MAT-3900

# Master's Thesis in Mathematics

# EXTENSIONS OF GROUPS AND MODULES

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Abstract. The main goal of this thesis is to build up detailed constructions and give complete proofs for the extension functors of modules and groups, which we define using cohomology functors. Further, we look at the relations that appear between these and short exact sequences of modules, respectively groups. We calculate also several concrete cohomology groups, and build extensions that are described by those cohomologies.

Contents



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*Key words and phrases.* Group extensions, module extensions, group cohomology, projective resolutions, injective coresolutions, pullback, pushout, short exact sequences, Baer sum, group ring.

## 0. Introduction

Most of the results in this thesis are known. Our goal was to put down on paper some longer technical proofs that are usually just sketched in the existing literature, and to build up a machinery that is easy to follow. There is a thread through the topics, which are revealed to be closely related.

In Part 1 we introduce the functors $Ext^n_R(-,-)$, $\overline{Ext}^n_R(-,-)$, defined for any non-negative integer $n$, using $n$-th cohomology functors, projective resolutions, and respectively, injective coresolutions, over some fixed ring $R$. One of their most interesting properties is that they are bifunctors (Theorem 3.4 and Theorem 4.4). Moreover, $Ext^n_R(-,-)$ and $\overline{Ext}^n_R(-,-)$ are isomorphic as bifunctors (Theorem 4.10).

We introduce another bifunctor, $E_R\,(C,A)$, the set of equivalence classes of short exact sequences of $R$-modules

$$0 \longrightarrow A \longrightarrow E \longrightarrow C \longrightarrow 0$$

with the Baer sum as an abelian group operation. Finally, we prove in Theorem 5.16 that the abelian groups $Ext^1_R(C,A)$ and $E_R(C,A)$ are naturally (on $C$ and $A$) isomorphic.

In Part 2, we define the functors $H^n(-,-)$, with first argument any group $G$ and second argument any $G$-module $A$, again using the $n$-th cohomology functors but now over the fixed group ring. For any action of $G$ on $A$, we can establish a set bijection between $H^n(G,A)$ and the set $E(G,A)$, consisting of equivalence classes of short exact sequences of groups

$$0 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

where $E, G$ are not necessarily abelian groups. Further, $E(G,A)$ turns out to be a group, and as a bifunctor from the category $PAIRS$ (as in Definition 6.6) to the category of abelian groups, it is isomorphic to $H^n(G,A)$ (Theorem 7.11).
What about the case when we do not restrict ourselves to an abelian kernel $A$? As described in Section 8, if an extension exists, it induces a triple called an abstract kernel $(A, G, \theta : G \longrightarrow Aut(A)/In(A))$. The other way, given an abstract kernel, it has an extension if and only if one of its obstructions is equal to 0 (considered as a 3-cochain of $Hom_{\mathbb{Z}G}(\mathbb{Z}^{trivial}_*, Z(A))$, where $\mathbb{Z}$ is the trivial $G$-module, and $Z(A)$ is the center of $A$). See Theorem 8.6.

In the last part, we specifically describe extensions of primary and the infinite cyclic group $\mathbb{Z}$ by primary and the infinite cyclic group $\mathbb{Z}$, see Theorem 9.3 and Theorem 9.4. Therefore, we shall have described all extensions of finitely generated abelian groups, as all such are a direct product of primary cyclic groups and of some rank. We have also shown that an abelian extension (an element of $E_{\mathbb{Z}}(G,A)$) may be embedded in $E(G,A)$, proved in Theorem 10.1. Specifically, when $G = \mathbb{Z}_m$, $m \geq 2$, we have that any extensions of $A$ by $\mathbb{Z}_m$ is an abelian extension, as shown in Theorem 10.2. We will also show that there exists extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p \times \mathbb{Z}_p$ that are not abelian, which follows from Theorem 10.14.

## 1. Preliminaries

We will now give some definitions and results that will be used frequently in the rest of the thesis. For any category $C$, we write $Ob(C)$ for the class of objects in $C$, and $Hom_C(A, B)$ the set of morphisms between any two objects $A, B \in Ob(C)$.

**Remark 1.1.** *A* ***class*** *is something larger than a set. A category $C$ is called* ***small*** *if $Ob(C)$ is a set, and* ***large*** *otherwise. Almost all categories in this Thesis are large.*

**Definition 1.2.** *([3] Chapter IX.1) A pre-additive category $C$ is a large category such that*

(1) *For any $A, B \in Ob(C)$, $Hom_C(A, B)$ is an abelian group,*
(2) *Composition of morphisms is distributive.*

**Lemma 1.3.** *([1] Chapter 5 Proposition 5.2) Fix any finite family of objects $\{A_i\}$ of a pre-additive category $C$. Whenever the coproduct of the $\{A_i\}'s$ exists, it is isomorphic to the product of the $\{A_i\}'s$, considered as objects of $C$.*

**Definition 1.4.** *([1] Chapter 5.1) Let $C$ and $C'$ be any two additive categories, and $F : C \longrightarrow C'$ a functor. $F$ is said to be additive if for any pair of morphisms $u, v \in Hom_C(A, B)$, we have*

$$F(u + v) = F(u) + F(v)$$

**Lemma 1.5.** *Let $C$ be any pre-additive category. The covariant and respectively contravariant functors $Hom_C(A, -)$, and $Hom_C(-, B)$, are additive functors.*

**Remark 1.6.** *Follows easily from Definition 1.2.*

**Lemma 1.7.** *Chain homotopies are preserved under covariant additive functors. Under a contravariant additive functor, chain homotopies are transformed to cochain homotopies.*

**Definition 1.8.** *An additive category $C$ is a pre-additive category where there exists coproducts of any finite family of objects of $C$.*

**Definition 1.9.** *([1] Chapter 5.4) An additive category $C$ is said to be pre-abelian if for any morphism $u \in Hom_C(A, B)$, there exists a $\ker(u)$ and $coker(u)$.*

**Definition 1.10.** *A pre-abelian category $C$ is called abelian if for any morphism $u \in Hom_C(A, B)$, we have an isomorphism between $Coim(u)$ and $\mathrm{Im}(u)$, where*

$$\begin{aligned} Coim(u) &= Coker(\ker(u)) \\ \mathrm{Im}(u) &= \ker(coker(u)) \end{aligned}$$

For the next three definitions, $C, D$ are two arbitrary categories.

**Definition 1.11.** *A covariant functor $T$ on $C$ to $D$ is a pair of functions: an object function and a mapping function. These assign to each object in $A$ an object $T(A)$ in $D$, and respectively, to any morphism $\gamma : A \longrightarrow B$ in $C$ a morphism $T(\gamma) : T(A) \longrightarrow T(B)$ in $D$. It preserves identities and composites, i.e.*

$$T(1_A) = 1_{T(A)}$$

*for all $A$ in $C$, and*

$$T(\beta\gamma) = T(\beta)T(\gamma)$$

*whenever $\beta\gamma$ is defined.*

**Definition 1.12.** *A contravariant functor $T$ from $C$ to $D$ is a covariant functor from $C^{opp}$ to $D$, where $C^{opp}$ is the category called the dual category to $C$, consisting of all objects of $C$, such that for any objects $A, B$ in $C^{opp}$, $Hom_{C^{opp}}(A, B) = Hom_C(B, A)$.*

**Definition 1.13.** *By* [3]*, a functor $T$, covariant in $B$ and contravariant in $A$, is a bifunctor if and only if for any $\alpha : A \longrightarrow A', \beta : B \longrightarrow B'$, the diagram*

$$\begin{tikzcd} T(A', B) \arrow[r, "T(\alpha, B)"] \arrow[d, "T(A', \beta)"'] & T(A, B) \arrow[d, "T(A, \beta)"] \\ T(A', B') \arrow[r, "T(\alpha, B')"] & T(A, B') \end{tikzcd}$$

*is commutative.*

Let us denote by $R$-mod, $AB$, $GR$, $Sets$, $Sets_*$, the frequently used categories of (left) $R$-modules, abelian groups, groups, sets, and pointed sets, respectively. We assume that all rings are associative and have multiplicative unity element. Given a chain complex of abelian groups $(X_*, d_*)$, let $Z_n = \ker d_{n-1}$, $B_n = d_n(X_{n+1})$. Elements of $Z_n$ are called $n$-cycles and elements of $B_n$ are called $n$-boundaries. As $X_n \xrightarrow{d_{n-1}} d_{n-1}X_n$ is an epimorphism, and $\ker d_{n-1} = Z_n$, it follows that $X_n/Z_n \simeq B_{n-1}$. Given a cochain complex $(X^*, d^*)$, let $Z^n = \ker d_n$, $B^n = d^{n+1}(X^{n+1})$. Elements of $Z^n$ are called $n$-cocycles, and elements of $B^n$ are called $n$-coboundaries.

**Definition 1.14.** *The $n$-th cohomology group of a cochain complex $(X^*, d^*)$ of abelian groups is the factor group $H^n(X) = Z^n/B^n$.*

**Definition 1.15.** *Let $(X^*, d^*)$ and $(Y^*, \delta^*)$ be cochain complexes. A cochain transformation $f : X \longrightarrow Y$ is a family of module homomorphisms $f^n : X^n \longrightarrow Y^n$, such that for any $n$,*

$$f^{n+1}d^n = \delta^n f^n$$

*A cochain homotopy $s$ between two cochain transformations $f, g : X \longrightarrow Y$ is a family of module homomorphisms $s^n : X^n \longrightarrow Y^{n-1}$ such that for any $n$,*

$$f^n - g^n = s^{n+1}d^n + \delta^{n-1}s^n$$

*We write $s : f \simeq g$. We say that $f$ is a homotopic equivalence if there exists a cochain transformation $g : Y \longrightarrow X$ and module homomorphisms $s : Y \longrightarrow Y$, $t : X \longrightarrow X$, such that*

$$s : fg \simeq 1_Y, \text{ and } t : gf \simeq 1_X.$$

We also have the notion of homology, chain transformation, chain homotopy, but we will, in this thesis, mostly be using the concept of cochain complex, cohomology, cochain transformation, cochain homotopy. We will simply write complex, transformation, homotopy, whenever their interpretation is clear from the context. A complex $(X^*, d^*)$ is said to be positive when $X^n = 0$ for $n < 0$.

**Proposition 1.16.** *$H^n$ becomes a covariant functor from the category of complexes of abelian groups (respectively $R$-modules) and transformations between them, to $AB$ (respectively $R$-mod).*

**Proposition 1.17.** *(*[3] *Theorem II.2.1) If $s : f \simeq g$*

$$H^n(f) = H^n(g) : H^n(X) \longrightarrow H^n(Y)$$

**Theorem 1.18.** *(Exact cohomology sequence,* [3] *Theorem II.4.1) For each short exact sequence of cochain complexes*

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

*we have a natural long exact sequence of cohomology:*

$$... \longrightarrow H^n(K) \longrightarrow H^n(L) \longrightarrow H^n(M) \longrightarrow H^{n+1}(K) \longrightarrow H^{n+1}(L) \longrightarrow ..$$

**Definition 1.19.** *A free $R$-module generated by a set $X$ consists of formal finite sums,*

$$\sum_{x\in X} n(x)\langle x\rangle\,,\ n(x)\in R$$

*and is denoted by $F(X)$.*

Clearly, $F(X) \simeq \oplus_{x\in X} F(\langle x\rangle) \simeq \oplus_{x\in X} R$.

**Definition 1.20.** *(*[3] *I.5) An $R$-module $P$ is projective if for any epimorphism $\sigma : A \longrightarrow B$, and any homomorphism $\gamma : P \longrightarrow B$, there exists a $\beta : P \longrightarrow A$ such that $\gamma = \sigma\beta$. An $R$-module $I$ is injective if for any monomorphism $\varkappa : A \longrightarrow B$, and any homomorphism $\mu : A \longrightarrow I$, there exists a $\rho : B \longrightarrow I$ such that $\rho\varkappa = \mu$. An $R$- module $M$ is divisible if for any $m \in M$, and every $r \in R$, there exists $m^{'} \in M$ such that $m = rm^{'}$.*

**Definition 1.21.** *Let $C$ be an $R$-module. A complex over $C$ is a positive complex $(X_*, d_*)$ and a transformation $\varepsilon$, to the trivial complex (i.e. concentrated in dimension zero) $C$. Write $(X_*, d_*) \xrightarrow{\varepsilon} C$. If all $X_n^{'}s$ are projective we say that $(X_*, d_*) \xrightarrow{\varepsilon} C$ is a projective complex over $C$. If $(X_*, d_*)$ has trivial homology in positive dimensions, while the induced mapping $\varepsilon : H_0(X) \longrightarrow C$ is an epimorphism, we say that $(X_*, d_*) \xrightarrow{\varepsilon} C$ is a resolution of $C$. A complex under $C$ is a transformation $\epsilon$ from the trivial complex $C$ to the positive complex $(Y^*, \delta^*)$. Write $C \xrightarrow{\epsilon} (Y^*, \delta^*)$. If all $Y_n^{'}s$ are injective, we say that $C \xrightarrow{\epsilon} (Y^*, \delta^*)$ is an injective complex under $C$. If $(Y^*, \delta^*)$ has trivial cohomology in positive dimensions, while $\epsilon : C \longrightarrow H^0(Y)$ is an isomorphism, we say that $C \xrightarrow{\epsilon} (Y^*, \delta^*)$ is a coresolution of $C$.*

**Lemma 1.22.** *(Comparison Lemma for projective resolutions,* [3] *Theorem III.6.1) Let $\gamma \in Hom_{R\text{-}Mod}(C, C^{'})$. If $(X_*, d_*) \xrightarrow{\varepsilon} C$ is a projective complex over $C$, and $(X_*^{'}, d_*^{'}) \xrightarrow{\varepsilon^{'}} C^{'}$ is a resolution of $C^{'}$, there is a transformation $f : X \longrightarrow X^{'}$ with $\varepsilon^{'} f = \gamma\varepsilon$, and any two such transformations are homotopic. We say that $f$ is a lifting of $\gamma$.*

**Lemma 1.23.** *(Comparison Lemma for injective coresolutions,* [3] *Theorem III.8.1) Let $\alpha \in Hom_{R\text{-}mod}(A, A^{'})$. If $A \xrightarrow{\varepsilon} (X^*, \delta^*)$ is a coresolution of $A$, and $A^{'} \xrightarrow{\varepsilon^{'}} (Y^*, \delta^*)$ is an injective complex under $A^{'}$, then there is a transformation $f : X \longrightarrow Y$ with $\varepsilon^{'}\alpha = f\varepsilon$, and any two such transformations are homotopic. We say that $f$ is a lifting of $\alpha$.*

**Definition 1.24.** *Let $G$ be a group and $R$ a ring. Define the group ring $RG$ as the free $R$-module generated by the symbols $\langle g\rangle, g \in G$, where multiplication is defined on the generators as $\langle g\rangle \langle h\rangle := \langle gh\rangle$, for any $g, h \in G$. So elements of $RG$ are formal (finite) sums $\sum_{g\in G} n(g)\langle g\rangle\,,\ n(g) \in R$.*

**Definition 1.25.** *A $G$-$R$-module is an $R$-module $A$ together with a group homomorphism $G \longrightarrow Aut_R(A)$. If $R = \mathbb{Z}$, we simply say that $A$ is a $G$-module.*

**Proposition 1.26.** *$A$ is a $G$-$R$-module if and only if $A$ is an $RG$-module.*

*Proof.* Take a $\varphi \in Hom_{GR}(G, Aut_R(A))$. Then $A$ becomes a $RG$-module through a function $\psi : RG \times A \longrightarrow A$ defined as

$$\psi(\sum n(g) \langle g \rangle a) = \sum_{g \in G} n(g)\varphi(g)(a), n(g) \in R.$$

Suppose we have a function $\psi$ that makes $A$ a $RG$-module. Define the function

$$\varphi(g)(a) = \psi(\langle g \rangle, a)$$

It can be shown that $\varphi \in Hom_{GR}(G, Aut_R(A))$. □

**Definition 1.27.** *([1] Chapter 2.1 ) Let $A$ be an object of the category $C$ and $I$ an arbitrary set of indices. We shall say that $A$ together with the family of morphisms $u_i : A \longrightarrow A_i$ is the direct product of $\{A_i\}_{i \in I}$ if for any object $B$ in $C$ and any family of morphisms $v_i : B \longrightarrow A_i$, there exists a unique morphism $v : B \longrightarrow A$ such that the diagrams*

$$\begin{tikzcd} B \arrow[r, "v"] \arrow[dr, "v_i"'] & A \arrow[d, "u_i"] \\ & A_i \end{tikzcd}$$

*are commutative.*

**Definition 1.28.** *([1] Chapter II.6) A kernel of a morphism $\sigma : K \longrightarrow L$ in an abelian category is a $\mu : J \longrightarrow K$ such that $\sigma\mu = 0$ and for any other $\tau$ such that $\sigma\tau = 0$, there exists a unique $\tau_0$ such that the diagram is commutative:*

$$\begin{tikzcd} J \arrow[r, "\mu"] & K \arrow[r, "\sigma"] & L \\ M \arrow[u, "\tau_0"] \arrow[ur, "\tau"'] & & \end{tikzcd}$$

**Definition 1.29.** *([2] II.(6.2))A pullback of two morphisms $\varphi : A \longrightarrow X$ and $\psi : B \longrightarrow X$ is a pair of morphisms $\alpha : Y \longrightarrow A$ and $\beta : Y \longrightarrow B$ such that $\varphi\alpha = \psi\beta$*

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd}$$

*and for any other pair $\gamma : Z \longrightarrow A$ and $\delta : Z \longrightarrow B$ such that $\varphi\gamma = \psi\delta$ there exists a unique $\xi : Z \longrightarrow Y$ such that $\alpha\xi = \gamma$ and $\beta\xi = \delta$.*

**Definition 1.30.** *([1] Chapter 3.1) Let $C$ be any category and $D$ be a small category, and let $F$ be a covariant functor $F : D \longrightarrow C$. An inverse limit of $F$ is an object $A$ in $C$ together with morphisms $u_X : A \longrightarrow F(X)$, one for each $X \in Ob(C)$, such that*

1. *For all $\alpha : X \longrightarrow Y$ in $D$, $F(\alpha)u_X = u_Y$;*
2. *For any other family $v_X : Z \longrightarrow F(X)$ such that $F(\alpha)v_X = v_Y$, there exists a unique $v : Z \longrightarrow A$ such that $u_X v = v_X$, for all $X \in Ob(D)$.*

**Proposition 1.31.** *Whenever they exist, kernels ([1] Chapter 3.1 Example 1), direct products ([1] Chapter 3.1 Example 2), and pullbacks ([2] II. Prop.6.1) are inverse limits.*

**Corollary 1.32.** *Inverse limits in general, as well as direct products, kernels and pullbacks in particular, are unique (up to an isomorphism).*

**Proposition 1.33.** *([1] Prop. 3.6) Let $C$ be any category and $A$ an object of $C$. The covariant functor $Hom_C(A,-)$ preserves inverse limits of functors from any small category.*

**Corollary 1.34.** *In an abelian category $C$, for any short exact sequence*

$$0 \longrightarrow B^{'} \longrightarrow B \longrightarrow B^{''} \longrightarrow 0$$

*and any $A \in Ob(C)$, we get the exact sequence*

$$0 \longrightarrow Hom_C(A, B^{'}) \longrightarrow Hom_C(A, B) \longrightarrow Hom_C(A, B^{''}).$$

**Remark 1.35.** *It follows from the Corollary above that the functor $Hom_C(A,-)$ is left exact.*

**Proposition 1.36.** *In $GR$ and $R$-mod, the pullback is $Y = \{(a,b) \in A \times B \mid \varphi(a) = \psi(b), a \in A, b \in B\}$, with the natural projections $\alpha = \pi_A, \beta = \pi_B$.*

**Definition 1.37.** *([1] Chapter 2.1) Let $A$ be an object of the category $C$ and $I$ an arbitrary set of indices. We shall say that $A$ together with the family of morphisms $u_i : A_i \longrightarrow A$ is the direct sum of $\{A_i\}_{i\in I}$ (also called coproduct), if for any object $B$ in $C$ and any family of morphisms $v_i : A_i \longrightarrow B$, there exists a unique morphism $v : A \longrightarrow B$ such that the diagram*

$$\begin{tikzcd} A_i \arrow[r, "u_i"] \arrow[dr, "v_i"'] & A \arrow[d, "v"] \\ & B \end{tikzcd}$$

*commutes.*

**Definition 1.38.** *([2] Chapter I.2) The cokernel of a morphism $\sigma : K \longrightarrow L$ in an abelian category is a $\mu : L \longrightarrow M$ such that $\mu\sigma = 0$ and for any other $\tau : L \longrightarrow M^{'}$ such that $\tau\sigma = 0$, there exists a unique $\tau_0$ such that the diagram is commutative:*

$$\begin{tikzcd} K \arrow[r, "\sigma"] & L \arrow[r, "\mu"] \arrow[dr, "\tau"'] & M \arrow[d, "\tau_0"] \\ & & M' \end{tikzcd}$$

**Definition 1.39.** *A pushout of two morphisms $\alpha : X \longrightarrow A$, $\beta : X \longrightarrow B$, is a pair of morphisms $f : A \longrightarrow Y$, $g : B \longrightarrow Y$ with $f\alpha = g\beta$,*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

*satisfying the universal property: for any $u : A \longrightarrow Z$, $v : B \longrightarrow Z$ such that $u\alpha = v\beta$, there exists a unique $\xi : Y \longrightarrow Z$ such that $u = \xi f$ and $v = \xi g$.*

**Proposition 1.40.** *In R-mod, the pushout of* $(X, \alpha, \beta)$ *is* $Y = A \oplus B / \langle (\alpha(x), -\beta(x)) : x \in X \rangle$, *where* $f = i_A$ *and* $g = i_B$ *are the canonical injections.*

**Definition 1.41.** *Let* $C$ *be any category and* $D$ *be a small category, and let* $F : D \longrightarrow C$ *be a covariant functor. A direct limit of* $F$ *is defined dually to the inverse limit of* $F$ *(as in Definition 1.30).*

**Proposition 1.42.** *Whenever they exist, direct sums, cokernels and pushouts are direct limits.*

**Proposition 1.43.** *Direct limits in general, as well as direct sums, cokernels and pushouts in particular, are unique (up to an isomorphism).*

**Proposition 1.44.** *Let* $C$ *be any category, and* $B$ *an object in* $C$*. The contravariant functor* $Hom_C(-, B)$ *carries direct limits of functors from any category into inverse limits.*

**Corollary 1.45.** *In an abelian category* $C$, *for any short exact sequence*

$$0 \longrightarrow A^{'} \longrightarrow A \longrightarrow A^{''} \longrightarrow 0$$

*and any* $B \in Ob(C)$, *we get the exact sequence*

$$0 \longrightarrow Hom_C(A^{''}, B) \longrightarrow Hom_C(A, B) \longrightarrow Hom_C(A^{'}, B)$$

**Remark 1.46.** *It follows from the Corollary above that the functor* $Hom_C(-, B)$ *is left exact.*

**Lemma 1.47.** *(Short Five Lemma,* [3] *Lemma I.3.1) Given any commutative diagram in* $GR$

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\gamma"'] & C \arrow[r] \arrow[d, "\beta"] & 1 \\
1 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 1
\end{tikzcd}$$

*where the rows are short exact sequences. If* $\alpha, \beta$ *are pairwise injective, surjective or isomorphisms, so is* $\gamma$.

**Lemma 1.48.** *(The* $3 \times 3$ *Lemma,* [3] *Lemma II.5.1) Suppose that in the following commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_3 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_3 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_3 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

*all three columns and the first (or last) two rows are exact. Then the third row is exact.*

**Lemma 1.49.** *(Ker-Coker sequence,* [3] *Lemma II.5.2) Given two short exact sequences in a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\varkappa} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\varkappa'} & B' & \xrightarrow{\sigma'} & C' & \longrightarrow & 0 \end{array}$$

*the sequence is exact*

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow coker(\alpha) \longrightarrow coker(\beta) \longrightarrow coker(\gamma) \longrightarrow 0$$

**Proposition 1.50.** *Let $A, B, C \in Ob(R\text{-}mod)$. For any short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*the sequences*

$$\begin{array}{rcl} 0 & \longrightarrow & Hom_R(P,A) \longrightarrow Hom_R(P,B) \longrightarrow Hom_R(P,C) \longrightarrow 0 \\ 0 & \longrightarrow & Hom_R(C,I) \longrightarrow Hom_R(B,I) \longrightarrow Hom_R(A,I) \longrightarrow 0 \end{array}$$

*are exact, for any projective module $P$ and injective module $I$.*

Let $R$ be any ring. Consider $R$ as the right $R$-module. $Hom_{\mathbb{Z}}(R,A)$ becomes a left $R$-module through

$$(rf)(s) = f(sr), s \in R, r \in R, f \in Hom_{\mathbb{Z}}(R,A)$$

**Definition 1.51.** *An $R$-module $C$ is called cofree if $C \simeq \prod_{j\in J} Hom_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})$, for some indexed set $J$.*

The $R$-module structure is given by

$$\left[r\pi_j\left(\prod_{j\in J} Hom_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})\right)\right](s) = (rg)(s),\ r \in R, j \in J, g \in Hom_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z}), s \in R$$

**Lemma 1.52.** *For any ring $R$, and any injective (divisible) abelian group $I$, $Hom_{\mathbb{Z}}(R,I)$ is an injective $R$-module.*

*Proof.* Let $\alpha \in Hom_R(A,B)$, be a monomorphism. For any $\gamma \in Hom_R(A, Hom_{\mathbb{Z}}(R,I))$, we must show that

$$Hom_R(B, Hom_{\mathbb{Z}}(R,I)) \xrightarrow{\alpha^*} Hom_R(A, Hom_{\mathbb{Z}}(R,I)), \alpha^*(g) = g\alpha$$

is an epimorphism. By ([2] Theorem III.7.2) we have the natural group homomorphism

$$Hom_R(N, Hom_{\mathbb{Z}}(M,I)) \simeq Hom_{\mathbb{Z}}(M \otimes_R N, I)$$

When we take $M = R$, we get the isomorphism

$$Hom_R(N, Hom_{\mathbb{Z}}(R,I)) \simeq Hom_{\mathbb{Z}}(N,I)$$

After letting $N = A$ and $N = B$, the problem translates into proving that

$$Hom_{\mathbb{Z}}(B,I) \longrightarrow Hom_{\mathbb{Z}}(A,I)$$

is an epimorphism, which is true, by the definition of the injective group $I$. □

**Corollary 1.53.** *For any ring $R$, $Hom_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})$ is an injective $R$-module.*

**Lemma 1.54.** *Any product of injective $R$-modules in an injective $R$-module, where $R$ is an arbitrary ring.*

*Proof.* For any short exact sequence in $R$-mod

$$0 \longrightarrow N^{'} \longrightarrow N \longrightarrow N^{''} \longrightarrow 0$$

Apply $Hom_R(-, \prod I_k)$, $k$ in some indexed set (finite or infinite), where each $I_k$ is an injective $R$-module, and get the left exact sequence:

$$\begin{array}{ccl} 0 & \longrightarrow & Hom_R(N^{''}, \prod I_k) \longrightarrow Hom_R(N, \prod I_k) \longrightarrow Hom_R(N^{'}, \prod I_k) \\ 0 & \longrightarrow & \prod Hom_R(N^{''}, I_k) \longrightarrow \prod Hom_R(N, I_k) \longrightarrow \prod Hom_R(N^{'}, I_k) \end{array}$$

Now, for each $k$, for any element of $\prod Hom_R(N^{'}, I_k)$, by the Axiom of Choice, it is possible to pick in $\prod Hom_R(N, I_k)$ exactly that $g_k$ such that $g_k \in \pi_k \left(\prod Hom_R(N, I_k)\right)$ and

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r, "\varkappa"] \arrow[d, "f_k"'] & N \arrow[dl, "g_k"] \\ & I_k & \end{tikzcd}$$

commutes. The sequence becomes exact, or equivalently, $\prod I_k$ is an injective $R$-module. $\square$

**Corollary 1.55.** *Any cofree module over any ring is injective.*

**Proposition 1.56.** *(*[2] *I.*(7.1)*) Let $R$ be a $PID$. An $R$-module is injective if and only if it is divisible.*

**Proposition 1.57.** *(*[2] *I.*(7.2)*) Let $R$ be a $PID$. A factor module of a divisible module is divisible.*

**Corollary 1.58.** *(*[2] *I.*(7.4)*) Any abelian group may be embedded in a divisible abelian group.*

## 2. Acknowledgement

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**Part** 1. **Extensions of modules**

## 3. The functors $Ext^n_R$

**Proposition 3.1.** *For any $R$-module $C$, there exists a projective resolution of $C$.*

*Proof.* Any $R$-module $C$ is a quotient of a free, hence projective module. Build the free $R$-module $F_0$ on the generators of $C$ and take the canonical epimorphism $\pi_0 : F_0 \longrightarrow C$. Build the free $R$-module $F_1$ on the generators of $\ker \pi$, and we get the canonical projection $\pi_1 : F_1 \longrightarrow \ker \pi_0$, and continue in this manner. Then we get a long exact sequence

$$... \xrightarrow{\pi_3} F_2 \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} C \longrightarrow 0$$

□

**Definition 3.2.** $Ext^n_R(C, A) := H^n(Hom_R(P_*, A))$, *where* $(P_*, d_*) \longrightarrow C$ *is any projective resolution of* $C$.

The definition of $Ext^n_R$ is correct (it is independent of the choice of projective resolution):

**Lemma 3.3.** *Given any two projective resolutions of* $C$, $(P_*, d_*) \xrightarrow{\varepsilon} C, (Q_*, \delta_*) \xrightarrow{\epsilon} C$, *and an $R$-module $A$, the following cohomology groups are naturally isomorphic:* $H^n(Hom_R(P_*, A)) \simeq H^n(Hom_R(Q_*, A))$.

*Proof.* Since $(P_*, d_*) \xrightarrow{\varepsilon} C$ is also a projective complex over $C$ and $(Q_*, \delta_*) \xrightarrow{\epsilon} C$ is a resolution of $C$, Lemma 1.22 gives that there exists a lifting $f : P \longrightarrow Q$ of $1_C$. Since $(Q_*, \delta_*) \xrightarrow{\epsilon} C$ is also a projective complex over $C$, and $(P_*, d_*) \xrightarrow{\varepsilon} C$ a resolution of $C$, so the same lemma gives that there exists a lifting $g : Q \longrightarrow P$ of $1_C$. Since the composition of two chain transformations is a chain transformation, we obtain two chain transformations $(gf) : P_* \longrightarrow P_*$ and $(fg) : Q_* \longrightarrow Q_*$ that satisfy

$$\begin{aligned} \varepsilon(gf) &= \epsilon f = \varepsilon \\ \epsilon(fg) &= \varepsilon g = \epsilon \end{aligned}$$

so they are homotopic to $1_{P_*}$ and $1_{Q_*}$, respectively. For any $R$-module $A$, applying the functor $Hom_R(-, A)$ gives the commutative diagram of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & Hom_R(C,A) \arrow[r, "\epsilon^*"] \arrow[d, "1_{Hom_R(C,A)}"'] & Hom_R(Q_0,A) \arrow[r, "{\delta_0}^*"] \arrow[d, shift right, "{f_0}^*"'] & Hom_R(Q_1,A) \arrow[r, "{\delta_1}^*"] \arrow[d, shift right, "{f_1}^*"'] & ... \\
0 \arrow[r] & Hom_R(C,A) \arrow[r, "\varepsilon^*"] & Hom_R(P_0,A) \arrow[r, "{d_0}^*"] \arrow[u, shift right, "{g_0}^*"'] & Hom_R(P_1,A) \arrow[r, "{d_1}^*"] \arrow[u, shift right, "{g_1}^*"'] & ...
\end{tikzcd}$$

where

$$\begin{aligned} f^*(u) &= uf, \ u \in Hom_R(P_*, A) \\ g^*v &= vg, \ v \in Hom_R(Q_*, A) \\ (f^*g^*)(v) &= f^*(vg) = v\,(gf) = (gf)^*(v) \\ (g^*f^*)(u) &= (g^*(uf) = ufg = (fg)^*(u) \\ d^*f^*(u) &= d^*(uf) = u\,(fd) = (u\delta)\,f = f^*(\delta u) = f^*\delta^* u \\ f_0^*\epsilon^* &= (\epsilon f_0)^* = \varepsilon^* \end{aligned}$$

By Lemma 1.7, $Hom_R(-,A)$ preserves homotopies, so

$$(gf) \simeq 1_{P_*} \implies f^*g^* = (gf)^* = Hom_R(gf,A) \simeq Hom_R(1_{P_*},A) = 1_{Hom_R(P_*,A)}$$
$$(fg) \simeq 1_{Q_*} \implies g^*f^* = (fg)^* = Hom_R(fg,A) \simeq Hom_R(1_{Q_*},A) = 1_{Hom_R(Q_*,A)}$$

Taking the covariant functor $H^n(-)$ we get

$$H^n(g^*f^*) = H^n(1_{Hom_R(Q_*,A)}) = 1_{H^n(Hom_R(Q_*,A))} = H^n(g^*)H^n(f^*)$$

So $H^n(f^*) : H^n(Hom_R(Q_*,A)) \longrightarrow H^n(Hom_R(P_*,A))$ is an isomorphism, with inverse $H^n(g^*)$. □

**Proposition 3.4.** *$Ext^n_R(-,-)$ is a bifunctor from $R$-mod$\times R$-mod to $AB$, for any $n \in \mathbb{Z}_{\geq 0}$.*

*Proof.* **Step 1.** We will show that $Ext^n_R(C,-)$ is a covariant functor. $Ext^n_R(C,A) := H^n(Hom_R(P_*,A))$, where $(P_*,\delta_*) \xrightarrow{\varepsilon} C$ is a projective resolution of $C$. Let $\alpha \in Hom_{R\text{-mod}}(A,B)$. It induces

$$\begin{tikzcd} 0 \arrow[r] & Hom_R(C,A) \arrow[r, "\varepsilon^*"] \arrow[d, "\alpha_*"'] & Hom_R(P_0,A) \arrow[r, "\delta_0^*"] \arrow[d, "\alpha_*"'] & Hom_R(P_1,A) \arrow[r, "\delta_1^*"] \arrow[d, "\alpha_*"'] & ... \\ 0 \arrow[r] & Hom_R(C,B) \arrow[r, "\varepsilon^*"] & Hom_R(P_0,B) \arrow[r, "\delta_0^*"] & Hom_R(P_1,B) \arrow[r, "\delta_1^*"] & ... \end{tikzcd}$$

where

$$\begin{aligned} \varepsilon^* l &= l\varepsilon,\ l \in Hom_R(C,A) \\ \delta^* h &= h\delta,\ h \in Hom_R(P_*,A) \\ \alpha_* l &= \alpha l \end{aligned}$$

The diagram is commutative:

$$\begin{aligned} \alpha_*\varepsilon^*(l) &= \alpha_*(l\varepsilon) = \alpha l\varepsilon = \varepsilon^*(\alpha l) = \varepsilon^*\alpha_*(l) \\ \alpha_*\delta_n^*(s) &= \alpha_*(s\delta_n) = \alpha s\delta_n = \delta_n^*(\alpha s) = \delta_n^*\alpha_*(s) \end{aligned}$$

So $\alpha_*$ becomes a transformation between the two complexes. Since $H^n$ is a covariant functor, we have

$$\begin{aligned} H^n(\alpha_*) &: H^n(Hom_R(P_*,A)) \longrightarrow H^n(Hom_R(P_*,B)) \\ H^n(\alpha_*) &: Ext^n_R(C,A) \longrightarrow Ext^n_R(C,B) \end{aligned}$$

If $\alpha = 1_A$, we simply get identity transformation on $Hom_R(P_*,A)$, and by functoriality of $H^n$, we get

$$H^n(1_{Hom_R(P_*,A)}) = 1_{H^n(Hom_R(P_*,A))}$$

A composition of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} D$ gives three complexes and two intertwining transformations (since composition of two transformations is a transformation:

$$\begin{tikzcd} 0 \arrow[r] & Hom_R(C,A) \arrow[r, "\varepsilon^*"] \arrow[d, "\alpha_*"'] & Hom_R(P_0,A) \arrow[r, "\delta_0^*"] \arrow[d, "\alpha_*"'] & Hom_R(P_1,A) \arrow[r, "\delta_1^*"] \arrow[d, "\alpha_*"'] & ... \\ 0 \arrow[r] & Hom_R(C,B) \arrow[r, "\varepsilon^*"] \arrow[d, "\beta_*"'] & Hom_R(P_0,B) \arrow[r, "\delta_0^*"] \arrow[d, "\beta_*"'] & Hom_R(P_1,B) \arrow[r, "\delta_1^*"] \arrow[d, "\beta_*"'] & ... \\ 0 \arrow[r] & Hom_R(C,D) \arrow[r, "\varepsilon^*"] & Hom_R(P_0,D) \arrow[r, "\delta_0^*"] & Hom_R(P_1,D) \arrow[r, "\delta_1^*"] & ... \end{tikzcd}$$

As

$$\begin{aligned} \beta_*\alpha_*(s) &= \beta_*(\alpha s) = (\beta\alpha)\, s = (\beta\alpha)_*(s) \\ H^n(\beta_*\alpha_*) &= H^n((\beta\alpha)_*) = H^n(\beta_*)H^n(\alpha_*) : Ext^n_R(C,A) \longrightarrow Ext^n_R(C,D), \end{aligned}$$

so $\beta\alpha$ gives composition $H^n(\beta_*)H^n(\alpha_*)$.

