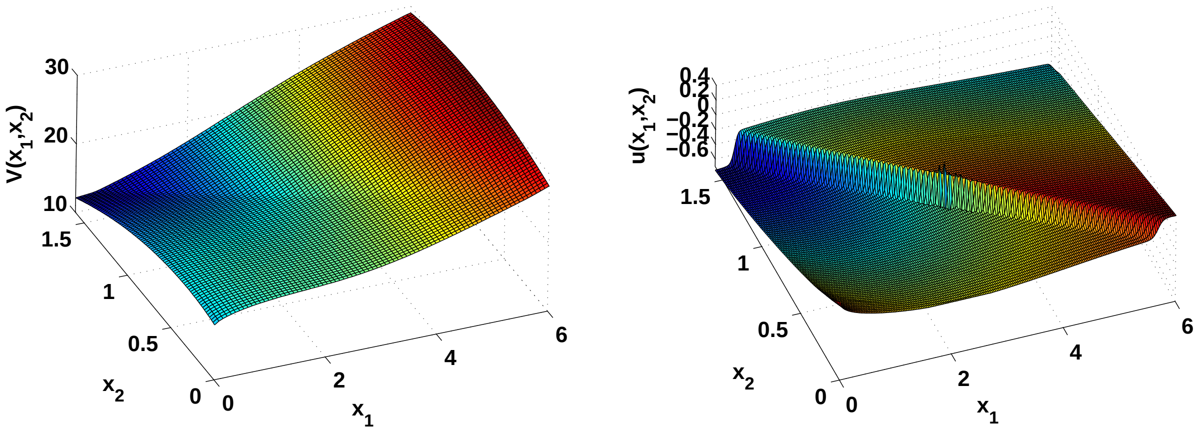
$$f(x,u) = \begin{pmatrix} x_2(t) - \sigma x_1(t) \\ u(t) \end{pmatrix}. \tag{3.16}$$

The state x_1 is capital stock, x_2 is investment, and the control, u, should be interpreted as change in investment. The parameters are $k_1 = 2, \quad k_2 = 0.0117, \quad c_1 = 0.75, \quad c_2 = 2.5, \quad \alpha = 12, \quad \sigma = 0.25$, and discount rate $\delta = 0.04$. Subsequently, we choose time-step $h = 1/20$.

The procedure is rather easy, but let us go through the first example step by step to convince all readers.

1. Let G_x be the 2D grid with $x_1 = \{0, 0.05, 0.1, ..., 10\}$ and $x_2 = \{0, 0.01, 0.02, ..., 3\}$ and let $h = 1/20$. This means that G_x is a very fine-meshed grid.
2. Start with a guess estimate for $V_h(x)$, e.g a zero-matrix on the whole x-grid, G_x.
3. Find $\nabla V_h(x)$ with equation (3.12). If the zero-matrix is chosen as a start value for $V(x)$, the first $\nabla V_h(x)$ will also be the zero-matrix.
4. On G_x, find the set of controls $u(x)$ that maximizes $V(x)$ according to equation (3.11). Since $g(x,u)$ is concave with respect to u the inner solution will be the maximum. Putting $\partial g/\partial u = -\alpha\, u$ into the equation gives $-\alpha\, u + \beta \nabla V_h(x)\, \partial f/\partial u = 0$, and consequently the algebraic solution is $u(x) = \beta\, \nabla V_h(x)/\alpha$.
5. Update $V_h(x)$ according to 3.11
6. Start again from point 1 with the updatet $V_h^1(x)$ as initial guess and repete the whole procedure until convergence

A matlab code that solves the problem numerically is given in appendix B. In the code one should regard that the finding of optimal policy according to equation (3.14) is done by a complete matrix operation. That is, instead of using if- and for-loops through the grid, G_x, the calculation of $u(x)$ is done on the whole G_x in a single operation.



(a) (b)

Figure 3.1: Example 1: (a) the optimal value function and (b) the optimal control on the state space grid.

Figure 3.1(b) shows a special characteristic of the optimal control. There are discontinuities in the politics along a line which, in literature, is referred to as a skibaline or a DNS line. (See Haunschmied 2003). The discontinuity slows down convergence, and for these kinds of problems adaptive grid schemes are ideal for treating the behavior of the optimal control. Utilization of the first-order conditions, however, allows selection of a fine-meshed grid G_x, and therefore we did not need an adaptive grid to find a satisfactory solution. (See Figures 3.1(a) and 3.1(b).)

3.3.2 Example 2: Optimal harvest of two species

The following example concerns management of a two-species fish resource. One of the species predates the other, and both of the species are harvested. Therefore this model is two-dimensional in both control and state space, and a discretization of the control space would have been even more numerically expensive than that for example 1.

The profit and dynamics to be putted into equation (3.1) and (3.2) respectively are

$$g(x,u) = \sum_{i=1}^{2} p_i\, u_i - \frac{c_i}{x_i}\, u^{\alpha_i} \tag{3.17}$$

and

$$f(x,u) = \left(\begin{array}{c} r_1 x_1 \, (1 - \frac{x_1}{k_1} - b_1 x_2) \\ r_2 x_2 \, (1 - \frac{x_1}{k_2} + b_2 x_1) \end{array} \right), \tag{3.18}$$

and the constants are $r_1 = r_2 = 1$, $p_1 = 1$, $p_2 = 25$, $c1 = c2 = 0.1$ $\alpha_1 = 2$, $\alpha_2 = 2.1$, $k_1 = k_2 = 1$, $b_1 = 0.001$, $b_2 = 0.0001$ and the discount rate $\delta = 0.05$.

Optimal controls, catch u_1 and u_2, are found directly from equation 3.11, and are given algebraically by

$$u_i = \left(\frac{x_i}{2} \, (p_i - \beta \frac{\partial V}{\partial x_i}) \right)^{1/(\alpha_i - 1)}. \tag{3.19}$$

Convergence to the optimal curves presented in Figures 3.2(a) and 3.2(b) is reached within half a minute when a x-grid, G_x of size 44×44, and time-step $h = 1/1000$ was chosen.

3.3.3 Example 3: Interpolation in the first-order conditions

The dynamics in this example is identical with that in example 3.3.2, but the object function is slightly changed to

$$g(x,u) = p_1 \, u_1 - \frac{c_1}{x_1} \, u^{\alpha_1} + (p_2 - p_3 u_2) u_2 - \frac{c_2}{x_2} \, u_2^{\alpha_3}. \tag{3.20}$$

The constants have the same values as in example 3.3.2, but $p_3 = 1$ and $\alpha_3 = 1.1$ are new to this problem.

The change of object function has consequences for the solving of equation 3.11, which is no longer analytically solvable for u_2. A direct numerical approach leads to very high throughput, and therefore we should search for a short path omitting that problem. Very often it is possible to find simple interpolated functions that are sufficiently close to $g(x,u)$ to be substituted into equation (3.11) (as in equaton (3.14)). These should be functions that make it possible to solve the equation either analytically or with fast numerical algorithms.

In this problem

$$\frac{\partial g(x,u)}{\partial u_2} = 25 - 2u_2 - \frac{0.11}{x_2} u_2^{0.1}. \tag{3.21}$$

This suggests that the linear function

$$h_1(i, u_2) = A(x(i)) + B(x(i)) u_2 \tag{3.22}$$

might be a sufficiently good approximation for $\frac{\partial g(x,u)}{\partial u_2}$. We have to test the quality, however. A graphical inspection shows that there is practically no difference between $\frac{\partial g(x,u)}{\partial u_2}$ and the interpolated function (see Figure 3.3.3). Only close to $x_2 = 0$ are there some

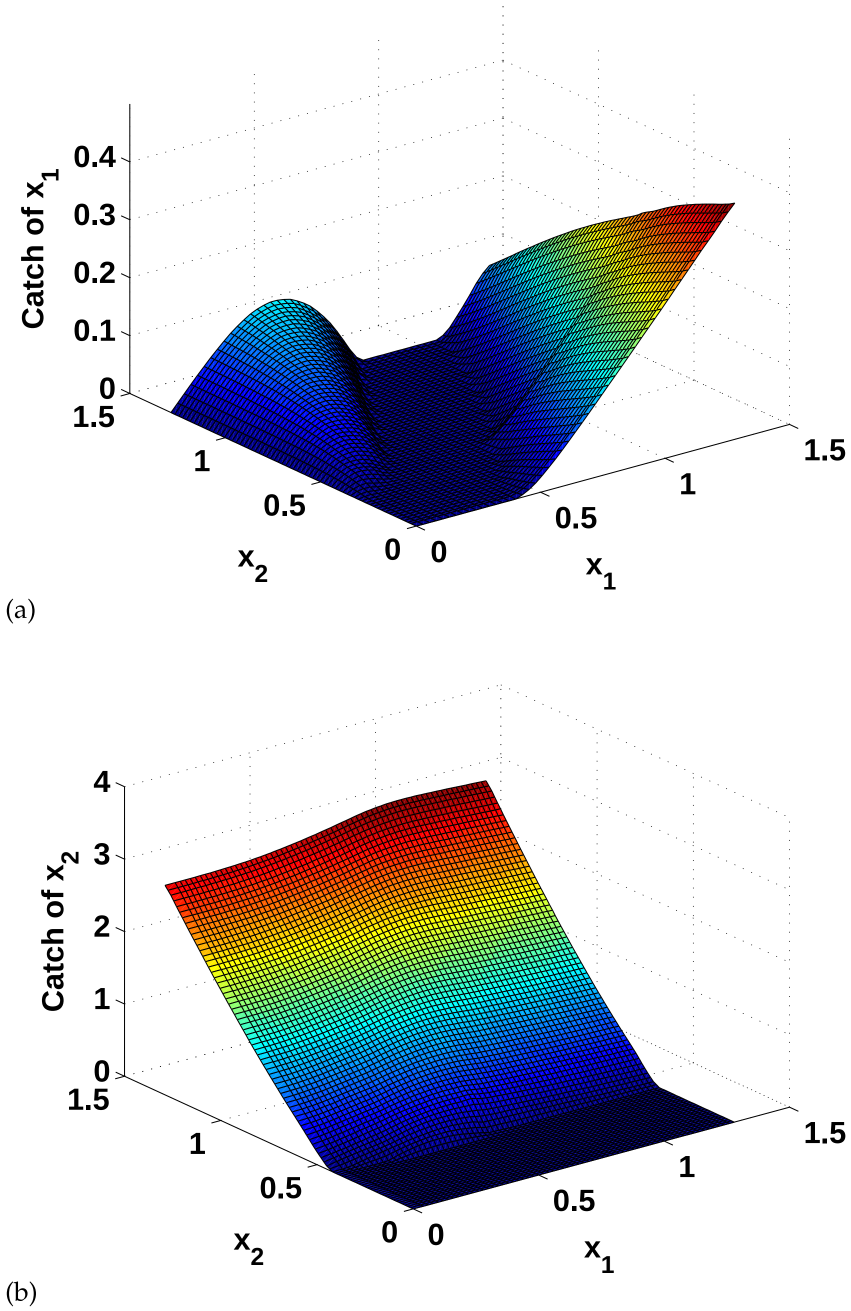


Figure 3.2: Example 2: The optimal catch of (a) species x_1 and (b) species x_2 on the state space grid G_x.

differences owing to extreme growth of the last term of eq. (3.21) when x_2 approaches zero. The lowest x_2-value in Figure 3.3.3 is 0.01.

When $\frac{\partial g(x,u)}{\partial u_2}$ is replaced with $x(i) + B(x(i))u_2$ in eq. (3.11), optimal catch is given by

$$u_2 = \left(\beta \frac{\partial V_h(x)}{\partial x_2} - A(x(i))\right) \Big/ B(x(i)). \tag{3.23}$$

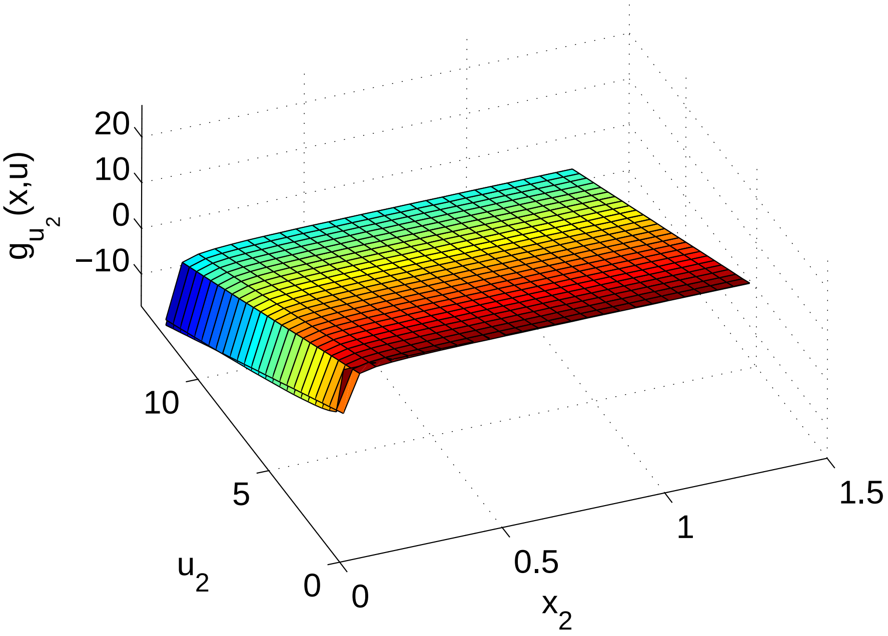


Figure 3.3: Example 3: The surfaces of $\frac{\partial g(x,u)}{\partial u_2}$ and the interpolated replacement function $h_1(i, u_2) = A(i) + B(i)u_2$ plotted in the same figure.

The optimal catches of the two species are plotted in Figures 3.4(a) and 3.4(b).

3.4 Range of use for the numerical procedure

The numerical procedure described in section 3.2.1 is not always effective. It is a necessary condition that there exists a method to find all solutions to equation (3.11) algebraically or numerically within a limited number of iterations.

For resource management models, where u typically is the harvest or the use of a certain resource, the function $f(x, u)$ is usually linear with respect to u. This simplifies equation (3.11) as it is reduced to

$$\frac{\partial g(x,u)}{\partial u} + \beta(\nabla V_h)^T(x) \cdot K = 0, \tag{3.24}$$

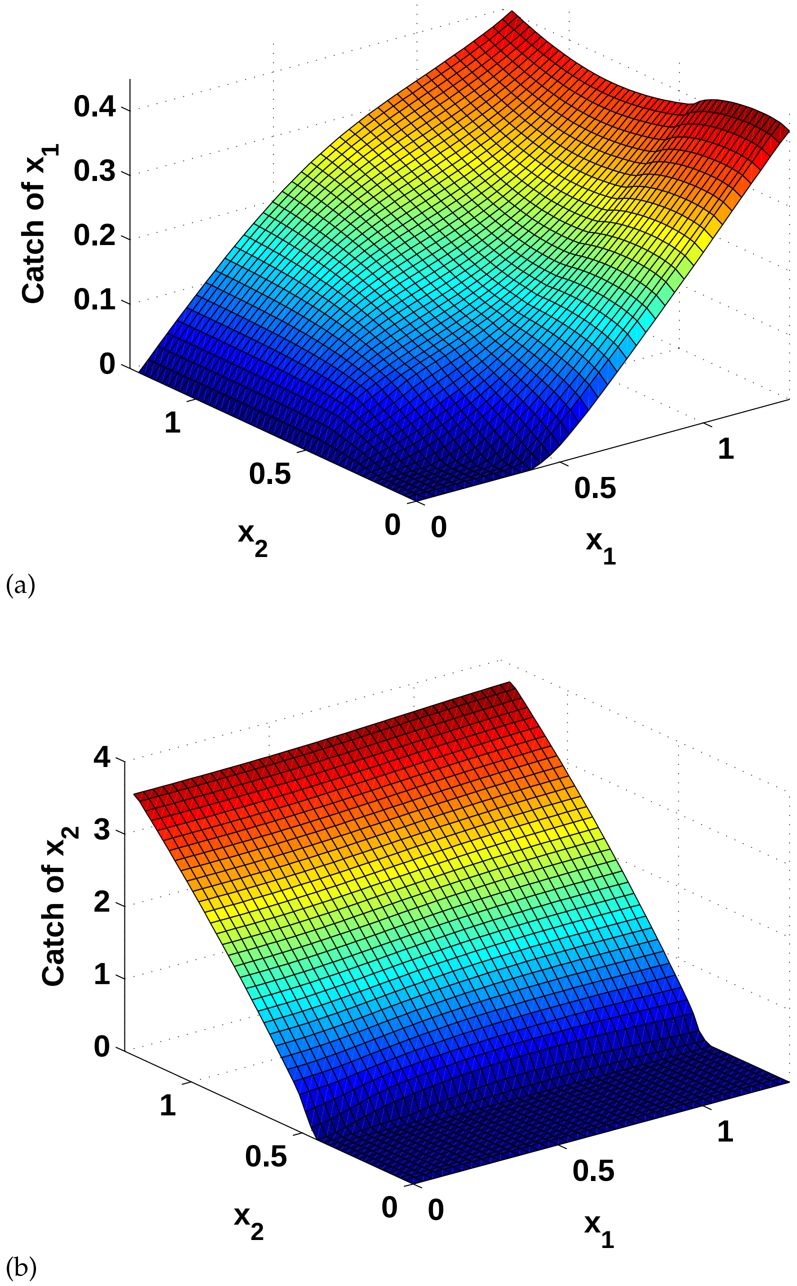


Figure 3.4: Example 3: The optimal catch of (a) species x_1 and (b)species x_2 on the state space grid G_x.

where K is some constant, usually $K = 1$. We will not go into any discussions about numerical solvability of such equations since it on its own is a special field written tons of literature about. However, when either $g(x, u)$ or $\partial g(x, u)/\partial u$ is quadratic, which often is the case in social economic and bioeconomic literature, it is simple to solve equation (3.24) algebraically. When that is the case, as in example 1 and example 2, the method is preferable. It should also be considered when (3.24) can be solved numerically within a reasonable number of iterations or when an approximation to this equation can be solved either algebraically, as in example 3, or numerically. This means that the method is very flexible for numerically experienced users.

How to numerically solve equation (3.10) effectively is a question that should be paid attention. In the problem of *example 1* the solution is characterized by a discontinuity in the control space, and it is already mentioned that an adaptive grid method is effective in that case. Since adaptive grids may easily be used in the solving of equation (3.10) our procedure should be combined with adaptive grids for problems of this kind. With an adaptive grid G_x is rather crude in the beginning but, depending on the sizes of local changes in each of the fixed-point iterations, the grid is refined around the nodes where the changes are largest. Adaptive grid is, however, not possible to implement without breaking up the matrix operations on G_x when finding optimal controls, u. Instead of finding $u(x)$ in a single matrix-operation (see matlab-code in appendix B), it is necessary to go through each element (node) of the matrixes by if- and for-loops. Although that increases the time to find each $u(x)$ on $G(x)$, the total time to solve the problem may decrease since the adaptive grid for most cases allows a lot cruder G_x (less nodes on G_x). Especially, this is the case when the optimal control, $u(x)$, is discontinuous.

3.5 Conclusions

In this work we have demonstrated the use of some efficiency-improving methods for the solving of optimal control problems with dynamic programming. These methods do not overcome the curse of dimensionality, but their efficiency in making dynamic programming solutions feasible and attractive for many problems with state space of up to four dimensions is unquestionable.

In the methods demonstrated, discretization in state space is employed (discretization methods). Subsequently, a combination of Taylor approximation of the optimal value function and first-order conditions with respect to the optimal controls is used to decide controls in accordance with the discrete Hamilton-Jacobi-Bellman equation. When an optimal control is not analytically solvable in the discrete two-stage problem, we may use interpolation and approximation techniques to find analytic solutions to related approximated problems that are good representations of the original problem.

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S. Aanestad, L.K. Sandal and S. Tjelmeland – *"Barents Sea Capelin and Cod Stock Scenarios. Testing biological consequences of economically-based management."*

Chapter 4

Essay III

Abstract

In this article we present a scenario model for the jointly Norwegian and Russian capelin and cod stocks in the Barents Sea. The model basically consists of two sub-models. The first is an autonomous management model which, based on profit functions and growth functions with coefficients calibrated from real data, finds a feedback rule for maximizing the Norwegian share of the fishery rent by using numerical dynamic programming. The second (Bifrost) is a biological multi-species growth model provided by the Institute of Marine Resources in Bergen. This model has higher resolution than the simple top-down growth used in the management model. With this we simulate possible scenarios resulting from employing the management model as a decision-making instrument in the real world. We concentrate on possible future catch and stock sizes rather than on economics. The objective is to find out whether the economical based management strategy is sustainable from a biological perspective.

Scenarios from six different strategies are tested, of which two are purely biological and inspired by existing management. Bifrost is applied direct to measure total allowable catch (TAC) in these strategies, whereas the four economic strategies are found from the management model. These strategies are compared with the biological strategies and the results are promising. The best economic strategies turn out to give higher long-term cod stocks than the biologic strategies.

4.1 Introduction

Scenario modeling in fishery economics is a relatively new topic. Originally it was developed for whaling, but it is now gaining popularity in fisheries (Schweder 2006, Link, Schneider and Tol 2004 and Eide 2007). In this article, for instance, it is used to combine economic optimizing with complex biological bottom-up multi-species modeling in the management of the fisheries of capelin and cod in the Barents Sea. The idea is to account for complex biology when evaluating a management models with purely economic objectives that is based on a quite simple 2D growth model. With a few exceptions (Hagen, Hatlebakk and Schweder 1998, Helstad 2000 and Eide and Flaaten

1998), one typically observes in the literature of fishery economics complex economic perspectives combined with quite simple biology. On the other hand biologists usually ignore the economics. Scenario modeling is a point of attack that makes it easier to deal with both economic optimizing and complex biological modeling at the same time.

Although there has been evolution in the field since Clark and Munro (1975) introduced optimal control theory to fishery management models in the mid-seventies, dynamic programming is never used when the biology is described by a complex multi-species growth model. The reason is that the size of the throughput becomes impossible to handle within reasonable computer time when the variable dimension of the problem arises. Basing the optimizing on a biological growth model with two interacting fish species requires dynamic optimizing in at least two dimensions, and adding in year classes or spatial locations for the fish species increases the dimensions further. Therefore, the management model that is tested in this work consists of a simple two-dimensional predator-prey growth model for cod (Gadus morhua) and capelin (Mallotus Villosus) that is combined with a corresponding simple economic model. The management model is previous presented in Agnarsson et. al. (2008), but is here verified with a new numerical solution.

The novelty in this work is that consequences of the model are tested when a much more complex biologic growth model is assumed to represent the real growth. From a social economic point of view single species economic models are of little value for optimal management of capelin since capelin might be more profitable as prey for cod than as catch for the fishing boats. The question is whether or not the two species management model, which is aware of the double role of capelin, is complex and robust enough to form a basis for sustainable management from a biological point of view.

The Barents Sea is not among the most complex ecosystems. It has an arctic climate with only a few species of importance. Among these are capelin and cod, which have nursery and feeding grounds here. This also applies to herring (*Clupea harengus*) (Dalpadado, Ingvaldsen and Hassel 2003), which plays a major role in the capelin-cod system. Hamre (2003) concludes that capelin and herring are the key prey species at fish level and that cod is their dominant predator. In addition to this, there is strong evidence for herring being one of the main causes of collapse in the capelin stock as it preys heavily on capelin fry. Other species of relevance are sea mammals (Schweder, Hatlebakk and Hagen 1998), sea birds, harp seals and some additional fish species. Harvest is also a major factor in the ecosystem (Hjermann, Ottersen and Stenseth 2004).

Although we cannot say for sure whether the top-down or the bottom-up approach is the best approximation to the true biomass variation for capelin and cod in the Barents Sea, we may be able to say something about the resilience and adaptability of the management model after simulating its consequences through another model, namely the Bifrost model. What are the consequences for the stocks of such management according to a simulation model with higher resolution? This is the main question of interest in this article. Will the capelin stock survive according to the simulations; will the size of the cod stock drop over time? These questions will be answered.

Even though we do not include herring when deciding TACs according to the management model, herring influence is accounted for when we simulate consequences on future biomasses and catches of capelin and cod. Bifrost, the simulation model used for testing consequences of the management model, is a multispecies assessment model involving herring as well as cod and capelin. This is a so-called bottom-up model starting with the small processes that influence the total marine ecosystem. Factors like capelin-cod interaction conditional on other food, herring effects on capelin recruitment and temperature effects on maturation of capelin and cod are all taken care of in this model. As far as herring influence is concerned, concurrent runs between Bifrost and the herring assessment model, SeaStar, are possible. In sec. 4.4.1 and sec. 4.4.2 we present simulations with SeaStar turned "off" and "on" respectively to find out whether a dynamic, more realistic, three-species approach is crucial to the simulations. Interesting differences between constant herring influence (SeaStar "off") and dynamical herring influence (SeaStar "on") are found, and this shows that more research should be carried out on joint management of the capelin and cod stocks in the Barents Sea and the herring stock further south. The findings in this work are rather promising on behalf of the two-dimensional management model tested, as it appears to be just as sustainable as other strategies that are based on existing management.

4.2 The Model

4.2.1 General considerations

The model consists of a set of two interacting fish biomasses $\big(x(t), y(t)\big)$ representing the joint Norwegian and Russian resources of capelin and cod in the Barents Sea. In the scenario model we investigate different management strategies for these stocks. Specifically, we examine an optimal feedback strategy designed to set TAC levels $(u_x(t), u_y(t))$ maximizing the Norwegian share of the economic rent given by the fishery.

As already mentioned, the scenario model consists of two models. It is the simulation model (Bifrost) provided by the Institute of Marine Resources in Bergen and it is the management model. The management model has its own biological growth model, which is quite different from the growth in the simulation model. The procedure is as follows:

1. Find optimal TACs from the management model according to the present stock sizes

2. Set optimal TACs in the simulation model

3. Simulate next year's stock sizes in accordance with the TACs (Bifrost)

4. Start again from point 1 with the new stock sizes as the starting-point.

Now, if we let the biomass growth in the simulation model be our representation of the real-world marine ecosystem, we investigate consequences for the stock sizes of following the feedback policy from the management model. Further, effects from three groups of management strategies (six strategies) are compared. These are

- $\tilde{A}_1$ and $\tilde{A}_2$: The management model with two different levels of the discount rate, 1% ($\tilde{A}_1$) and 5% ($\tilde{A}_2$). (See sec. 4.3.1).

- $\tilde{B}_1$ (1% discount rate) and $\tilde{B}_2$ (5% discount rate): Strategies revealed from strategy $\tilde{A}$, but capelin catch is more aware. (See sec.4.3.2)

- $\tilde{C}_1$ and $\tilde{C}_2$: two pure biologically based strategies associated with the existing management regime for capelin and cod. (See sec. 4.3.2)

Strategies $\tilde{A}_1$ and $\tilde{A}_2$ are based on finding feedback TAC-levels that are maximizing all discounted future income from the two fisheries, and strategy $\tilde{B}_1$ and $\tilde{B}_2$ are ad hoc strategies intended to improve performance of respectively $\tilde{A}_1$ and $\tilde{A}_2$.

Strategies $\tilde{C}_1$ and $\tilde{C}_2$ are based on the existing multi-species management of capelin and single species management of cod. A target spawning level of capelin and F-value of cod are chosen to maximize long-term catches of the two species. No direct optimizing has been done to find these sizes, but through repeated trials sensible results are reached.

Strategies $\tilde{A}_1$ and $\tilde{A}_2$ use simple relations between stock sizes and profit and between stock sizes and stock growth. Let us have a look at the management model that produces them.

4.2.2 The optimal management model

The management model is the same as that used by Agnarsson et. al. (2008), but the model is resolved with other algorithms. It consists of an economic objective function, which is to be maximized under the constraints of two biological growth functions for the stocks of capelin and cod in the Barents Sea. That is, the principal form of the management model is given by

$$V(x_0, y_0) \equiv \max_{u_x, u_y} \int_0^\infty \Pi\big(x, y, u_x, u_y\big) e^{-\delta\, t}\, dt, \qquad \text{when} \tag{4.1}$$

$$\dot{x} = f_1(x, y) - u_x \quad \text{and} \quad x(0) = x_0 \tag{4.2}$$

$$\dot{y} = f_2(x, y) - u_y \quad \text{and} \quad y(0) = y_0, \tag{4.3}$$

and δ is the discount rate. In the following we will go further into the economics represented by the flow of profit $\Pi(\cdot)$ and the biological growth represented by $f_1(\cdot)$ and $f_2(\cdot)$.

Economic profit

The Norwegian profit function is of the form

$$\Pi(x, u) = p(u)\, \alpha u - C(x, \alpha u) \tag{4.4}$$

for both the species. Here α is the pre-decided Norwegian share of the TAC, $p(u)$ represents the (inverse) demand function and $C(x, u)$ is the associated cost. Price elasticity and cost dependent on stock size are, however, only assumed for cod. As far as the prices are concerned, the joint Norwegian and Russian catch is large enough to influence the world market, whereas capelin is price-inelastic since it is easily replaced by other species on the market. The specific inverse demand (price) functions for capelin and cod are therefore given by

$$\begin{aligned} p_x &= p_1 \\ p_y(u_y) &= p_2 - p_3\, u_y \end{aligned}$$

and with Norwegian catches of $\alpha_1\, u_x$ and $\alpha_2\, u_y$ the cost functions are

$$\begin{aligned} c_x(\alpha_1 u_x) &= c_1\, (\alpha_1 u_x)^{1.4} \\ c_y(y, \alpha_2 u_y) &= \frac{c_2}{y}\, (\alpha_2 u_y)^{1.1}. \end{aligned}$$

The stock dependence of the cost function for cod implies that the cost approaches infinity as the stock approaches zero. This means that cod is economically protected. There is no such protection associated with capelin.

With no economic interspecies interactions in the fish market, total Norwegian profit flow from the fisheries of capelin and cod is given by the sum of profit from each of the fisheries

$$\Pi(x, y, u_x, u_y) = p_1 \alpha_1 u_x - c_1 (\alpha_1 u_x)^{c_2} + (p_2 - p_3 u_y) \alpha_2 u_y - \frac{c_3}{y} (\alpha_2 u_y)^{c_4},$$

where all the coefficients $\alpha_1, \alpha_2, p_1, p_2, p_3, c_1, c_2, c_3 > 0$.

