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Some new iterated Hardy-type inequalities: the case $\theta = 1$

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Abstract

In this paper we characterize the validity of the Hardy-type inequality

$$\left\| \left\| \int_s^\infty h(z)\,dz \right\|_{p,u,(0,t)} \right\|_{q,w,(0,\infty)} \le c\|h\|_{1,v,(0,\infty)},$$

where $0 < p < \infty$, $0 < q \le +\infty$, u, w and v are weight functions on $(0,\infty)$. It is pointed out that this characterization can be used to obtain new characterizations for the boundedness between weighted Lebesgue spaces for Hardy-type operators restricted to the cone of monotone functions and for the generalized Stieltjes operator.

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1 Introduction

Throughout the paper, we assume that $I := (a,b) \subseteq (0,\infty)$. By $\mathcal{M}(I)$ we denote the set of all measurable functions on I. The symbol $\mathcal{M}^+(I)$ stands for the collection of all $f \in \mathcal{M}(I)$ which are non-negative on I, while $\mathcal{M}^+(I;\downarrow)$ is used to denote the subset of those functions which are non-increasing on I. The family of all weight functions (also called just weights) on I, that is, locally integrable non-negative functions on $(0,\infty)$, is denoted by $\mathcal{W}(I)$.

For $p \in (0,+\infty]$ and $w \in \mathcal{M}^+(I)$, we define the functional $\|\cdot\|_{p,w,I}$ on $\mathcal{M}(I)$ by

$$\|f\|_{p,w,I} := \begin{cases} \left(\int_I |f(x)|^p w(x)\,dx\right)^{1/p} & \text{if } p < +\infty, \\ \operatorname{ess\,sup}_I |f(x)|w(x) & \text{if } p = +\infty. \end{cases}$$

If, in addition, $w \in \mathcal{W}(I)$, then the weighted Lebesgue space $L^p(w,I)$ is given by

$$L^p(w,I) = \left\{ f \in \mathcal{M}(I) : \|f\|_{p,w,I} < +\infty \right\}$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,w,I}$.

When $w \equiv 1$ on I, we write simply $L^p(I)$ and $\|\cdot\|_{p,I}$ instead of $L^p(w,I)$ and $\|\cdot\|_{p,w,I}$, respectively.

Everywhere in the paper, u, v and w are weights. We denote by

$$U(t) := \int_0^t u(s)\,ds, \qquad V(t) := \int_0^t v(s)\,ds \quad \text{for every } t \in (0,\infty),$$

and assume that $U(t) > 0$ for every $t \in (0, \infty)$.

In this paper we characterize the validity of the inequality

$$\left\| \left\| \int_s^\infty h(z)\,dz \right\|_{p,u,(0,t)} \right\|_{q,w,(0,\infty)} \le c\|h\|_{\theta,v,(0,\infty)}, \tag{1.1}$$

where $0 < p < \infty$, $0 < q \le +\infty$, $\theta = 1$, u, w and v are weight functions on $(0, \infty)$. Note that inequality (1.1) was considered in the case $p = 1$ in [1] (see also [2]), where the result was presented without proof, in the case $p = \infty$ in [3] and in the case $\theta = 1$ in [4] and [5], where the special type of a weight function v was considered, and recently in [6] in the case $0 < p < \infty$, $0 < q \le +\infty$, $1 < \theta \le \infty$.

We pronounce that the characterization of inequality (1.1) is important because many inequalities for classical operators can be reduced to this form. Just to illustrate this important fact, we give two applications of the obtained results in Section 5. Firstly, we present some new characterizations of weighted Hardy-type inequalities restricted to the cone of monotone functions (see Theorems 5.3 and 5.4). Secondly, we point out boundedness results in weighted Lebesgue spaces concerning the weighted Stieltjes transform (see Theorems 5.6 and 5.7). Here, we also need to prove some reduction theorems of independent interest (see Theorems 5.1, 5.2 and 5.5).

Our approach is based on discretization and anti-discretization methods developed in [4, 7, 8] and [6]. Some basic facts concerning these methods and other preliminaries are presented in Section 2. In Section 3 discretizations of inequalities (1.1) are given. Anti-discretization of the obtained conditions in Section 3 and the main results (Theorems 4.1, 4.2 and 4.3) are stated and proved in Section 4. Finally, the described applications can be found in Section 5.

2 Notations and preliminaries

Throughout the paper, we always denote by c or C a positive constant, which is independent of the main parameters but it may vary from line to line. However, a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$ ($b \gtrsim a$) we mean that $a \le \lambda b$, where $\lambda > 0$ depends only on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. Throughout the paper, we use the abbreviation LHS($*$) (RHS($*$)) for the left (right) hand side of the relation ($*$). By χ_Q we denote the characteristic function of a set Q. Unless a special remark is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals.

Convention 2.1 (i) Throughout the paper, we put $1/(+\infty) = 0$, $(+\infty)/(+\infty) = 0$, $1/0 = (+\infty)$, $0/0 = 0$, $0 \cdot (\pm\infty) = 0$, $(+\infty)^\alpha = +\infty$ and $\alpha^0 = 1$ if $\alpha \in (0, +\infty)$.

(ii) If $p \in [1, +\infty]$, we define p' by $1/p + 1/p' = 1$. Moreover, we put $p^* = \frac{p}{1-p}$ if $p \in (0, 1)$ and $p^* = +\infty$ if $p = 1$.

(iii) If $I = (a, b) \subseteq \mathbb{R}$ and g is a monotone function on I, then by $g(a)$ and $g(b)$ we mean the limits $\lim_{x \to a+} g(x)$ and $\lim_{x \to b-} g(x)$, respectively.

In this paper we shall use the Lebesgue-Stieltjes integral. To this end, we recall some basic facts.

Let φ be a non-decreasing and finite function on the interval $I := (a,b) \subseteq \mathbb{R}$. We assign to φ the function λ defined on subintervals of I by

$$\lambda\big([\alpha,\beta]\big) = \varphi(\beta+) - \varphi(\alpha-), \tag{2.1}$$

$$\lambda\big([\alpha,\beta)\big) = \varphi(\beta-) - \varphi(\alpha-), \tag{2.2}$$

$$\lambda\big((\alpha,\beta]\big) = \varphi(\beta+) - \varphi(\alpha+), \tag{2.3}$$

$$\lambda\big((\alpha,\beta)\big) = \varphi(\beta-) - \varphi(\alpha+). \tag{2.4}$$

The function λ is a non-negative, additive and regular function of intervals. Thus (*cf.* [9], Chapter 10), it admits a unique extension to a non-negative Borel measure λ on I.

Formula (2.2) implies that

$$\int_{[\alpha,\beta)} d\varphi = \varphi(\beta-) - \varphi(\alpha-). \tag{2.5}$$

Note also that the associated Borel measure can be determined, *e.g.*, only by putting

$$\lambda\big([y,z]\big) = \varphi(z+) - \varphi(y-) \quad \text{for any } [y,z] \subset I$$

(since the Borel subsets of I can be generated by subintervals $[y,z] \subset I$).

If $J \subseteq I$, then the Lebesgue-Stieltjes integral $\int_J f\,d\varphi$ is defined as $\int_J f\,d\lambda$. We shall also use the Lebesgue-Stieltjes integral $\int_J f\,d\varphi$ when φ is non-increasing and finite on the interval I. In such a case, we put

$$\int_J f\,d\varphi := -\int_J f\,d(-\varphi).$$

We conclude this section by recalling an integration by parts formula for Lebesgue-Stieltjes integrals. For any non-decreasing function f and a continuous function g on $\mathbb{R}$, the following formula is valid for $-\infty < \alpha < \beta < \infty$:

$$\int_{[\alpha,\beta)} f(t)\,d\big(g(t)\big) = f(\beta-)g(\beta) - f(\alpha-)g(\alpha) + \int_{[\alpha,\beta)} g(t)\,d\big(-f(t-)\big). \tag{2.6}$$

Remark 2.1 Let $I = (a,b) \subseteq \mathbb{R}$. If $f \in C(I)$ and φ is a non-decreasing, right-continuous and finite function on I, then it is possible to show that for any $[y,z] \subset I$, the Riemann-Stieltjes integral $\int_{[y,z]} f\,d\varphi$ (written usually as $\int_y^z f\,d\varphi$) coincides with the Lebesgue-Stieltjes integral $\int_{(y,z]} f\,d\varphi$. In particular, if $f,g \in C(I)$ and φ is non-decreasing on I, then the Riemann-Stieltjes integral $\int_{[y,z]} f\,d\varphi$ coincides with the Lebesgue-Stieltjes integral $\int_{(y,z]} f\,d\varphi$ for any $[y,z] \subset I$.

Let us now recall some definitions and basic facts concerning discretization and anti-discretization which can be found in [7, 8] and [4].

Definition 2.1 Let $\{a_k\}$ be a sequence of positive real numbers. We say that $\{a_k\}$ is geometrically increasing or geometrically decreasing and write $a_k \uparrow\uparrow$ or $a_k \downarrow\downarrow$ when

$$\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \quad \text{or} \quad \sup_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} < 1,$$

respectively.

Definition 2.2 Let U be a continuous strictly increasing function on $[0,\infty)$ such that $U(0) = 0$ and $\lim_{t\to\infty} U(t) = \infty$. Then we say that U is admissible.

