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# Combinatorics of Reflection Groups and Real Algebraic Geometry

Invariant Sums of Squares, Nonnegative Forms and Specht Ideals

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# Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at UiT The Arctic University of Norway. The research presented here was conducted at UiT The Arctic University of Norway under the supervision of Professor Cordian Riener and Associate Professor Hugues Verdure. This work was supported by the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement 813211 (POEMA).

The thesis is a collection of three papers, presented in an order which relates to their topics. They all concern the symmetries of reflection groups, and the study of nonnegative and sums of squares polynomials. The papers are preceded by an introductory chapter that relates them to each other and provides background information and motivation for the work. The first paper is a joint work with Cordian Riener. The second paper is a joint work with Jose Acevedo, Greg Blekherman and Cordian Riener. The third paper is a joint work with Philippe Moustrou, Cordian Riener and Hugues Verdure.

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# List of Papers

## Paper I

S. Debus and C. Riener "Reflection groups and cones of sums of squares". *arXiv preprint arXiv:2011.09997*, submitted for publication.

## Paper II

J. Acevedo, G. Blekherman, S. Debus and C. Riener "At the limit of symmetric nonnegative forms".

## Paper III

S. Debus, P. Moustrou, C. Riener and H. Verdure "The poset of Specht ideals for hyperoctahedral groups". *arXiv preprint arXiv:2206.08925*, submitted for publication.

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Chapter 1

# Introduction

In this thesis we study symmetries in real algebraic geometry. We study the sets of invariant nonnegative and sums of squares polynomials with a motivation coming from polynomial optimization. Our focus lies on invariant polynomials with respect to the action of a finite reflection group. We discuss efficient representation of invariant sums of squares and study the sets of symmetric nonnegative and sum of squares polynomials in countably infinitely many variables. Moreover, we explore the combinatorial structure of ideals corresponding to the irreducible representations of the hyperoctahedral group.

## 1.1 Representation and invariant theory

*Representation theory* concerns the study of groups by representing the elements of a group as linear transformations on a vector space. Every representation of a group on a vector space of dimension $n$ naturally induces an action of the group on a polynomial ring in $n$ variables. *Invariant theory* is the study of polynomials which are fixed under the action of a representation of the group. Both topics are classical subjects lying in the intersections of algebra, combinatorics, linear algebra and geometry. An introduction to the topic of representation theory of finite groups can be found in [Mus93; Ser+77].
A *linear representation* of a group $G$ is a pair $(\rho, V)$ where $V$ is a vector space over a field $\mathbb{K}$ and $\rho\colon G \times V \to V$ a map which satisfies the following properties for all $\sigma, \tau \in G, v \in V$. We write $\sigma \cdot v$ for $\rho(\sigma, v)$.

1. The map $V \to V,\ w \mapsto \sigma \cdot w$ is linear,
2. $\mathrm{id} \cdot v = v$,
3. $\sigma \cdot (\tau \cdot v) = (\sigma\tau) \cdot v$.

A linear representation of $G$ is also called a *$G$-module* and the map $\rho$ is usually understood from the context. We also say that $G$ acts on $V$. We only consider the case of characteristic zero and usually suppose $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The case of positive characteristic is called modular representation theory. There is another way to present a representation more concretely as an embedding $G \hookrightarrow \mathrm{GL}(V)$ which is equivalent to the definition above. Examples of linear representations of a finite group $G$ are the *trivial representation* $\rho(G) = \{\mathrm{id}\}$ and the *regular representation* where $V = \mathbb{K}[G]$ is the group algebra equipped with vector space structure on which $G$ acts via left multiplication. A linear isomorphism $\phi\colon V \to W$ between $G$-modules $V$ and $W$ is a *$G$-isomorphism* if $\phi(\sigma \cdot v) = \sigma \cdot \phi(v)$ for all $v \in V$ and all $\sigma \in G$.
Suppose that $G$ acts on $\mathbb{K}^n$. Let $X_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{K}^n$ be the vector

whose $i$-th coordinate is 1 and all others are 0. The *induced* action on the polynomial ring is given via

$$\sigma \cdot f(X_1, \dots, X_n) = f(\sigma \cdot X_1, \dots, \sigma \cdot X_n).$$

For instance, the symmetric group $\mathcal{S}_n$ acts on $\mathbb{K}[X_1, \dots, X_n]$ via permutation of variables. We have $(1,2) \cdot X_1 - X_2 = X_2 - X_1$. A polynomial is called *invariant* if and only if it is fixed under any element of the group. The underlying group action should be understandable from the context. For example, constant polynomials and polynomials of the form $\sum_{\sigma \in G} \sigma \cdot f(X_1, \dots, X_n)$ are invariant for every group $G$. The set of all invariant polynomials is denoted by $\mathbb{K}[X_1, \dots, X_n]^G$ and has the structure of a ring as products and sums of invariant polynomials are again invariant. Therefore, it is called the *invariant ring*. A fundamental theorem of Hilbert from 1890 states that the invariant ring of a finite group is finitely generated.
A $G$-module $V$ is called *irreducible* if and only if $V$ does not contain a linear subspace $U \notin \{\{0\}, V\}$ which is closed under the action of $G$, i.e., $\sigma \cdot u \in U$ for all $\sigma \in G$ and for all $u \in U$. For a finite group the number of irreducible representation equals the number of conjugacy classes. A fundamental theorem by Maschke is that any $G$-module decomposes into a direct sum of irreducible modules, if the characteristic of $\mathbb{K}$ is 0 [Mas98; Mas99]. The decomposition of $V$ into its direct sum of all the irreducible submodules up to $G$-isomorphism is called *isotypic decomposition*.
In applications, objects often have some inherent symmetries which can be exploited using techniques from representation theory. This can yield complexity reduction and simplifications.
Papers I-III deal with different aspects of representation theory. Paper I exploits the symmetries to reduce complexity in computations which are used in Paper II as a starting point for further theoretical investigations. In Paper III we consider the combinatorial relation of ideals which are generated by $G$-modules.