**Step 2.** We will show that $Ext^n_R(-,A)$ is a contravariant functor. Given a $f \in Hom_{R\text{-mod}}(K,C)$, fix a projective resolution of $K$, $(K_*,\zeta_*) \xrightarrow{\epsilon} K$. By Lemma 1.22, there exists a lifting $t : K_* \longrightarrow P_*$. Applying $Hom_R(-,A)$ for any $R$-module $A$, we get the commutative diagram of cochains and cochain transformations $t_*$

$$\begin{tikzcd} 0 \arrow[r] & Hom_R(C,A) \arrow[r, "\varepsilon^*"] \arrow[d, "f^*"'] & Hom_R(P_0,A) \arrow[r, "\delta_0^*"] \arrow[d, "{t^*}_0"'] & Hom_R(P_1,A) \arrow[r, "\delta_1^*"] \arrow[d, "{t^*}_1"'] & ... \\ 0 \arrow[r] & Hom_R(K,A) \arrow[r, "\epsilon^*"] & Hom_R(K_0,A) \arrow[r, "\zeta_0^*"] & Hom_R(K_1,A) \arrow[r, "\zeta_1^*"] & ... \end{tikzcd}$$

$$\begin{aligned} t^*\varepsilon^*(s) &= t^*(s\varepsilon) = s\,(\varepsilon t) = s(f\epsilon) = \epsilon^*(sf) = \epsilon^* f^*(s) \\ t^*_{n+1}\delta^*_n(l) &= t^*(l\delta_0) = l\,(\delta_n t_{n+1}) = (lt_n)\,\zeta_n = \zeta^*_n(lt_n) = \zeta^*_n t^*_n(l), n \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Applying $H^n$ gives

$$\begin{aligned} H^n(t_*) &: H^n(Hom_R(P_*,A)) \longrightarrow H^n(Hom_R(K_*,A)) \\ H^n(t_*) &: Ext^n_R(C,A) \longrightarrow Ext^n_R(K,A) \end{aligned}$$

If $f = 1_C$, $t_* = 1_{Hom_R(P_*,A)}$, and taking $H^n$ gives

$$1_{H^n(Hom_R(P_*,A))} = 1_{Ext^n_R(C,A)}$$

Look at the composition of any two morphisms $L \xrightarrow{g} K \xrightarrow{f} C$. Fix a projective resolution of $L$.

$$\begin{tikzcd} ... \arrow[r, "\gamma_2"] & L_2 \arrow[r, "\gamma_1"] & L_1 \arrow[r, "\gamma_0"] & L_0 \arrow[r, "\rho"] & L \arrow[r] \arrow[d, "g"'] & 0 \\ ... \arrow[r, "\zeta_2"] & K_2 \arrow[r, "\zeta_1"] & K_1 \arrow[r, "\zeta_0"] & K_0 \arrow[r, "\epsilon"] & K \arrow[r] \arrow[d, "f"'] & 0 \\ ... \arrow[r, "\delta_2"] & P_2 \arrow[r, "\delta_1"] & P_1 \arrow[r, "\delta_0"] & P_0 \arrow[r, "\varepsilon"] & C \arrow[r] & 0 \end{tikzcd}$$

By Lemma 1.22, there exists a lifting $s : L_* \longrightarrow K_*$. Then we get the lifting $ts : L_* \longrightarrow C_*$. Apply the functor $Hom_R(-,A)$ (for any fixed $R$-module $A$). We get the commutative diagram of cochain complexes and cochain transformations

$$\begin{tikzcd} 0 \arrow[r] & Hom_R(C,A) \arrow[r, "\varepsilon^*"] \arrow[d, "f^*"'] & Hom_R(P_0,A) \arrow[r, "\delta_0^*"] \arrow[d, "t^*"'] & Hom_R(P_1,A) \arrow[r, "\delta_1^*"] \arrow[d, "t^*"'] & ... \\ 0 \arrow[r] & Hom_R(K,A) \arrow[r, "\epsilon^*"] \arrow[d, "g^*"'] & Hom_R(K_0,A) \arrow[r, "\zeta_0^*"] \arrow[d, "s^*"'] & Hom_R(K_1,A) \arrow[r, "\zeta_1^*"] \arrow[d, "s^*"'] & ... \\ 0 \arrow[r] & Hom_R(L,A) \arrow[r, "\rho^*"] & Hom_R(L_0,A) \arrow[r, "\gamma_0^*"] & Hom_R(L_1,A) \arrow[r, "\gamma_1^*"] & ... \end{tikzcd}$$

Apply $H^n$ and get

$$\begin{aligned} H^n(s^*t^*) &= H^n((ts)^*) = H^n(s^*)H^n(t^*) : H^n(Hom_R(P_*,A)) \longrightarrow H^n(Hom_R(L_*,A)) \\ H^n((ts)^*) &= H^n(s^*)H^n(t^*) : Ext^n_R(C,A) \longrightarrow Ext^n_R(L,A) \end{aligned}$$

**Step 3.** We will establish that $Ext^n_R(-,-)$ is a bifunctor. Since the composition $Ext^n_R(C,A) \xrightarrow{t^*} Ext^n_R(K,A) \xrightarrow{\alpha_*} Ext^n_R(K,B)$ is equal to $Ext^n_R(C,A) \xrightarrow{\alpha_*} Ext^n_R(C,B) \xrightarrow{t^*} Ext^n_R(K,B)$

$$\alpha_* t^*(s) = \alpha_*(st) = (\alpha s)\, t = t^*(\alpha s) = t^* \alpha_*(s)$$

$Ext^n_R(-,-)$ is a bifunctor. □

**Proposition 3.5.** $Ext^0_R(C,A) \simeq Hom_R(C,A)$.

*Proof.* Let $(P_*, d_*) \xrightarrow{\varepsilon} C$ be a projective resolution of $C$. Apply $Hom_R(-,A)$ and get the complex

$$0 \longrightarrow Hom_R(C,A) \xrightarrow{\varepsilon^*} Hom_R(P_0,A) \xrightarrow{d_0^*} Hom_R(P_1,A) \xrightarrow{d_1^*} ...$$

which by Corollary 1.45 is exact at $Hom_R(C,A)$ and $Hom_R(P_0,A)$. Since $d_{-1} = 0$, $d^*_{-1} = 0$,

$$Hom_R(C,A) \simeq \ker d_0^* = \frac{\ker d_0^*}{\operatorname{Im} d^*_{-1}} = H^0(Hom_R(P_*,A)) = Ext^0_R(C,A)$$

□

**Proposition 3.6.** *Given a short exact sequence of $R$-modules*

$$0 \longrightarrow K \xrightarrow{\varkappa} L \xrightarrow{\sigma} M \longrightarrow 0$$

*For any $R$-module $C$, we get a long exact sequence of $Ext^n_R$ :*

$$\begin{array}{lll} 0 & \longrightarrow & Hom_R(C,K) \longrightarrow Hom_R(C,L) \longrightarrow Hom_R(C,M) \longrightarrow Ext^1_R(C,K) \longrightarrow Ext^1_R(C,L) \longrightarrow \\ & \longrightarrow & Ext^1_R(C,M) \longrightarrow Ext^2_R(C,K) \longrightarrow Ext^2_R(C,L) \longrightarrow Ext^2_R(C,M) \longrightarrow \\ & \longrightarrow & ... \longrightarrow Ext^n_R(C,M) \longrightarrow Ext^{n+1}_R(C,K) \longrightarrow Ext^{n+1}_R(C,L) \longrightarrow ... \end{array}$$

*Proof.* Fix a projective resolution of $C$ :

$$... \xrightarrow{\delta_2} P_2 \xrightarrow{\delta_1} P_1 \xrightarrow{\delta_0} P_0 \xrightarrow{\varepsilon} C \longrightarrow 0$$

We get a commutative diagram of three complexes and transformations $\varkappa_*$ and $\sigma_*$ :

$$\begin{tikzcd}
0 \arrow[r] & Hom_R(C,K) \arrow[r, "\varepsilon^*"] \arrow[d, "\varkappa_*"'] & Hom_R(P_0,K) \arrow[r, "\delta_0^*"] \arrow[d, "\varkappa_*"'] & Hom_R(P_1,K) \arrow[r, "\delta_1^*"] \arrow[d, "\varkappa_*"'] & ... \\
0 \arrow[r] & Hom_R(C,L) \arrow[r, "\varepsilon^*"] \arrow[d, "\sigma_*"'] & Hom_R(P_0,L) \arrow[r, "\delta_0^*"] \arrow[d, "\sigma_*"'] & Hom_R(P_1,L) \arrow[r, "\delta_1^*"] \arrow[d, "\sigma_*"'] & ... \\
0 \arrow[r] & Hom_R(C,M) \arrow[r, "\varepsilon^*"] & Hom_R(P_0,M) \arrow[r, "\delta_0^*"] & Hom_R(P_1,M) \arrow[r, "\delta_1^*"] & ...
\end{tikzcd}$$

We get a short exact sequence of complexes

$$0 \longrightarrow Hom_R(P_*,K) \xrightarrow{\varkappa_*} Hom_R(P_*,L) \xrightarrow{\sigma_*} Hom_R(P_*,M) \longrightarrow 0$$

since $Hom_R(P_*,-)$ converts kernels to kernels, and at each dimension, since $P_*$ is projective, $Hom_R(P_*,L) \xrightarrow{\sigma_*} Hom_R(P_*,M)$ is surjective. By Theorem 1.18, we get the long exact sequence

$$\begin{array}{lll} 0 & \longrightarrow & H^0(Hom_R(P_*,K)) \longrightarrow H^0(Hom_R(P_*,L)) \longrightarrow H^0(Hom_R(P_*,M)) \longrightarrow H^1(Hom_R(P_*,K)) \\ & \longrightarrow & H^1(Hom_R(P_*,L)) \longrightarrow H^1(Hom_R(P_*,M)) \longrightarrow H^2(Hom_R(P_*,K)) \longrightarrow H^2(Hom_R(P_*,L)) \longrightarrow ... \end{array}$$

which is isomorphic to

$$\begin{array}{lll} 0 & \longrightarrow & Hom_R(C,K) \longrightarrow Hom_R(C,L) \longrightarrow Hom_R(C,M) \longrightarrow Ext^1_R(C,K) \longrightarrow Ext^1_R(C,L) \longrightarrow \\ & \longrightarrow & Ext^1_R(C,M) \longrightarrow Ext^2_R(C,K) \longrightarrow Ext^2_R(C,L) \longrightarrow ... \end{array}$$

□

**Proposition 3.7.** *Given a short exact sequence of $R$-modules*

$$0 \longrightarrow K \xrightarrow{\varkappa} L \xrightarrow{\sigma} M \longrightarrow 0$$

*For any $R$-module $A$, we get a long exact sequence of $Ext^n_R$ :*

$$\begin{array}{lll} 0 & \longrightarrow & Hom_R(M,A) \longrightarrow Hom_R(L,A) \longrightarrow Hom_R(K,A) \longrightarrow Ext^1_R(M,A) \longrightarrow Ext^1_R(L,A) \longrightarrow \\ & \longrightarrow & Ext^1_R(K,A) \longrightarrow Ext^2_R(M,A) \longrightarrow Ext^2_R(L,A) \longrightarrow Ext^2_R(K,A) \longrightarrow ... \longrightarrow Ext^n_R(K,A) \\ & \longrightarrow & Ext^{n+1}_R(M,A) \longrightarrow Ext^{n+1}_R(L,A) \longrightarrow ... \end{array}$$

*Proof.* Fix projective resolutions of $K$, and $M$.

$$\begin{tikzcd}
\cdot \arrow[d, "\delta_1"'] & & \cdot \arrow[d, "d_1"] \\
P_1 \arrow[r] \arrow[d, "\delta_0"'] & P_1 \oplus Q_1 \arrow[r] & Q_1 \arrow[d, "d_0"] \\
P_0 \arrow[r] \arrow[d, "\varepsilon"'] & P_0 \oplus Q_0 \arrow[r] & Q_0 \arrow[d, "\epsilon"] \\
K \arrow[r, "\mu"] & L \arrow[r, "\sigma"] & M
\end{tikzcd}$$

Start building a projective resolution of $L$, that makes the diagram commutative. Since

$$Hom_R(P_0 \oplus Q_0, L) \simeq Hom_R(P_0, L) \times Hom_R(Q_0, L)$$

any $h : P_0 \oplus Q_0 \longrightarrow L$ can be written as

$$h(p,q) = f(p) + g(q), \text{ where } f : P_0 \longrightarrow L, g : Q_0 \longrightarrow L$$

Such a $g$ exists since $Q_0$ is projective. We need that

$$\begin{array}{rcl} hi & = & \mu\delta_{-1} \wedge \sigma h = d_{-1}\pi. \\ hi(p) & = & h(p,0) = f(p) + g(0) = f(p). \end{array}$$

Define $f(p) = \mu\delta_{-1}$.

$$\sigma h(p,q) = \sigma(f(p) + g(q)) = \sigma f(p) + \sigma g(q) = \sigma(\mu\delta_{-1}(p)) + \sigma g(q) = \sigma g(q)$$

Define $g(q)$ as $\sigma g(q) = d_{-1}\pi(p,q) = d_{-1}(q)$. Let $(K_1, e_K)$, $(L_1, e_L)$, $(M_1, e_M)$ be the kernels of $\varepsilon, h$, and $\epsilon$, respectively. By Lemma 1.49, since $coker(\varepsilon)$ is 0, we have a short exact sequence

$$0 \longrightarrow K_1 \longrightarrow L_1 \longrightarrow M_1 \longrightarrow 0$$

We have new maps $\alpha$, $\beta$, which are the (unique) maps satisfying

$$e_L\alpha = i_0 e_K, e_M\beta = \pi_0 e_L$$

(follows from the definition of the kernel, Definition 1.28). We have built the commutative diagram of short exact sequences

$$\begin{tikzcd}
K_1 \arrow[r, "\alpha"] \arrow[d, "e_K"'] & L_1 \arrow[r, "\beta"] \arrow[d, "e_L"'] & M_1 \arrow[d, "e_M"] \\
P_0 \arrow[r, "i_0"] \arrow[d, "\varepsilon"'] & P_0 \oplus Q_0 \arrow[r, "\pi_0"] \arrow[d, "h"'] & Q_0 \arrow[d, "\epsilon"] \\
K \arrow[r, "\mu"] \arrow[d] & L \arrow[r, "\sigma"] \arrow[d] & M \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

Since

$$\varepsilon\delta_0 = 0, \epsilon d_0 = 0$$

there exists unique homomorphisms $u : P_1 \longrightarrow K_1$ and $v : Q_1 \longrightarrow M_1$, such that

$$e_K u = \delta_0, e_M v = d_0$$

Since we have built $\alpha u : P_1 \longrightarrow L_1$, there exists a homomorphism $k : P_1 \oplus Q_1 \longrightarrow L_1$. Any such $k$ can be described as

$$k(p, q) = s(p) + t(q)$$

We now require that

$$ki_1 = \alpha u, v\pi_1 = \beta k$$

Since

$$ki_1(p, q) = k(p, 0) = s(p) \implies \text{define } s(p) = \alpha u$$

Now,

$$\begin{aligned}
\beta k(p, q) &= \beta(s(p) + t(q)) = \beta s(p) + \beta t(q) = \beta\alpha u(p) + \beta t(q) = \beta t(q) \\
&\implies \text{define } t(q) \text{ as } \beta t(q) = v\pi_1(p, q) = v(q)
\end{aligned}$$

The only thing remaining is to check exactness

$$P_1 \oplus Q_1 \xrightarrow{e_L k} P_0 \oplus Q_0 \xrightarrow{h} L$$

$$he_L k = 0, \text{ so } \mathrm{Im}(e_L k) \subseteq \ker(h)$$

To prove the other way, it is enough to show that $k$ is surjective on $L_1$. Using Lemma 1.47, it is enough to show that $u$ and $v$ are surjective. But this follows, since

$$\begin{aligned}
\mathrm{Im}\,\delta_0 &= \mathrm{Im}\,e_K u = \ker\varepsilon \implies u \text{ is surjective} \\
\mathrm{Im}\,d_0 &= \mathrm{Im}\,e_M v = \ker\varepsilon \implies v \text{ is surjective}
\end{aligned}$$

So we have developed the commutative diagram

$$\begin{tikzcd}
P_1 \arrow[r, "i_1"] \arrow[d, "\delta_0"'] & P_1 \oplus Q_1 \arrow[r, "\pi_1"] \arrow[d, "e_L k"'] & Q_1 \arrow[d, "d_0"] \\
P_0 \arrow[r, "i_0"] \arrow[d, "\varepsilon"'] & P_0 \oplus Q_0 \arrow[r, "\pi_0"] \arrow[d, "h"'] & Q_0 \arrow[d, "\epsilon"] \\
K \arrow[r, "\mu"] & L \arrow[r, "\sigma"] & M
\end{tikzcd}$$

Continue this procedure, i.e. take $(K_2, e_{K_2})$, $(L_2, e_{L_2})$, $(M_2, e_{M_2})$ kernels of $\delta_0$, $e_L k$, and $d_0$. We will get a new short exact sequence

$$0 \longrightarrow P_2 \longrightarrow P_2 \oplus Q_2 \longrightarrow Q_2 \longrightarrow 0$$

which together with a homomorphisms $(\delta_1, l, d_1)$, make a larger commutative diagram. We then get a projective resolution $P_* \oplus Q_*$ of $L$, and we get a short exact sequence of complexes

$$0 \longrightarrow P_* \xrightarrow{i_*} P_* \oplus Q_* \xrightarrow{\pi_*} Q_* \longrightarrow 0$$

where, for each $n \geq 0$, we have a split exact sequence

$$0 \longrightarrow P_n \xrightarrow{i_n} P_n \oplus Q_n \xrightarrow{\pi_n} Q_n \longrightarrow 0$$

where $i_n$ is the natural inclusion and $\pi_n$ is the natural projection. For any $R$-module $A$, apply $Hom_R(-, A)$ to the short exact sequence of complexes. We get a short exact sequence of cochain complexes

$$0 \longrightarrow Hom_R(P_*, A) \longrightarrow Hom_R(P_* \oplus Q_*, A) \longrightarrow Hom_R(Q_*, A) \longrightarrow 0$$

For each $n \geq 0$, it is split exact, since $Hom_R(P_n \oplus Q_n, A) \simeq Hom_R(P_n, A) \times Hom_R(Q_n, A)$. Apply $H^n$ to get the long exact sequence of cohomology

$$\begin{aligned} 0 \quad \longrightarrow \quad & H^0(Hom_R(P_*, A)) \longrightarrow H^0(Hom_R(P_* \oplus Q_*, A)) \longrightarrow H^0(Hom_R(Q_*, A)) \longrightarrow \\ \longrightarrow \quad & H^1(Hom_R(P_*, A)) \longrightarrow H^1(Hom_R(P_* \oplus Q_*, A)) \longrightarrow ... \end{aligned}$$

which is isomorphic to

$$0 \longrightarrow Hom_R(K, A) \longrightarrow Hom_R(L, A) \longrightarrow Hom_R(M, A) \longrightarrow Ext^1_R(K, A) \longrightarrow Ext^1_R(L, A) \longrightarrow ...$$

□

**Proposition 3.8.** *$Ext^n_R(P, A) = 0$, $P$ projective module, $n \geq 1$.*

*Proof.*

$$... \longrightarrow 0 \longrightarrow 0 \longrightarrow P \xrightarrow{1_P} P \longrightarrow 0$$

is a projective resolution of our projective module $P$, where $1_P$ is as isomorphism of $P$. By Corollary 1.45, taking $Hom_R(-, A)$ converts cokernels to kernels, and we obtain the complex

$$0 \longrightarrow Hom_R(P, A) \overset{(1_P)_*}{\twoheadrightarrow} Hom_R(P, A) \xrightarrow{0_*} 0 \xrightarrow{0_*} 0 \xrightarrow{0_*} ...$$

So $\operatorname{Im}((1_P)_*) = \ker 0_* = Hom_R(P, A)$, so $(1_P)_*$ is an isomorphism, and we have

$$H^1(Hom_R(P_*, A)) = \frac{\{0\}}{\{0\}} \simeq 0 = H^2(Hom_R(P_*, A)) = .. = H^n(Hom_R(P_*, A)) = ..$$

for any $n \geq 1$, which gives

$$Ext^n_R(P, A) = 0, \ n \geq 1.$$

□

**Proposition 3.9.** *Given the short exact sequence of $R$-modules*

$$E : 0 \longrightarrow S \longrightarrow P \longrightarrow C \longrightarrow 0$$

*where $P$ is projective (also called a projective presentation of $C$),*

$$Ext^1_R(C, A) \simeq coker(Hom_R(P, A) \longrightarrow Hom_R(S, A))$$

*Also,*

$$Ext^i_R(S,A) \simeq Ext^{i+1}_R(C,A), i \geq 1.$$

*Proof.* Using Proposition 3.7, we get the long exact sequence where $Ext^n_R(P,A) = 0$, for $n \geq 1$

$$\begin{aligned} 0 \quad \longrightarrow \quad & Hom_R(C,A) \longrightarrow Hom_R(P,A) \longrightarrow Hom_R(S,A) \longrightarrow Ext^1_R(C,A) \longrightarrow 0 \longrightarrow \\ \longrightarrow \quad & Ext^1_R(S,A) \longrightarrow Ext^2_R(C,A) \longrightarrow 0 \longrightarrow Ext^2_R(S,A) \longrightarrow Ext^3_R(C,A) \longrightarrow 0 \longrightarrow .. \end{aligned}$$

Exactness gives surjectivity on $Ext^1(C,A)$, so by the description of cokernel in $R$-mod, $Ext^1(C,A) \simeq coker(Hom_R(P,A) \longrightarrow Hom_R(R,A))$. Also, $Ext^i_R(S,A) \simeq Ext^{i+1}_R(C,A)$, $i \geq 1$. □

**Proposition 3.10.** *$H^n(Hom_R(X,I)) \simeq Hom_R(H_n(X),I)$, where $X$ is a complex, $I$ an injective module.*

*Proof.* Fix the complex

$$... \xrightarrow{d_{n+1}} X_{n+1} \xrightarrow{d_n} X_n \xrightarrow{d_{n-1}} X_{n-1} \xrightarrow{d_{n-2}} ...$$

Apply $Hom_R(-,I)$ and get the cochain complex

$$\begin{aligned} ... \quad \longrightarrow \quad & Hom_R(X_{n-1},I) \xrightarrow{d^*_{n-1}} Hom_R(X_n,I) \xrightarrow{d^*_n} Hom_R(X_{n+1},I) \longrightarrow .. \\ d^*_n(f) \quad = \quad & fd_n, \text{ for any } n \in \mathbb{Z}. \end{aligned}$$

For any $f \in Z^n$, we can find a morphism to $Hom_R(H_n(X),I)$

$$f{\in}Z^n{\in}Hom_R(Xn,I) \xrightarrow{restriction} Hom_R(Hn(X),I)$$

Define this homomorphism $\zeta : Z^n \longrightarrow Hom_R(H_n(X),I)$. Will show that $\zeta$ is an epimorphism with kernel $B_n$. Given an $g : H_n(X) \longrightarrow I$, i.e.

$$\begin{aligned} g \quad &: \quad Z_n \longrightarrow I \\ \ker(g) \quad &= \quad d_n X_{n+1} \\ g(x) \quad &| \quad d_{n-1}x = 0 \end{aligned}$$

Let $i : Z_n \longrightarrow X_n$ be the canonical injection homomorphism.

$$\begin{tikzcd} Z_n \arrow[r, "i"] \arrow[d, "g"'] & X_n \\ I & \end{tikzcd}$$

Since $I$ is an injective module, there exists an

$$h : X_n \longrightarrow I \mid hi = g$$

$h$ is a $n$-cocycle since

$$d^*_n h(x) = h(d_n(x)) = g(d_n(x)) = 0.$$

Take $h \in B^n$, i.e. $h = sd_{n-1}$. Take any $x \in H_n(X)$.

$$h(x) = sd_{n-1}(x) = s(0) = 0 \implies B^n \in \ker \zeta.$$

Take any $f \in \ker \zeta$, so

$$\begin{array}{rcl} f(x) & = & 0, \forall x \in H_n(x) \implies f(x) = 0, \forall x \in Z_n \implies \exists \widetilde{f} : X_n/Z_n \simeq B_{n-1} \longrightarrow I \\ \widetilde{f} & = & gd_{n-1} \in B_n,\ g : X_n \longrightarrow I. \\ \widetilde{f} & \in & \ker \zeta : \widetilde{f}(x) = gd_{n-1}(x) = g(0) = 0, \forall x \in H_n(x). \text{ So } B^n \in \ker \zeta. \end{array}$$

□

**Proposition 3.11.** *$Ext^n_R(C, I) = 0$, $I$ injective module, $n \geq 1$.*

*Proof.* For any resolution of $C$, $.. \longrightarrow P_1 \xrightarrow{\delta^1} P_0 \overset{\varepsilon}{\twoheadrightarrow} C \longrightarrow 0$ we get using Lemma 3.10 that

$$H^n(Hom_R(P_*, I)) \simeq Hom_R(H_n(P_*), I) = Hom_R(0, I) = 0$$

for $n \geq 1$. □

**Proposition 3.12.** *$Ext^n_{\mathbb{Z}}(C, A) = 0$, $C$ and $A$ abelian groups, $n \geq 2$.*

*Proof.* Since any abelian group is isomorphic to a quotient of a free abelian group, we get the short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow C \longrightarrow 0$$

which is a resolution of $C$ (since any subgroup of a free abelian group is itself a free abelian group). For any $R$-module $A$, by Propositions 3.5, 3.9 and 3.8, we have:

$$\begin{array}{rcl} Ext^0_{\mathbb{Z}}(C, A) & \simeq & Hom_{\mathbb{Z}}(C, A) \\ Ext^1_{\mathbb{Z}}(C, A) & \simeq & coker(Hom_{\mathbb{Z}}(F, A) \longrightarrow Hom_{\mathbb{Z}}(K, A)) \\ Ext^{i+1}_{\mathbb{Z}}(C, A) & \simeq & Ext^i_{\mathbb{Z}}(K, A) = 0, i \geq 2 \end{array}$$

□

## 4. The functors $\overline{Ext}^n_R$

**Proposition 4.1.** *For any $R$-module $A$, there exists an injective coresolution of $A$.*

*Proof.* Will show that any module can be embedded in an injective module. We have the $R$-module monomorphism $\gamma : A \longrightarrow Hom_{\mathbb{Z}}(R, A)$ as

$$\begin{array}{rcl} (\gamma(a))(r) & = & f(r) = ra \\ ra & = & 0,\ \forall r \in R \implies a = 0 \end{array}$$

By Corollary 1.58, there exists an injective group homomorphism $j : A \longrightarrow I$, where $I$ is an injective $\mathbb{Z}$-module. We have the short exact sequence

$$0 \longrightarrow A \longrightarrow I \longrightarrow K_{AI} \longrightarrow 0$$

Apply $Hom_{\mathbb{Z}}(R, -)$ which by Corollary 1.45 preserves kernels, so we get the left exact sequence of $R$-modules

$$0 \longrightarrow Hom_{\mathbb{Z}}(R, A) \xrightarrow{j_*} Hom_{\mathbb{Z}}(R, I) \longrightarrow Hom_{\mathbb{Z}}(R, K_{AI}),\ j_* f = jf$$

and the composition of $R$-module monomorphisms $j_*\gamma : A \longrightarrow Hom_{\mathbb{Z}}(R, I)$. Since $Hom_{\mathbb{Z}}(R, I)$ is an injective $R$-module,we have found the first step of a coresolution of $A$. Let $C = coker(j^*\gamma) \simeq Hom_{\mathbb{Z}}(R, I)/\operatorname{Im}(j_*\gamma)$. We have the $R$-module monomorphism $\alpha : C \longrightarrow Hom_{\mathbb{Z}}(R, C)$. There exists an injective group homomorphism $\beta : C \longrightarrow J$, for some divisible abelian group $J$. Apply $Hom_{\mathbb{Z}}(R, -)$ on the short exact sequence

$$0 \longrightarrow C \longrightarrow J \longrightarrow K_{CJ} \longrightarrow 0$$

and get the left exact sequence

$$0 \longrightarrow Hom_{\mathbb{Z}}(R, C) \xrightarrow{\beta_*} Hom_{\mathbb{Z}}(R, J) \longrightarrow Hom_{\mathbb{Z}}(R, K_{CJ}),\ \beta_* f = \beta f$$

and the composition of the two $R$-module monomorphisms $\beta_*\alpha : C \longrightarrow Hom_{\mathbb{Z}}(R, J)$, which gives an $R$-module homomorphism from $Hom_{\mathbb{Z}}(R, I) \longrightarrow Hom_{\mathbb{Z}}(R, J)$ with kernel the image of $j_*\gamma$, so we have build the left exact sequence

$$0 \longrightarrow A \longrightarrow Hom_{\mathbb{Z}}(R, I) \longrightarrow Hom_{\mathbb{Z}}(R, J)$$

Repeat this step, and we will get a injective coresolution of $A$. $\square$

**Definition 4.2.** $\overline{Ext}^n_R(C, A) := H^n(Hom_R(C, I^*))$, *where* $A \xrightarrow{\varepsilon} (I^*, d^*)$ *is an injective coresolution of* $A$.

The definition of $\overline{Ext}^n_R$ is correct (it is independent of the choice of injective coresolution):

**Lemma 4.3.** *Given any two injective coresolutions of* $A$,

$$0 \longrightarrow A \xrightarrow{\varepsilon} (I^*, d^*),\ 0 \longrightarrow A \xrightarrow{\varepsilon'} (J^*, \delta^*)$$

*and an* $R$*-module* $C$*, the following cohomology groups are naturally isomorphic:* $H^n(Hom_R(C, I^*)) \simeq H^n(Hom_R(C, J^*))$.

*Proof.* Use Lemma 1.23, with $\alpha = 1_A$. We get two liftings $f : I^* \longrightarrow J^*$ and $g : J^* \longrightarrow I^*$. Since the composition of two liftings is a lifting, $(fg) : J^* \longrightarrow J^*$ and $(gf) : I^* \longrightarrow I^*$ are liftings. Since any two such liftings are homotopic, we get $(fg) \simeq 1_{J^*}$ and $(gf) \simeq 1_{I^*}$. By Lemma 1.7, the additive covariant functor $Hom_R(C, -)$ preserves homotopies. We get

$$\begin{aligned} fg &\simeq 1_{J_*} \implies f_*g_* = (fg)_* = Hom_R(C, fg) \simeq Hom_R(C, 1_{J_*}) = 1_{Hom_R(C,J_*)} \\ gf &\simeq 1_{I_*} \implies g_*f_* = (gf)_* Hom_R(C, gf) \simeq Hom_R(C, 1_{I_*}) = 1_{Hom_R(C,I_*)} \end{aligned}$$

where

$$\begin{aligned} f_*(v) &= fv,\ f_* : Hom_R(C, I_*) \longrightarrow Hom_R(C, J_*),\ v \in Hom_R(C, I^*), \\ g_*(u) &= gu,\ g_* : Hom_R(C, J_*) \longrightarrow Hom_R(C, I_*),\ u \in Hom_R(C, J^*) \\ (f_*g_*)(u) &= f_*(gu) = (fg)\, u = (fg)_*(u) \\ (g_*f_*)(v) &= g_*(fv) = (gf)\, v = (gf)_*(v) \end{aligned}$$

Using Proposition 1.17,

$$\begin{aligned} H^n\left((fg)_*\right) &= H^n(f_*)H^n(g_*) = H^n(1_{Hom_R(C,J^*)}) = 1_{H^n(Hom_R(C,J^*))} \\ H\left(^n(gf)_*\right) &= H^n(g_*)H^n(f_*) = H^n(1_{Hom_R(C,I^*)}) = 1_{H^n(Hom_R(C,I^*))} \end{aligned}$$

so $H^n(f_*) : H^n(Hom_R(C, I^*)) \longrightarrow H^n(Hom_R(C, J^*))$ is an isomorphism (with inverse $H^n(g_*)$). $\square$

**Proposition 4.4.** $\overline{Ext}^n_R$ *is a bifunctor from* $R$*-mod*$\times R$*-mod to* $AB$*, for any* $n \in \mathbb{Z}_{\geq 0}$.

*Proof.* **Step 1.** We will establish that $\overline{Ext}^n_R(-,A)$ is a contravariant functor from $R$-mod to $AB$. Given a morphism $g : D \longrightarrow C$. Fix $A$, and an injective coresolution of $A$.

$$\begin{tikzcd} & D \arrow[r] & C \arrow[d, "f"'] & & & \\ 0 \arrow[rr] & & A \arrow[r, "\varepsilon"] & I^0 \arrow[r, "\delta^0"] & I^1 \arrow[r, "\delta^1"] & I^2 \arrow[r, "\delta^2"] & \dots \end{tikzcd}$$

We then have commutativity at each level of the two induced left exact complexes:

$$\begin{tikzcd} 0 \arrow[r] & Hom_R(C,A) \arrow[r, "\varepsilon_*"] \arrow[d, "g^*"'] & Hom_R(C,I^0) \arrow[r, "\delta^0_*"] \arrow[d, "g^*"'] & Hom_R(C,I^1) \arrow[r, "\delta^1_*"] \arrow[d, "g^*"'] & Hom_R(C,I^2) \dots \arrow[d, "g^*"'] \\ 0 \arrow[r] & Hom_R(D,A) \arrow[r, "\varepsilon_*"] & Hom_R(D,I^0) \arrow[r, "\delta^0_*"] & Hom_R(D,I^1) \arrow[r, "\delta^1_*"] & Hom_R(D,I^2) \dots \end{tikzcd}$$

where

$$\begin{aligned} \varepsilon_*(s) &= \varepsilon s, s \in Hom_R C, A) \\ g^*(u) &= ug, u \in Hom_R(C,A) \\ \delta_*(v) &= \delta v, v \in Hom_R(C,I^*) \\ g^*\varepsilon_*(s) &= g^*(\varepsilon s) = \varepsilon\,(sg) = \varepsilon_*(sg) = \varepsilon_* g^*(s) \\ g^*\delta_*(v) &= g^*(\delta v) = \delta\,(vg) = \delta_*(vg) = \delta_* g^*(v) \end{aligned}$$

Hence $g^*$ is a cochain transformation and applying $H^n$ gives:

$$\begin{aligned} H^n(g^*) &: H^n(Hom_R(C,I^*)) \longrightarrow H^n(Hom_R(D,I^*)) \\ H^n(g^*) &: \overline{Ext}^n_R(C,A) \longrightarrow \overline{Ext}^n_R(D,A) \end{aligned}$$

If $g = 1_C$, then we get the identity $1_{Hom_R(C,I^*)}$, which gives

$$H^n(1_{Hom_R(C,I^*)}) = 1_{H^n(Hom_R(C,I^*))} = 1_{\overline{Ext}^n_R(C,A)}$$

Now, let's look at the composition $E \xrightarrow{h} D \xrightarrow{g} C$. We get three complexes and two intertwining transformations $g^*$ and $h^*$ :

$$\begin{tikzcd} 0 \arrow[r] & Hom_R(C,A) \arrow[r, "\varepsilon_*"] \arrow[d, "g^*"'] & Hom_R(C,I^0) \arrow[r, "\delta^0_*"] \arrow[d, "g^*"'] & Hom_R(C,I^1) \arrow[r, "\delta^1_*"] \arrow[d, "g^*"'] & Hom_R(C,I^2) \dots \arrow[d, "g^*"'] \\ 0 \arrow[r] & Hom_R(D,A) \arrow[r, "\varepsilon_*"] \arrow[d, "h^*"'] & Hom_R(D,I^0) \arrow[r, "\delta^0_*"] \arrow[d, "h^*"'] & Hom_R(D,I^1) \arrow[r, "\delta^1_*"] \arrow[d, "h^*"'] & Hom_R(D,I^2) \dots \arrow[d, "h^*"'] \\ 0 \arrow[r] & Hom_R(E,A) \arrow[r, "\varepsilon_*"] & Hom_R(E,I^0) \arrow[r, "\delta^0_*"] & Hom_R(E,I^1) \arrow[r, "\delta^1_*"] & Hom_R(E,I^2) \dots \end{tikzcd}$$

$$\begin{aligned} h^*g^*(u) &= h^*(ug) = u\,(gh) = (gh)^*(u) \\ H^n(h^*g^*) &= H^n((gh)^*) = H^n(h^*)H^n(g^*) : H^n(Hom_R(C,I^*)) \longrightarrow H^n(Hom_R(E,I^*)) \\ H^n(h^*g^*) &: \overline{Ext}^n_R(C,A) \longrightarrow \overline{Ext}^n_R(E,A) \end{aligned}$$

**Step 2.** We will show that $\overline{Ext}^n_R(C,-)$ is a covariant functor from $R$-mod to $AB$. Fix $C$. Suppose $\alpha \in Hom_{R\text{-mod}}(A,B)$. Fix an injective coresolution of $A$ and $B$, $(I^*,\delta^*) \xrightarrow{\epsilon} A$ and $(J^*,\zeta^*) \xrightarrow{\varepsilon} B$, respectively. By Lemma 1.23, there exists a lifting $f : I^* \longrightarrow J^*$. Take $Hom_R(C, -)$ and get following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_R(C,A) & \xrightarrow{\varepsilon_*} & Hom_R(C,I^0) & \xrightarrow{\delta^0_*} & Hom_R(C,I^1) & \xrightarrow{\delta^1_*} & ... \\ & & \alpha_* \downarrow & & f^0_* \downarrow & & f^1_* \downarrow & & \\ 0 & \longrightarrow & Hom_R(C,B) & \xrightarrow{\epsilon_*} & Hom_R(C,J^0) & \xrightarrow{\zeta^0_*} & Hom_R(C,J^1) & \xrightarrow{\zeta^1_*} & ... \end{array}$$

where

$$\begin{aligned} \alpha_*(u) &= \alpha u, u \in Hom_R(C,A) \\ f_*(v) &= fv, v \in Hom_R(C,I^*) \\ \delta^*(v) &= v\delta \\ \zeta^*(s) &= s\zeta, s \in Hom_R(C,J^*) \\ \epsilon_*(u) &= \varepsilon u \\ \varepsilon_*(t) &= \varepsilon t, t \in Hom_R(C,B) \end{aligned}$$

The diagram is commutative (which gives that $f_* : Hom_R(C, I^*) \longrightarrow Hom_R(C, J^*)$ is a transformation):

$$\begin{aligned} f_*\epsilon_*(u) &= f_*(\varepsilon u) = (f\varepsilon)\, u = \varepsilon\,(\alpha u) = \varepsilon_*(\alpha u) = \varepsilon_*\alpha_*(u) \\ f^{n+1}_*\delta^n_*(v) &= f^{n+1}_*(\delta^n v) = \left(f^{n+1}\delta^n\right) v = \zeta^n\,(f^n v) = \zeta^n_*(f^n v) = \zeta^n_n f^n_*(v), n \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

Apply $H^n$ :

$$\begin{aligned} H^n(f_*) &: H^n(Hom_R(C,I^*) \longrightarrow H^n(Hom_R(C,J^*)) \\ H^n(f_*) &: \overline{Ext}^n_R(C,A) \longrightarrow \overline{Ext}^n_R(C,B) \end{aligned}$$

If $\alpha = 1_A$, then we would get the identity transformation on the complex $Hom_R(C, I^*)$, and applying $H^n$ gives

$$H^n(1_{Hom_R(C,I^*)}) = 1_{H^n(Hom_R(C,I^*))} = 1_{\overline{Ext}^n_R(C,A)}$$

Let's look at the composition $A \xrightarrow{\alpha} B \xrightarrow{\beta} D$. Let $(K^*, \rho^*) \xrightarrow{\xi} D$ be a coresolution of $D$. Lemma 1.23 gives the existence of a lifting $g : J^* \longrightarrow K^*$. Then the composition $gf : I^* \longrightarrow K^*$ is a lifting too. Apply $Hom_R(C, -)$ and get the commutativity conditions:

$$\begin{aligned} (g_0f_0)_*\epsilon_*(u) &= (g_0f_0)_*(\epsilon u) = g_0f_0\epsilon u = \xi(\beta\alpha)u = \zeta_*(\beta\alpha u) = \zeta_*(\beta\alpha)_*(u) \\ (gf)_*\delta_*(v) &= (gf)_*(\delta v) = (gf\delta)\, v = \rho(gf)v = \rho_*(gfv) = \rho_*(gf)_*(v) \end{aligned}$$

So $(gf)_* : Hom_R(C, I^*) \longrightarrow Hom_R(C, K^*)$ becomes a transformation. Also,

$$(gf)_*(v) = g\,(fv) = g_*(f_*(v) = g_*f_*(v)$$

As $g_*$ and $f_*$ are transformations, applying $H^n$ gives:

$$\begin{aligned} H^n((gf)_*) &= H^n(g_*f_*) = H^n(g^*)H^n(f^*) : H^n(Hom_R(C,I^*)) \longrightarrow H^n(Hom_R(C,K^*)) \\ H^n((gf)_*) &= H^n(g^*)H^n(f^*) : \overline{Ext}^n_R(C,A) \longrightarrow \overline{Ext}^n_R(C,D) \end{aligned}$$

**Step 3.** We must check whether the compositions $\overline{Ext}^n_R(C, A) \longrightarrow \overline{Ext}^n_R(C, B) \longrightarrow \overline{Ext}^n_R(D, B)$ and $\overline{Ext}^n_R(C, A) \longrightarrow \overline{Ext}^n_R(D, A) \longrightarrow \overline{Ext}^n_R(D, B)$, are equal, for any $n \in \mathbb{Z}_{\geq 0}$. Take a $k \in \overline{Ext}^n_R(C, A)$. Using the notation of this proof, the first gives

$$g^*(f^n_* k) = g^*(f^n k) = f^n kg$$

and the second gives

$$f_*^n(g^*k) = f_*^n(kg) = f^n kg$$

so $\overline{Ext}_R^n(-,-)$ is a bifunctor. $\square$

**Proposition 4.5.** $\overline{Ext}_R^0(C,A) \simeq Hom_R(C,A)$.

*Proof.* $\overline{Ext}_R^0(C,A) := H^0(Hom_R(C,I^*))$, where $A \xrightarrow{\varepsilon} (I^*,\delta^*)$, is any injective coresolution of $A$. Applying $Hom_R(C,-)$ gives the complex

$$0 \longrightarrow Hom_R(C,A) \xrightarrow{\varepsilon_*} Hom_R(C,I^0) \xrightarrow{\delta_*^0} Hom_R(C,I^1) \xrightarrow{\delta_*^1} Hom_R(C,I^2) \xrightarrow{\delta_*^2} ...$$

which is exact at $Hom_R(C,A)$ and $Hom_R(C,I^0)$, since $Hom_R(C,-)$ preserves kernels. $\overline{Ext}_R^0(C,A) = \frac{\ker \delta_*^0}{\{0\}} \simeq \operatorname{Im}\varepsilon_* \simeq Hom_R(C,A)$. $\square$

**Proposition 4.6.** *Given short exact sequence of R-modules*

$$0 \longrightarrow K \xrightarrow{\varkappa} L \xrightarrow{\sigma} M \longrightarrow 0$$

*For any R-module A, we get a long exact sequence of* $\overline{Ext}_R^n$ :

$$\begin{aligned} 0 \quad &\longrightarrow \quad Hom_R(M,A) \longrightarrow Hom_R(L,A) \longrightarrow Hom_R(K,A) \longrightarrow \overline{Ext}_R^1(M,A) \longrightarrow \overline{Ext}_R^1(L,A) \longrightarrow \\ \overline{Ext}_R^1(K,A) \quad &\longrightarrow \quad \overline{Ext}_R^2(M,A) \longrightarrow \overline{Ext}_R^2(L,A) \longrightarrow ... \longrightarrow \overline{Ext}_R^n(K,A) \longrightarrow \overline{Ext}_R^{n+1}(M,A) \longrightarrow ... \end{aligned}$$