Let U be an admissible function. We say that a function φ is U-quasiconcave if φ is equivalent to an increasing function on $(0,\infty)$ and $\frac{\varphi}{U}$ is equivalent to a decreasing function on $(0,\infty)$. We say that a U-quasiconcave function φ is non-degenerate if

$$\lim_{t\to 0+} \varphi(t) = \lim_{t\to\infty} \frac{1}{\varphi(t)} = \lim_{t\to\infty} \frac{\varphi(t)}{U(t)} = \lim_{t\to 0+} \frac{U(t)}{\varphi(t)} = 0.$$

The family of non-degenerate U-quasiconcave functions is denoted by Ω_U. We say that φ is quasiconcave when $\varphi \in \Omega_U$ with $U(t) = t$. A quasiconcave function is equivalent to a concave function. Such functions are very important in various parts of analysis. Let us just mention that, *e.g.*, the Hardy operator $Hf(x) = \int_0^x f(t)\,dt$ of a decreasing function, the Peetre K-functional in interpolation theory and the fundamental function $\|\chi_E\|_X$, X is a rearrangement invariant space, all are quasiconcave.

Definition 2.3 Assume that U is admissible and $\varphi \in \Omega_U$. We say that $\{x_k\}_{k\in\mathbb{Z}}$ is a discretizing sequence for φ with respect to U if

(i) $x_0 = 1$ and $U(x_k) \uparrow\uparrow$;
(ii) $\varphi(x_k) \uparrow\uparrow$ and $\frac{\varphi(x_k)}{U(x_k)} \downarrow\downarrow$;
(iii) there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$ such that $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$ and for every $t \in [x_k, x_{k+1}]$,

$$\varphi(x_k) \approx \varphi(t) \quad \text{if } k \in \mathbb{Z}_1,$$
$$\frac{\varphi(x_k)}{U(x_k)} \approx \frac{\varphi(t)}{U(t)} \quad \text{if } k \in \mathbb{Z}_2.$$

Let us recall [7, Lemma 2.7] that if $\varphi \in \Omega_U$, then there always exists a discretizing sequence for φ with respect to U.

Definition 2.4 Let U be an admissible function, and let ν be a non-negative Borel measure on $[0,\infty)$. We say that the function φ defined by

$$\varphi(t) = U(t) \int_{[0,\infty)} \frac{d\nu(s)}{U(s) + U(t)}, \quad t \in (0,\infty),$$

is the fundamental function of the measure ν with respect to U. We also say that ν is a representation measure of φ with respect to U.

We say that ν is non-degenerate with respect to U if the following conditions are satisfied for every $t \in (0,\infty)$:

$$\int_{[0,\infty)} \frac{d\nu(s)}{U(s) + U(t)} < \infty, \quad t \in (0,\infty) \quad \text{and} \quad \int_{[0,1]} \frac{d\nu(s)}{U(s)} = \int_{[1,\infty)} d\nu(s) = \infty.$$

We recall from Remark 2.10 of [7] that

$$\varphi(t) \approx \int_{[0,t]} d\nu(s) + U(t) \int_{[t,\infty)} U(s)^{-1}\, d\nu(s), \quad t \in (0,\infty).$$

Lemma 2.1 ([8, Lemma 1.5]) *Let $p \in (0,\infty)$, u, w be weights and φ be defined by*

$$\varphi(t) = \operatorname*{ess\,sup}_{s\in(0,t)} U(s)^{\frac{1}{p}} \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{w(\tau)}{U(\tau)^{\frac{1}{p}}}, \quad t \in (0,\infty). \tag{2.7}$$

Then φ is the least $U^{\frac{1}{p}}$-quasiconcave majorant of w, and

$$\sup_{t\in(0,\infty)} \varphi(t)\left(\frac{1}{U(t)}\int_0^t \left(\int_s^\infty h(z)\,dz\right)^p u(s)\,ds\right)^{\frac{1}{p}} = \operatorname*{ess\,sup}_{t\in(0,\infty)} w(t)\left(\frac{1}{U(t)}\int_0^t \left(\int_s^\infty h(z)\,dz\right)^p u(s)\,ds\right)^{\frac{1}{p}}$$

for any non-negative measurable h on $(0,\infty)$. Further, for $t \in (0,\infty)$,

$$\varphi(t) = \operatorname*{ess\,sup}_{\tau\in(0,\infty)} w(\tau)\min\left\{1, \left(\frac{U(t)}{U(\tau)}\right)^{\frac{1}{p}}\right\} = U(t)^{\frac{1}{p}} \operatorname*{ess\,sup}_{s\in(t,\infty)} \frac{1}{U(s)^{\frac{1}{p}}} \operatorname*{ess\,sup}_{\tau\in(0,s)} w(\tau),$$

$$\varphi(t) \approx \operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\left(\frac{U(t)}{U(s)+U(t)}\right)^{\frac{1}{p}}.$$

Theorem 2.1 ([7, Theorem 2.11]) *Let $p, q, r \in (0,\infty)$. Assume that U is an admissible function, ν is a non-negative non-degenerate Borel measure on $[0,\infty)$, and φ is the fundamental function of ν with respect to U^q and $\sigma \in \Omega_{U^p}$. If $\{x_k\}$ is a discretizing sequence for φ with respect to U^q, then*

$$\int_{[0,\infty)} \frac{\varphi(t)^{\frac{r}{q}-1}}{\sigma(t)^{\frac{r}{p}}}\, d\nu(t) \approx \sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{\frac{r}{q}}}{\sigma(x_k)^{\frac{r}{p}}}.$$

Lemma 2.2 ([7, Corollary 2.13]) *Let $q \in (0,\infty)$. Assume that U is an admissible function, $f \in \Omega_U$, ν is a non-negative non-degenerate Borel measure on $[0,\infty)$ and φ is the fundamental function of ν with respect to U^q. If $\{x_k\}$ is a discretizing sequence for φ with respect to U^q, then*

$$\left(\int_{[0,\infty)} \left(\frac{f(t)}{U(t)}\right)^q d\nu(t)\right)^{\frac{1}{q}} \approx \left(\sum_{k\in\mathbb{Z}} \left(\frac{f(x_k)}{U(x_k)}\right)^q \varphi(x_k)\right)^{\frac{1}{q}}.$$

Lemma 2.3 ([7, Lemma 3.5]) *Let $p, q \in (0,\infty)$. Assume that U is an admissible function, $\varphi \in \Omega_{U^q}$ and $g \in \Omega_{U^p}$. If $\{x_k\}$ is a discretizing sequence for φ with respect to U^q, then*

$$\sup_{t\in(0,\infty)} \frac{\varphi(t)^{\frac{1}{q}}}{g(t)^{\frac{1}{p}}} \approx \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{\frac{1}{q}}}{g(x_k)^{\frac{1}{p}}}.$$

We shall use some Hardy-type inequalities in this paper. Define

$$\begin{aligned} &\underline{v}(a,b) := \operatorname*{ess\,sup}_{s\in I} v(s)^{-1}, \\ &B(a,b) := \sup_{h\in\mathcal{M}^+(I)} \left\| \int_s^b h(z)\,dz \right\|_{p,u,I} \Big/ \|h\|_{1,v,I}. \end{aligned} \tag{2.8}$$

Lemma 2.4 *We have the following Hardy-type inequalities*:
(a) *Let* $1 \le p < \infty$. *Then the inequality*

$$\left\| \int_s^b h(z)\,dz \right\|_{p,u,I} \le c\|h\|_{1,v,I} \tag{2.9}$$

holds for all $h \in \mathcal{M}^+(I)$ *if and only if*

$$\sup_{t\in I} \left(\int_a^t u(z)\,dz \right)^{\frac{1}{p}} \underline{v}(t,b) < \infty,$$

and the best constant $c = B(a,b)$ *in* (2.9) *satisfies*

$$B(a,b) \approx \sup_{t\in I} \left(\int_a^t u(z)\,dz \right)^{\frac{1}{p}} \underline{v}(t,b). \tag{2.10}$$

(b) *Let* $0 < p < 1$. *Then inequality* (2.9) *holds for all* $h \in \mathcal{M}^+(I)$ *if and only if*

$$\left(\int_a^b \left(\int_a^t u(z)\,dz \right)^{p^*} u(t)\underline{v}(t,b)^{p^*}\,dt \right)^{\frac{1}{p^*}} < \infty,$$

and

$$B(a,b) \approx \left(\int_a^b \left(\int_a^t u(z)\,dz \right)^{p^*} u(t)\underline{v}(t,b)^{p^*}\,dt \right)^{\frac{1}{p^*}}.$$

These well-known results can be found in Maz'ya and Rozin [10], Sinnamon [11], Sinnamon and Stepanov [5] (*cf.* also [12] and [13]).

We shall also use the following fact (*cf.* [14, p.188]):

$$C(a,b) := \sup_{h\in\mathcal{M}^+(I)} \|h\|_{1,I}/\|h\|_{1,v,I} \approx \underline{v}(a,b). \tag{2.11}$$

Finally, if $q \in (0,+\infty]$ and $\{w_k\} = \{w_k\}_{k\in\mathbb{Z}}$ is a sequence of positive numbers, we denote by $\ell^q(\{w_k\},\mathbb{Z})$ the following discrete analogue of a weighted Lebesgue space: if $0 < q < +\infty$, then

$$\ell^q\big(\{w_k\},\mathbb{Z}\big) = \left\{ \{a_k\}_{k\in\mathbb{Z}} : \|a_k\|_{\ell^q(\{w_k\},\mathbb{Z})} := \left(\sum_{k\in\mathbb{Z}} |a_k w_k|^q \right)^{\frac{1}{q}} < +\infty \right\}$$

and

$$\ell^\infty\big(\{w_k\},\mathbb{Z}\big) = \left\{ \{a_k\}_{k\in\mathbb{Z}} : \|a_k\|_{\ell^\infty(\{w_k\},\mathbb{Z})} := \sup_{k\in\mathbb{Z}} |a_k w_k| < +\infty \right\}.$$

If $w_k = 1$ for all $k \in \mathbb{Z}$, we write simply $\ell^q(\mathbb{Z})$ instead of $\ell^q(\{w_k\}, \mathbb{Z})$.