## Symmetric polynomials, partitions and Young tableaux

The study of symmetric polynomials has a solid algebraic and combinatorial foundation. We call a polynomial $f \in \mathbb{K}[X_1, \dots, X_n]$ *symmetric* if and only if $f(X_1, \dots, X_n) = f(X_{\sigma(1)}, \dots, X_{\sigma(n)})$ for any permutation $\sigma \in \mathcal{S}_n$. There exist various families of symmetric polynomials in $\mathbb{K}[X_1, \dots, X_n]$. Examples of symmetric polynomials are:

- For $1 \leq k \leq n$ the *elementary symmetric polynomial* $e_k^{(n)}(X) = \sum_{I \subset [n]:|I|=n} \prod_{i \in I} X_i$, $e_0^{(n)}(X) = 1$ and $e_k^{(n)}(X) = 0$ for $k > n$.
- For $1 \leq k \leq n$ the *power sum polynomial* $p_k^{(n)}(X) = \sum_{i=1}^{n} X_i^k$, $p_0^{(n)}(X) = 1$ and $p_k^{(n)}(X) = 0$ for $k > n$.
- For $\alpha \in \mathbb{N}^n$ the *monomial symmetric polynomial* $m_\alpha^{(n)}(X) = \sum_{\sigma \in \mathcal{S}_n} \sigma \cdot X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

Elementary symmetrics and power sums form a polynomial generator system of the ring of symmetric polynomials. Newton's identities,

$$ke_k^{(n)} = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}^{(n)} p_i^{(n)} \tag{1.1}$$

for all $k \in \mathbb{N}$, provide a polynomial map between those families [Mea92]. We observe, for a given symmetric polynomial $f(X_1, \ldots, X_{n+k})$ we can set the last $k$ variables equal to 0 and obtain a symmetric polynomial $f(X_1, \ldots, X_n, 0, \ldots, 0)$ which we regard as a symmetric polynomial in $n$ variables. Similarly, for $n \geq m$ we embed $\mathcal{S}_m \hookrightarrow \mathcal{S}_n$.

A *symmetric function* $f$ is a formal power series in countably infinitely many variables which is invariant under the action of the group $\mathcal{S}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ and for which the set of degrees of the monomials in $f$ is finite. The analogous to elementary symmetric and power sum polynomials are the power sum and elementary symmetric functions:

$$\mathfrak{e}_k = \sum_{I\subset\mathbb{N}, |I|=k} \prod_{i\in I} X_i \text{ and } \mathfrak{p}_k = \sum_{i\in\mathbb{N}} X_i^k. \tag{1.2}$$

Newton's identities (1.1) provide polynomial transition maps between the first $k$ elementary symmetric and power sum functions, for all $k \in \mathbb{N}$ and any symmetric function $f$ is a polynomial in elementary symmetric (resp. power sum) functions. The ring of symmetric functions $\mathbb{R}[X_1, X_2, \ldots]^{\mathcal{S}_\infty}$ can either be constructed as the inverse limit or the direct limit of the rings of symmetric polynomials ([Mac98, §I.2]). We usually consider $\mathbb{R}[X_1, X_2, \ldots]^{\mathcal{S}_\infty}$ as the inverse limit with respect to the transistion maps

$$\mathbb{R}[X]^{\mathcal{S}_{n+1}} \to \mathbb{R}[X]^{\mathcal{S}_n}, \ f(X_1, \ldots, X_{n+1}) \mapsto f(X_1, \ldots, X_n, 0). \tag{1.3}$$

Note for $n \geq d$ we have

$$f(X_1, \ldots, X_{n+1}) = g(p_1^{(n+1}, \ldots, p_d^{(n+1)}) \mapsto f(X_1, \ldots, X_n, 0) = g(p_1^{(n)}, \ldots, p_d^{(n)}),$$

where the power sums are in a different number of variables.

A *partition* $\lambda \vdash n$ is a sequence of nonnegative, non-increasing integers which sum equals $n$. We identify partitions whose non-zero entries are all equal. For example, we identify $(3, 2)$ and $(3, 2, 0, 0)$. Partitions are fundamental in the representation theory of the symmetric group since they correspond to the conjugacy classes and therefore to irreducible representations of $n$. The *dominance order* denoted by $\trianglelefteq$ defines a partial order on the set of partitions of $n$ via $\mu \trianglelefteq \lambda$ if and only if $\sum_{i=1}^k \mu_i \leq \sum_{i=1}^k \lambda_i$ for all $k \in \mathbb{N}$. The dominance order occurs in algebraic combinatorics and in the representation theory of the symmetric group. For instance, the only irreducible representations that occur in the permutation module $M^\mu = 1 \uparrow_{\mathcal{S}_\mu}^{\mathcal{S}_n}$ are those corresponding to a partition $\lambda \vdash n$ with $\mu \trianglelefteq \lambda$ ([Lam77, Theorem 1]).

A partition can also be uniquely represented by its diagram. A *Ferrers diagram* or a *diagram* associated with $\lambda \vdash n$ is a sequence of ordered boxes, where the $i$-th row starting from the top contains $\lambda_i$ many boxes. For instance, the diagram associated with $(4, 3, 3, 1)$ is

$$\ydiagram{4,3,3,1}\ .$$

A filling of a diagram of shape $\lambda \vdash n$ with all the integers in the set $[n] = \{1, 2, \dots, n\}$ is called a *Young tableau* or *tableau.* Those fillings were first introduced by Young in 1901 [You19]. We call a tableau *standard* if and only if the entries in every row and column are increasing. For instance, the tableau

$$\begin{ytableau} 1 & 3 & 6 & 9 \\ 2 & 5 & 7 \\ 4 & 8 & 11 \\ 10 \end{ytableau} \tag{1.4}$$

is standard. It turns out that the number of standard tableaux of shape $\lambda$ equals the dimension of its associated irreducible representation ([Sag01, Theorem 2.6.5]). Although the concept of diagrams and tableaux appears surprisingly simple these objects have a rich impact on algebraic combinatorics and representation theory (see e.g. [FF97; Sag90; Yon07]).
A *bipartition* of $n$ is a pair of partitions $(\lambda, \mu)$ such that $\lambda \vdash k$ and $\mu \vdash n - k$. Bipartitions naturally occur in the representation theory of the hyperoctahedral group $\mathcal{B}_n$. There are various generalizations of the dominance order to a partial order on bipartitions, for instance the ones introduced in [AMP81; DJM95]. In Paper III we present a partial order on the set of bipartitions which combinatorially explains the inclusion of certain associated $\mathcal{B}_n$-modules. This partial order could be seen as a generalization of the dominance order on partitions.
Analogously to the tableaux of partitions we define bitableau for bipartitions. A *Young bitableau* or *bitableau* of shape $(\lambda, \mu)$ is a pair of tableaux of shape $\lambda$ and $\mu$ such that all the integers in $[|\lambda| + |\mu|]$ occur precisely once.