*Proof.* Pick an injective coresolution of $A$, $A \xrightarrow{\varepsilon} (I^*,d^*)$. It induces the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Hom_R(M,A) \arrow[r, "\varepsilon_*"] \arrow[d, "\sigma^*"'] & Hom_R(M,I^0) \arrow[r, "d^0{}_*"] \arrow[d, "\sigma^*"'] & Hom_R(M,I^1) \arrow[r, "d^1{}_*"] \arrow[d, "\sigma^*"'] & ... \\
0 \arrow[r] & Hom_R(L,A) \arrow[r, "\varepsilon_*"] \arrow[d, "\varkappa^*"'] & Hom_R(L,I^0) \arrow[r, "d^0{}_*"] \arrow[d, "\varkappa^*"'] & Hom_R(L,I^1) \arrow[r, "d^1{}_*"] \arrow[d, "\varkappa^*"'] & ... \\
0 \arrow[r] & Hom_R(K,A) \arrow[r, "\varepsilon_*"] & Hom_R(K,I^0) \arrow[r, "d^0{}_*"] & Hom_R(K,I^1) \arrow[r, "d^1{}_*"] & ...
\end{tikzcd}$$

since $\sigma^*$ and $\varkappa^*$ are transformations such that

$$\sigma^*\varepsilon_* = \varepsilon_*\sigma^*, \ \varkappa^*\varepsilon_* = \varepsilon_*\varkappa^*$$

Then, $\varkappa^*\sigma^* = (\sigma\varkappa)^* = 0^* : Hom_R(M,I^*) \longrightarrow Hom_R(K,I^*)$, so,

$$\operatorname{Im}\sigma^* \subseteq \ker\varkappa^*; \ f \in \ker\varkappa^* \iff \varkappa^* f = f\varkappa = 0$$

By the universal property of the cokernel

$$(\exists! s : M \longrightarrow I^* \mid f = s\sigma) \implies f = \sigma^* s \iff f \in \operatorname{Im}\sigma^*.$$

So the sequence

$$0 \longrightarrow Hom_R(M,I^*) \xrightarrow{\sigma^*} Hom_R(L,I^*) \xrightarrow{\varkappa^*} Hom_R(K,I^*) \longrightarrow 0$$

is a short exact sequence of cochain complexes. By Theorem 1.18, we get the induced long exact sequence of cohomology:

$$\begin{aligned} 0 \quad &\longrightarrow \quad H^0(Hom_R(M,I^*)) \longrightarrow H^0(Hom_R(L,I^*)) \longrightarrow H^0(Hom_R(K,I^*)) \longrightarrow H^1(Hom_R(M,I^*)) \longrightarrow \\ &\longrightarrow \quad H^1(Hom_R(L,I^*) \longrightarrow ... \end{aligned}$$

which is isomorphic to:

$$0 \longrightarrow Hom_R(M,A) \longrightarrow Hom_R(L,A) \longrightarrow Hom_R(K,A) \longrightarrow \overline{Ext}_R^1(M,A) \longrightarrow \overline{Ext}_R^1(L,A) \longrightarrow ...$$

□

**Proposition 4.7.** *Given a short exact sequence of R-modules*

$$0 \longrightarrow A^{'} \xrightarrow{\varkappa} A \xrightarrow{\sigma} A^{''} \longrightarrow 0$$

*For any R-module C, we get a long exact sequence of* $\overline{Ext}_R^n$ :

$$\begin{array}{rl} 0 \longrightarrow & Hom_R(C, A^{'}) \longrightarrow Hom_R(C, A), \longrightarrow Hom_R(C, A^{''}) \longrightarrow \overline{Ext}_R^1(C, A^{'}) \longrightarrow \overline{Ext}_R^1(C, A) \longrightarrow \\ \longrightarrow & \overline{Ext}_R^1(C, A^{''}) \longrightarrow \overline{Ext}_R^2(C, A^{'}) \longrightarrow ... \longrightarrow \overline{Ext}_R^n(C, A^{''}) \longrightarrow \overline{Ext}_R^n(C, A^{'}) \longrightarrow ... \end{array}$$

*Proof.* Fix injective coresolution of $A^{'}$ and $A^{''}$,

$$A^{'} \xrightarrow{\varepsilon} (I^*, d^*)$$
$$A^{''} \xrightarrow{\epsilon} (J^*, \delta^*)$$

We start building a particular injective coresolution of $A$,

$$A \longrightarrow (I^* \oplus J^*, \text{some homomorphisms})$$

As

$$Hom(A, I^n \oplus J^n) \simeq Hom(A, I^n) \times Hom(A, J^n), \forall n \in \mathbb{Z}_{n \geq 0}$$

Since $I_0$ is an injective module and $\varkappa$ is monomorphism, we have a

$$k : A \longrightarrow I^0 \mid k\varkappa = \varepsilon$$

so we automatically get an

$$h : A \longrightarrow I^0 \oplus J^0 \mid h(a) = k(a) + \epsilon\sigma(a)$$

$h$ makes the first step of the diagram commutative:

$$\begin{aligned} h\varkappa(a^{'}) &= k(\varkappa(a^{'})) + \epsilon\sigma(\varkappa(a^{'})) = \varepsilon(a^{'}) = (\varepsilon(a^{'}), 0) = i_0(\varepsilon(a^{'})) \\ \pi_0 h(a) &= \pi_0(k(a) + \epsilon\sigma(a)) = \epsilon\sigma(a) \end{aligned}$$

Let $(C^{'}, \pi_{A^{'}}), (C, \pi_A), (C^{''}, \pi_{A^{''}})$ be the cokernels of $\varepsilon, h$ and $\epsilon$, respectively. Since

$$(\pi_A i_0)\,\varepsilon = \pi_A(h\varkappa) = 0$$

by the Definition 1.38 of the cokernel,

$$\exists! u : C^{'} \longrightarrow C \mid \pi_A i_0 = u\pi_{A^{'}}$$

Also, since

$$(\pi_{A^{''}}\pi_0)h = \pi_{A^{''}}(\epsilon\sigma) = 0 \implies \exists! v : C \longrightarrow C^{''} \mid \pi_{A^{''}}\pi_0 = v\pi_A$$

Lemma 1.48 gives that the *coker* row sequence is exact. Since

$$d^0\varepsilon = 0 \implies \exists! e_{C^{'}} : C^{'} \longrightarrow I^1 \mid e_{C^{'}}\pi_{A^{'}} = d^0$$

Also, since

$$\delta^0\epsilon = 0 \implies \exists! e_{C^{''}} : C \longrightarrow J^1 \mid e_{C^{''}} : C \longrightarrow J^1$$

$$
\begin{tikzcd}
A' \arrow[r, "\varkappa"] \arrow[d, "d^{-1}"'] & A \arrow[r, "\sigma"] \arrow[d, "h"'] & A'' \arrow[d, "\delta^{-1}"] \\
I^0 \arrow[r, "i_0"] \arrow[d, "\pi_{A'}"'] & I^0 \oplus J^0 \arrow[r, "\pi_0"] \arrow[d, "\pi_A"'] & J^0 \arrow[d, "\pi_{A''}"] \\
C' \arrow[r, "u"] \arrow[d, "e_{C'}"'] & C \arrow[r, "v"] & C'' \arrow[d, "e_{C''}"] \\
I^1 \arrow[r, "i_1"] \arrow[d, "d^1"'] & I^1 \oplus J^1 \arrow[r, "\pi_1"] & J^1 \arrow[d, "\delta^1"'] \\
\vdots & & \vdots
\end{tikzcd}
$$

Since $I^1$ is injective, there exists

$$\exists s : I^0 \oplus J^0 \mid si_0 = d^0$$

Trivially, we have

$$\delta^0 \pi_0 : I^0 \oplus J^0 \longrightarrow J^1$$

Define

$$
\begin{aligned}
k &: I^0 \oplus J^0 \longrightarrow I^1 \oplus J^1 \\
k(i,j) &= s(i,j) + \delta^0 \pi_0(i,j) = s(i,j) + \delta^0(j)
\end{aligned}
$$

This homomorphism makes the whole diagram commutative:

$$
\begin{aligned}
ki_0(i) &= k(i,0) = s(i,0) = si_0(i) = d^0(i) = (d^0(i),0) = i_1(d^0(i)) \\
\pi_1 k(i,j) &= \delta^0(j) = \delta^0(\pi_1(i,j))
\end{aligned}
$$

Continue in this manner: take $(C_1^{'}, \pi_{A_1^{'}}), (C, \pi_{A_1}), (C^{''}, \pi_{A_1^{''}})$ as the cokernels of $d^0, k$, and $\delta^0$, respectively. In this manner we get the specific desired injective coresolution of $A$. Then we get a short exact sequence of complexes

$$0 \longrightarrow I^* \xrightarrow{i_*} I^* \oplus J^* \xrightarrow{\pi_*} J^* \longrightarrow 0$$

where $i_*$ is the natural injection and $\pi_*$ is the natural projection. It is not split exact (since the middle map is not $d^0 \oplus \delta^0$, but some twisted homomorphism $k_*$). But the sequence is split exact for each $n \geq 0$. For any $R$-module $C$, take $Hom_R(C, -)$ and get a short exact sequence of complexes (since any $f : C \longrightarrow J^*$ induces $i_{J^*} f : C \longrightarrow I^* \oplus J^*$):

$$0 \longrightarrow Hom_R(C, I^*) \longrightarrow Hom_R(C, I^* \oplus J^*) \longrightarrow Hom_R(C, J^*) \longrightarrow 0$$

By Theorem 1.18, we get a long exact sequence of cohomology:

$$
\begin{aligned}
0 &\longrightarrow H^0(Hom_R(C, I^*)) \longrightarrow H^0(Hom_R(C, I^* \oplus J^*)) \longrightarrow H^0(Hom_R(C, J^*)) \longrightarrow \\
&\longrightarrow H^1(Hom_R(C, I^*)) \longrightarrow H^1(Hom_R(C, I^* \oplus J^*)) \longrightarrow ...
\end{aligned}
$$

which is isomorphic to

$$
\begin{aligned}
0 &\longrightarrow Hom_R(C, A^{'}) \longrightarrow Hom_R(C, A) \longrightarrow Hom_R(C, A^{''}) \longrightarrow \overline{Ext}_R^1(C, A^{'}) \longrightarrow \\
&\longrightarrow \overline{Ext}_R^1(C, A) \longrightarrow ...
\end{aligned}
$$

□

**Proposition 4.8.** $\overline{Ext}^n_R(C,I) = 0$, $n \geq 1$, *when $I$ is an injective $R$-module.*

*Proof.* $... \longrightarrow 0 \longrightarrow 0 \longrightarrow I \xrightarrow{1_I} I \longrightarrow 0$, is an injective coresolution of $I$, which is equivalent to the isomorphism $1 : I \longrightarrow I$. Since $Hom_R(C,-)$ is a functor, we get $1_{Hom_R(C,I)}$, an isomorphism, which gives the long left exact sequence

$$0 \longrightarrow Hom_R(C,I) \longrightarrow Hom_R(C,I) \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$$

Since all $Hom_R(C,I^n) = 0,\ n \geq 1$, we see that we have

$$\overline{Ext}^n_R(C,I) = H^n(Hom_R(C,I^*)) = 0,\ n \geq 1.$$

□

**Proposition 4.9.** *Let*

$$0 \longrightarrow A \longrightarrow I \longrightarrow K \longrightarrow 0$$

*be short exact sequence of abelian groups, where the middle module $I$ is injective. Then,*

$$\begin{array}{rcl} \overline{Ext}^1_R(C,A) & \simeq & coker(Hom_R(C,I) \longrightarrow Hom_R(C,K)) \\ \overline{Ext}^i_R(C,K) & \simeq & \overline{Ext}^{i+1}_R(C,A), i \geq 1 \end{array}$$

*Proof.* Using Proposition 4.7, the short exact sequence induces a long exact sequence of $\overline{Ext}^n_R$ :

$$\begin{array}{rl} 0 \longrightarrow & Hom_R(C,A) \longrightarrow Hom_R(C,I) \longrightarrow Hom_R(C,K) \longrightarrow \overline{Ext}^1_R(C,A) \longrightarrow \overline{Ext}^1_R(C,I) \longrightarrow \\ \longrightarrow & \overline{Ext}^1_R(C,K) \longrightarrow \overline{Ext}^2_R(C,A) \longrightarrow .. \longrightarrow \overline{Ext}^n_R(C,I) \longrightarrow \overline{Ext}^n_R(C,K) \longrightarrow \overline{Ext}^{n+1}_R(C,A) \\ \longrightarrow & \overline{Ext}^{n+1}_R(C,I) \longrightarrow ... \end{array}$$

By Proposition 4.8, we get

$$\overline{Ext}^1_R(C,A) \simeq coker(Hom_R(C,I) \longrightarrow Hom_R(C,K))$$

Also,

$$\overline{Ext}^i_R(C,K) \simeq \overline{Ext}^{i+1}_R(C,A), i \geq 1.$$

□

**Theorem 4.10.** $Ext^n_R \simeq \overline{Ext}^n_R$ *as bifunctors $R$-mod$\times R$-mod to $AB$, for each positive integer $n$.*

*Proof.* For any $R$-module $C$, we have

$$Ext^0_R(C,A) \simeq Hom_R(C,A) \simeq \overline{Ext}^0_R(C,A)$$

Define a projective presentation of $C$

$$0 \longrightarrow S \longrightarrow P \longrightarrow C \longrightarrow 0$$

We will prove the claim by induction. Suppose

$$\overline{Ext}^i_R(S,A) \simeq Ext^i(S,A),\ \text{for some } i \geq 1$$

By Proposition 4.9, we get

$$\overline{Ext}^i_R(S,A) \simeq \overline{Ext}^{i+1}_R(C,A),$$

Then we also have

$$Ext_R^n(S, A) \simeq Ext_R^{n+1}(C, A)$$

and so we get $\overline{Ext}_R^{n+1}(C, A) \simeq Ext_R^{n+1}(C, A)$. $\square$

## 5. The group $E_R(C, A)$

**Definition 5.1.** *Let $C$ and $A$ be $R$-modules. Denote by $E_R(C, A)$ the set of equivalence classes of extensions (short exact sequences) of the form*

$$0 \longrightarrow A \xrightarrow{\varkappa} B \xrightarrow{\sigma} C \longrightarrow 0$$

*where two such extensions are called equivalent if there exists a homomorphism (hence an isomorphism) $\beta : B \longrightarrow B^{'}$ making the diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"'] & B \arrow[r] \arrow[d, "\gamma"'] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

*commutative.*

We see that the direct sum extension

$$0 \longrightarrow A \overset{a\to(a,0)}{\longrightarrow} A \oplus C \overset{(a,c)\to c}{\longrightarrow} C \longrightarrow 0$$

is an element of the set. Fix a ring $R$. Given an element in $E_R(C, A)$ and a homomorphism $\alpha : C^{'} \longrightarrow C$, define the derived extension we get by taking the pullback $PB$ of $(C, \alpha, \sigma)$ using Lemma 5.2, namely

$$\begin{tikzcd} A \arrow[r, "\varkappa'"] \arrow[d, "1_A"'] & PB \arrow[r, "\pi_{C'}"] \arrow[d, "\pi_B"'] & C' \arrow[d, "\alpha"] \\ A \arrow[r, "\varkappa"] & B \arrow[r, "\sigma"] & C \end{tikzcd}$$

**Lemma 5.2.** *If $\sigma$ is surjective, so is $\pi_{C'}$. Also,* $\ker \pi_{C'} \simeq \ker \sigma$.

*Proof.* Take any $c^{'} \in C^{'}$. Take $\alpha(c^{'}) = c$, for some $c \in C$. Since $\sigma$ is surjective,

$$\begin{aligned} \exists b \quad &\in \quad B \mid \sigma(b) = c = \alpha(c^{'}) \implies (c^{'}, b) \in PB \\ &\implies \exists (c^{'}, b) \in PB \mid \pi_{C'}(c^{'}, b) = c^{'}, \forall c^{'} \in C^{'} \end{aligned}$$

$$\begin{aligned} (b, c^{'}) \quad &\in \quad \ker \pi_{C'} \iff \pi_{C'}(b, c^{'}) = 0 \iff c^{'} = 0 \implies ((b, 0) \in PB) \in \ker \pi_{C'} \\ &\iff \sigma(b) = \alpha(0) = 0 \implies b \in \ker \sigma = \operatorname{Im} \varkappa \\ &\implies \ker \pi_{C'} = ((\varkappa(a), 0), a \in A) = i(\varkappa(a)), \text{ where } i \text{ is the canonical injection.} \end{aligned}$$

$\square$

Define this element in $E_R(C^{'}, A)$ as the image of the map

$$\alpha^* : E_R(C, A) \longrightarrow E_R(C^{'}, A)$$

In detail,

$$\begin{aligned} \alpha^*(0 \quad &\longrightarrow \quad A \xrightarrow{\varkappa} B \xrightarrow{\sigma} C \longrightarrow 0) = 0 \longrightarrow A \xrightarrow{i} PB \xrightarrow{p} C^{'} \longrightarrow 0 \\ PB \quad &= \quad \{(b, c^{'}) \mid \sigma(b) = \alpha(c^{'}), b \in B, c^{'} \in C^{'}\} \\ i(a) \quad &= \quad (\varkappa(a), 0), \ p(b, c^{'}) = c^{'}. \end{aligned}$$

$\alpha^*$ is well-defined. Suppose

$$\begin{aligned} 0 &\longrightarrow A \xrightarrow{\varkappa'} B' \xrightarrow{\sigma'} C \longrightarrow 0 \in \left[0 \longrightarrow A \xrightarrow{\varkappa} B \xrightarrow{\sigma} C \longrightarrow 0\right] \\ &\Longleftrightarrow \exists \zeta : B' \longrightarrow B \mid \sigma\zeta = \sigma' \wedge \zeta\varkappa' = \varkappa \end{aligned}$$

Will show that $\alpha^*(0 \longrightarrow A \xrightarrow{\varkappa'} B' \xrightarrow{\sigma'} C \longrightarrow 0) \in \left[0 \longrightarrow A \xrightarrow{i} PB \xrightarrow{p} C' \longrightarrow 0\right]$.
Define

$$\begin{aligned} \beta &: PB' \longrightarrow PB \text{ as } \beta(b, c') = (\zeta(b), c') \\ \text{since } \sigma(\zeta(b)) &= \sigma'(b) = \alpha(c') \end{aligned}$$

$\beta$ makes the diagram

$$\begin{tikzcd} A \arrow[r, "i'"] \arrow[d, "1_A"'] & PB' \arrow[r, "p'"] \arrow[d, "\beta"'] & C' \arrow[d, "1_C"] \\ A \arrow[r, "i"] & PB \arrow[r, "p"] & C' \end{tikzcd}$$

commutative:

$$\begin{aligned} \beta i'(a) &= \beta(\varkappa'(a), 0) = (\zeta\varkappa'(a), 0) = (\varkappa(a), 0) = i(a) \\ p\beta(b, c') &= p(\zeta(b), c') = c' = p'(b, c') \end{aligned}$$

**Proposition 5.3.** *$\alpha^*$ makes $E_R(-, A)$ into a contravariant functor from R-mod to Sets.*

*Proof.* Take $\alpha = 1_C : C \longrightarrow C$. It induces in $E_R(C, A)$

$$\begin{aligned} 0 &\longrightarrow A \xrightarrow{i} PB \xrightarrow{p} C \longrightarrow 0 \\ i(a) &= (\varkappa(a), 0),\ p(b, \sigma(b)) = \sigma(b) \end{aligned}$$

which is equivalent to our original extension through a homomorphism

$$\begin{aligned} \beta &: PB \longrightarrow B, \beta(b, \sigma(b)) = b \\ &\Longrightarrow \beta i(a) = \beta(\varkappa(a), 0) = \varkappa(a) \\ \sigma\beta(b, \sigma(b) &= \sigma(b) = p(b) \end{aligned}$$

So we get

$$\alpha^*(1_C) = 1_{E_R(C,A)}$$

Now, given two homomorphisms

$$\alpha' : C'' \longrightarrow C', \alpha : C' \longrightarrow C$$

The pullback of $(\sigma, \alpha)$ gives an element in $E(C', A)$, where the middle module

$$\begin{aligned} B' &= \{(b, c') \mid \sigma(b) = \alpha(c')\} \\ \pi_{C'} &: B' \longrightarrow C' \text{ as } \pi_{C'}(b, c') = c' \end{aligned}$$

Taking the pullback of this $(\pi_{C'}, \alpha')$ gives an extension in $E(C'', A)$, where the middle module

$$\begin{aligned} B'' &= \{(b', c'') \mid \pi_{C'}(b, c') = \alpha'(c'')\} \\ \pi_{C''} &: B'' \longrightarrow C'' \text{ as } \pi_{C''}(b', c'') = c'' \end{aligned}$$

We have the commutative diagram:

$$\begin{tikzcd} B'' \arrow[r, "\pi_{C''}"] \arrow[d, "\pi_{B'}"'] & C'' \arrow[d, "\alpha'"] \\ B' \arrow[r, "\pi_{C'}"] \arrow[d, "\pi_B"'] & C' \arrow[d, "\alpha"] \\ B \arrow[r, "\sigma"] & C \end{tikzcd}$$

Since

$$(\alpha\alpha^{'})\pi_{C''} = \alpha(\alpha^{'}\pi_{C''}) = \alpha(\pi_{C'}\pi_{B'}) = (\alpha\pi_{C'})\pi_{B'} = (\sigma\pi_B)\pi_{B'} = \sigma(\pi_B\pi_{B'})$$

$(B^{''}, \pi_B\pi_{B'}, \pi_{C''})$ may be the pullback of $(\sigma, \alpha\alpha^{'})$. For any module $R$-module $Z$, take any $f : Z \longrightarrow B$, $g : Z \longrightarrow C^{''}$, such that

$$\sigma f = \alpha\alpha^{'} g \iff \sigma f = \alpha(\alpha^{'} g), \alpha^{'} g : Z \longrightarrow C^{'}$$

Since $B^{'}$ is the pullback of $(\sigma, \alpha)$, there exists

$$!\gamma : Z \longrightarrow B^{'} \mid \pi_{C'}\gamma = \alpha^{'} g \text{ and } \pi_B\gamma = f.$$

Since $B^{''}$ is the pullback of $(\pi_{C'}, \alpha^{'})$, there exists

$$!u : Z \longrightarrow B^{''} \mid \pi_{B'}u = \gamma \text{ and } \pi_{C''}u = g$$

Then we have that there exists

$$!u : Z \longrightarrow B^{''} \mid \pi_B(\pi_{B'}u) = \pi_B\gamma = f \text{ and } \pi_{C''}u = g,$$

which is exactly the universal property of the pullback of $(\sigma, \alpha\alpha^{'})$. This gives, using our notation, that we may write

$$(\alpha\alpha^{'})^* : E(C, A) \longrightarrow E(C^{''}, A), (\alpha\alpha^{'})^* = (\alpha^{'})^*\alpha^*$$

which makes $E_R(-, A)$ into a contravariant functor. □

Given an element in $E_R(C, A)$ and a homomorphism $\beta : A \longrightarrow A^{'}$, define the derived extension we get by taking the pushout of $(A, \beta, \varkappa)$ using Lemma 5.4, namely:

$$\begin{tikzcd} A \arrow[r, "\varkappa"] \arrow[d, "\beta"'] & B \arrow[r, "\sigma"] \arrow[d, "h"'] & C \arrow[d, "1_C"] \\ A' \arrow[r, "\varkappa'"] & PO \arrow[r, "\sigma'"] & C \end{tikzcd}$$

**Lemma 5.4.** *If $\varkappa$ is injective, so is $\varkappa^{'}$. Also, coker$(\varkappa^{'}) \simeq$ coker$(\varkappa)$.*

*Proof.*

$$\begin{aligned} a^{'} &\in \ker\varkappa^{'} \iff (a^{'}, 0) = (\beta(a), -\varkappa(a)), a \in A. \\ &\implies a = 0 \implies \beta(0) = 0 = a^{'}. \end{aligned}$$

Define the map $\sigma^{'} : PO \longrightarrow C$ as $\sigma^{'}((a^{'}, b) + L) = \sigma(b)$. It is correct:

$$\begin{aligned} (a^{'}, b) &\sim (c^{'}, d) \iff \exists a \in A \mid (c^{'}, d) = (a^{'}, b) + (\beta(a), -\varkappa(a)) = (a^{'} + \beta(a), b - \varkappa(a)) \\ \sigma^{'}((c^{'}, d) + L) &= \sigma(d) = \sigma(b - \varkappa(a)) = \sigma(b) - \sigma\varkappa(a) = \sigma(b) \end{aligned}$$

It is an homomorphism with kernel $\varkappa^{'}$:

$$\begin{aligned} \sigma^{'}((a^{'},b)+(c^{'},d)+L) &= \sigma^{'}((a^{'}+c^{'},b+d)+L)=\sigma(b+d)=\sigma(b)+\sigma(d) \\ (a^{'},b)+L &\in \ker\sigma^{'} \iff \sigma(b)=0 \iff b=\varkappa(a) \implies (a^{'},\varkappa(a))+L\in\ker\sigma^{'} \\ &\iff \Big((a^{'}+\beta(a),0)+L\Big)=i(a^{'}+\beta(a)\in\ker\sigma^{'} \end{aligned}$$

□

Define this element in $E_R(C,A^{'})$ as the image of the map $\beta_*: E_R(C,A)\longrightarrow E_R(C,A^{'})$. In detail, we have:

$$\begin{aligned} \beta_*(0 &\longrightarrow A\xrightarrow{\varkappa}B\xrightarrow{\sigma}C\longrightarrow 0)=0\longrightarrow A^{'}\xrightarrow{i}PO\xrightarrow{p}C\longrightarrow 0 \\ PO &= A^{'}\oplus B/\langle(\beta(a),-\varkappa(a)):a\in A\rangle = A^{'}\times B/L \\ i(a^{'}) &= (a^{'},0)+L,\ p((a^{'},b)+L)=\sigma(b) \end{aligned}$$

$\beta_*$ is well-defined.

**Proposition 5.5.** *The map $\beta_*$ makes $E_R(C,-)$ into a covariant functor from R-mod to Sets.*

*Proof.* Take $\beta=1_A:A\longrightarrow A$. Then

$$\begin{aligned} \beta_*(0 &\longrightarrow A\xrightarrow{\varkappa}B\xrightarrow{\sigma}C\longrightarrow 0)=(0\longrightarrow A\xrightarrow{i}PO\xrightarrow{p}C\longrightarrow 0) \\ PO &= A\times B/\langle(a,-\varkappa(a)):a\in A\rangle \end{aligned}$$

Define the homomorphism $\gamma:PO\longrightarrow B$ as

$$\begin{aligned} \gamma(a,b) &= \varkappa(a)+b \\ \gamma(a,-\varkappa(a)) &= \varkappa(a)-\varkappa(a)=0,\forall a\in A \end{aligned}$$

It makes the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "1_A"'] & PO \arrow[r, "p"] \arrow[d, "\gamma"'] & C \arrow[d, "1_C"] \\ A \arrow[r, "\varkappa"] & B \arrow[r, "\sigma"] & C \end{tikzcd}$$

commutative:

$$\begin{aligned} \gamma(i(a)) &= \gamma(a,0)=\varkappa(a)+0=\varkappa(a) \\ \sigma(\gamma((a,b)+L)) &= \sigma(\varkappa(a)+b)=\sigma(b) \end{aligned}$$

So $(1_A)_*=1_{E_R(C,A)}$. Given any two homomorphisms $\beta:A\longrightarrow A^{'}$, $\beta^{'}:A^{'}\longrightarrow A^{''}$. Taking pushout $(B^{'},i_{A'},i_B)$ of $(A,\varkappa,\beta)$, gives an element in $E_R(C,A^{'})$ where the middle module is

$$\begin{aligned} B^{'} &= A^{'}\oplus B/\langle(\beta(a),-\varkappa(a)):a\in A\rangle = A^{'}\oplus B/L \\ i_{A'}(a^{'}) &= (a^{'},0)+L \\ i_B(b) &= (0,b)+L \end{aligned}$$

Taking the pushout $(B^{''}, i_{A^{''}}, i_Y)$ of $(A^{'}, \beta^{'}, i_{A^{'}})$ gives an element in $E_R(C, A^{''})$, where the middle module is

$$\begin{aligned} B^{''} &= A^{''} \oplus B^{'} / \left\langle \left(\beta^{'}(a^{'}), -i^{'}_A(a^{'})\right) : a^{'} \in A^{'} \right\rangle = A^{''} \oplus B^{'}/L^{'} \\ i_{A^{''}}(a^{''}) &= (a^{''}, 0, 0) + L \\ i_{B^{'}}((0, b) + L) &= (0, 0, b) + L^{'} \end{aligned}$$

and we get the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varkappa"] \arrow[d, "\beta"'] & B \arrow[r, "\sigma"] \arrow[d, "i_B"] & C \\ A' \arrow[r, "i'_A"] \arrow[d, "\beta'"'] & B' \arrow[d, "i'_Y"] & \\ A'' \arrow[r, "i''_A"] & B'' & \end{tikzcd}$$

So $(B^{''}, i_{B^{'}} i_B, i_{A^{''}})$ is a candidate for the pushout of $(\varkappa, \beta^{'}\beta)$. We only have to show the universal property, i.e. that for any $R$-module $Z$, any $f : B \longrightarrow Z$, $g : A^{''} \longrightarrow Z$ such that

$$f\varkappa = g(\beta^{'}\beta) \implies \exists! u : B^{''} \longrightarrow Z \mid u i_{B^{'}} i_B = f \text{ and } u i_{A^{''}} = g$$

Since $B^{'}$ is the pushout of$(\varkappa, \beta)$,and

$$f\varkappa = (g\beta^{'})\beta \implies \exists! \gamma : B^{'} \longrightarrow Z \mid f = \gamma i_B \text{ and } g\beta^{'} = \gamma i_{A^{'}}$$

For this $\gamma$, and $g$, since $B^{''}$ is the pushout of $(i_{A^{'}}, \beta^{'})$

$$\exists! u : B^{''} \longrightarrow Z \mid u i_{A^{''}} = g \text{ and } \gamma = u i_{B^{'}}$$

Then,

$$\gamma i_B = f = (u i_{B^{'}}) i_B = u(i_{B^{'}} i_B) \text{ and } u i_{A^{''}} = g,$$

as desired. We get

$$(\beta^{'})_* \beta_* = (\beta^{'}\beta)_* : E_R(C, A) \longrightarrow E_R(C, A^{''})$$

So $E_R(C, -)$ is a covariant functor. □

**Proposition 5.6.** *$E_R(C, A)$ is a bifunctor from $R$-mod$\times R$-mod to Sets.*

*Proof.* We must show that this diagram is commutative:

$$\begin{tikzcd} E(C, A) \arrow[r, "E(C,\beta)"] \arrow[d, "E(\alpha, A)"'] & E(C, A') \arrow[d, "E(\alpha, A')"] \\ E(C', A) \arrow[r, "E(C',\beta)"] & E(C', A') \end{tikzcd}$$

Pick any element in $E_R(C, A)$. Compute first $E(\alpha, A^{'}) \circ E(C, \beta)$. We get

$$\begin{tikzcd}
A \arrow[r, "\varkappa"] \arrow[d, "\alpha"'] & B \arrow[r, "\sigma"] \arrow[d, "i_B"'] & C \arrow[d, "1_C"] \\
A' \arrow[r, "i_{A'}"] & PO \arrow[r, "\sigma\pi_B"] & C' \\
A' \arrow[r, "i_{A'}"] \arrow[u, "1_{A'}"] & PB \arrow[r, "\pi_{C'}"] \arrow[u, "\pi_{PO}"] & C' \arrow[u, "\beta"']
\end{tikzcd}$$

where $i_{A'}$ and $\pi$ are the canonical injections and projections. Now,

$$\begin{aligned}
PO &= A^{'} \oplus B / \langle((\alpha(a), -\varkappa(a)) : a \in A\rangle \\
PB &= \left\{\left((a^{'}, b), c^{'}\right) \in PO \oplus C^{'} \mid \beta(c^{'}) = \sigma(b), \forall b \in B, c^{'} \in C^{'}\right\}
\end{aligned}$$

The other way, compute $E(C^{'}, \beta) \circ E(\alpha, A)$. We get

$$\begin{tikzcd}
A \arrow[r, "\varkappa"] & B \arrow[r, "\sigma"] & C \\
A \arrow[r, "i\varkappa"] \arrow[u, "1_A"] \arrow[d, "\alpha"'] & Pb \arrow[r, "\pi_{C'}"] \arrow[u, "\pi_B"'] \arrow[d, "i_{PB}"'] & C' \arrow[u, "\beta"'] \arrow[d, "1_{C'}"] \\
A' \arrow[r, "i_{A'}"] & Po \arrow[r, "\pi_{C'}"] & C'
\end{tikzcd}$$

where $i$ and $\pi$ represent the canonical injections and projections. Now

$$\begin{aligned}
Pb &= \{(b, c) \in B \oplus C \mid \sigma(b) = \beta(c^{'})\} \\
Po &= A^{'} \oplus Pb / \langle(\alpha(a), \varkappa(a), 0) : a \in A\rangle
\end{aligned}$$

So $Po = PB$, they both contain the same elements. Choose the isomorphism $1_{PB}$ : $Po \longrightarrow PB$ in

$$\begin{tikzcd}
A' \arrow[r, "i_{A'}"] \arrow[d, "1_{A'}"'] & Po \arrow[r, "\pi_{C'}"] \arrow[d, "\zeta"'] & C' \arrow[d, "1_{C'}"] \\
A' \arrow[r, "i_{A'}"] & PB \arrow[r, "\pi_{C'}"] & C'
\end{tikzcd}$$

Since the diagram is commutative, we have that the extensions are equivalent, and $E_R(-,-)$ is a bifunctor from $R$-mod$\times R$-mod to *Sets*. □

**Definition 5.7.** *The diagonal homomorphism for a module $C$ is $\triangle = \triangle_C : C \longrightarrow C \oplus C$, $\triangle(c) = (c, c)$.*

**Definition 5.8.** *The codiagonal homomorphism for a module $A$ is $\nabla = \nabla_A : A \oplus A \longrightarrow A$, $\nabla(a_1, a_2) = a_1 + a_2$.*

Then, for any two $f, g : C \longrightarrow A$, we may write $f + g = \nabla_A(f \oplus g)\triangle_C$, where $\alpha \oplus \beta(a, b) = (\alpha(a), \beta(b))$ (when $\alpha(a)$ and $\beta(b)$ are defined)

**Definition 5.9.** *Given two extensions $\left\{E_i : A_i \xrightarrow{\varkappa_i} B_i \xrightarrow{\sigma_i} C_i\right\}_{i=1,2}$ we define their direct sum to be the extension*

$$E_1 \oplus E_2 : 0 \longrightarrow A_1 \oplus A_2 \xrightarrow{\varkappa_1 \oplus \varkappa_2} B_1 \oplus B_2 \xrightarrow{\sigma_1 \oplus \sigma_2} C_1 \oplus C_2 \longrightarrow 0$$

This is indeed a short exact sequence.

$$\begin{aligned}\sigma_1 \oplus \sigma_2(\varkappa_1 \oplus \varkappa_2(a_1, a_2) &= \sigma_1 \oplus \sigma_2(\varkappa_1(a_1), \varkappa_2(a_2)) = (\sigma_1(\varkappa_1(a_1)), \sigma_2(\varkappa_2(a_2)) = (0,0)\\ &\Longrightarrow \operatorname{Im} \varkappa_1 \oplus \varkappa_2 \subseteq \ker \sigma_1 \oplus \sigma_2\end{aligned}$$

The other way, take

$$\begin{aligned}(b_1, b_2) \quad &| \quad (\sigma_1 \oplus \sigma_2)(b_1, b_2) = (0,0) \implies \sigma_1(b_1) = 0 \wedge \sigma_2(b_2) = 0\\ &\Longleftrightarrow \quad b_1 \in \operatorname{Im}(\varkappa_1) \wedge b_2 \in \operatorname{Im}(\varkappa_2) \implies (b_1, b_2) \in \operatorname{Im}(\varkappa_1 \oplus \varkappa_2).\end{aligned}$$

**Definition 5.10.** *Define a binary operation on $E_R(C, A)$, called the Baer sum of two extensions $E_1$ and $E_2$ as $E_1 + E_2 = \nabla_A(E_1 \oplus E_2)\triangle_C = \nabla_*(\triangle^*(E_1 \oplus E_2))$.*

**Lemma 5.11.** *There exists a well-defined mapping $\varphi_{C,A} : E_R(C, A) \longrightarrow Ext^1(C, A)$.*

*Proof.* Let $[\epsilon]$ := class of equivalent extensions of $(0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0)$. Choose a projective resolution of $C$ :

$$... \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow C \longrightarrow 0.$$

Lemma 1.22 gives the existence of a lifting $f_0 : P_0 \longrightarrow B$ and $f_1 : P_1 \longrightarrow A$ that satisfies

$$pf_0 = \alpha,\ f_0 d_0 = if_1,\ f_1 d_1 = 0 \implies f_1 \in Ext^1(C, A)$$

Define

$$\begin{aligned}\varphi_{C,A} \quad &: \quad E_R(C, A) \longrightarrow Ext^1(C, A)\\ \varphi_{C,A}([\epsilon]) \quad &= \quad f_1\end{aligned}$$

Since any two lifting homomorphisms are chain homologous, they induce equal cohomology homomorphisms, so $\varphi_{C,A}$ does not depend on the choice of lifting $f_*$. $\varphi$ is a well-defined map. Let two elements $\left\{E_i : A \xrightarrow{\varkappa_i} B_i \xrightarrow{\sigma_i} C\right\}_{i=1,2}$ of $E_R(C, A)$ be equivalent by a homomorphism $\beta : B_1 \longrightarrow B_2$. Let $E_1$ induce $f_1 \in Ext^1_R(C, A)$.

$$\begin{tikzcd}
 & & A \arrow[r, "\varkappa_2"] & B_2 \arrow[r, "\sigma_2"] & C \\
 & & A \arrow[u, "1_A"] \arrow[r, "\varkappa_1"] & B_1 \arrow[u, "\beta"] \arrow[r, "\sigma_1"] & C \arrow[u, "1_C"'] \\
... \arrow[r, "d_2"] & P_2 \arrow[r, "d_1"] & P_1 \arrow[u, "f_1"] \arrow[r, "d_0"] & P_0 \arrow[u, "f_0"] \arrow[r, "\varepsilon"] & C \arrow[u, "1_C"']
\end{tikzcd}$$

The same $f_1$ is also induced by $E_2$, since we have $\beta f_0 : P_0 \longrightarrow B_2$, and all squares are commutative. By Lemma 3.3, $\varphi_{C,A}$ does not depend on the choice of projective resolution of $C$. ☐

**Lemma 5.12.** *There exists a well-defined mapping $\psi_{C,A} : Ext^1_R(C, A) \longrightarrow E_R(C, A)$.*

*Proof.* Fix a projective resolution of $C$ :

$$... \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow C \longrightarrow 0.$$

Define $\psi(f) = [0 \longrightarrow A \longrightarrow pushout(f, d_0) \longrightarrow C]$. We must first show that this is indeed a short exact sequence, i.e. that $C$ is the cokernel follows from the isomorphism with $coker(d_0)$ (by Lemma 5.4). $\psi_{C,A}$ is well-defined. Let $f, f^{'}$ be two cohomologous cochains, so there exists a $g : P_0 \longrightarrow A$ such that

$$f^{'} - f = gd_0$$

In the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "1_A"'] & pushout(f, d_0) \arrow[r, "p"] & C \arrow[d, "1_C"] \\ A \arrow[r, "i'"] & pushout(f', d_0) \arrow[r, "p'"] & C \end{tikzcd}$$

we may define map $\gamma : pushout(f, d_0) \longrightarrow pushout(f^{'}, d_0)$ as

$$\begin{aligned} \gamma(a, p_0) &= (a - gp_0, p_0),\ p_0 \in P_0 \\ \gamma(f(p_1), -d_0(p_1)) &= (f(p_1) - g(-d_0(p_1)), -d_0(p_1)) = (f(p_1) + gd_0(p_1), -d_0(p_1)) \\ &= (f^{'}(p_1), -d_0(p_1)),\ p_1 \in P_1 \end{aligned}$$

It is a homomorphism $A \times P_0 \longrightarrow A \times P_0$ :

$$\begin{aligned} \gamma((a, p_0) + (b, p)) &= \gamma(a + b, p_0 + p) = ((a + b) - g(p_0 + p), p_0 + p) \\ &= (a - g(p_0), p_0) + (b - g(p), p) = \gamma(a, p_0) + \gamma(b, p) \end{aligned}$$

$\gamma$ gives that the two extensions are equivalent, as ($p_1 \in P_1$):

$$\begin{aligned} \gamma(i(a)) &= \gamma((a, 0) + \langle f(p_1), -d_0(p_1)\rangle) = (a - g(0), 0) + \Big\langle \Big(f^{'}(p_1), -d_0(p_1)\Big) \\ &= (a, 0) + \Big\langle \Big(f^{'}(p_1), -d_0(p_1)\Big) : p_1 \in P_1 \Big\rangle = i^{'}(a) \\ p^{'}\gamma((a, p_0) + \langle (f(p_1), -d_0(p_1)) : p_1 \in P_1\rangle) &= p^{'}((a - g(p_0), p_0)) + \langle (f(p_1), -d_0(p_1)) : p_1 \in P_1\rangle) \\ &= p_0 = p((a, p_0) + \langle (f(p_1), -d_0(p_1)) : p_1 \in P_1\rangle \end{aligned}$$