We quote some known results. Proofs can be found in [15] and [16].

Lemma 2.5 *Let $q \in (0, +\infty]$. If $\{\tau_k\}_{k\in\mathbb{Z}}$ is a geometrically decreasing sequence, then*

$$\left\| \tau_k \sum_{m\le k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathbb{Z})}$$

and

$$\left\| \tau_k \sup_{m\le k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathbb{Z})}$$

for all non-negative sequences $\{a_k\}_{k\in\mathbb{Z}}$.

Let $\{\sigma_k\}_{k\in\mathbb{Z}}$ be a geometrically increasing sequence. Then

$$\left\| \sigma_k \sum_{m\ge k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathbb{Z})}$$

and

$$\left\| \sigma_k \sup_{m\ge k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathbb{Z})}$$

for all non-negative sequences $\{a_k\}_{k\in\mathbb{Z}}$.

We shall use the following inequality, which is a simple consequence of the discrete Hölder inequality:

$$\left\|\{a_k b_k\}\right\|_{\ell^q(\mathbb{Z})} \le \left\|\{a_k\}\right\|_{\ell^\rho(\mathbb{Z})} \left\|\{b_k\}\right\|_{\ell^p(\mathbb{Z})}, \tag{2.12}$$

where $\frac{1}{\rho} = (\frac{1}{q} - \frac{1}{p})_+$.[a]

Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and if the natural embedding of X in Y is continuous.

The following two lemmas are discrete versions of the classical Landau resonance theorems. Proofs can be found, for example, in [7].

Proposition 2.1 ([7, Proposition 4.1]) *Let $0 < p, q \le \infty$, and let $\{v_k\}_{k\in\mathbb{Z}}$ and $\{w_k\}_{k\in\mathbb{Z}}$ be two sequences of positive numbers. Assume that*

$$\ell^p\big(\{v_k\}, \mathbb{Z}\big) \hookrightarrow \ell^q\big(\{w_k\}, \mathbb{Z}\big). \tag{2.13}$$

(i) *If $0 < p \le q \le \infty$, then*

$$\left\|\left\{w_k v_k^{-1}\right\}\right\|_{\ell^\infty(\mathbb{Z})} \le C,$$

where C stands for the norm of inequality (2.13).

(ii) *If* $0 < q \le p \le \infty$, *then*

$$\left\| \left\{ w_k v_k^{-1} \right\} \right\|_{\ell^r(\mathbb{Z})} \le C,$$

where $1/r := 1/q - 1/p$ *and* C *stands for the norm of inequality* (2.13).

3 Discretization of inequalities

In this section we discretize the inequalities

$$\left(\int_0^\infty \left(\frac{1}{U(t)} \int_0^t \left(\int_s^\infty h(z)\,dz \right)^p u(s)\,ds \right)^{\frac{q}{p}} w(t)\,dt \right)^{\frac{1}{q}} \le c \int_0^\infty h(z) v(z)\,dz \tag{3.1}$$

and

$$\sup_{t\in(0,\infty)} w(t) \left(\frac{1}{U(t)} \int_0^t \left(\int_s^\infty h(z)\,dz \right)^p u(s)\,ds \right)^{\frac{1}{p}} \le c \int_0^\infty h(z) v(z)\,dz. \tag{3.2}$$

We start with inequality (3.1). At first we do the following remark.

Remark 3.1 Let φ be the fundamental function of the measure $w(t)\,dt$ with respect to $U^{\frac{q}{p}}$, that is,

$$\varphi(x) := \int_0^\infty \mathcal{U}(x,s)^{\frac{q}{p}} w(s)\,ds \quad \text{for all } x \in (0,\infty), \tag{3.3}$$

where

$$\mathcal{U}(x,t) := \frac{U(x)}{U(t) + U(x)}.$$

Assume that $w(t)\,dt$ is non-degenerate with respect to $U^{\frac{q}{p}}$. Then $\varphi \in \Omega_{U^{\frac{q}{p}}}$, and therefore there exists a discretizing sequence for φ with respect to $U^{\frac{q}{p}}$. Let $\{x_k\}$ be one such sequence. Then $\varphi(x_k) \uparrow\uparrow$ and $\varphi(x_k) U^{-\frac{q}{p}} \downarrow\downarrow$. Furthermore, there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$, $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$ such that for every $k \in \mathbb{Z}_1$ and $t \in [x_k, x_{k+1}]$, $\varphi(x_k) \approx \varphi(t)$ and for every $k \in \mathbb{Z}_2$ and $t \in [x_k, x_{k+1}]$, $\varphi(x_k) U(x_k)^{-\frac{q}{p}} \approx \varphi(t) U(t)^{-\frac{q}{p}}$.

Next, we state a necessary lemma which is also of independent interest.

Lemma 3.1 *Let* $0 < q < \infty$, $0 < p < \infty$, $1/\rho = (1/q - 1)_+$, *and let* u, v, w *be weights. Assume that* u *is such that* U *is admissible and the measure* $w(t)\,dt$ *is non-degenerate with respect to* $U^{\frac{q}{p}}$. *Let* $\{x_k\}$ *be any discretizing sequence for* φ *defined by* (3.3). *Then inequality* (3.1) *holds for every* $h \in \mathcal{M}^+(0,\infty)$ *if and only if*

$$A := \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} B(x_{k-1}, x_k) \right\} \right\|_{\ell^\rho(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} < \infty, \tag{3.4}$$

and the best constant in inequality (3.1) *satisfies*

$$c \approx A.$$

Proof By using Lemma 2.2 with

$$dv(t) = w(t)\,dt \quad \text{and} \quad f(t) = \int_0^t \left(\int_s^\infty h(z)\,dz\right)^p u(s)\,ds,$$

we get that

$$\text{LHS(3.1)} \approx \left\| \left\{ \left\| \int_s^\infty h(z)\,dz \right\|_{p,u,(0,x_k)} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})}.$$

Moreover, by using Lemma 2.5,

$$\begin{aligned}
\text{LHS(3.1)} &\approx \left\| \left\{ \left\| \int_s^\infty h(z)\,dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\
&\approx \left\| \left\{ \left\| \int_s^{x_k} h(z)\,dz + \int_{x_k}^\infty h(z)\,dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\
&\approx \left\| \left\{ \left\| \int_s^{x_k} h(z)\,dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\
&\quad + \left\| \left\{ \left\| \int_{x_k}^\infty h(z)\,dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\
&\approx \left\| \left\{ \left\| \int_s^{x_k} h(z)\,dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\
&\quad + \left\| \left\{ \int_{x_k}^\infty h(z)\,dz \|1\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})},
\end{aligned}$$

where $I_k := (x_{k-1}, x_k)$, $k \in \mathbb{Z}$. By now, using the fact that

$$\|1\|_{p,u,I_k} = \int_{x_{k-1}}^{x_k} u(s)\,ds = U(x_k) - U(x_{k-1}) \approx U(x_k),$$

we find that

$$\text{LHS(3.1)} \approx \left\| \left\{ \left\| \int_s^{x_k} h(z)\,dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \int_{x_k}^\infty h(z)\,dz \right\} \right\|_{\ell^q(\mathbb{Z})}.$$

Consequently, by using Lemma 2.5 on the second term,

$$\begin{aligned}
\text{LHS(3.1)} &\approx \left\| \left\{ \left\| \int_s^{x_k} h(z)\,dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\
&\quad + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \int_{x_k}^{x_{k+1}} h(z)\,dz \right\} \right\|_{\ell^q(\mathbb{Z})} := I + II. \qquad (3.5)
\end{aligned}$$

To find a sufficient condition for the validity of inequality (3.1), we apply to I locally (that is, for any $k \in \mathbb{Z}$) the Hardy-type inequality

$$\left\| \int_s^{x_k} h(z)\,dz \right\|_{p,u,I_k} \le B(x_{k-1}, x_k)\|h\|_{1,v,I_k}, \quad h \in \mathcal{M}^+(I_k). \tag{3.6}$$

Thus, in view of inequality (2.12), we have that

$$\begin{aligned} I &\le \left\| \left\{ B(x_{k-1}, x_k) \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \|h\|_{1,v,I_k} \right\} \right\|_{\ell^q(\mathbb{Z})} \\ &\le \left\| \left\{ B(x_{k-1}, x_k) \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \left\| \left\{ \|h\|_{1,v,I_k} \right\} \right\|_{\ell^1(\mathbb{Z})} \\ &= \left\| \left\{ B(x_{k-1}, x_k) \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \|h\|_{1,v,(0,\infty)}. \end{aligned} \tag{3.7}$$

For II, by inequalities (2.11) and (2.12), we get that

$$\begin{aligned} II &= \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \int_{x_k}^{x_{k+1}} h(z)\,dz \right\} \right\|_{\ell^q(\mathbb{Z})} \\ &\le \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \|h\|_{1,v,I_{k+1}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\ &\le \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \left\| \left\{ \|h\|_{1,v,I_{k+1}} \right\} \right\|_{\ell^1(\mathbb{Z})} \\ &= \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \|h\|_{1,v,(0,\infty)}. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), in view of (3.5), we obtain that

$$\begin{aligned} &\text{LHS(3.1)} \\ &\lesssim \left(\left\| \left\{ B(x_{k-1}, x_k) \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \right) \text{RHS(3.1)}. \end{aligned} \tag{3.9}$$

Consequently, (3.1) holds provided that $A < \infty$ and $c \le A$.