Biological growth

The growth functions describe the dynamics of the capelin-cod system before catch is added. It is set to

$$f_1(x,y) = b_1x^2 + b_2x^3 + b_3xy \tag{4.5}$$

$$f_2(x,y) = b_4y^2 + b_5y^4 + b_6xy, \tag{4.6}$$

where the last term of each function represents predation and other interspecies relationships between the two species. Both the coefficients for the profit functions and the coefficients for the growth functions are found by calibration with real historical data. (see Tables 4.1 and 4.2). For further information about the properties of the biological model, refer to Agnarsson et. al. (2008).

Table 4.1: Coefficients for the (inverse) demand and cost functions.

	Species	Function	Parameters	t-statistic	R^2
Price/ inverse demand	Capelin	$p(h) = 1$			
	Cod	$p(h) = p_2 - p_3\, u_y$	$p_2 = 12.65$	9.7	$R^2 = 0.59$
			$p_3 = 0.00839$	3.94	$F = 15.6$
Cost	Capelin	$C(x,u) = c_1(u)^{1.4}$	$c_1 = 0.07$	32.12	$R^2 = 0.98$
	Cod	$C(y,u) = c_2\,\frac{u^{1.1}}{y}$	$c_2 = 5848.1$	44.7	$R^2 = 0.95$

Table 4.2: Parameter values and statistical properties for the biological functions.

Species	Function	Parameters	t-statistic	R^2
Capelin	$f(x,y) = b_1x^2 - b_2x^3 - b_3xy$	$b_1 = 0.00018$	4.9	$R^2 = 0.59$
		$b_2 = 1.19E-8$	3.1	
		$b_3 = 0.00021$	3.4	
Cod	$f(x,y) = b_4y^2 - b_5y^4 + b_6xy$	$b_4 = 0.00022$	8.4	$R^2 = 0.5$
		$b_5 = 3.49E-11$	4.2	
		$b_6 = 1.82E-5$	2.6	

4.2.3 Solving the management model

As already mentioned, the solution of the problem defined in equation (4.1-4.3) is found in feedback form. That is, the optimal catch of capelin and cod is a function of stock sizes only. Therefore we can write the optimal catch as

$$[u_x^\star, u_y^\star] \equiv [\tilde{u}_x(x,y), \tilde{u}_y(x,y)]. \tag{4.7}$$

The problem is not, however, algebraically solvable. The solution must be derived numerically via the optimal value function, $V(x,y)$. This function can be found in various ways. Letting the discrete dynamics be defined by the time step $0 < h << 1$, the initial value $(x_h(0), y_h(0)) = (x,y)$ and

$$x_h(i+1) = \varphi_1(x,y,u_x) \equiv x_h(i) + h\left(f_1\big(x_h(i), y_h(i)\big) - u_x\right) \tag{4.8}$$

$$y_h(i+1) = \varphi_2(x,y,u_y) \equiv y_h(i) + h\left(f_2\big(x_h(i), y_h(i)\big) - u_y\right), \tag{4.9}$$

we solve the equation

$$V_h(x,y) = \max_{u_x,u_y}\left\{h\Pi(x,y,u_x,u_y) + \beta V_h\big(\varphi_1(x,y,u_x), \varphi_2(x,y,u_y)\big)\right\} \quad \text{with} \tag{4.10}$$

$$\beta \equiv 1 - \delta h \tag{4.11}$$

by fixed-point iteration. This can be done according to the procedure described in Grüne and Semmler (2004).

4.2.4 The simulation model - Bifrost

Bifrost is an assessment tool developed at the Institute of Marine Research in Bergen, Norway. It was originally designed for the Barents Sea capelin, but is now considerably expanded. It has developed into a simulator for experimenting with harvesting control rules in the capelin-cod-herring system of the Barents Sea. In the simulation process it considers cod-capelin interactions conditional on other food, herring effects on capelin recruitment, temperature effects on recruitment of capelin and cod, capelin consumption affecting growth of cod, temperature influence on the maturation of cod and finally predation by harp seal on both capelin and cod.

Concurrent runs between SeaStar, an assessment tool for herring, and Bifrost are possible. In this work simulations are both performed with constant herring influence on the capelin cod system (SeaStar turned "off") and with a dynamical herring stock (SeaStar turned "on"). [1]

4.3 Three/six management strategies

4.3.1 Strategy A: Optimal management with feedback-Policy

In this work the optimal value function represents the discounted sum of all future economic rent from the fishery of capelin and cod in the Barents Sea. Despite this practical interpretation, it is mainly a theoretical size used to find optimal TAC levels for

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[1] For more detailed information about the Bifrost simulator, refer to the website *http://www.assessment.imr.no/*.

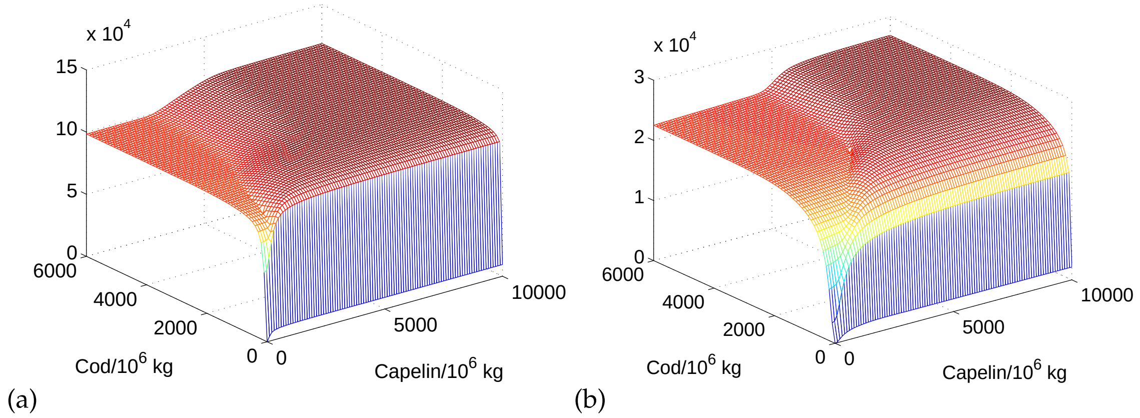
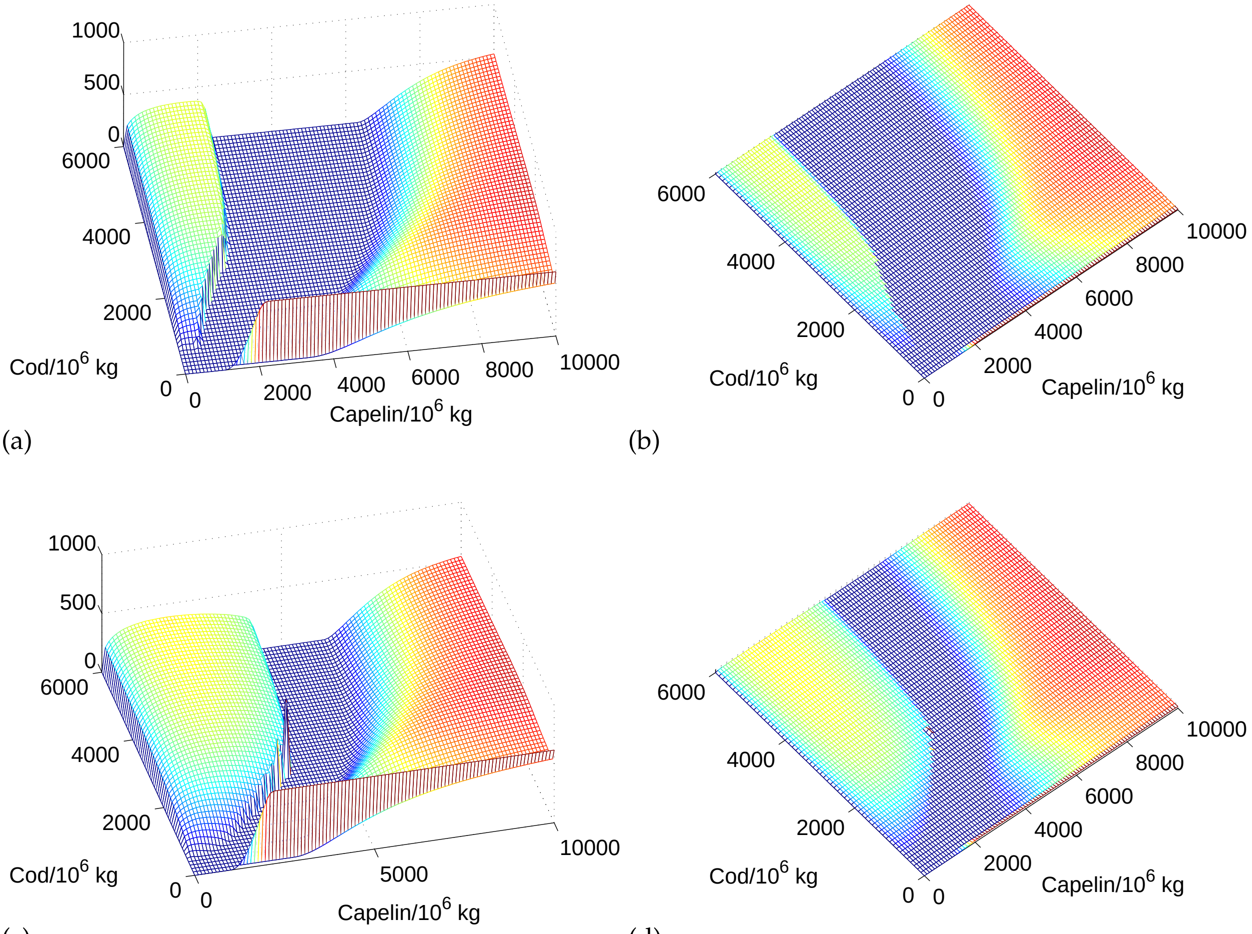


Figure 4.1: The optimal value function when discount rate is respectively (a) $\delta = 0.01$ and (b) $\delta = 0.05$.

capelin and cod. Its shape on the capelin-cod grid, however, (see Figure 4.1(a) and Figure 4.1(b)), is of sufficient interest for a presentation, as it is strongly linked to the optimal TAC levels.

Two discount rates are used in the simulations. The level of these appears to have minor influence on the shape of the optimal value function, but of course the total discounted value is higher with lower discount rate since more future profit is added. Another noticeable difference is that the first plateau expands slightly more to the right (ends at higher capelin stocks) for 5% discount rate. Also, the step up to the next plateau is steeper and shorter than for 1% discount rate. The term *plateau* is somewhat inaccurate, however, since the optimal value function grows with the biomasses of capelin and cod on most of the grid, but with the scaling used this growth is not very visible.

Optimal TACs of capelin show both expected and unexpected behavior on the capelin-cod grid. First of all we observe an expected discontinuity visible as a wall rising up along the capelin axis. This discontinuity is between zero-level of cod, which gives optimal TACs on "bliss-level", the level that maximizes current profit, and the first non-zero level of cod giving considerably lower optimal TACs along the capelin axis. Further on, for low levels of capelin along the cod axis, there is a ridge in the TAC-levels. (See figures 4.2(a) and 4.2(c)). In figures 4.2(b) and 4.2(d) we can see that the corridor to the right of the ridge is wider for low discount rate (1%) than for high discount rate (5%). The stock levels on the curve constituting the verge between the ridge and the zero corridor coincide with the stock levels on the curve constituting the beginning of the step from the first plateau to the second plateau in respectively Figure 4.1(a) and 4.1(b). As far as TAC sizes are concerned, the top of the ridge is below four hundred thousand tons, which is not a very high harvest from a historical perspective.



(a)

(b)

(c)

(d)

Figure 4.2: Optimal catch of capelin as a function of stock sizes for discount rate $\delta = 0.01$ in (a) and (c) (same figure watched from different angles) and discount rate $\delta = 0.05$ in (b) and (d).

In the optimal harvest of cod surface, the main trend is that TACs rise with rising stock levels and the size of the capelin stock seems to have minor influence. (See Figure 4.3(a) and 4.3(b)). This picture is heavily interrupted, however, for certain combinations of capelin and cod. A peak or a ridge emerges on the area of the capelin-cod plane where zero-harvest of capelin is optimal. This interesting irregularity is a strong expression of multispecies effects in the model.

According to the biological model, the optimal feedback policies are based on stock combinations to the left (lower capelin stocks) of the corridor in Figure 4.2(a)-4.2(c) will result in extinction of capelin as the trajectories end up with an equilibrium level of around 2.5 million tons of cod and the absence of capelin. (See Figure 4.4.) Beyond

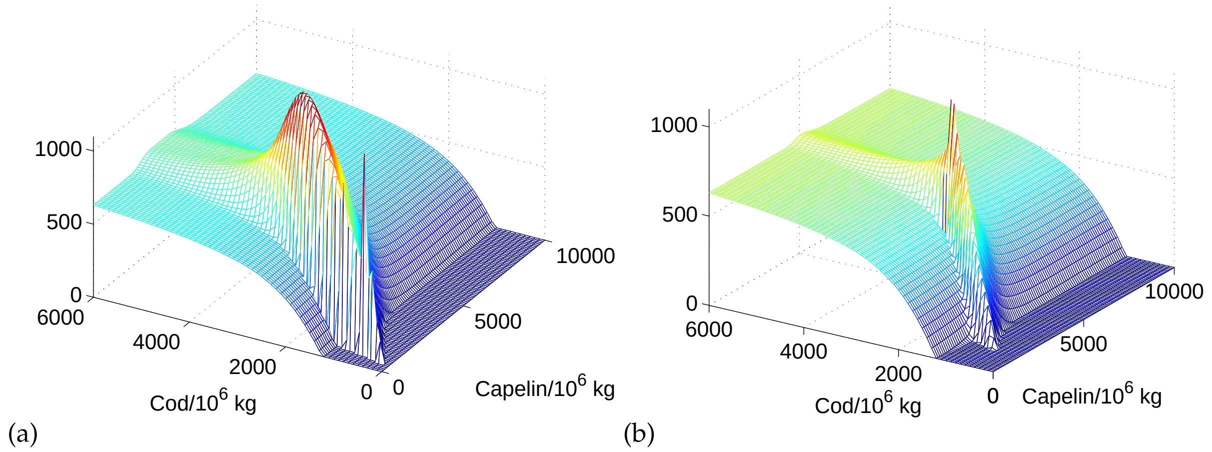


Figure 4.3: Optimal catch of cod as a function of stock size of both capelin and cod when discount rate is (a) $\delta = 0.01$ and (b) $\delta = 0.05$.

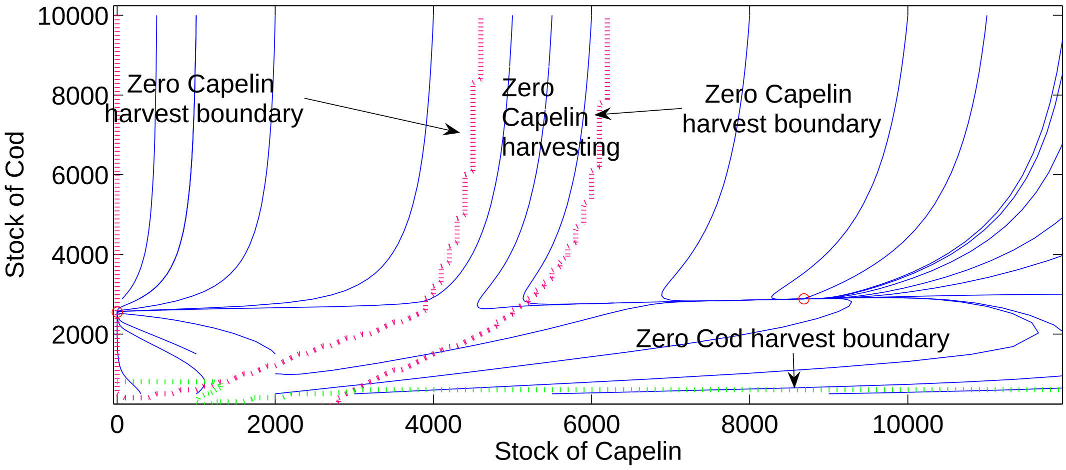
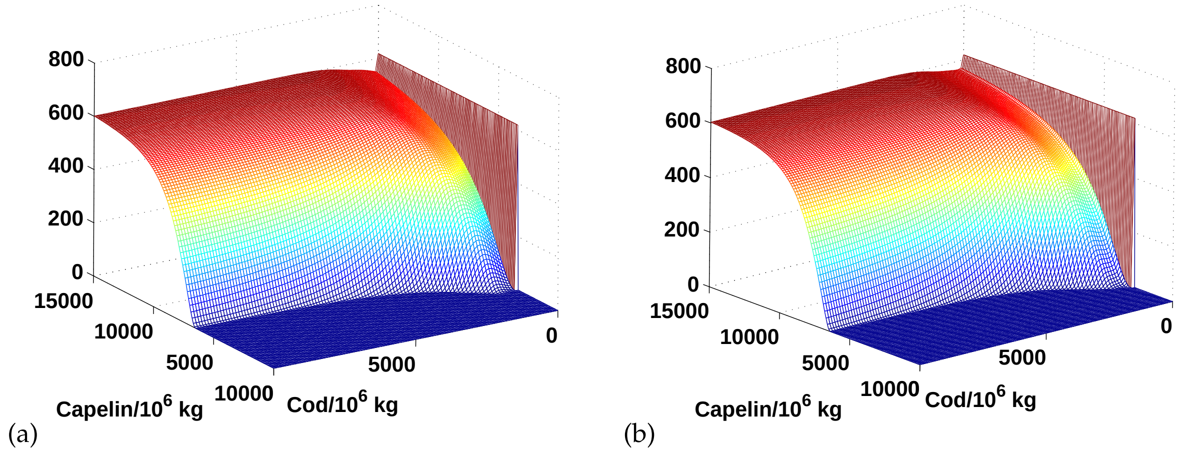


Figure 4.4: Optimal paths and equilibriums from Norwegian feedback policy when discount rate is 5%. These paths are based on the biological sub model used in the management model.

that, we see that there is another equilibrium for about 8.5 million tons of capelin. (Red rings.) When discount rate is lower (1%) the equilibrium stock sizes are slightly lower. (See overview in Figure 4.3.)

Table 4.3: Equilibriums of capelin and cod respectively resulting from optimal harvest according to the biological sub-model used in the management model. (Units in million tons.)

Discount rate	Interior equilibrium	Extinct capelin
$\delta = 0.01$	$(8.727, 2.887)$	$(0, 2.552)$
$\delta = 0.05$	$(8.678, 2, 885)$	$(0, 2.550)$



(a) (b)

Figure 4.5: A suboptimal extinction-aware catch of capelin.

4.3.2 Alternative sub-optimal strategies

Strategy B: A modified optimal strategy

The second strategy presented is based on the optimal strategy (strategy $\tilde{A}$). TACs of cod are the same, but the TAC of capelin is a more cautious figure. When the capelin stock is high, the TAC of capelin is also the same, but the ridge for low levels of capelin is removed. (See Figure 4.5(a) and 4.5(b)).

There are three aspects behind this modification. From an economic perspective the economic model puts no protection on capelin as the cost function does not depend on the stock size (see Figure 4.5). Although capelin is a schooling species, it may be too optimistic to assume that the fishing costs do not rise with smaller stocks. Second, from a biological point of view, the capelin growth might be under- or over-estimated for low levels of capelin and high levels of cod. Also, influence from the capelin-predator herring is not directly measured in the growth, and both of these aspects of uncertainty suggest a more cautious management of capelin. Finally, there are also ethical aspects concerning a potential planned capelin extinction.

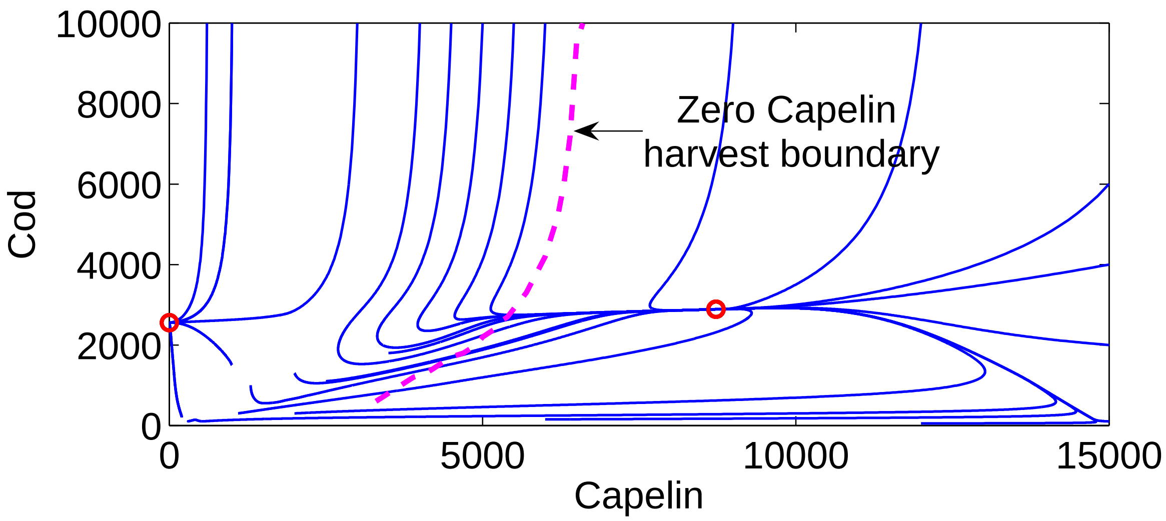


Figure 4.6: Paths and equilibriums when discount rate is 1%. These paths are based on the biological sub-model used in the management model.

Strategy C: A purely biologically based reference strategy

The reference strategies $\tilde{C}_1$ and $\tilde{C}_2$ are both inspired by existing management rules for capelin and cod. Management of capelin has been pioneering because of its multi-species approach. Today it is managed according to a rule allowing a maximum 5% probability for the spawning stock to become smaller than 200 thousand tons. In both strategy $\tilde{C}_1$ and $\tilde{C}_2$ we choose a target spawning level of 500 thousand tons, which should correspond quite well to the existing management rule.

Strategies $\tilde{C}_1$ and $\tilde{C}_2$ differ in the management of cod. Whereas the fully-exploited fishing mortality level of cod, the F-value, is 0.65 in $\tilde{C}_1$, it is 1.0 in $\tilde{C}_2$.

4.4 Simulations with Bifrost

4.4.1 Simulations assuming constant herring influence

In the Bifrost simulations presented below herring influence is assumed constant as it is represented by a biomass of 0.5 million tons throughout the whole simulation period.

We will first present simulations with each of the management strategies, and then we will compare the strengths and weaknesses of the economic management strategies with those of the biological strategies. Combinations of historic spawning stock biomasses (SSBs) and TACs from 1972 to 2005 and corresponding prognostics for the

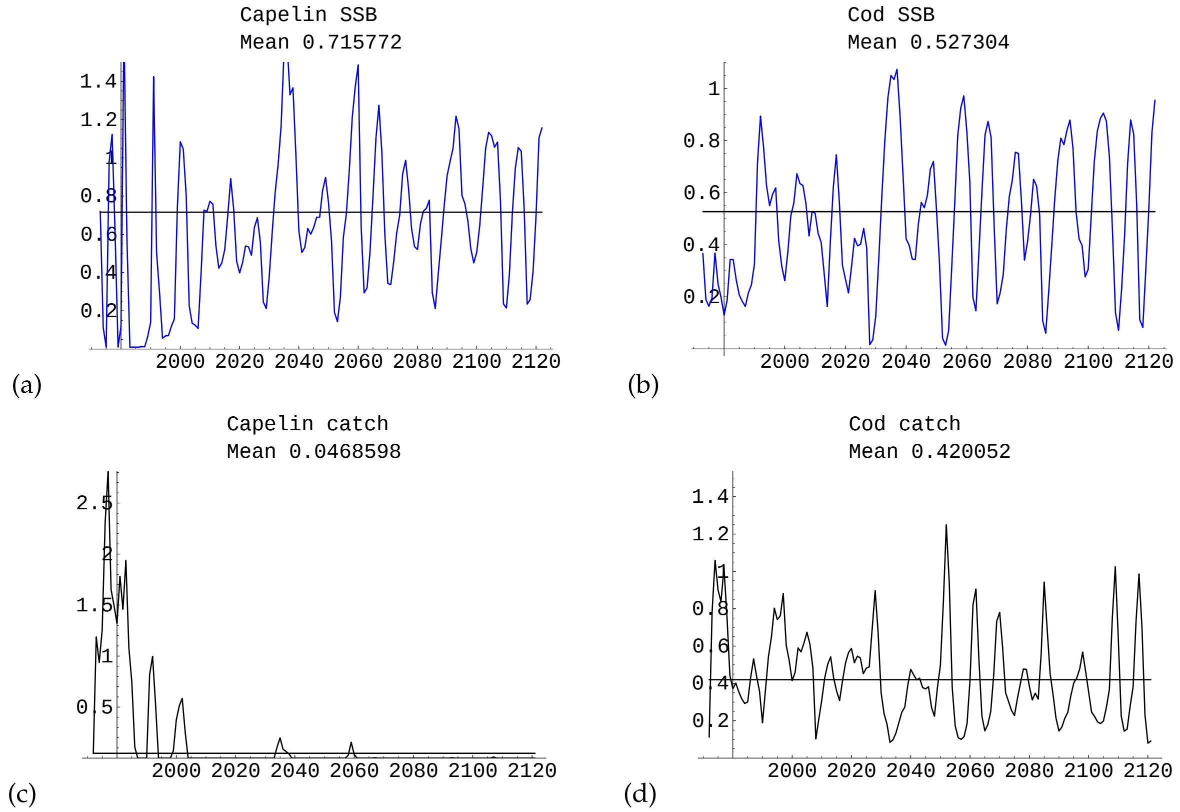
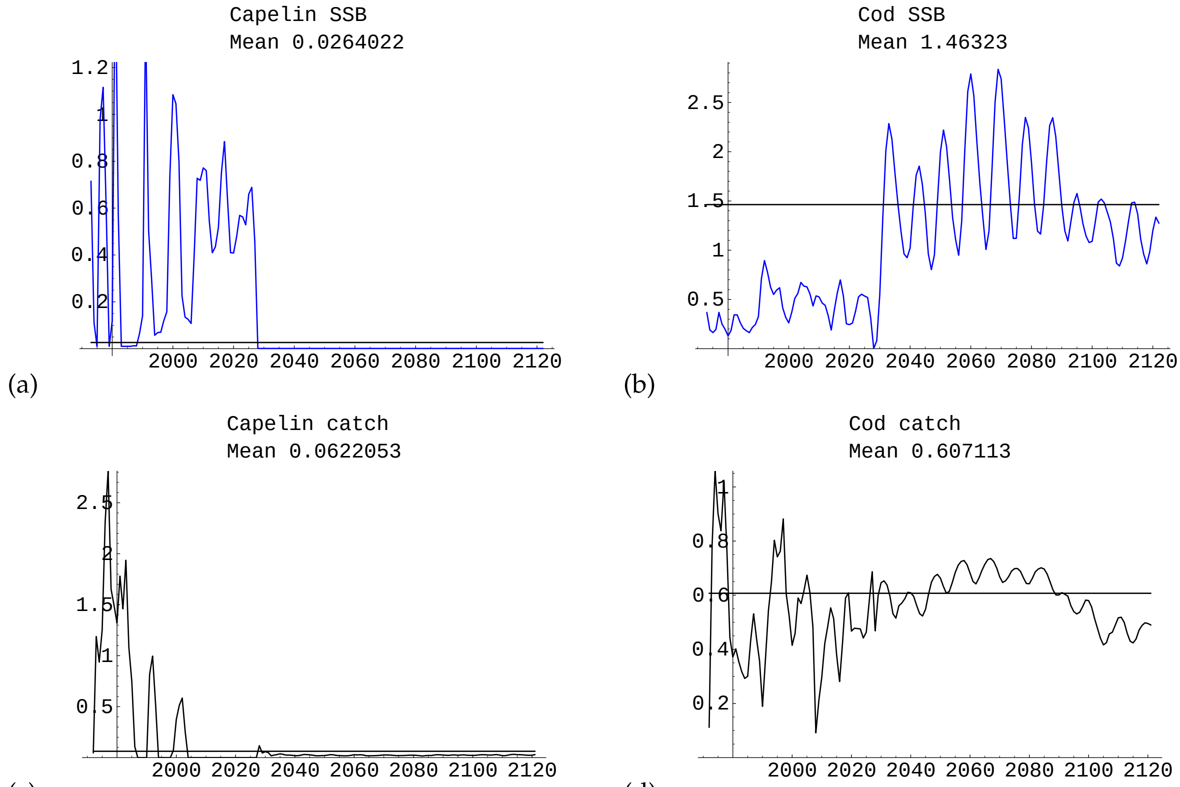


Figure 4.7: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from strategy $\tilde{A}_1$ (1% discount rate).

years 2006 to 2120 may be viewed in figures 4.7 to4.11. In each of these figures a horizontal line indicates the average of the 600-year period from 1972 to 2571. Table 4.4 shows basic statistics as mean and standard deviation (SD) for the 965-year prognosis of total biomasses (TBs) and TACs.

The scenario resulting from strategy $\tilde{A}_1$ can be viewed in Figure 4.7. SSB of capelin turns out to be somewhat lower in the first years (historic results) than for the prognostic period, but the variability is high (see Figure 4.7(a)). Total biomass has a mean of 3.09 million tons and a standard deviation of 0.66 for the prognostic years (see Table 4.4). The total biomass average for cod (1.21 million tons) is much lower than for capelin, as is also the standard deviation (0.17). This might be a little surprising since the SSB numbers as appear to fluctuate just as much for cod in Figure 4.7(b) as for capelin in Figure 4.7(a). These figures, however, only show the first hundred years of the prognostics and there may in addition be differences in the variability of the SSB and TB.

The fishery of capelin (see Figure 4.7(c)) is characterized by total closures interrupted by small positive TACs for a few years when the biomass is high enough. TACs of cod



(a) (b) (c) (d)

Figure 4.8: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from optimal strategy $\tilde{A}_2$ (5% discount rate) when a constant Barents Sea herring stock of 0.5 million tons is assumed.