$\square$

**Corollary 5.13.** *$\varphi_{C,A}$ and $\psi_{C,A}$ as defined in Lemmas 5.11 and 5.12, respectively, are inverse mappings.*

*Proof.* Choose a lifting $f_*$ and get the class $[0 \longrightarrow A \longrightarrow pushout(f_1, d_0) \longrightarrow C]$. Then the result follows easily, since we can choose any projective resolution of $C$, and we can take $f_1$ (as we can freely chose any lifting). And we get $\varphi_{C,A} \circ \psi_{C,A}(f_1) = f_1$, i.e. $\varphi_{C,A} \circ \psi_{C,A} = 1_{Ext^1(C,A)}$. Now we will show that $\psi_{C,A} \circ \varphi_{C,A} = 1_{E_R(C,A)}$. Start with $E : 0 \longrightarrow A \xrightarrow{\varkappa} B \xrightarrow{\sigma} C \longrightarrow 0$. Fix a lifting $f_*$. Take pushout of $(f_1, d_0)$. Will show that the extension we get is equivalent to $E$, i.e. there exists an isomorphism $h : PO \longrightarrow B$ that makes the diagram commutative:

$$\begin{tikzcd} \ldots \arrow[r, "d_2"] & P_2 \arrow[r, "d_1"] & P_1 \arrow[r, "\varepsilon"] \arrow[d, "f_1"'] & P_0 \arrow[d, "f_0"'] \arrow[dr, "d_{-1}"] & \\ & & A \arrow[r, "\varkappa"] \arrow[d, "1_A"'] & B \arrow[r, "\sigma"] & C \arrow[d, "1_C"'] \\ & & A \arrow[r, "i_A"] & PO \arrow[r, "\varepsilon"] & C \end{tikzcd}$$

Define

$$h(a, p_0) = \varkappa(a) + f_0(p_0) \in B$$

It is well-defined homomorphism. When $(a, p_0) \sim (b, q_0)$, there exists $p_1 \in P_1$ such that

$$a - b = f_1(p_1) \text{ and } p_0 - q_0 = -d_0(p_1)$$

So

$$\begin{aligned} h((a,p_0) - (b,q_0)) &= h(a-b, p_0 - q_0) = \varkappa(a-b) + f_0(p_0 - q_0) \\ &= \varkappa f_1(p_1) + f_0(-d_0(p_1)) = \varkappa f_1(p_1) - f_0 d_0(p_1) = 0 \\ &\Longrightarrow h(a,p_0) = h(b,q_0) \end{aligned}$$

So there exists a homomorphism $h : PO \longrightarrow B$. Need only to check commutativity:

$$\begin{aligned} hi_A(a) &= h(a,0) = \varkappa(a) + f_0(0) = \varkappa(a) \\ \sigma h(a,p_0) &= \sigma(\varkappa(a) + f_0(p_0)) = \sigma f_0(p_0) = \varepsilon(p_0) \end{aligned}$$

so the pushout extension is equivalent the original one. This gives

$$\psi_{C,A} \circ \varphi_{C,A} = 1_{E_R(C,A)}$$

□

**Corollary 5.14.** *$\varphi_{C,A}$ (and $\psi_{C,A}$) is a natural transformation of bifunctors.*

*Proof.* We must show that for any $\gamma : K \longrightarrow C$, $\alpha : A \longrightarrow B$, the following diagram is commutative

$$\begin{tikzcd} Ext^1(C,A) \arrow[r, "\varphi_{(C,A)}"] \arrow[d, "Ext^1(\gamma,\alpha)"'] & E_R(C,A) \arrow[d, "E_R(\gamma,\alpha)"] \\ Ext^1(K,B) \arrow[r, "\varphi_{(K,B)}"] & E_R(K,B) \end{tikzcd}$$

Fix a projective resolution of $C$, $(P_*, d_*) \xrightarrow{\varepsilon} C$. Start with a 1-cocycle $f_1$ of $Ext^1(C,A)$, so $f_1 d_1 = 0$. Take pushout $PO_C$ of $(f_1, d_0)$. Then you get the commutative diagram

$$\begin{tikzcd} \dots \arrow[r, "d_1"] & P_1 \arrow[r, "d_0"] \arrow[d, "f_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "i_{P_0}"'] & C \arrow[d, "1_C"] \\ & A \arrow[r, "i_A"] & PO_C \arrow[r, "\varepsilon\pi_{P_0}"] & C \end{tikzcd}$$

$$PO_C = \{(a,p_0), a \in A, p_0 \in P_0 \mid (f_1(p_1), -d_0(p_1)) = (0,0)\}$$

Take pullback $PB_K$ of $(\varepsilon\pi_{P_0}, \gamma)$ :

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "1_A"'] & PB_K \arrow[r, "\pi_K"] \arrow[d, "\pi_{PO_C}"'] & K \arrow[d, "\gamma"] \\ A \arrow[r, "i_A"] & PO_C \arrow[r, "\varepsilon\pi_{P_0}"] & C \end{tikzcd}$$

$$PB_K = \{(k,a,p_0) \mid \gamma(k) = \varepsilon\pi_{P_0}(a,p_0) = \varepsilon(p_0), (a,p_0) \in PO_C\}$$

Now take pushout $PO_B$ of $(\alpha, i_A)$:

$$PO_B = \{(b,k,a,p_0) \mid (k,a,p_0) \in PB_K, (\alpha(a),0,0,0) = (0,0,a,0), \forall a \in A\}$$

Then you get the commutative diagram

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "\alpha"'] & PB_K \arrow[r, "\pi_K"] \arrow[d, "i_{PB_K}"'] & K \arrow[d, "1_K"] \\ B \arrow[r, "i_B"] & PO_B \arrow[r, "\pi_K"] & K \end{tikzcd}$$

We have found an element of $E_R(K,B)$, and stop here. Now take $f_1 \in Ext^1(C,A)$, and follow the diagram the other way. Fix the projective resolution of $(Q_*, \delta_*) \xrightarrow{\epsilon} K$. By Lemma 1.22, there exists a lifting $t_* : Q_* \longrightarrow P_*$, such that

$$\gamma\epsilon = \varepsilon t_0, t\delta = dt, \alpha f_1 t \in Ext^1(K,B)$$

since

$$\delta_1(\alpha f_1 t(p_1)) = \alpha f_1 t(p_1) = \alpha(0) = 0$$

Take now pushout $PO$ of $(\alpha f_1 t_1, \delta_0)$,

$$\begin{array}{ccccccc} \ldots & \xrightarrow{\delta_1} & Q_1 & \xrightarrow{\delta_0} & Q_0 & \xrightarrow{\epsilon} & K \\ & \alpha f_1 t \downarrow & & i_{Q_0} \downarrow & & & \downarrow 1_K \\ & & B & \xrightarrow{i_B} & PO & \xrightarrow{\varepsilon\pi_Q} & K \end{array}$$

$$PO = \{(b, q_0), \forall b \in B, q_0 \in Q_0 \mid (\alpha f_1 t(q_1), 0) = (0, \delta_0(q_1)), \forall q_1 \in Q_1\}$$

Define $h : PO \longrightarrow PO_B$ by

$$h(b, q_0) = b + \epsilon(q_0) + t_0(q_0) = (b, \epsilon(q_0), 0, t_0(q_0))$$

It is a well-defined homomorphism. Suppose two elements $(b, q_0) \sim (b', q_0')$ of $PO$ are equivalent, i.e. their difference is equal to $(\alpha f_1 t(q_1), -\delta_0(q_1))$, for some $q_1 \in Q_1$. Then

$$\begin{aligned} h(\alpha f_1 t(q_1), -\delta_0(q_1)) &= \alpha f_1 t(q_1) + \epsilon(-\delta_0(q_1)) + t_0(-\delta_0(q_1)) = \\ &= \alpha f_1 t(q_1) - t_0(\delta_0(q_1)) = \alpha f_1 t(q_1) - d_0 t_1(q_1) = \alpha f_1(p_1) - d_0(p_1) = \\ &= (\alpha f_1(p_1), 0, 0, -d_0(p_1)) = (0, 0, f_1(p_1), -d_0(p_1)) = (0,0,0,0), \\ h((b, q_0) - (b', q_0')) &= 0 \implies h((b, q_0)) = h((b', q_0')). \end{aligned}$$

This is a homomorphism that makes the diagram commutative:

$$\begin{array}{ccccc} B & \xrightarrow{i_{B_{PO}}} & PO & \xrightarrow{\epsilon\pi_Q} & K \\ 1_B \downarrow & & h \downarrow & & \downarrow 1_K \\ B & \xrightarrow{i_B} & PO_B & \xrightarrow{\pi_K} & K \end{array}$$

$$\begin{aligned} h i_{B_{P_0}}(b) &= h(b, 0) = b = i_B(b) \\ \pi_K h(b, q_0) &= \pi_K(b + \epsilon(q_0) + t_0(q_0)) = \pi_K(\epsilon(q_0), 0, t_0(q_0)) = \epsilon(q_0) = \epsilon(\pi_Q(b, q_0)) \end{aligned}$$

So the two extensions in out previous diagram are equivalent, hence $\varphi$ is a functorial isomorphism of bifunctors $Ext^1_R(C,A)$ and $E_R(C,A)$, from $R$-mod$\times R$-mod to $Sets_*$. $\square$

**Lemma 5.15.** $\varphi_{C,A} : E_R(C,A) \longrightarrow Ext^1_R(C,A)$ *is group homomorphism.*

*Proof.* Take any two elements of $E_R(C,A) : \left\{ E_i : A_i \xrightarrow{\varkappa_i} B_i \xrightarrow{\sigma_i} C_i \right\}_{i=1,2}$. Let $f_*$ be a lifting for $E_1$, and $g_*$ a lifting for $E_2$. Take it step by step.

$$\begin{tikzcd}
\dots \arrow[r, "d_2"] & P_2 \arrow[r, "d_1"] & P_1 \arrow[dl, "1_{P_1}"'] \arrow[d, "d_0"] \\
& P_1 \arrow[dl, "f_1 \oplus g_1"'] & P_0 \arrow[dl, "f_0 \oplus g_0"'] \arrow[d, "\epsilon"] \\
A \oplus A \arrow[r, "\varkappa_1 \oplus \varkappa_2"] \arrow[d, "1_A"'] & PB \arrow[r, "\pi_C"] \arrow[d, "\pi_{B_1 B_2}"'] & C \arrow[d, "\triangle_C"'] \\
A \oplus A \arrow[r, "\varkappa_1 \oplus \varkappa_2"] \arrow[d, "\nabla_A"'] & B_1 \oplus B_2 \arrow[r, "\sigma_1 \oplus \sigma_2"] & C \oplus C \\
A \arrow[r, "i_A"] & PO \arrow[r, "\pi_C"] & C
\end{tikzcd}$$

By Proposition 1.36,

$$PB = \{(b_1, b_2, c) \mid (c, c) = (\sigma(b_1), \sigma(b_2), b_1 \in B_1, b_2 \in B_2\} = \{(b_1, b_2, \sigma(b_1)) \mid \sigma(b_1) = \sigma(b_2)\}$$

By Proposition 1.40,

$$\begin{aligned}
PO &= \{(a, b_1, b_2, \sigma(b_1)), \forall a \in A, \forall p_0 \in P_0, (b_1, b_2, \sigma(b_1)) \in PB \mid \\
(a_1 + a_2, -\varkappa_1(a_1), -\varkappa_2(a_2), 0) &= 0, \forall a_1, a_2 \in A\}
\end{aligned}$$

First,

$$\operatorname{Im} f_0 \oplus g_0 \subseteq PB \text{ since } \sigma(f_0(p_0)) = \sigma(g_0(p_0)) = \varepsilon$$

Further,

$$\begin{aligned}
\pi_C(f_0 \oplus g_0(p_0)) &= \pi_C(f_0(p_0), g_0(p_0), \varepsilon(p_0)) = \varepsilon(p_0). \\
\varkappa_1 \oplus \varkappa_2(f_1 \oplus g_1(p_1)) &= (\varkappa_1(f_1(p_1)), \varkappa_2(g_1(p_1))) = (f_0 d_0(p_1), g_0 d_0(p_1)) = f_0 \oplus g_0(d_0(p_1)) \\
(f_1 \oplus g_1) d_1(p_2) &= (f_1 d_1(p_2), g_1 d_1(p_2)) = (0, 0)
\end{aligned}$$

So $f_* \oplus g_*$ is a lifting. Claim that

$$(i_{PB}(f_0 \oplus g_0), \nabla_A(f_1 \oplus g_1))$$

is a lifting for $E_1 + E_2$ :

$$\begin{aligned}
\pi_C(i_{PB}(f_0 \oplus g_0(p_0))) &= \pi_C(0, f_0(p_0), g_0(p_0), \varepsilon(p_0)) = \varepsilon(p_0) \\
\nabla_A(f_1 \oplus g_1)(p_1) &= \nabla_A(f_1(p_1), g_1(p_1)) = f_1(p_1) + g_1(p_1) = (f_1 + g_1)(p_1) \\
&= f_1(p_1) + g_1(p_1) \\
&\implies \varphi(E_1 + E_2) = \varphi(E_1) + \varphi(E_2).
\end{aligned}$$

□

**Theorem 5.16.** *$E_R(C,A)$ is an abelian group with operation given by the Baer sum. Also, $E_R(C,A) \simeq Ext^1_R(C,A)$ as bifunctors*

$$R\text{-}mod \times R\text{-}mod \longrightarrow AB$$

Look at the class of the split exact sequence. Since we are free to choose lifting homomorphisms, we choose

$$\begin{tikzcd} \dots \arrow[r, "d_1"] & P_1 \arrow[r, "d_0"] \arrow[d, "f_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & A \oplus C \arrow[r, "\pi"] & C \arrow[r] & 0 \end{tikzcd}$$

where $f_0(p_0) = (0, \varepsilon(p_0))$ and $f_1 = 0$. We may choose these since $\pi f_0(p_0) = \varepsilon(p_0)$ and $f_0(d_0(p_1)) = 0 = i(0)$. So $\varphi([0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0]) = 0$, thus the zero element in $E_R(C, A)$ is the class of the split exact sequence.

# Part 2. Extensions of groups

## 6. Cohomology of groups

**Definition 6.1.** *Given a left $G$-module $A$, define for $n \geq 0$, the $n$-th cohomology group*

$$H^n(G, A) := Ext^n_{\mathbb{Z}G}(\mathbb{Z}^{triv}, A)$$

*where $\mathbb{Z}$ is the trivial left $\mathbb{Z}G$-module.*

We need a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Take $F_0$ to be the free $\mathbb{Z}G$-module on one generator, the symbol $[]$. Define $\varepsilon : F_0 \longrightarrow \mathbb{Z}$ as $\varepsilon([]) = 1$, so $\varepsilon(\sum n(g) \langle g \rangle []) = \sum n(g) \langle g \rangle \varepsilon([]) = \sum n(g) \langle g \rangle \cdot 1 = \sum n(g)$. Define $F_1$ to be the free $\mathbb{Z}G$-module on $[g_1]$, for all $g_1 \in G$. Define $d_1([g_1]) = \langle g_1 \rangle [] - []$. Build $F_2$ to be the free $\mathbb{Z}G$-module on all $[g_1 \mid g_2]$, for all $g_1, g_2 \in G$. Define $d_2([g_1 \mid g_2]) = \langle g_1 \rangle [g_2] - [g_1 g_2] + [g_1]$.

Build $F_3$ to be the free $\mathbb{Z}G$-module on $[g_1 \mid g_2 \mid g_3]$, for all $g_1, g_2, g_3 \in G$. Define $d_3([g_1 \mid g_2 \mid g_3]) = \langle g_1 \rangle [g_2 \mid g_3] - [g_1 g_2 \mid g_3] + [g_1 \mid g_2 g_3] - [g_1 \mid g_2]$.

Continue in this manner. For any $n > 0$, $F_n$ is the free $\mathbb{Z}G$-module on $[g_1 \mid g_2 \mid ... \mid g_n]$, for all $g_1, g_2, ..., g_n \in G$. The differential $d_n : F_n \longrightarrow F_{n-1}$ is defined as

$$d_n([g_1 \mid g_2 \mid ... \mid g_n]) = \langle g_1 \rangle [g_2 \mid .. \mid g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1 \mid .. \mid g_i g_{i+1} \mid .. \mid g_n] + (-1)^n [g_1 \mid .. \mid g_{n-1}]$$

Define also the $\mathbb{Z}G$-module homomorphisms $s_n : F_n \longrightarrow F_{n+1}$ $s_n(\langle g \rangle [g_1 \mid g_2 \mid .. \mid g_n]) = [g \mid g_1 \mid .. \mid g_n]$, whenever $n \geq 0$. Define $s_{-1}(1) = []$. Define this long sequence of free $\mathbb{Z}G$-modules as $B_G(\mathbb{Z})$, the bar resolution of the trivial $\mathbb{Z}\dot{G}$-module $\mathbb{Z}$.

**Proposition 6.2.** *Fix the ring $\mathbb{Z}G$. $B_G(\mathbb{Z})$ is a projective resolution over $\mathbb{Z}$.*

*Proof.* We have

$$... \underset{s_2}{\overset{d_3}{\rightleftarrows}} F_2 \underset{s_1}{\overset{d_2}{\rightleftarrows}} F_1 \underset{s_0}{\overset{d_1}{\rightleftarrows}} F_0 \underset{s_{-1}}{\overset{\varepsilon}{\rightleftarrows}} Z^{triv} \longrightarrow 0$$

It follows:

$$\begin{aligned}
\varepsilon s_{-1}(1) &= \varepsilon([]) = 1 \\
(s_{-1}\varepsilon + d_1 s_0)(\langle g \rangle []) &= s_{-1}\varepsilon(\langle g \rangle []) + d_1 s_0(g[]) = s_{-1}(g \cdot 1) + d_1([g]) \\
&= s_{-1}(1) + \langle g \rangle [] - [] = [] + \langle g \rangle [] - [] = \langle g \rangle []
\end{aligned}$$

$$
\begin{aligned}
s_{n-1}d_n(\langle g\rangle[g_1 \quad | \quad & g_2 \mid .. \mid g_n] \\
= \quad & s_{n-1}(\langle g\rangle(\langle g_1\rangle[g_2 \mid .. \mid g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1 \mid .. \mid g_i g_{i+1} \mid .. \mid g_n] + \\
+(-1)^n[g_1 \quad | \quad & .. \mid g_{n-1}]) \\
= \quad & [gg_1 \mid g_2 \mid .. \mid g_n] + \sum_{i=1}^{n-1}[g \mid g_1 \mid .. \mid g_i g_{i+1} \mid .. \mid g_n] + \\
+(-1)^n[g \quad | \quad & g_1 \mid .. \mid g_{n-1}] \\
d_{n+1}s_n(g[g_1 \quad | \quad & g_2 \mid .. \mid g_n]) = d_n[g \mid g_1 \mid .. \mid g_n] \\
= \quad & \langle g\rangle[g_1 \mid .. \mid g_n] + \sum_{i=1}^{n}(-1)^i[h_1 \mid h_2 .. \mid h_i h_{i+1} \mid .. \mid h_n] + (-1)^{n+1}[g \mid g_1 \mid .. \mid g_{n-1}]
\end{aligned}
$$

$$
\begin{aligned}
s_{n-1}d_{n-1} + d_n s_n \quad = \quad & 2\langle g\rangle[g_1 \mid .. \mid g_n] - [gg_1 \mid g_2 \mid ... \mid g_n] + (-1)^n[g \mid g_1 \mid .. \mid g_{n-1}] + \\
(-1)^{n+1}[g \quad | \quad & g_1 \mid .. \mid g_{n-1}] \\
= \quad & [gg_1 \mid g_2 \mid .. \mid g_n] + \langle g\rangle[g_1 \mid .. \mid g_n] - [gg_1 \mid g_2 \mid ... \mid g_n] = \langle g\rangle[g_1 \mid .. \mid g_n]
\end{aligned}
$$

So we have

$$
\begin{aligned}
\varepsilon s_{-1} &= 1_{\mathbb{Z}} \\
s_{-1}\varepsilon + d_1 s_0 &= 1_{F_0} \\
s_{n-1}d_n + d_{n+1}s_n &= 1_{F_n}, \forall n \geq 1.
\end{aligned}
$$

So if such a sequence exists, it splits as a sequence of abelian groups, hence is is exact as a sequence of $\mathbb{Z}G$-modules. Now we show that we can build such a sequence. Given the homomorphisms $s_n$, $n \geq -1$ as $s_{-1}(1) = []$ and $s_n(\langle g\rangle[g_1 \mid .. \mid g_n]) = [g \mid g_1 \mid .. \mid g_n]$. Let $F_0$ be the free $\mathbb{Z}G$-module on $[]$. We can recursively construct $\varepsilon$ and $d_n, n \geq 1$, and the free modules $F_n, n \geq 0$, from the three equations above. $F_{n+1}$, as a $\mathbb{Z}G$-module, is equal to the submodule $s_n F_n$, for $n \geq 0$.

$$
\begin{aligned}
\text{Need } \varepsilon s_{-1}(1) &= 1 \implies \varepsilon s_{-1}(1) = \varepsilon([]) = 1 \implies \text{define } \varepsilon([]) = 1 \\
\text{Need } d_1 s_0(\langle g\rangle[]) &= \langle g\rangle[] - s_{-1}\varepsilon(\langle g\rangle[]) = \langle g\rangle[] - s_{-1}(1) = \langle g\rangle[] - [] \implies d_1([g]) = \langle g\rangle[] - []
\end{aligned}
$$

Build $F_1$, the free $\mathbb{Z}G$-module on $[g], g \in G$.

$$d_{n+1}s_n(\langle g\rangle[g_1 \mid .. \mid g_n]) = \langle g\rangle[g_1 \mid .. \mid g_n] - s_{n-1}d_n(\langle g\rangle[g_1 \mid .. \mid g_n], n \geq 1$$

From this equation we can recursively build $F_n$ and $d_n, n \in \mathbb{Z}_{>1}$, and so far we have some long sequence of free modules over $\mathbb{Z}$. It turns out to be a complex. We have:

$$\varepsilon d_1([g]) = \varepsilon(\langle g\rangle[] - []) = g\varepsilon([]) - \varepsilon([]) = \varepsilon[] - \varepsilon[] = 0$$

Use now induction on the claim $P_n : d_n d_{n+1} = 0$.

$$
\begin{aligned}
d_1 d_2([g_1 \quad | \quad & g_2]) = d_1(\langle g_1\rangle[g_2] - [g_1 g_2] + [g_1]) = \langle g_1\rangle d_1([g_2]) - d_1([g_1 g_2]) + d_1[g_1] = \\
= \quad & \langle g_1\rangle(\langle g_2\rangle[] - []) - (\langle g_1 g\rangle_2[] - []) + \langle g_1\rangle[] - [] = 0
\end{aligned}
$$

So $P_1$ is correct. Suppose that for $n > 3, P_n$ is correct.

$$
\begin{aligned}
d_n(d_{n+1}s_n) &= d_n(1_{F_n} - s_{n-1}d_n) = d_n - (d_n s_{n-1})d_n = d_n - (1_{F_n} - s_{n-2}d_{n-1})d_n \\
&= d_n - d_n + s_{n-2}(d_{n-1}d_n) = 0
\end{aligned}
$$

When we build $F_{n+1}$ as the free $\mathbb{Z}G$-module on $\operatorname{Im} s_n$, we get that we build a chain complex of free $\mathbb{Z}G$-modules, i.e. a free chain complex of abelian groups with a contractive homotopy $s : 1_{F_*} \sim 0_{F_*}$. Since $H_n(1_{F_*}) = H_n(0_{F_*}) = 0$, we get a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. □

Take $Hom_{\mathbb{Z}G}(-, A)$, for any $\mathbb{Z}G$-module $A$.

$$0 \longrightarrow Hom_{\mathbb{Z}G}(\mathbb{Z}, A) \xrightarrow{\varepsilon^*} Hom_{\mathbb{Z}G}(F_0, A) \xrightarrow{d_1^*} Hom_{\mathbb{Z}G}(F_1, A) \xrightarrow{d_2^*} .. \longrightarrow Hom_{\mathbb{Z}G}(F_n, A) \xrightarrow{d_n^*} ...$$

So we have the codifferential $\delta^{n-1} = d_n^*$, $n \geq 1$. As a set, $Hom_{\mathbb{Z}G}(F_n, A)$ is equal to the set of functions $f : \underbrace{G \times G \times ... \times G}_{n} \longrightarrow A$, and adding the $\mathbb{Z}G$-module homomorphism structure gives

$$\begin{aligned} \delta^n f(g_1, g_2, .., g_n, g_{n+1}) &= d_{n+1}^* f = f d_{n+1}([g_1 \mid g_2 \mid .. \mid g_n \mid g_{n+1}]) \\ &= g_1 f(g_2, g_3, .., g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, .., g_i g_{i+1}, .., g_{n+1}) + \\ (-1)^{n+1} f(g_1, g_2, ..g_n) &= h(g_1, g_2, .., g_{n+1}) \end{aligned}$$

Let's take a closer look at the lowest cohomology groups. We know $H^0(G, A) \simeq Hom_{\mathbb{Z}G}(\mathbb{Z}, A)$, so it is given by $\mathbb{Z}G$-module homomorphisms $f(1) = a$ for those $a \in A$ such that $f(1) = f(g \cdot 1) = gf(1) = ga = a$, that $G$ acts trivially on. Denote this group as $A^G$. 1-cocycles are given by those functions $f : G \times G \longrightarrow A$ such that

$$\begin{aligned} \delta^1 f(g_1, g_2) &= 0 = f(d_2[g_1, g_2]) = f(\langle g_1 \rangle [g_2] - [g_1, g_2] + [g_1]) \\ &= g_1 f(g_2) - f(g_1 g_2) + f(g_1) \implies f(g_1 g_2) = g_1 f(g_2) + f(g_1) \end{aligned}$$

We call these homomorphisms for crossed homomorphisms. They would necessarily satisfy $f(1) = 0$. 1-coboundaries are given by

$$\delta^0 f(g) = f d_1([g]) = f(\langle g \rangle [] - []) = gf([]) - f[] = ga - a = h_a(g), \text{ for any } a \in A.$$

We call these homomorphisms for principal homomorphisms. They would necessarily satisfy $h(1) = 0$. So we have that $H^1(G, A)$ is the factor group of the group of crossed homomorphisms modulo the subgroup of principal homomorphisms. 2-cocycles are given by those $f : G \times G \longrightarrow A$ such that

$$\begin{aligned} \delta^2 f(g_1, g_2, g_3) &= 0 = f(d_3[g_1 \mid g_2 \mid g_3]) \\ &= f(\langle g_1 \rangle [g_2 \mid g_3] - [g_1 g_2 \mid g_3] + [g_1 \mid g_2 g_3] - [g_1 \mid g_2]) \\ &= g_1 f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = h(g_1, g_2, g_3) \\ &\implies f(g_1, g_2) = g_1 f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) \end{aligned}$$

2-coboundaries lie in the image of $\delta^1$

$$\delta^1 f(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1) = h(g_1, g_2)$$

**Remark 6.3.** *Since $Ext^n_{\mathbb{Z}G}$ is independent of the choice of projective resolution, we may also work with the normalized bar resolution. Denote $\overline{F_n}$ the factor module of the free module on $[g_1 \mid g_2 \mid .. \mid g_n]$, modulo the submodule generated by $[g_1 \mid g_2 \mid .. \mid g_n]$, if any of the $\{g_i\}_{i=1}^n = 1$. The homomorphisms $\varepsilon, d, s_{-1}$ still hold, need only to check that $d_n[g_1 \mid g_2 \mid .. \mid g_n] = 0$ if any one of the $g_i, i = 1, .., n$, are equal to 1. This is easily seen from the formula of $d_n : F_n \longrightarrow F_{n-1}$. The normalized bar resolution $\overline{B_G(\mathbb{Z})}$ obtained in the same manner as for $B_G(\mathbb{Z})$, with the extra condition, is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then any of the $n$-cochains will satisfy* *the normalisation condition,* $f(g_1, g_2, .., g_n) = 0$ *if any one of the* $g_i, i = 1, .., n$*, are equal to* 1.

**Proposition 6.4.** *For any* $n \in \mathbb{N}$, $H^n(G, -)$ *is a covariant functor from* $G$*-mod to* $AB$.

*Proof.* By Proposition 3.4, $Ext^n_{\mathbb{Z}G}(\mathbb{Z}^{trivial}, -)$ is a covariant functor. Since $H^n(G, A) = Ext^n_{\mathbb{Z}G}(\mathbb{Z}^{trivial}, A)$, we have proved the claim. □

**Proposition 6.5.** *For any* $n \in \mathbb{N}$, $H^n(-, A)$ *is a contravariant functor from* $GR$ *to* $AB$.

*Proof.* Suppose $\gamma \in Hom_{GR}(K, G)$. Let $A$ be any $\mathbb{Z}G$-module. $A$ becomes a $K$-module through its $G$-module structure: $ka = \gamma(k)a$. Define a projective resolution of $K$ as in the bar resolution for $K$, and denote its free modules by $\{K_i\}_{i=0}^N$. By the universal property of free modules, there exists the family of $\mathbb{Z}K$-module homomorphisms $f_* : B_K(\mathbb{Z}) \longrightarrow B_G(\mathbb{Z})$ defined as

$$\begin{aligned} f_0([]) &= [],\ f_1([k]) = [\gamma(k)] \\ f_n([k_1 &\mid k_2 \mid .. \mid k_n]) = [\gamma(k_1) \mid \gamma(k_2) \mid ... \mid \gamma(k_n)], n \in \mathbb{Z}_{\geq 1}. \end{aligned}$$

They make each square of

$$\begin{array}{ccccccccc} ... & \xrightarrow{d_3} & K_2 & \xrightarrow{d_2} & K_1 & \xrightarrow{d_1} & K_0 & \xrightarrow{\varepsilon} & Z \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1_Z} \\ ... & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\varepsilon} & Z \end{array}$$

commutative:

$$f_0(\varepsilon([])) = f_0([]) = [] = \varepsilon([])$$

$$\begin{aligned} f_{n-1}d_n([k_1 &\mid k_2 \mid .. \mid k_n]) = f_{n-1}(\langle k_1 \rangle [k_2 \mid ... \mid k_n] + \sum_{i=1}^{n-1}(-1)^i[k_1 \mid .. \mid k_ik_{i+1} \mid .. \mid k_n] \\ +(-1)^n[k_1 &\mid .. \mid k_{n-1}]) \\ &= \langle k_1 \rangle f_{n-1}([k_2 \mid ... \mid k_n]) + \sum_{i=1}^{n-1}(-1)^i f_{n-1}([k_1 \mid .. \mid k_ik_{i+1} \mid .. \mid k_n]) \\ +(-1)^n f_{n-1}([k_1 &\mid .. \mid k_{n-1}]) \\ &= \langle \gamma(k_1) \rangle [\gamma(k_2) \mid ... \mid \gamma(k_n)] + \sum_{i=1}^{n-1}(-1)^i[\gamma(k_1) \mid .. \mid \gamma(k_ik_{i+1}) \mid .. \mid \gamma(k_n)] \\ +(-1)^n[\gamma(k_1) &\mid .. \mid \gamma(k_{n-1})]) \\ d_nf_n([k_1 &\mid ... \mid k_n]) \\ &= d_n([\gamma(k_1) \mid ... \mid \gamma(k_n)] = \langle \gamma(k_1) \rangle [\gamma(k_2) \mid ... \mid \gamma(k_n)] \\ + \sum_{i=1}^{n-1}(-1)^i[\gamma(k_1) &\mid .. \mid \gamma(k_ik_{i+1}) \mid .. \mid \gamma(k_n)] + (-1)^n[\gamma(k_1) \mid .. \mid \gamma(k_{n-1})]) \\ &= f_{n-1}d_n([k_1 \mid k_2 \mid .. \mid k_n]), n \in \mathbb{Z}_{\geq 1}. \end{aligned}$$

Hence the family $\{f_n\}_{n=0}$ is a lifting. Take $Hom_{\mathbb{Z}G}(-,A)$, and get the commutative diagram of complexes of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_{ZG}(Z,A) & \xrightarrow{\varepsilon^*} & Hom_{ZG}(F_0,A) & \xrightarrow{{d_1}^*} & Hom_{ZG}(F_1,A) & \xrightarrow{{d_2}^*} & ... \\ & & \downarrow{\scriptstyle 1_{Hom_{ZG}(Z,A)}} & & \downarrow{\scriptstyle {f_0}^*} & & \downarrow{\scriptstyle {f_1}^*} & & \\ 0 & \longrightarrow & Hom_{ZK}(Z,A) & \xrightarrow{\varepsilon^*} & Hom_{ZK}(K_0,A) & \xrightarrow{{d_1}^*} & Hom_{ZK}(K_1,A) & \xrightarrow{{d_2}^*} & ... \end{array}$$

$$\begin{aligned} (f_0^*\varepsilon^*(s))([]) &= f_0^*(s\varepsilon)([]) = (s\varepsilon f_0)([]) = s(\varepsilon([])) = s([]) \\ (\varepsilon^* s)([]) &= (s\varepsilon)([]) = s(\varepsilon[]) = s([]) \\ f_n^* d_{n-1}^*(t) &= f_n^*(td_{n-1}) = (td_{n-1})f_n = t(d_{n-1}f_n) = t(f_{n-1}d_{n-1}) = (tf_{n-1})d_{n-.1} \\ &= d_{n-1}^*(f_{n-1}^*(t)) = d_{n-1}^* f_{n-1}^*(t), \forall t \in Hom_{\mathbb{Z}G}(F_{n-1},A), n \in \mathbb{Z}_{\geq 1}. \end{aligned}$$

Hence $f^* : Hom_{\mathbb{Z}G}(F_*,A) \longrightarrow Hom_{\mathbb{Z}K}(K_*,A)$ is a cochain transformation. Applying $H^n$ on $f^*$ gives:

$$\begin{aligned} H^n(f^*) &= f_* : H^n(G,A) \longrightarrow H^n(K,A) \\ f_*(l + d_n^*\,(Hom_{\mathbb{Z}G}(F_{n-1},A)) &= f_n^*(l) + d_n^*\,(Hom_{\mathbb{Z}K}(K_{n-1},A)) \end{aligned}$$

When $\gamma = 1_G$ we get $f_n([g_1 \mid g_2 \mid .. \mid g_n]) = ([g_1 \mid g_2 \mid .. \mid g_n])$ and:

$$f_*(l+d_n^*((Hom_{\mathbb{Z}G}(F_{n-1},A))\,(g_1,g_2,..,g_n) = f_n^*(l)([g_1 \mid g_2 \mid .. \mid g_n])+d_n^*\,(Hom_{\mathbb{Z}G}(F_{n-1},A))$$

$$\begin{aligned} &= l(f_n([g_1 \mid g_2 \mid .. \mid g_n]) + d_n^*\,(Hom_{\mathbb{Z}G}(F_{n-1},A)) = l([g_1 \mid g_2 \mid .. \mid g_n]) + d_n^*\,(Hom_{\mathbb{Z}G}(F_{n-1},A)) \\ &= 1_{H^n(G,A)} \end{aligned}$$

Look at the pair of morphisms $\beta : S \longrightarrow K$, $\gamma : K \longrightarrow G$. $A$ becomes an $S$-module through its $K$-module structure: $sa = \beta(s)a$. Define a projective resolution of $S$ as in the bar resolution for $S$, and denote its free modules by $\{S_i\}_{i=0}^N$. By the universal property of free modules, there exists the family of $\mathbb{Z}S$-module homomorphisms $g_* : B_S(\mathbb{Z}) \longrightarrow B_K(\mathbb{Z})$ defined as

$$\begin{aligned} g_0([]) &= [],\ g_1([s]) = [\beta(s)] \\ g_n([s_1 &\mid s_2 \mid .. \mid s_n]) = [\beta(s_1) \mid \beta(s_2) \mid ... \mid \beta(s_n)], n \in \mathbb{Z}_{\geq 1}. \end{aligned}$$

They make each square of

$$\begin{array}{ccccccccc} ... & \xrightarrow{d_3} & S_2 & \xrightarrow{d_2} & S_1 & \xrightarrow{d_1} & S_0 & \xrightarrow{\varepsilon} & Z \\ & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle 1_Z} \\ ... & \xrightarrow{d_3} & K_2 & \xrightarrow{d_2} & K_1 & \xrightarrow{d_1} & K_0 & \xrightarrow{\varepsilon} & Z \end{array}$$

commutative. Take $Hom_{\mathbb{Z}K}(-,A)$, and get the commutative diagram of complexes of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_{ZK}(Z,A) & \xrightarrow{\varepsilon^*} & Hom_{ZK}(K_0,A) & \xrightarrow{{d_1}^*} & Hom_{ZK}(K_1,A) & \xrightarrow{{d_2}^*} & ... \\ & & \downarrow{\scriptstyle 1_{Hom_{ZG}(Z,A)}} & & \downarrow{\scriptstyle {g_0}^*} & & \downarrow{\scriptstyle {g_1}^*} & & \\ 0 & \longrightarrow & Hom_{ZS}(Z,A) & \xrightarrow{\varepsilon^*} & Hom_{ZS}(S_0,A) & \xrightarrow{{d_1}^*} & Hom_{ZS}(S_1,A) & \xrightarrow{{d_2}^*} & ... \end{array}$$

Take the covariant functor $H^n$, and get the group homomorphism

$$\begin{aligned} H^n(g^*) &= g_* : H^n(K,A) \longrightarrow H^n(S,A) \\ g_*(u + d_n^*\,(Hom_{\mathbb{Z}K}(K_{n-1},A)) &= g_n^*(u) + d_n^*\,(Hom_{\mathbb{Z}S}(S_{n-1},A)) \end{aligned}$$

Then we have the composition homomorphism

$$\begin{aligned} g_*f_* &: H^n(G,A) \longrightarrow H^n(S,A) \\ (g_*f_*)(l + d_n^*\,(Hom_{\mathbb{Z}G}(F_{n-1},A))\,[g_1,g_2,..,g_n] &= g_*(l(f_n([g_1 \mid g_2 \mid .. \mid g_n]) + d_n^*\,(Hom_{\mathbb{Z}K}(K_{n-1},A)) \end{aligned}$$

$$\begin{aligned} g_*(l([\gamma(g_1) \mid .. \mid \gamma(g_n)]) + d_n^*\,(Hom_{\mathbb{Z}K}(K_{n-1},A)) &= l(g_n([\gamma(g_1) \mid .. \mid \gamma(g_n)]) + d_n^*\,(Hom_{\mathbb{Z}S}(S_{n-1},A)) \\ &= l([\beta(\gamma(g_1)) \mid .. \mid \beta(\gamma(g_n))]) + d_n^*\,(Hom_{\mathbb{Z}S}(S_{n-1},A)) \\ &= l([(\beta\gamma)(g_1) \mid .. \mid (\beta\gamma)(g_n)]) + d_n^*\,(Hom_{\mathbb{Z}S}(S_{n-1},A)) \end{aligned}$$

We also have the $\mathbb{Z}G$-module homomorphisms $g_nf_n = k_n$ defined as

$$\begin{aligned} k_0([]) &= [] \\ k_n([s_1 \mid s_2 \mid .. \mid s_n]) &= [(\beta\gamma)(s_1) \mid (\beta\gamma)(s_2) \mid .. \mid (\beta\gamma)(s_n)], n \in \mathbb{Z}_{\geq 1}. \end{aligned}$$

we get the commutative diagram

$$\begin{tikzcd} ... \arrow[r, "d_3"] & S_2 \arrow[r, "d_2"] \arrow[d, "k_2"'] & S_1 \arrow[r, "d_1"] \arrow[d, "k_1"'] & S_0 \arrow[r, "\varepsilon"] \arrow[d, "k_0"'] & Z \arrow[d, "1_Z"'] \\ ... \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r, "\varepsilon"] & Z \end{tikzcd}$$

Applying $Hom_{\mathbb{Z}G}(-,A)$, we get the commutative diagram of cochain complexes

$$\begin{tikzcd} 0 \arrow[r] & Hom_{ZG}(Z,A) \arrow[r, "\varepsilon^*"] \arrow[d, "1_{Hom_{ZG}(Z,A)}"'] & Hom_{ZG}(F_0,A) \arrow[r, "{d_1}^*"] \arrow[d, "{k_0}^*"'] & Hom_{ZG}(F_1,A) \arrow[r, "{d_2}^*"] \arrow[d, "{k_1}^*"'] & ... \\ 0 \arrow[r] & Hom_{ZS}(Z,A) \arrow[r, "\varepsilon^*"] & Hom_{ZS}(S_0,A) \arrow[r, "{d_1}^*"] & Hom_{ZS}(S_1,A) \arrow[r, "{d_2}^*"] & ... \end{tikzcd}$$