Next we prove that condition (3.4) is also necessary for the validity of inequality (3.1). Assume that inequality (3.1) holds with $c < \infty$. By (2.8), there are $h_k \in \mathcal{M}^+(I_k)$, $k \in \mathbb{Z}$, such that

$$\|h_k\|_{1,v,I_k} = 1 \tag{3.10}$$

and

$$\frac{1}{2} B(x_{k-1}, x_k) \le \left\| \int_s^{x_k} h_k(z)\,dz \right\|_{p,u,I_k} \quad \text{for all } k \in \mathbb{Z}. \tag{3.11}$$

Define g_k, $k \in \mathbb{Z}$, as the extension of h_k by 0 to the whole interval $(0, \infty)$ and put

$$g = \sum_{k \in \mathbb{Z}} a_k g_k, \tag{3.12}$$

where $\{a_k\}_{k\in\mathbb{Z}}$ is any sequence of positive numbers. We obtain that

$$\begin{aligned}\text{LHS(3.1)} &\gtrsim \left\| \left\{ \left\| \int_s^{x_k} \sum_{m\in\mathbb{Z}} a_m g_m \right\|_{p,u,I_k} \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\ &\gtrsim \left\| \left\{ a_k B(x_{k-1}, x_k) \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})}. \end{aligned} \tag{3.13}$$

Moreover,

$$\text{RHS(3.1)} = c \left\| \sum_{m\in\mathbb{Z}} a_m g_m \right\|_{1,\nu,(0,\infty)} = c \left\| \{a_k\} \right\|_{\ell^1(\mathbb{Z})}. \tag{3.14}$$

Therefore, by (3.1), (3.13) and (3.14), we arrive at

$$\left\| \left\{ a_k B(x_{k-1}, x_k) \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^q(\mathbb{Z})} \lesssim c \left\| \{a_k\} \right\|_{\ell^1(\mathbb{Z})}, \tag{3.15}$$

and Proposition 2.1 implies that

$$\left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} B(x_{k-1}, x_k) \right\} \right\|_{\ell^\rho(\mathbb{Z})} < c. \tag{3.16}$$

On the other hand, there are $\psi_k \in \mathcal{M}^+(I_k)$, $k \in \mathbb{Z}$, such that

$$\|\psi_k\|_{1,\nu,I_k} = 1 \tag{3.17}$$

and

$$\|\psi_k\|_{1,I_{k+1}} \geq \frac{1}{2} C(x_k, x_{k+1}) \quad \text{for all } k \in \mathbb{Z}. \tag{3.18}$$

Define f_k, $k \in \mathbb{Z}$, as the extension of ψ_k by 0 to the whole interval $(0, \infty)$ and put

$$f = \sum_{k\in\mathbb{Z}} b_k f_k, \tag{3.19}$$

where $\{b_k\}_{k\in\mathbb{Z}}$ is any sequence of positive numbers. We obtain that

$$\begin{aligned}\text{LHS(3.1)} &\geq \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \int_{x_k}^{x_{k+1}} \sum_{m\in\mathbb{Z}} b_m f_m \right\} \right\|_{\ell^q(\mathbb{Z})} \\ &\gtrsim \left\| \left\{ b_k \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \right\} \right\|_{\ell^q(\mathbb{Z})}. \end{aligned}$$

Note that

$$\text{RHS(3.1)} = c \left\| \sum_{m\in\mathbb{Z}} b_m f_m \right\|_{1,\nu,(0,\infty)} = c \left\| \{b_k\} \right\|_{\ell^1(\mathbb{Z})}.$$

Then, by (3.1) and previous two inequalities, we have that

$$\left\| \left\{ b_k \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \right\} \right\|_{\ell^q(\mathbb{Z})} \lesssim c \left\| \{b_k\} \right\|_{\ell^1(\mathbb{Z})}.$$

Proposition 2.1 implies that

$$\left\| \left\{ \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} < c. \tag{3.20}$$

Inequalities (3.16) and (3.20) prove that $A \lesssim c$. □

Before we proceed to inequality (3.2), we make the following remark.

Remark 3.2 Suppose that $\varphi(x) < \infty$ for all $x \in (0, \infty)$, where φ is defined by (2.7). Let φ be non-degenerate with respect to $U^{\frac{1}{p}}$. Then, by Lemma 2.1, $\varphi \in \Omega_{U^{\frac{1}{p}}}$, and therefore there exists a discretizing sequence for φ with respect to $U^{\frac{1}{p}}$. Let $\{x_k\}$ be one such sequence. Then $\varphi(x_k) \uparrow\uparrow$ and $\varphi(x_k) U^{-\frac{1}{p}} \downarrow\downarrow$. Furthermore, there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$, $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$ such that for every $k \in \mathbb{Z}_1$ and $t \in [x_k, x_{k+1}]$, $\varphi(x_k) \approx \varphi(t)$ and for every $k \in \mathbb{Z}_2$ and $t \in [x_k, x_{k+1}]$, $\varphi(x_k) U(x_k)^{-\frac{1}{p}} \approx \varphi(t) U(t)^{-\frac{1}{p}}$.

The following lemma is proved analogously, and for the sake of completeness, we give the full proof.

Lemma 3.2 *Let $0 < p < \infty$, and let u, v, w be weights. Assume that u is such that $U^{\frac{1}{p}}$ is admissible. Let φ, defined by* (2.7), *be non-degenerate with respect to $U^{\frac{1}{p}}$. Let $\{x_k\}$ be any discretizing sequence for φ. Then inequality* (3.2) *holds for every $h \in \mathcal{M}^+(0, \infty)$ if and only if*

$$D := \left\| \left\{ \frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} B(x_{k-1}, x_k) \right\} \right\|_{\ell^\infty(\mathbb{Z})} + \left\| \left\{ \varphi(x_k) C(x_k, x_{k+1}) \right\} \right\|_{\ell^\infty(\mathbb{Z})} < \infty, \tag{3.21}$$

and the best constant in inequality (3.2) *satisfies $c \approx D$.*

Proof Using Lemma 2.1, Lemma 2.3, Lemma 2.5, we obtain for the left-hand side of (3.2) that

$$\begin{aligned}
\mathrm{LHS}(3.2) &= \sup_{t \in (0,\infty)} \frac{\varphi(t)}{U(t)^{\frac{1}{p}}} \left\| \int_s^\infty h(z)\, dz \right\|_{p,u,(0,t)} \\
&\approx \left\| \left\{ \frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} \left\| \int_s^\infty h(z)\, dz \right\|_{p,u,(0,x_k)} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \\
&\approx \left\| \left\{ \frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} \left\| \int_s^\infty h(z)\, dz \right\|_{p,u,I_k} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \\
&\approx \left\| \left\{ \frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} \left\| \int_s^{x_k} h(z)\, dz \right\|_{p,u,I_k} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \\
&\quad + \left\| \left\{ \varphi(x_k) \int_{x_k}^{x_{k+1}} h(z)\, dz \right\} \right\|_{\ell^\infty(\mathbb{Z})} := III + IV.
\end{aligned} \tag{3.22}$$

To find a sufficient condition for the validity of inequality (3.2), we apply to *III* locally Hardy-type inequality (3.6). Thus

$$\begin{aligned} III &\le \left\| \left\{ B(x_{k-1},x_k)\frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}}\|h\|_{1,\nu,I_k} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \\ &\le \left\| \left\{ B(x_{k-1},x_k)\frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \left\| \left\{ \|h\|_{1,\nu,I_k} \right\} \right\|_{\ell^1(\mathbb{Z})} \\ &= \left\| \left\{ B(x_{k-1},x_k)\frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \|h\|_{1,\nu,(0,\infty)}. \end{aligned} \tag{3.23}$$

For IV we have that

$$\begin{aligned} IV &= \left\| \left\{ \varphi(x_k)\int_{x_k}^{x_{k+1}} h(z)\,dz \right\} \right\|_{\ell^\infty(\mathbb{Z})} \\ &\le \left\| \left\{ \varphi(x_k)C(x_k,x_{k+1})\|h\|_{1,\nu,I_{k+1}} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \\ &\le \left\| \left\{ \varphi(x_k)C(x_k,x_{k+1}) \right\} \right\|_{\ell^\infty(\mathbb{Z})} \left\| \left\{ \|h\|_{1,\nu,I_{k+1}} \right\} \right\|_{\ell^1(\mathbb{Z})} \\ &= \left\| \left\{ \varphi(x_k)C(x_k,x_{k+1}) \right\} \right\|_{\ell^\infty(\mathbb{Z})} \|h\|_{1,\nu,(0,\infty)}. \end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24), in view of (3.22), we get that

$$\begin{aligned} &\text{LHS(3.2)} \\ &\lesssim \left(\left\| \left\{ B(x_{k-1},x_k)\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\infty(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}}C(x_k,x_{k+1}) \right\} \right\|_{\ell^\infty(\mathbb{Z})} \right) \text{RHS(3.2)}. \end{aligned}$$

Consequently, inequality (3.2) holds provided that $D < \infty$ and $c \lesssim D$.