(mean of about 415 thousand tons) are a lot higher than for capelin, but the standard deviation is very low for the TACs of both (six thousand tons for capelin and 92 thousand tons for cod).

Though the feedback curves in strategy $\tilde{A}_1$ and strategy $\tilde{A}_2$ are quite similar (see figures 4.2 and 4.3), the simulation results from Bifrost differ very much when discount rate rises from 1% to 5%. With strategy $\tilde{A}_2$ a drastic shift occurs between the year 2025 and the year 2030 (see figures 4.8(a) and 4.8(b)). Then the SSB of cod grows heavily and the capelin stock falls dramatically. At the same time the fishery of capelin opens from a total closure (see Figure 4.8(c)).

Average TACs of both capelin and cod are higher with strategy $\tilde{A}_2$ than with strategy $\tilde{A}_1$, but the standard deviation in the TACs is much lower (see Table 4.4). All this shows that, according to Bifrost, strategy $\tilde{A}_2$ is better than strategy $\tilde{A}_1$.

The simulation with strategy $\tilde{B}_1$ is very similar to the simulation with strategy $\tilde{A}_1$. The only difference is that TACs of capelin are a little lower for a few years. For the

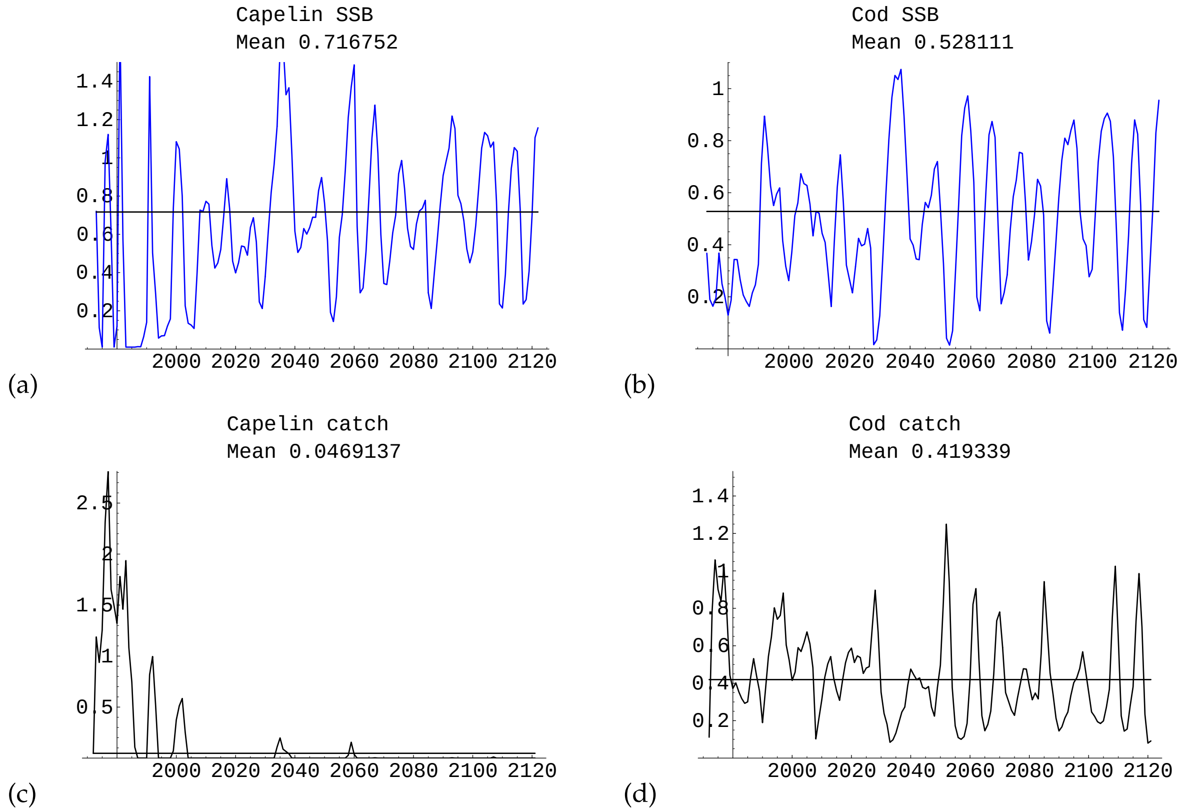
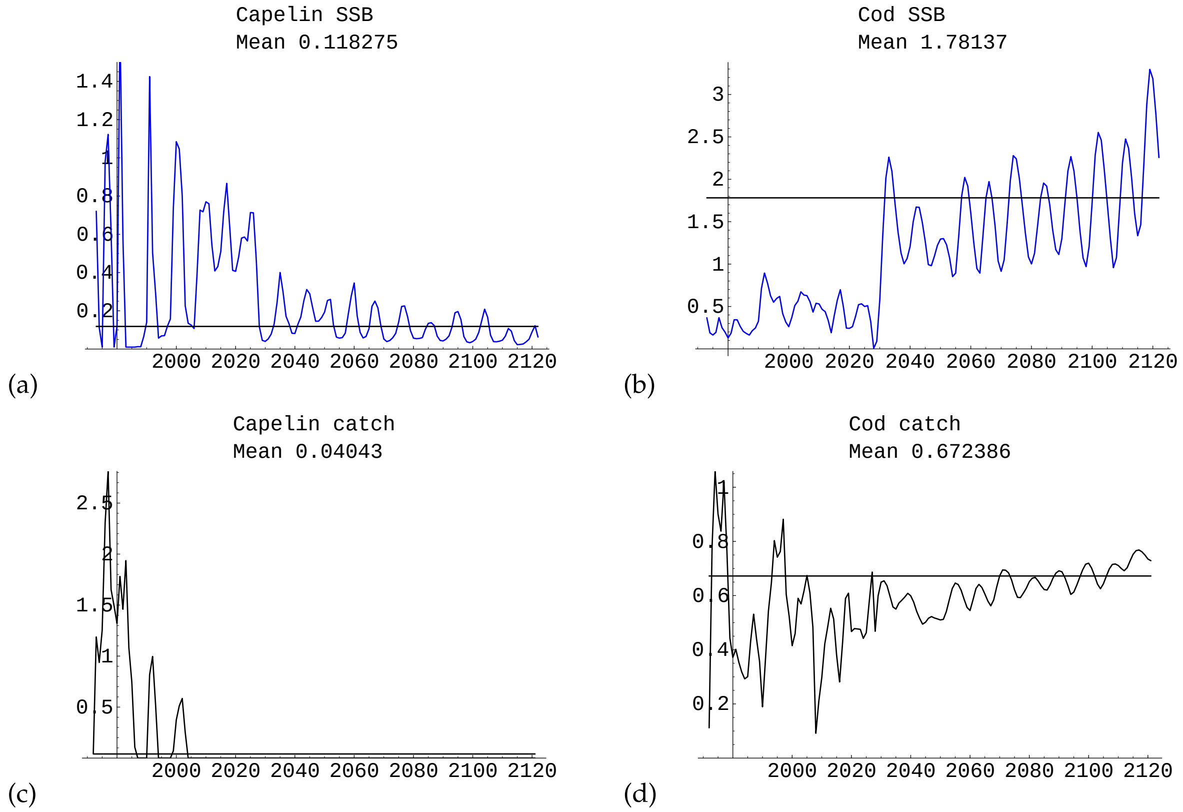


Figure 4.9: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from strategy $\tilde{B}_1$ (1% discount rate) when a constant Barents Sea herring stock of 0.5 million tons is assumed.

years visible in figures 4.7 and 4.9 TACs and SSBs are identical.

In the same way as there are many similarities between the simulation of strategy $\tilde{A}_1$ and $\tilde{B}_1$, the similarities between $\tilde{A}_2$ and $\tilde{B}_2$ are obvious. Both simulations have a sudden rise in the cod stock and a fall in the capelin stock between 2025 and 2030, and both simulations have very high TACs of cod and low TACs of capelin. In strategy $\tilde{B}_2$ the fishery of capelin is completely closed for the whole period of the simulations (see prognostics in Table 4.4). Another characteristic is that TACs of cod are very high, with rather low standard deviation. TACs of cod are on average above 702 thousand tons with a standard deviation of around 77 thousand tons. The very high mean in TB of about 3.38 million tons makes the high TACs possible.

The purely biologically based management strategies $\tilde{C}_1$ and $\tilde{C}_2$ differ from each other only with respect to the fully-exploited fishing mortality level of cod. Nevertheless, the resulting TB mean and TACs differ dramatically both for capelin and cod. The simulation of $\tilde{C}_1$ gives a capelin average of 1.946 million tons and a cod average of 1.967



million tons. This result is different from all the other strategies, which either give high stocks of capelin and low stocks of cod or low stocks of capelin and high stocks of cod. Average TACs with strategy $\tilde{A}_2$ and $\tilde{C}_1$ are, however, rather close. The main difference is that TACs of cod are a little higher and have much lower variation with strategy $\tilde{A}_2$. In both cases TACs on capelin are very low, but the total biomass of capelin is much higher with strategy $\tilde{C}_1$.

The simulations with strategy $\tilde{C}_2$ differ considerably from the rest of the simulations. The main difference is that both the total biomass average (4.104 million tons) and TAC average (0.967 million tons) of capelin is very high, but at the same time both TB and catch of cod is much lower than for all the other strategies. An interesting observation is that it takes some time before the SSB of capelin becomes high and the SSB of cod becomes low with strategy $\tilde{C}_2$. In the first part of the prognostics (the part visible in figures 4.11(a) and 4.11(b)) we see that the SSB is mainly lower than the mean for capelin and higher than the mean for cod).

In the second run with the Bifrost simulator, the biomass of herring is at a constant level of 800 million tons. Under such circumstances both the capelin and cod stock biomasses get lower according to both figures 4.12-4.17 and Table 4.5. Still, capelin does

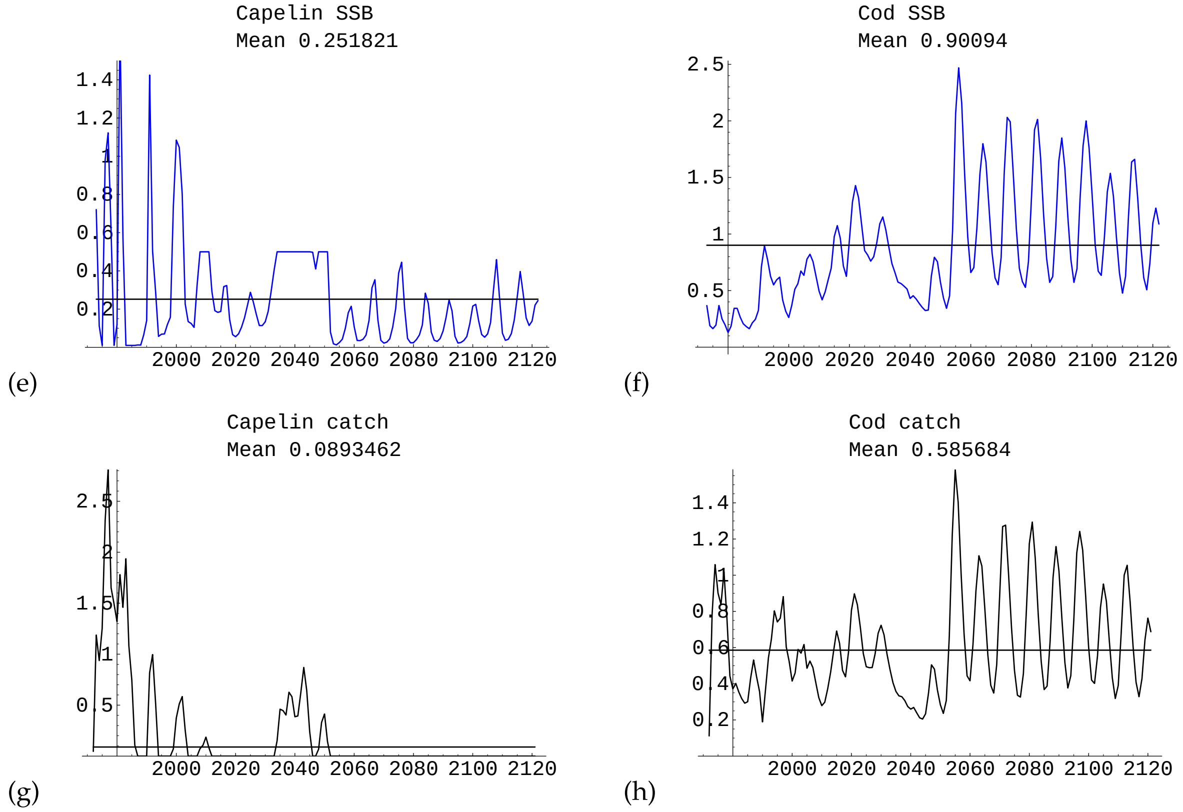


Figure 4.10: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from biologically based strategy $\tilde{C}_1$. Relative F on cod $= 0.65$. Target SSB capelin $= 0.5$. Constant Barents Sea capelin of 0.5 million tons

not get exterminated with any of the strategies even though it is very close with strategy $\tilde{A}_2$.

4.4.2 Simulations with SeaStar integrated in Bifrost

In the simulations presented below a dynamic herring influence is accounted for in Bifrost through the assessment tool SeaStar. Two different herring managements are tested. The first leads to a long-term average Barents Sea fraction of 0.5 million tons (see figures 4.18-4.23 and Table 4.6), and the second gives an average of 0.8 million tons. (See Table 4.7).

It is hard to see any system in the changes taking place when we switch from a constant to a dynamic herring influence. The most dramatic change comes under strategy $\tilde{A}_1$. When constant herring influence from a Barents Sea herring biomass of 0.5 million tons is assumed, the total biomass of capelin has a mean value of about 3.09 million

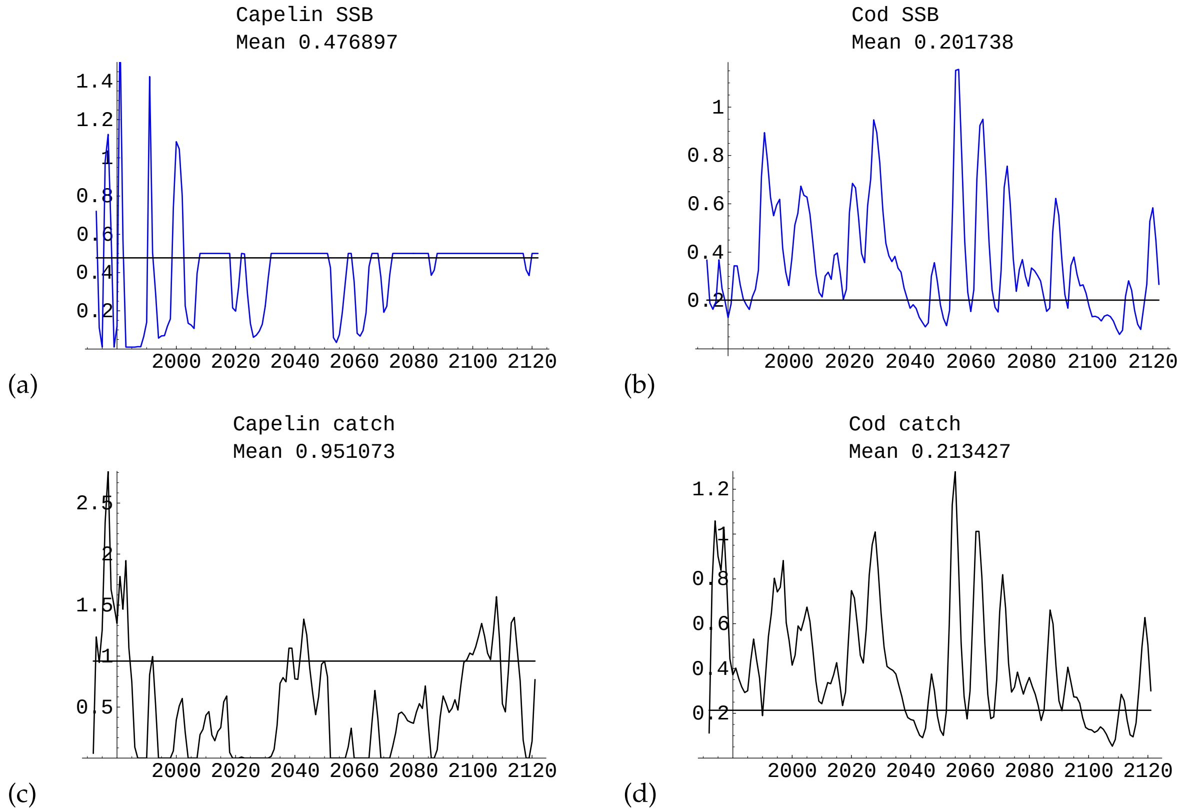


Figure 4.11: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from biologically-based strategy. Relative F on cod $= 1.0$. Target SSB capelin $= 0.5$. Constant Barents Sea herring stock of 0.5 million tons.

tons, but with dynamic herring this mean value falls to about 0.10 million tons. Similarly, with a stronger herring influence (0.8 million tons), the mean TB of capelin falls from 2.21 million tons under constant herring influence to 0.21 million tons when the herring stock is dynamical. Conversely, the TB of cod grows considerably from about 1.21 million tons to about 2.6 million tons when the long-term mean of herring is 0.5 million tons and from 0.892 million tons to 2.63 million tons when the mean of herring is 0.8 million tons.

With a mean of 0.8 million tons of herring in the system, simulations with dynamical herring give more desirable results than the simulations with a constant herring influence. When the mean biomass of herring is lower (0.5 million tons), the picture is more complex. In addition to strategy $\tilde{A}_1$, Strategy $\tilde{B}_1$ also appears better when herring influence is dynamical. Even though the mean TB of the capelin stock falls slightly, both TACs of capelin and TACs of cod grow. For the rest of the strategies, constant herring gives better results than dynamical herring of mean 0.5 million tons, since stocks and

Table 4.4: Mean and standard deviation (SD) for prognostics of total biomass and TACs of capelin (autumn stock) and cod (january) when a constant Barents Sea herring stock of 0.5 million tons each year is assumed. Each run covers a 965 years period with 2006 as the starting year.

Strategy	Capelin stock		Cod stock	
	Mean	SD	Mean	SD
A, $\delta = 0.01$	3.091630272	0.662100941	1.20874704	0.174171855
A, $\delta = 0.05$	0.143890648	0.407358485	2.615785602	0.540907652
B, $\delta = 0.01$	3.090494807	0.662041799	1.208038903	0.174585102
B, $\delta = 0.05$	1.150288865	0.451976511	3.378947033	0.729517508
C_1	1.945762907	0.756340991	1.966642941	0,646805015
C_2	4.104687943	0.899346301	0.496878953	0.446232054

Strategy	Capelin TAC		Cod TAC	
	Mean	SD	Mean	SD
A, $\delta = 0.01$	0.005443101	0.024460816	0.410329623	0.24766933
A, $\delta = 0.05$	0.023567972	0.00623093	0.601002962	0.092304818
B, $\delta = 0.01$	0.005071674	0.023421135	0,411132836	0.248686928
B, $\delta = 0.05$	0.0	0.0	0.702030157	0.076879613
C_1	0.051854613	0.158286673	0.587451550	0.248665038
C_2	0.965346285	0.474029877	0,192500381	0.184195318

TACs of cod are higher in the first case. With strategy $\tilde{B}_2$ particularly the difference is significant, as mean TACs are respectively about 0.702 and 0.637 million tons. Yet strategy $\tilde{B}_2$ is still the strategy giving the highest mean TACs of cod.

Looking at long-term maximum sustainable TACs of cod, we see that the simulations with constant herring influence (0.5 million tons and 0.8 million tons) give the following ranking of the strategies: 1.$\tilde{B}_2$, 2.$\tilde{A}_2$, 3.$\tilde{C}_1$, 4.$\tilde{A}_1$, 5.$\tilde{B}_1$ and 6.$\tilde{C}_2$. With dynamic herring influence strategy the ranking is: 1.$\tilde{B}_2$, 2.$\tilde{A}_1$, 3.$\tilde{A}_2$, 4.$\tilde{C}_1$, 5.$\tilde{B}_1$ and 6.$\tilde{C}_2$.

4.5 Discussions

4.5.1 The characteristics of the management model

When looking at the optimal value function in Figure 4.1(a) we should be more concerned about the shape of the surface than the specific values on it. Whereas the values are of little relevance to practical perspectives, the shape of the surface may help us to better understand the properties of both the biological and economic parts of the man-

Table 4.5: Mean and standard deviation (SD) for prognostics of total biomass and TACs of capelin and cod when a constant Barents Sea herring stock of 0.8 million tons each year is assumed. Each run with the Bifrost-simulator covered a 565-year period with 2006 as the starting year. Biomass estimates of capelin and cod respectively correspond to total autumn stock and total stock on January 1.

Strategy	Capelin stock		Cod stock	
	Mean	SD	Mean	SD
A, $\delta = 0.01$	2.214898711	0.42006007	0.891694836	0.147040637
A, $\delta = 0.05$	0.085405304	0.234210912	2.490765284	0.458312948
B, $\delta = 0.01$	2.219912272	0.42183278	0.892104416	0.147830299
B, $\delta = 0.05$	0.867700142	0.272603932	2.997248327	0.635147445
C_1	1.509473215	0.570326057	1.629396401	0.66140552
C_2	2.593762349	0.571576056	0.484616247	0.426159468

Strategy	Capelin TAC		Cod TAC	
	Mean	SD	Mean	SD
A, $\delta = 0.01$	0	0	0.318495806	0.220373323
A, $\delta = 0.05$	0.018942894	0.015747563	0.575867007	0.090659615
B, $\delta = 0.01$	0	0	0.317066717	0.223027975
B, $\delta = 0.05$	0	0	0.658436597	0.08312256
C_1	0.044380072	0.121733895	0.487169386	0.23482143
C_2	0.423687288	0.277196976	0,187882981	0.176577218

agement model.

First of all we observe that the surface goes through the origin. This is intuitively understandable since an extinction of both capelin and cod would give no future surplus from the fishery of these stocks. Second, with some capelin and no cod present, the biological model gives good growth conditions for capelin and absence of cod for the future. Since cod is more valuable than capelin in the economic model, future profit from such a scenario is considerably lower than with some cod present. This is obviously because of the discontinuity creating a steep wall rising up from the edge of the surface along the capelin axis.

If we have a closer look at the surface edges leading out of the origin along the capelin and cod axis, we see that they rise for small values of the stocks and flatten out for large values. The reason is two-fold: according to the biological growth model extreme stock sizes are not sustainable, and according to the concave nature of the economic model, higher catches would lead to lower prices, and therefore give limited profit.

From an economic perspective, the purpose of capelin is more or less to feed the cod

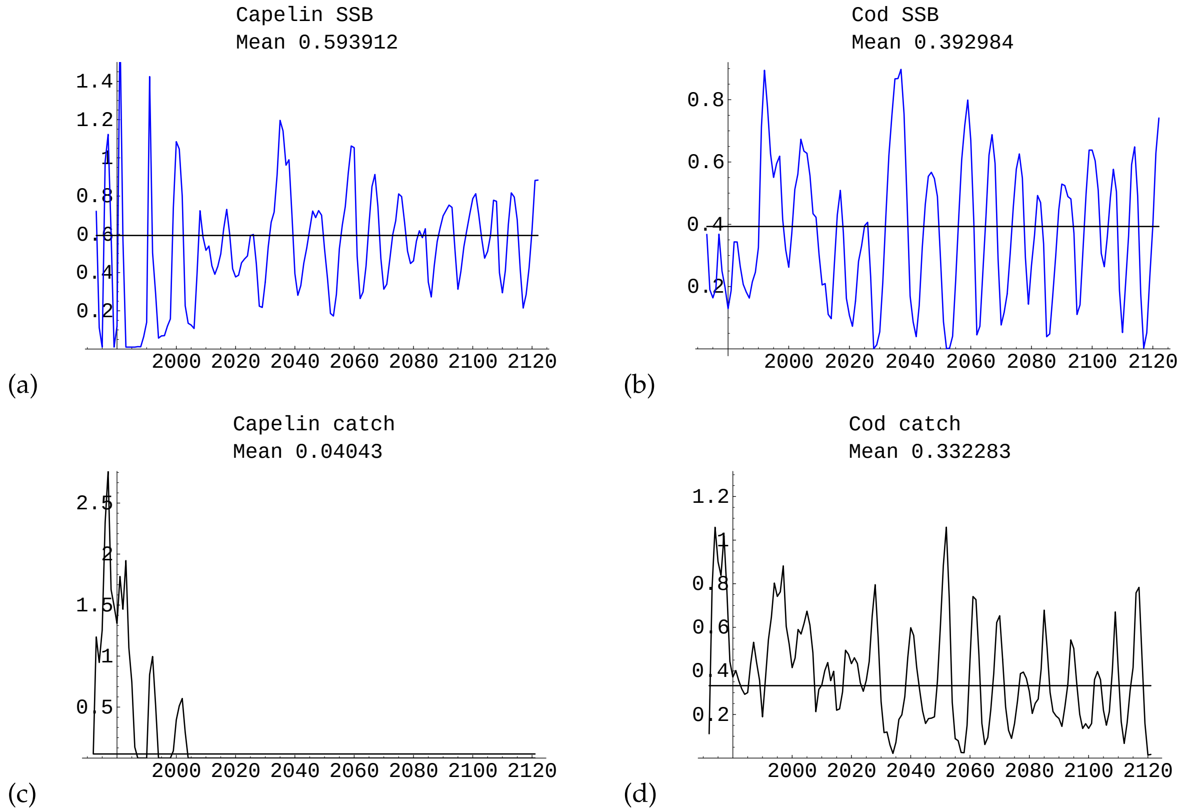


Figure 4.12: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from strategy $\tilde{A}_1$ (1% discount rate) when a Barents Sea herring stock of 0.8 million tons is assumed.

stock, which is the most profitable species, but a small harvest of capelin may in some years add profit without harming the growth of cod too much. Specifically, if the cod stock were totally depleted, optimal harvest would mean much heavier exploitation of the capelin stock since its main predator would have been absent for all future. This is the reason for the wall rising up from the capelin axis of Figure 4.2(a) and 4.2(c).

The pattern of the optimal catch of capelin is rather surprising. (See figures 4.2(a)-4.2(d)). The surprising part is the ridge parallel to the cod axis. Zero-level of capelin means of course absence of capelin harvest, but why do small levels of capelin give higher TACs of capelin than fairly high levels of the stock?

This is not easy to explain, but according to the biological management model capelin will contribute little to further cod growth when there are high levels of cod (roughly two million tons and more) and little capelin. At the same time, capelin is in a situation where it will become totally depleted by predation from cod. In such a situation it might be profitable to catch the capelin and get some immediate surplus. With a lot

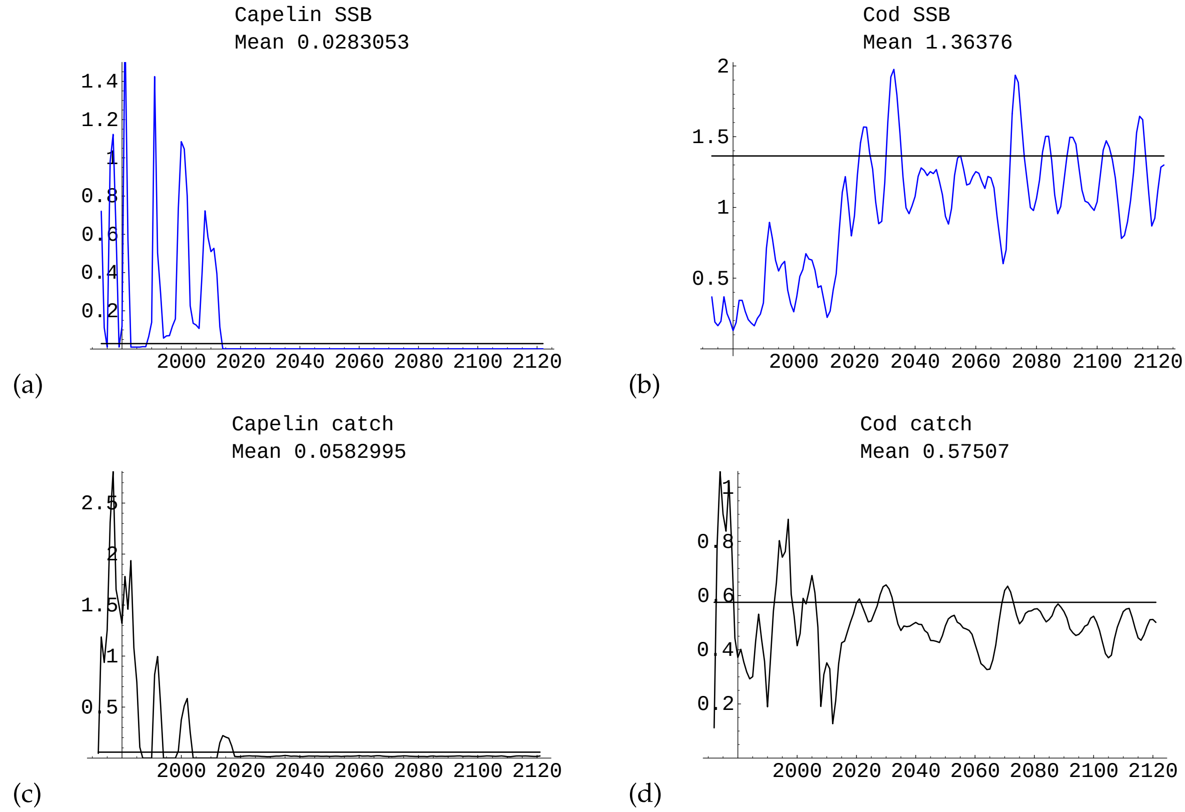
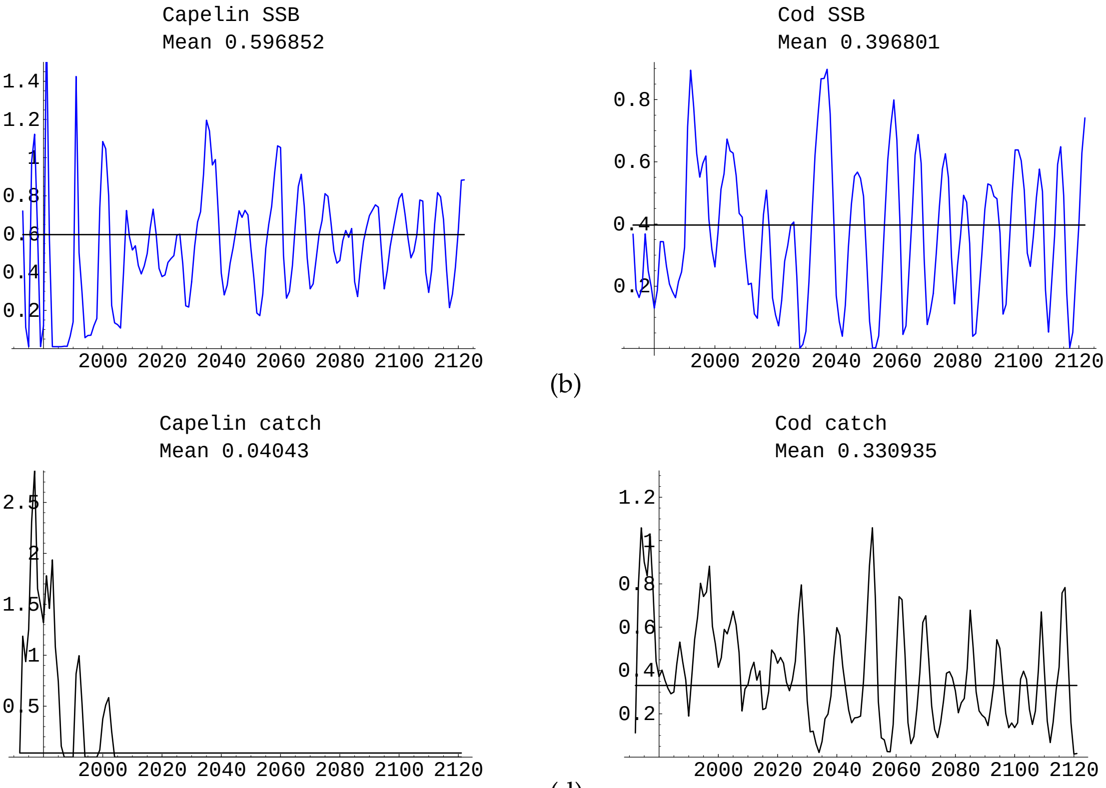


Figure 4.13: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from optimal strategy $\tilde{A}_2$ (5% discount rate) when a constant Barents Sea herring stock of 0.8 million tons is assumed.

more capelin present, the capelin stock may still be under pressure from cod, but will not become depleted. Then no catch of capelin and heavy catch of cod is the best harvest policy. Figures 4.3(a) and 4.3(b) show that cod should be heavily exploited on the area of the capelin-cod plane where zero harvest of capelin is optimal. Specifically, there is a peak in the cod TACs on the borderline between the ridge in the capelin TAC and the zero corridor. This leads us to conclude that heavy exploitation of cod is a rescue operation to secure future stocks of capelin. The ridge in the TACs of capelin covers an area where optimal management leads to extinction of capelin. One could ask, however, whether the simple biological model underestimates the survival ability of the capelin stock. If that is the case, the ridge close to the cod axis in Figure 4.2(a) and 4.2(c) should be replaced with zero levels as optimal harvest of capelin. This is a strong argument for the introduction of strategy $\tilde{B}_1$ and $\tilde{B}_1$.