## Finite reflection groups

Reflection groups have a rich and well understood algebraic, combinatorial and geometric theory (see e.g. [Hum90; Kan01; LT09]). For all integers $n$ we suppose that the real vector space $\mathbb{R}^n$ is equipped with the euclidean inner product $\langle \cdot, \cdot \rangle$. We say that a group $G$ is a *finite reflection group* if the group $G$ acts on $\mathbb{R}^n$ for some $n$ and the group of linear transformations is generated by reflections. A *reflection* is a linear map $s_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ of the form $x \mapsto x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$ for some $\alpha \in \mathbb{R}^n$. We usually just say that $G$ is a *reflection group* and mean that $G$ is a finite reflection group. A generalization to complex vector spaces $\mathbb{C}^n$ are so called *pseudoreflections*, which are linear maps $\ell \colon \mathbb{C}^n \to \mathbb{C}^n$ fixing a hyperplane pointwise and satisfying $\ell^m = 1$ for some $m \in \mathbb{N}$. A reflection group is called *essential* if and only if no linear subspace of $V$ is fixed pointwise. Every reflection

group is isomorphic to a direct product of essential reflection groups. A complete combinatorial classification of reflection groups can be given through their *Dynkin diagrams* based on a study of root systems of Lie algebras [Hum90]. It turns out that there are four infinite series of essential reflection groups $A_n, B_n, D_n, I_{2k}$ and six exceptional $H_3, H_4, F_4, E_6, E_7$ and $E_8$. We have $\mathcal{S}_n \simeq A_{n-1}$ as groups, i.e., $A_{n-1}$ is the symmetric group acting on $\mathbb{R}^n/\mathbb{R}(1,\ldots,1)$, and $B_n \simeq \{\pm 1\} \wr \mathcal{S}_n$ is the *hyperoctahedral group* which acts on $\mathbb{R}[X_1,\ldots,X_n]$ via permutation of variables and switching of signs. We write $\mathcal{B}_n$ for the hyperoctahedral group when we do not regard $B_n$ in the specific context of infinite series of reflection groups. Note that the reflections $s_\alpha$ of the symmetric group $\mathcal{S}_n$ come from those $\alpha \in \mathbb{R}^n$ with exactly two non-zero coordinates $\alpha_i = 1$ and $\alpha_j = -1$. The additional reflections of the hyperoctahedral group $\mathcal{B}_n$ are of the form $s_\alpha$ with $\alpha$ has exactly one non-zero entry $\alpha_i = 1$. The group $D_n$ is a subgroup of $\mathcal{B}_n$ of index 2 which is generated by all permutations and all the sign changes which switch an even number of signs.
Representation theory of reflection groups has some remarkable and elegant properties.
First, there is the Chevalley-Shephard-Todd theorem: the invariant ring of a group is isomorphic to a polynomial ring if and only if the group is a reflection group. The theorem was initially proven by Shephard and Todd for each essential reflection group separately [ST54] and a uniform proof was given shortly afterwards [Che55]. The polynomial generators of the invariant ring of reflection groups are not unique. However, the generators are homogeneous polynomials and the multiset of their degrees is unique. For instance, the invariant ring of the symmetric group $\mathcal{S}_n$ is a polynomial ring in the first $n$ power sums or in the elementary symmetric functions and the degrees of the generators are $1, 2, \ldots, n$.
Second, let $J_+ \subset \mathbb{R}[X_1,\ldots,X_n]$ be the ideal generated by invariant polynomials of positive degree. The *coinvariant algebra* $\mathbb{R}[X_1,\ldots,X_n]_G := \mathbb{R}[X_1,\ldots,X_n]/J_+$ inherits the structure of a $G$-module. Whereas these algebras can be defined and studied for all reductive groups, it was shown by Chevalley ([Che55, Theorem (B)]) that $\mathbb{R}[X_1,\ldots,X_n]_G \simeq \mathbb{R}[G]$, i.e., the coinvariant algebra is isomorphic to the regular representation of $G$ if and only if $G$ is reflection group.
In Paper I we use these properties of reflection group as a starting point for our investigations.

## Higher Specht polynomials

In general computing the isotypic decomposition of a $G$-module is a computationally challenging problem. However, formulas based on linear algebra are known which require knowledge of the characters of the irreducible representations ([Ser+77, Section 2.7]). Although, it is well known that for finite groups the number of irreducible representations equals the number of conjugacy classes, it seems unreasonable that there exist natural bijections between the set of conjugacy classes and characters [DJ86]. However, there are few exceptions. For instance, for the symmetric group $\mathcal{S}_n$ there exists a natural bijection between the *partitions* of $n$ and the irreducible representations.

The conjugacy classes of $\mathcal{S}_n$ are indexed by partitions. The construction of the irreducible representations was first investigated in [Spe37b]. Specht presented explicit representations in the polynomial ring. Today, the polynomials which he constructed are known as Specht polynomials. For a partition $\lambda$ and a tableau $T$ the *Specht polynomial* associated with $T$ is the product of the Vandermonde determinants of all columns of $T$ and is denoted by $\operatorname{sp}_T(X)$. For instance, the Specht polynomial associated with the tableau (1.4) equals $\operatorname{sp}_T(X) = \prod_{i<j\in\{1,2,4,10\}}(X_i - X_j)\prod_{i<j\in\{3,5,8\}}(X_i - X_j)\prod_{i<j\in\{6,7,11\}}(X_i - X_j)$. A presentation of the representation theory of the symmetric group can be found in [FF97; Mus93; Sag01]. Less known is that Specht also provided explicit representations of the irreducible representations of the hyperoctahedral group $\mathcal{B}_n$ [Spe37a]. The Specht polynomial associates with a bitableau $(T, S)$ is defined as $\operatorname{sp}_{(T,S)}(X) = \operatorname{sp}_T(X^2)\operatorname{sp}_S(X^2)\prod_{i\in S} X_i$, where $X^2 := (X_1^2, \dots, X_n^2)$.
The construction of Specht gives irreducible representations in the polynomial ring. The vector space $\langle \operatorname{sp}_T : T \text{ is a tableau of shape } \lambda\rangle_{\mathbb{R}}$ is irreducible and called the *Specht module* $\mathbb{S}^\lambda$. However, Specht's construction provides only one representation for any partition. A generalization of Specht's result to an explicit decomposition of the coinvariant algebra $\mathbb{K}[X_1, \dots, X_n]_{\mathcal{S}_n}$ was constructed in [MY98]. Morita and Yamada introduce *higher Specht polynomials* which give all irreducible representations. They provide a combinatorial algorithm and thus explicit polynomials which decompose the coinvariant algebra. The construction has been further generalized to pseudoreflection groups of type $G(r, p, n)$ [MY98]. We recover the reflection groups $\mathcal{B}_n = G(2, 1, n)$ and $D_n = G(2, 2, n)$.
Paper I uses the higher Specht polynomials to show that the isotypic decomposition stabilizes for the groups $\mathcal{S}_n$, $\mathcal{B}_n$ and $D_n$ acting in an increasing number of variables on the subspace of the polynomial ring in a fixed degree (see Theorem I.3.21). This was already known for the symmetric group ([Rie+13, Theorem 4.7.]) but their proof uses different methods.