Next we prove that condition (3.21) is also necessary for the validity of inequality (3.2). Assume that inequality (3.2) holds with $c < \infty$. By (3.10), (3.11) and (3.12), we obtain that

$$\begin{aligned} \text{LHS(3.2)} &\gtrsim \left\| \left\{ \left\| \int_s^{x_k} \sum_{m\in\mathbb{Z}} a_m g_m \right\|_{p,u,I_k} \frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \\ &\gtrsim \left\| \left\{ a_k B(x_{k-1},x_k)\frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\infty(\mathbb{Z})}. \end{aligned} \tag{3.25}$$

Moreover,

$$\text{RHS(3.2)} = c\left\| \sum_{m\in\mathbb{Z}} a_m g_m \right\|_{1,\nu,(0,\infty)} = c\left\| \{a_k\} \right\|_{\ell^1(\mathbb{Z})}. \tag{3.26}$$

Therefore, by (3.2), (3.25) and (3.26),

$$\left\| \left\{ a_k B(x_{k-1},x_k)\frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} \right\} \right\|_{\ell^\infty(\mathbb{Z})} \lesssim c\left\| \{a_k\} \right\|_{\ell^1(\mathbb{Z})}, \tag{3.27}$$

and Proposition 2.1 implies that

$$\left\| \left\{ \frac{\varphi(x_k)}{U(x_k)^{\frac{1}{p}}} B(x_{k-1},x_k) \right\} \right\|_{\ell^\infty(\mathbb{Z})} \lesssim c. \tag{3.28}$$

On the other hand, accordingly to (3.17), (3.18) and (3.19), we obtain that

$$\mathrm{LHS}(3.2) \gtrsim \left\| \left\{ \varphi(x_k) \int_{x_k}^{x_{k+1}} \sum_{m\in\mathbb{Z}} b_m f_m \right\} \right\|_{\ell^\infty(\mathbb{Z})} \gtrsim \left\| \left\{ b_k \varphi(x_k) C(x_k, x_{k+1}) \right\} \right\|_{\ell^\infty(\mathbb{Z})}.$$

Since

$$\mathrm{RHS}(3.2) = c \left\| \sum_{m\in\mathbb{Z}} b_m f_m \right\|_{1,v,(0,\infty)} = c \left\| \{b_k\} \right\|_{\ell^1(\mathbb{Z})},$$

in view of (3.2) and previous two inequalities, we have that

$$\left\| \left\{ b_k \varphi(x_k) C(x_k, x_{k+1}) \right\} \right\|_{\ell^\infty(\mathbb{Z})} \lesssim c \left\| \{b_k\} \right\|_{\ell^1(\mathbb{Z})}.$$

Proposition 2.1 implies that

$$\left\| \left\{ \varphi(x_k) C(x_k, x_{k+1}) \right\} \right\|_{\ell^\infty(\mathbb{Z})} \lesssim c. \tag{3.29}$$

Finally, inequalities (3.28) and (3.29) imply that $D \lesssim c$. □

Remark 3.3 In view of (2.11) and Lemma 2.5, it is evident now that

$$\left\| \left\{ \varphi(x_k)^{\frac{1}{q}} C(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \approx \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \approx \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k, \infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})}.$$

Monotonicity of $\underline{v}(t,\infty)$ implies that

$$\left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \geq \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \lim_{t\to\infty} \underline{v}(t,\infty).$$

Since $\{\varphi(x_k)^{\frac{1}{q}}\}$ is geometrically increasing, we obtain that

$$\left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \geq \varphi(\infty)^{\frac{1}{q}} \lim_{t\to\infty} \underline{v}(t,\infty).$$

This inequality shows that $\lim_{t\to\infty} \underline{v}(t,\infty)$ must be equal to 0, because $\varphi(\infty)$ is always equal to ∞ by our assumptions on the function φ. Therefore, in the remaining part of the paper, we consider weight functions v such that

$$\lim_{t\to\infty} \underline{v}(t,\infty) = 0.$$

4 Anti-dicretization of conditions

In this section, we anti-discretize the conditions obtained in Lemmas 3.1 and 3.2. We distinguish several cases.

The case $0 < p < 1$, $0 < q < \infty$. We need the following lemma.

Lemma 4.1 *Let $0 < q < \infty$, $0 < p < 1$, $1/\rho = (1/q - 1)_+$, and let u, v, w be weights. Assume that u is such that U is admissible and the measure $w(t)\,dt$ is non-degenerate with respect to $U^{\frac{q}{p}}$. Let $\{x_k\}$ be any discretizing sequence for φ defined by* (3.3). *Then*

$$A \approx A_1,$$

where

$$A_1 := \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,\infty)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})}.$$

Proof By Lemma 2.4, in this case it yields that

$$B(x_{k-1},x_k) \approx \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,x_k)^{p^*}\,dt \right)^{\frac{1}{p^*}}.$$

Therefore, in view of (2.11), Lemma 3.1, we have that

$$\begin{aligned} A \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,x_k)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k,x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} A_1 \lesssim{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,x_k)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \underline{v}(x_k,\infty) \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t)\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ ={}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,x_k)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \underline{v}(x_k,\infty) \left(\int_{x_{k-1}}^{x_k} u(t)\,dt \right)^{\frac{1}{p}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,x_k)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,x_k)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k,x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \approx A. \end{aligned}$$

On the other hand,

$$\begin{aligned} A \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,x_k)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k,x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t,x_k)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \end{aligned}$$

$$+\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\underline{v}(x_k,x_{k+1})\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t}u(s)\,ds\right)^{p^*}u(t)\,dt\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$

$$\lesssim\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t}u(s)\,ds\right)^{p^*}u(t)\underline{v}(t,\infty)^{p^*}\,dt\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}=A_1.$$

□

Lemma 4.2 *Assume that the conditions of Lemma* 4.1 *are fulfilled. Then*

$$A_1\approx A_2,$$

where

$$A_2:=\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{x_{k-1}}^{x_k}U(t)^{p^*}u(t)\underline{v}(t,\infty)^{p^*}\,dt\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Proof Evidently, $A_1\le A_2$. Using integrating by parts formula (2.6), we have that

$$\begin{aligned}
A_2&\approx\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{[x_{k-1},x_k)}\underline{v}(t,\infty)^{p^*}\,d\big(U(t)^{\frac{p^*}{p}}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\lesssim\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{[x_{k-1},x_k)}U(t)^{\frac{p^*}{p}}\,d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\quad+\left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\lesssim\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{[x_{k-1},x_k)}\left(\int_{x_{k-1}}^{t}u(s)\,ds\right)^{\frac{p^*}{p}}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\quad+\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}U(x_{k-1})^{\frac{1}{p}}\left(\int_{[x_{k-1},x_k)}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\quad+\left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\lesssim\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{[x_{k-1},x_k)}\left(\int_{x_{k-1}}^{t}u(s)\,ds\right)^{\frac{p^*}{p}}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\quad+\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}U(x_{k-1})^{\frac{1}{p}}\underline{v}(x_{k-1}-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\quad+\left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\lesssim\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{[x_{k-1},x_k)}\left(\int_{x_{k-1}}^{t}u(s)\,ds\right)^{\frac{p^*}{p}}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\quad+\left\|\left\{\varphi(x_{k-1})^{\frac{1}{q}}\underline{v}(x_{k-1}-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}+\left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\approx\left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{[x_{k-1},x_k)}\left(\int_{x_{k-1}}^{t}u(s)\,ds\right)^{\frac{p^*}{p}}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}\\
&\quad+\left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}.
\end{aligned}$$

Again integrating by parts, we have that

$$\begin{aligned}
A_2 &\lesssim \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \underline{v}(t-,\infty)^{p^*} d\left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{p^*}{p}} \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k-,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&= \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t-,\infty)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k-,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&= A_1 + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k-,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k-,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad = \left\| \left\{ \varphi(x_{k-1})^{\frac{1}{q}} \underline{v}(x_{k-1}-,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad \approx \left\| \left\{ \frac{\varphi(x_{k-1})^{\frac{1}{q}}}{U(x_{k-1})^{\frac{1}{p}}} \underline{v}(x_{k-1}-,\infty) \left(\int_{x_{k-2}}^{x_{k-1}} \left(\int_{x_{k-2}}^{t} u(s)\,ds \right)^{p^*} u(t)\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad \le \left\| \left\{ \frac{\varphi(x_{k-1})^{\frac{1}{q}}}{U(x_{k-1})^{\frac{1}{p}}} \left(\int_{x_{k-2}}^{x_{k-1}} \left(\int_{x_{k-2}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t-,\infty)\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad = \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{p^*} u(t) \underline{v}(t-,\infty)^{p^*}\,dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} = A_1, \qquad (4.1)
\end{aligned}$$

we obtain that

$$A_2 \lesssim A_1. \qquad \square$$

Lemma 4.3 *Assume that the conditions of Lemma* 4.1 *are fulfilled. Then*

$$A_2 \approx A_3,$$

where

$$\begin{aligned}
A_3 &:= \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{[x_{k-1},x_k)} U(t)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*} \right) dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k-,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})}.
\end{aligned}$$

Proof Integrating by parts, in view of inequality (4.1) and Lemma 4.2, we have that

$$\begin{aligned}
A_3 &\le \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{x_{k-1}}^{x_k} \underline{v}(t,\infty)^{p^*} d\left(U(t)^{\frac{p^*}{p}} \right) dt \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad + \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} U(x_{k-1})^{\frac{1}{p}} \underline{v}(x_{k-1}-,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k-,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})}
\end{aligned}$$

$$\lesssim A_2 + \left\|\left\{\varphi(x_{k-1})^{\frac{1}{q}}\underline{v}(x_{k-1}-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})} + \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\approx A_2 + \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})} \lesssim A_2 + A_1 \approx A_2.$$

On the other hand, again integrating by parts, we get that

$$A_2 = \left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{x_{k-1}}^{x_k}\underline{v}(t,\infty)^{p^*}d\big(U(t)^{\frac{p^*}{p}}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\lesssim \left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{[x_{k-1},x_k)}U(t)^{\frac{p^*}{p}}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$+ \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})} = A_3. \qquad \square$$