(a) (b) (c) (d)

Figure 4.14: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from strategy $\tilde{B}_1$ (1% discount rate) when a constant Barents Sea herring stock of 0.8 million tons is assumed.

4.5.2 Simulations with Bifrost

The two-species nature of the biology in the management model is a simplification of real nature, but it should be superior to a one-species model. In the same way a three-species model would have been a simplification, but less so than a two-species model. In the two-species herring growth influence is indirectly reflected through the parameters of the growth function since the regression analysis performed to find these parameters is based on data/calculations found from survey data and assessment calculations regarding, among other factors, influence from herring. With the uncertainty about the sufficiency of a two-species model in mind, however, we believe that it is a good strategy to test the two-species management model when SeaStar is turned both "on" and "off" in the Bifrost simulations. When SeaStar is "off", herring is set to a constant level, as in sec. 4.4.1, and when SeaStar is on, herring varies according to this assessment tool, as in the simulations of sec. 4.4.2.

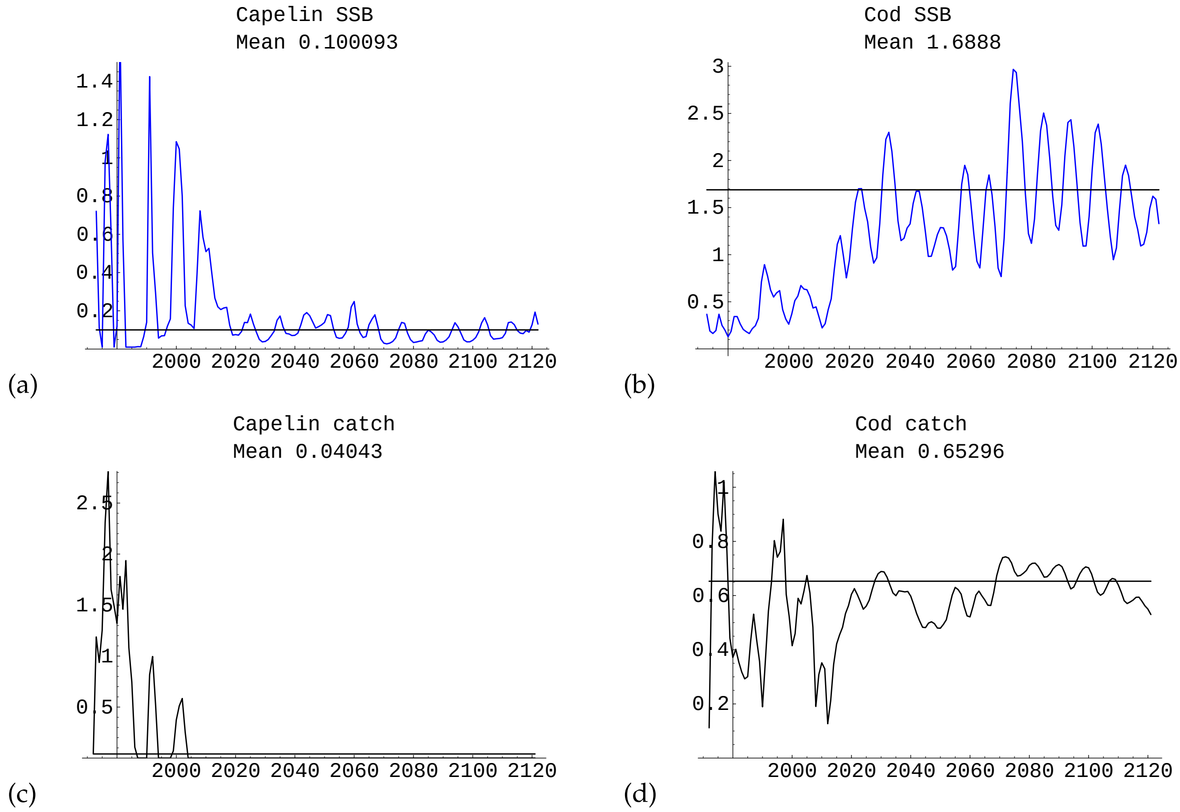


Figure 4.15: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from optimal strategy $\tilde{B}_2$ (5% discount rate) when a constant Barents Sea herring stock of 0.8 million tons is assumed.

With SeaStar turned "off" the strategies with low discount rate ($\tilde{A}_1$) and ($\tilde{B}_1$) result in a much lower average biomass of cod than those with high discount rate. At the same time the biomass average of capelin is higher, actually almost three times higher, than it is with strategy $\tilde{A}_2$ and the capelin extinction-aware strategy $\tilde{B}_2$ (see table 4.4). These findings may be explained by the feedback management. When the discount rate is low, cod harvest is close to bliss, and even above bliss, in parts of the stock area where the capelin stock is under pressure (see figures 4.3(a) and 4.3(b)). In other words, the capelin rescue operation from the management model also works in the simulations. It even works better there than in the management model itself. Although vanishing low biomasses of capelin are found in simulations with strategy $\tilde{A}_2$, the stock survives in all the simulations, as it does when SeaStar is on. The survival of capelin in all the simulations might seem a little strange since, according to the management model, low biomasses of capelin would result in extinction. It is obvious that the survival ability of the capelin stock must be considered stronger in the simulation model than in the

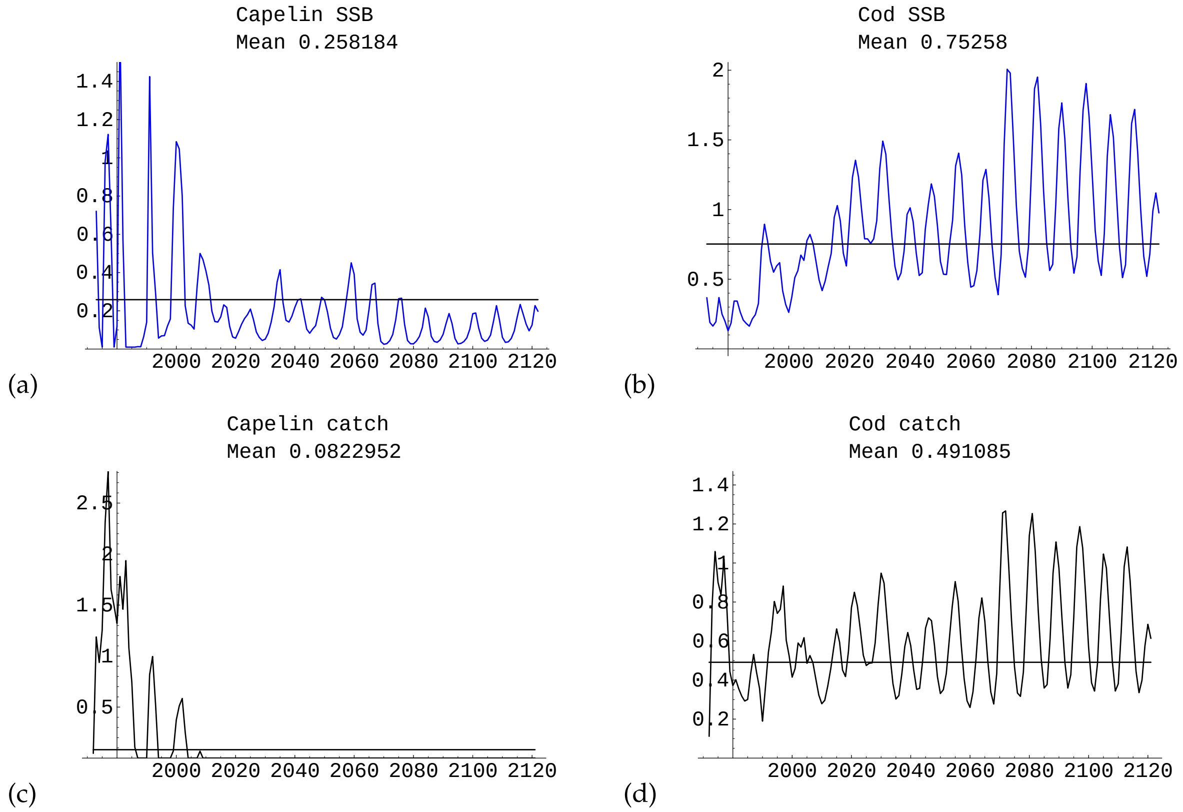
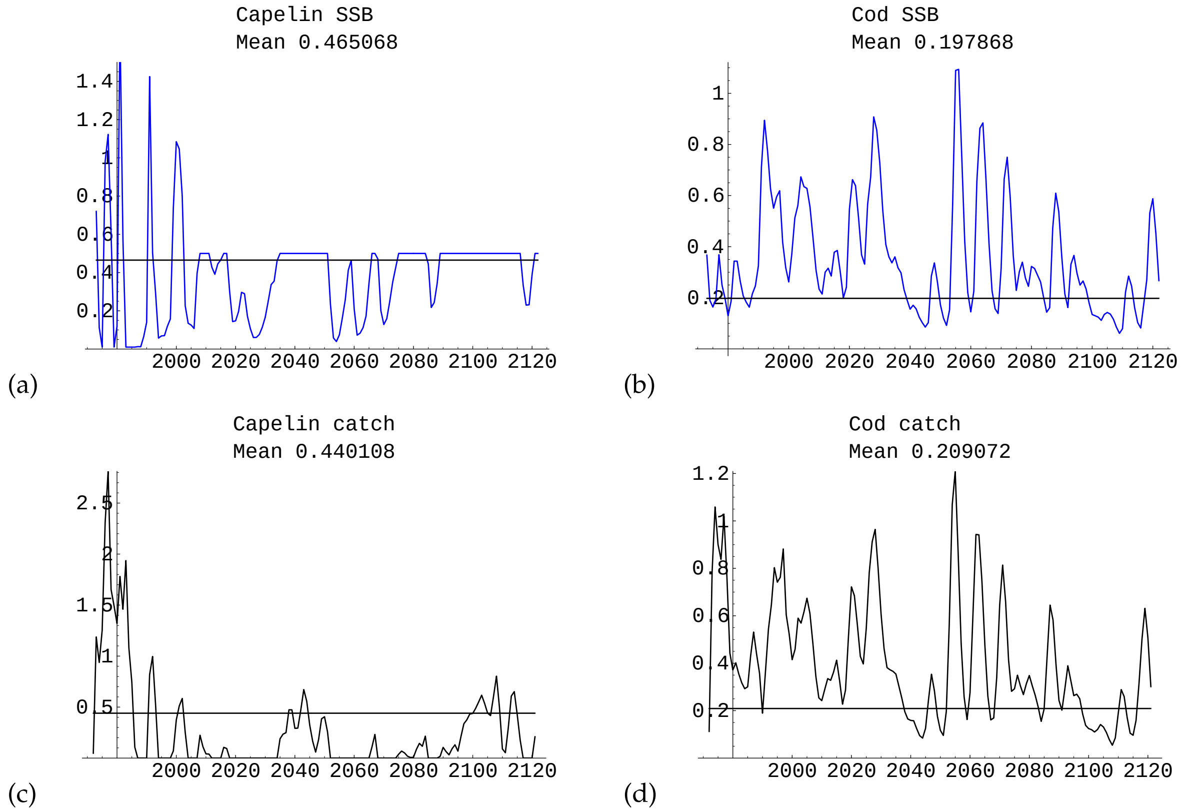


Figure 4.16: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from biologically based strategy $\tilde{C}_1$. Relative F on cod $= 0.65$. Target SSB capelin $= 0.5$. Constant Barents Sea capelin of 0.8 million tons

management model. Yet none of the strategies give capelin levels near the management model equilibriums (see Table 4.3.) Neither the capelin stock, nor the cod stock seems to have the same strong growth potential for high stocks in the simulation model as it does in the management model.

The discount rate seems very important to the results in terms of average catches and standard deviation are concerned. Strategy $\tilde{A}_2$ and $\tilde{B}_2$ (both with 5% discount rate) give higher cod TAC means with much lower standard deviations than all the other strategies when herring influence is constant. (See Table 4.4 and 4.5.) When herring influence is dynamical strategy $\tilde{A}_1$ with low discount rate also gives high TACs of cod with low standard deviation. (See tables 4.6 and 4.7.) The low standard deviation in these strategies is connected with low capelin levels. The capelin levels are, however, acceptable in the simulations with the capelin aware strategies $\tilde{B}_1$ and $\tilde{B}_2$.

Surprisingly, when herring influence is constant (sec. 4.4.1) strategy $\tilde{A}_2$ gives a higher capelin TAC mean than do strategy $\tilde{A}_1$. The reason is that strategy $\tilde{A}_1$ leads



(a) (b) (c) (d)

Figure 4.17: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from biologically-based strategy. Relative F on cod $= 1.0$. Target SSB capelin $= 0.5$. Constant Barents Sea herring stock of 0.8 million tons.

to a capelin stock that most of the time is found somewhere in the wide fishing closure tunnel of Figure 4.2(b), whereas under strategy $\tilde{A}_2$ the capelin stock vanishes low and stays there since very high cod stocks are combined with low, but too high, capelin TACs. This means that the misfit between the extremely low growth for small capelin biomasses in the management model and the higher growth in the simulation model leads to a capelin fishery that is probably suboptimal according to the simulations. The cod TACs following from strategy $\tilde{B}_2$ are both higher and more stable than those following strategy $\tilde{A}_2$ (see Table 4.4).

In sec. 4.4.2 we document a dramatic difference between results from simulations with constant herring influence and simulations with a dynamic herring stock when strategy $\tilde{A}_1$ is followed. For the rest of the strategies there is only minor differences, and it is difficult to explain why the mean capelin biomass is so much lower with strategy $\tilde{A}_1$ when SeaStar is on. Of course, heavy predation from a much higher cod stock is an important reason, but why does the cod stock become higher? Probably, variation

Table 4.6: Mean and standard deviation (SD) for prognostics of total biomass and TACs of capelin and cod when a dynamic Barents Sea herring stock with mean 0.5 million tons each year is integrated through SeaStar assessment. Each run with the Bifrost-simulator covered a 565-year period with 2006 as the starting year. Biomass estimates of capelin and cod respectively responds to total autumn stock and total stock on January 1.

Strategy	Capelin stock		Cod stock	
	Mean	SD	Mean	SD
A, $\delta = 0.01$	0.104942418	0.211483267	2.601265362	0.530627246
A, $\delta = 0.05$	0.104653328	1.124502037	2.498446365	0.470974904
B, $\delta = 0.01$	2.709593392	1.952733583	1.639118427	0.9733563
B, $\delta = 0.05$	1.3106663	0.76442343	2.87125168	0.6587045
C_1	2.193543783	1.60508793	1.906429291	0.600704853
C_2	4.570881257	2.817617425	0.484917141	0.426078118

Strategy	Capelin TAC		Cod TAC	
	Mean	SD	Mean	SD
A, $\delta = 0.01$	0.024383397	0.011183115	0.59004937	0.0976793800
A, $\delta = 0.05$	0.025217851	0.242822457	0,577847113	0.099450130
B, $\delta = 0.01$	0.051455493	0.120846468	0.492555017	0.318010448
B, $\delta = 0.05$	0.0	0.0	0.636517844	0.098382788
C_1	0.1327351	0.363901952	0.569615536	0.231219606
C_2	1.231926845	1.074320779	0.187904156	0.177654534

with a low herring stock in some following years allows the capelin stock to maintain a rather high level (see Figure 4.18(a)), which in turn gives rise to a higher cod stock (Figure 4.18(b)). When such a cod stock interferes with a couple of years with a stronger herring stock, the capelin stock falls below the level where it is sustainable according to the management model. Then continued fishing of capelin prevents a recovery of the stock.

An interesting feature with the Bifrost simulations is the occurrence of sudden changes after a long period with smaller fluctuations. The stocks appear to be fluctuating around an equilibrium, but change to a completely different level within a few years. Such changes are visible in e.g. Figure 4.8, where the SSB of cod has a sudden rise in year 2027 and the capelin stock collapses at the same time. Similar changes can be found in Figure 4.10(b), 4.10(f), 4.13(b), 4.15, 4.18 and 4.19. In addition, traces of such changes occurring after the year 2120, which is the last year plotted in the figs., can be found in figures 4.11, 4.17, 4.20(a) and 4.23. In all these figures the long-term average (marked with a horizontal line) is very different from the average for the whole simulation period. This indicates that there might be similar sudden, but lasting, changes in the stocks

Table 4.7: Mean and standard deviation (SD) for prognostics of total biomass and TACs of capelin and cod when a dynamic Barents Sea herring stock with mean 0.8 million tons each year is integrated through SeaStar assessment. Each run with the Bifrost-simulator covered a 565-year period with 2006 as the starting year. Biomass estimates of capelin and cod respectively correspond to total autumn stock and total stock on January 1.

Strategy	Capelin stock		Cod stock	
	Mean	SD	Mean	SD
A, $\delta = 0.01$	0.104588139	0.211689948	2.630227166	0.534135135
A, $\delta = 0.05$	0.105859949	0.212403585	2.517932311	0.453666864
B, $\delta = 0.01$	2.425595884	1.917156855	1.977312934	1.179767557
B, $\delta = 0.05$	1.114681689	0.683799163	3.195746753	0.640288498
C_1	2.170154644	1.594883362	1.805945447	0.624974832
C_2	4.372569005	2.521841766	0.466860171	0.409037602

Strategy	Capelin TAC		Cod TAC	
	Mean	SD	Mean	SD
A, $\delta = 0.01$	0.02411249	0.010830534	0.595820109	0.098188791
A, $\delta = 0.05$	0.024825422	0.01275787	0.580031794	0.088291182
B, $\delta = 0.01$	0.041258541	0.108957671	0.53026835	0.320719587
B, $\delta = 0.05$	0	0	0.682807604	0.085331068
C_1	0.150639218	0.407289171	0.539808212	0.232475843
C_2	1.157060003	0.993375262	0.181215568	0.169439547

appearing after the year 2120. To understand these sudden changes in the stock sizes one needs to remember that Bifrost is a year class model. For Example, small recruitment of capelin in some subsequent years contemporaneously with a couple of good year classes of cod reaching maturity may lead to very high biomasses of cod and collapse of the capelin stock. Conversely, a slightly too high catch of cod may bring the cod stock to very low levels and give favorable conditions for capelin, which is the case when strategy $\tilde{C}_2$ is followed. In figures 4.11, 4.17 and 4.23 we can see that the capelin stock is generally much lower than the average and the cod stock is much higher than the average until the year 2120. To shift the average large changes must occur in the next 500 years.

The potential capelin collapse when a large cod stock interferes with strong year classes of herring and over-high capelin fishery suggests that greater attention should be paid to multi-species aspects in the management of capelin and cod. Prognostics of the size of the herring inflow to the Barents Sea basin may become important for future management. Meanwhile, an "aware" management of the capelin stock is the best approach according to the simulations in this work. Strategy $\tilde{B}_2$ gives a higher sustain-

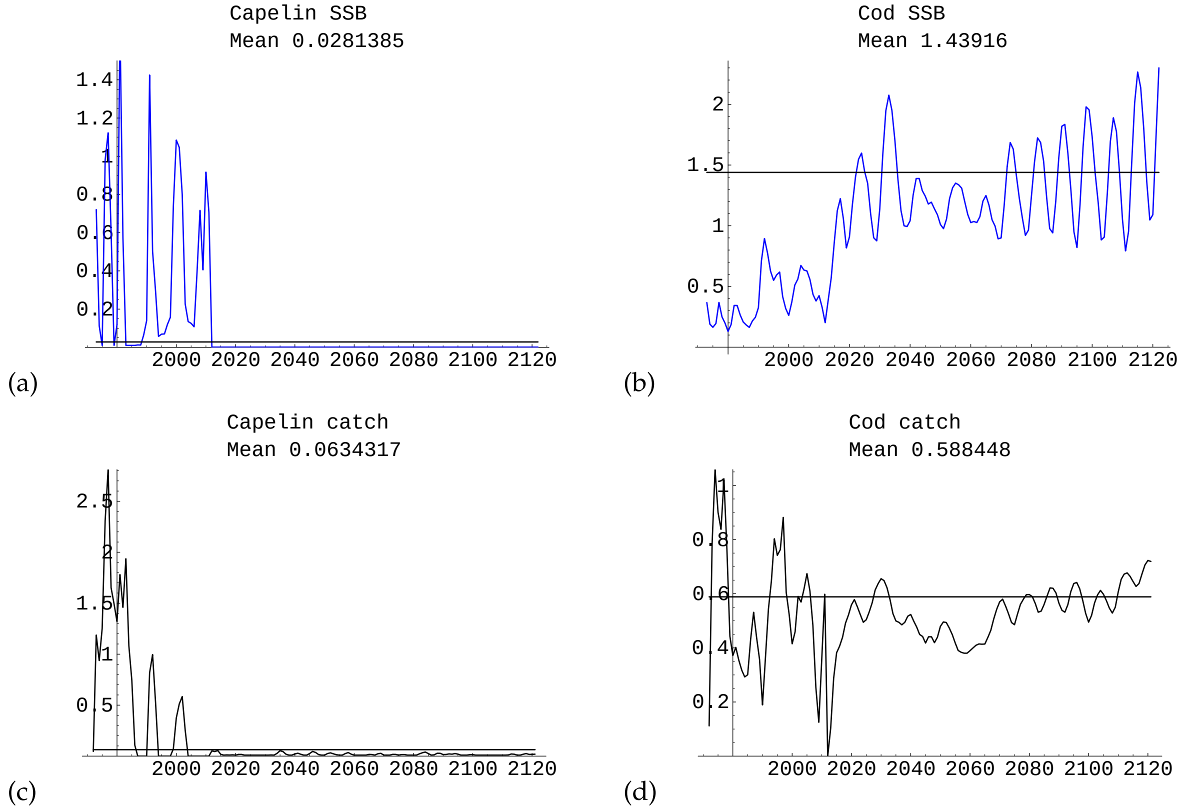


Figure 4.18: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from strategy $\tilde{A}_1$ (1% discount rate) when a dynamic Barents Sea herring stock with mean 0.5 million tons is assumed.

able cod TAC than all the other strategies in all the simulations performed. The best of the purely biologically-based strategies, strategy $\tilde{C}_1$, also gives rather good results with higher capelin TACs than the purely economic strategies. Nevertheless, when the herring stock is dynamical and we focus on mean cod TACs, this strategy is only ranked as number four with regard to the size of the cod TACs.

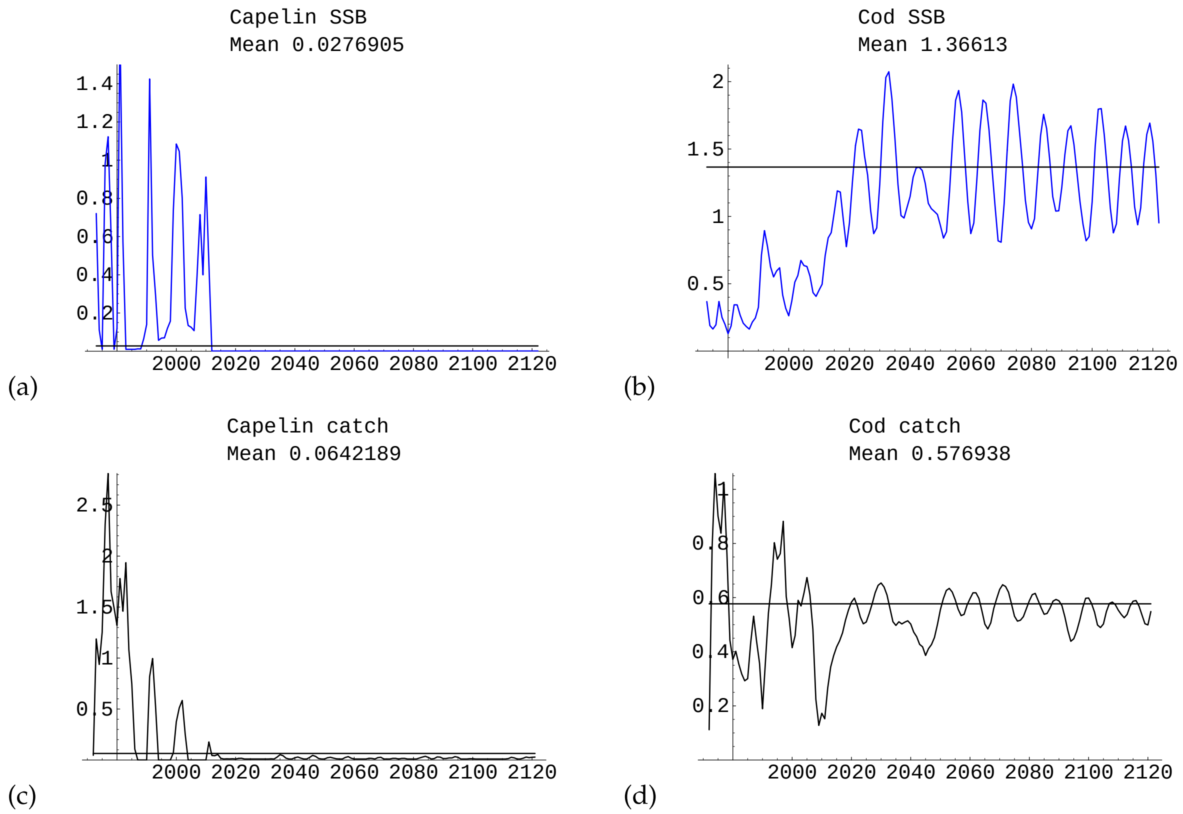


Figure 4.19: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from optimal strategy $\tilde{A}_2$ (5% discount rate) when a dynamic Barents Sea herring stock with mean 0.5 million tons is assumed.