## Specht ideals

Suppose that a group $G \in \{\mathcal{S}_n, \mathcal{B}_n\}$ acts on the polynomial ring $\mathbb{K}[X_1, \dots, X_n]$ and $V \subset \mathbb{K}[X_1, \dots, X_n]$ is an irreducible representation. Then, the ideal generated by $V$ is closed under the action of $G$. For a partition $\lambda \vdash n$ we call

$$I_\lambda = (\operatorname{sp}_T : T \text{ tableau of shape } \lambda) \subset \mathbb{K}[X_1, \dots, X_n]$$

the *Specht ideal* of $\lambda$. Analogously, we write $I_{(\lambda,\mu)}$ for the Specht ideal associated with a bipartition $(\lambda, \mu)$. The definition of a Specht ideal does not depend on the field and can also be applied in positive characteristic. The varieties of Specht ideals are intersections of hyperplane arrangements of type $A$ or $B$, i.e., the hyperplanes are the reflection hyperplanes of the groups $A_n$ and $\mathcal{B}_n$. The Specht ideals of the symmetric group have been studied in combinatorial commutative algebra and in subspace arrangements [BPS05; Bro+16]. However, applications in the solution of symmetric systems of polynomial equations exist as well [MRV21].

Woo investigated in his doctoral thesis [Woo05] among other results algebraic properties of Specht ideals. He proved that ideal inclusion of Specht ideals is encoded by dominance of their associated partitions, characterized the Specht varieties as disjoint unions of orbit sets and followed an unpublished proof of Haiman to show the radicality of Specht ideals. Moreover, Woo proved that the Specht polynomials contained in a Specht ideal form a universal Gröbner basis. It seems that his results have been unknown to a wider mathematical audience and several results have been rediscovered independently. In general, a Specht ideal is not Cohen-Macaulay. The few cases of partitions for which its associated Specht ideal is Cohen-Macaulay have been classified in [Yan21]. Free resolutions for certain types of Cohen-Macaulay Specht ideals were found in [SY20] and their regularity was examined in [SY21]. Recently, a new proof of the radicality of Specht ideals has been given [MOY22] and Gröbner fans of Specht ideals were investigated [OY22]. Motivated by algorithmic purposes in solving systems of equations the aforementioned results of Woo have been rediscovered and extended in [MRV21]. Additionally, the authors proved which Specht ideals are contained in a symmetric ideal $I$ satisfying a sparsity condition. This allows to obtain bounds on the number of different coordinates of elements contained in the affine variety of $I$.
Paper III studies the Specht ideals of the hyperoctahedral group. Notice that the $\mathcal{B}_n$-Specht ideals and their varieties form a poset with respect to inclusion. We introduce a partial order on bipartitions in Definition III.4.1 and classify the covering cases in Theorem III.4.3 which can be seen as generalization of ([Bry73, Proposition 2.3]). The partial order on bipartitions allows us to prove analogous assertions to Theorem 1 and Corollary 1 in [MRV21]. We show that the poset of Specht ideals is equivalent to the poset of bipartitions with respect to bidominance in Theorem III.5.1, introduce an orbit type for $\mathcal{B}_n$ in Definition III.6.2 and prove a decomposition of Specht varieties in Theorem III.6.6.

## 1.2 Real algebraic geometry

*Real algebraic geometry* concerns the study of solutions to polynomial equations and inequalities in real closed fields. A *real closed* field is an ordered field $(\mathcal{R}, \leq)$ whose field extension $\mathcal{R}(\sqrt{-1})$ is algebraically closed. A real closed field can be considered as a generalization of the real numbers. The subject is linked to inner mathematical areas such as analytic geometry, algebraic topology, analysis and real algebra, but also has applications towards moment problems and convex optimization. In particular, polynomial optimization problems can be solved using methods from real algebraic geometry. We refer to [BCR13; BPC07] for a comprehensive presentation of topics in real algebraic geometry.
The basic objects of study are *semialgebraic* sets, i.e., finite unions of sets of the form

$$\{x \in \mathcal{R}^n : f(x) = 0, g_1(x) > 0, \ldots, g_r(x) > 0\},$$

for polynomials $f, g_1, \ldots, g_r \in \mathcal{R}[X_1, \ldots, X_n]$. In contrast to classical algebraic geometry, the projection of a semialgebraic set is again semialgebraic which

follows by the famous Tarski-Seidenberg theorem [Sei54; Tar98]. Moreover, the following property of semialgebraic sets is a consequence of Tarski and Seidenberg's work. Let $\mathcal{R}'$ denote a field extension of $\mathcal{R}$ which is also a real closed field and $f, g_1, \ldots, g_r \in \mathcal{R}[X_1, \ldots, X_n]$. Then, the semialgebraic set $\{x \in \mathcal{R}^n : f(x) = 0, g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ is non-empty if and only if the semialgebraic set $\{x \in \mathcal{R}'^n : f(x) = 0, g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ is non-empty. We usually consider $\mathcal{R} = \mathbb{R}$ the real numbers.
*Real algebra* is the algebraic foundation of real algebraic geometry in the sense that algebraic geometry builds up on commutative algebra. For instance, there is the notation of a real radical ideal which can be seen as the analogue to the radical ideal in algebraic geometry. Let $p, g_1, \ldots, g_r \in \mathbb{R}[X_1, \ldots, X_n]$ be polynomials and $J = (g_1, \ldots, g_r) \subset \mathbb{R}[X_1, \ldots, X_n]$ denote the ideal generated by $g_1, \ldots, g_r$. Then the *real radical ideal* generated by $g_1, \ldots, g_r$ is denoted by $\mathrm{rrad}(g_1, \ldots, g_r)$ and equals

$$\{f \in \mathbb{R}[X_1, \ldots, X_n] : f^2 + \sum_{i=1}^{m} h_i^2 \in J, m \in \mathbb{N}, h_1, \ldots, h_m \in \mathbb{R}[X_1, \ldots, X_n]\}.$$