Lemma 4.4 *Assume that the conditions of Lemma* 4.1 *are fulfilled. Then*

$$A_3 \approx A_4,$$

where

$$A_4 := \left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\left(\int_{[x_{k-1},x_k)}U(t)^{\frac{p^*}{p}}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$+ \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\left(\int_{[x_k,x_{k+1})}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Proof By Lemma 2.5, in view of Remark 3.3, we have that

$$\left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k-,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\approx \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\left(\sum_{i=k}^{\infty}\big[\underline{v}(x_i-,\infty)^{p^*} - \underline{v}(x_{i+1}-,\infty)^{p^*}\big]\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$+ \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\lim_{t\to\infty}\underline{v}(t,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\approx \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\big(\underline{v}(x_k-,\infty)^{p^*} - \underline{v}(x_{k+1}-,\infty)^{p^*}\big)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\approx \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\left(\int_{[x_k,x_{k+1})}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}. \qquad \square$$

Lemma 4.5 *Assume that the conditions of Lemma* 4.1 *are fulfilled. Then*

$$A_4 \approx A_5,$$

where

$$A_5 := \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\left(\int_{[0,\infty)}\mathcal{U}(t,x_k)^{\frac{p^*}{p}}d\big(-\underline{v}(t-,\infty)^{p^*}\big)\right)^{\frac{1}{p^*}}\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Proof By Lemma 2.5, we have that

$$A_4 \approx \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\int_{[0,x_k)} U(t)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \left(\int_{[x_k,\infty)} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})}.$$

Hence,

$$\begin{aligned} A_4 &\approx \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \left(\int_{[0,x_k)} \mathcal{U}(t,x_k)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &\quad + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \left(\int_{[x_k,\infty)} \mathcal{U}(t,x_k)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &\approx \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \left(\int_{[0,\infty)} \mathcal{U}(t,x_k)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} = A_5. \end{aligned}$$

□

We are now in a position to state and prove our first main theorem.

Theorem 4.1 *Let $0 < p < 1$, $0 < q < \infty$, and let u, v, w be weights. Assume that u is such that U is admissible and the measure $w(t)\,dt$ is non-degenerate with respect to $U^{\frac{q}{p}}$.*

(i) *Let $1 \le q < \infty$. Then inequality* (3.1) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$I_1 := \sup_{x\in(0,\infty)} \left(\int_0^\infty \mathcal{U}(x,s)^{\frac{q}{p}} w(s)\,ds \right)^{\frac{1}{q}} \left(\int_{[0,\infty)} \mathcal{U}(t,x)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{1}{p^*}} < \infty.$$

Moreover, the best constant c in (3.1) *satisfies $c \approx I_1$.*

(ii) *Let $0 < q < 1$. Then inequality* (3.1) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$I_2 := \left(\int_0^\infty \left(\int_0^\infty \mathcal{U}(x,s)^{\frac{q}{p}} w(s)\,ds \right)^{q^*} \left(\int_{[0,\infty)} \mathcal{U}(t,x)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{q^*}{p^*}} w(x)\,dx \right)^{\frac{1}{q^*}} < \infty.$$

Moreover, the best constant c in (3.1) *satisfies $c \approx I_2$.*

Proof (i) The proof of the statement follows by using Lemmas 3.1, 4.1-4.5 and 2.3.

(ii) The proof of the statement follows by combining Lemmas 3.1, 4.1-4.5 and Theorem 2.1. □

The case $1 \le p < \infty$, $0 < q < \infty$. The following lemma is true.

Lemma 4.6 *Let $1 \le p < \infty$, $0 < q < \infty$, and let u, v, w be weights. Assume that u is such that U is admissible and the measure $w(t)\,dt$ is non-degenerate with respect to $U^{\frac{q}{p}}$. Let $\{x_k\}$ be any discretizing sequence for φ defined by* (3.3). *Then*

$$A \approx B_1,$$

where

$$B_1 := \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,\infty) \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})}.$$

Proof By Lemma 2.4, in this case we find that

$$B(x_{k-1},x_k) \approx \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,x_k).$$

By using (2.11), in view of Lemma 3.1, we have that

$$\begin{aligned} A \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,x_k) \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k,x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})}. \end{aligned}$$

Obviously,

$$\begin{aligned} B_1 \lesssim{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,x_k) \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \underline{v}(x_k,\infty) \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ ={}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,x_k) \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \underline{v}(x_k,\infty) \left(\int_{x_{k-1}}^{x_k} u(s)\,ds \right)^{\frac{1}{p}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,x_k) \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,x_k) \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{v}(x_k,x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} = A. \end{aligned}$$

On the other hand,

$$\begin{aligned} A \approx{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,x_k) \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \underline{v}(x_k,x_{k+1}) \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ \lesssim{}& \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \left(\sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{v}(t,\infty) \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} := B_1. \end{aligned}$$

□

Lemma 4.7 *Assume that the conditions of Lemma* 4.6 *are fulfilled. Then*

$$B_1 \approx B_2,$$

where

$$B_2 := \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \Big(\sup_{x_{k-1}<t<x_k} U(t)^{\frac{1}{p}} \underline{\nu}(t,\infty) \Big) \right\} \right\|_{\ell^\rho(\mathbb{Z})}.$$

Proof Obviously,

$$B_1 \le B_2.$$

Since

$$\begin{aligned}
&\left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{\nu}(x_k,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad \approx \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \underline{\nu}(x_k,\infty) \left(\int_{x_{k-1}}^{x_k} u(s)\,ds \right)^{\frac{1}{p}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad = \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \underline{\nu}(x_k,\infty) \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\quad \le \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{\frac{1}{p}} \underline{\nu}(t,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} = B_1, \qquad (4.2)
\end{aligned}$$

we obtain that

$$\begin{aligned}
B_2 &\le B_1 + \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} U(x_{k-1})^{\frac{1}{p}} \sup_{x_{k-1}<t<x_k} \underline{\nu}(t,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&= B_1 + \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} U(x_{k-1})^{\frac{1}{p}} \underline{\nu}(x_{k-1},\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&\lesssim B_1 + \left\| \left\{ \varphi(x_{k-1})^{\frac{1}{q}} \underline{\nu}(x_{k-1},\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\
&= B_1 + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{\nu}(x_k,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \lesssim B_1. \qquad \square
\end{aligned}$$

Lemma 4.8 *Assume that the conditions of Lemma* 4.6 *are fulfilled. Then*

$$B_2 \approx B_3,$$

where

$$B_3 := \left\| \left\{ \frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}} \Big(\sup_{x_{k-1}<t<x_k} U(t)^{\frac{1}{p}} \underline{\nu}(t,\infty) \Big) \right\} \right\|_{\ell^\rho(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{\frac{1}{q}} \underline{\nu}(x_k,\infty) \right\} \right\|_{\ell^\rho(\mathbb{Z})}.$$

Proof Obviously,

$$B_2 \le B_3.$$

On the other hand, by (4.2), we get that

$$\left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k,x_{k+1})\right\}\right\|_{\ell^\rho(\mathbb{Z})} \approx \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})} \lesssim B_1 \lesssim B_2.$$

Thus

$$B_3 := B_2 + \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\underline{v}(x_k,\infty)\right\}\right\|_{\ell^\rho(\mathbb{Z})} \lesssim B_2. \qquad \square$$

Lemma 4.9 *Assume that the conditions of Lemma* 4.6 *are fulfilled. Then*

$$B_3 \approx B_4,$$

where

$$B_4 := \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\Big(\sup_{t\in(0,\infty)} U(t,x_k)^{\frac{1}{p}}\underline{v}(t,\infty)\Big)\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Proof By Lemma 2.5, we get that

$$\begin{aligned} B_3 &\approx \left\|\left\{\frac{\varphi(x_k)^{\frac{1}{q}}}{U(x_k)^{\frac{1}{p}}}\Big(\sup_{0<t<x_k} U(t)^{\frac{1}{p}}\underline{v}(t,\infty)\Big)\right\}\right\|_{\ell^\rho(\mathbb{Z})} \\ &\quad + \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\Big(\sup_{x_k<t<\infty}\underline{v}(t,\infty)\Big)\right\}\right\|_{\ell^\rho(\mathbb{Z})} \\ &\approx \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\Big(\sup_{0<t<x_k} U(t,x_k)^{\frac{1}{p}}\underline{v}(t,\infty)\Big)\right\}\right\|_{\ell^\rho(\mathbb{Z})} \\ &\quad + \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\Big(\sup_{x_k<t<\infty} U(t,x_k)^{\frac{1}{p}}\underline{v}(t,\infty)\Big)\right\}\right\|_{\ell^\rho(\mathbb{Z})} \\ &\approx \left\|\left\{\varphi(x_k)^{\frac{1}{q}}\Big(\sup_{t\in(0,\infty)} U(t,x_k)^{\frac{1}{p}}\underline{v}(t,\infty)\Big)\right\}\right\|_{\ell^\rho(\mathbb{Z})} = B_4. \end{aligned} \qquad \square$$

Our next main result reads as follows.

Theorem 4.2 *Let* $1 \le p < \infty$, $0 < q < \infty$, *and let* u, v, w *be weights. Assume that* u *is such that* U *is admissible and the measure* $w(t)\,dt$ *is non-degenerate with respect to* $U^{\frac{q}{p}}$.