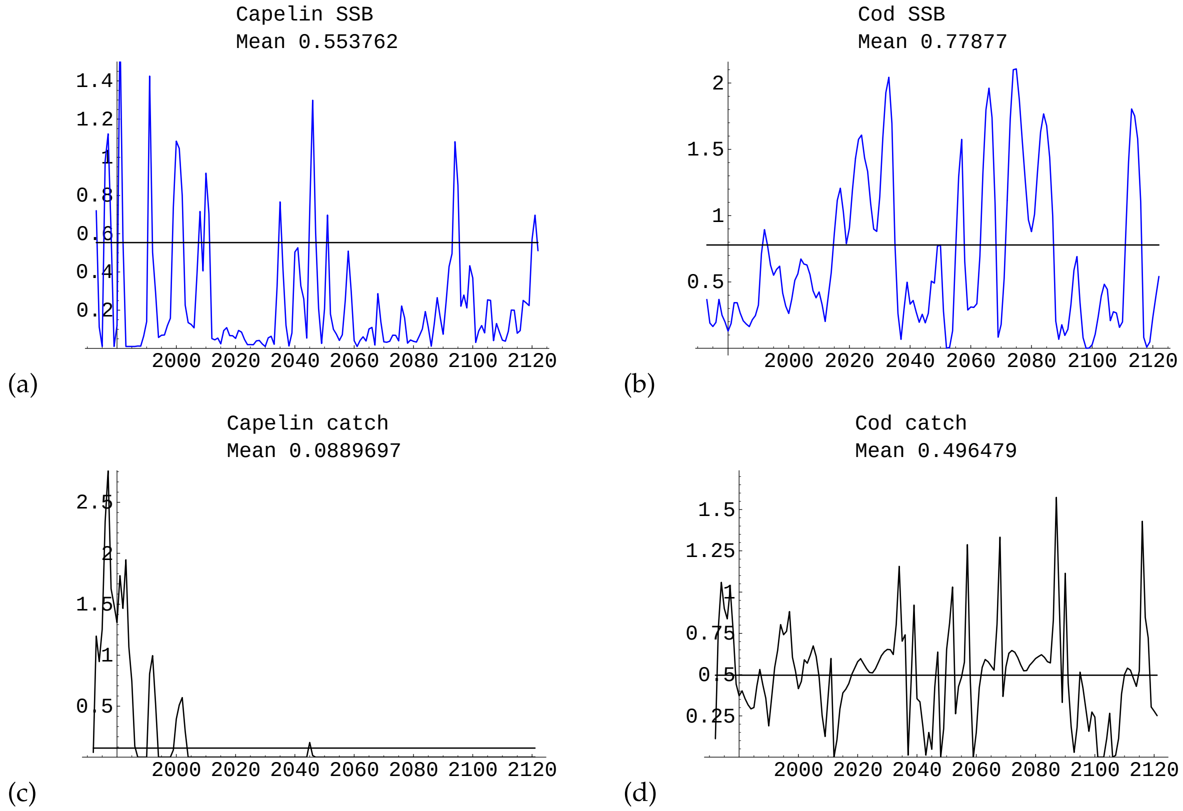


Figure 4.20: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from strategy $\tilde{B}_1$ (1% discount rate) when a dynamic Barents Sea herring stock with mean 0.5 million tons is assumed.

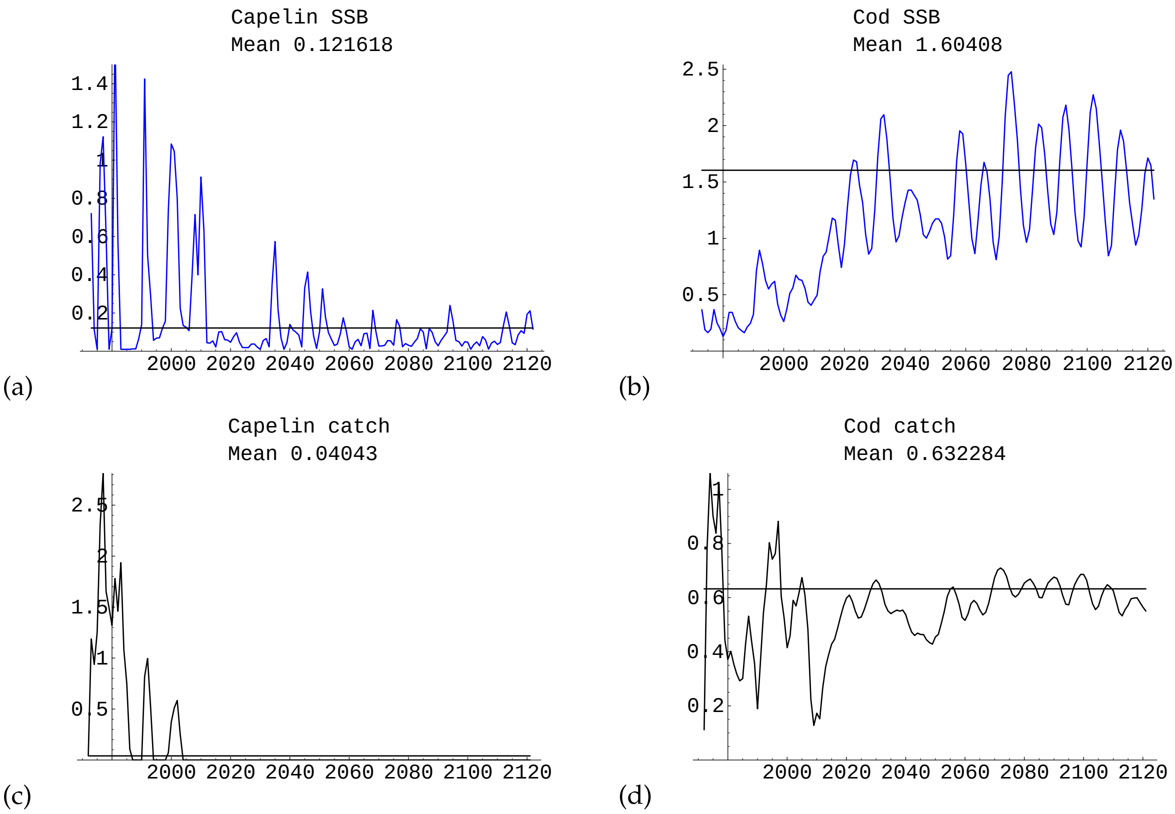


Figure 4.21: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from optimal strategy $\tilde{B}_2$ (5% discount rate) when a dynamic Barents Sea herring stock with mean 0.5 million tons is assumed.

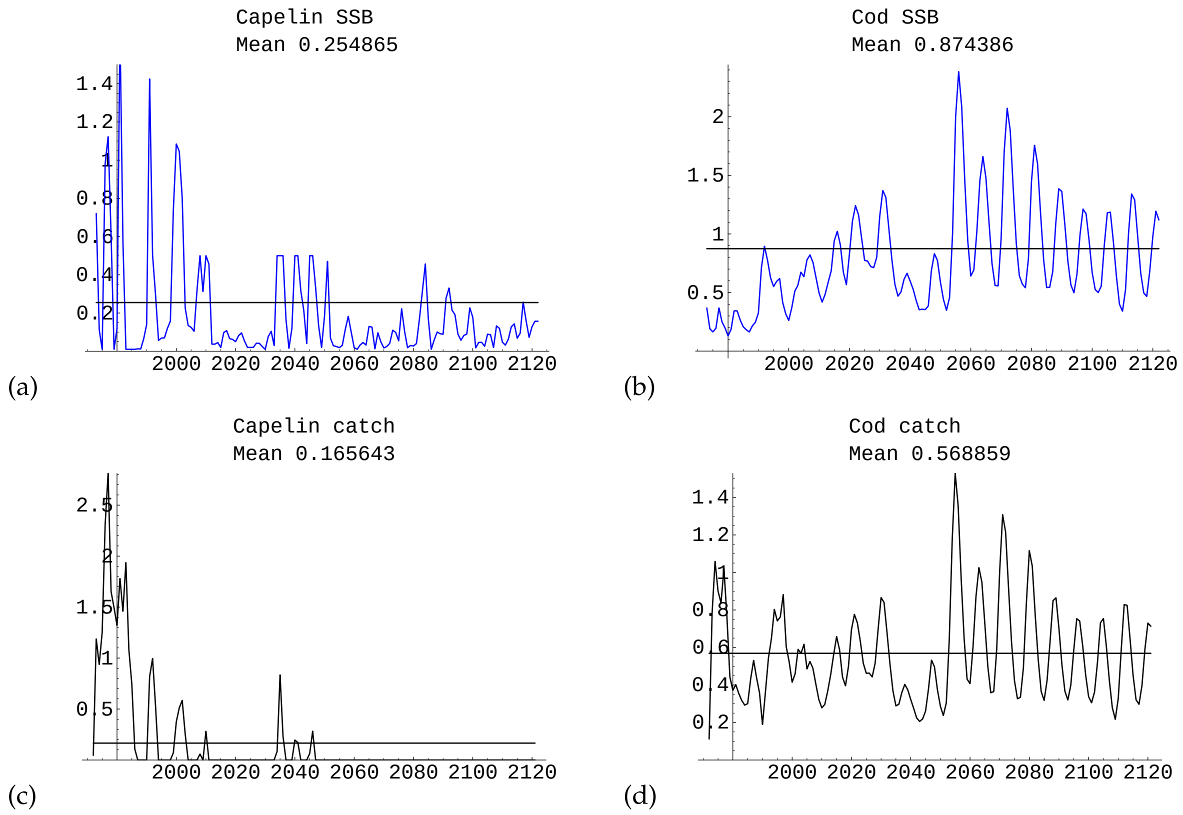


Figure 4.22: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from biologically based strategy $\tilde{C}_1$. Relative F on cod $= 0.65$. Target SSB capelin $= 0.5$. Dynamic Barents Sea capelin stock with mean 0.5 million tons.

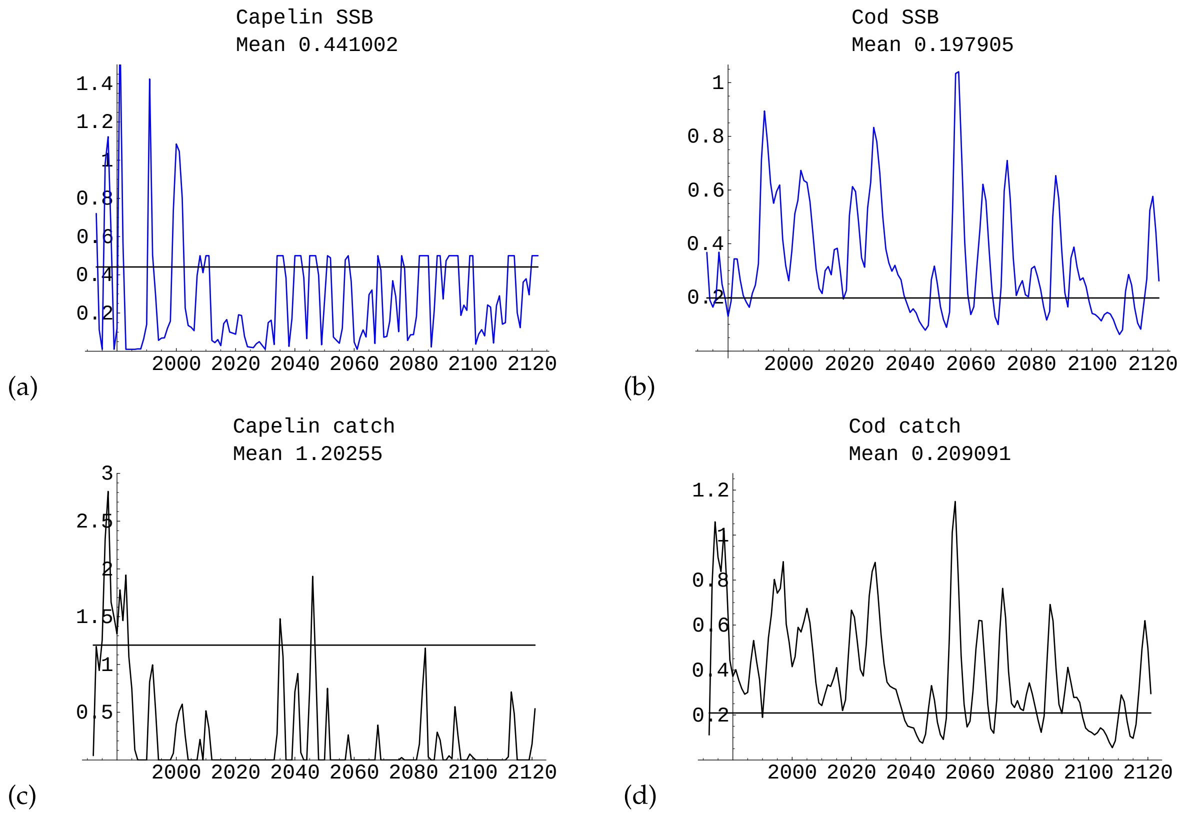


Figure 4.23: Spawning stock biomass (SSB) and catch prognostics for capelin and cod resulting from biologically-based strategy. Relative F on cod $= 1.0$. Target SSB capelin $= 0.5$. Dynamic Barents Sea herring stock with mean 0.5 million tons.

4.6 Conclusions

In this work a feedback model for optimal management of the joint Norwegian resources of capelin and cod in the Barents Sea is presented and tested. The management is based on purely economic objectives, and the simulated scenarios test the biological consequences of adopting the proposed management as prevailing harvest control. The question is whether the management model is sustainable from a biological perspective.

Four feedback strategies are tested and compared with two different reference strategies, both revealed by existing management. The feedback strategies differ in discount rate and capelin-awareness, and the reference strategies differ in the mortality rate of the cod stock. All the strategies are tested under different conditions. Constant herring influence to the capelin-cod system gives other results than a dynamical herring influence simulated from the herring assessment model SeaStar. On both occasions the results from the feedback-management are promising. Larger long-term mean TACs of cod than those resulting from the best reference strategy are found in the best feedback strategies. Because of rather high cod stock biomasses following these strategies, the capelin stock gets a little lower, but does not become extinct. The stocks also tend to have a lower variability (lower standard deviation) in the simulations with the feedback-management.

The simulations assuming a dynamical herring stock give quite different results from those assuming constant herring. This suggests that the three-species perspective (capelin-cod and herring) is important. Further investigations should be carried out concerning multi-species management of capelin and cod, and integrating herring prognostics in the management is a natural extension of the management model tested in this work.

The capelin TAC chosen in the second strategy differs from the first strategy only for low levels of capelin and high levels of cod. Specifically, the difference is that the ridge in the TACs of capelin for low levels of the stock is removed. (See figures 4.5(a) and 4.5(b)). This results in a more aware TAC-setting for capelin. The fishery will only be open for either very high stocks of capelin or for combinations of low cod stocks and rather high capelin stocks. Moreover, the discount rate has little influence on the capelin TACs, but with high discount rate (5%) the fishery is opened for a little lower stocks.

Economical perspectives are not discussed in this work since the purpose is to test biological consequences of economical based management. The conclusions about lower variations in the stocks and a higher cod stock with the economical based strategies suggest, however, that these strategies are superior to the reference strategies as far as economy is concerned. Between capelin and cod, cod is the economically most important species by far. Moreover, in theory and also according to the economical based management model, a higher cod stock increase the efficiency in the cod fishery as it reduce the cost per unit catch.

Nomenclature

Parameters and Variables

x and y	Stock levels of capelin and cod respectively
u and v	Catch levels of capelin and cod respectively
δ	Discount rate
α_1 and α_2	Norwegian shares of TAC for capelin and cod respectively
h	Discrete time jump
β	Parameter in the discrete HJB equation
c_1, c_2, c_3 and c_4	Cost components in the profit function
a_1, a_2 and a_3	Price and income components in the profit function
b_1, b_2, b_3, b_4, b_5 and b_6	Growth functions parameters

Functions

$V(x,y)$	The optimal value function
$\Pi(x,y,u,v)$	Current profit
$f_i(x,y)$	Biological growth function for species i
$CaP(u_1)$	Capelin profit
$CoP(x_2,u_2)$	Cod profit
$P_2(u)$	Price for a unit of cod

4.7 References

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Chapter 5

Essay IV

Abstract

This paper presents a bioeconomic model for the Barents Sea capelin and cod fisheries including juvenile herring in the biological model, as the young herring influence the cod-capelin system. The objective of the study is to reveal new perspectives on management of capelin and cod given by a multi-species angle of attack. Balancing the model-complexity of biology and economics is stressed in the investigation of deterministic optimal catch strategies. Optimality is measured as the maximum economic rent in the fishery.

The three species, capelin, cod and juvenile herring, constitute a very dynamic system, also because prey-predation relations are functions of ages within each stock. The biological model relates total biomass of each species through growth functions that include predation effects. The growth functions are based on assessment data from the Institute of Marine Research in Bergen, Norway, and the estimation is done by multivariate regression analysis assuming logistic like growth. Optimal fishing strategies are defined by employing a numerical feedback rule for optimal fishing through dynamic programming. The feedback rule suggests that previous TAC (total allowable catch) levels on average have been too large for both capelin and cod over the past 30 years, according to the management objectives assumed in the study. Moreover, presence of some herring in the system is important for the economic yield although the herring fishery is closed. This indicates that a focus only on the capelin-predator role of herring is too narrow, as herring are also an important prey for cod. Moreover, very interesting multispecies effects on the optimal catch strategy are discovered.

5.1 Introduction

The Barents Sea is one of the most productive ocean areas in the world (O'Brien et. al. 2004). It represents a highly diverse arctic ecosystem (Larsen et.al. 2001), whereof three fish species are of major commercial importance. The key species are the plankton feeders capelin (*Mallotus villosus*) and herring (*Clupea harengus*), as well as their main fish predator, North-East Arctic Cod (*Gadus morhua*, from now on *cod*). The relationships between the three species are highly dynamic, and the cod-capelin relationship is particularly important in the Barents Sea ecosystem (Bogstad, Hauge and Ulltang 1997).

Therefore we choose a growth model consisting of capelin, cod and herring as foundation for the decision-making on total allowable catches (TACs) for capelin and cod.

The herring stock has shown substantial natural fluctuations during the last century (Dragesund, Hamre and Ulltang 1980, Toresen and Østvedt 2000), and a high harvest pressure and unfavorable climatic conditions led to stock depletion around the 1970s (Engelhard and Heino 2004, Røttingen 2003). Although herring is mainly a Norwegian Sea species, the juvenile part of the stock occasionally enters the Barents Sea area. This inflow of young herring into the Barents Sea basin has a substantial impact on the system; the cod-capelin relation is particularly affected. Strong year classes of herring move into the Barents Sea by larval drift and feed in that area until they move back into the Norwegian Sea at 3 to 4 years of age (Huse, Railsback and Fernö 2002), but the stock is not harvested during its stay in the Barents Sea.

Predation from juvenile herring in the Barents Sea causes mass death of capelin fry, which has major impacts on other capelin dependent species, as at the end of the 1980s when there was a rather large cod stock (Mehl 1991) and herring had recovered from the collapse in the late 1960's. A strong year class of herring occurred in the south of the Barents Sea in 1983 to 1986, causing a capelin stock collapse in 1985 to 1986. According to Hamre (2003) this collapse led to cod starvation and had a negative impact also on other fish predators like sea birds and marine mammals. The capelin stock is the largest pelagic fish stock in the Barents Sea and also potentially the largest capelin stock in the world (Gjøsæther 1998, Gjøsæther and Bogstad 1998), and is of crucial importance for the growth of juvenile cod (Dalpadado and Bogstad 2004).

The strong dependence between the biomasses of the three species in our model calls for a management regime that is aware of it. Single-species models fail to explain the large fluctuations in the stocks, and therefore feedback strategies for optimal catch based on single-species growth functions misses the target of optimality.

This work presents a 3D (three-species) biological model where the catch is optimized on the basis of a multi-species feedback strategy. The unique element is that the feedback strategy, which maximizes economic rent from the fishery, is based on the size of all three stocks. With this model it is possible to identify multi-species effects which give greater insights than a single-species or two-species model can do. Identifying such insights is the main goal of this work.

It is not an easy task to include capelin, cod and Barents Sea herring in a limited multi-species growth model and get good statistical results. The picture is complicated by other factors such as primary production and predators not included in our model. Especially when it comes to herring, stock size and catch of the mature population further south are of great importance to the extent of new recruits in the Barents Sea. Year class dynamics within each species is also very important for the fluctuations in stock biomass. Nevertheless, limitation is essential for solvability of the bioeconomic model within the frames of numerical dynamic programming, which is the instrument of use in this article. It is very computer-intensive, and we have therefore limited ourselves to work with a three-dimensional state space (capelin, cod and herring). Although higher dimensions are possible in theory (Grüne and Semmler 2004), such a rise would result in a dramatic growth of throughput, and is therefore beyond the scope selected for this article. It would, however, be interesting as a future extension to include time-dependence as a cyclic driving force. The time-dependence could be represented by a fourth dimension of the growth model (Capelin, cod, herring and time), some of which could be put into lump stoachastic terms and mainly influence herring growth. The model would then constitute a stochastic optimal dynamic programming problem.

Very interesting multi-species perspectives concerning the optimal catch of capelin are revealed in this work. Also, applied to historical stock data, the model suggests that the catches of both capelin and cod have been much too high. Finally, the interpretation of herring as a tragedy affecting the capelin-cod system is partly challenged by the optimal feedback strategy.

5.2 Model description

This section presents the bioeconomic control model and all of its assumptions. A solution procedure is also sketched.

5.2.1 *Management model*

Our bioeconomic control model is based on a biological multi-species growth model created for this article and on an economic profit model used in a previous article (Agnarsson et al. (2008)).The system of the two models constitutes an optimal control problem, where the objective is to find optimal total Norwegian and Russian harvest levels from a Norwegian point of view when the Norwegian and Russian fraction of total harvest is pre-decided. If we assume $\underline{x}$ is stock biomasses and $\underline{u}$ is corresponding harvest, the continuous time Hamilton-Jacobi-Bellman equation with discount rate δ can be written as (see Kushner and Dupuis 2001)

$$\delta V(\underline{x}) = \max_{\underline{u}} \left\{ \Pi(\underline{x}, \underline{u}) + \nabla V(\underline{x})(\underline{F}(\underline{x}) - \underline{u}) \right\}, \tag{5.1}$$

where the optimal value function $V(\underline{x})$ represents total future discounted profit from optimal harvest and ∇ is the gradient operator. This equation describes the optimal balance between immediate harvest and saving of stocks for future exploitation, and gives necessary conditions for optimum (see Kamien and Schwarz(1991) or Bertsekas (2001)).$\Pi(\underline{x}, \underline{u})$ represents immediate profit (economic model) and $(\underline{F}(\underline{x}) - \underline{u})$ represents biological growth of the stocks. From equation (5.1) it is obvious that as far as optimality is concerned, both the biological and economic submodels play an important role in the solution. That is, we are primarily interested in determining a reasonable economical and biological balanced harvest strategy through harvest feedback policies taking the stock estimates at any particular time into consideration.

5.2.2 *Biological growth model*

The complexity of the nature can not be fully covered in a model used for management purposes, but a good model should be consistent with some of the main features of real marine life. In this article, predation and other multi-species interaction are given the greatest priority.

The three stocks in our growth model, namely capelin, cod and juvenile herring aged one to three, have historically shown a cyclic behavior, which includes several collapses in the capelin stock owing to high inflow of herring fry to the Barents Sea and good growth conditions for both cod and herring (Hamre 2003). It is reasonable to assume that there exists for all the stocks a saturation level affected by food supply, predation pressure and other external effects like marine temperatures and ocean currents. We focus on saturation and predation, and choose logistic like growth modified by predation as functional form for both capelin and cod. Furthermore, the predation reflects the well-documented fact that cod seems to prefer capelin as prey, but turns over to herring and cannibalism in years of capelin scarcity (Hamre 2004).

The growth function assumed for herring is more complicated than the capelin and cod growth. Since only juvenile herring are found in the Barents Sea, it is somewhat difficult to implement herring growth in a closed model for this area. Historical stock sizes of the Barents Sea fraction of the population are not available. Assessment estimates are available, however, for each year class of herring, and it is known that only the three- to four-year-old first year classes are found in the Barents Sea, and that those aged one to three heavily predate on capelin when overlapping between the two species takes place (Hjermann, Ottersen and Stenseth 2004). Therefore, in the growth of capelin and cod, these age classes represent herring influence. The growth of juvenile herring itself is fully explained by three factors: First, its own stock size, second, multi-species interactions with capelin and cod and, finally, external inflow of herring fry from the North Sea.

A major argument against including older herring in the model is that the emphasis should be on the joint Norwegian and Russian resources of the Barents Sea. From that perspective, to endogenize older year classes of the herring stock, which is located in the North Sea, destroys the possibility of a closed model since those stocks are shared with other partners than Russia. Moreover, it would have been difficult to find the connection between the size of the mature fraction in the south and the juvenile fraction in the north, and it should also be mentioned that the curse of dimensionality is a very convincing argument against including a fourth stock in the growth model.

Assume stock biomasses are given by the vector $\underline{x} = [x_1, x_2, x_3]$ and harvest by $\underline{u} = [u_1, u_2, u_3]$, where the indexes 1, 2 and 3 represent capelin, cod and herring respectively. The state equations with deterministic evolution are given by

$$\dot{x}_i = F_i(\underline{x}) - u_i, \quad i = 1, 2, 3 \tag{5.2}$$

where F_i is surplus growth of species i. Since herring is not exploited in the Barents Sea, $u_3 = 0$. This means that herring in principle could be removed from the model and be replaced by $\dot{x}_3 = 1$. Then, instead of being herring, x_3 is a direct measure of time. Another possibility is to let $\dot{x}_3 = F_3(x_3)$ be cyclic and measure time-dependent inflow of herring. These are good alternative forms if cyclic time-dependence is more important to the model results than predation-prey relationships with herring, and a lot of literature claims that long-term biomass cycles play an important role in the Barents Sea ecosystem (e.g. Yndestad and Stene (2002), Yndestad (2003)). Unfortunately, we did not succeed in finding such forms that fitted well with our data.

The forms of the growth functions were based on assumptions about the overall structure of the growth dynamics and on numerous statistical fitness-testing with relative growth for stock data. The growth functions are

$$\begin{aligned} F_1(\underline{x}) &= r_1 x_1 \big(1 - \frac{x_1}{k_1} - b_1(x_2 + x_3)\big) \\ F_2(\underline{x}) &= r_2 x_2 \big(1 - \frac{x_2}{k_2} + b_2(1 + x_1)\sqrt{x_3}\big) \\ F_3(\underline{x}) &= \tilde{r}_3(x_1, x_2) x_3 \big(1 - \frac{x_3}{\tilde{k}_3(x_1, x_2)} + b_3 \frac{x_1(100 + x_1 + x_2)}{1 + x_3}\big) + C, \text{ where} \end{aligned} \tag{5.3}$$

$$\tilde{r}_3(x_1, x_2) = \frac{r_3}{100 + x_1 + x_2}, \tag{5.4}$$

$$\tilde{k}_3(x_1, x_2) = k_3 \tilde{r}_3(x_1, x_2) \tag{5.5}$$

$$r_i, b_i, k_i, C > 0 \text{ for all } i.$$

Generally the growth function for each species is concave with respect to its own species. The growth functions for capelin and cod have straight forward logarithmic forms with r_1 and r_2 representing the intrinsic growth rate. The parameters k_1 and k_2 are sustainable biomass capacities when multi-species interactions with endogenized species are disregarded. Multi-species interactions adjust the sustainable biomass capacities. For capelin the capacity is reduced below k_1 according to the last term of the capelin growth ($b_1 > 0$). Also, this term implies that biomass of cod affects capelin in the same way as biomass of herring. Cod growth is improved by multi-species interactions ($b_2 > 0$). The term $(1 + x_1)\sqrt{x_3}$ implies for the first that biomass of capelin is more important to cod growth than biomass of herring. Second, it implies that with capelin extinct from the Barents sea ($x_1 = 0$), herring biomass is still beneficial for cod growth.

The growth of herring is somewhat different from that of capelin and cod. It is presented in a quasi-logarithmic form to fit with the forms used for capelin and cod in equation (5.3). An alternative form, which will be referred to later in the article in connection with the parameter estimation, is

$$F_3(\underline{x}) = x_3\big(C(7)\,\frac{1}{100 + x_1 + x_2} + C(8)\,x_3 + C(9)\,\frac{x_1}{1 + x_3}\big) + C. \tag{5.6}$$

The interpretation of $\tilde{r}_3(\cdot)$ as intrinsic growth rate and $\tilde{k}_3(\cdot)$ as carrying capacity for herring is a little debatable owing to the dependences on the biomass of capelin and cod. Let us therefore turn to (5.6). The parameter C reflects the positive inflow of herring (year class 0) from the Norwegian Sea to the Barents Sea. The fact that $F_3\big([x_1, x_2, 0]\big) = C$ implies that herring cannot become extinct. This characteristic is reasonable since mature herring is not a part of the model.

The first term of (5.6) is a growth term ($C(7) > 0$ is assumed) depending on the total biomass of the other endogen species. This growth term is reduced by large biomasses of capelin and cod. The number 100 in the denominator prevents excessive growth when capelin and cod biomasses are both low. The second term secures concavity with respect to herring if $C(8) < 0$, and the third term is a growth term representing multi-species interactions between herring and capelin (herring predation).

5.2.3 *Economic model*

The optimal value function represents the maximum discounted future profit from the fishery. If $\mathcal{U}$ is the set of admissible controls, the optimal value function is given by

$$V(\underline{x}) \equiv \max_{\underline{u} \in \mathcal{U}} \int_0^\infty \Pi\big(\underline{x}, \underline{u}, \underline{\alpha} \cdot \underline{u}\big) e^{-\delta t}\, dt, \tag{5.7}$$

under the conditions in (5.2) with $u_3 = 0$.

Here δ is the social time discount rate, $\underline{u}$ is total landings, $\underline{\alpha}$ is the Norwegian fraction of total landings and Π is the current profit of the fishery.

The formulation implies that the foreign (Russian) catch could be controlled, which is partly compatible with the joint Norwegian and Russian management of the Barents Sea fisheries. Moreover, it is assumed that there are no interactions in the market for fish, meaning that profit from cod and capelin can be added together. The same is true for the cost functions since capelin and cod are not fished jointly. Further, it is assumed that there is no economic protection on capelin; that is, landing costs are considered independent of the stock size (finding costs are neglected). Beyond that, capelin is not price-elastic, but Russian and Norwegian landings of cod are large enough to influence the price level, $P_2(u_2)$. The Norwegian profit function can be written

$$\Pi(\underline{x}, \underline{u}, \underline{\alpha u}) = CaP(\alpha_1 u_1) + CoP(x_2, u_2, \alpha_2 u_2), \text{ where} \tag{5.8}$$

$$CaP(\alpha_1 u_1) = p_1 \cdot \alpha_1 u_1 - c_1(\alpha_1 u_1)^{c_2}, \tag{5.9}$$

$$CoP(x_2, u_2, \alpha_2 u_2) = (p_2 - p_3 u_2)\alpha_2 u_2 - \frac{c_3}{x_2}(\alpha_2 u_2)^{c_4}, \tag{5.10}$$

and all the coeffisients $\underline{\alpha}, p_1, p_2, p_3, c_1, c_2, c_3$ and $c_4 > 0$.