The *real Nullstellensatz* says that a polynomial $f$ is the constant 0 function on the real algebraic variety $\{x \in \mathbb{R}^n : g_i(x) = 0, 1 \leq i \leq r\}$ if and only if $f$ is contained in $\mathrm{rrad}(g_1, \ldots, g_r)$ [Mar08]. We call a polynomial $p \in \mathbb{R}[X_1, \ldots, X_n]$ *nonnegative* if and only if $p$ attains only nonnegative values, i.e., if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. Since the reals are equipped with an order one can ask for Positivstellensätze in analogy to Hilbert's Nullstellensatz over the complex numbers. Such theorems exist and can be used to provide a certificate for nonnegativity of a polynomial on a semialgebraic set. The Krivine–Stengle Positivstellensatz was proven in [Kri64] and independently rediscovered in [Ste74] which then received more recognition internationally. We write $\Sigma_n = \sum \mathbb{R}[X_1, \ldots, X_n]^2$ for the set of *sums of squares*, i.e., those polynomials that can be written as a sum of squares of polynomials. We denote by

$$P(g_1, \ldots, g_r) := \left\{ \sum_{\alpha \in \{0,1\}^r} h_\alpha g_1^{\alpha_1} \cdots g_r^{\alpha_r} : h_\alpha \in \Sigma_n \right\}$$

the *preorder* defined by $g_1, \ldots, g_r \in \mathbb{R}[X_1, \ldots, X_n]$. Then $f \geq 0$ on the semialgebraic set $\{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ if and only if there exist $q_1, q_2 \in P(g_1, \ldots, g_r), s \in \mathbb{N}$ such that

$$q_1 f = f^{2s} + q_2.$$

Sometimes, this theorem is called the *Nichtnegativstellensatz*. We can also replace $f \geq 0$ by $f > 0$ and set $s = 0$. This theorem is known as the *Positivstellensatz*. Algorithmically motivated, the search for algebraic nonnegativity certificates leads to classifications that do not use denominators. Such certificates were proven by Schmüdgen and Putinar [Put93; Sch91]. There have been several articles on bounding the degrees in Schmüdgen's and Putinar's Positivstellensätze [BM21; NS07; Sch04].

### Semidefinite programming

Let $\mathrm{Sym}^n(\mathbb{R})$ denote the set of real symmetric $n \times n$-matrices and $\mathrm{Sym}^n_+(\mathbb{R})$ denote the cone of positive semidefinite $n \times n$ matrices. The intersection of an affine halfplane with the set $\mathrm{Sym}^n_+(\mathbb{R})$ is called a *spectrahedron*. *Semidefinite programming* is a generalization of linear programming where a linear function is optimized over a spectrahedron. Although spectrahedra are more complex than polyhedra, a semidefinite program can be solved in polynomial time to a given accuracy using numerical inner-point algorithms [EOL98]. Describing the feasible sets of semidefinite programming is an ongoing research topic in applied real algebraic geometry (see e.g. [BPT12]). Several combinatorial optimization problems such as computing the Lovász number and sphere packings can be formalized as semidefinite programs [Ali95; RG95]. If the dimension is small, exact algorithms can also be applied [HNS19]. The complexity status of exact algorithms is not yet known [Ram97].

### Nonnegative polynomials and sums of squares

An application of real algebraic geometry lies in polynomial optimization where one is interested in computing the global minima of a polynomial or the constrained minima on a semialgebraic set.
A homogeneous polynomial is called a *form* and we denote the real vector space of $n$-ary forms of degree $d$ by $H_{n,d}$. We write $\mathcal{P}_{n,2d}$ for the set of nonnegative forms in $n$ variables of degree $2d$. In general, testing nonnegativity of a polynomial in more than two variables is an NP-hard problem even for quartics [Blu+98; MK85]. For instance, a priori it is not clear that $g(X) = X_1^4 - 2X_1^3X_3 + X_1^2X_3^2 + 2X_1^2X_2^2 - 2X_1X_3X_2^2 + X_2^4$ is nonnegative. However, we have $g(X) = (X_1^2 - X_1X_3 + X_2^2)^2$ which immediately shows $f \in \mathcal{P}_{3,4}$. We call a polynomial $p$ a *sum of squares* if and only if $p$ can be written as a sum of squares of polynomials. We write $\Sigma_{n,2d}$ for the set of homogeneous sums of squares in $n$ variables of degree $2d$. Since $\Sigma_{n,2d} \subset \mathcal{P}_{n,2d}$ it is natural to ask if the reverse inclusion is also true. We remark that a polynomial is nonnegative (a sum of squares) if and only if its homogenization is nonnegative (a sum of squares) [Mar08]. The sets $\Sigma_{n,2d}$ and $\mathcal{P}_{n,2d}$ are closed pointed convex cones. Thus, one can use *convex geometry* to study these sets and there dual cones $\Sigma^*_{n,2d}$ and $\mathcal{P}^*_{n,2d}$. We recall that for a set $C \subset V$ the dual cone of $C$ is the set $C^* = \{\ell \in V^* : \ell(x) = 0 \text{ for all } x \in C\}$ which is a closed convex cone.
In 1888 Hilbert showed in a remarkable paper [Hil88] that there are only very few cases of equality. Namely, if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$. Actually, $n = 2$ follows from the fundamental theorem of algebra for univariate polynomials and $2d = 2$ follows from the diagonalization of quadratic forms. The equality between the sets $\mathcal{P}_{3,4}$ and $\Sigma_{3,4}$ is more difficult to prove. Hilbert's proof was non-constructive and a constructive, but internationally not well recognized proof of Hilbert's results for $(n, 2d) = (4, 6)$ was given in [Ter39]. Despite for the simplicity of the formulation of the question it took almost 80 years until the first recognized example of a nonnegative but not sum of squares polynomial

appeared in the literature. The Motzkin polynomial [The65]

$$X_1^4X_2^2 + X_1^2X_2^4 - 3X_1^2X_2^2 + 1$$

was proven to be nonnegative and not a sum of squares. Its nonnegativity follows from the arithmetic–geometric mean inequality. That the polynomial is not a sums of squares can be proven by studying its *Newton polytope*. It turns out that if a form is a sum of squares, then it is a sum of squares of forms of half its degree. Finding nonnegative but not sums of squares polynomials has been investigated since then by various authors. The Robinson polynomial [Rob69] is an example of a form in 4 variables of degree 4. In a series of papers Choi and Lam investigated nonnegative but not sums of squares biquadratics, the cones of nonnegative and sums of squares forms, studied the extremal rays, and produced further examples of nonnegative but not sums of squares polynomials [Cho75; CL77a; CL77b]. More insight and examples have been provided in [Sch79] and [Rez89].