(i) *Let* $1 \le q < \infty$. *Then inequality* (3.1) *holds for every* $h \in \mathcal{M}^+(0,\infty)$ *if and only if*

$$I_3 := \sup_{x>0}\left(\int_0^\infty U(x,t)^{\frac{q}{p}}w(t)\,dt\right)^{\frac{1}{q}} U(x)^{-\frac{1}{p}} \sup_{t\in(0,x)} U(t)^{\frac{1}{p}}\underline{v}(t,\infty) < \infty.$$

Moreover, the best constant c *in* (3.1) *satisfies that* $c \approx I_3$.

(ii) *Let* $0 < q < 1$. *Then inequality* (3.1) *holds for every* $h \in \mathcal{M}^+(0,\infty)$ *if and only if*

$$I_4 := \left(\int_0^\infty\left(\int_0^\infty U(x,t)^{\frac{q}{p}}w(t)\,dt\right)^{q^*} U(x)^{-\frac{q^*}{p}}\Big(\sup_{t\in(0,x)} U(t)^{\frac{q^*}{p}}\underline{v}(t,\infty)^{q^*}\Big)w(x)\,dx\right)^{\frac{1}{q^*}} < \infty.$$

Moreover, the best constant c *in* (3.1) *satisfies that* $c \approx I_4$.

Proof (i) The proof of the statement follows by combining Lemmas 4.6-4.9, 2.3 and 2.1.

(ii) The proof of the statement follows by using Lemmas 4.6-4.9, 2.1 and Theorem 2.1. □

The case $0 < p < \infty$, $q = \infty$. The following lemma is true.

Lemma 4.10 *Let $0 < p < \infty$, and let u, v, w be weights. Assume that u is such that U is admissible. Let φ, defined by* (2.7), *be non-degenerate with respect to $U^{\frac{1}{p}}$. Let $\{x_k\}$ be any discretizing sequence for φ.*

(i) *If $0 < p < 1$, then*

$$D \approx \left\| \left\{ \varphi(x_k) \left(\int_{[0,\infty)} U(t,x_k)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{1}{p^*}} \right\} \right\|_{\ell^\infty(\mathbb{Z})}.$$

(ii) *If $1 \le p < \infty$, then*

$$D \approx \left\| \left\{ \varphi(x_k) \left(\sup_{t\in(0,\infty)} U(t,x_k)^{\frac{1}{p}} \underline{v}(t,\infty) \right) \right\} \right\|_{\ell^\infty(\mathbb{Z})}.$$

Proof (i) The proof of the statement follows by using Lemmas 3.2, 2.4 and 4.1-4.5.

(ii) The proof of the statement follows by combining Lemmas 3.2, 2.4 and 4.6-4.9. □

Now we are in a position to formulate our last main result.

Theorem 4.3 *Let $0 < p < \infty$, and let u, v, w be weights. Assume that u is such that U is admissible. Let φ, defined by* (2.7), *be non-degenerate with respect to $U^{\frac{1}{p}}$.*

(i) *Let $0 < p < 1$. Then inequality* (3.2) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$I_5 := \sup_{x\in(0,\infty)} \left(\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s) U(x,s)^{\frac{1}{p}} \right) \left(\int_{[0,\infty)} U(t,x)^{\frac{p^*}{p}} d\left(-\underline{v}(t-,\infty)^{p^*}\right) \right)^{\frac{1}{p^*}} < \infty.$$

Moreover, the best constant c in (3.2) *satisfies that $c \approx I_5$.*

(ii) *Let $1 \le p < \infty$. Then inequality* (3.2) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$I_6 := \sup_{x\in(0,\infty)} \left(\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s) U(x,s)^{\frac{1}{p}} \right) U(x)^{-\frac{1}{p}} \sup_{t\in(0,x)} U(t)^{\frac{1}{p}} \underline{v}(t,\infty) < \infty.$$

Moreover, the best constant c in (3.2) *satisfies that $c \approx I_6$.*

Proof Both statements of the theorem follow by using Lemmas 3.2, 4.10, 2.3 and 2.1. □

5 Some applications

In this section, we give some applications of the obtained results. We start with the weighted Hardy inequality on the cone of non-increasing functions. Denote by H_u the weighted Hardy operator

$$H_u f(x) := \frac{1}{U(x)} \int_0^x f(t) u(t)\, dt, \quad x \in (0,\infty).$$

Note that the characterization of the weighted Hardy inequality on the cone of non-increasing functions

$$\|H_u f\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)}, \quad f \in \mathcal{M}^+(0,\infty;\downarrow), \tag{5.1}$$

has been obtained in [17] and [4].

The following reduction theorem is true.

Theorem 5.1 *Let $0 < p,q < \infty$, and let u, v, w be weights. Then inequality* (5.1) *holds for every $f \in \mathcal{M}^+(0,\infty;\downarrow)$ if and only if the inequality*

$$\left(\int_0^\infty \left(\frac{1}{U(x)}\int_0^x \left(\int_t^\infty h\right)^{\frac{1}{p}} u(t)\,dt\right)^q w(x)\,dx\right)^{\frac{p}{q}} \le C\int_0^\infty h(t)V(t)\,dt \tag{5.2}$$

holds for all $h \in \mathcal{M}^+(0,\infty)$. Moreover, the best constants c and C in (5.1) *and* (5.2), *respectively, satisfy $C \approx c^p$.*

Proof It is well known that every non-negative, non-increasing function f is the pointwise limit of an increasing sequence of functions of the form $\int_s^\infty h$ for $h \ge 0$ (*cf.* [5, p.97]). Since f is non-increasing if and only if f^p is non-increasing, by the monotone convergence theorem, (5.1) is equivalent to

$$\left(\int_0^\infty \left(\frac{1}{U(x)}\int_0^x \left(\int_t^\infty h\right)^{\frac{1}{p}} u(t)\,dt\right)^q w(x)\,dx\right)^{\frac{p}{q}} \le c^p \int_0^\infty \left(\int_t^\infty h\right) v(t)\,dt, \quad h \in \mathcal{M}^+(0,\infty),$$

which, by Fubini's theorem, is equivalent to

$$\left(\int_0^\infty \left(\frac{1}{U(x)}\int_0^x \left(\int_t^\infty h\right)^{\frac{1}{p}} u(t)\,dt\right)^q w(x)\,dx\right)^{\frac{p}{q}} \le c^p \int_0^\infty h(t)V(t)\,dt, \quad h \in \mathcal{M}^+(0,\infty).$$

□

Analogously, the following theorem can be proved.

Theorem 5.2 *Let $0 < p < \infty$, and let u, v, w be weights. Then the inequality*

$$\|H_u f\|_{\infty,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)} \tag{5.3}$$

holds for every $f \in \mathcal{M}^+(0,\infty;\downarrow)$ if and only if the inequality

$$\operatorname*{ess\,sup}_{x\in(0,\infty)} w(x)^p \left(\frac{1}{U(x)}\int_0^x \left(\int_t^\infty h\right)^{\frac{1}{p}} u(t)\,dt\right)^p \le C\int_0^\infty h(t)V(t)\,dt \tag{5.4}$$

holds for all $h \in \mathcal{M}^+(0,\infty)$. Moreover, for the best constants c and C in (5.3) *and* (5.4), *respectively, it yields that $C \approx c^p$.*

Combining Theorem 5.1 with Theorems 4.1 and 4.2, we obtain the following statement.

Theorem 5.3 *Let u, v, w be weights. Assume that u is such that U is admissible and the measure $w(t)\,dt$ is non-degenerate with respect to U^q.*

(i) *Let $0 < p \le 1$, $p \le q < \infty$. Then inequality* (5.1) *holds for every $f \in \mathcal{M}^+(0,\infty;\downarrow)$ if and only if*

$$C_1 := \sup_{x\in(0,\infty)} \left(\int_0^\infty \mathcal{U}(x,t)^q w(t)\,dt\right)^{\frac{1}{q}} U(x)^{-1}\Big(\sup_{t\in(0,x)} U(t)V(t)^{-\frac{1}{p}}\Big) < \infty.$$

Moreover, the best constant c in (5.1) *satisfies that $c \approx C_1$.*

(ii) *Let $0 < p \le 1$, $0 < q < p$. Then inequality* (5.1) *holds for every $f \in \mathcal{M}^+(0,\infty;\downarrow)$ if and only if*

$$C_2 := \left(\int_0^\infty \left(\int_0^\infty \mathcal{U}(x,t)^q w(t)\,dt\right)^{\frac{q}{p-q}} U(x)^{\frac{pq}{q-p}} \Big(\sup_{t\in(0,x)} U(t)^{\frac{pq}{p-q}} V(t)^{\frac{q}{q-p}}\Big) w(x)\,dx\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, the best constant c in (5.1) *satisfies that $c \approx C_2$.*

(iii) *Let $1 < p \le q < \infty$. Then inequality* (5.1) *holds for every $f \in \mathcal{M}^+(0,\infty;\downarrow)$ if and only if*

$$C_3 := \sup_{x\in(0,\infty)} \left(\int_0^\infty \mathcal{U}(x,t)^q w(t)\,dt\right)^{\frac{1}{q}} \left(\int_0^\infty \mathcal{U}(t,x)^{p'} \frac{v(t)}{V(t)^{p'}}\,dt\right)^{\frac{1}{p'}} < \infty.$$

Moreover, the best constant c in (5.1) *satisfies that $c \approx C_3$.*

(iv) *Let $1 < p < \infty$, $0 < q < p$. Then inequality* (5.1) *holds for every $f \in \mathcal{M}^+(0,\infty;\downarrow)$ if and only if*

$$C_4 := \left(\int_0^\infty \left(\int_0^\infty \mathcal{U}(x,t)^q w(t)\,dt\right)^{\frac{q}{p-q}} \left(\int_0^\infty \mathcal{U}(t,x)^{p'} \frac{v(t)}{V(t)^{p'}}\,dt\right)^{\frac{q(p-1)}{p-q}} w(x)\,dx\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, the best constant c in the (5.1) *satisfies that $c \approx C_4$.*

Combining Theorems 5.2 and 4.3, we arrive at the following statement.