5.2.4 *Parameter Estimation*

The parameters of the growth, functions (5.3), (5.4) and (5.5), are fitted to the discrete relative growth

$$\underline{Y}(t) \equiv \big(\underline{x}(t+1) - \underline{x}(t) - \underline{u}(t)\big)/\underline{x}(t), \tag{5.11}$$

which is given by historical data from 1973 to 2005 (see Table 5.4 in Appendix 5.6). These data are assumed stationary, but this cannot be confirmed or disconfirmed with the amount of data available. If, however, the stationarity assumption is true co-integration and unit root are not a problem.

The residuals or error terms from the data fitting are given by $\underline{R}_t = \big|\underline{Y}(t) - \underline{F}(\underline{x}(t))/\underline{x}\big|$. Parameters for the three growth functions are found simultaneously with the *Full information maximum likelihood method* to minimize the sum of residuals $\sum_{t=1973}^{2005} R_t$. The residuals are plotted in Figure 5.6.

The statistical results for each of the species are listed in Table 5.2 along with the coefficients $C(1) - C(9)$. The growth functions in equation (5.3) are the same functions that are used in the regression analysis, but in equation (5.3) they are factorized in a manner that stresses their logarithmic behavior. The value of the growth function coefficients and their connections with the regression analysis coefficients are listed in Table 5.1, where $C(i), \quad i \in [1, 9]$, refers to the coefficients in Table 5.2. The parameter C in the herring growth (see equation (5.3)) representing exogenous inflow of juvenile herring is a priory given the value 50 (50 thousand tons). This value should be reasonable although C is a number that depends on the time of year that is the basis for the data used.

All coefficients have signs corresponding with the fact that cod predate both capelin and herring whereas herring predate capelin. This implies that b_1, b_2, b_3, k_1, k_2 and k_3 are greater than zero, which is in accordance with the model formulation (see equation (5.3)).

All the economic parameters apart from the discount rate $\delta = 0.05$ are collected from (Agnarsson et al. (2008)). They are summarized in Table 5.3, but for further information about the economic model refer to the original source in (Agnarsson et al (2008)).

5.2.5 *Solution procedure*

The optimal control problem defined in eq. (5.2) and (5.7) is solved by numerical dynamic programming. The solution is found on feedback form, which implies that optimal catches of capelin and cod are functions of stock sizes only, and we can write the optimal harvest policy as $\underline{u}^\star \equiv \tilde{u}^\star(\underline{x})$. The problem, however, is not algebraically solvable, and the solution must be obtained by numerical techniques.

The main procedure used to find the optimal policy starts out as described in Grüne and Semmler (2004), but the approach differs somewhat. A discrete representation of the continuous-time problem in (5.2) and (5.7) could be expressed by a discrete first-order approximation given by

$$V_h(\underline{x}) \approx \max_{u} h \sum_{i=0}^{\infty} \beta^i \Pi(\underline{x}_h(i), u(i)) \tag{5.12}$$

$$\underline{x}_h(0) = \underline{x}, \quad \underline{x}_h(i+1) = \underline{\varphi}(\underline{x}, \underline{u}) \equiv \underline{x}_h(i) + \big(\underline{F}\big(\underline{x}_h(i)\big) - \underline{u}(i)\big)h, \tag{5.13}$$

where $0 < h << 1$ is the discrete time step and $\beta = 1 - \delta h$ is the corresponding approximation to the discrete discount rate as shown by Grüne and Semmler (2004).

By inserting (5.13) into (5.12) and extracting profit of the first period from the summation sign, we get the discrete version of the Hamilton-Jacobi-Bellman equation

$$V_h(\underline{x}) = \max_{\underline{u}} \Big\{ h\Pi(\underline{x}, \underline{u}) + \beta V_h\big(\varphi(\underline{x}, \underline{u})\big) \Big\}. \tag{5.14}$$

Optimal value is obtained by solving this equation for every possible stock combination on a selected stock-grid.

Now we define the linear operator

$$L(u)(V_h) \equiv h\Pi(\underline{x}, \underline{u}) + \beta V_h\big(\varphi(\underline{x}, \underline{u})\big), \tag{5.15}$$

and the dynamic programming operator

$$T_h(V_h)(\underline{x}) \equiv \max_{u \in \mathcal{U}} \big\{ L(u)(V_h) \big\}. \tag{5.16}$$

V_h being the solution to the fixed-point equation

$$V_h(\underline{x}) = T_h(V_h)(\underline{x}). \tag{5.17}$$

We use the dynamic programming operator (5.16) to solve the Hamilton-Jacobi-Bellman equation (5.14) with fixed point iteration starting with the initial value $V(\underline{x}_0) = 0$, where $\underline{x}_0$ is the $\underline{x}-$grid chosen. To speed up convergence we switch from this policy iteration to less numerically expensive value iterations. In the value-iterations we fix the policy, $\underline{u}^f(\underline{x})$, and therefore also the profit $\Pi(\underline{x}, \underline{u}^f)$ when using the linear value-iteration operator (5.15). When the value-iterations

$$V_h^{n+1} = L(u^f)V_h^n \tag{5.18}$$

stabilize, we shift to policy-iterations in accordance with equation (5.16), before returning again to value-iteration. The alternation between value and policy iteration continues until convergence.

Utilization of first-order conditions

The Hamilton-Jacobi-Belmann equation (5.14) can be solved in various ways, but in three-dimensional space through-put and computer-time become critical points. In the iterative process there are only two limitations in policy space. The feasible policy, that is the harvest on each species, is non-negative and no higher than the size available. Instead of working with a discrete control space, $\tilde{u} = (\underline{\tilde{u}}_0, \underline{\tilde{u}}_2, \cdots, \underline{\tilde{u}}_n)$, and for each fixed-point iteration doing a crude search for the "argmax" to (5.16) among all of the $\underline{\tilde{u}}_i$'s, we check only boundary values and the inner point solution to equation (5.16) when $V_h\varphi(\underline{x}, \underline{u})$ is replaced by its first-order Taylor approximation to reduce the number of terms.

$$V_h\big(\varphi(\underline{x}, \underline{u})\big) \approx V_h(\underline{x}) + (\nabla V_h)^T(\underline{x})(\underline{F}(\underline{x}) - \underline{u})h. \tag{5.19}$$

That is, inserting (5.19) into (5.16) reduces the problem to

$$V_h(\underline{x}) = \frac{h}{1-\beta} \max_{\underline{u}} \left\{ \Pi(\underline{x}, \underline{u}) + \beta(\nabla V_h)^T(\underline{x})(\underline{F}(\underline{x}) - \underline{u}) \right\}, \tag{5.20}$$

which may be tested for three control values only, namely the lower bound $\underline{u} = \underline{0}$, the upper bound and the interior solution solving

$$\frac{\partial \Pi(\underline{x}, \underline{u})}{\partial \underline{u}} - \beta \nabla V = \underline{0} \tag{5.21}$$

with respect to $\underline{u}$. Optimal $\underline{u}$-value in each fixed point iteration step will be the one that gives the highest value to (5.16).

5.3 Results

In this section a brief description of the results will be given.

5.3.1 *The optimal value function*

The optimal value function, $V(\underline{x})$, is a theoretical function relating the present situation of the biological state space (biomass of capelin, cod and herring) to current value of all future profit from the fishery. This function is used to deduce optimal TAC levels in the management model. The estimated size of the function, however, being a measure of the total future value of the fishery as a function of stock sizes, also gives some interesting interpretations. Since a simultaneous graphical presentation of this function on the whole capelin, cod and herring stock range would imply a four-dimensional figure, which is hard to depict, it is presented with one of the stocks fixed in Figure (5.2(a)-5.2(d)). This means that only three dimensions of the four-dimensional value functions are presented in the figures.

The shapes of the figures verify this rather intuitive result. Absence of stocks of both capelin and cod means zero profit, and a rise in one or both of them means a rise in future profit and therefore also rises in the value function. The value function grows less, however, for extreme values of the stocks.

Since extinction of either capelin or cod means that the extinct stock will never return, it also implies a low optimal value function. If either capelin or cod is depleted, there is a dramatic fall in the value function. On the surfaces in Figure 5.2(a) - 5.2(d) this can be clearly observed, as the edges are very steep from zero stock value to the first non-zero value along both the capelin axis and the cod axis.

According to the growth model, herring predate capelin. The economic trace of this effect is interesting to observe in the cod/herring grid of Figure 5.2(d). Without cod (along the herring axis) the optimal value function approaches zero for high levels of herring. With some cod in the model the influence from herring is also clearly negative for the optimal value function, but this trend declines when the cod biomass is high. Figure 5.2(c) shows minor influence from herring when the cod stock is 3.5 million tons. Owing to the large scale of the Z-axis in this figure it is impossible to see the exact influence from herring. A smaller scale, e.g. $2.7 \cdot 10^6$ - $2.9 \cdot 10^6$, would have shown positive influence from herring when the herring stock is not too high and the capelin stock is high.

5.3.2 *Optimal Catches*

In this section we focus on total Norwegian and Russian landings of capelin and cod. The optimality condition maximizes discounted Norwegian profit from the fishery when the Norwegian fraction of the total quota is $\alpha_1 = 0.6$ and $\alpha_2 = 0.5$ for capelin and cod respectively. New TACs are decided once a year.

Optimal TAC levels for capelin are presented for several different initial conditions in Figure 5.3(a)-5.3(f). The main trend is that optimal TAC of capelin grows with the capelin stock, but the surfaces also show clear dependence on the levels of cod and juvenile herring. In Figure 5.3(a) we observe that TACs of capelin fall slightly with growing cod stock for low levels of cod. There is even a small discontinuity between zero-level of cod and the first non-zero level. This is a discontinuity between "bliss", which is the TAC level that maximizes current profit, and lower TAC levels in the presence of cod. The discontinuity becomes more explicit with more herring present, as in Figure 5.3(b). The trend that more cod means lower capelin TACs is only confirmed for low levels of cod in that figure. For very high levels of cod it might be more valuable to catch more of the capelin instead of leaving them for cod and herring predation.

The optimal TAC levels for capelin when cod are extinct show a very interesting pattern. When there is much capelin the herring stock alone is no threat to the capelin stock and capelin should be harvested at bliss level. When the capelin stock is low, however, and the herring stock is high, the TAC should be lower or set to zero (see Figure 5.3(c)). The exception is when the capelin stock is below one million tons at the same time as the herring stock is above two million tons. Then capelin should be harvested even though the stock is low. (See the ridge in the TAC for capelin when the stock is low.) This pattern is very interesting, specially since it disappears when the cod stock is high (see Figure 5.3(e) and 5.3(f)). When the cod stock is very low, e.g 150 000 tons as in Figure 5.3(d), the ridge is still visible, but it has been displaced towards higher herring biomasses.

There is an intuitive correlation between the cod biomass and the optimal catch of cod, since cod is the top predator in the model. This intuitive correlation is confirmed in Figure 5.4(a)-5.4(d). The sizes of the capelin and herring stocks do not seem to be very important for optimal TAC of cod. Only in the absence of capelin are there interesting changes. The optimal TAC of cod then gets lower. This is visible in both in Figure 5.4(a) and 5.4(b)). An even more interesting observation is that a very low level of capelin combined with high levels of both herring and cod gives a small peak in the optimal TAC of cod. This peak is visible as a slim ridge in the upper left of Figure 5.4(b).

Optimal catch versus historic catch

In Figure 5.5(a) and 5.5(b) optimal catch (given historical stock data) of capelin and cod respectively is plotted together with actual catch for the period 1973-2005. The capelin curves reveal a dramatic overfishing from the beginning of the period until the closure of the capelin fishery in 1987. In later years the fishery of capelin has been more in accordance with optimal TACs.

The results for cod are similar. The actual harvest has been dramatically higher than what is optimal according to these simulations.

For both the capelin and cod catches it is obvious that variations in the optimal TACs are much lower than the variation in the historic catches. Moreover, the optimal TACs show a tendency to forestall the historical catch within a year or two.

Given the observations in Figure 5.5(a) and 5.5(b) it is obvious that the historical biomasses of cod from 1973 onwards would have been much higher with optimal management. Figure 5.6(a)-5.6(d) show how past stock biomass development and catch could have been given use of the optimal control rule and trust on the biological growth model (see equation 5.3). As expected, Figure 5.6(c) paints a picture of a much higher cod stock than the last 35 years have shown. As far as the capelin stock is concerned, the optimal management would have given a biomass between three and five million tons, which is rather high given the high cod stocks in the optimal scenario. (See Figure 5.6(a).) The falling trend of capelin must be seen in the light of the very high stocks in 1973, which was probably a result of the herring collapse a few years earlier.

The optimal catch of capelin is rather low compared with the actual catch between 1973 and the mid-eighties, but after that the optimal average is bigger. (See Figure 5.6(b).) For cod, the optimal catch becomes higher than the actual catch in 1978. There are also, however, two periods after 1978 when the actual catch is bigger but owing to much lower biomasses in the sea the cost corresponding to actual catch is much higher than it would have been with optimal catches. For the period as a whole (1973-2005), assuming biological growth in accordance with equation (5.3) and use of the optimal feedback rule, average actual catch of cod would have been almost 13 % lower than historic numbers, but the average biomass would have been more than double the historical average.

5.3.3 *Paths towards equilibrium*

Our deterministic growth model with optimal management implies that the stocks will move towards equilibrium levels. These theoretical levels have not been observed in the real marine ecosystem, and there are several reasons for this discrepancy. A lot of varying factors and stochasticity cause huge fluctuations in the stocks. Moreover, the management system in place has not managed the stocks according to our model and hence it has created a different path. Yet, paths towards equilibrium may be interesting from a theoretical point of view, and it is possible that the real biomasses would have fluctuated around these paths with management governed by our prescribed harvest policy.

The paths towards equilibrium resulting from a total closure of the capelin and cod fishery may tell us much about the biological growth model used. In the 2D curves of Figure 5.7(a) - 5.7(c), where in each figure one of the stocks is suppressed, we can see the three equilibriums of the biological model. Note that the external inflow of young herring from the Norwegian Sea means that there is no equilibrium with the absence of herring. Therefore, with initial absence of herring as in figure 5.7(a), the phase-plot shows evolution towards an interior equilibrium (all model stocks present) of $(2806, 4021, 2040)$. In addition $(0, 3192, 2666)$ is an equilibrium without capelin. Note that the equilibrium without capelin leaves cod in a poorer situation than the interior equilibrium with all stocks present.

The two equilibriums shown in figure 5.7(a) are the same two that are shown in figure 5.7(b). The only difference is that the axes have changed.

In the absence of cod there are two biological equilibriums as well, but only one of them is visible in 5.7(c). The visible equilibrium shows a very high capelin stock (more than six million tons) and a medium high herring stock. The equilibrium not shown in this figure has a very low capelin stock and a very high herring stock.

The management that optimizes the model of course gives other equilibriums and other paths towards them. Figure 5.8 is a 3D phase-plot showing paths towards equilibrium (marked with a red ring) from different initial conditions when new TACs are set each year according to the optimal management strategy. Although all the initial stock size combinations in figure 5.8 are at the edge, and also outside the edge, of a stock space that is reasonable based on historical biomass estimates, most of the combinations end up in the same interior equilibrium with a capelin stock of about 2.7 million tons, a cod stock of 3.2 million tons and a herring stock of almost two million tons. In addition to the interior equilibrium there is also an equilibrium without capelin. The cod stock is about the same for this equilibrium, but the herring stock is a little higher. The equilibrium without cod (when initial cod biomass is zero) is not shown in the figure.

In figure 5.8 we also observe over- and undershooting for all three stocks before settlement in equilibrium. The overshooting for herring is extreme for the initial condition with low level of capelin and cod and high level of herring. The curve with these initial conditions disappears out of the figure because the herring stock grows so much. It comes down again, however, and settles in the equilibrium without capelin.

Figures 5.9(a)-5.9(c), 5.10(a)-5.10(b), and 5.11(a)-5.11(c) illustrate more detailed phase plots in two dimensions when the harvest is according to the optimal control rule. In each of these figures one of the stocks is suppressed from the figure because of the dimension reduction. The suppressed stock, however, works indirectly in the development of the other two stocks.

Initial absence of capelin (see figure 5.10(a)) may give another equilibrium than the equilibrium without capelin shown in figure 5.8, where capelin is extinct owing to predation pressure and harvest. There are at least two equilibriums without capelin and with both cod and herring. The interior equilibrium in 5.10(b) is the same as the interior equilibrium in figure 5.8, namely $(2.73, 3.18, 1.98)$ million tons.

The most interesting observation from figure 5.11(a) - 5.11(c) is all the over and undershooting in the capelin-herring plane. The high capelin equilibrium of about 5.5 million tons should also be noted in figure (5.11(a). In addition, there is also another equilibrium not shown in this figure with a very high herring stock and a very low capelin stock.

The spiraling behavior of the paths towards the interior equilibrium (see Figure 5.8), means that the stocks are fluctuating before settlement in equilibrium. This fluctuation is confirmed in Figure 5.12(a)-5.12(c). These figures show evolution towards equilibrium on the time-stock plane.

5.4 Discussions

5.4.1 *Biological model*

A limitation in the model is the fact that we have to choose between an autonomous (time-independent) growth model with three species (capelin, cod and herring) or a time-dependent growth model with only two species (capelin and cod). A series of tests with regression analysis for a number of different growth functions, indicated that the data were best represented when herring were included instead of cyclic time-dependent fluctuations. It will be interesting to further investigate time-dependent fluctuations in future works, but it ought to be done in a more general stochastic setting.

The choice of the specific growth has some model-technical basis as a closed system is essential for optimizing when sizes of external factors beyond control are unknown. (Catch quotas of mature capelin, ocean currents determining inflow of herring fry to the Barents area etc...). Such factors are very difficult to quantify and recognize, and are for the time being not ready for use in this model. The growth in this article, however, is consistent with expected predation relationships and gives a better representation than a purely stochastic inflow, which cannot explain the forces of importance. At least this model aims to explain some of the variability in the herring stock, and independently of the relevance of the predation terms, the growth functions should cover for some of the stock dynamics. As far as the herring growth is concerned, the regression analysis performed indicates that the stock of capelin and cod to some extent is a proxy. On the other hand, herring can be interpreted as playing the role of a time-dependency in the capelin-cod system representing effects of both herring and other non-autonomities.

One might argue that the growth function for herring is not satisfactory from a biological point of view, as the growth is only passed for and based on the immature part of the stock in addition to predation relationships. It may be a little too much to claim that these factors are the most important for growth of juvenile herring in the marine ecosystem, but it is very difficult to quantify others.

Although we include inflow of juvenile herring in the herring growth, we disregard dependency between the size of this inflow and the size of the mature fraction of herring in the North Sea and the Norwegian Sea. That dependency must be expected to be important to herring recruitment (Hamre and Hatlebakk 1998) in spite the fact that biomass of spawners has turned out to be only a weak indicator of egg production in some populations. (Marshall et. al 1999, Fiksen and Slotte 2002.) We use a growth model, however, not a recruitment model. That is, the aim is not to describe bottom-up factors in recruitment and mortality, but rather to explain some of the top-down stock dynamics which is relevant for more realistic economical and informational optimization. Given the very high uncertainty about stock data, the error made when using a constant factor $C = 50$ (50 thousand tons) in equation (5.3) to represent external inflow of herring should not detract from the validity of the biological model.

The best argument for the biological growth model is that all the parameters from the regression analysis are reasonable. The signs of b_1, b_2 and b_3 are positive, and this is in accordance with the assumed predation pattern for all the species. (Cod predate capelin and herring whereas herring predate capelin.) Moreover, the biological carrying capacity for capelin and cod of respectively 7.89 and 3.19 million tons (see Table 5.1) makes sense. Given the extremely fluctuating nature of the system described and the high inaccuracy in the available assessment data, the general agreement between commonsense and the size of the parameters found in the regression analysis is a very strong argument for the growth model used.

Moreover, the statistical results are acceptable as herring are fitted with only three parameters. The significance of some of the parameters could have been more convincing and the residuals could have been lower, but we should remember that herring have shown very large fluctuation and faced a collapse at the beginning of the period our data are built upon. Therefore we cannot expect them to be lower.

As far as the biological equilibriums are concerned, there are several equilibriums depending on the initial stock sizes. The formulation implies that herring will never go extinct and that cod may only do that as a result of over-fishing. Too much herring and cod might be a catastrophe for capelin owing to predation. That is obvious from the equilibrium $(0, 3192, 3316)$, where capelin have died out.

An interior biological equilibrium is $(2261, 4042, 1769)$, where all three stocks are in good condition. One might argue that the cod level in this equilibrium is too low compared to historically gigantic cod levels reported, but extreme cod levels do not agree very well with the data this analysis is built upon. Also, since the cod stock has a fluctuating nature, the highest sizes reported should probably not be regarded as close to a natural no-fishing equilibrium level.

5.4.2 *The optimal value function*

If we keep in mind that the value function represents the total future discounted profit from harvest of capelin and cod, it is clear that the surfaces must go through the origin between the capelin and cod axes. Absence of both capelin and cod means zero profit. Subsequently, absence of capelin or cod in one moment of time implies that the stock is exterminated for all future, whereas small positive initial level leads to higher future levels under optimal policy. The difference in economic rent between these two situations explains the discontinuity between zero-levels of capelin and cod and the first non-zero levels.

Generally, a rise in one or both stocks means a rise in future profit and therefore rises in the value function as well. Extreme values are not sustainable, however, and if the stocks get very high, they will fall soon independently of the fishing policy and leave little extra profit. Remember also that the profit function (see eq. (5.8)) is concavely related to harvest. Consequently, extreme harvest in a limited moment of time will not result in extreme profit.

It is interesting that although herring is not harvested in the model a modest presence has a positive influence on the economics. This might not be very surprising since herring meat is on the cod-menu, but it contradicts the quite common one-track interpretation of herring as juvenile capelin predator and a tragedy to the capelin-cod system. Nevertheless, for certain initial conditions herring affects the capelin-cod system in a negative way in this model as well. With a low initial stock size of capelin, high herring biomasses may lead to capelin extinction. With low initial levels of cod the biological model gives a very high growth of herring, and, as seen from Figure 5.2(d), this gives a very poor fishery for both capelin and cod. Yet the main impression when we study the optimal value function is that herring, in spite of its predation of capelin, contributes to cod growth. Therefore, when the stock is not too big, herring has a positive influence on the profit in the model. This finding finds support in Helstad (2000), who also concludes that the rebuilding of the herring stock after its collapse in the late 1960s has been optimal from an economic point of view, when considering the three species of relevance in this article.

5.4.3 *Optimal catch policy*

From a multi-species perspective there are many interesting questions concerning optimal catch. On the fishmarket the value of cod is a lot higher than the value of capelin. Since capelin is the most important food for juvenile cod it is not obviously that capelin should be harvested at all. The important role of herring in the capelin-cod system distorts the situation even more, and according to the biological model capelin will become extinct given certain initial conditions. That is obviously not a desirable scenario. Given the worst possible initial conditions, however, it is probably not possible to prevent it in a real situation. Within the frames of this model it would have been possible, allowing for harvest of juvenile herring, but that strategy is not investigated in this work.

If the initial conditions given are of a kind that makes the extinction of capelin unpreventable, what is the best response from an economic point of view? Is it to let cod and herring alone finish their job of hunting down the capelin stock or is it to contribute to the extinction by fishing as much capelin as possible to earn at least something on the remaining stock? Given that it is assumed no finding costs (costs are independent of stock size) in the capelin fishery (see equation 5.9) the latter alternative is quite likely. Therefore it is not surprising that this alternative is confirmed in Figure 5.3(b), 5.3(c) and 5.3(d).

What is seen in these figures is that capelin is heavily harvested for certain initial conditions where the stock itself is so low that extinction is the only outcome according to the biological model. Figure 5.3(c) illustrates this very clearly. If the survival of the stock is not vulnerable (enough capelin and not too much herring) the optimal catch is rather high, but for slightly different biomass combinations where the predation pressure on capelin becomes critical, the fishery is closed. Yet, with even worse initial conditions for capelin, the fishery is open. The conclusion is that it is better to profit from the remaining capelin stock by heavy fishing than let herring and cod finish the extinction job alone. This is, however, a rather theoretical conclusion. In reality it is probably unlikely that capelin, from almost any initial conditions, could be driven to extinction by cod and herring without continuous contribution from man all the way. In this model, though, it is possible. That is mainly a result of high herring growth when both the capelin and cod stock is low. Then the term with $C(7)$ as coefficient in equation (5.6) dominates the other terms - also for rather high stocks of herring. The reason is that $C(7)$ is quite big according to the parameter estimation (see Table 5.1). This explains the high catch when there is little capelin and the stock of cod is extinct (along the capelin axis for cod$=0$) in figure 5.3(b) and it explains the high catch of capelin for very high levels of herring and very low levels of capelin in figure 5.3(d) (lower right corner).

Except for the very special features just mentioned, the findings illustrated in figure 5.3(a)-5.3(f) are pretty straight-forward. In the absence of cod there is no pressure on the capelin stock. This explains the discontinuities visible in figure 5.3(a) and 5.3(b) between the capelin catch given by zero cod stock and the catch given by the first non-zero cod stock level. Actually, absence of cod leads to capelin TACs on "bliss-level" for most levels of herring when the capelin stock is above a certain lower limit. This is shown in figure 5.3(b), where the herring stock is four million tons, which is a very high biomass level for juvenile herring.

The main trend in figure 5.3(a)-5.3(f), that optimal capelin-TACs increase with the capelin stock, is very intuitive and could be assumed without a multi-species model. The trend that optimal TAC of capelin is negatively correlated with the herring stock is also intuitive (see Figure 5.3(e) and 5.3(f)). A high herring stock leads to a high predation pressure on capelin and therefore the TAC of capelin must be kept low to maintain the stock.

Although cod biomass affects capelin growth just as much as herring according to equation (5.3), the TAC of capelin and the cod biomass is not as negatively correlated as TAC of capelin and herring biomass. To understand this we must remember that cod predates herring. A high cod stock therefore prevents herring growth, and is consequently not as critical to capelin as herring.

Cod is economically the most important species and the top predator of this model. Moreover, profit from cod is concave as a function of harvest. One should thus expect stable and high, but sustainable, optimal TACs of cod. Furthermore, a rather weak dependence of the other two species should be expected. Both expectations are confirmed in figure 5.4(a)-5.4(d). There are only a few discrepancies with these main findings. The first is that TACs of cod fall slightly in the absence of capelin. This can be observed along the cod-axis in figure 5.4(b). The reason is that the future growth conditions for cod are poorer in absence of capelin and the situation demands a lower cod harvest. The most interesting multi-species effect in the TAC of cod is the ridge visible on the left of figure 5.4(b). For large cod stocks there is a peak in the TAC on the first non-zero level of capelin. For smaller cod stocks the ridge shifts to a little "dale". The key to understanding this phenomenon is to be aware of the fact that the capelin stock is under very strong pressure when there is little capelin and very much cod and herring. What is observed is a rescue operation of the capelin stock. As can be seen from figure 5.9(c), the capelin stock might be saved for most stock combinations of capelin and cod even when the initial herring stock is above five million tons (six million tons in that figure).

Cod affects capelin growth both directly and indirectly through the growth functions in equation (5.3). For high cod stocks the direct negative effect on capelin is beyond doubt the most important. A high cod TAC that reduces the stock may therefore ease the pressure on the capelin stock. This explains the ridge in the capelin TAC when high levels of cod coincide with the first non-zero level of capelin. For low levels of cod, however, a rise in the cod stock weakens herring growth very much, according to equation (5.3). This direct effect on herring has an indirect positive effect on capelin growth, which is more important than the direct negative effect from cod. This explains the small (hardly visible) "dale" in the TAC of cod for low levels of cod and low levels of capelin.

When comparing our results with actual historical catches we find that optimal catches of cod are much lower. Similar results have been documented before (Arnason, Sandal, Steinshamn and Vestergaard 2004), (Kugarajh, Sandal and Berge 2006), and the magnitude of the discrepancy should be a wake-up call. Optimal catches of capelin are also much lower than actual historical catches, but the management of capelin has changed. Before the 1990s the fishery was open all year, but lately it has only been open in a short post-spawning period. Catches of mature capelin after spawning may not harm the capelin stock too much since a high fraction of the capelin population spawns only once (Yndestad and Stene (2002)). The capelin management seems to have improved after the collapse in the late eighties.

Although catches of both capelin and cod have been much too high, there is a clear correlation between actual catch and optimal catch. The positive correlation of 0.75 for capelin and 076 for cod is rather large. We typically observe, however, that changes in actual catch are a little delayed compared with changes in optimal catch. (See Figure 5.5(a) and 5.5(b)). This indicates that multi-species feedback rules can improve the stability of the stocks and reduce the fluctuation.

The sizes of the optimal equilibrium catch curves for capelin and cod in Figure 5.5(a) and 5.5(b) are respectively about 36 % and 34 % of actual average catches in the last thirty years. At the same time, equilibrium level yields stock sizes considerably larger than average stock sizes, especially for capelin and herring. Average stock size of capelin, cod and juvenile herring year 0-2 in the period from 1973 to 2005 is respectively 3.017, 1.522 and 1.084 (million tons), which is considerably lower than the interior equilibrium with the capelin stock of about 2.7 million tons, a cod stock of 3.2 million tons and a herring stock of almost two million tons shown in Figure 5.8. The conclusion is that lower harvest pressure on both capelin and cod would give a much higher cod stock and improve the efficiency of the cod fishery.

Given the growth functions in equation (5.3) and management by the feedback rule suggested, historical stock and catch curves would have been very different. Whereas Figure 5.5(a) and 5.5(b) shows historical optimal catch given the stock biomass assumed from Table 5.4 each year, figure 5.6(a) - 5.6(d) shows historical optimal stocks and catches given by the growth functions in (5.3) and the feedback rule. The discrepancies between historical data and how they could have looked with management from the feedback rule are enormous. The effect of the lower catch rates on both capelin and cod is that the cod stock gets much bigger whereas capelin biomass is kept rather low owing to higher predation pressure from cod. Nevertheless, the optimal biomass of capelin in Figure 5.6(a) is on average more than 10 % higher than the historical biomass. The very low optimal catch of capelin shown in both Figure 5.5(a) and 5.6(b) is a clear sign of multi-species effects. The direct costs in the capelin fishery are not very high, and therefore a single-species cost/utility analysis of the capelin fishery results in rather a high harvest and a low biomass, but when cod and herring are included in the analysis, the benefit of a more conservative capelin management is illumined.