The 17th Problem in Hilbert's list of influential problems for mathematical development in the 20th century at the International Congress of Mathematicians in 1900 was whether every nonnegative polynomial is a sum of squares of rational functions. This was proven in [Art27] and Artin's proof set the foundations of modern real algebra. A consequence of Artin's solution is that verifiability of nonnegativity of rational polynomials is decidable. This follows from simultaneously searching for a decomposition into a sum of sums of squares of rational functions and evaluating the polynomial at rational points.

As mentioned earlier, real algebraic geometry provides fruitful methods for solving polynomial optimization problems. Let $p \in \mathbb{R}[X_1, \ldots, X_n]$ be a polynomial. Then

$$\min_{x \in \mathbb{R}^n} p(x) = \max_{c \in \mathbb{R}} p - c \text{ is nonnegative on } \mathbb{R}^n.$$

Lasserre proposed a hierarchy of semidefinite optimization programs to solve polynomial optimization problems [Las01] based on the Positivstellensätze. Using semidefinite programming it can be tested if a polynomial is a sum of squares [CKP20]. In Paper II we apply semidefinite optimization to obtain certificates for a polynomial being a sum of squares or not. Since nonnegativity is an immediate consequence of being a sum of squares it is of interest to explore the relations of the sets $\Sigma_{n,2d}$ and $\mathcal{P}_{n,2d}$. Despite the complexity in finding nonnegative polynomials which are not sums of squares Blekherman proved that for a fixed degree there are significantly more nonnegative polynomials than sums of squares [Ble06]. He measures the sizes of these cones by considering conical compact bases and comparing the ratio of their volumes. He proves that the difference between the sets of nonnegative and sums of squares forms grows in the number of variables and asymptotic bounds on the ratio of the volumes of these sets are given. It is shown that the difference between these sets grows in the number of variables. On the contrary, there are results stating that for a fixed number of variables on the hypercube $[-1, 1]^n$ the sums of squares polynomials are dense in the nonnegatives with respect to the $\ell^1$-norm [BCR76; LN07].

It turns out that Hilbert's classification does not necessarily remain valid if we restrict to invariant polynomials under the action of a group $G$. We write $\mathcal{P}^G_{n,2d}, \Sigma^G_{n,2d}$ for the intersections of $\mathcal{P}_{n,2d}, \Sigma_{n,2d}$ with the set of invariant forms. Again, the sets $\mathcal{P}^G_{n,2d}, \Sigma^G_{n,2d}$ are closed pointed convex cones in the vector space of invariant forms of degree $2d$ denoted by $H^G_{n,2d}$. We denote their dual cones by $\mathcal{P}^{G,*}_{n,2d}, \Sigma^{G,*}_{n,2d}$. Although Hilbert's classification remains valid for $G = \mathcal{S}_n$ the symmetric group [CL77a; GKR16], it does not for $G = \mathcal{B}_n$ the hyperoctahedral group. Harris proved equality for $(n, 2d) = (3, 8)$ and $(n, 4)$ for all $n$ [Har99], but in any other non-trivial case we have $\Sigma^{\mathcal{B}_n}_{n,2d} \subsetneq \mathcal{P}^{\mathcal{B}_n}_{n,2d}$ [GKR17].
In Paper I we discuss how representation theory can be exploited to determine if $\mathcal{P}^G_{n,2d} = \Sigma^G_{n,2d}$. As applications we provide a new simple proof of Harris' equality case for even symmetric ternary octics (see Corollary I.4.2) and classify all the equality cases for the reflection group $D_n$ (see Theorem I.4.26). The classification for the groups $\mathcal{B}_n$ and $D_n$ turn out to be the same.
Exploiting symmetries in semidefinite optimization problems was first investigated in [GP04] and further work on symmetries in semidefinite optimization [DS10] and polynomial optimization [Rie+13] has been done. There exist various computational applications, e.g., complexity reduction can be used to improve bounds on spherical packings with symmetries [DGV+17]. In contrast to the aforementioned work we focus on reflection groups and the higher Specht polynomials allow a uniform treatment of the infinite series of groups $(\mathcal{S}_n)_n, (\mathcal{B}_n)_n$ and $(D_n)_n$.
Due to Blekherman's findings [Ble06] that there are significantly more nonnegative forms than sums of squares for a fixed degree it is of interest to explore if this remains true for invariant polynomials when the number of variables tends towards infinity. In Paper II we study whether the sets of nonnegative and sums of squares homogeneous symmetric functions are equal. We prove that this is not the case for any non-trivial pair $(n, 2d)$, and that the same is true in the even symmetric setup (see Theorem II.5.1). We consider the ring of symmetric functions as the inverse limit of the rings of symmetric polynomials with respect to the transition maps (1.3). If we restrict to forms of degree $2d$ and $n \geq 2d$ then the transition maps induce isomorphism. For all $n \geq 2d$ the dimension of the vector space of symmetric $n$-ary forms of degree $2d$ equals the number of partitions of $2d$ and we have nested chains

$$\mathcal{P}^{\mathcal{S}_{2d}}_{2d,2d} \supset \mathcal{P}^{\mathcal{S}_{2d+1}}_{2d+1,2d} \supset \cdots \supset \bigcap_{n\geq 2d} \mathcal{P}^{\mathcal{S}_n}_{n,2d} =: \mathfrak{P}^{\mathcal{S}}_{2d} \ ;$$

$$\Sigma^{\mathcal{S}_{2d}}_{2d,2d} \supset \Sigma^{\mathcal{S}_{2d+1}}_{2d+1,2d} \supset \cdots \supset \bigcap_{n\geq 2d} \Sigma^{\mathcal{S}_n}_{n,2d} =: \mathfrak{S}^{\mathcal{S}}_{2d} \ .$$

The nestedness follows from the observation that if $f(X_1, \ldots, X_{n+1})$ is nonnegative (a sum of squares) then $f(X_1, \ldots, X_n, 0)$ is nonnegative (a sum of squares). The sets $\mathfrak{P}^{\mathcal{S}}_{2d}, \mathfrak{S}^{\mathcal{S}}_{2d}$ are called the *limit sets* of symmetric nonnegative forms and symmetric sums of squares forms of degree $2d$, which we also call the *sets of nonnegative* and *sums of squares symmetric homogeneous functions*. Analogously, we introduce the limit sets $\mathfrak{P}^{\mathcal{B}}_{2d}$ and $\mathfrak{S}^{\mathcal{B}}_{2d}$ of *even symmetric functions*.