Applying $H^n$ on the cochain transformation $k^*$, we get

$$\begin{aligned} H^n(k^*) &= k_* : H^n(G,A) \longrightarrow H^n(S,A) \\ k_*(l + d_n^*(Hom_{\mathbb{Z}G}(F_{n-1},A)) &= k_n^*(l) + d_n^*(Hom_{\mathbb{Z}S}(S_{n-1},A)) \end{aligned}$$

$$\begin{aligned} &= l(k_n([g_1 \mid . \mid g_n])) + d_n^*(Hom_{\mathbb{Z}S}(S_{n-1},A)) \\ &= l([(\beta\gamma)(s_1) \mid (\beta\gamma)(s_2) \mid .. \mid (\beta\gamma)(s_n)] + d_n^*(Hom_{\mathbb{Z}S}(S_{n-1},A)) \\ &= (g_*f_*)(l + d_n^*\,(Hom_{\mathbb{Z}G}(F_{n-1},A))\,([g_1,g_2,..,g_n]) \end{aligned}$$

So $H^n(-,A)$ is a contravariant functor. □

**Definition 6.6.** *Define the pairs $(G,A)$ where $G$ is any group and $A$ is any $G$-module. For any $\varphi \in Hom_{GR}(K,G), \psi \in Hom_{AB}(A,B)$, define a morphism $(\varphi,\psi) : (G,A) \longrightarrow (K,B)$ as $\psi(\varphi(k)a) = k\psi(a)$. We have described a category which we will denote $PAIRS$.*

**Proposition 6.7.** *$H^n(-,-)$ is a bifunctor from $PAIRS$ to $AB$.*

**Remark 6.8.** *Actually, $H^n$ is not a bifunctor in a proper sense, since the variables $G$ and $A$ are not independent.*

*Proof.* For any $G$-module homomorphism $\alpha : A \longrightarrow A^{'}$, and group homomorphism $\gamma : G^{'} \longrightarrow G$, the diagram

$$\begin{tikzcd} H^n(G,A) \arrow[r, "\alpha^*"] \arrow[d, "\gamma^*"'] & H^n(G,A') \arrow[d, "\gamma^*"] \\ H^n(G',A) \arrow[r, "\alpha^*"] & H^n(G',A') \end{tikzcd}$$

is commutative. Start with the bar resolution for $G$. We may take $Hom_{\mathbb{Z}G}(B_G(\mathbb{Z}), A)$. $\alpha$ induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Hom_{ZG}(Z,A) & \xrightarrow{\varepsilon^*} & Hom_{ZG}(F_0,A) & \xrightarrow{d_1^*} & Hom_{ZG}(F_1,A) & \xrightarrow{d_2^*} & ... \\
 & & \alpha_*\big\downarrow & & \alpha_*\big\downarrow & & \alpha_*\big\downarrow & & \\
0 & \longrightarrow & Hom_{ZG}(Z,A') & \xrightarrow{\varepsilon^*} & Hom_{ZG}(F_0,A') & \xrightarrow{d_1^*} & Hom_{ZG}(F_1,A') & \xrightarrow{d_2^*} & ...
\end{array}$$

$$\begin{aligned}
\alpha_*\varepsilon^*(s) &= \alpha_*(s\varepsilon) = \alpha(s\varepsilon) \\
\varepsilon^*\alpha_*(s) &= \varepsilon^*(\alpha s) = (\alpha s)\varepsilon \\
\alpha_* d_n^*(t) &= \alpha_*(td_n) = \alpha(td_n) = d_n^*(\alpha t) = d_n^*(\alpha_* t), n \in \mathbb{Z}_{\geq 1}.
\end{aligned}$$

So $\alpha_*$ is a cochain transformation between our two cochain complexes $Hom_{\mathbb{Z}G}(B_G(\mathbb{Z}), A)$ and $Hom_{\mathbb{Z}G}(B_G(\mathbb{Z}), A^{'})$. Apply $H^n(\alpha_*) = \alpha_*$ and get the group homomorphism

$$\begin{aligned}
\alpha_* \quad &: \quad H^n(G, A) \longrightarrow H^n(G, A^{'}) \\
\alpha_*(l + d_n^*(Hom_{\mathbb{Z}G}(F_{n-1}, A))) \quad &= \quad \alpha_* l + d_n^*(Hom_{\mathbb{Z}G}(F_{n-1}, A^{'}))
\end{aligned}$$

Define a projective resolution of $G^{'}$ as in the bar resolution for $G^{'}$, and denote its free modules by $\{G_i^{'}\}_{i=0}^N$. By the universal property of free modules, there exists the family of $\mathbb{Z}G^{'}$-module homomorphisms $f_* : B_{G'}(\mathbb{Z}) \longrightarrow B_G(\mathbb{Z})$ defined as

$$\begin{aligned}
f_0([]) \quad &= \quad [], \; f_1([g^{'}]) = [\gamma(g^{'})] \\
f_n([g_1^{'} \quad &| \quad g_2^{'} \mid .. \mid g_n^{'}]) = [\gamma(g_1^{'}) \mid \gamma(g_2^{'}) \mid ... \mid \gamma(g_n^{'})], n \in \mathbb{Z}_{\geq 1}.
\end{aligned}$$

They make each square of

$$\begin{array}{ccccccccc}
... & \xrightarrow{d_3} & G'_2 & \xrightarrow{d_2} & G'_1 & \xrightarrow{d_1} & G'_0 & \xrightarrow{\varepsilon} & Z \\
 & & f_2\big\downarrow & & f_1\big\downarrow & & f_0\big\downarrow & & 1_Z\big\downarrow \\
... & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\varepsilon} & Z
\end{array}$$

commutative. Applying $Hom_{\mathbb{Z}G}(B_G(\mathbb{Z}), A)$ induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Hom_{ZG}(Z,A') & \xrightarrow{\varepsilon^*} & Hom_{ZG}(F_0,A') & \xrightarrow{d_1{}^*} & Hom_{ZG}(F_1,A') & \xrightarrow{d_2{}^*} & ... \\
 & & 1_{Hom_{ZG}(Z,A')}\big\downarrow & & f_0{}^*\big\downarrow & & f_1{}^*\big\downarrow & & \\
0 & \longrightarrow & Hom_{ZG'}(Z,A') & \xrightarrow{\varepsilon^*} & Hom_{ZG'}(G'_0,A') & \xrightarrow{d_1{}^*} & Hom_{ZG'}(G'_1,A') & \xrightarrow{d_2{}^*} & ...
\end{array}$$

and the cochain transformations $\{f_n^*\}_{n=0}$. Take $H^n(f^*) = f_*$ as

$$\begin{aligned}
f_* \quad &: \quad H^n(G, A^{'}) \longrightarrow H^n(G^{'}, A^{'}) \\
f_*(s + d_n^*(Hom_{\mathbb{Z}G}(F_{n-1}, A^{'})) \quad &= \quad f_n^*(s) + d_n^*(Hom_{\mathbb{Z}G'}(G_{n-1}^{'}, A^{'}))
\end{aligned}$$

The composition yields

$$\begin{aligned}
f_*\alpha_* \quad &: \quad H^n(G, A) \longrightarrow H^n(G^{'}, A^{'}) \\
f_*\alpha_*(l + d_n^*(Hom_{\mathbb{Z}G}(F_{n-1}, A))) \quad &= \quad f_*(\alpha_* l + d_n(Hom_{\mathbb{Z}G}(F_{n-1}, A^{'})))
\end{aligned}$$

$$= f_*(\alpha l + d_n(Hom_{\mathbb{Z}G}(F_{n-1}, A^{'}))) = (\alpha l)f_n + d_n^*(Hom_{\mathbb{Z}G'}(G_{n-1}^{'}, A^{'}))$$

Following the anti- clockwise direction in our diagram, we get

$$\begin{aligned} f_* &: H^n(G,A) \longrightarrow H^n(G^{'},A) \\ f_*(l + d_n^*(Hom_{\mathbb{Z}G}(F_{n-1},A)) &= f_n^*(l) + d_n^*(Hom_{\mathbb{Z}G'}(G^{'}_{n-1},A)) \\ \alpha_* &: H^n(G,A^{'}) \longrightarrow H^n(G^{'},A^{'}) \\ \alpha_*(s + d_n^*(Hom_{\mathbb{Z}G}(F_{n-1},A))) &= \alpha_* s + d_n^*(Hom_{\mathbb{Z}G}(G^{'}_{n-1},A^{'})) \end{aligned}$$

Their composition gives

$$\begin{aligned} \alpha_* f_*(l + d_n^*(Hom_{\mathbb{Z}G}(F_{n-1},A)) &= \alpha_*(f_n^*(l) + d_n^*(Hom_{\mathbb{Z}G'}(G^{'}_{n-1},A))) \\ &= \alpha_*(f_n^*(l)) + d_n^*(Hom_{\mathbb{Z}G'}(G^{'}_{n-1},A^{'}))) \\ &= \alpha_*(lf_n) + d_n^*(Hom_{\mathbb{Z}G'}(G^{'}_{n-1},A^{'}))) \\ &= \alpha(lf_n) + d_n^*(Hom_{\mathbb{Z}G'}(G^{'}_{n-1},A^{'}))) \end{aligned}$$

Hence $H^n(G,A)$ is a bifunctor: $\alpha_* f_* = f_* \alpha_* : H^n(G,A) \longrightarrow H^n(G^{'},A^{'})$ □

## 7. Extensions with abelian kernel

7.1. **Description using cocycles.** Look at a short exact sequence

$$\varepsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 0$$

where $A$ is abelian. We will write $+$ for the binary operation on $A$ and $E$ ($E$ is not necessarily abelian), and multiplicatively for a group $G$. $E$ acts on itself by conjugation. Since $A \simeq \varkappa A$, and $E/\varkappa A \simeq G$, where $\varkappa A = \ker \sigma$, $\varkappa A$ is a normal subgroup of $E$, and $A$ is isomorphic to a normal subgroup of $E$ (we will write $a$ for $\varkappa a$ when it is clear from the context what we mean). Therefore, $E$ acts on $A$ by conjugation: there exists a group homomorphism $\varphi^{'} : E \longrightarrow Aut(A)$ given by $\varphi^{'}(e)(a) = e + \varkappa a - e$. Since $\varkappa(A) \subseteq \ker \varphi^{'}$, there exists $\varphi : E/\varkappa A \simeq G \longrightarrow Aut(A)$. So $A$ is a $G$-module. . The action defined on a set of representatives $\langle g \rangle$ of $\{g\}_{g \in G}$ in $E$, such that $\sigma(\langle g \rangle) = g$, is

$$\varphi(g)(a) = \langle g \rangle + a - \langle g \rangle \iff \varphi(g)(a) + \langle g \rangle = \langle g \rangle + a$$

**Definition 7.1.** *Let $G$ be a group and $A$ be a $G$-module, with the fixed action $\varphi$ of $G$ on $A$. Denote by $E(G, A^{\varphi})$ the set of equivalence classes of short exact sequences of groups (extensions)*

$$0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1,$$

*such that*

$$\varphi\left(g\right)\left(a\right) = e + a - e \mid e \in \sigma^{-1}\left(g\right), g \in G, a \in A,$$

*where two extensions are called equivalent if there exists a group homomorphism $h : E_1 \longrightarrow E_2$ (hence isomorphism), such that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "1_A"'] & E_1 \arrow[r] \arrow[d, "h"'] & G \arrow[d, "1_G"] \\ A \arrow[r] & E_2 \arrow[r] & G \end{tikzcd}$$

*commutes.*

Let $\langle\ \rangle : G \longrightarrow E$ be a function (not a homomorphism) such that $\sigma(\langle g\rangle) = g$. We choose $\langle 1\rangle = 0$. Any $e \in E$ belongs to the right $A$-coset $A + \langle\sigma(e)\rangle$, and can therefore be written as $e = a + \langle g\rangle, a \in A, g = \sigma(e) \in G$. Let's look at the operation on $E$ :

$$(a + \langle g\rangle) + (b + \langle h\rangle = a + (\langle g\rangle + b) + \langle h\rangle = a + gb + \langle g\rangle + \langle h\rangle$$

Now,

$$\sigma(\langle g\rangle + \langle h\rangle - \langle gh\rangle) = \sigma(\langle g\rangle)\sigma(\langle h\rangle)\sigma(\langle gh\rangle)^{-1} = gh(gh)^{-1} = ghh^{-1}g = 1$$

$$\begin{aligned} \langle g\rangle + \langle h\rangle - \langle gh\rangle &\in \ker\sigma = \operatorname{Im}\varkappa \\ f(g,h) &= \langle g\rangle + \langle h\rangle - \langle gh\rangle \in \operatorname{Im}\varkappa \\ \langle g\rangle + \langle h\rangle &= f(g,h) + \langle gh\rangle \end{aligned}$$

for some function $f : G \times G \longrightarrow A$. It follows

$$\begin{aligned} f(1,g) + \langle g\rangle &= \langle 1\rangle + \langle g\rangle \implies f(1,g) = 0 \\ f(g,1) + \langle g\rangle &= \langle g\rangle + \langle 1\rangle \implies f(g,1) = 0 \end{aligned}$$

So:

$$(a + \langle g\rangle) + (b + \langle h\rangle) = a + gb + f(g,h) + \langle gh\rangle$$

For simplicity, define $(a,g) := a + \langle g\rangle$ . Then

$$(a,g) + (b,h) = (a + gb + f(g,h), gh)$$

We see that the right hand side gives that $E$ is some 'twisted' semi- direct product of $A$ and $G$. If $f(g,h) = 0, \forall g,h \in G$, then $E = A \rtimes_\varphi G$. We should have a group structure in $E$ :

- There exists a unique zero element $(0,1)$

$$\begin{aligned} (b,h) + (a,g) &= (a,g) \\ &= (b + ha + f(h,g), hg) \\ hg &= g \implies h = 1 \\ b + ha + f(h,g) &= a \implies b + 1 \cdot a + f(1,g) = a \\ &\implies b + a + 0 = a \implies b + a = a \iff b = 0 \end{aligned}$$

- The right inverse element:

$$\begin{aligned} (a,g) + (b,h) &= (0,1) \\ &= (a + gb + f(g,h), gh) \\ gh &= 1 \implies h = g^{-1} \\ a + gb + f(g,g^{-1}) &= 0 \implies gb = -f(g,g^{-1}) - a \implies b = -g^{-1}f(g,g^{-1}) - g^{-1}a \end{aligned}$$

The left inverse element:

$$\begin{aligned} (b,h) + (a,g) &= (-f(1,g),1) \\ &= (b + ha + f(h,g), hg) \\ hg &= 1 \implies h = g^{-1} \\ b + g^{-1}a + f(g^{-1},g) &= 0 \implies b = -g^{-1}a - f(g^{-1},g) \end{aligned}$$

The inverse must be unique so this condition must yield

$$-f(g^{-1},g) = -g^{-1}f(g,g^{-1}) \iff f(g^{-1},g) = g^{-1}f(g,g^{-1})$$

So $(a,g)^{-1} = (-g^{-1}a - f(g^{-1},g), g^{-1})$.

- $E$ must be associative

$$\begin{aligned}
\{(a,g)+(b,h)\}+(c,k) &= (a+gb+f(g,h),gh)+(c,k) \\
&= (a+gb+f(g,h)+ghc+f(gh,k),ghk) \\
(a,g)+\{(b,h)+(c,k)\} &= (a,g)+(b+hc+f(h,k),hk)
\end{aligned}$$

$$\begin{aligned}
&= && (a+g(b+hc+f(h,k))+f(g,hk),ghk) = (a+gb+ghc+gf(h,k)+f(g,hk),ghk) \\
&\Longrightarrow && f(g,h)+f(gh,k) = gf(h,k)+f(g,hk) \\
&\Longleftrightarrow && gf(h,k)+f(g,hk)-f(gh,k)-f(g,h) = 0 \Longleftrightarrow \delta^2 f(g,h,k) = 0
\end{aligned}$$

So $f$ is a 2-cocycle.

Claim that the set $H = \{(a,1) : a \in A\} \simeq \{(\varkappa(a),1) : a \in A\}$ is a normal subgroup of $E$ :

(1) $(0,1) \in H$
(2) $(a,1)^{-1} = (a,1) \in A$
(3) it is closed under addition: $(a,1)+(a^{'},1) = (a+1\cdot a^{'}+f(1,1),1) = (a+a^{'},1) \in A$
(4)
$$(b,h)+(a,1)-(b,h) = (b+ha+f(h,1),h)-(b,h)$$

$$\begin{aligned}
&= && (a+ha,h)+(-hb-f(1,h),h^{^{-1}}) = (a+ha,h)+(-hb,h^{^{-1}}) \\
&= && (a+ha-hhb+f(h,h^{^{-1}}),hh^{^{-1}}) = (a+ha-h^2b+f(h,h^{^{-1}}),1) \in H
\end{aligned}$$

Define the function $i : A \longrightarrow E$ as $i(a) = (a,1)$. It is a group isomorphism $A \simeq i(A) \lhd E$ :

$$\begin{aligned}
i(a+b) &= (a+b,1) = (a,1)+(b,1) \\
a &\in \ker i \Longleftrightarrow i(a) = (a,1) = (0,1) \Longrightarrow a = 0
\end{aligned}$$

Define the function $p : E \longrightarrow G$ as $p(a,g) = g$. It is a group epimorphism:

$$\begin{aligned}
p((a,g)+(b,h)) &= p(a+gb+f(g,h),gh) = gh = p(a,g)p(b,h) \\
\forall g &\in G, \exists (0,g) \in E \mid p(0,g) = g
\end{aligned}$$

Its kernel is

$$\ker p = \{(a,g) \mid p(a,g) = 1\} = \{(a,1), a \in A\} = i(A)$$

In the beginning, we chose a set of representatives for the elements of $G$, and especially $\langle 1\rangle = 0$. Let $\{g\}_{g\in G}$ be another set of representatives. Then the two extensions $\left(0 \longrightarrow A \xrightarrow{i} E_{\langle g\rangle} \xrightarrow{p} G \longrightarrow 1\right)$ and $\left(0 \longrightarrow A \xrightarrow{i^{'}} E_{\{g\}} \xrightarrow{p^{'}} G \longrightarrow 1\right)$ are equivalent by a homomorphism $\zeta : E_{\langle g\rangle} \longrightarrow E_{\{g\}}$ defined as $\zeta(a+\langle g\rangle) = a+\{g\}$ :

$$\begin{aligned}
\zeta((a+\langle g\rangle)+(b+\langle h\rangle)) &= \zeta(a+gb+f(g,h)+\langle gh\rangle) = a+gb+f(g,h)+\{gh\} \\
\zeta i(a) &= \zeta(a+\langle 1\rangle) = a+\{1\} = i^{'}(a) \\
p^{'}\zeta(a+\langle g\rangle) &= p^{'}(a+\{g\}) = g = p(a+\langle g\rangle)
\end{aligned}$$

Independently of which representative of elements of $G$ we choose, we get an extension in the same equivalence class. Claim that $\varepsilon$ is equivalent to this extension

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

Choose $\langle 1 \rangle = 0$. Take $\zeta : E \longrightarrow E$ as $\zeta\left(a + \langle g \rangle\right) = a + \langle g \rangle$. Use that $(a + \langle g \rangle) = (a + \langle 1 \rangle) + (0 + \langle g \rangle)$.

$$\begin{aligned}
\zeta((a + \langle g \rangle) + (b + \langle h \rangle)) &= \zeta(a + gb + f(g,h) + \langle gh \rangle) = a + gb + f(g,h) + \langle gh \rangle \\
\zeta(a + \langle g \rangle) + \zeta(b + \langle h \rangle) &= (a + \langle g \rangle) + (b + \langle h \rangle) = a + gb + f(g,h) + \langle gh \rangle \\
\zeta(i(a)) &= \zeta(\varkappa(a) + \langle 1 \rangle) = \varkappa(a) + \langle 1 \rangle = \varkappa(a) \\
\sigma\zeta(a + \langle g \rangle) &= \sigma(a + \langle g \rangle) = \sigma(\varkappa(a))\sigma(\langle g \rangle) = g = p(a + \langle g \rangle)
\end{aligned}$$

What can we say about two equivalent extensions? Suppose $\zeta : E \longrightarrow E^{'}$ in

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[dr, "i'"'] & E \arrow[r, "p"] \arrow[d, "\zeta"'] & G \arrow[r] & 1 \\
& & E' \arrow[ur, "p'"'] & &
\end{tikzcd}$$

is a homomorphism that makes the diagram commutative. Then,

$$\begin{aligned}
\zeta(a,g) &= \zeta\left((a,1) + (0,g)\right) = \zeta(a,1) + \zeta(0,g) \\
p^{'}\zeta(0,g) &= p(0,g) = g \implies \zeta(0,g) = (\alpha(g), g) \implies \zeta(a,g) = (a,1) + (\alpha(g), g) = (a + \alpha(g), g)
\end{aligned}$$

for some function $\alpha : G \longrightarrow A$. As

$$\begin{aligned}
\zeta(0,1) &= (\alpha(1), 1) \implies \alpha(1) = 0 \\
\zeta((a,g) + (b,h)) &= \zeta(a + gb + f(g,h), gh) = (a + gb + f(g,h) + \alpha(gh), gh) \\
&\equiv \zeta(a,g) + \zeta(b,h) = (a + \alpha(g), g) + (b + \alpha(h), h) \\
&= (a + \alpha(g) + g(b + \alpha(h)) + f^{'}(g,h), gh) = (a + gb + \alpha(g) + g\alpha(h) + f^{'}(g,h), gh) \\
&\implies f^{'}(g,h) - f(g,h) = \alpha(g) + g\alpha(h) - \alpha(gh) = \delta^1\alpha(g,h)
\end{aligned}$$

So the factor sets of equivalent extensions are equal modulo 2-coboundaries. Given these factor sets modulo coboundaries, and a fixed action $\varphi : G \longrightarrow Aut(A)$, we can recover all elements of $E(G, A^{\varphi})$. Also, given an extension in $E(G, A^{\varphi})$, will give that its factor sets are 2-cocycles, which are equal for all the elements in the equivalence class. So, we have proved the following Proposition:

**Proposition 7.2.** *For any $G$, and any $G$-module $A$, there exists a bijection of pointed sets $E(G,A) \longrightarrow H^2(G,A)$.*

The semi-direct extension has 0 as its factor set, and 0 as a factor set gives the semi-direct extension.

**Lemma 7.3.** *Let $\alpha : A \longrightarrow A^{'}$ be a morphism of $G$-modules. There exists a well-defined mapping of pointed sets:*

$$\alpha_* : E(G,A) \longrightarrow E(G,A^{'})$$

*Proof.* Start with an element $\left(\varepsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1\right) \in E(G,A)$. Let $\langle 1 \rangle = 0$. When $G$ acts on $A^{'}$, $E$ acts on $A^{'}$ as $ea^{'} = \sigma(e)a^{'}$.

$$\begin{aligned}
1a^{'} &= \sigma(1)a^{'} = a^{'} \\
g(ha^{'}) &= g(\sigma(h)a^{'}) = \sigma(g)(\sigma(h)a^{'}) \overset{G\text{-module}}{=} (\sigma(g)\sigma(h))a^{'} = \sigma(gh)a^{'} = (gh)a^{'}
\end{aligned}$$

The set $S = \{(\alpha(a), -\varkappa(a)) : a \in A\}$ is a normal subgroup of $A' \rtimes E$ :

$$\begin{aligned}
a &= 0 \implies (0,0) \in S \\
(\alpha(a), -\varkappa(a)) + (\alpha(b), -\varkappa(b)) &= (\alpha(a) + \sigma(\varkappa(-a))\alpha(b), -\varkappa(a) - \varkappa(b)) = (\alpha(a) + \alpha(b), -\varkappa(a+b)) \\
&= (\alpha(a+b), -\varkappa(a+b)) \in S \\
-(\alpha(a), -\varkappa(a)) &= (-\sigma(-\varkappa(a))^{-1}\alpha(a), \varkappa(a)) = (-\sigma(\varkappa(-a))^{-1}\alpha(a), \varkappa(a)) \\
&= (-\alpha(a), \varkappa(a)) = (\alpha(-a), -\varkappa(-a)) \in S \\
(a', e) + (\alpha(a), -\varkappa(a)) - (a', e) &= (a' + ea(a), e - \varkappa(a)) + (-(-e)a', -e)
\end{aligned}$$

$$\begin{aligned}
&= (a' + ea(a) + (e - \varkappa(a))(-(-e)(a')), e - \varkappa(a) - e) = (a' + \alpha(ea) - (e - \varkappa(a) - e)a', e(-\varkappa(a))) \\
&= (a' + \alpha(ea) - a', -e\varkappa(a)) = (\alpha(ea), -\varkappa(ea)) \in S
\end{aligned}$$

Take the factor group $E' = A' \times E / \langle(\alpha(a), -\varkappa(a)) : a \in A\rangle$ . $A'$ is isomorphic to a normal subgroup of $E'$ by the map $a' \overset{\varkappa'}{\hookrightarrow} \overline{(a', 0)}$:

$$\begin{aligned}
\varkappa'(0) &= (0,0) \in S \\
\varkappa'(a' + b') &= (a' + b', 0) = (a', 0) + (b', 0) = \varkappa(a') + \varkappa(b') \\
a' &\in \ker \varkappa' \implies (a', 0) \in S \iff \exists a \in A \mid -\varkappa(a) = 0 \implies a = 0 \wedge \alpha(a) = a' \\
&\implies a' = 0 \implies \varkappa' \text{ is a monomorphism.} \\
(b', e) + (a', 0) - (b', e) &= (b' + ea', e) + (-(e^{-1}a), -e) = (b' + ea' - a, 0) = \varkappa(b' + ea' - a)
\end{aligned}$$

Define a map $\sigma' : E' \longrightarrow G$ as $\sigma'(a', e) = \sigma(e)$.

$$\begin{aligned}
\sigma'(\alpha(a), -\varkappa(a)) &= \sigma(-\varkappa(a)) = \sigma(\varkappa(-a)) = (\sigma\varkappa)(-a) = 1, \forall a \in A. \\
\sigma((a', e) + (b', f)) &= \sigma(a' + eb', e + f) = \sigma(e + f) = \sigma(e)\sigma(f) = \sigma'(a', e)\sigma'(b', f)
\end{aligned}$$

$\sigma'$ is an epimorphism:

$$\begin{aligned}
\forall g &\in G, \exists e \in E \mid \sigma(e) = g. \\
\text{Suppose } (0, e) &\in S \iff \exists a \in A \mid \alpha(a) = 0 \wedge -\varkappa(a) = e \implies e = (-\varkappa(a), 1), \sigma(e) = 1. \\
&\implies \forall g \in G, \exists (0, e) \in A' \rtimes E \mid \sigma'(0, e) = \sigma(e) = g.
\end{aligned}$$

The kernel of $\sigma'$ is $\varkappa'(A')$ :

$$\begin{aligned}
(a', e) &\in \ker \sigma' \iff \sigma(e) = 1 \implies e \in \operatorname{Im} \varkappa, e = \varkappa(a) \implies \{(a', \varkappa(a)) : a' \in A', a \in A\} \in \ker \sigma' \\
(a', \varkappa(a)) &= (a', 0) + (0, \varkappa(a)) = (a', 0) + (\alpha(a), 0) = (a' + \alpha(a), 0) = \varkappa'(a' + \alpha(a))
\end{aligned}$$

Together with the canonical injections $i_{A'} : A' \longrightarrow E', i_E : E \longrightarrow E'$, we have built the commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\varkappa"] \arrow[d, "\alpha"'] & E \arrow[r, "\sigma"] \arrow[d, "i_E"'] & G \arrow[d, "1_G"'] \\
A' \arrow[r, "i_{A'}"] & E' \arrow[r, "\pi_G"] & G
\end{tikzcd}$$

Define this element in $E(G, A')$ as $\alpha_*(\varepsilon)$. It is well-defined since if

$$\varepsilon' : 0 \longrightarrow A \longrightarrow E'' \longrightarrow G \longrightarrow 1$$

is an equivalent extension to $\varepsilon$ by the homomorphism $\psi : E'' \longrightarrow E$, we have that there exists the homomorphism $i_E\psi : E'' \longrightarrow E'$ that $\alpha_*(\varepsilon') = \alpha_*(\varepsilon)$. Suppose the sequence $\varepsilon$ we started with, splits by a homomorphism $s : G \longrightarrow E$. Then the homomorphism $v = i_{E'} s : G \longrightarrow E^{'}$, satisfies $\sigma^{'} v = 1_G$ by the commutativity condition. Therefore, the induced sequence in $E(G, A^{'})$ splits. Since $\alpha_*(\varepsilon)$ is an exact sequence, $E^{'}$ acts on $A^{'}$ by conjugation, hence $G$ acts on $A^{'}$ by conjugation

$$ga^{'} = e^{'} + a^{'} - e^{'} \mid \sigma^{'}(e^{'}) = \sigma^{'}(a,e) = \sigma(e) = g.$$

□

**Lemma 7.4.** *$E(G,-)$ is a covariant functor from $G$-mod to Sets$_\star$.*

*Proof.* Start with $\left(\varepsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1\right) \in E(G,A)$. Let $\alpha = 1_A$. We get

$$\begin{array}{rcl} \alpha_*(\varepsilon) & : & 0 \longrightarrow A \xrightarrow{i} E^{'} \xrightarrow{p} G \longrightarrow 1 \\ E^{'} & = & A \rtimes E / \langle (a, -\varkappa(a)) : a \in A \rangle \\ i(a) & = & (a,0),\ p(a,e) = \sigma(e). \end{array}$$

Define the mapping $\zeta : A \rtimes E / \langle (a, -\varkappa(a)) : a \in A \rangle \longrightarrow E$ as $\zeta(a,e) = \varkappa(a) + e$. It is a homomorphism:

$$\begin{array}{rcl} \zeta(a, -\varkappa(a)) & = & \varkappa(a) - \varkappa(a) = 0, (\forall a \in A) \\ \zeta((a,e) + (b,f)) & = & \zeta(a + eb, e + f) = \varkappa(a + eb) + e + f = \varkappa(a + \sigma(e)b) + e + f \\ \zeta(a,e) + \zeta(b,f) & = & \varkappa(a) + e + \varkappa(b) + f = \varkappa(a) + \sigma(e)b + e + f = \varkappa(a + \sigma(e)b) + e + f \end{array}$$

The diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "1_A"'] & E' \arrow[r, "p"] \arrow[d, "\zeta"'] & G \arrow[d, "1_G"'] \\ A \arrow[r, "\varkappa"] & E \arrow[r, "\sigma"] & G \end{tikzcd}$$

is commutative:

$$\begin{array}{rcl} \zeta(i(a)) & = & \zeta(a,0) = \varkappa(a) \\ \sigma\zeta(a,e) & = & \sigma(\varkappa a + e) = (\sigma\varkappa(a))\sigma(e) = p(a,e) \end{array}$$

Hence the two extensions are equivalent, $(1_A)_* = 1_{E(G,A)}$. Given a pair of morphisms, $\alpha : A \longrightarrow A^{'}, \alpha^{'} : A^{'} \longrightarrow A^{''}$, will show that $(\alpha^{'}\alpha)_* = \alpha^{'}_*\alpha_*$ (they give equivalent extensions in $E(G, A^{''})$).

$$\begin{array}{rcl} \alpha_*(\varepsilon) & = & 0 \longrightarrow A^{'} \xrightarrow{i} E^{'} \xrightarrow{p} G \longrightarrow 1 \\ E^{'} & = & A^{'} \rtimes E / \langle (\alpha(a), -\varkappa(a)) : a \in A \rangle \\ i(a^{'}) & = & (a^{'}, 0) + \langle (\alpha(a), -\varkappa(a)) \rangle,\ p((a^{'}, e) + \langle (\alpha a, -\varkappa(a)) \rangle) = \sigma(e) \\ \alpha^{'}_*(\alpha_*(\varepsilon)) & = & 0 \longrightarrow A^{''} \xrightarrow{i^{'}} E^{''} \xrightarrow{p^{'}} G \longrightarrow 1 \\ E^{''} & = & A^{''} \rtimes E^{'} / \left\langle (\alpha^{'}(a^{'}), -i(a^{'})) : a^{'} \in A^{'} \right\rangle \\ i^{'}(a^{''}) & = & (a^{''}, 0) + \left\langle (\alpha^{'}(a^{'}), -i(a^{'})) \right\rangle,\ p^{'}\left((a^{''}, e^{'}) + \left\langle (\alpha^{'}(a^{'}), -i(a^{'})) \right\rangle\right) = p(e^{'}) \end{array}$$

$$
\begin{aligned}
(\alpha^{'}\alpha)_*(\varepsilon) &= 0 \longrightarrow A^{''} \xrightarrow{i^{''}} F \xrightarrow{p^{''}} G \longrightarrow 1 \\
F &= A^{''} \rtimes E / \Big\langle ((\alpha^{'}\alpha)(a), -\varkappa(a)) : a \in A \Big\rangle \\
i^{''}(a^{''}) &= (a^{''}, 0) + \Big\langle ((\alpha^{'}\alpha)(a), -\varkappa(a)) \Big\rangle, \; p^{''}\Big((a^{''}, e) + \Big\langle ((\alpha^{'}\alpha)(a), -\varkappa(a)) \Big\rangle\Big) = \sigma(e)
\end{aligned}
$$

In

$$
\begin{tikzcd}
A \arrow[r, "i'"] \arrow[d, "1_A"'] & E'' \arrow[r, "p'"] \arrow[d, "\zeta"'] & G \arrow[d, "1_G"'] \\
A \arrow[r, "i''"] & F \arrow[r, "p''"] & G
\end{tikzcd}
$$

let $\zeta : E^{''} \longrightarrow F$ be the mapping defined as $\zeta(a^{''}, (a^{'}, e)) = (a^{''} + \alpha^{'}(a^{'}), e)$. It is a map of pointed sets:

$$
\begin{aligned}
\zeta(0, \alpha(a), -\varkappa(a)) &= (\alpha^{'}(\alpha(a)), -\varkappa(a)) \in \Big\langle ((\alpha^{'}\alpha)(a), -\varkappa(a)) \Big\rangle \\
\zeta(\alpha^{'}(a^{'}), -a^{'}, 0) &= \alpha^{'}(a^{'}) + \alpha^{'}(-a^{'}) = 0
\end{aligned}
$$

It is a group homomorphism:

$$
\begin{aligned}
\zeta((a^{''}, a^{'}, e) + (b^{''}, b^{'}, f)) &= \zeta(a^{''} + (a^{'}, e)b^{''}, (a^{'}, e) + (b^{'}, f) = \zeta(a^{''} + eb^{''}, a^{'} + eb^{'}, e + f) \\
\text{as } (a^{'}, e) &\in E^{'} \text{ acts on } A^{''} \text{ as } p^{'}(a^{'}, e) = e \iff E \text{ acts on } A^{''} \\
&= (a^{''} + eb^{''} + \alpha^{'}(a^{'} + eb^{'}), e + f) \\
\zeta((a^{''}, a^{'}, e) + \zeta(b^{''}, b^{'}, f)) &= (a^{''} + \alpha^{'}(a^{'}), e) + (b^{''} + \alpha^{'}(b^{'}), f) = (a^{''} + \alpha^{'}(a^{'}) + e(b^{''} + \alpha^{'}(b^{'}), e + f) \\
&= (a^{''} + \alpha^{'}(a^{'}) + eb^{''} + e\alpha^{'}(b^{'}), e + f) \\
\text{Since } e\alpha^{'}(b^{'}) &= \sigma(e)\alpha^{'}(b^{'}) = \alpha^{'}(\sigma(e)b^{'}) = \alpha^{'}(eb^{'}) \\
&= (a^{''} + \alpha^{'}(a^{'}) + eb^{''} + \alpha^{'}(eb^{'}), e + f) = (a^{''} + eb^{''} + \alpha^{'}(a^{'} + eb^{'}), e + f)
\end{aligned}
$$

□

**Theorem 7.5.** *The functors $E(G,-)$ and $H^2(G,-)$ are naturally isomorphic as functors from G-mod to Sets$_*$.*

*Proof.* We will show that the diagram

$$
\begin{tikzcd}
E(G,A) \arrow[r, "\zeta"] \arrow[d, "\alpha_*"'] & H^2(G,A) \arrow[d] \\
E(G,A') \arrow[r, "\zeta"] & H^2(G,A')
\end{tikzcd}
$$

is commutative for all $G, \alpha : A \longrightarrow A^{'}$. Take any element $\varepsilon$ of $E(G,A)$,

$$
\varepsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1
$$

Given the factor set for an extension $f(x,y)$, it corresponds to a normalized cocycle modulo normalized coboundaries in $H^2(G,A)$. It induces a function $\alpha f : G \times G \longrightarrow$

$A^{'}$, which is also a 2-cocycle:

$$\begin{aligned}
(\alpha f)(x,y) &= \alpha(f(x,y)) = \alpha(xf(y,z) - f(xy,z) + f(x,yz)) \\
&= \alpha(xf(y,z)) - \alpha(f(xy,z) + \alpha(f(x,yz)) \\
&= x(\alpha f)(y,z) - (\alpha f)(xy,z) + (\alpha f)(x,yz)
\end{aligned}$$

since $\alpha$ is a homomorphism of $G$-modules, $\alpha(ga) = g\alpha(a)$. So $\alpha f$ is an element in $H^2(G, A^{'})$. Now, the other way. By $\alpha_*$ we obtain an element of $E(G, A^{'})$. Choose representatives for $g \in G$ in $E$, $[g]$. Choose representatives for $g \in G$ in $E^{'}$ as $i_E([g]) = (0, g)$.

$$\begin{aligned}
i_E([g]) + i_E([h]) - i_E([gh]) &= i_E([g] + [h] - [gh]) = i_E(f(g,h)) \\
&= i_E(\varkappa(f(g,h)) = (\varkappa^{'}\alpha)f(g,h) = \varkappa^{'}(\alpha f(g,h)) \\
&\equiv \alpha f(g,h)
\end{aligned}$$

and we get the same element in $H^2(G, A)$. □

**Lemma 7.6.** *Let $\gamma : G^{'} \longrightarrow G$ be a group homomorphism. There exists a well-defined mapping of pointed sets*

$$\gamma^* : E(G, A) \longrightarrow E(G^{'}, A)$$

*Proof.* Given $A$ is a $G$-module, it induces that $A$ is a $G^{'}$-module with the action given by $g^{'}a = \gamma(g^{'})a$. Fix $\varepsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1$, and a $\gamma : G^{'} \longrightarrow G$. Take pullback $PB$ of $\sigma$ and $\gamma$,

$$PB = \left\{(e, g^{'}) \mid \sigma(e) = \gamma(g^{'}), e \in E, g^{'} \in G^{'}\right\}$$

$A$ is isomorphic normal subgroup of $PB$ by the injection map $i(a) = (\varkappa(a), 1)$ :

$$\begin{aligned}
i(a) &\in PB\ (\sigma(\varkappa(a)) = 1 = \gamma(1)) \\
(0,1) &\in i(a) \text{ by } a = 0 \\
i(a+b) &= (\varkappa(a+b), 1) = (\varkappa(a) + \varkappa(b), 1) = (\varkappa(a), 1) + (\varkappa(b), 1) = i(a) + i(b) \\
a &\in \ker \varkappa \Leftrightarrow i(a) = (\varkappa(a), 1) = (0, 1) \implies \varkappa(a) = 0 \implies a = 0 \\
(e, g^{'}) + (\varkappa(a), 1) - (e, g^{'}) &= (e + \varkappa(a), g^{'}) + (-e, (g^{'})^{-1}) = (e + \varkappa(a) - e, g^{'}(g^{'})^{-1}) \\
&= (g(\varkappa(a)), 1) = (\varkappa(ga), 1) \in PB\ (\sigma(\varkappa(ga)) = 1 = \gamma(1))
\end{aligned}$$

This normal subgroup is the kernel of the projection homomorphism $\pi_{G'} : PB \longrightarrow G^{'}$ :

$$\begin{aligned}
(e, g^{'}) \quad &\in \quad \ker \pi_{G'} \Longleftrightarrow \pi_{G'}(e, g^{'}) = g^{'} = 1 \implies \{(e, 1) \mid \sigma(e) = 1, e \in E\} \in \ker \pi_{G'} \\
\implies \quad & e \in \varkappa(a) \implies \ker \pi_{G'} = \operatorname{Im} i.
\end{aligned}$$

Together with the canonical projections $\pi_E : PB \longrightarrow E, \pi_{G'} : PB \longrightarrow G^{'}$, we have built the commutative diagram of short exact sequences:

$$\begin{tikzcd}
A \arrow[r, "\varkappa'"] \arrow[d, "1_A"'] & PB \arrow[r, "\pi_{G'}"] \arrow[d, "\pi_E"'] & G' \arrow[d, "\sigma"] \\
A \arrow[r, "\varkappa"] & E \arrow[r, "\sigma"] & G
\end{tikzcd}$$

$$\begin{aligned}
\sigma\pi_E(e, g^{'}) &= \sigma(e) = \gamma(g^{'}) = \gamma(\pi_G(e, g^{'})) \\
\pi_E(i(a)) &= \pi_E(\varkappa(a), 1) = \varkappa(a)
\end{aligned}$$