As far as cod is concerned, low optimal catch is not a result of a multi-species effect.

It is rather a question of cost-effectiveness. Large catches give lower cod stocks and lower cod stocks increase the cost of each unit fished. The actual biomass average for the cod stock from 1973 to 2005 is only 49 % of what it could have been according to the growth equations (5.3) and the feedback rule.

5.5 Conclusions

This article is a demonstration of optimal dynamic programming as a useful technique in multi-species management modeling. So far multi-species management has been mostly concerned with stock goals, and less with economics and other important objectives utilizing information outside the biological sphere. We have tried to balance model-complexity of biology and economics in the search for optimal feedback catch-curves for capelin and cod. The aim has been to identify multi-species effects in the optimal feedback. The importance of a multi-species basis in the management of the Barents Sea capelin has certainly been demonstrated. Stocks like capelin, which are more important as prey for other species than as harvest for man, should always be managed in a multi-species context.

As far as the concrete results are concerned, the principles of the findings are intuitive. They show that optimal catch of capelin is very dependent on the size of both the cod and herring stock as well as of the capelin stock itself. Furthermore, optimal catch of cod is less dependent on capelin and herring, but a fully depleted capelin stock will result in less cod growth and should therefore also lead to lower cod landings.

Comparisons between historical catches and long-term optimal catches show that the former have been drawn much too heavily from both capelin and cod, but the management of capelin appears improved after the collapse in the late 1980s. The extent of overharvesting is almost incredible. According to the analysis, the feedback solution would have given a cod stock average that is more than twice the size of the actual average between the years 1973 and 2005.

Perhaps the most interesting observation from a comparison between historical catch curves and the feedback solution is that the feedback solutions appear to forestall future actual catches. Typically the feedback catch curves drop a couple of years before the actual catch curves are forced to drop owing to low biomasses. This is a very strong argument for management that is aware of multi-species effects.

Another important finding is that the presence of some herring in the Barents Sea system apparently gives higher economical yield from capelin and cod than the absence of herring would have done.

Our work should not be considered to be the definitive answer to the best management of the Norwegian and Russian resources of capelin and cod. Rather, it is a first attempt at combining complicated biology with economics and giving a clear-cut answer to the optimal response to the biological model and the economic models relied upon. In practical management of the fisheries, however, it is rather obvious that the kind of feedback-policy suggested in this work is better for the economy than the pure biological based existing management strategies for capelin and cod. Along with more biological ecosystem based models to test consequences of the feedback solution, a management model of this kind could be interesting to practical politics.

Nomenclature

Variables

$\underline{x}$	Stock levels
$\underline{u}$	Catch levels
δ	Discount rate
$[\alpha_1, \alpha_2]$	Norwegian share of TAC for capelin and cod

Functions

$V(\underline{x})$	The optimal value function
$\Pi(\underline{x}, \underline{u})$	Current profit
$F_i(\underline{x})$	Biological growth function for species i
$CaP(u_1)$	Capelin profit
$CoP(x_2, u_2)$	Cod profit
$P_2(u_2)$	Price for a unit of cod
$C_1(u_1)$	Cost function for catch of capelin
$C_2(x_2, u_2)$	Cost function for catch of cod

Parameters

p_1, p_2 and p_3	Income parameters
c_1, c_2, c_3 and c_4	Cost parameters

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Yndestad, H and A. Stene. 2002. "System dynamics of the Barents Sea capelin." ICES Journal of Marine Science 59:1155-1166.

Table 5.1: Coefficients for the growth functions, $C(\cdot)$, are found in the regression analysis results summarized in table 5.2.

	r_1	r_2	r_3	k_1	k_2	k_3
Formula	$r_1 = C(1)$	$r_2 = C(4)$	$r_3 = C(7)$	$k_1 = -\frac{C(1)}{C(2)}$	$k_2 = -\frac{C(4)}{C(5)}$	$k_3 = -\frac{C(7)}{C(8)}$
Value	1.8515	0.5490	1380.7	7890.1	3191.3	$1.0572 \cdot 10^7$

	b_1	b_2	b_3
Formula	$b_1 = -\frac{C(3)}{r_1}$	$b_2 = \frac{C(6)}{C(4)}$	$b_3 = \frac{C(9)}{C(7)}$
Value	$1.1142 \cdot 10^{-4}$	$2.30 \cdot 10^{-6}$	$1.87 \cdot 10^{-6}$

Table 5.2: Regression analysis of relative growth for capelin, cod and herring. The analysis is based on official stock and catch data from ICES.
Estimation Method: Full Information Maximum Likelihood (Marquardt)
Sample: 1973 2004
Included observations: 32
Total system (balanced) observations 96

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	1.851524	1.080013	1.714354	0.0865
C(2)	-0.000233	0.000210	-1.115061	0.2648
C(3)	-0.000211	0.000208	-1.015541	0.3098
C(4)	0.548985	0.131277	4.181881	0.0000
C(5)	-0.000172	7.51 E-05	-2.289525	0.0220
C(6)	1.26E-06	4.52E-07	2.794831	0.0052
C(7)	1380.662	1186.601	1.163544	0.2446
C(8)	-0.000131	0.000438	-0.298199	0.7656
C(9)	0.002591	0.005169	0.501171	0.6163
Log Likelihood		-74.99661		
Determinant Residual Covariance		0.021789		

Eq.: $Y_1(t) = C(1) + C(2)\, x_1(t) + C(3)\, (x_2(t) + x_3(t))$
Obs.: 32

R-squared	0.244569	Mean dependent var	0.578742
Adjusted R-squared	0.192470	S.D. dependent var	1.192886
S.E. of regression	1.071959	Sum squared resid	33.32379

Eq.: $Y_2(t) = C(4) + C(5)\, x_2(t) + C(6)\, x_1(t)(1 + \sqrt{x_3(t)})$
Obs.: 32

R-squared	0.462158	Mean dependent var	0.36970
Adjusted R-squared	0.425066	S.D. dependent var	0.155888
S.E. of regression	0.118201	Sum squared resid	0.405173

Eq.: $Y_3(t) = C(7)\, \frac{1}{100+x_1(t)+x_2(t)} + C(8)\, x_3(t) + C(9)\, \frac{x_1(t)}{1+x_3(t)} + \frac{50}{x_3(t)}$
Obs.: 32

R-squared	0.849223	Mean dependent var	1.004244
Adjusted R-squared	0.838825	S.D. dependent var	3.410217
S.E. of regression	1.369087	Sum squared resid	54.35754

Table 5.3: Economic parameters.

	demand parameters	cost parameters
capelin	$p_1 = 1$	$c_1 = 0.07, \quad c_2 = 1.4$
cod	$p_2 = 12.65, \quad p_3 = 0.00839$	$c_3 = 5848.1, \quad c_4 = 1.1$

Table 5.4: Offical stock and catch data from ICES. All numbers are in million kg.

Year	Capelin	Cod	herring	Catch of capelin	Catch of cod
1973	5144	2421	166	1336	793
1974	5733	2250	251	1149	1102
1975	7806	2055	196	1439	829
1976	6417	1962	215	2587	867
1977	4796	1971	196	2987	905
1978	4247	1578	185	1916	699
1979	4162	1131	287	1783	441
1980	6715	877	219	1648	380
1981	3895	993	58	1986	399
1982	3779	753	3	1760	364
1983	4230	854	65	2358	290
1984	2964	1031	94	1477	278
1985	860	1179	34	868	308
1986	120	1348	43	123	430
1987	101	1135	77	0	523
1988	428	922	553	0	435
1989	864	933	1420	0	332
1990	5831	1081	1479	0	212
1991	7287	1776	774	929	319
1992	5150	2110	1145	1123	513
1993	796	2462	3159	586	582
1994	200	2240	3179	0	771
1995	193	1909	1331	0	740
1996	503	1795	95	0	732
1997	911	1637	172	1	762
1998	2056	1300	196	1	593
1999	2776	1200	468	105	485
2000	4273	1183	789	410	415
2001	3630	1448	1375	575	426
2002	2210	1649	5803	659	535
2003	533	1695	6370	282	552
2004	628	1700	3404	0	606
2005	324	1634	1970	1	641

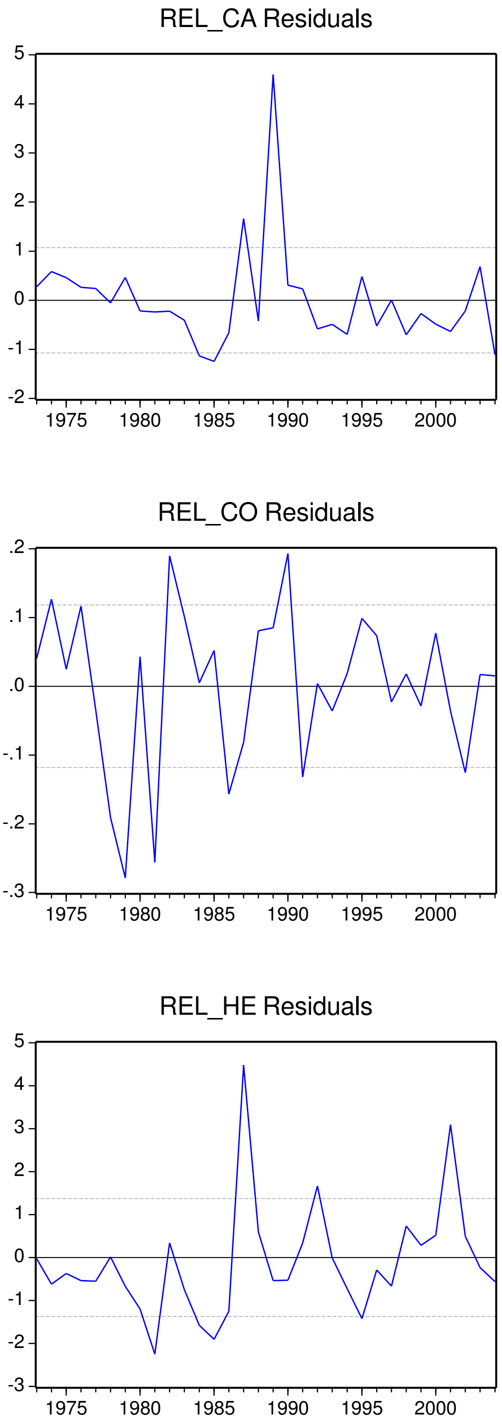
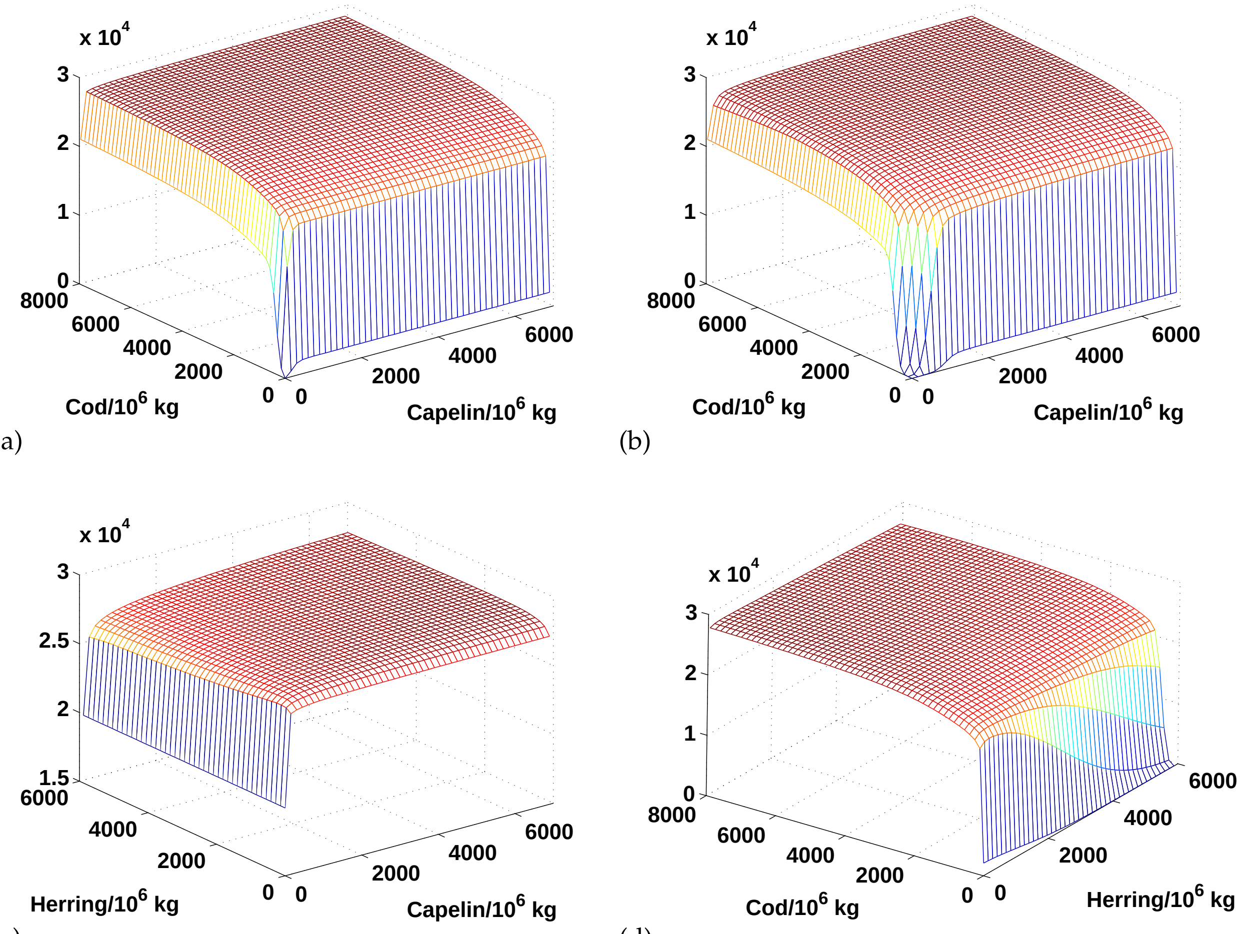


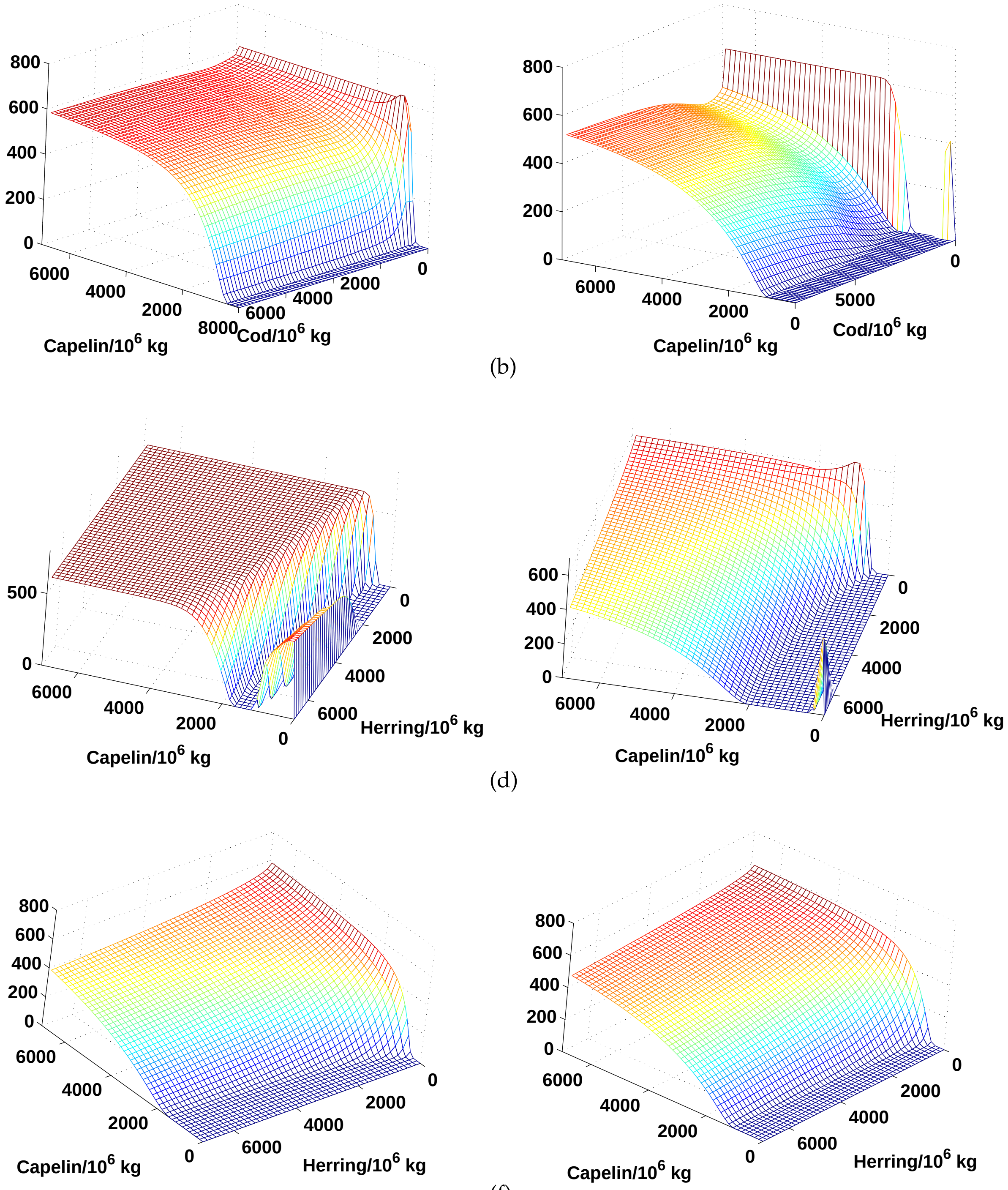
Figure 5.1: The residuals (error terms) of the biological fitting.



(a) (b)

(c) (d)

Figure 5.2: The optimal value function (a) with low herring level (280 000 tons), (b) with 4 nillion tons of juvenile herring, (c) with 3.5 million tons of cod and (d) with 1/2 million tons of capelin present.



(a) (b) (c) (d) (e) (f)

Figure 5.3: Optimal TACs (thousand tons) of capelin. Juvenile herring is fixed in figure (a)-(b) and in figure (c)-(f) cod is fixed. (a) 280 000 tons of herring, (b) 4 million tons of herring, (c) absence of cod, (d) 150 000 tons of cod, (e) 1.5 million tons of cod and (f) 4.5 million tons of cod.

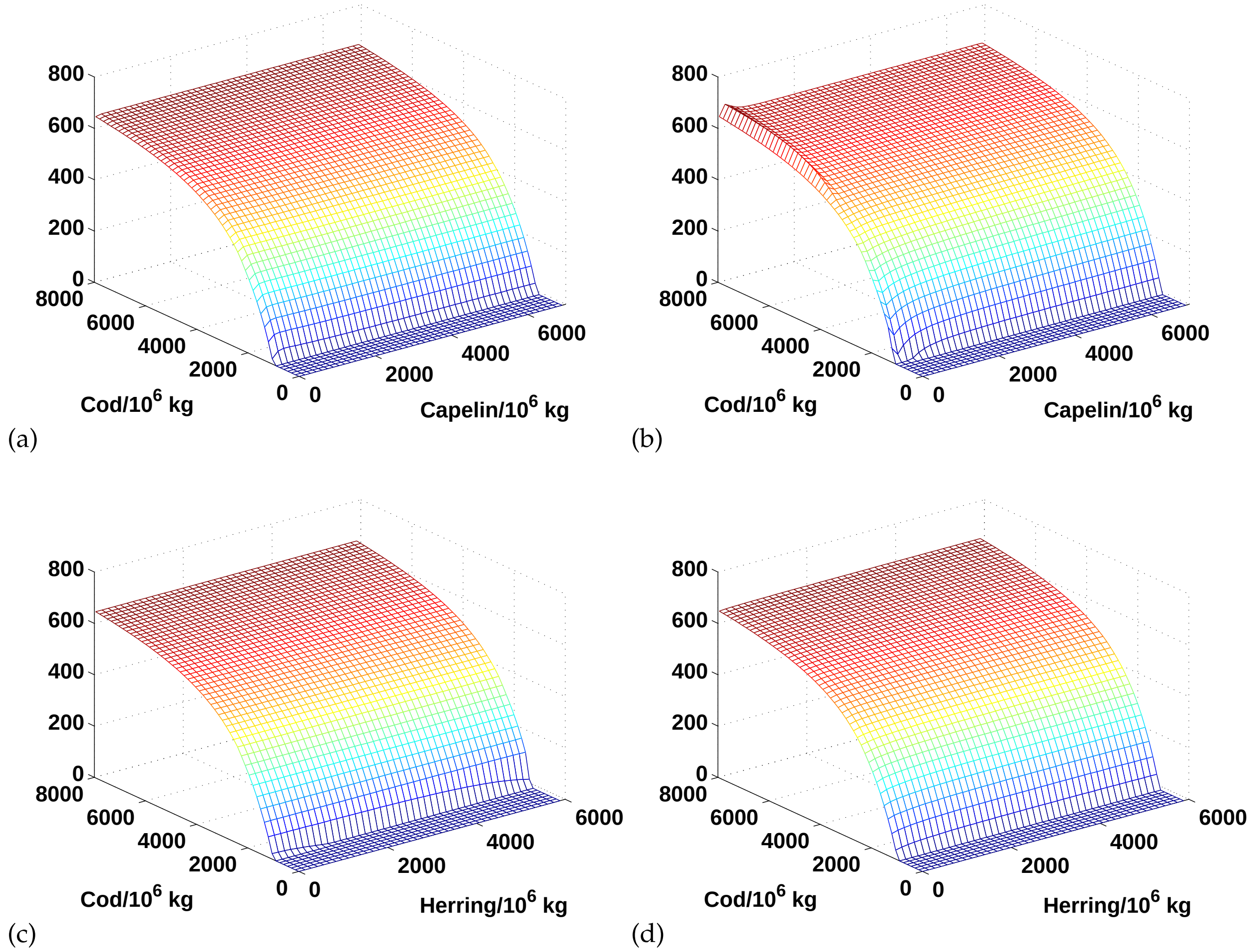
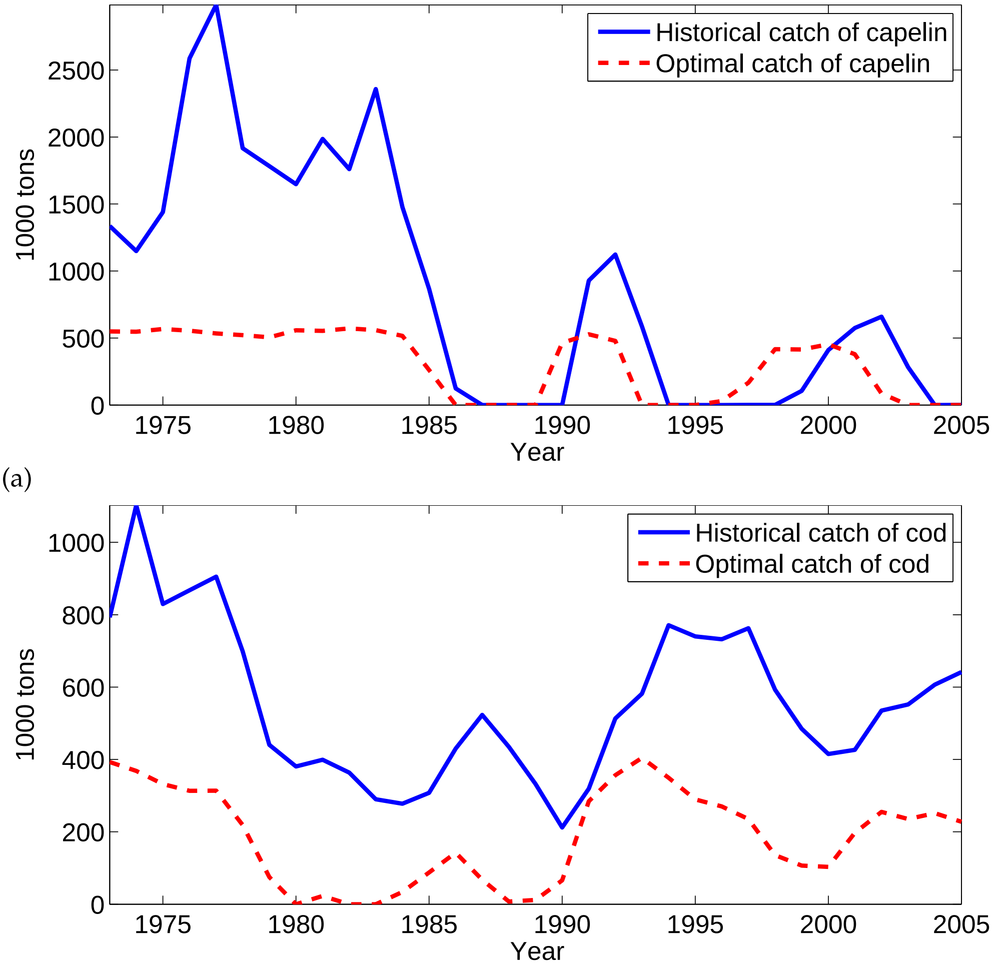


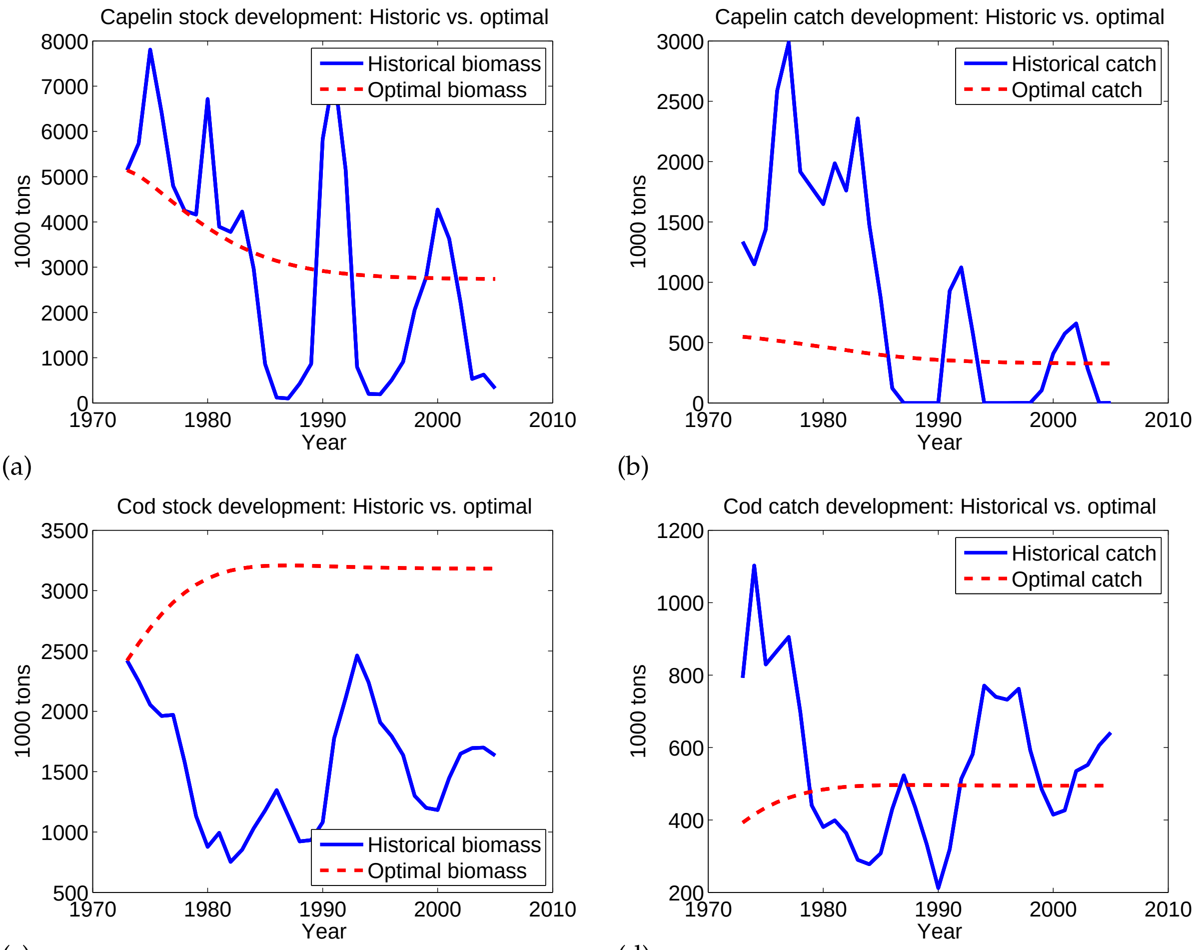
Figure 5.4: Optimal TACs (thousand tons) of cod as a function of capelin and cod with (a) 280 000 tons herring present, with (b) 5 million tons of juvenile herring present, and optimal catch of cod as a function of cod and herring under (c) absence of capelin and with (d) 3 million tons of capelin present.



(a)

(b)

Figure 5.5: Historical catches vs. optimal catches given historical biomasses for (a) Capelin and (b) Cod.



(a) (b) (c) (d)

Figure 5.6: Historical biomasses and catches vs. how they could have been with optimal management. (a) Capelin stock, (b) Capelin catch, (c) Cod stock and (d) Cod catch.

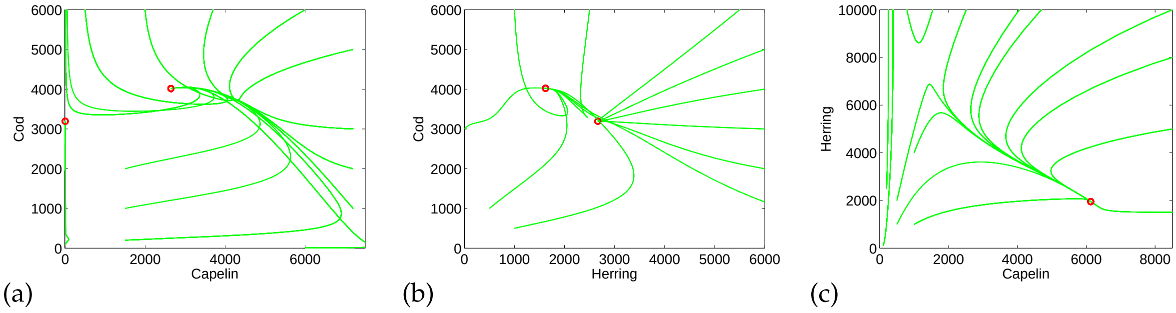


Figure 5.7: Paths towards the biological equilibriums (red rings) resulting from a total closure of the capelin and cod fishery. (a) Initial absence of herring, (b) absence of capelin and (c) absence of cod.