Theorem 5.4 *Let u, v, w be weights. Assume that u is such that U is admissible. Let φ, defined by*

$$\varphi(t) := \operatorname*{ess\,sup}_{s\in(0,t)} U(s) \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{w(\tau)}{U(\tau)}, \quad t \in (0,\infty),$$

be non-degenerate with respect to U.

(i) *Let $0 < p \le 1$. Then inequality* (5.3) *holds for every $f \in \mathcal{M}^+(0,\infty;\downarrow)$ if and only if*

$$C_5 := \sup_{x\in(0,\infty)} \Big(\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)\Big) U(x)^{-1}\Big(\sup_{t\in(0,x)} U(t)V(t)^{-\frac{1}{p}}\Big) < \infty.$$

Moreover, the best constant c in (5.3) *satisfies that $c \approx C_5$.*

(ii) *Let* $1 < p < \infty$. *Then inequality* (5.3) *holds for every* $f \in \mathcal{M}^+(0,\infty;\downarrow)$ *if and only if*

$$C_6 := \sup_{x\in(0,\infty)} \Big(\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)\Big)\bigg(\int_0^\infty \mathcal{U}(t,x)^{p'} \frac{v(t)}{V(t)^{p'}}\,dt\bigg)^{\frac{1}{p'}} < \infty.$$

Moreover, the best constant c *in* (5.3) *satisfies that* $c \approx C_6$.

Now we consider the generalized Stieltjes transform S defined by

$$(Sh)(x) = \int_0^\infty \frac{h(t)\,dt}{U(x) + U(t)}$$

for all $h \in \mathcal{M}^+(0,\infty)$; the usual Stieltjes transform is obtained by putting $U(x) \equiv x$. In the case $U(x) \equiv x^\lambda$, $\lambda > 0$, the boundedness of the operator S between weighted L^p and L^q spaces was investigated in [18] (when $1 \le p \le q \le \infty$) and in [19, 20] (when $1 \le q < p \le \infty$). This problem was also considered in [21] and [22], where a completely different approach was used, based on the so call 'gluing lemma' (see also [23]).

The following reduction theorem is true.

Theorem 5.5 *Let* $0 < q \le \infty$, $1 \le p \le \infty$, *and let* u, v, w *be weights*. *Then the inequality*

$$\|Sh\|_{q,w,(0,\infty)} \le c\|h\|_{p,v,(0,\infty)}, \quad h \in \mathcal{M}^+(0,\infty), \tag{5.5}$$

holds if and only if

$$\bigg\|H_u\bigg(\int_t^\infty h\bigg)\bigg\|_{q,w,(0,\infty)} \le c\|hU\|_{p,v,(0,\infty)}, \quad h \in \mathcal{M}^+(0,\infty), \tag{5.6}$$

holds.

Proof Evidently, inequality (5.5) is equivalent to the following inequality:

$$\big\|S(hU)\big\|_{q,w,(0,\infty)} \le c\|hU\|_{p,v,(0,\infty)}, \quad h \in \mathcal{M}^+(0,\infty).$$

It is easy to see that

$$S(hU)(x) \approx \frac{1}{U(x)}\int_0^x \bigg(\int_t^\infty h(s)\,ds\bigg)u(t)\,dt, \quad h \in \mathcal{M}^+(0,\infty).$$

Indeed, by Fubini's theorem, we have that

$$\begin{aligned}
\int_0^x \bigg(\int_t^\infty h(s)\,ds\bigg)u(t)\,dt &= \int_0^x \bigg(\int_t^x h(s)\,ds + \int_x^\infty h(s)\,ds\bigg)u(t)\,dt \\
&= \int_0^x \int_0^s u(t)\,dt h(s)\,ds + \int_x^\infty h(s)\,ds \int_0^x u(t)\,dt \\
&= \int_0^x U(s)h(s)\,ds + U(x)\int_x^\infty h(s)\,ds \\
&\approx U(x)\int_0^\infty \frac{U(s)}{U(x)+U(s)}h(s)\,ds = U(x)S(hU)(x),
\end{aligned}$$

that is,

$$S(hU)(x) \approx H_u\left(\int_t^\infty h\right)(x), \quad x \in (0,\infty).$$

Hence, we see that inequality (5.5) is equivalent to inequality (5.6). □

Combining Theorem 5.5 with Theorems 4.2, 4.3 and Theorems 3.1, 3.2 in [6], we obtain the following statements.

Theorem 5.6 *Let u, v, w be weights. Assume that u is such that U is admissible and the measure $w(t)\,dt$ is non-degenerate with respect to U^q. Let $p, q \in (0,\infty]$. When $q < p < \infty$, we set $r = \frac{pq}{p-q}$.*

(i) *Let $p = 1$, $1 \le q < \infty$. Then inequality* (5.5) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$S_1 := \sup_{x\in(0,\infty)} \left(\int_0^\infty \mathcal{U}(x,t)^q w(t)\,dt\right)^{\frac{1}{q}} U(x)^{-1} \sup_{t\in(0,x)} U(t) \operatorname*{ess\,sup}_{s\in(t,\infty)} \big(U(s)v(s)\big)^{-1} < \infty.$$

Moreover, the best constant c in (5.5) *satisfies that $c \approx S_1$.*

(ii) *Let $p = 1$, $0 < q < 1$. Then inequality* (5.5) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$S_2 := \left(\int_0^\infty \left(\int_0^\infty \mathcal{U}(x,t)^q w(t)\,dt\right)^{q^*} U(x)^{-q^*} \times \left(\sup_{t\in(0,x)} U(t)^{q^*} \operatorname*{ess\,sup}_{s\in(t,\infty)} \big(U(s)v(s)\big)^{-q^*}\right) w(x)\,dx\right)^{\frac{1}{q^*}} < \infty.$$

Moreover, the best constant c in (5.5) *satisfies that $c \approx S_2$.*

(iii) *Let $1 < p \le q < \infty$. Then inequality* (5.5) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$S_3 := \sup_{x\in(0,\infty)} \left(\int_0^\infty \mathcal{U}(x,t)^q w(t)\,dt\right)^{\frac{1}{q}} \left(\int_0^\infty \mathcal{U}(t,x)^{p'} U(t)^{-p'} v(t)^{1-p'}\,dt\right)^{\frac{1}{p'}} < \infty.$$

Moreover, the best constant c in (5.5) *satisfies that $c \approx S_3$.*

(iv) *Let $1 < p < \infty$, $0 < q < p$. Then inequality* (5.5) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$S_4 := \left(\int_0^\infty \left(\int_0^\infty \mathcal{U}(x,t)^q w(t)\,dt\right)^{\frac{r}{p}} \left(\int_0^\infty \mathcal{U}(t,x)^{p'} U(t)^{-p'} v(t)^{1-p'}\,dt\right)^{\frac{r}{p'}} w(x)\,dx\right)^{\frac{1}{r}} < \infty.$$

Moreover, the best constant c in (5.5) *satisfies that $c \approx S_4$.*

(v) *Let $p = \infty$, $0 < q < \infty$. Then inequality* (5.5) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$S_5 := \left(\int_0^\infty \left(\mathcal{U}(t,x) U(t)^{-1} \frac{dt}{v(t)}\right)^q w(x)\,dx\right)^{\frac{1}{q}} < \infty.$$

Moreover, the best constant c in (5.5) *satisfies that $c \approx S_5$.*

Theorem 5.7 *Let u, v, w be weights. Assume that u is such that U is admissible. Let φ, defined by*

$$\varphi(t) := \operatorname*{ess\,sup}_{s\in(0,t)} U(s) \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{w(\tau)}{U(\tau)}, \quad t \in (0,\infty),$$

be non-degenerate with respect to U.

(i) *Let $p = 1$. Then the inequality*

$$\|Sh\|_{\infty,w,(0,\infty)} \le c\|h\|_{p,v,(0,\infty)} \tag{5.7}$$

holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if

$$S_6 := \sup_{x\in(0,\infty)} \Big(\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)\Big) U(x)^{-1} \sup_{t\in(0,x)} U(t) \operatorname*{ess\,sup}_{s\in(t,\infty)} \big(U(s)v(s)\big)^{-1} < \infty.$$

Moreover, the best constant c in (5.7) *satisfies that $c \approx S_6$.*

(ii) *Let $1 < p < \infty$. Then inequality* (5.7) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$S_7 := \sup_{x\in(0,\infty)} \Big(\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)\Big) \left(\int_0^\infty \mathcal{U}(t,x)^{p'} U(t)^{-p'} v(t)^{1-p'}\,dt\right)^{\frac{1}{p'}} < \infty.$$

Moreover, the best constant c in (5.7) *satisfies that $c \approx S_7$.*

(iii) *Let $p = \infty$. Then inequality* (5.7) *holds for every $h \in \mathcal{M}^+(0,\infty)$ if and only if*

$$S_8 := \sup_{x\in(0,\infty)} \Big(\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)\Big) \left(\int_0^\infty \mathcal{U}(t,x) U(t)^{-1} \frac{dt}{v(t)}\right) < \infty.$$

Moreover, the best constant c in (5.7) *satisfies that $c \approx S_8$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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Endnote
[a] For any $a \in \mathbb{R}$, define $a_+ = a$ when $a > 0$ and $a_+ = 0$ when $a \le 0$.

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