Define $\gamma^*(\varepsilon)$ to be the top sequence. It is well-defined since if

$$\epsilon : 0 \longrightarrow A \xrightarrow{\varkappa'} E' \xrightarrow{\sigma'} G \longrightarrow 1$$

lies in the same equivalence class as $\varepsilon$, then there exists a homomorphism $\psi : E \longrightarrow E'$ such that $\psi\varkappa = \varkappa'$, $\sigma'\psi = \sigma$. The pullback $PB'$ of $p$ and $\gamma$ would contain

$$PB' = \left\{(e', g') \mid \sigma'(e') = \gamma(g'), e' \in E', g' \in G'\right\}$$

We then get the sequence

$$\gamma^*(\epsilon) : 0 \longrightarrow A \xrightarrow{i'} PB' \xrightarrow{\pi_{G'}} G' \longrightarrow 1$$

Define a map $\beta : PB \longrightarrow PB'$ as

$$\beta(e, g') = (\psi(e), g') \text{ (since } (\psi(e), g') \in PB' : \sigma'(\psi(e)) = (\sigma'\psi)(e) = \sigma(e) = \gamma(g'))$$

It is a homomorphism:

$$\begin{aligned} \beta((e, g') + (u, g)) &= \beta(e + u, g'g) = (\psi(e + u), g'g) = (\psi(e) + \psi(u), g'g) \\ &= (\psi(e), g') + (\psi(u), g) = \beta(e, g') + \beta(u, g) \\ \beta(0, 1) &= (\psi(0), 1) = (0, 1) \end{aligned}$$

Also,

$$\begin{aligned} \beta i(a) &= \beta(\varkappa(a), 1) = (\psi\varkappa(a), 1) = (\varkappa'(a), 1) = i'(a) \\ \pi_{G'}\beta(e, g') &= \pi_{G'}(\psi(e), g') = g' = \pi'_G(e, g') \end{aligned}$$

So the diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & PB & \xrightarrow{\pi_{G'}} & G' \\ \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_{G'}} \\ A & \xrightarrow{i'} & PB' & \xrightarrow{\pi_{G'}} & G' \end{array}$$

is commutative, and $\gamma^*(\varepsilon) \sim \gamma^*(\epsilon)$. The map is well-defined. Suppose $\varepsilon$ splits. Then there exists a homomorphism $v : G \longrightarrow E$ such that $\sigma v = 1_G$. $\gamma^*(\varepsilon)$ splits iff

$$\begin{aligned} \exists s &: G' \longrightarrow PB \mid \{\pi_{G'} s = 1_{G'}\} \\ &\Updownarrow \\ \exists t &: G' \longrightarrow E, u : G' \longrightarrow G' \mid \left\{s = (t, u) \wedge \pi_{G'}(t(g'), u(g')) = u(g') = g' \wedge \sigma t(g') = \gamma(u(g'))\right\}. \end{aligned}$$

If we let

$$\begin{aligned} u &= 1_{G'},\ t = v\gamma \\ &\Longrightarrow \sigma t = (\sigma v)\gamma = \gamma \wedge \gamma u = \gamma \\ &\Longrightarrow (t(g), u(g)) \in PB. \\ \pi_{G'}(t(g), u(g)) &= u(g) = 1_{G'} \end{aligned}$$

So $\gamma^*(\varepsilon)$ splits too. □

**Proposition 7.7.** *$E(-, A)$ is a contravariant functor in the first variable, from $GR$ to $Sets_*$.*

*Proof.* Start with an $\varepsilon \in E(G, A)$,

$$\varepsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1$$

Take $\gamma = 1_G$.

$$\begin{array}{rcl} (1_G)^* & = & 0 \longrightarrow A \xrightarrow{i} PB \xrightarrow{\pi_G} G \longrightarrow 1 \\ PB & = & \{(e, g), e \in E, g \in G \mid \sigma(e) = g\} \\ i(a) & = & (\varkappa(a), 1),\ \pi_G(e, \sigma(e)) = \sigma(e) \end{array}$$

The canonical projection $\pi_E : PB \longrightarrow E$ is an isomorphism (need only to show injectivity):

$$\pi_E(e, \sigma(e)) = e \implies (e, \sigma(e)) \in \ker \pi_E \iff e = 0 \implies \sigma(e) = 1 \implies \ker \pi_E = (0, 1)$$

which gives that $(1_G)^* = 1_{E(G,A)}$. Now take any pair of morphisms $\beta : G^{''} \longrightarrow G^{'}, \gamma : G^{'} \longrightarrow G$. We will show that $(\gamma\beta)^* = \beta^*(\gamma^*)$. $\beta^*(\gamma^*)$ is the top extension in the diagram

$$\begin{tikzcd} A \arrow[r, "i'"] \arrow[d, "1_A"'] & PB' \arrow[r, "\pi_{G''}"] \arrow[d, "\pi_{PB}"'] & G'' \arrow[d, "\beta"] \\ A \arrow[r, "i"] \arrow[d, "1_A"'] & PB \arrow[r, "\pi_{G'}"] \arrow[d, "\pi_E"'] & G' \arrow[d, "\gamma"] \\ A \arrow[r, "\varkappa"] & E \arrow[r, "\sigma"] & G \end{tikzcd}$$

where

$$\begin{array}{rcl} i(a) & = & (\varkappa(a), 1),\ i^{'}(a) = (\varkappa(a), 1, 1) \\ PB & = & \{(e, g^{'}), e \in E, g \in G \mid \sigma(e) = \gamma(g^{'})\} \\ PB^{'} & = & \{(e, g^{'}, g^{''}), (e, g^{'}) \in PB, g^{''} \in G^{''} \mid \pi_{G^{'}}(e, g^{'}, g^{''}) = g^{'} = \beta(g^{''})\} \end{array}$$

$$\begin{array}{rcl} (\gamma\beta)^* & = & 0 \longrightarrow A \xrightarrow{i''} PB^{''} \xrightarrow{\pi_{G''}} G^{''} \longrightarrow 1 \\ i^{''}(a) & = & (\varkappa(a), 1),\ \pi_{G''}(e, g^{''}) = g^{''} \\ PB^{''} & = & \{(e, g^{''}), e \in E, g^{''} \in G^{''} \mid \sigma(e) = (\gamma\beta)(g^{''})\} \end{array}$$

We define the mapping $\zeta : PB^{'} \longrightarrow PB^{''}$ as

$$\zeta(e, g^{'}, g^{''}) = (e, g^{''}) \text{ (since } (e, g^{''}) \in PB^{''} : \sigma(e) = \gamma(g^{'}) = \gamma(\beta(g^{''})) = (\gamma\beta)(g^{''}))$$

It is the canonical projection on $PB^{''}$. Its kernel is

$$(e, g^{'}, g^{''}) \in \ker \zeta \iff e = 0 \wedge g^{''} = 1 \implies \beta(g^{''}) = \beta(1) = 1 = g^{'} \implies \ker \zeta = \{0, 1, 1\}$$

Since $\zeta$ makes the diagram

$$\begin{tikzcd} A \arrow[r, "i'"] \arrow[d, "1_A"'] & PB' \arrow[r, "\pi_{G''}"] \arrow[d, "\zeta"'] & G'' \arrow[d, "1_{G''}"] \\ A \arrow[r, "i''"] & PB'' \arrow[r, "\pi_{G''}"] & G'' \end{tikzcd}$$

commutative:

$$\pi_{G''}\zeta(e, g^{'}, g^{''}) = \pi_{G''}(e, g^{''}) = g^{''} \implies \zeta\varkappa^{'}(a) = \zeta(\varkappa(a), 1, 1) = (\varkappa(a), 1) = \varkappa^{''}(a)$$

the extensions are equivalent, which concludes our proof. $\square$

**Theorem 7.8.** *$E(-,A)$ and $H^2(-,A)$ are naturally isomorphic as functors from $GR$ to $Sets_\star$.*

*Proof.* We will show that

$$\begin{array}{ccc} E(G,A) & \xrightarrow{\zeta} & H^2(G,A) \\ {\scriptstyle\gamma^*}\downarrow & & \downarrow \\ E(G',A) & \xrightarrow{\zeta} & H^2(G',A) \end{array}$$

is commutative for any $\gamma : G^{'} \longrightarrow G$ and $A$. Pick an element of $E(G,A)$ :

$$\varepsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1$$

It has the factor set $\zeta(\varepsilon) = f(x,y)$ which is a 2-cocycle in $H^2(G,A)$. Look at the normalized bar resolution for $G^{'}$ and $G$ respectively:

$$\begin{array}{ccccccc} \mathbb{Z} \overset{\epsilon^{'}}{\twoheadleftarrow} F_0^{'} & \leftarrow & F_1^{'} \longleftarrow F_2^{'} \longleftarrow F_3^{'} \longleftarrow ... \\ \mathbb{Z} \overset{\epsilon}{\twoheadleftarrow} F_0 & \leftarrow & F_1 \longleftarrow F_2 \longleftarrow F_3 \longleftarrow ... \end{array}$$

We have the morphisms between that $\mathbb{Z}G^{'}$ and $\mathbb{Z}G$ modules:

$$\begin{array}{ccl} \gamma & : & F_1^{'} \longrightarrow F_1 \mid \gamma[g^{'}] = [\gamma(g^{'})] \\ \gamma_\bullet & : & F_2^{'} \longrightarrow F_2 \mid \gamma_\bullet([g^{'},h^{'}]) = [\gamma(g^{'}),\gamma(h^{'})] \end{array}$$

$\gamma_\bullet^* : H^2(G,A) \longrightarrow H^2(G^{'},A)$ is induced, and hence the 2-cocycle

$$\gamma_\bullet^* f(g^{'},h^{'}) = f\gamma_\bullet(g^{'},h^{'}) = f(\gamma(g^{'}),\gamma(h^{'})) : G^{'} \times G^{'} \longrightarrow A.$$

The other way:$\gamma^*(\varepsilon) = (0 \longrightarrow A \xrightarrow{i} PB \xrightarrow{p} G^{'} \longrightarrow 1)$. Choose representatives $\Big([\gamma(g^{'})], g^{'}\Big)$ for $g^{'}$.

$$\Big([\gamma(g^{'})],g^{'}\Big)+\Big([\gamma(h^{'})],h^{'}\Big)-\Big([\gamma(g^{'}h^{'})],g^{'}h^{'}\Big) = \Big([\gamma(g^{'})]+[\gamma(h^{'})]-[\gamma(g^{'}h^{'})],g^{'}h^{'}(g^{'}h^{'})^{-1}\Big)$$

$$= \Big([\gamma(g^{'})]+[\gamma(h^{'})]-[\gamma(g^{'})\gamma(h^{'})],1\Big) = \Big(f(\gamma(g^{'}),\gamma(h^{'})),1\Big) = i(f(\gamma(g^{'}),\gamma(h^{'}))) \equiv f(\gamma(g^{'}),\gamma(h^{'}))$$

□

**Proposition 7.9.** *$E(G,A)$ is a bifunctor from $PAIRS$ $(G,A)$ to $Sets_\star$.*

*Proof.* Since we have the commutativity of the whole diagram, and the peripheral

$$\begin{array}{ccccccc} H^2(G,A) & & & \longrightarrow & & & H^2(G',A) \\ \downarrow & \nwarrow & & & & \nearrow & \downarrow \\ & & E(G,A) & \longrightarrow & E(G',A) & & \\ & & \downarrow & & \downarrow & & \\ & & E(G,A') & \longrightarrow & E(G',A') & & \\ \downarrow & \swarrow & & & & \searrow & \downarrow \\ H^2(G,A') & & & \longrightarrow & & & H^2(G',A') \end{array}$$

squares, the middle square is commutative, which is equivalent to that $E(-,-)$ is a bifunctor. □

Suppose we have two elements

$$\begin{aligned} E_1 &: \quad 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1 \\ E_2 &: \quad 0 \longrightarrow A \xrightarrow{\varkappa'} E' \xrightarrow{\sigma'} G \longrightarrow 1 \end{aligned}$$

of $E(G,A)$. Look at the set $\left\{(\triangle_A(a_1,a_2), -\varkappa \otimes \varkappa'(a_1,a_2)) : a_1, a_2 \in A\right\}$. It is a normal subgroup of $A \rtimes PB$, where $PB$ is the pullback of $(\sigma \times \sigma')$ and $\bigtriangledown_G$ :

$$\begin{aligned} a_1 &= a_2 = 0 \implies 0 \equiv (0,0,1) \in PB \in PO \\ -(a+a', -\varkappa(a), -\varkappa'(a'), 1) &= (-a-a', \varkappa(a), \varkappa'(a'), 1) \in PO \iff a_1 = -a, a_2 = -a' \end{aligned}$$

$$\begin{aligned} & (a,e,e',g) + (a_1+a_2, -\varkappa(a_1), -\varkappa'(a_2), 1) - (a,e,e',g) \\ = \; & (a + g(a_1+a_2), e - \varkappa(a_1), e' - \varkappa'(a_2), g) + (-g^{-1}a, -e, -e', g^{-1}) \\ = \; & (a + ga_1 + ga_2 + g(-g^{-1}a), e - \varkappa(a_1) - e, e' - \varkappa'(a_2) - e', gg^{-1}) \\ = \; & (a + ga_1 + ga_2 - a, g(-a_1), g(-a_2), 1) = (ga_1 + ga_2, -ga_1, -ga_2, 1) \\ \equiv \; & ((ga_1+ga_2, -g(\varkappa a_1), -g(\varkappa' a_2), 1) = ((ga_1+ga_2, -\varkappa(ga_1), -\varkappa'(ga_2), 1) \in \left\langle (\triangle_A, -\varkappa \otimes \varkappa') \right\rangle \end{aligned}$$

since $\varkappa, \varkappa'$ are homomorphisms of $G$-modules. Define the Baer product of $E_1$ and $E_2$ to be as in the Baer sum for $R$-modules, only that $PO$ here is not the pushout of $\triangle_A$ and $\varkappa \otimes \varkappa'$ in the category $GR$, just the factor group as described above. The morphisms are unchanged.

**Proposition 7.10.** *$E(G,A)$ is an abelian group with operation given by the Baer product.*

*Proof.* For any $E \in E(G,A)$, we have the one-to-one correspondence with $H^2(G,A)$ given by $\zeta(E) = f$, where $f$ is the factor system for the extension $E$. Suppose $[g] \in E$ and $\langle g \rangle \in E'$ are representatives for $g \in G$. Suppose $f$ is a factor set for $E$, and $f'$ is a factor set for $E'$, i.e.

$$\begin{aligned} f(g,h) &= [g] + [h] - [gh] \\ f'(g,h) &= \langle g \rangle + \langle h \rangle - \langle gh \rangle \end{aligned}$$

Look at the direct product extension:

$$0 \longrightarrow A \times A \overset{\varkappa \otimes \varkappa'}{\longrightarrow} E \times E' \overset{\sigma \otimes \sigma'}{\longrightarrow} G \times G \longrightarrow 1$$

Choose $([g], \langle h \rangle)$ as representatives for $(g,h) \in G \times G$ in $E \times E'$.

$$\begin{aligned} ([g_1], \langle h_1 \rangle) + ([g_2], \langle h_2 \rangle) - ([g_1g_2], \langle h_1h_2 \rangle) &= ([g_1] + [g_2] - [g_1g_2], \langle h_1 \rangle + \langle h_2 \rangle - \langle h_1h_2 \rangle) \\ &= (f(g_1,g_2), f'(h_1,h_2)) \end{aligned}$$

So we get that the factor set of the direct product extension is $f \times f'(g_1,g_2,h_1,h_2)$ : $G \times G \times G \times G \longrightarrow A \times A$. Further, take the pullback of $\sigma \otimes \sigma'$ and $\triangle_G$and get the element

$$0 \longrightarrow A \times A \xrightarrow{\varkappa \otimes \varkappa'} PB \xrightarrow{\pi_G} G \longrightarrow 1$$

of $E(G, A \times A)$ where

$$PB = \{(e,e',g) \mid \sigma(e) = \sigma'(e') = g, e \in E, e' \in E'\}$$

Choose representatives for $g$ in $PB : ([g], \langle g \rangle, g)$.

$$\begin{aligned}([g], \langle g \rangle, g) + ([h], \langle h \rangle, h) - ([gh], \langle gh \rangle, gh) &= ([g] + [h] - [gh], \langle g \rangle + \langle h \rangle - \langle gh \rangle, gh(gh)^{-1}) \\ &= (f(g,h), f^{'}(g,h), 1)\end{aligned}$$

So $(f \oplus f^{'})\bigtriangledown_{G\times G}$ is a function from $G \times G \longrightarrow A \times A$ such that

$$\|g\| + \|h\| - \|gh\| = (f \oplus f^{'})\bigtriangledown_{G\times G}(g,h)$$

where $\|g\| = ([g], \langle g \rangle, g)$ is a representative of $g$. Further, take $PO = A \times PB / \langle (\bigtriangledown_A, -\varkappa \otimes \varkappa) \rangle$ and get the commutative diagram

$$\begin{tikzcd} A\otimes A \arrow[r] \arrow[d, "\bigtriangledown_A"'] & PB \arrow[r] \arrow[d] & G \arrow[d] \\ A \arrow[r] & PO \arrow[r] & G \end{tikzcd}$$

Choose representatives for $g$ in $PO : (0, [g], \langle g \rangle, g)$. Then:

$$\begin{aligned} & (0, [g], \langle g \rangle, g) + (0, [h], \langle h \rangle, h) - (0, [gh], \langle gh \rangle, gh) \\ = \; & (0 + g \cdot 0, ([g], \langle g \rangle, g) + ([h], \langle h \rangle, h)) + (-gh \cdot 0, -[gh], -\langle gh \rangle, gh) \\ = \; & (0, [g] + [h], \langle g \rangle + \langle h \rangle, gh) + (0, -[gh], -\langle gh \rangle, (gh)^{-1}) \\ = \; & (0, [g] + [h] - [gh], \langle g \rangle + \langle h \rangle - \langle gh \rangle, gh(gh)^{-1}) = (0, [g] + [h] - [gh], \langle g \rangle + \langle h \rangle - \langle gh \rangle, 1) \\ = \; & (0, f(g,h), f^{'}(g,h), 1) = (f(g,h) + f^{'}(g,h), 0, 0, 1) \end{aligned}$$

So $f + f^{'} = \bigtriangledown_A(f \oplus f^{'})\triangle_{G\times G}$ is a function from $G \times G \longrightarrow A$ such that

$$\{g\} + \{h\} - \{gh\} = (f + f^{'})(g,h)$$

where $\{g\} = (0, [g], \langle g \rangle, g)$ is a representative for $g$. So we get that $\zeta(E_1 + E_2) = \zeta(E_1) + \zeta(E_2)$. $\zeta$ becomes a group homomorphism from $E(G,A)$ to $H^2(G,A)$, which is an abelian group Therefore, $E(G,A)$ is an abelian group with operation given by the Baer product. Since we have that $\zeta(A \longrightarrow A \rtimes G \longrightarrow G) = 0$, we have found that the class of the split exact extension is the zero element in $E(G,A)$. The factor set of the inverse element of $\varepsilon \in E(G,A)$ is just $-\zeta(\varepsilon)$ (since $\zeta$is a group homomorphism), which then gives a complete description of $-\varepsilon \in E(G,A)$. $\square$

**Theorem 7.11.** *$E(G,A)$ and $H^2(G,A)$ are isomorphic as functors from $PAIRS\ (G,A)$ to $AB$.*

*Proof.* $\zeta$ is actually an isomorphism. For any cocycle in $H^2(G,A)$, we can obtain an extension in $E(G,A)$ by taking that cocycle to be its factor system. Let $E \in \ker \zeta$. That means that the factor system of $E$ is a coboundary in $C^2(G,A)$.

$$\zeta(E) = \delta^1 f(g,h) = gf(h) - f(gh) + f(g)$$

Now, the extension with the factor system $s(g,h) = \delta^1 f(g,h)$ is equivalent to the semi- direct extension by a $\beta$ in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "j"'] & A\rtimes_{\varphi} G \arrow[r, "p"] \arrow[d, "\beta"'] & G \\ & E_s \arrow[ur, "\pi"'] & \end{tikzcd}$$

defined as $\beta(a,g) = (a - f(g), g)$. $\beta$ is a homomorphism:

$$\begin{aligned}
\beta((a,g)+(b,h)) &= \beta(a+gb, gh) = (a+gb-f(gh), gh)\\
\beta(a,g)+\beta(b,h) &= (a-f(g), g) + (b-f(h), h)\\
&= (a - f(g) + g(b-f(h)) + s(g,h), gh)\\
&= (a - f(g) + gb - gf(h) + gf(h) - f(gh) + f(g), gh) = (a+gb-f(gh), gh)
\end{aligned}$$

And

$$\begin{aligned}
\beta i(a) &= \beta(a,1) = (a - f(1), 1) = (a,1) = j(a)\\
\pi\beta(a,g) &= \pi(a-f(g), g) = G = p(a,g)
\end{aligned}$$

□

7.2. **Characteristic class of an extension.** Look at the short exact sequence of free abelian groups:

$$0 \longrightarrow I(G) \xrightarrow{i} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $I(G)$ is the kernel of the augmentation map $\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$.

**Proposition 7.12.** *$I(G)$ is a free abelian group on $\{\langle g\rangle - \langle 1\rangle\}, \forall g \in G\backslash\{1\}$.*

*Proof.* Since $i$ is a $\mathbb{Z}G$ module homomorphism, it is a group homomorphism. So $I(G)$ is isomorphic to a normal subgroup of $\mathbb{Z}G$, which is free abelian. As numbers of generators, it has $|G|-1$ many. First, $\{\langle g\rangle - \langle 1\rangle, g \in G\} \in I(G)$ since $\varepsilon(\langle g\rangle - \langle 1\rangle) = 0$. The set $\{\langle g\rangle - \langle 1\rangle, g \in G\}$ is linearly independent (by induction on $n$):

$$\begin{aligned}
a(\langle g\rangle - \langle 1\rangle) &= a\langle g\rangle - a\langle 1\rangle = 0 \implies a\langle g\rangle = a\langle 1\rangle \implies (g \neq 1)a = 0 \ (\mathbb{Z}G \text{ free abelian})\\
(1)\ \ 0 &= a_1(\langle g_1\rangle - \langle 1\rangle) + a_2(\langle g_2\rangle - \langle 1\rangle) + \ldots + a_n(\langle g_n\rangle - \langle 1\rangle) \implies \{a_i\}_{i=1}^n = 0\\
(2)\ \ 0 &= a_1(\langle g_1\rangle - \langle 1\rangle) + a_2(\langle g_2\rangle - \langle 1\rangle) + \ldots + a_n(\langle g\rangle - \langle 1\rangle) + a_{n+1}(\langle g_{n+1}\rangle - \langle 1\rangle)
\end{aligned}$$

Take $(2)-(1)$, and since elements in $\mathbb{Z}G$ commute, we are left with

$$a_{n+1}((\langle g_{n+1}\rangle - \langle 1\rangle) = 0 \implies a_{n+1} = 0.$$

Now we only need to show that any element of $I(G)$ can be written as a linear combination of $\{\langle g\rangle - \langle 1\rangle, g \in G\}$.

$$\sum_{g\in G} a(g)\langle g\rangle \in I(G) \iff \sum_{g\in G} a(g) = 0$$

By writing out the expression and using the above, we conclude the proof:

$$\sum_{g\in G} a(g)(\langle g\rangle - \langle 1\rangle) = \sum_{g\in G} a(g)\langle g\rangle - \left(\sum_{g\in G} a(g)\right)\langle 1\rangle = \sum_{g\in G} a(g)\langle g\rangle$$

□

For any $\mathbb{Z}G$-module $A$, it induces a long exact sequence of $Ext^n_{\mathbb{Z}G}$ :

$$\begin{aligned}
0 &\longrightarrow Hom_{\mathbb{Z}G}(\mathbb{Z}, A) \longrightarrow Hom_{\mathbb{Z}G}(\mathbb{Z}G, A) \longrightarrow Hom_{\mathbb{Z}G}(I(G), A) \longrightarrow Ext^1_{\mathbb{Z}G}(\mathbb{Z}, A) \longrightarrow\\
Ext^1_{\mathbb{Z}G}(\mathbb{Z}G, A) &\longrightarrow Ext^1_{\mathbb{Z}G}(I(G), A) \longrightarrow Ext^2_{\mathbb{Z}G}(\mathbb{Z}, A) \longrightarrow Ext^2_{\mathbb{Z}G}(\mathbb{Z}G, A) \longrightarrow \ldots
\end{aligned}$$

Since $\mathbb{Z}G$ is a free (hence projective) $\mathbb{Z}G$-module, we get an isomorphism between

$$Ext^1_{\mathbb{Z}G}(I(G), A) \simeq Ext^2_{\mathbb{Z}G}(\mathbb{Z}, A) = H^2(G, A)$$

and since

$$Ext^1_{\mathbb{Z}G}(I(G), A) \simeq E_{\mathbb{Z}G}(I(G), A)$$

we get that $H^2(G,A)$ is isomorphic to the group of extensions of $A$ by $I(G)$.

$$0 \longrightarrow A \longrightarrow H \longrightarrow I(G) \longrightarrow 1$$

Now, to find $Ext^1_{\mathbb{Z}G}(I(G),A)$ we must choose a projective resolution of $I(G)$. Take

$$.. \xrightarrow{\mathfrak{d}_2} Q_2 \xrightarrow{\mathfrak{d}_1} Q_1 \xrightarrow{\mathfrak{d}_0} Q_0 \overset{\varepsilon}{\twoheadrightarrow} I(G) \longrightarrow 0$$

where $Q_i = F_{i+1}$,and $\mathfrak{d}_i = d_{i+2}$, for $i = 0, 1, ..$, for the free $\mathbb{Z}G$-modules $F_i$ and the module homomorphisms $d_i$ from the normalized bar resolution. We get

$$0 \longrightarrow Hom(I(G),A) \xrightarrow{\varepsilon^*} Hom(Q_0,A) \xrightarrow{\mathfrak{d}_0^*} Hom(Q_1,A) \xrightarrow{\mathfrak{d}_1^*} Hom(Q_2,A) \longrightarrow ..$$

So $Ext^1_{\mathbb{Z}G}(I(G),A) = \ker \mathfrak{d}_1^* / \operatorname{Im} \mathfrak{d}_0^* = \ker \delta^2 / \operatorname{Im} \delta^1 = H^2(G,A)$, so it contains factor sets, as many factor sets as elements of $E_\varphi(G,A)$. By Lemma 5.12, we find the correspondent element of $E_\varphi(I(G),A)$ by taking the middle module as $PO = A \times F_1 / \langle (f - \mathfrak{d}_1) \rangle$ and get the short exact sequence

$$\begin{aligned} 0 &\longrightarrow A \xrightarrow{i} PO \xrightarrow{p} I(G) \longrightarrow 0 \\ i(a) &= (a,1) \\ p(a,g) &= \mathfrak{d}_0(g) = \langle g \rangle - \langle 1 \rangle \end{aligned}$$

**Proposition 7.13.** *Fix an element of $E(G,A)$*

$$\epsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1$$

*Let $ML$ be the factor module of the free $\mathbb{Z}G$-module on $[e], e \in E, [0] = 0$ modulo the submodule generated by*

$$\langle 1 \rangle [e_1 + e_2] - \langle \sigma(e_1) \rangle [e_2] - \langle 1 \rangle [e_1] : e, e_1, e_2 \in E, [0] = 0$$

*The morphisms $\alpha : A \longrightarrow ML$ and $\beta : ML \longrightarrow I(G)$ are $\mathbb{Z}G$-module homomorphisms*

$$\begin{aligned} \alpha(a) &= \overline{[\varkappa(a)]} \\ \beta(\overline{[b]}) &= \langle \sigma(b) \rangle - \langle 1 \rangle \end{aligned}$$

*which give that the sequence splits as a sequence of abelian groups:*

$$0 \longrightarrow A \xrightarrow{\alpha} ML \xrightarrow{\beta} I(G) \longrightarrow 0$$

*Proof.*

$$\begin{aligned} \alpha(a+b) &= [\varkappa(a+b)] + L = [\varkappa(a) + \varkappa(b)] + L = [\varkappa(a)] + \langle \sigma(\varkappa(a)) \rangle [\varkappa(b)] + L \\ &= [\varkappa(a)] + [\varkappa(b)] + L = ([\varkappa(a)] + L) + ([\varkappa(b)] + L) = \alpha(a) + \alpha(b) \\ \alpha\left( \sum_{g \in G} n(g) \langle g \rangle \cdot a \right) &= \alpha\left( \sum_{g \in G} n(g)(ga) \right) = \left[ \varkappa\left( \sum_{g \in G} n(g)(ga) \right) \right] + L = \left[ \sum_{g \in G} n(g) \varkappa(ga) \right] + L \\ &= [n(g_1)\varkappa(g_1 a) + n(g_2)\varkappa(g_2 a) + .. + n(g_k)\varkappa(g_k a)] + L \\ &= [n(g_1)\varkappa(g_1 a) + n(g_2)\varkappa(g_2 a) + .. n(g_{k-1})\varkappa(g_{k-1} a)] + [n(g_k)\varkappa(g_k a)] + L \end{aligned}$$

By the definition of $L$, each term in both brackets can be decomposed to $n(g_s)$ terms of form $[\varkappa(g_s a)], s = 1, 2, .., k$, obtaining:

$$\begin{aligned}\alpha\left(\sum_{g\in G} n(g)\langle g\rangle \cdot a\right) &= \sum_{g\in G} n(g)[\varkappa(ga)] + L = \sum_{g\in G} n(g)[g(\varkappa a)] + L \\ &= \sum_{g\in G} n(g)\langle g\rangle [\varkappa(a)] + L = \left(\sum_{g\in G} n(g)\langle g\rangle\right)\alpha(a)\end{aligned}$$

since (as $L$ is a submodule). We defined $M$ as the free $\mathbb{Z}G$-module on $[e], e \in L$. Then, using the universal property of free modules with the functions $i : [e] \longrightarrow [e], f([e]) = \langle\sigma(e)\rangle - \langle 1\rangle$ , we get that there exists a $\mathbb{Z}G$-module homomorphism $\beta([e]) = \langle\sigma(e)\rangle - \langle 1\rangle$ .

$$\begin{aligned}\beta(L) &= \beta([e_1+e_2] - \langle\sigma(e_1)\rangle [e_2] - \langle 1\rangle [e_1]) = \beta([e_1+e_2]) - \langle\sigma(e_1)\rangle \beta([e_2]) - \langle 1\rangle \beta([e_1]) \\ &= \langle\sigma(e_1+e_2)\rangle - \langle 1\rangle - \langle\sigma(e_1)\rangle \{\langle\sigma(e_2)\rangle - \langle 1\rangle\} - \langle 1\rangle \{\langle\sigma(e_1)\rangle - \langle 1\rangle\} \\ &= \langle\sigma(e_1+e_2)\rangle - \langle 1\rangle - \langle\sigma(e_1)\rangle\langle\sigma(e_2)\rangle + (-\langle\sigma(e_1)\rangle)(-\langle 1\rangle) - \langle 1\rangle\langle\sigma(e_1)\rangle + (-\langle 1\rangle)(-\langle 1\rangle) \\ &= \langle\sigma(e_1+e_2)\rangle - \langle 1\rangle - \langle\sigma(e_1)\sigma(e_2)\rangle + \langle\sigma(e_1)\rangle - \langle\sigma(e_1)\rangle + \langle 1\rangle = 0\end{aligned}$$

$$\beta\alpha(a) = \beta(\overline{[\varkappa(a)]}) = \langle\sigma(\varkappa(a))\rangle - \langle 1\rangle = 0$$

So the sequence is a complex of abelian groups. Define $s : I(G) \longrightarrow ML$ as

$$s(\langle g\rangle - \langle 1\rangle) = [\{g\}] + L$$

(where $\sigma(\{g\}) = g$, $\{g\}$ is a chosen representative for $g$ in $E$ where $\{1\} = 0$). By the universal property of free abelian groups (which are free $\mathbb{Z}$-modules), in defining the functions

$$i(\langle g\rangle - \langle 1\rangle) = \langle g\rangle - \langle 1\rangle\,,\ f(\langle g\rangle - \langle 1\rangle) = [\{g\}] + L$$

we get $si = f$ and $s$ is a group homomorphism. By the universal property of the free $\mathbb{Z}G$-module $F$, we have a $\mathbb{Z}G$-module homomorphism $v : M \longrightarrow A$ as $v(\langle g\rangle [e]) = h(g, e)$ when taking the functions

$$i([e]) = g[e],\ f(\langle g\rangle [e]) = \{g\} + e - \{g\sigma(e)\} = h(g, e)$$

When we look at $v$ as a group homomorphism, we get $v(L) = 0$ :

$$\begin{aligned}v([e_1+e_2] - \langle\sigma(e_1)\rangle [e_2] - [e_1]) &= v([e_1+e_2]) - v(\langle\sigma(e_1)\rangle [e_2]) - v([e_1]) \\ h(1, e_1+e_2) &= \{1\} + (e_1+e_2) - \{\sigma(e_1+e_2)\} \quad (1) \\ h(\sigma(e_1), e_2) &= \{\sigma(e_1)\} + e_2 - \{\sigma(e_1)\sigma(e_2)\} \quad (2) \\ h(1, e_1) &= \{1\} + e_1 - \{\sigma(e_1)\} \quad (3)\end{aligned}$$

$$\begin{aligned}(1)-(2)-(3) \quad &: \quad [(e_1+e_2) - \{\sigma(e_1+e_2)\}] + [\{\sigma(e_1)\sigma(e_2)\} - e_2 - \{\sigma(e_1)\}] + [\{\sigma(e_1)\} - e_1] \\ &= (e_1+e_2) - e_2 - e_1 = 0\end{aligned}$$

As a chain complex of abelian groups,

$$0 \longrightarrow A \xrightarrow{\alpha} ML \xrightarrow{\beta} I(G) \longrightarrow 0$$

has a contracting homotopy $(s, v)$ :

$$\begin{aligned}(v\alpha)(a) &= v([\varkappa(a)] + L) = h(1, \varkappa(a)) = \{1\} + \varkappa(a) - [\sigma(\varkappa(a))] = \varkappa(a) \simeq a \\ \beta v(\langle g\rangle - \langle 1\rangle) &= \beta([\{g\}] + L) = \langle\sigma(\{g\})\rangle - \langle 1\rangle = \langle g\rangle - \langle 1\rangle \\ (\alpha v + s\beta)([e] + L) &= \alpha(v([e] + L)) + s\beta([e] + L) = \alpha(h(1, e)) + s(\langle\sigma(e)\rangle - \langle 1\rangle)\end{aligned}$$

$$\begin{aligned} &= [h(1, e)] + L + [\{\sigma(e)\}] + L = [e - \{\sigma(e)\}] + L + [\{\sigma(e)\}] + L \\ &= [e] + \langle\sigma(e)\rangle\, [-\{\sigma(e)\}] + L + [\{\sigma(e)\}] + L = [e] + (\langle\sigma(e)\rangle\, [-\{\sigma(e)\}] + [\{\sigma(e)\}]) + L \\ &= [e] + [-\{\sigma(e)\} + \{\sigma(e)\}] + L = [e] + L\end{aligned}$$

So the complex is split exact as a complex as abelian groups, hence it is exact as a complex of groups. □

Call this element in $H^2(G, A)$, the characteristic class of the original extension $\epsilon$.

**Proposition 7.14.**

$$0 \longrightarrow A \xrightarrow{i} PO \xrightarrow{p} I(G) \longrightarrow 0$$

*and*

$$0 \longrightarrow A \xrightarrow{\alpha} ML \xrightarrow{\beta} I(G) \longrightarrow 0$$

*are equivalent extensions of $A$ by $I(G)$.*

*Proof.* In defining a group homomorphism $\gamma : PO \longrightarrow ML$, we must define group homomorphisms

$$\gamma_A : A \longrightarrow ML,\ \gamma_{F_1} : F_1 \longrightarrow ML$$

such that

$$\gamma_A(f(g, h) + \gamma_{F_1}(-\delta_2[g \mid h]) \in L \iff \beta(\gamma_A(f(g, h) + \gamma_{F_1}(\delta_2[g \mid h]) = 0$$

Define

$$\gamma_A(a) = \alpha(a) = [\varkappa(a)] + L,\ \gamma_{F_1}([g]) = [\{g\}]$$

where we choose a set of representatives $\{g\}$ in $E$, for each $g \in G$, and $\{1\} = 0$. So

$$\gamma(a, g) = [\varkappa(a)] + [\{g\}] + L$$

is a $\mathbb{Z}G$-homomorphism (hence also a group homomorphism).

$$\begin{aligned}\gamma(f(g, h), -d_2[g \mid h]) &= [\varkappa f(g, h)] - \langle g\rangle\, [\{h\}] + [\{gh\}] - [\{g\}] + L \\ \beta(\gamma(f(g, h), -d_2[g \mid h])) &= \beta([\varkappa f(g, h)] + L) - \langle g\rangle\, \beta[\{h\}] + L) + \beta([\{gh\}] + L) - \beta([\{g\}] + L) \\ &= 0 - \langle g\rangle\, (\langle h\rangle - \langle 1\rangle) + \langle gh\rangle - \langle 1\rangle - (\langle g\rangle - \langle 1\rangle) = 0\end{aligned}$$

So $\gamma \in Hom_{GR}(PO, ML)$. It commutes with both squares:

$$\begin{aligned}\gamma(i_A(a)) &= \gamma(a, 1) = [\varkappa(a)] + [\{1\}] + L = [\varkappa(a)] + L \\ \beta(\gamma(a, g)) &= \beta([\varkappa(a)] + [\{g\}] + L) = \beta([\varkappa(a)] + L) + \beta([\{g\}] + L) \\ &= \langle 1\rangle - \langle 1\rangle + \langle\sigma(\{g\})\rangle - \langle 1\rangle = \langle g\rangle - \langle 1\rangle = p(a, g)\end{aligned}$$

□

## 8. Extensions with non-abelian kernel

Look at the exact sequence of groups,

$$\varepsilon : 0 \longrightarrow A \xrightarrow{\varkappa} E \xrightarrow{\sigma} G \longrightarrow 1$$

where $A$ is not necessarily abelian. It induces a group homomorphism $\theta^{'} : E \longrightarrow Aut(A)$, $\theta^{'}(e)(a) = ea = e + \varkappa a - e$. We also have a group homomorphism $\psi : A \longrightarrow In(A)$, where $In(A) \subseteq Aut(A)$ is the subgroup of inner automorphisms, $\psi(a)(b) = a + b - a$. So we have a group homomorphism $\theta : E/\varkappa(A) \simeq G \longrightarrow Aut(A)/In(A)$ given by $\varphi(g)(a) = \langle g \rangle + a - \langle g \rangle$ ,where $\sigma(\langle g \rangle) = g$ and $\varphi(g)$ is a representative in the factor group $Aut(A)/In(A)$. So for each $e \in E$, the automorphism $\theta^{'}(e)$ is in the automorphism class of $\theta(\sigma(e))$. We say that $\varepsilon$ has **conjugation class** $\theta$. Conversely, we say that the triple $(G, A, \theta : G \longrightarrow Aut(A)/In(A))$ is an **abstract kernel**.

**Lemma 8.1.** *Equivalent extensions have the same conjugation class.*

*Proof.* Given two equivalent extensions

$$\begin{tikzcd} A \arrow[r, "\varkappa'"] \arrow[dr, "\varkappa"'] & E' \arrow[r, "\sigma'"] \arrow[d, "\rho"] & G \\ & E \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Let the top extension induce a $\theta : G \longrightarrow Aut(A)/In(A)$, and the bottom one a $\zeta : G \longrightarrow Aut(A)/In(A)$. $\overline{\theta\,(g)} \in Aut(A)/In(A)$ is given by

$$\theta\,(g)\,(a) = (\varkappa')^{-1}\,(e' + \varkappa'\,(a) - e')$$

where $e' \in \sigma'^{-1}\,(g)$. To define $\overline{\zeta\,(g)} \in Aut(A)/In(A)$, we need a representative $e \in \sigma^{-1}\,(g)$. Since $\sigma\rho = \sigma'$, we can choose $e = \rho\,(e')$. Finally, using $\rho\varkappa' = \varkappa$,

$$\begin{aligned} \zeta\,(g)\,(a) &= \varkappa^{-1}\,(e + \varkappa\,(a) - e) = \\ &= \varkappa^{-1}\,(\rho\,(e) + \rho\varkappa'\,(a) - \rho\,(e)) = \\ &= \varkappa^{-1}\,(\rho\,(e + \varkappa'\,(a) - e)) = \\ &= (\varkappa')^{-1}\,(e' + \varkappa'\,(a) - e') = \theta\,(g)\,(a)\,. \end{aligned}$$

□

Pick a representative $[g] \in E$, $\sigma\,([g]) = g$, for each $g \in G - \{1\}$, and define $[1] := 0$. Then

$$[g] + a - [g] = \varphi(g)(a) \iff [g] + a = \varphi(g)(a) + [g]$$

for some element $\varphi(g)$ of the automorphism class of $\theta(g)$. Since

$$\sigma([g] + [h] - [gh]) = 1$$

So we have factor set $f(g, h) \in A$ such that

$$f(g, h) + [gh] = [g] + [h]$$

In order that the representatives should make a group, associativity must hold:

$$\begin{aligned}
([g]+[h])+[k] &= (f(g,h)+[gh])+[k] = f(g,h)+([gh]+[k]) \\
&= f(g,h)+f(gh,k)+[ghk] \\
[g]+([h]+[k]) &= ([g]+f(h,k))+[hk] = \varphi(g)(f(h,k))+([g]+[hk]) \\
&= \varphi(g)(f(h,k))+f(g,hk)+[ghk] \\
&\Longrightarrow f(g,h)+f(gh,k)+[ghk] = \varphi(g)(f(h,k))+f(g,hk)+[ghk] \\
f(g,h)+f(gh,k) &= \varphi(g)(f(h,k))+f(g,hk) \\
0 &= \varphi(g)(f(h,k))+f(g,hk)-f(gh,k)-f(g,h)
\end{aligned}$$

We see that if $A$ were abelian, we would get the 2-cocycle condition on $f$.