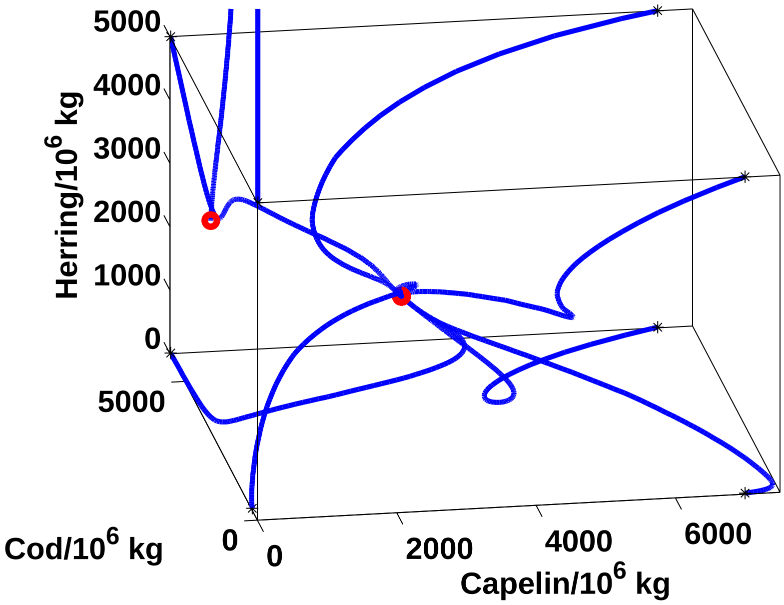


Figure 5.8: Paths towards equilibrium in the 3D capelin-cod-herring plane for different initial stock sizes of capelin, cod and herring. Equilibrium marked with a red ring, and initial positions marked with "$*$".

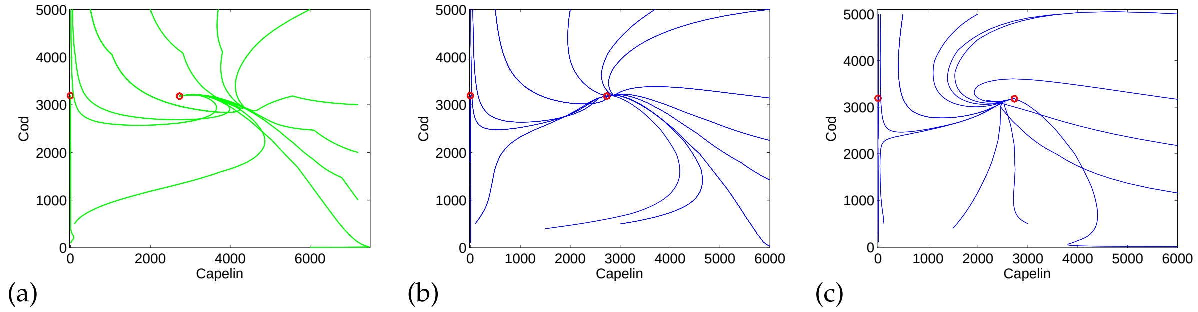


Figure 5.9: Paths towards equilibrium stock sizes on the capelin-cod plane for different initial conditions of capelin, cod and herring. (a) Absence of herring and initial herring stock of (b) 1.5 million tons and (c) 6 million tons.

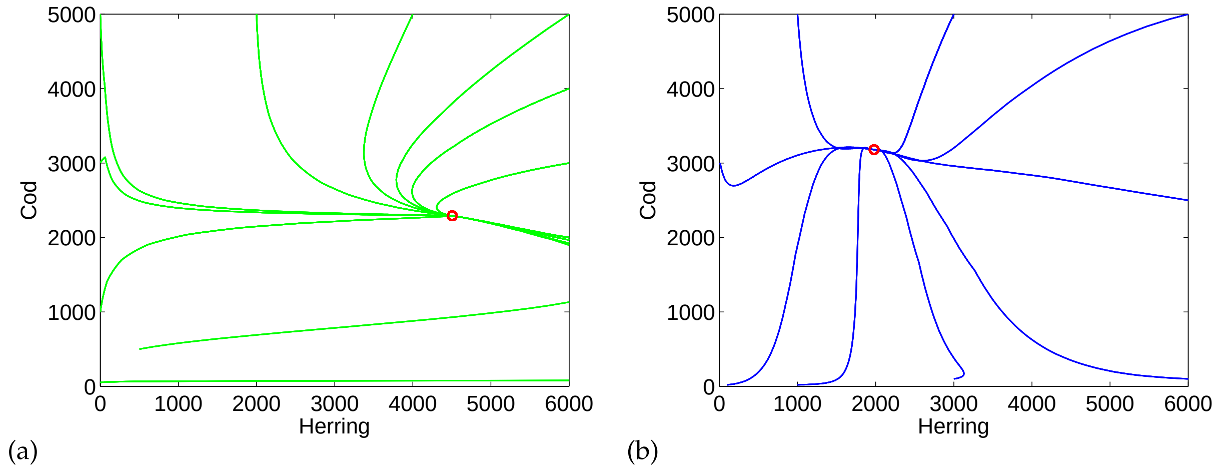
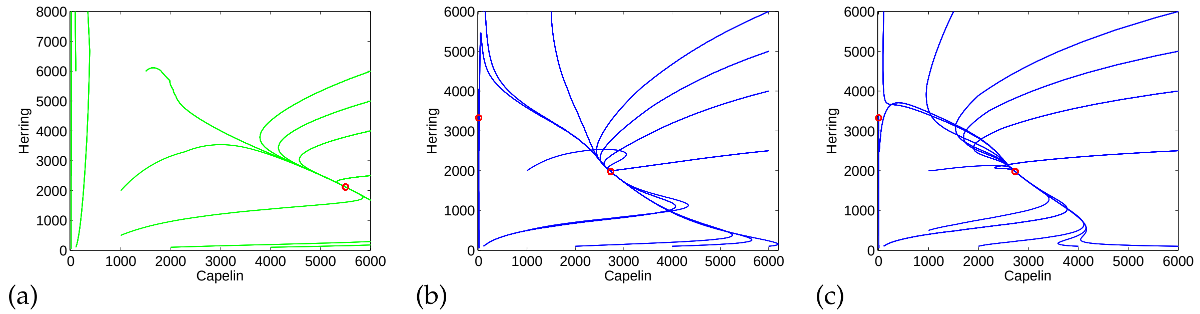
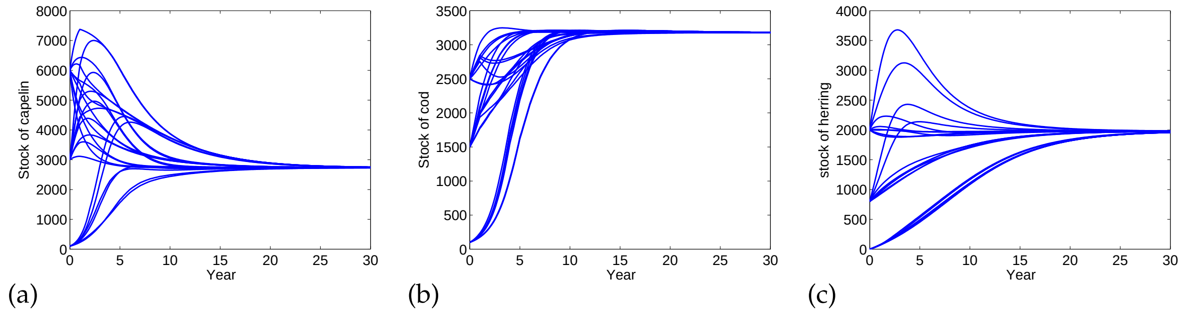


Figure 5.10: Paths towards equilibrium on the herring/cod plane for different initial biomasses of capelin, cod and herring. (a) Absence of capelin and (b) initial capelin stock of 2.5 million tons



(a) (b) (c)

Figure 5.11: Paths towards equilibrium on the capelin/herring plane for different initial biomasses of capelin, cod and herring. (a) Absence of cod, initial cod stock of (b) 1 million tons and (c) 5 million tons.



(a) (b) (c)

Figure 5.12: Stocks of capelin, herring and cod approaching equilibrium from several different initial conditions.

S. Aanestad, L.K. Sandal and A. Eide – *"Optimal Fishing Policy for Two Species in a Three-Species Predator-Prey Model*
The case of Capelin, Cod and Juvenile Herring in the Barents Sea."

Appendix A

Appendix essay I

A.1 Sosialt optimum

Vi skal her vise at $PS + CS + GS = \pi(a, y) - D(a)$. (Se fig. 2.1). Vi har

$$PS = y^\star \cdot C_p(a, y^\star) - \int_0^{y^\star} C_p(a, x)\, dx. \tag{A.1}$$

$$CS = \int_0^{y^\star} P(a, x)\, dx - y^\star\big(C_p(a, y^\star) + \tau(a, y^\star)\big) \tag{A.2}$$

$$GS = y^\star\tau - D(a) - \int_0^{y^\star} \big(C_s(a, x) - C_p(a, x)\big)\, dx. \tag{A.3}$$

Dermed får vi

$$\begin{aligned} PS + CS + GS &= y^\star \cdot C_p(a, y^\star) - \int_0^{y^\star} C_p(a, x)\, dx \\ &\quad + \int_0^{y^\star} P(a, x)\, dx - y^\star\big(C_p(a, y) + \tau(a, y^\star)\big) \\ &\quad + y^\star\tau - D(a) - \int_0^{y^\star} \big(C_s(a, x) - C_p(a, x)\big)\, dx \\ &= \int_0^{y^\star} \big(P(a, x) - C_s(a, x)\big)\, dx - D(a) \\ &= \pi(a, y^\star) - D(a). \end{aligned} \tag{A.4}$$

A.2 Eksistensbevis

Vi skal her vise at det alltid eksisterer løsning for optimeringsproblemet definert i (2.6) og (2.7). I den sammenheng begrenser vi oss til å vise at betingelsene i *Filipov-Cesaris eksistensteorem*, som vi finner som teorem 6.18 i Seierstad og Sydsæter (1987), er oppfylte.

Kravet om at mengden $N(a, y, t)$ er konveks, utgjør sammen med kravet om en øvre grense for tilstandsvariablene, de eneste ikke-trivielle betingelser stilt i dette teoremet.

Når det gjelder det siste kravet har vi at $\dot{a} + \dot{s} = y - f(a) + (-y) = -f(a) \leq 0$, slik at $0 \leq a + s \leq a_0 + s_0$. Følgelig er dette tilfredsstilt.

Teoremet forutsetter at mengden

$$N(a, y, t) = \left\{(e^{-rt}[\pi(a, y) - D(a)] + \gamma, y - f(a), -y) : \gamma \leq 0, y \in [0, \hat{y}]\right\} \tag{A.5}$$

er konveks for alle $(a, t) \in R \times [0, T]$. (I (A.5) er $\hat{y}$ gitt ved $\pi_y(a, \hat{y}) = 0$).

Bevis: Fikser (a, t), la $z_i \equiv e^{-rt}[\pi(a, y_i) - D(a)] + \gamma$ for $i = 1, 2$, og la y_3, z_3 være en konveks kombinasjon av y_1, y_2 og z_1, z_2.

Vi får $\lambda(z_1, y_1 - f(a), -y_1) + (1 - \lambda)(z_2, y_2 - f(a), -y_2) = (\lambda z_1 + (1 - \lambda)z_2, \lambda y_1 + (1 - \lambda)y_2 - f(a), -\lambda y_1 - (1 - \lambda)y_2) = (z_3, y_3 - f(a), -y_3)$. Konkaviteten til $\pi(a, y)$ gir $z_3 = e^{-rt}[\lambda\pi(a, y_1) + (1 - \lambda)\pi(a, y_2) - D(a)] + \lambda\gamma_1 + (1 - \lambda)\gamma_2 \leq e^{-rt}[\pi(a, y_3) - D(a)] + \lambda\gamma_1 + (1 - \lambda)\gamma_2 \Rightarrow \gamma_3 \leq \lambda\gamma_1 + (1 - \lambda)\gamma_2 \leq 0 \quad \Rightarrow \quad (z_3, y_3 - f(a), -y_3) \in N(a, y, T)$.

A.3 Formulering av modell

I det følgende skisserer vi først kort i seksjon A.3.1 hvordan vi kommer frem til en differensialligning som gir en feedback-form for den optimale kontrollen, $y = y(a)$. Deretter gjør vi i seksjon A.3.2 rede for hvordan vi benytter oss av denne differensialligningen til å finne en numerisk løsning av modellen.

Optimeringsproblemet i ligningene (2.6) og (2.7) kan formuleres slik at det strekker seg over to perioder - før og etter teknologiskiftet. Kontrollvariabelen er uttaket, y, og vi har et tilpasningsproblem i tidspunktet T.

Hamiltonfunksjonen med tilstandsligninger, multiplikatorligninger og det indre optimalitetskravet er oppsummert i tab. A.1.

Beskrivelse	Første periode	Sluttperiode
Tid	$0 \leq t \leq T$	$T < t \leq \infty$
produksjon	$y > 0, \quad y(T) = y_T > 0$	$y = 0$
Sosial velferd	$\pi(a, y) - D(a)$	$\widehat{\pi}(t) - D(a)$
Tilstandligning 1	$\dot{a} = y - f(a)$	$\dot{a} = -f(a)$
Tilstandsligning 2	$\dot{s} = -y$	
Hamilton	$\mathcal{H} = \pi(a, y) - D(a) + m(y - f(a)) - ny$	$\mathcal{H} = \widehat{\pi}(t) - D(a) - mf(a)$
Indre optimum	$\mathcal{H}_y = 0 \Leftrightarrow m - n = -\pi_y(a, y)$	
Kofaktor 1	$\dot{m} = (r + f'(a))m - \pi_a(a, y) + D'(a)$	$\dot{m} = (r + f'(a))m + D'(a)$
Kofaktor 2	$\dot{n} = rn$	$\dot{n} = rn, ns = 0$ og $n, s \geq 0$

Table A.1: Førsteordens betingelser.

Vi krever at denne Hamilton-funksjonen, tilstandsvariablene og multiplikatorene er kontinuerlige i $t = T$.

Kofaktor 2: Løser vi ligningen for kofaktor 2 i tab. A.1, får vi

$$n(t) = n_0 e^{rt}, \quad \text{der} \quad n_0 = n(0). \tag{A.6}$$

Dette kombinerer vi med (2.13) og får to muligheter for skyggeprisen $n(t)$:
Tilfelle 1: $s(T) > 0$ og $n(t) = 0$ for alle t.
Tilfelle 2: $s(T) = 0$ og $n(t) = n_0 e^{rt}$.

A.3.1 Feedback-formulering

I det følgende skal vi skal vi uten å gå inn på detaljer skissere hvordan vi utleder en 1. ordens differensialligning for feedback-kurven til y for tilfelle 1 og tilfelle 2. For begge tilfellene vil den dynamiske egenskapen

$$\frac{d}{dt}\mathcal{H} = r(\mathcal{H}_a \dot{a} + \mathcal{H}_s \dot{s}) \tag{A.7}$$

bli brukt. (En utledning av dette resultatet finnes blant annet i Sandal og Berge (2000)). Videre setter vi kravet om indre løsning, se tab. A.1, inn i Hamilton-funksjonen. Dette gir

$$\mathcal{H}(a, y) = \pi(a, y) - D(a) - y\pi_y - mf(a). \tag{A.8}$$

Tilfelle 1: $s(t) > 0$ og $n(t) = 0$ for alle t. Det at $n = 0$ forenkler ligningssystemet (A.7) og (A.8). Vi kommer frem til differensialligningen

$$y'(a)(y - f)\pi_{yy} = \pi_a - D' - (y - f)\pi_{ay} + (r + f')\pi_y \tag{A.9}$$

ved å løse dette systemet. Dette er vist i Aanestad, Sandal og Berge (2003).

Tilfelle 2: $s(T) = 0$ og $n(t) = n_0 e^{rt}$. I dette tilfellet blir feedback-formen mer komplisert. I tillegg til (A.7) og (A.8) må vi trekke inn differensialligningen for kofaktor m. (Se tab. A.1). Da kommer vi etter noen tekniske detaljer frem til en feedback-ligning. Istedenfor å presentere denne, presenterer vi den tidseksplisitte versjon som er noe enklere rent notasjonsmessig. Vi har

$$\frac{dY}{dt} = \begin{bmatrix} \dot{a} \\ \dot{y} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} y - f(a) \\ G_1(a, y, m)(y - f(a)) \\ (r + f'(a))m + D'(a) \end{bmatrix}, \tag{A.10}$$

med

$$G_1(a, y, m) \equiv y'(a) = \frac{\pi_a - D' - y\pi_a y - mf)(y - f) + r(y\pi_y + mf)}{y\pi_y y(y - f)}.$$

Detaljene for hvordan vi kommer frem til denne ligningen er presentert i Aanestad, Sandal og Berge (2003).

A.3.2 To randverdiproblem

I seksjon A.3.1 fant vi differensialligninger for optimale løsningskurver for hvert av de to spesialtilfellene av optimeringsproblemet vårt. Nå skal vi finne tilhørende randbetingelser tilstrekkelige for å finne løsningskurvene for hvert av tilfellene.

Fremgangsmåten blir i praksis at man først antar at der ikke er noen ressursknapphet, altså tilfelle 1. Man finner tilhørende optimale løsningskurver for dette tilfellet og kontrollerer at uttaket gir $s(T) \geq 0$. Deretter søker man en løsning hvor ressursen brukes helt opp. (Tilfelle 2). Hvis der eksisterer potensielle løsninger for begge de to tilfeller, må man numerisk sammenligne overskuddet knyttet til løsningene.

Tilfelle 1: $s_T > 0$ og $n(t) = 0$ for alle t. I dette tilfellet er randkravene initialbetingelsen $a(0) = a_0$, samt overgangsbetingelsene i $t = T$. Siden $n(t) = 0$, er Hamiltonfunksjonen vår $\mathcal{H} = \pi(a, y) - D(a) + m(y - f(a))$.

Ligningen for kofaktor 1 sammen med betingelsen $\lim_{t\to\infty} a(t) = 0$, det indre optimalitetskravet og kravet om kontinuerlig hamilton-funksjon gir oss følgende betingelser i teknologiskiftetidspunktet: (Se Aanestad, Sandal og Berge (2003) for en utledning av disse betingelsene).[1]

$$m_T = m(\infty) + \int_0^{a_T} \frac{dm}{da}\, da \equiv G(a_T), \quad t \geq T \tag{A.11}$$

$$\mathcal{H}_y = 0 \Rightarrow \pi_y(a_T, y_T) + m_T = 0 \tag{A.12}$$

$$\widehat{\pi}(T) + \gamma(a_T) y_T^2 = 0. \tag{A.13}$$

Størrelsene som skal bestemmes er T, a_T, y_T og m_T. I tillegg til de tre overgangsbetingelsene i teknologiskiftetidspunktet, (A.11)-(A.13), krever vi at initialbetingelsen er oppfylt. Vi har dermed fire randkrav og fire ukjente størrelser.

Siden vi har eksplisitt tidsavhengighet i den alternative profittfunksjonen, $\widehat{\pi}(t)$, lønner det seg å bruke eksplisitt tidsavhengighet også når vi skal finne de optimale løsningskurvene for y og a. Vi har

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} \dot{a} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} (y - f(a)) \\ y'(a)\big(y - f(a)\big) \end{bmatrix}, \tag{A.14}$$

der $y'(a)$ er gitt ved (A.9) og vi har randkravene (A.11) - (A.13) samt $a(0) = a_0$.

Det viser seg at $\frac{d\mathbf{Y}}{dt}$ er numerisk ustabil når man integrerer fra $t = 0$ til $t = T$. Går man motsatt vei fra $t = T$ til $t = 0$ blir systemet stabilt.

Siden man har randkrav både i $t = 0$ og $t = T$, er det ikke et trivielt problem å løse (A.14) med tilhørende randkrav. Vi ble nødt for å utvikle en egen shooting-metode som kunne håndtere dette randverdiproblemet numerisk.

---

[1] Vi har i det følgende brukt $a(T) = a_T$, $y(T) = y_T$, $s(T) = s_T$ og $m(T) = m_T$.

Tilfelle 2: $n(t) = n_0 e^{rt}$ og $s(T) = 0$. Vi bruker følgende randkrav:

$$\begin{aligned} &a(0) - a_0 = 0 \\ &\widehat{\pi}(T) + \gamma(a_T) y_T^2 = 0 \\ &m_T - M(a_T) = 0 \\ &s(T) = 0. \end{aligned} \tag{A.15}$$

Dessuten innfører vi parameteren $\Omega(t)$ med initialkrav $\Omega(0) = 0$. Denne betegner samlet neddiskontert overskudd ved tidspunkt t, og er tatt med fordi differensialligningene våre med tilhørende randkrav i noen tilfeller kan ha flere løsninger slik at vi må sammenligne dem numerisk for å skille ut hvilken som er den beste. Antall mulige løsninger vil avhenge av a-avhengigheten i γ. (Se andre ligning i (A.15)). Dette vil vi imidlertid ikke komme noe mer inn på her.

I tillegg til de nevnte randkravene har vi $m_T - n_T = -\pi_y(a_T, y_T)$, men dette kravet bidrar ikke med noe nytt siden det innfører en ny ukjent, nemlig n_T.

Differensialligningssytem (A.10) utvider vi med uttrykk for $\dot{s}$ og $\dot{\Omega}$. Dermed får vi:

$$\frac{dY}{dt} = \begin{bmatrix} \dot{a} \\ \dot{y} \\ \dot{m} \\ \dot{s} \\ \dot{\Omega} \end{bmatrix} = \begin{bmatrix} y - f(a) \\ G_1(a,y,m)(y - f(a)) \\ (r + f'(a))m + D'(a) \\ -y \\ e^{-rt}\big(\pi(a,y) - D(a)\big) \end{bmatrix}, \tag{A.16}$$

Vi ser at vi har fem randkrav og fem differensialligninger. Dette randverdiproblemet er like komplisert som tilsvarende problem under *tilfelle 1*. Vi brukte en såkalt kollokasjonsmetode for å løse problemet numerisk.

Appendix B

Appendix essay II

This appendix displays the numerical solver used to solve example 1 in essay II. The solver consists of a matlab-file that is runned with the command *toDimEks2(dx,dy,delta,feil_tol,iterasjoner)*, where *dx, dy, ...* etc. are numerical values.

```
function [X0,Y0,V0,u0,feil_u,feil_V]= ...
    toDimEks2(dx,dy,delta,feil_tol,iterasjoner);

%Solves the 2D-model in ex. 5.3 of Grüne and Semmler (2004)
tic %Cmd that registers the time the prgm starts
epsilon=1e-10; %Used to avoid problems with 1/0.

h=1/200; beta=1-delta.*h;
xmin=0;ymin=0; xmax=7;ymax=2;
X=xmin:dx:xmax; Y=ymin:dy:ymax;
DX=[dx diff(X)]; DY=[dy diff(Y)];
mx=length(X); my=length(Y)

[X0,Y0]=meshgrid(X,Y);

% Initial conditions for the value function V0,
% the politics u0 and so on ...
V0=zeros(my,mx); %The value funct. along the state space grid
V_I=V0; Vxder=zeros(my,mx); Vyder=zeros(my,mx);
u0=zeros(my,mx);
x_ny=X0+fx(X0,Y0).*h;  y_ny=Y0+fy(u0);
feil_u=1;feil_V=1;feil=1; % Initiate 3 error terms

V_I=V0; Vxder=zeros(my,mx); Vyder=zeros(my,mx);
Vzder=Vyder; u0=zeros(my,mx);

x_ny=min(xmax,X0+fx(X0,Y0).*h); %Demand a closed state-space
```

```
y_ny=Y0+fy(u0).*h;

% A little procedure to make use of previous estimates
% (if such exist). These might be used as a
% better initial condition
try
    load toDim; %Download toDim.mat
    vv=size(Vgammel);
    if size(V0)==size(Vgammel)
        V0=Vgammel;
    else
        V0=interp2(Xgammel,Ygammel,Vgammel,X0,Y0);
    end
end

% Estimate partial derivatives of the value function in x
% and y-direction and make use of them to find starting
% values for the optimal controls.

[Vxder,Vyder]=gradient(V0); %Gradients (x and y-direction)
Vxder=Vxder./ (ones(my,1)*DX);
Vyder=Vyder./(DY'*ones(1,mx));

% Get u-values from sub-function "fangst"
u0=fangst(X0,Y0,Vxder,Vyder,beta,h);

% Find new x- and y-values corresponding (to u0)
x_ny=min(xmax,X0+fx(X0,Y0).*h);
y_ny=Y0+fy(u0).*h;

% Interpolate V-value outside Gx for x1=x_ny and x2=y_ny
V_I=interp2(X0,Y0,V0,x_ny,y_ny,'linear');
% The first policy it. is finished

% A combination of policy and value iterations until
% convergence or till maximum numb. of it. is exceeded
feil_old=1; V0_old=V0; it=1;
while  it<iterasjoner && feil>feil_tol
   it=it+1;
```

```
    if mod(it,20)==0  %A procedure to display number of it.
        disp('iterasjoner');
        disp(it);
        disp(feil_u);
        disp(feil_V);
    end

    % Value it: Fixed point it. on the opt. value func.
    [V0,feil_V]=verdiIt(X0,Y0,V0,x_ny,y_ny,u0, ...
        V_I,beta,h,feil_tol,feil_V,epsilon);

    %policy iterations
   [u0,V0,V_I,x_ny,y_ny,feil_u,feil_V]=policyit(X0,Y0,u0, ...
       V0,feil_tol,feil_u, beta,h,mx,my,DX,DY,epsilon);

    feil=max(feil_u,feil_V); % Update size of error
end

%___________________________________________________________%
% Have reached a solution. Save it for later use and
% display errors and time to convergence.
disp(['iterasjoner: ' 'rel_feil_u: ' 'rel_feil_V:' 'tid:']);
disp([it feil_u feil_V toc]);
Vgammel=V0; Xgammel=X0; Ygammel=Y0;
x=X0; y=Y0;u=u0;

save toDim Xgammel Ygammel Vgammel;

% Select a cruder grid to be plotted
x=0:2.5*dx:6;  y=0:2*dy:1.6;
[x,y]=meshgrid(x,y);
V=interp2(X0,Y0,V0,x,y,'linear'); %Interpol on new grid
u=interp2(X0,Y0,u0,x,y,'linear');

%% Figures
figure(1) % The value function V(x,y)
box off; legend('boxoff')
surf(x,y,V);
set(gca,'FontSize',14,'FontWeight','bold')
```

```
xlabel('x_1'); ylabel('x_2'); zlabel('V(x_1,x_2)')

figure(2) % Optimal policy
surf(x,y,u);
box off; legend('boxoff')
set(gca,'FontSize',14,'FontWeight','bold');
xlabel('x_1'); ylabel('x_2'); zlabel('u(x_1,x_2)');

%% All sub-functions
%%- State eq. 1
function tilst1=fx(x,y)
sigma=0.25;
tilst1=y-sigma.*x;

%%-State eq. 2
function tilst2=fy(u)
tilst2=u;

%%- Utility/profit function
function out=g(x,y,u)
k1=2;k2=0.0117;c1=0.75;c2=2.5;alfa=12;
out=k1.*x.^0.5-x./(1+k2.*x.^4)-c1.*y-c2.*y.^2./2-alfa.*u.^2./2;

%%- Function for optimal control, u
function u=fangst(x,y,Vxder,Vyder,beta,h) %
alfa=12;
u=beta.*Vyder./alfa; % F.o.c. (inner solution)
u=max(-1,u);
u=max(u,-y./h);
u=min(1,u);

%%- Value iterations
function [V0,feil_V]= ...
    verdiIt(X0,Y0,V0,x_ny,y_ny,u0,V_I,beta,h,feil_tol, ...
    feil_V_old,epsilon)

    fiksPi=g(X0,Y0,u0); %Utility function
    feil_V=feil_V_old; V0_old=V0;it=1;
```

```
    % Use g(x,u_fixed) to est. val. func. with fixed pnt. it
    V0=h.*fiksPi+beta.*V_I;
    % Improve old V_I
    V_I=interp2(X0,Y0,V0,x_ny,y_ny,'linear');

    while feil_V>feil_tol && it<45

        % Use g(x,u_fixed) to estimate value func.
        V0=max(0,h.*fiksPi+beta.*V_I);
        V_I=interp2(X0,Y0,V0,x_ny,y_ny,'linear');

        %Rel. change val. func.
        feil_V=max(max(abs(V0_old-V0)./(V0+epsilon)));
        V0_old=V0;
        it=it+1;
    end

%%- Policy-iterations
function [u0,V0,V_I,x_ny,y_ny,feil_u,feil_V]= ...
    policyit(X0,Y0,u0,V0,feil_tol,feil_old,beta,h, ...
    mx,my,DX,DY,epsilon)

    xmax=6;feil_u=1;feil_V=1;it=0;

    while  (feil_u>feil_tol || it<1) && it<6
        it=it+1;
        u_0=u0; V_0=V0;

        [Vxder,Vyder]=gradient(V0);
        Vxder=Vxder./(ones(my,1)*DX);
        Vyder=Vyder./(DY'*ones(1,mx));
        u0=fangst(X0,Y0,Vxder,Vyder,beta,h);
        x_ny=min(xmax,X0+fx(X0,Y0).*h);
        y_ny=Y0+fy(u0).*h;

        % Est. val. func. for new x-vals.
        V_I=interp2(X0,Y0,V0,x_ny,y_ny,'linear');

        % Updts val. funct. for old x-vals.
```

```
        V0=max(0,h.*g(X0,Y0,u0)+beta.*V_I);

        % Rel. change u
        feil_u=max(max(max(abs(u_0-u0)./(u_0+1e-14))));
        % Rel. change of V
        feil_V=max(max(abs(V_0-V0)./(V_0+epsilon)));
    end
```