These sets define again pointed closed convex cones in the vector space of (even) symmetric homogeneous functions of degree $2d$ (see Theorem II.2.5). The study of the limit comes with additional complexity compared to the case of finitely many variables. By Theorem II.5.8 the sets $\mathfrak{P}_{2d}^{\mathcal{S}}$ and $\mathfrak{P}_{2d}^{\mathcal{B}}$ are not semialgebraic for $2d \geq 6$, although the sets $\mathcal{P}_{n,2d}^{\mathcal{S}_{2d}}$ and $\mathcal{P}_{n,2d}^{\mathcal{B}_{2d}}$ are semialgebraic for all $n$ and all $d$. Moreover, in the *multisymmetric* setup we prove containment of elements in the limit cones is not computationally traceable, i.e., it is undecidable if a given multisymmetric function is contained in this cone (see Theorem II.6.1). This is in sharp contrast to the case of finitely many variables, where determining validity of nonnegativity of any polynomial is decidable.
A univariate real polynomial $p \in \mathbb{R}[t]$ of degree $d$ is called *hyperbolic* if $p$ has only real roots $z_1, \ldots, z_d$. We suppose $p$ is monic, then

$$p(t) = \prod_{i=1}^{d}(t - z_i) = t^d - e_1(z)t^{d-1} + e_2(z)t^{d-2} \mp \ldots + (-1)^d e_d(z),$$

where $z = (z_1, \ldots, z_d)$. Thus, the set of monic hyperbolic polynomials of degree $d$ can be identified with the set $(e_1, \ldots, e_d)(\mathbb{R}^d)$. The *Vandermonde map* in $n$ variables of degree $d$ is the function

$$\mathbb{R}^n \to \mathbb{R}^d, \quad x \mapsto (p_1(x), \ldots, p_d(x))$$

and we denote its image by $\mathcal{M}_{n,d}$. It follows from Newton's identities that the sets $(e_1, \ldots, e_d)(\mathbb{R}^n)$ and $\mathcal{M}_{n,d}$ are images of each other under a polynomial diffeomorphism for all $n \geq d$. Analogously, we define the *even Vandermonde map*

$$\mathbb{R}^n \to \mathbb{R}^d, \; x \mapsto (p_2(x), \ldots, p_{2d}(x))$$

and denote its image by $\mathcal{N}_{n,d}$. These maps are essential in Paper II. Let $\nu_d : \mathbb{R}^d \to \mathbb{R}^{\pi(d)}, (x_1, \ldots, x_d) \mapsto (x_1^d, x_1^{d-2}x_2, \ldots, x_d)$ then the dual cones to the nonnegative forms can be presented as follows

$$\mathcal{P}_{n,2d}^{\mathcal{S}_n,*} = \operatorname{cone}(\nu_{2d}(\mathcal{M}_{n,2d})) \text{ and } \mathcal{P}_{n,2d}^{\mathcal{B}_n,*} = \operatorname{cone}(\nu_d(\mathcal{N}_{n,d})) \;,$$

where $\operatorname{cone}(S) = \{\sum_{i=1}^m \lambda_i v_i : m \in \mathbb{N}, v_i \in S, \lambda_i \in \mathbb{R}_{\geq 0}\}$ denotes the *convex conical hull* of $S \subset \mathbb{R}^m$. Moreover, we study the images of *Vandermonde maps at infinity*, i.e., the sets

$$\mathcal{M}_d := \operatorname{cl}\left(\bigcup_{n \in \mathbb{N}} \mathcal{M}_{n,d}\right) \text{ and } \mathcal{N}_d := \operatorname{cl}\left(\bigcup_{n \in \mathbb{N}} \mathcal{N}_{n,d}\right),$$

and note

$$\mathfrak{P}_{2d}^{\mathcal{S},*} = \operatorname{cone}(\nu_{2d}(\mathcal{M}_{2d})) \text{ and } \mathcal{P}_{2d}^{\mathcal{B},*} = \operatorname{cone}(\nu_d(\mathcal{N}_d)).$$

The boundaries of $\mathcal{M}_{n,d}$ and $(e_1, \ldots, e_d)(\mathbb{R}^n)$ have been studied by various authors [Arn86; Giv87; Kos89; Kos99; Meg92]. The elements $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ that are mapped to the boundary of $\mathcal{M}_{n,d}$ are uniquely defined by their *multiplicity vectors* of equal coordinates (see [Kos89, Theorem 1.14]). Since the Vandermonde map is symmetric, we can restrict to points $x \in \mathbb{R}^n$ with $x_1 \leq x_2 \leq \ldots \leq x_n$. Similarly, we characterize the boundary of $\mathcal{N}_{n,d}$ in terms of the multiplicity vector of elements that are mapped to the boundary in Theorem II.3.6. We explicitly parameterize the planar boundary of $\mathcal{N}_{n,3} \cap \{p_2 = 1\}$ (see Theorem II.3.7) and $\mathcal{N}_3 \cap \{p_2 = 1\}$ (see Corollary II.3.10). Kostov investigated the set $\widetilde{\mathcal{M}}_d = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}(e_1, \ldots, e_d)(\mathbb{R}^n)\right)$ and focused on explicit parametrizations of the boundary for degree $d = 4$ [Kos04; Kos07]. He motivates his study since the set $\widetilde{M}_d$ can be considered as the closure of the set of monic univariate polynomials of degree $d$ which can be extended to a hyperbolic polynomial. A monic univariate polynomial $p \in \mathbb{R}[t]$ of degree $d$ can be *extended to a hyperbolic polynomial* if there exists a nonnegative integer $k$ and a polynomial $q \in \mathbb{R}[t]$ of degree $\leq k-1$ such that $p(t)t^k + q(t)$ is hyperbolic. Using Kostov's results on degree 4 and analysing the extremal rays of the cone $\mathfrak{S}_4^{\mathcal{S},*}$ we prove $\mathfrak{S}_4^{\mathcal{S}} \subsetneq \mathfrak{P}_4^{\mathcal{S}}$ (see Theorem II.5.3). Moreover, we construct test sets to check if a symmetric homogeneous function of degree 4 is nonnegative or a sum of squares (Theorem II.5.4) and provide a uniform example of a nonnegative but not sum of squares symmetric polynomial in any number of variables. We show in Theorem II.5.5 that for all $n \geq 4$ we have

$$4p_1^4 - 5p_2p_1^2 - \frac{139}{20}p_3p_1 + 4p_2^2 + 4p_4 \in \mathcal{P}_{n,4}^{\mathcal{S}_n} \setminus \Sigma_{n,4}^{\mathcal{S}_n}.$$

The question of nonnegativity versus sums of squares for limits of (even) symmetric forms has been studied before in a related version. Namely, through a slight modification of the transition maps (1.3) to

$$\frac{1}{n+1}\sum_{i=1}^{n+1} X_i^k \mapsto \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

we obtain the ring of *normalized symmetric functions* as the inverse limit. In contrast to symmetric functions we note that normalized symmetric functions can be evaluated at the all one vector $(1, 1, \ldots)$. It was observed in [BR21] that the normalized symmetric limit sets are equal in degree 4 and the authors conjectured that this is true for all degrees. This was proven to be false in [AB22] not only for the symmetric group and degree $2d \geq 6$ but also in the even symmetric setup and degree $2d \geq 10$.