**Remark 8.2.** *Let*

$$Z(A) = \{a \in A \mid \forall b \in A\,(ab = ba)\}$$

*be the center of $A$. It is well-known that the center is a characteristic subgroup, i.e. it is invariant under any automorphism. Therefore, if $\overline{\theta(g)} \in Aut(A)/In(A)$, then*

$$\theta(g)(Z(A)) = Z(A).$$

*Moreover, if $a \in Z(A)$, and if $\xi \in In(A)$, i.e. $\xi(x) = bxb^{-1}$, for some $b \in A$, then*

$$bab^{-1} = bb^{-1}a = a.$$

*It follows that $In(A)$ acts trivially on $Z(A)$. Therefore, the action of $G$ on $Z(A)$ given by*

$$ga := \theta(g)(a)$$

*is well-defined.*

Now, conjugation by $[g]+[h]$ and by $f(g,h)+[gh]$ should be the same:

$$([g]+[h])+a-([g]+[h]) = [g]+\varphi(h)(a)+[h]-([g]+[h])$$

$$\begin{aligned}
&= [g]+\varphi(h)(a)+[h]-(f(g,h)+[gh]) = [g]+\varphi(h)(a)+\varphi(h)(-f(g,h))+[h]-[gh] \\
&= ([g]+\varphi(h)(a))-\varphi(h)(f(g,h))+[h]-[gh] \\
&= \varphi(g)(\varphi(h)(a))+[g]-\varphi(h)(f(g,h))+[h]-[gh] \\
&= \varphi(g)(\varphi(h)(a))+\varphi(g)(-\varphi(h)(f(g,h)))+[g]+[h]-[gh] \\
&= \varphi(g)(\varphi(h)(a))-\varphi(g)(\varphi(h)(f(g,h)))+f(g,h) \\
&= \varphi(g)\varphi(h)(a-f(g,h))+f(g,h) = \varphi(g)\varphi(h)(a-f(g,h))+f(g,h)
\end{aligned}$$

$$(f(g,h)+[gh])+a-(f(g,h)+[gh]) = f(g,h)+([gh]+(a-f(g,h))-[gh]$$

$$f(g,h)+\varphi(gh)(a-f(g,h))+[gh]-[gh] = f(g,h)+\varphi(gh)(a-f(g,h))$$

$$\Longrightarrow \varphi(g)\varphi(h)(a-f(g,h))+f(g,h) = f(g,h)+\varphi(gh)(a-f(g,h))$$

$$\begin{aligned}
\varphi(g)\varphi(h)(a-f(g,h)) &= f(g,h)+\varphi(gh)(a-f(g,h))-f(g,h) \\
\varphi(g)\varphi(h) &= i\psi(f(g,h))\varphi(gh)
\end{aligned}$$

So $i\psi$ measures the extend that $\varphi$ fails to be a homomorphism from $G$ to $Aut_{GR}(A)$.

**Proposition 8.3.** *Given $A, G$, functions $\varphi : G \longrightarrow Aut_{GR}(A)$, $f : G \times G \longrightarrow A$, with the properties*

(1) $f(g,1) = f(1,h) = 0$
(2) $0 = \varphi(g)(f(h,k))+f(g,hk)-f(gh,k)-f(g,h)$

(3) $\varphi(g)\varphi(h) = i\psi(f(g,h))\varphi(gh)$

*we can construct an extension of groups*

$$0 \longrightarrow A \xrightarrow{i} E' \xrightarrow{p} G \longrightarrow 1$$

*where* $E'$ *is the set of all pairs* $(a,g), a \in A, g \in G,$ *with*

$$\begin{aligned} (a,g)+(b,h) &= (a+\varphi(h)b+f(g,h),gh) \\ i(a) &= (a,1),\ p(a,g)=g \end{aligned}$$

*Call* $E'$ *for the crossed product group, and this extension for the crossed product extension.*

*Proof.* $i$ is a homomorphism:

$$\begin{aligned} i(a+b) &= (a+b,1) \\ i(a)+i(b) &= (a,1)+(b,1) = (a+\varphi(1)b+f(1,1),1) = (a+b,1) \end{aligned}$$

$p$ is a homomorphism:

$$\begin{aligned} p(a,g)p(b,h) &= gh \\ p((a,g)+(b,h)) &= p(a+\varphi(g)b+f(g,h),gh) = gh \end{aligned}$$

The zero element is $(0,1)$ :

$$\begin{aligned} (a,g)+(b,h) &= (a+\varphi(g)(b)+f(g,h),gh) = (a,g) \\ &\Longrightarrow h=1, \varphi(g)(b)+f(g,h)=0 \implies \varphi(g)(b)=0, \forall g \\ (b,h)+(a,g) &= (b+\varphi(h)(a)+f(h,g),hg) = (a,g) \\ &\Longrightarrow h=1, b+\varphi(h)(a)+f(h,g)=a \\ b+\varphi(1)a &= a \implies b=0 \end{aligned}$$

The inverse element $-(a,g)$ is $(-f(g^{-1},g)-\varphi(g)^{-1}(a),g^{-1})$ :

$$\begin{aligned} (a,g)+(b,h) &= (a+\varphi(g)(b)+f(g,h),gh) = (0,1) \\ &\Longrightarrow h=g^{-1}, a+\varphi(g)(b)+f(g,h)=0 \implies a+\varphi(g)(b)+f(g,g^{-1})=0 \\ &\Longrightarrow \varphi(g)(b) = -a-f(g,g^{-1}) \\ (b,h)+(a,g) &= (b+\varphi(h)(a)+f(h,g),hg) = (0,1) \\ &\Longrightarrow h=g^{-1}, b+\varphi(g^{-1})(a)+f(g^{-1},g)=0 \implies b=-f(g^{-1},g)-\varphi(g^{-1})(a) \\ &\Longrightarrow \varphi(g)(b) = -\varphi(g)(f(g^{-1},g))-\varphi(g)\varphi(g^{-1})(a) \\ \varphi(g)(b) &= -\varphi(g)(f(g^{-1},g))-\varphi(1)(a) = -\varphi(g)(f(g^{-1},g))-a \\ &\Longrightarrow b=\varphi(g)^{-1}(-\varphi(g)(f(g^{-1},g)))-\varphi(g)^{-1}(a) \implies b=-f(g^{-1},g)-\varphi(g)^{-1}(a) \end{aligned}$$

The set $E$ is associative:

$$\begin{aligned} \{(a,g)+(b,h)\}+(c,k) &= (a+\varphi(g)(b)+f(g,h),gh)+(c,k) \\ &= (a+\varphi(g)(b)+f(g,h)+\varphi(gh)(c)+f(gh,k),ghk) \\ (a,g)+\{(b,h)+(c,k)\} &= (a,g)+(b+\varphi(h)(c)+f(h,k),hk) \end{aligned}$$

$$\begin{aligned} &= (a+\varphi(g)(b+\varphi(h)(c)+f(h,k))+f(g,hk),ghk) \\ &= (a+\varphi(g)(b)+\varphi(g)\varphi(h)(c)+\varphi(g)(f(h,k))+f(g,hk),ghk) \\ &= (a+\varphi(g)(b)+f(g,h)+\varphi(gh)(c)-f(g,h)+\varphi(g)(f(h,k))+f(g,hk),ghk) \\ &= (a+\varphi(g)(b)+f(g,h)+\varphi(gh)(c)+f(gh,k),ghk) = \{(a,g)+(b,h)\}+(c,k) \end{aligned}$$

The sequence is exact:

$$\begin{array}{rcl} a & \in & \ker i \iff i(a) = (a,1) = (0,1) \implies a = 0. \\ (a,g) & \in & \ker p \iff p(a,g) = g = 1 \implies (a,1) \in \ker p, \forall a \in A \implies \ker p = \operatorname{Im} i \end{array}$$

□

**Corollary 8.4.** *If any automorphism $\varphi(g) \in \zeta(g)$ satisfies $\varphi(1) = 1_A$, then any extension of the abstract kernel $(A, G, \zeta)$ is congruent to a crossed product extension $(A, G, \varphi, f)$ with the given function $\varphi$.*

*Proof.* Suppose there exists an extension $\varepsilon : 0 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$, with all $\varphi(g) \in \zeta(g), \varphi(1) = 1_A$. All elements of $E$ are of form $a + [g], a \in A, g \in G$. We have:

$$[g] + [h] = f(g,h) + [gh], \text{ for some } f(g,h) \in A.$$

$E$ must be a group, closed under addition:

$$(a + [g]) + (b + [h]) = a + \varphi(g)(b) + f(g,h) + [gh]$$

Simplify $a + [g] \longrightarrow (a, g)$.

$$\begin{array}{rcl} \sigma(a + [g]) & = & \sigma(\varkappa(a))\sigma([g]) = g \\ b + [h] & \in & \ker \sigma \iff \sigma([h]) = 1 \iff [h] = [1] \\ \text{If we choose } [1] & = & 0, \text{ we get an equivalent extensions} \\ & \implies & \ker \sigma = \{b + [1], b \in A\} \implies \varkappa(a) = a + [1] = (a, 1) \text{ is defined.} \end{array}$$

□

Suppose we are given an abstract kernel $(G, A, \zeta)$. In each automorphism class $\zeta(g)$, pick an automorphism $\varphi(g)$ such that $\varphi(1) = 1$.

$$\begin{array}{rcl} \varphi^{-1}\varphi(k)(a) & = & \varphi^{-1}([k] + a - [k]) = a \implies \varphi^{-1}(k)(a) = -[k] + a + [k] \\ \left[\varphi(g)\varphi(h)\varphi^{-1}(gh)\right](a) & = & \varphi(g)\varphi(h)(-[gh] + a + [gh]) \end{array}$$

$$\begin{array}{rl} = & \varphi(g)([h] - [gh] + a + [gh] - [h]) = ([g] + [h] - [gh]) + a + ([gh] - [h] - [g]) \\ = & ([g] + [h] - [gh]) + a - ([g] + [h] - [gh]) = \varkappa(e) + a - \varkappa(e). \\ \implies & \varphi(g)\varphi(h)\varphi^{-1}(gh)(a) = f(g,h) + a - f(g,h) \in In(A) \\ \iff & \varphi(g)\varphi(h) = i\psi f(g,h)\varphi(gh) \end{array}$$

for some function $f : G \times G \longrightarrow A$ satisfying

$$\begin{array}{rcl} \varphi(1)\varphi(h)\varphi^{-1}(h)(a) & = & f(1,h) + a - f(1,h) \\ a & = & f(1,h) + a - f(1,h) \end{array}$$

We may pick $f(1,h) = f(g,1) = 1$. Now, $\varphi(g)\varphi(h)\varphi(z)$ should be associative:
$\varphi(g)\left[\varphi(h)\varphi(k)(a)\right]$ :

$$\begin{array}{rl} = & \varphi(g)\left[f(h,k) + \varphi(hk)(a) - f(h,k)\right] = \varphi(g)(f(h,k)) + \varphi(g)\varphi(hk)(a) - \varphi(g)(f(h,k)) \\ = & \varphi(g)(f(h,k)) + f(g,hk) + \varphi(ghk)(a) - f(g,hk) - \varphi(g)(f(h,k)) \\ = & (\varphi(g)(f(h,k)) + f(g,hk)) + \varphi(ghk)(a) - (\varphi(g)(f(h,k)) + f(g,hk)) \\ = & i\psi(\varphi(g)(f(h,k)) + f(g,hk))\varphi(ghk)(a) \end{array}$$

$[\varphi(g)\varphi(h)]\,(\varphi(k)(a)):$

$$\begin{aligned}
&= f(g,h)+\varphi(gh)\varphi(k)(a)-f(g,h)=f(g,h)+f(gh,k)+\varphi(ghk)(a)-f(gh,k)-f(g,h)\\
&= (f(g,h)+f(gh,k))+\varphi(ghk)(a)-(f(g,h)+f(gh,k))=i\psi\,(f(g,h)+f(gh,k))\,(\varphi(ghk)(a))
\end{aligned}$$

Gives:

$$\begin{aligned}
i\psi(\varphi(g)(f(h,k))+f(g,hk)) &= i\psi\,(f(g,h)+f(gh,k))\\
\varphi(g)(f(h,k))+f(g,hk)-(f(g,h)+f(gh,k)) &\in \ker i\psi\\
\varphi(g)(f(h,k))+f(g,hk)-(f(g,h)+f(gh,k)) &= \mathcal{O}(g,h,k)\in Z(A)\\
\varphi(g)(f(h,k))+f(g,hk) &= \mathcal{O}(g,h,k)+f(g,h)+f(gh,k)
\end{aligned}$$

where $\mathcal{O}:G\times G\times G\longrightarrow Z(A)$ is a (normalized) function:

$$\mathcal{O}(1,h,k)=\mathcal{O}(g,1,k)=\mathcal{O}(g,h,1)=0$$

So we can regard $\mathcal{O}$ as a 3-cochain of the normalized bar resolution of $G$ with coefficients in $Z(A)$.

**Proposition 8.5.** *Any obstruction of an abstract kernel $(A,G,\zeta)$ is a* 3*-cocycle of* $\overline{B_G(\mathbb{Z})}$.

*Proof.* We will show that $\delta^3\mathcal{O}(g,h,k,l)=0$.

$$\begin{aligned}
g\mathcal{O}(h,k,l)-\mathcal{O}(gh,k,l)+\mathcal{O}(g,hk,l)-\mathcal{O}(g,h,kl)+\mathcal{O}(g,h,k) &= \delta^3\mathcal{O}(g,h,k,l)\\
\varphi(h)f(k,l)+f(h,kl)-f(hk,l)-f(h,k) &= \mathcal{O}(h,k,l)\\
\varphi(g)[\varphi(h)f(k,l)+f(h,kl)-f(hk,l)-f(h,k)] &= g\mathcal{O}(h,k,l)\\
f(g,h)+\varphi(gh)(f(k,l))-f(g,h)+\varphi(g)(f(h,kl))-\varphi(g)(f(hk,l))-\varphi(g)(f(h,k)) &= g\mathcal{O}(h,k,l) \quad (1)\\
\varphi(gh)(f(k,l))+f(gh,kl)-f(ghk,l)-f(gh,k) &= \mathcal{O}(gh,k,l)\\
f(gh,k)+f(ghk,l)-f(gh,kl)-\varphi(gh)((f(k,l)) &= -\mathcal{O}(gh,k,l) \quad (2)\\
\varphi(g)f(hk,l)+f(g,hkl)-f(ghk,l))-f(g,hk) &= \mathcal{O}(g,hk,l) \quad (3)\\
\varphi(g)(f(h,kl))+f(g,hkl)-f(gh,kl)-f(g,h) &= \mathcal{O}(g,h,kl)\\
f(g,h)+f(gh,kl)-f(g,hkl)-\varphi(g)(f(h,kl)) &= -\mathcal{O}(g,h,kl) \quad (4)\\
\varphi(g)f(h,k)+f(g,hk)-f(gh,k)-f(g,h) &= \mathcal{O}(g,h,k) \quad (5)
\end{aligned}$$

Take $[(1)+(5)]+(4)+(3)+(2):$

$$\begin{aligned}
&\{f(g,h)+\varphi(gh)(f(k,l))-f(g,h)+\varphi(g)(f(h,kl))-\varphi(g)(f(hk,l))-\varphi(g)(f(h,k))\}\\
&+\{\varphi(g)f(h,k)+f(g,hk)-f(gh,k)-f(g,h)\}+\{f(g,h)+f(gh,kl)-f(g,hkl)-\varphi(g)(f(h,kl))\}\\
&+\{\varphi(g)f(hk,l)+f(g,hkl)-f(ghk,l))-f(g,hk)\}+\{f(gh,k)+f(ghk,l)-f(gh,kl)-\varphi(gh)((f(k,l))\}
\end{aligned}$$

$$\begin{aligned}
= \;& f(g,h)+\varphi(gh)(f(k,l))-f(g,h)+\varphi(g)(f(h,kl))-\varphi(g)(f(hk,l))+f(g,hk)\\
&-f(gh,k)+f(gh,kl)-f(g,hkl)-\varphi(g)(f(h,kl))+[\varphi(g)f(hk,l)+f(g,hkl)-f(ghk,l))-f(g,hk)]\\
&+[f(gh,k)+f(ghk,l)-f(gh,kl)-\varphi(gh)((f(k,l))]
\end{aligned}$$

The elements in brackets are $\mathcal{O} \in Z(A)$, so they commute with any of the single elements of $A$ in the expression, so we get:

$$\begin{aligned}
& f(g,h)+\varphi(gh)(f(k,l))-f(g,h)+\varphi(g)(f(h,kl))-\varphi(g)(f(hk,l))+ \\
& [\varphi(g)f(hk,l)+f(g,hkl)-f(ghk,l))-f(g,hk)]+f(g,hk) \\
& -f(gh,k)+[f(gh,k)+f(ghk,l)-f(gh,kl)-\varphi(gh)((f(k,l))]+f(gh,kl)-f(g,hkl)-\varphi(g)(f(h,kl)) \\
= \;& f(g,h)+\varphi(gh)(f(k,l))-f(g,h)+\varphi(g)(f(h,kl))+f(g,hkl)-f(ghk,l)-f(g,hk) \\
& +f(g,hk)+f(ghk,l)-f(gh,kl)-\varphi(gh)((f(k,l))+f(gh,kl)-f(g,hkl)-\varphi(g)(f(h,kl)) \\
= \;& f(g,h)+\varphi(gh)(f(k,l))-f(g,h)+\{\varphi(g)(f(h,kl))+f(g,hkl)-f(gh,kl)\}-\varphi(gh)((f(k,l)) \\
& +\{f(gh,kl)-f(g,hkl)-\varphi(g)(f(h,kl))\} \\
= \;& f(g,h)+\varphi(gh)(f(k,l))-f(g,h)+\mathcal{O}(g,h,kl)+f(g,h)-\varphi(gh)((f(k,l))-f(g,h)-\mathcal{O}(g,h,kl) \\
= \;& f(g,h)+\varphi(gh)(f(k,l))-f(g,h)+f(g,h)-\varphi(gh)((f(k,l))-f(g,h)-\mathcal{O}(g,h,kl)+\mathcal{O}(g,h,kl)=0
\end{aligned}$$

□

**Theorem 8.6.** *An abstract kernel $(A, G, \zeta)$ has an extension if and only if one of its obstructions is equal to* 0.

*Proof.* $\Longleftarrow$If $\mathcal{O} = 0$, then we get the associativity condition

$$\varphi(g)f(h,k)+f(g,hk)=f(g,h)+f(gh,k)$$

and we build the crossed product extension as in (8.3).

$\Longrightarrow$ By choosing $[1] = 0$, we get $\varphi(1) = 1$, and using Proposition 8.3, we get

$$\varphi(g)(f(h,k))+f(g,hk)-f(gh,k)-f(g,h)=0 \implies \mathcal{O}=0.$$

□

**Lemma 8.7.** *Given $(A, G, \zeta)$. Fix $\varphi(g) \in \zeta(g)$. If we change $f$ to another function $f^{'}$ that satisfies*

$$\begin{aligned}
0 &= \varphi(g)(f(h,k))+f(g,hk)-f(gh,k)-f(g,h) \\
\varphi(g)\varphi(h) &= i\psi(f(g,h))\varphi(gh)
\end{aligned}$$

*then we are replacing $\mathcal{O}$ by a cohomologous cocycle. By suitably changing $f$, $\mathcal{O}$ may be replaced by any cohomologous cocycle.*

*Proof.* We choose another element $f^{'}(g,h) \in A$ such that

$$\begin{aligned}
\varphi(g)\varphi(h)\varphi^{-1}(gh) &= i\psi(f(g,h))=i\psi(f^{'}(g,h)) \implies f^{'}(g,h)-f(g,h)\in \ker i\psi = Z(A) \\
f^{'}(g,h)-f(g,h) &= s(g,h) \iff f^{'}(g,h)=s(g,h)+f(g,h)
\end{aligned}$$

for some normalized function $s : G \times G \longrightarrow Z(A)$ (since we chose $f, f^{'}$ to be normalized). So we may look at $s$ as a 2-cochain of the bar resolution of $G$ with coefficients in $Z(A)$. Actually, it is a 2-cocycle:

$$\delta^2 s(g,h,k)=gs(h,k)-s(gh,k)+s(g,hk)-s(g,h)=0$$

$gs(h,k):$

$$\begin{aligned}
= \;& \varphi(g)(f^{'}(h,k)-f(h,k))=\mathcal{O}+f^{'}(g,h)+f^{'}(gh,k)-f^{'}(g,hk)+f(g,hk)-f(gh,k)-f(g,h)-\mathcal{O} \\
= \;& f^{'}(g,h)+f^{'}(gh,k)-f^{'}(g,hk)+f(g,hk)-f(gh,k)-f(g,h)
\end{aligned}$$

$-s(gh,k)+s(g,hk)-s(g,h):$

$$= \left[f(gh,k)-f^{'}(gh,k)\right]+\left[f^{'}(g,hk)-f(g,hk)\right]+\left[f(g,h)-f^{'}(g,h)\right]$$

Use that the elements in the brackets commute with all $f, f^{'}$ and get:

$$\begin{aligned} & \left[f(g,h)-f^{'}(g,h)\right]+f^{'}(g,h)+f^{'}(gh,k)-f^{'}(g,hk)+\left[f^{'}(g,hk)-f(g,hk)\right]+f(g,hk) \\ & -f(gh,k)+\left[f(gh,k)-f^{'}(gh,k)\right]-f(g,h) \\ = \quad & f(g,h)+f^{'}(gh,k)-f(g,hk)+f(g,hk)-f^{'}(gh,k)-f(g,h)=0 \end{aligned}$$

Further,

$$\begin{aligned} \varphi(g)f^{'}(h,k)+f^{'}(g,hk) &= \mathcal{O}^{'}(g,h,k)+f^{'}(g,h)+f^{'}(gh,k) \\ \varphi(g)\left[s(h,k)+f(g,h)\right]+s(g,hk)+f(g,hk) &= \mathcal{O}^{'}(g,h,k)+s(g,h)+f(g,h)+s(g,hk)+f(g,hk) \\ \varphi(g)\left[f(g,h)+s(g,h)\right]+f(g,hk)+s(g,hk) &= \mathcal{O}^{'}(g,h,k)+s(g,h)+s(g,hk)+f(g,h)+f(g,hk) \\ \varphi(g)f(g,h)+f(g,hk)+\varphi(g)s(g,h)+s(g,hk) &= \mathcal{O}^{'}(g,h,k)+s(g,h)+s(g,hk)+f(g,h)+f(g,hk) \end{aligned}$$

So

$$\varphi(g)f(g,h)+f(g,hk)=(\mathcal{O}^{'}(g,h,k)-gs(g,h)+s(gh,k)-s(g,hk)+s(g,h))+f(g,h)+f(g,hk)$$

$$\begin{aligned} \mathcal{O}^{'}(g,h,k)-gs(g,h)+s(gh,k)-s(g,hk)+s(g,h) &= \mathcal{O}(g,h,k) \\ \mathcal{O}^{'}(g,h,k) &= \mathcal{O}(g,h,k)+\delta^3 s(g,h,k) \end{aligned}$$

Thus we have replaced $\mathcal{O}$ by $\mathcal{O}^{'}$, a cohomologous cocycle. As we may choose any normalized 2-cochain $s$, we reach any cohomologous cocycle by definition of cohomologous cocycles. $\square$

**Lemma 8.8.** *A change in the choice of the automorphisms $\varphi(g)$ may be followed by a the choice of such an $f$ such that the obstruction remains unchanged.*

*Proof.* Change $\varphi(g)$ to $\varphi^{'}(g)$ such that $\varphi^{'}(1)=1$ in $\zeta(g)$. Then their difference is an inner automorphism of $A$

$$\varphi^{'}(g)(a)=\gamma(g)+\varphi(g)(a)-\gamma(g)$$

where $\gamma: G \longrightarrow A$ is a function. Calculate $\varphi^{'}(g)(\varphi^{'}(h)(a)):$

$$\begin{aligned} = \quad & \varphi^{'}(g)\left[\gamma(h)+\varphi(h)(a)-\gamma(h)\right] \\ = \quad & \gamma(g)+\varphi(g)\left[\gamma(h)+\varphi(h)(a)-\gamma(h)\right]-\gamma(g) \\ = \quad & \gamma(g)+\varphi(g)(\gamma(h))+\varphi(g)\varphi(h)(a)-\varphi(g)(\gamma(h))-\gamma(g) \\ = \quad & \gamma(g)+\varphi(g)(\gamma(h))+f(g,h)+\varphi(gh)(a)-f(g,h)-\varphi(g)(\gamma(h))-\gamma(g) \\ = \quad & \gamma(g)+\varphi(g)(\gamma(h))+f(g,h)-\gamma(gh)+\varphi^{'}(gh)(a)+\gamma(gh)-f(g,h)-\varphi(g)(\gamma(h))-\gamma(g) \\ = \quad & \left[\gamma(g)+\varphi(g)(\gamma(h))+f(g,h)-\gamma(gh)\right]+\varphi^{'}(gh)(a)-\left[\gamma(g)+\varphi(g)(\gamma(h))+f(g,h)-\gamma(gh)\right] \end{aligned}$$

Denote the element of $A$ in the brackets as

$$f^{'}(g,h)=\gamma(g)+\varphi(g)(\gamma(h))+f(g,h)-\gamma(gh) \implies \varphi^{'}(g)\varphi^{'}(h)=i\psi(f^{'}(g,h))\varphi^{'}(gh)$$

This gives

$$\varphi^{'}(g)f^{'}(h,k)+f^{'}(g,hk)=\mathcal{O}^{'}(g,h,k)+f^{'}(g,h)+f^{'}(gh,k)$$

Left side gives:

$$
\begin{aligned}
&= \gamma(g) + \varphi(g)(\gamma(h)) + f(g,h) + \varphi(gh)(\gamma(k)) - f(g,h) + \varphi(g)(f(h,k)) - \varphi(g)(\gamma(hk)) - \gamma(g) \\
&\quad + \gamma(g) + \varphi(g)(\gamma(hk)) + f(g,hk) - \gamma(ghk) \\
&= \gamma(g) + \varphi(g)(\gamma(h)) + f(g,h) + \varphi(gh)(\gamma(k)) - f(g,h) + \varphi(g)(f(h,k)) + f(g,hk) - \gamma(ghk) \\
&= \mathcal{O} - \mathcal{O} + \gamma(g) + \varphi(g)(\gamma(h)) + f(g,h) + \varphi(gh)(\gamma(k)) - f(g,h) + \varphi(g)(f(h,k)) + f(g,hk) \\
&\quad - \gamma(ghk) \\
&= \mathcal{O} + (\gamma(g) + \varphi(g)(\gamma(h)) + f(g,h)) - ob + \varphi(gh)(\gamma(k)) - f(g,h) + \varphi(g)(f(h,k)) + f(g,hk) \\
&\quad - \gamma(ghk) \\
&= \mathcal{O} + f^{'}(g,h) + \gamma(gh) - ob + \varphi(gh)(\gamma(k)) - f(g,h) + \varphi(g)(f(h,k)) + f(g,hk) - \gamma(ghk) \\
&= \mathcal{O} + f^{'}(g,h) + \gamma(gh) + \varphi(gh)(\gamma(k)) - \mathcal{O} - f(g,h) + \varphi(g)(f(h,k)) + f(g,hk) - \gamma(ghk) \\
&= \mathcal{O} + f^{'}(g,h) + \gamma(gh) + \varphi(gh)(\gamma(k)) - f(g,h) + \varphi(g)(f(h,k)) + f(g,hk) - (f(g,h) + f(gh,k)) \\
&\quad + (f(g,h) + f(gh,k)) - \mathcal{O} - \gamma(ghk) \\
&= \mathcal{O} + f^{'}(g,h) + \gamma(gh) + \varphi(gh)(\gamma(k)) - f(g,h) \\
&\quad + \mathcal{O} - \mathcal{O} + (f(g,h) + f(gh,k)) - \gamma(ghk) \\
&= \mathcal{O} + f^{'}(g,h) + \gamma(gh) + \varphi(gh)(\gamma(h)) + f(gh,k) - \gamma(ghk) = \mathcal{O}(g,h,k) + f^{'}(g,h) + f^{'}(gh,k)
\end{aligned}
$$

so the new obstruction is identical to the old one. □

**Theorem 8.9.** *Fix an abstract kernel* $(A, G, \zeta)$*. The map*

$$
\begin{aligned}
obs &: (A, G, \zeta) \longrightarrow \mathcal{H}^3(G, Z(A)) \\
obs(\mathcal{O}) &= \overline{\mathcal{O}}
\end{aligned}
$$

*where* $\mathcal{O}$ *is any one of its obstructions, is well-defined.* $(A, G, \zeta)$ *has an extension if and only if* $\overline{\mathcal{O}} = 0$.

*Proof.* By Lemma 8.7, all obstructions are (cohomologous, i.e.) equal modulo 3-coboundaries, so the map gives an unique element in $H^3(G, Z(A))$.

If $\overline{\mathcal{O}} = 0$, then there exists a 3-cochain $l$ such that $\mathcal{O} = \delta^3 l$. Using the same lemma, there exists such a shift in $f$, $f^{'}$, such that $\mathcal{O}$ is replaced by a cohomologous cocycle, 0. Then, by Theorem 8.6, there exists an extension corresponding to that kernel.

The other way, the kernel has an extension if and only if one of its obstructions is cochain identical to 0, and since the map is well-defined, we get $\overline{\mathcal{O}} = 0$. □

# Part 3. Calculations

## 9. Abelian extensions

**Lemma 9.1.** *$E_{\mathbb{Z}}(\mathbb{Z}_m, A) \simeq A/mA$, $m$ is a positive integer.*

*Proof.* Since $E_{\mathbb{Z}}(\mathbb{Z}_m, A) \simeq Ext^1_{\mathbb{Z}}(\mathbb{Z}_m, A)$, we pick the projective resolution of $\mathbb{Z}_n$ :

$$\begin{aligned} 0 \quad &\longrightarrow \quad \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_m \longrightarrow 0 \\ i(1) \quad &= \quad 1,\ \pi(1) = 1 \bmod m \end{aligned}$$

Any 1-cocycle $f : \mathbb{Z} \longrightarrow A$ is a group homomorphism, thus it is totally described by $f(1) = a,\ a \in A$. Through the group homomorphism $\phi : Z^1 \longrightarrow A$ defined as $\phi(f) = f(1)$, we get that $Z^1 \simeq A$. The 1-coboundaries $g : \mathbb{Z} \longrightarrow A$ are defines by $gi(1) = g(m) = mg(1) = ma, a \in A$. Through that group homomorphism $\Phi : B^1 \longrightarrow A$ defines as $\Phi(g) = mg(1)$, we get that $B^1 \simeq mA$. Hence $E_{\mathbb{Z}}(\mathbb{Z}_m, A) \simeq Z^1/B^1 \simeq A/mA$. □

**Proposition 9.2.** *Fix the ring $\mathbb{Z}$. Let $p, q$ be distinct primes, $i, j$ positive integers.*

$$E_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = E_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_{p^i}) = E_{\mathbb{Z}}(\mathbb{Z}_{p^i}, \mathbb{Z}_{q^j}) = 0$$

*Proof.* Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module, hence projective, $Ext^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0 \simeq E_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$. So all extensions of $\mathbb{Z}$ by $\mathbb{Z}$ are equivalent to the direct sum extension. Also, all extensions of $\mathbb{Z}_{p^i}$ by $\mathbb{Z}$ are equivalent to the direct sum extension. $E_{\mathbb{Z}}(\mathbb{Z}_{p^i}, \mathbb{Z}_{q^j}) = \mathbb{Z}_{q^j}/p^i\mathbb{Z}_{q^j} \simeq \mathbb{Z}_{q^j}/\mathbb{Z}_{q^j} = 0$ since $\gcd(p^i, q^j) = \gcd(p, q) = 1$. □

**Theorem 9.3.** ***a)*** *$E_{\mathbb{Z}}(\mathbb{Z}_{p^i}, \mathbb{Z}) \simeq \mathbb{Z}_{p^i}$;*
***b)*** *Given $a \in \mathbb{Z}_{p^i}$, the corresponding extension has the form*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}_{p^i} \longrightarrow \mathbb{Z}_{p^i} \longrightarrow 0$$

*if $a = 0$, the form*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_{p^i} \longrightarrow 0$$

*if* $\gcd(a, p) = 1$*, and the form*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^i} \longrightarrow 0$$

*if $a = bp^k$, $gcd(b, p) = 1$.*

*Proof.* **a)** By Lemma 9.1, $E_{\mathbb{Z}}(\mathbb{Z}_{p^i}, \mathbb{Z}) \simeq \mathbb{Z}/p^i\mathbb{Z} \approx \mathbb{Z}_{p^i}$.
**b)** From Lemma 5.12, we obtain the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}/\left\langle (a, -p^i) \right\rangle \longrightarrow \mathbb{Z}_{p^i} \longrightarrow 0$$

In the middle module we get the relations matrix $\begin{bmatrix} a \\ -p^i \end{bmatrix}$. Suppose

$$\gcd(a, -p^i) = p^k \iff a = bp^k, p \nmid b$$

for some $k \in \{0, 1, ..., i\}$, so our middle module is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}_{p^k}$. In the first case, we have the extensions

$$0 \longrightarrow \mathbb{Z} \overset{1 \longrightarrow p^i}{\longrightarrow} \mathbb{Z} \overset{1 \longrightarrow g}{\longrightarrow} \mathbb{Z}_{p^i} \longrightarrow 0$$

where $g$ is any generator of $\mathbb{Z}_{p^i}$. There are $p^{i-1}(p-1) = p^i - p^{i-1}$ such distinct $g^{'}s$. For the second case,we must define the homomorphisms in

$$0 \longrightarrow \mathbb{Z} \overset{1 \longrightarrow (c,d)}{\longrightarrow} \mathbb{Z} \times \mathbb{Z}_{p^k} \overset{(x,y) \longrightarrow ux+vy}{\longrightarrow} \mathbb{Z}_{p^i} \longrightarrow 0$$

Look at the relations matrix $\begin{bmatrix} 0 & c \\ p^k & d \end{bmatrix}$. We require that by a number of elementary row/ column operations, we can transform the proceeding matrix to $\begin{bmatrix} 1 & 0 \\ 0 & p^i \end{bmatrix}$, which can be done according to ([4] Ex.9.15). This is equivalent to requiring

$$\gcd(c, d, p^k) = 1, cp^k = p^i \implies c = p^{i-k}, p \not| d$$

i.e. $d$ is a generator $g$ of $\mathbb{Z}_{p^k}$ so we have $p^{k-1}(p-1)$ choices for $d$. We want that $up^{i-k} + vg \equiv 0 \bmod p^i$. Pick $u = g, v = -p^{i-k}$. Since $\gcd(u, v) = 1$, it is an epimorphism. The kernel of this epimorphism consists of those $(x, y)$ such that

$$gx - p^{i-k}y \equiv 0 \bmod p^i \implies p^{i-k} \mid x, x = lp^{i-k} \implies y \equiv gl \bmod p^k$$

i.e. $(p^{i-k}l, gl) = l(p^{i-k}, g)$. If we sum the number of extensions $\sum_{k=1}^{i} p^{k-1}(p-1) = \sum_{m=0}^{i-2}(p-1)p^m = \frac{(p-1)(1-p^{i-1})}{(1-p)} = p^{i-1} - 1$. Together with the direct sum extension

$$0 \longrightarrow \mathbb{Z} \overset{1 \longrightarrow (1,0)}{\longrightarrow} \mathbb{Z} \oplus \mathbb{Z}_{p^i} \overset{(a,b) \longrightarrow b}{\longrightarrow} \mathbb{Z}_{p^i} \longrightarrow 0$$

so we have described all equivalence classes in $E_{\mathbb{Z}}(\mathbb{Z}_{p^i}, \mathbb{Z})$. □

**Theorem 9.4.** ***a)*** *$E_{\mathbb{Z}}(\mathbb{Z}_{p^i}, \mathbb{Z}_{p^j}) \simeq \mathbb{Z}_{\gcd(p^i, p^j)} = \mathbb{Z}_{p^{\min(i,j)}}$;*
***b)*** *Given $a \in \mathbb{Z}_{p^i}$, the corresponding extension has the form*

$$0 \longrightarrow \mathbb{Z}_{p^j} \longrightarrow \mathbb{Z}_{p^j} \times \mathbb{Z}_{p^i} \longrightarrow \mathbb{Z}_{p^i} \longrightarrow 0$$

*if $a = 0$, the form*

$$0 \longrightarrow \mathbb{Z}_{p^j} \longrightarrow \mathbb{Z}_{p^{i+j}} \longrightarrow \mathbb{Z}_{p^i} \longrightarrow 0$$

*if $\gcd(a, p) = 1$, and the form*

$$0 \longrightarrow \mathbb{Z}_{p^j} \longrightarrow \mathbb{Z}_{p^{i+j-k}} \times \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^i} \longrightarrow 0$$

*if $a = bp^k$, $gcd(b, p) = 1$.*

*Proof.* **a)** Using Lemma 9.1 we get

$$E_{\mathbb{Z}}(\mathbb{Z}_{p^i}, \mathbb{Z}_{p^j}) \simeq \begin{cases} \mathbb{Z}_{p^j}/p^i\mathbb{Z}_{p^j} \simeq \mathbb{Z}_{p^j}/\mathbb{Z}_{p^{j-i}} \simeq \mathbb{Z}_{p^i} & i < j \\ \mathbb{Z}_{p^j}/p^i\mathbb{Z}_{p^j} \simeq \mathbb{Z}_{p^j} & i \geq j \end{cases} \simeq \mathbb{Z}_{\gcd(p^i, p^j)}$$

**b)** As in Lemma 5.12, we obtain

$$0 \longrightarrow \mathbb{Z}_{p^j} \longrightarrow \mathbb{Z}_{p^j} \times \mathbb{Z}/\left\langle (a, -p^i) \right\rangle \longrightarrow \mathbb{Z}_{p^i} \longrightarrow 0$$

We can represent the $\mathbb{Z}$-module $\mathbb{Z}_{p^j} \times \mathbb{Z}/\left\langle (a, -p^i) \right\rangle$ as a matrix of relations $M = \begin{bmatrix} p^j & a \\ 0 & -p^i \end{bmatrix}$. By ([4] Ex.9.15), we can, by a series of row and column transformations, transform the matrix to

$$M' = \left\{ \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}, u_1 = d_1, u_2 = \frac{d_2}{d_1} \right.$$

where $d_i$ is the gcd of the minors of size $i$ in $M$, i.e.,

$$\begin{aligned} d_1 &= \gcd(p^j, a, -p^i) = p^k, a = bp^k, p \nmid b, k \in \{0, 1, .., \min(i, j)\} \\ d_2 &= \frac{\gcd(-p^i p^j)}{p^k} = -p^{i+j-k} \end{aligned}$$

We get the relations matrix $M^{'} = \left\{ \begin{bmatrix} p^k & 0 \\ 0 & -p^{i+j-k} \end{bmatrix} \right.$ which gives the $[\min(i,j)+1]$ non- isomorphic middle modules $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{i+j-k}},\ k \in \{0,1,..,\min(i,j)\}$. Let's explicitly define the maps in the short exact sequence:

$$0 \longrightarrow \mathbb{Z}_{p^j} \overset{1\longrightarrow(c,d)}{\longrightarrow} \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{i+j-k}} \overset{(x,y)\longrightarrow(ux+vy)}{\longrightarrow} \mathbb{Z}_{p^i} \longrightarrow 0$$

We have the relations matrix $\begin{bmatrix} p^k & 0 & c \\ 0 & p^{i+j-k} & d \end{bmatrix}$ which we need to be transformed to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & p^i & 0 \end{bmatrix}$ by elementary row/ column operations. This means that $\gcd(c,p^k) = 1, \gcd(d,p^{i+j-k}) = p^i$, i.e. $p \nmid c$ and $d \mid p^{i+j-k}$. Since any element of $\mathbb{Z}_{p^k}$ has order a divisor of $p^j$, we can take $c = g$, where $g$ is any generator of $\mathbb{Z}_{p^k}$. We have $p^{k-1}(p-1)$ many distinct choices for $c$. Since $p^j d = sp^{i+j-k}$, for some integer $s$, we pick $d = p^{i-k}$. In defining the epimorphism, we wish that $ug + p^{i-k} = wp^i$, for some integer $w$. Choose $u = p^{i-k}, v = -g$. Since $\gcd(u,v) = 1$, we have defined an epimorphism $(x,y) \longrightarrow \big(p^{i-k}x - gy\big) \bmod p^i$. Look at

$$\begin{aligned} \big(p^{i-k}x - gy\big) \quad &\equiv \quad 0 \bmod p^i \iff p^{i-k} \mid gy \iff p^{i-k} \mid y,\ y = lp^{i-k} \implies p^{i-k}x - glp^{i-k} \equiv 0 \bmod p^i \\ &\implies \quad x \equiv gl \bmod p^k \implies (x,y) = (gl, lp^{i-k}) = l(g, p^{i-k}) \end{aligned}$$

So the sequence is exact. We have found, in total

$$\sum_{k=1}^{\min(i,j)} p^{k-1}(p-1) = \sum_{m=0}^{\min(i,j)-1} p^m(p-1) = (p-1)\frac{(1-p^{\min(i,j)-1})}{(1-p)} = p^{\min(i,j)-1} - 1$$

extensions. When $k = 0$ we have the short exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^j} \overset{1\longrightarrow c}{\longrightarrow} \mathbb{Z}_{p^{i+j}} \overset{x\longrightarrow ux}{\longrightarrow} \mathbb{Z}_{p^i} \longrightarrow 0$$

$c$ should be an element of order $p^j$, so we pick $c = p^i$. Let's find the epimorphism. We want that $up^i \equiv 0 \bmod p^i$. Pick $u = g$, where $g$ is any generator of $\mathbb{Z}_{p^{\min(i,j)}}$, so we have $p^{\min(i,j)} - p^{\min(i,j)-1}$ choices for $g$. Altogether we have $p^{\min(i,j)} - 1$ extensions, and together with the direct sum extensions, we have found all. □

**Lemma 9.5.** *Let $G$ be the finite cyclic group of order $m$, with generator $x$. Fix the ring $\mathbb{Z}G$.*

$$... \xrightarrow{N_*} \mathbb{Z}G \xrightarrow{D_*} \mathbb{Z}G \xrightarrow{N_*} \mathbb{Z}G \xrightarrow{D_*} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

*is a free resolution of $\mathbb{Z}^{triv}$, with the homomorphisms given by*

$$\varepsilon(\sum_{i=0}^{m-1} a_i x^i) = \sum_{i=0}^{m-1} a_i,\ D_*u = Du,\ D = x-1,\ N_*u = Nu,\ N = 1 + x + ... + x^{m-1}$$

*Proof.* Look at Example 10.5 in the next subsection. □

Apply $Hom_{\mathbb{Z}G}(-,A)$, for an arbitrary $\mathbb{Z}G$-module $A$, :and get the left complex

$$0 \longrightarrow Hom_{\mathbb{Z}G}(\mathbb{Z},A) \xrightarrow{\varepsilon^*} Hom_{\mathbb{Z}G}(\mathbb{Z}G,A) \xrightarrow{D^*} Hom_{\mathbb{Z}G}(\mathbb{Z}G,A) \xrightarrow{N^*} Hom_{\mathbb{Z}G}(\mathbb{Z}G,A) \xrightarrow{D^*} ..$$

which is exact at the first two non-zero terms. Since $\ker D^* = \{a \mid ga = a\} = A^G$, and $Hom_{\mathbb{Z}G}(\mathbb{Z}G,A) \simeq A$ as abelian groups, we get

$$0 \longrightarrow A^G \xrightarrow{i} A \xrightarrow{D^*} A \xrightarrow{N^*} A \xrightarrow{D^*} ...,\ i(a) = a$$

$$\begin{aligned} D^*a &= D^*f(1) = f(D_*(1)) = f(D \cdot 1) = f(t-1) = tf(1) - f(1) = ta - a \\ N^*a &= N^*(f(1)) = f(N_*(1)) = f(1 + t + ..t^{m-1}) = f(1) + tf(1) + ... + t^{m-1}f(1) = \sum_{i=0}^{m-1} t^i a \end{aligned}$$

This gives:

**Theorem 9.6.** *Let $G = \langle x \rangle$ be a finite cyclic group of order $m$, with generator $xt$. For any $G$-module $A$, we have the following cohomology groups:*

$$\begin{aligned} H^0(G, A) &= \{a \in A \mid ta = a\} \\ H^{2n+1}(G, A) &= \{a \in A \mid N^*a = 0\}/D^*A, n \in \mathbb{Z}_{\geq 0} \\ H^{2n}(G, A) &= A^G/N^*A, n \in \mathbb{Z}_{>0}. \end{aligned}$$

**Corollary 9.7.** *Let $G = \langle x \rangle$ be a finite cyclic group of order $m$, with generator $x$. For any trivial $G$-module $A$ we have the following cohomology groups:*

$$\begin{aligned} H^0(G, A) &= A \\ H^{2n+1}(G, A) &= \{a \in A \mid ma = 0\}, n \in \mathbb{Z}_{\geq 0} \\ H^{2n}(G, A) &= A/mA, n \in \mathbb{Z}_{>0}. \end{aligned}$$

## 10. Results connecting abelian extensions to non-abelian extensions

### 10.1. $E_{\mathbb{Z}}(G, A)$ is a subgroup of $E(G, A^{trivial})$.

**Theorem 10.1.** *There exists an injective group homomorphism from $E_{\mathbb{Z}}(G, A)$ to $E(G, A^{trivial})$.*

*Proof.* Remember that $E_{\mathbb{Z}}(G, A) \simeq Ext^1_{\mathbb{Z}}(G, A)$. Let $\mathbb{Z}[G]$ denote the factor group of the free abelian group on $[g], g \in G$, on the subgroup generated by $[1]$, and $\mathbb{Z}[G \times G]$ the factor group of the free abelian group on $[g, h], g, h \in G$, modulo the subgroup generated by $[1, h], h \in G$ and $[g, 1], g \in G$, and so on. We can construct the projective resolution

$$\begin{aligned} ... &\longrightarrow \mathbb{Z}[G \times G \times G] \oplus \mathbb{Z}[G \times G] \oplus F \xrightarrow{\mathfrak{d}_1} \mathbb{Z}[G \times G] \xrightarrow{\mathfrak{d}_0} \mathbb{Z}[G] \xrightarrow{\varepsilon} G \longrightarrow 1 \\ \varepsilon([g]) &= g \\ \mathfrak{d}_0([g, h]) &= [g] + [h] - [gh] \\ \mathfrak{d}_1([g, h, k]) &= [h, k] - [gh, k] + [g, hk] - [g, h] \\ \mathfrak{d}_1([g, h]) &= [g, h] - [h, g] \end{aligned}$$

By the universal property of free modules, $\varepsilon, \mathfrak{d}_0, \mathfrak{d}_1$ are $\mathbb{Z}$-module homomorphisms. $F$ is a free abelian group that is attached to make the sequence exact.

$$\begin{aligned} \varepsilon\mathfrak{d}_0([g, h]) &= \varepsilon([g] + [h] - [gh]) = gh(gh)^{-1} = 1 \\ \mathfrak{d}_0(\mathfrak{d}_1([g, h, k])) &= \mathfrak{d}_0([h, k] - [gh, k] + [g, hk] - [g, h]) \\ &= [h] + [k] - [hk] - [gh] - [k] + [ghk] + [g] + [hk] - [ghk] - [g] - [h] + [gh] = 0 \\ \mathfrak{d}_0(d_1([g, h])) &= \mathfrak{d}_0([g, h] - [h, g]) = [g] + [h] - [gh] - [h] - [g] + [hg] \\ &= -[gh] + [hg] = -[gh] + [gh] = 0 \end{aligned}$$

We have exactness at $\mathbb{Z}[G]$. Any $a \in \ker \varepsilon$ has the form

$$a = a_1[g_1] + a_2[g_2] + .. + a_r[g_r], \; g_1^{a_1} g_2^{a_2} .. g_r^{a_r} = 1$$

Claim that the kernel is generated by

$$\begin{aligned}\langle s,g\rangle &= s[g]-[g^s]\\ \langle g,h\rangle &= [g]+[h]-[gh],\ s=\{0,1,..,\mathrm{ord}(g)\}, g,h\in\mathbb{Z}\end{aligned}$$

which are both elements of $\mathbb{Z}[G]$ .Now

$$\begin{aligned}a^{'} &= a-\langle a_1,g\rangle-\langle a_2,g\rangle-...-\langle a_r,g\rangle=[g_1^{a_1}]+[g_2^{a_2}]+...+[g_r^{a_r}]\\ a^{''} &= a^{'}-\langle g_1^{a_1},g_2^{a_2}\rangle=[g_1^{a_1}g_2^{a_2}]+[g_3^{a_3}]+...+[g_r^{a_r}]\\ a^{'''} &= a^{''}-\langle g_1^{a_1}g_2^{a_2},g_3^{a_3}\rangle=[g_1^{a_1}g_2^{a_2}g_3^{a3}]+...+[g_r^{a_r}]\end{aligned}$$

Continue in this manner and get that $a$ minus a linear combination of $\langle s,g\rangle$ and $\langle g,h\rangle$ is equal to

$$[g_1^{a_1}g_2^{a_2}...g_r^{a_r}]=[1]=0$$

Hence the sequence is exact at $\mathbb{Z}[G]$. We do not need to continue the projective resolution to the left, we just know it can be done. We obtain an element of $E_{\mathbb{Z}}(G,A)$ by picking an element $f\in Ext^1_{\mathbb{Z}}(G,A)$ and taking pushout of $(\mathfrak{d}_0,f)$, as in Lemma 5.12. We have found a factor system for the extension: $\varphi(g,h)=f([g,h])\in A$. Then

$$\begin{aligned}\varphi(g,h)-\varphi(h,g) &= f([g,h])-f([h,g])=f([g,h]-[h,g])=f\mathfrak{d}_1([g,h])=0\implies\varphi(g,h)=\varphi(h,g)\\ f\mathfrak{d}_1([g,h,k]) &= f([h,k])-f([gh,k])+f([g,hk])-f([g,h])=0\end{aligned}$$

$$\implies f([g,h])=f([h,k])-f([gh,k])+f([g,hk])\iff\varphi(g,h)=\varphi(h,k)-\varphi(gh,k)+\varphi(g,hk)$$

So $\varphi\in Z^2(G,A),.\varphi$ is a 2-cocycle. Define the map $\lambda:Ext^1_{\mathbb{Z}}(G,A)\longrightarrow H^2(G,A)$ as $\lambda(f)=\varphi$. It is well-defined. Fix the above resolution over $\mathbb{Z}$. Given two cochain homologous elements $f,l\in Hom_{\mathbb{Z}}(P_1,A)$, their difference is a 0-cochain,

$$f([g,h])-l([g,h])=s([g])+s([h])-s([gh]),s\in Hom_{\mathbb{Z}}(P_0,A)$$

Let $\psi(g,h)=l([g,h])\in A$ be the factor system in the extension given by pushout of $(l,\mathfrak{d}_0)$. By the universal property of free modules, there exists a $\zeta\in Hom_{\mathbb{Z}G}(F_1,A)$, where $F_1$ is the projective module in the normalized bar resolution such that

$$\zeta(g)=s([g])$$

Then we have

$$\varphi(g,h)-\psi(g,h)=\zeta(g)+\zeta(h)-\zeta(gh)=\zeta\mathfrak{d}_1([g,h])=\mathfrak{d}_1^*\zeta([g,h])$$

So $\lambda$ maps the cohomologous chains to the same element in $H^2(G,A)$. Suppose $\lambda(f)=0$ :

$$\varphi(g,h)=\mathfrak{d}_1^*\zeta([g,h])=\zeta\mathfrak{d}_1([g,h])=\zeta(g)+\zeta(h)-\zeta(gh)$$

By the universal property of free modules, there exists an $s\in Hom_{\mathbb{Z}}(P_0,A)$ defined as

$$s([g])=\zeta(g)$$

This gives

$$\begin{aligned}\varphi(g,h) &= f([g,h])=s([g])+s([h])-s([gh])=s([g]+[h]-[gh])\\ &= s(\mathfrak{d}_1([g,h]))=\mathfrak{d}_1^*s([g,h])\end{aligned}$$

So $\lambda$ is a monomorphism. $\lambda$ is not (in general) an epimorphism since in $H^2(G,A)$, there is no condition that the cocycles should be symmetric:

$$\begin{array}{rcl} \varphi(g,h) & = & \varphi(h,k)-\varphi(gh,k)+\varphi(g,hk) \\ & \shortparallel & \\ \varphi(h,g) & = & \varphi(g,l)-\varphi(hg,l)+\varphi(h,gl) \end{array}$$

We have the isomorphism $\beta : H^2(G,A) \longrightarrow E(G,A)$ given by cocycles in $H^2(G,A)$ give extension with that cocycle as factor set. Hence we have a monomorphism $E_{\mathbb{Z}}(G,A) \longrightarrow E(G,A)$. ☐

10.2. **The case $G$ is finite cyclic.**

**Theorem 10.2.** *$E_{\mathbb{Z}}(\mathbb{Z}_m,A) \simeq E(\mathbb{Z}_m,A)$ as abelian groups.*

*Proof.* We must show that the composition $A/mA \simeq Ext^1_{\mathbb{Z}}(\mathbb{Z}_m,A) \hookrightarrow Ext^2_{\mathbb{Z}[\mathbb{Z}_m]}(\mathbb{Z}^{trivial},A) \simeq A/mA$ gives identity. Start with $Ext^1_{\mathbb{Z}}(\mathbb{Z}_m,A)$. Fix the projective resolution of $\mathbb{Z}_m$

$$\begin{array}{rcl} 0 & \longrightarrow & \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_m \longrightarrow 0 \\ i(1) & = & m,\ \pi(1)=1 \end{array}$$

By the comparison lemma we have

$$\begin{tikzcd} & Z \arrow[r, "i"] & Z \arrow[r, "\pi"] & Z_m \\ ...Z[Z_m\times Z_m\times Z_m]\times F \arrow[r, "d_1"] & Z[Z_m\times Z_m] \arrow[u, "f_1"] \arrow[r, "d_0"] & Z[Z_m] \arrow[u, "f_0"] \arrow[r, "\epsilon"] & Z_m \arrow[u, "1_{Z_m}"'] \end{tikzcd}$$

Set $[i]=[j] \iff i \equiv j \bmod m$.

$$\begin{array}{rcl} \pi f_0([j]) & = & \epsilon([j])=j \\ f_0 d_0([j,k]) & = & f_0([j]+[k]-[j+k]) = i f_1([j,k]) = m f_1([j,k]) \\ f_1(d_1([j,k,l])) & = & f_1([k,l]-[j+k,l]+[j,k+l]-[j,k]) = 0 \end{array}$$

So let's define such a family $\{f_i\}_{i=0,1}$ :

$$f_0[j]=j \implies j+k-(j+k) \bmod m = \left\{ \begin{array}{ll} 0 & j+k<m \\ m & j+k \geq m \end{array} \right. \implies f_1([j,k]) = \left\{ \begin{array}{ll} 0 & j+k<m \\ 1 & j+k \geq m \end{array} \right.$$

Let $a$ represent $\overline{a} \in A/mA$.

$$\varphi(j,k) := f_1([j,k])a = \left\{ \begin{array}{ll} 0 & j+k<m \\ a & j+k \geq m \end{array} \right.$$

is a 1-cocycle:

$$\begin{array}{rcl} \varphi d_1([j,k,l]) & = & \varphi([k,l]-[j+k,l]+[j,k+l]-[j,k]) \\ & = & f_1([k,l])a - f_1([j+k,l])a + f_1([j,k+l])a - f_1([j,k])a \\ & = & f_1([k,l]-[j+k,l]+[j,k+l]-[j,k])a = f_1 d_1([j,k,l])a = 0a = 0 \end{array}$$

and a 2-cocycle in $H^2(\mathbb{Z}_m,A)$. For the $E(\mathbb{Z}_m,A)$, we have the specific resolution of the finite cyclic group $\mathbb{Z}_m$ and the bar resolution, so there exists a lifting $g$ such that the diagram

$$\begin{array}{ccccccccc}
...Z[Z_m] & \xrightarrow{D_*} & Z[Z_m] & \xrightarrow{N_*} & Z[Z_m] & \xrightarrow{D_*} & Z[Z_m] & \xrightarrow{\varepsilon} & Z \\
g_3\uparrow & & g_2\uparrow & & g_1\uparrow & & g_0\uparrow & & \uparrow 1_{Z_m} \\
\dots\quad F_3 & \xrightarrow{\delta_2} & F_2 & \xrightarrow{\delta_1} & F_1 & \xrightarrow{\delta_0} & F_0 & \xrightarrow{\epsilon} & Z
\end{array}$$

commutes. Set $\langle j\rangle = \langle k\rangle \iff j \equiv k \bmod m$.

$$\begin{aligned}
N_*(\langle j\rangle) &= (\langle 0\rangle + \langle 1\rangle + ... + \langle m-1\rangle)\,\langle j\rangle = \langle j\rangle + \langle j+1\rangle + ... + \langle j+m-1\rangle \\
&= \langle 0\rangle + \langle 1\rangle + ... + \langle m-1\rangle\,, \text{ independent of } j \in \mathbb{Z}_m. \\
D_*(\langle j\rangle) &= (\langle 1\rangle - \langle 0\rangle)\,\langle j\rangle = \langle j+1\rangle - \langle j\rangle
\end{aligned}$$

Let's define the $g_0, g_1, g_2$ from the commutativity conditions:

$$\begin{aligned}
\varepsilon(g_0[]) &= \epsilon([]) = [] \implies g_0([]) = [] \\
g_0(\delta_0[j]) &= g_0(\langle j\rangle\,[] - []) = \langle j\rangle\, g_0([]) - g_0([0]) = \langle j\rangle\,[] - [] = \langle j\rangle - \langle 0\rangle \\
D_*(g_1([j]) &= D_*\left(\sum_{j\in\mathbb{Z}_m} c(j)\,\langle j\rangle\right) = \sum_{j\in\mathbb{Z}_m} c(j) D_*(\langle j\rangle) = \sum_{j\in\mathbb{Z}_m} c(j)\,(\langle j+1\rangle - \langle j\rangle) \equiv \langle j\rangle - \langle 0\rangle\,, c(j) \in \mathbb{Z} \\
&\implies g_1([j]) = \langle 0\rangle + \langle 1\rangle + ... + \langle j-1\rangle \text{ may be chosen}
\end{aligned}$$

$$\begin{aligned}
& & g_1(\delta_1([j,k])) &= g_1(\langle j\rangle\,[k] - [j+k] + [j]) \\
& & \langle j\rangle\, g_1([k]) &= \langle j\rangle\,(\langle 0\rangle + \langle 1\rangle + ... + \langle k-1\rangle) \\
&(1) & &= \langle j\rangle + \langle j+1\rangle + ... + \langle j+k-1\rangle \\
& & -g_1([j+k]) &= -(\langle 0\rangle + \langle 1\rangle + ... + \langle j+k-1\rangle) \\
&(2) & &= -\langle 0\rangle - \langle 1\rangle - ... - \langle j+k-1\rangle \\
&(3) & g_1([j]) &= \langle 0\rangle + \langle 1\rangle + ... + \langle j-1\rangle
\end{aligned}$$

If $j + k < m$, we get from $[(3) + (1)] + (2)$ that

$$g_1\delta_1([j,k]) = \langle 0\rangle + \langle 1\rangle + ... + \langle j-1\rangle + \langle j\rangle + \langle j+1\rangle + ... + \langle j+k-1\rangle - \langle 0\rangle - \langle 1\rangle - ... - \langle j+k-1\rangle = 0$$

Suppose $j + k \geq m$. Let $j + k = m + s$, $s \in \mathbb{Z}_m$.

$$\begin{aligned}
&(1) \quad \langle j\rangle + \langle j+1\rangle + ... + \langle j+k-1\rangle \\
= \ &\langle j\rangle + \langle j+1\rangle + .. + \langle m-1\rangle + \langle 0\rangle + \langle 1\rangle + .. + \langle s-1\rangle \\
&(2) \quad -\langle 0\rangle - \langle 1\rangle - ... - \langle s-1\rangle \\
&(3) \quad \langle 0\rangle + \langle 1\rangle + ... + \langle j-1\rangle
\end{aligned}$$

Take $[(3) + (1)] + (2)$ :

$$\begin{aligned}
& \langle 0\rangle + \langle 1\rangle + ... + \langle j-1\rangle + \langle j\rangle + \langle j+1\rangle + ... + \langle m-1\rangle + \langle 0\rangle + \langle 1\rangle + .. + \langle s-1\rangle \\
& -\langle 0\rangle - \langle 1\rangle - ... - \langle s-1\rangle \\
= \ & \langle 0\rangle + \langle 1\rangle + ... + \langle j-1\rangle + \langle j\rangle + \langle j+1\rangle + ... + \langle m-1\rangle = \langle 0\rangle + \langle 1\rangle + ... + \langle m-1\rangle \equiv N_*(\langle 0\rangle) \\
\implies \ & g_2([j,k]) = \begin{cases} 0 & j+k < m \\ \langle 0\rangle & j+k \geq m \end{cases}
\end{aligned}$$

Since

$$\begin{aligned} N_* g_2 \delta_2 &= (N_* D_*)\, g_3 = 0 \implies g_2\delta_2 \in \ker N_* = \operatorname{Im} D_* \\ g_2\delta_2([j,k,l]) &= D_*(b), b = \sum_{i\in\mathbb{Z}_m} a(i)\,\langle i\rangle \in \mathbb{Z}[\mathbb{Z}_m] \\ &= \sum_{i\in\mathbb{Z}_m} a(i)\,(\langle i+1\rangle - \langle i\rangle) = \sum_{i\in\mathbb{Z}_m} a(i)\,\langle i+1\rangle - \sum_{i\in\mathbb{Z}_m} a(i)\,\langle i\rangle \end{aligned}$$

Then for any $\mathbb{Z}[\mathbb{Z}_m]$-module homomorphism $h(\langle j\rangle) = a, \forall i \in \mathbb{Z}_m$,define:

$$\psi(j,k) = h(g_2([j,k])) = \begin{cases} h(0) & j+k<m \\ h(\langle 0\rangle) & j+k\geq m \end{cases} = \begin{cases} 0 & j+k<m \\ a & j+k\geq m \end{cases}$$

$\psi$ is a 2-cocycle:

$$\begin{aligned} \psi\delta_2([j,k,l]) &= hg_2\delta_2([j,k,l]) = h(D_*(b)) = h\left(\sum_{i\in\mathbb{Z}_m} a(i)\,\langle i+1\rangle - \sum_{i\in\mathbb{Z}_m} a(i)\,\langle i\rangle\right) \\ &= \sum_{i\in\mathbb{Z}_m} a(i) h\,(\langle i+1\rangle) - \sum_{i\in\mathbb{Z}_m} a(i) h(\langle i\rangle) = 0 \\ (\varphi-\psi)([i,j]) &= \begin{cases} 0-0=0 & j+k<m \\ a-a=0 & j+k\geq m \end{cases} \end{aligned}$$

so $\varphi$ and $\psi$ are cochain cohomologous, and give the equivalent extensions in $E(G, A)$. So $E_{\mathbb{Z}}(\mathbb{Z}_m, A) \simeq E(\mathbb{Z}_m, A)$. □

10.3. **The case $G$ is $\mathbb{Z}_p \times \mathbb{Z}_p$.**

**Lemma 10.3.** *Let $R$ be a ring, and let $\mathbf{P} = \left(P_* \xrightarrow{r} C\right)$ be a complex over a (left or right) $R$-module $C$. We consider $C$ as a trivial complex concentrated in dimension 0. All the homomorphisms below are $R$-module homomorphisms.*

***a)*** *For $\mathbf{P}$ being a resolution, it is **sufficient** that there exist a homomorphism*

$$q : C \longrightarrow P_0$$

*and a homotopy*

$$S_n : P_n \longrightarrow P_{n+1}, n \geq 0,$$

*such that*

$$\begin{aligned} rq &= \mathbf{1}_C, \\ S &: \mathbf{1}_{P_*} \backsimeq qr. \end{aligned}$$

***b)*** *If both $P_n$, $n \geq 0$, and $C$, are projective then the existence of such a $q$ and an $S$ is a **necessary** condition.*

**Remark 10.4.** *In other words, the lemma above means:* ***a)*** *if $P_* \longrightarrow C$ is a chain homotopy equivalence, then $P_* \longrightarrow C$ is a resolution;* ***b)*** *If $P_* \longrightarrow C$ is a projective resolution, and $C$ is projective, then $P_* \longrightarrow C$ is a chain homotopy equivalence.*

*Proof.* **a)** $rq = \mathbf{1}_C$ implies that $r$ is an epimorphism. Let now $x \in \ker r \subseteq P_0$. It follows that

$$x = x - qr\,(x) = (\mathbf{1} - qr)\,(x) = dS\,(x)\,,$$

i.e. $x$ is a boundary. Let $x \in \ker d \subseteq P_n$, $n > 0$. It follows that

$$x = (\mathbf{1})\,(x) = (dS + Sd)\,(x) = dS\,(x)\,,$$

i.e. $x$ is a boundary.

**b)** Since $C$ is projective, and $r$ is an epimorphism, there exists a $q : P_0 \longrightarrow C$ with $rq = \mathbf{1}_C$. Consider now two chain transformations:

$$\mathbf{1}, qr : P_* \longrightarrow P_*.$$

Since $r(qr) = r$, both are liftings of the identity homomorphism $C \longrightarrow C$. It follows immediately from Lemma 1.22 that $\mathbf{1}$ and $qr$ are chain homotopic, via some homotopy $S$. $\square$

**Example 10.5.** *Let $G$ be a cyclic group with $m$ elements. The group ring $R = \mathbb{Z}G$ is isomorphic to $\mathbb{Z}[x]/\langle x^m - 1\rangle$. The following is a projective resolution of $\mathbb{Z}^{triv}$ (see* [3]*, Theorem IV.7.1):*

$$... \longrightarrow R \longrightarrow R \longrightarrow R \longrightarrow R \xrightarrow{r} \mathbb{Z}^{triv} \longrightarrow 0$$

*where $r(x) = 1$, and where $d_s : P_{s+1} \longrightarrow P_s$ is the multiplication by $x-1$ when $s$ is even, and the multiplication by*

$$N_x = \frac{x^m - 1}{x-1} = 1 + x + x^2 + ... + x^{m-1}$$

*when $s$ is odd. Consider the resolution above as a $\mathbb{Z}$-module resolution. All abelian groups involved are free, $\mathbb{Z}^{triv}$ with one generator $1$, and $R$ with $m$ generators $1, x, x^2, ..., x^{m-1}$. Lemma 10.3**b)** implies that there exist a group homomorphism $q : \mathbb{Z}^{triv} \longrightarrow R$, and a homotopy (over $\mathbb{Z}$)*

$$S_n : R \longrightarrow R, n \geq 0,$$

*such that*

$$\begin{aligned} rq &= \mathbf{1}_{\mathbb{Z}^{triv}}, \\ \mathbf{1}_{P_0} - qr &= dS, \\ \mathbf{1}_{P_n} &= dS + Sd, n > 0. \end{aligned}$$

*However, we can construct $q$ and $S$ independently of* [3]*. It will follow from Lemma 10.3**a)**, applied to $\mathbb{Z}$-modules, that the sequence above is indeed a resolution of $\mathbb{Z}^{triv}$.*

*Let $q(1) = 1$ and let*

$$\begin{aligned} S_{2k}\left(x^i\right) &= \frac{x^i - 1}{x-1} = \begin{cases} 1 + x + x^2 + ... + x^{i-1} & \textit{if} \quad i > 0 \\ 0 & \textit{if} \quad i = 0 \end{cases}, k \geq 0, \\ S_{2k+1}\left(x^i\right) &= \begin{cases} 1 & \textit{if} \quad i = m-1 \\ 0 & \textit{if} \quad i \neq m-1 \end{cases}, k \geq 0. \end{aligned}$$

*Then:*

$$\begin{aligned} rq(1) &= r(1) = 1 \implies rq = \mathbf{1}_{\mathbb{Z}^{triv}}, \\ (\mathbf{1} - qr)\left(x^i\right) &= x^i - 1 = (x-1)\frac{x^{i-1}}{x-1} = dS\left(x^i\right), x^i \in R = P_0, \\ dS\left(x^i\right) + Sd\left(x^i\right) &= \begin{cases} \frac{x^m-1}{x-1} - \frac{x^{m-1}-1}{x-1} = \frac{x^m - x^{m-1}}{x-1} = x^{m-1} & \textit{if} \quad i = m-1 \\ \frac{x^{i+1}}{x-1} - \frac{x^i}{x-1} = \frac{x^{i+1}-x^i}{x-1} = x^i & \textit{if} \quad i \neq m-1 \end{cases} = \\ &= x^i = \mathbf{1}\left(x^i\right), x^i \in R = P_{2k+1}, \\ dS\left(x^i\right) + Sd\left(x^i\right) &= (x-1)\frac{x^i - 1}{x-1} + 1 = x^i = \mathbf{1}\left(x^i\right), x^i \in R = P_{2k}, k > 0. \end{aligned}$$

**Definition 10.6.** *Consider two positive complexes $(P_*, d_*)$ of right $R$-modules and $(Q_*, \delta_*)$ of left $R$-modules. Let $V_{st} = P_s \otimes_R Q_t$. We hope that no confusion arises if we denote by the same letters*

$$\begin{aligned} d_{st} &: = d_s \otimes \mathbf{1}_{Q_t} : V_{st} \longrightarrow V_{s-1,t}, \\ \delta_{st} &: = (-1)^s \mathbf{1}_{P_s} \otimes \delta_t : V_{st} \longrightarrow V_{s,t-1}. \end{aligned}$$

*Clearly $dd = 0$, $\delta\delta = 0$, $d\delta + \delta d = 0$, even for $s = 0$ or $t = 0$, since we have assumed that $d_{-1} = 0$ and $\delta_{-1} = 0$.*

*Let*

$$W_m = \bigoplus_{s=0}^{m} V_{s,m-s},$$

*and let*

$$D_m : W_{m+1} \longrightarrow W_m, m \geq 0,$$

*be given by*

$$\begin{aligned} D(w) &= dw + \delta w, \\ w &\in V_{s,m-s} \subseteq W_m, \\ dw &\in V_{s-1,m-s} \subseteq W_{m-1}, \\ \delta w &\in V_{s,m-1-s} \subseteq W_{m-1.} \end{aligned}$$

*It follows that*

$$DD = dd + d\delta + \delta d + \delta\delta = 0 + 0 + 0 = 0,$$

*and $(W_*, D_*)$ is a complex. That complex is called the* ***tensor product*** *of complexes $P_*$ and $Q_*$:*

$$W_* := P_* \otimes_R Q_*.$$

**Remark 10.7.** *If $R$ is commutative, then $P_s \otimes_R Q_t$ become $R$-modules (projective if $P_s$ and $Q_t$ were projective). If $R$ is arbitrary, then we can only claim that $P_s \otimes_R Q_t$ are $\mathbb{Z}$-modules.*

**Remark 10.8.** *We will write $\otimes$ instead of $\otimes_R$ when no confusion arises.*

**Proposition 10.9.** *Let $P_*$ and $U_*$ be positive complexes of right $R$-modules, and let $Q_*$ and $V_*$ be positive complexes of left $R$-modules. Let further*

$$\begin{aligned} f, f' &: P_* \longrightarrow U_*, \\ g, g' &: Q_* \longrightarrow V_*, \end{aligned}$$

*be pairwise homotopic chain transformations:*

$$S : f \backsimeq f', T : g \backsimeq g'.$$

*Then the transformations $f \otimes g$ and $f' \otimes g'$ are homotopic.*

*Proof.* Roughly speaking, $S \otimes g$ gives a homotopy between $f \otimes g$ and $f' \otimes g$, while $f' \otimes T$ gives a homotopy between $f' \otimes g$ and $f' \otimes g'$. The desired homotopy between $f \otimes g$ and $f' \otimes g'$ is given by $S \otimes g + f' \otimes T$. The only problem is to choose the correct signs.

Consider first $f \otimes g$ and $f' \otimes g$. For $x \otimes y \in P_s \otimes Q_t$, let

$$A(x \otimes y) = S(x) \otimes g(y).$$

Then

$$\begin{aligned} & \left(f\otimes g-f'\otimes g\right)(x\otimes y) \\ = \ & \left(f(x)-f'(x)\right)\otimes g(y)=\left(\left(dS+Sd\right)(x)\right)\otimes g(y), \end{aligned}$$

while

$$\begin{aligned} & (DA+AD)(x\otimes y) \\ = \ & D\left(S(x)\otimes g(y)\right)+A\left(dx\otimes y+(-1)^{s}x\otimes\delta y\right)= \\ = \ & dS(x)\otimes g(y)+S(x)\otimes(-1)^{s+1}\delta g(y)+Sd(x)\otimes g(y)+(-1)^{s}S(x)\otimes g\delta(y)= \\ = \ & \left(\left(dS+Sd\right)(x)\right)\otimes g(y), \end{aligned}$$

since $\delta g=g\delta$. Therefore, $A$ gives a homotopy between $f\otimes g$ and $f'\otimes g$.

Analogously, let

$$B(x\otimes y)=(-1)^{s}f'(x)\otimes T(y).$$

Then

$$\begin{aligned} & \left(f'\otimes g-f'\otimes g'\right)(x\otimes y) \\ = \ & f'(x)\otimes\left(g(y)-g'(y)\right)=f'(x)\otimes\left(\left(dT+Td\right)y\right), \end{aligned}$$

while

$$\begin{aligned} & (DB+BD)(x\otimes y) \\ = \ & (-1)^{s}D\left(f'(x)\otimes T(y)\right)+B\left(dx\otimes y+(-1)^{s}x\otimes\delta y\right)= \\ = \ & (-1)^{s}df'(x)\otimes T(y)+f'(x)\otimes\delta T(y)+(-1)^{s-1}f'd(x)\otimes T(y)+f'(x)\otimes T\delta(y)= \\ = \ & f'(x)\otimes\left(\left(dT+Td\right)y\right), \end{aligned}$$

since $df'=f'd$. Therefore, $B$ gives a homotopy between $f'\otimes g$ and $f'\otimes g'$.

Finally, $A+B$ gives a homotopy between $f\otimes g$ and $f'\otimes g'$. □

**Corollary 10.10.** *If, in the conditions of the above Proposition, $f$ and $g$ are homotopy equivalences, then*

$$f\otimes_{R}g:P_{*}\otimes_{R}Q_{*}\longrightarrow U_{*}\otimes_{R}V_{*}$$

*is a homotopy equivalence.*

*Proof.* It follows from the Proposition, that

$$\begin{aligned} \left(f^{-1}\otimes g^{-1}\right)(f\otimes g) &= f^{-1}f\otimes g^{-1}g\backsimeq 1_{P_{*}\otimes Q_{*}}, \\ (f\otimes g)\left(f^{-1}\otimes g^{-1}\right) &= ff^{-1}\otimes gg^{-1}\backsimeq 1_{U_{*}\otimes V_{*}}, \end{aligned}$$

□

**Theorem 10.11.** *(simplified Künneth formula) Let $C$ be a projective right $R$-module, $D$ be a projective left $R$-module, and let $e:(P_{*},d_{*})\longrightarrow C$ and $\varepsilon:(Q_{*},\delta_{*})\longrightarrow D$ be projective resolutions. Then*

$$e\otimes_{R}\varepsilon:(P_{*},d_{*})\otimes_{R}(Q_{*},\delta_{*})\longrightarrow C\otimes_{R}D$$

*is a resolution.*

*Proof.* Due to Lemma 10.3**b)**, $e:P_{*}\longrightarrow C$ and $\varepsilon:Q_{*}\longrightarrow D$ are homotopy equivalences. Corollary 10.10 guarantees that $e\otimes_{R}\varepsilon$ is a homotopy equivalence as well. Due to Lemma 10.3**a)**, $e\otimes\varepsilon$ is a resolution. □

**Lemma 10.12.** *Let $G$ be a cyclic group with $m$ elements, and $H$ be cyclic with $n$ elements. Then*

$$R = \mathbb{Z}[G \times H] \approx \mathbb{Z}[x,y]/\langle x^m - 1, y^n - 1\rangle .$$

*Consider the following complex $(U_*, d_*)$ of free $R$-modules ($[i, m-i]$ are symbolic generators for free modules):*

$$\begin{aligned} U_m &= \bigoplus_{i=0}^{m} R[i, m-i], \\ d([s, m-s]) &= \left\{ \begin{bmatrix} (x-1)[s-1, m-s] - (y-1)[s, m-s-1] & \text{if} & s \text{ and } m-s \text{ odd} \\ N_x[s-1, m-s] + (y-1)[s, m-s-1] & \text{if} & s \text{ even and } m-s \text{ odd} \\ (x-1)[s-1, m-s] - N_y[s, m-s-1] & \text{if} & s \text{ odd and } m-s \text{ even} \\ N_x[s-1, m-s] + N_y[s, m-s-1] & \text{if} & s \text{ and } m-s \text{ even} \end{bmatrix} \right. \end{aligned}$$

*where*

$$\begin{aligned} N_x &= \frac{x^m - 1}{x - 1} = 1 + x + x^2 + \ldots + x^{m-1}, \\ N_y &= \frac{y^n - 1}{y - 1} = 1 + y + y^2 + \ldots + y^{n-1}. \end{aligned}$$

*Let further $\pi : U_0 \longrightarrow \mathbb{Z}^{triv}$ be given by $\pi([0,0]) = 1$. Then*

$$\pi : U_* \longrightarrow \mathbb{Z}^{triv}$$

*is a projective resolution over $R$.*

*Proof.* Let

$$e : P_* \longrightarrow \mathbb{Z}^{triv}$$

be a projective resolution from Example 10.5. Here

$$P_s = R_1 = \mathbb{Z}[G] \approx \mathbb{Z}[x]/\langle x^m - 1\rangle .$$

Analogously, apply Example 10.5 to the cyclic group $H$, and obtain a projective resolution

$$\varepsilon : Q_* \longrightarrow \mathbb{Z}^{triv}$$

where

$$Q_t = R_2 = \mathbb{Z}[H] \approx \mathbb{Z}[y]/\langle y^n - 1\rangle .$$

Forget temporarily about $G$- and $H$-module structures, and consider the two resolutions as free $\mathbb{Z}$-module resolutions of a free $\mathbb{Z}$-module $\mathbb{Z}^{triv}$. Using Theorem 10.11, construct a free $\mathbb{Z}$-module resolution

$$U_* = P_* \otimes_{\mathbb{Z}} Q_* \longrightarrow \mathbb{Z}^{triv}.$$

It is easy to check that this resolution is actually a free resolution over the ring $\mathbb{Z}[G \times H]$, because

$$R_1 \otimes_{\mathbb{Z}} R_2 \approx \mathbb{Z}[G \times H]$$

as abelian groups, while all differentials $D_*$ in the complex, as well as the projection $\pi : U_0 \longrightarrow \mathbb{Z}^{triv}$, are in fact $G \times H$-module homomorphisms. $\square$

**Lemma 10.13.** $E(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p) \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

*Proof.* By taking $Hom_{\mathbb{Z}[G\times H]}(-,A)$ on the projective resolution in Lemma 10.12, we get

$$0 \longrightarrow Hom_{\mathbb{Z}[G\times H]}(\mathbb{Z}^{trivial},A) \longrightarrow A \longrightarrow A\times A \longrightarrow A\times A\times A \longrightarrow A\times A\times A\times A \longrightarrow ..$$

where the differentials are some maps consisting of $D_* \langle g\rangle$ and $N_* \langle g\rangle$. When $A$ is the trivial $G\times H$-module, $D_* \langle g\rangle (a) = 0, N_* \langle g\rangle (a) = ord(g)a$. When $G = H = A = \mathbb{Z}_p$, we get that $N_* \langle g\rangle (a) = \overline{0} \in \mathbb{Z}_p$, so all differentials in between the $A's$ are the zero map. Hence, $H^2(G,A) \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. $\square$

**Theorem 10.14.** *The natural homomorphism*

$$E_{\mathbb{Z}}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p) \longrightarrow E(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p)$$

*is a monomorphism, but not an isomorphism.*

*Proof.* $E_{\mathbb{Z}}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p) \approx \mathbb{Z}_p \times \mathbb{Z}_p$, while $E(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p) \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. $\square$

**Remark 10.15.** *It is well-known that there are two non-isomorphic non-abelian groups of order $p^3$. Let us denote them $G\left(p^3\right)$ and $H\left(p^3\right)$. The center of both is a cyclic subgroup with p elements. The group $E(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p)$ consists of $p^3$ elements, and describes central extensions*

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow E \longrightarrow \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow 1.$$

*The zero element of $E(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p)$ corresponds to the case $E \approx \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Some $p^2 - 1$ elements of $E(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p)$ correspond to the case $E \approx \mathbb{Z}_p \times \mathbb{Z}_{p^2}$. The remaining $p^3 - p^2$ elements are subdivided into two classes, corresponding to the two cases $E \approx G\left(p^3\right)$ and $E \approx H\left(p^3\right)$.*

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