## 1.3 Tropicalization

Tropicalization is usually used in algebraic geometry to replace an algebraic variety by its "combinatorial shadow". On the one hand tropicalization loses information but keeps combinatorial structures which may be simpler to investigate. Tropicalization has applications in several fields of mathematics such as algebraic geometry [MS21], intersection theory [AR10], moduli spaces [Cav+16; Uli15], matroid theory and Hodge theory [Huh16]. We refer to [Bru+15;

DS04; MS21] for an introduction to tropical geometry.
We use the definition of tropicalization as *logarithmic limits*. Log-limits have first been studied in [Ber71] for complex algebraic varieties and were introduced on semialgebraic sets in [Ale13]. For a set $S \subset \mathbb{R}^n_{\geq 0}$ we define its *tropicalization* as

$$\operatorname{trop}(S) = \lim_{t \to \infty} \{(\log_t(x_1), \dots, \log_t(x_n)) : x = (x_1, \dots, x_n) \in S \cap \mathbb{R}^n_{>0}\}.$$

Alessandrini proved that $\operatorname{trop}(S)$ is always a closed cone ([Ale13, Proposition 2.2]). More general, tropicalization can also be defined using valuations on the field. We write $\overline{\mathbb{R}} = (\mathbb{R} \uplus \{-\infty\}, \oplus, \otimes)$ for the *tropical semiring* with tropical addition $a \oplus b := \max\{a, b\}$ and tropical multiplication $a \otimes b := a + b$. The natural element with respect to addition is $-\infty$ and the natural element with respect to multiplication is 0. We refer to [DS04] for background information.
The study of tropicalizations of convex cones which are contained in the nonnegative orthant was investigated in [Ble+22b] to study problems in extremal combinatorics. This was further developed in [BR22] to study binomial inequalities of graph homomorphisms. Moreover, tropicalization has been applied in real algebra to study the sets of nonnegative and sums of squares polynomials, and their dual cones. [Ble+22a] concerns the study of truncated moments and pseudomoments on semialgebraic sets. The authors show that tropicalization provides new insights into limitations of sums of squares approximations of nonnegative polynomials. Tropicalization is also used in [AB22] to study the nonnegativity versus sums of squares question for normalized limits of symmetric and even symmetric forms. In [AB22; Ble+22b; BR22] the authors focus on convex cones which satisfy the Hadamard property. A set $S \subset \mathbb{R}^n$ has *Hadamard property* if it is closed under coordinatewise multiplication of elements in $S$. For instance, a spectahedron defined as the positive semidefinite locus of a symmetric matrix whose coefficients are monomials has Hadamard property. Tropicalizations of sets with Hadamard property have a nice structure, since they are a closed convex cone ([Ble+22b, Lemma 2.1]). The *tropical convex hull* of a set $S \subset \mathbb{R}^n$ is defined as

$$\operatorname{tconv}(S) := \{a_1 \odot x_1 \oplus \ldots \oplus a_l \odot x_l : l \in \mathbb{N}, a_1, \dots, a_l \in \mathbb{R}, x_1, \dots, x_m \in S\}.$$

A set is called *tropical convex* if it equals its tropical convex hull. Thus, any tropical convex set $S \subset \mathbb{R}^n$ contains the all one vector $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ in its linearity space, i.e., $s + \lambda \mathbf{1} \in S$ for all $s \in S$ and all $\lambda \in \mathbb{R}$. In tropical convex geometry one usually studies the quotients of tropical convex sets in $\mathbb{R}^n / \mathbb{R}\mathbf{1}$.
In Section 7 in Paper II we apply *tropical convex geometry* to study the limit sets of even symmetric nonnegatives and sums of squares which is an approach independently from the one used in the other sections in the paper. However, tropicalization allows quantification of the difference of the sets of nonnegative and sums of squares even symmetric limit forms. We show that the minimal degree for which $\operatorname{trop}(\mathfrak{P}^{\mathcal{B},*}_{2d})$ and $\operatorname{trop}(\mathfrak{S}^{\mathcal{B},*}_{2d})$ are different is $2d = 10$ (see Theorem II.7.1), although the sets are already different for $2d \in \{6, 8\}$. More general, we show how $\operatorname{trop}(\mathfrak{S}^{\mathcal{B},*}_{2d})$ (see Lemma II.7.3) and $\operatorname{trop}(\mathfrak{P}^{\mathcal{B},*}_{2d})$ (see Proposition

II.7.23) can be computed. Lemma II.7.3 can be seen as an expansion of ([Ble+22b, Theorem 4.4.]) for spectrahedra defined by a matrix whose entries are monomials. Computing $\mathrm{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*})$ is challenging and we make a detour over $\mathcal{N}_d$. We show that the set $\mathcal{N}_d$ has Hadamard property. So we can use many of the techniques in [Ble+22b; BR22]. Although the description of the image of the Vandermonde map at infinity is challenging, we prove that $\mathrm{trop}(\mathcal{N}_d)$ is a rational polyhedral cone whose defining linear inequalities can be given uniformly in the number of variables (see II.7.2). The defining linear inequalities actually arise from two families of binomial inequalities in power sums. In general it is not known if

$$\mathrm{trop}(\mathrm{cone}(S)) = \mathrm{tconv}(S) \tag{1.5}$$

for all sets $S \subset \mathbb{R}^n_{\geq 0}$. (1.5) is known for semialgebraic sets ([AGS19, Lemma 8]), but since we want to apply (1.5) to $S = \nu_d(\mathcal{N}_d)$ which is not semialgebraic, we prove the technical Proposition II.7.13 which shows (1.5) is also true for sets $S$ having Hadamard property. This allows us to deduce the computational description of $\mathrm{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*})$. The strategy of decomposing the tropicalization was already applied in [AB22] for normalized limits. A similar question to (1.5) has been investigated in [HLS19], where the authors prove that tropical convex hull and ordinary convex hull commute in two dimensions but not in higher dimensions.
The extremal rays in $\mathrm{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*})$ that are not contained in $\mathrm{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ correspond to inequalities in power sum symmetric functions. It was observed in [AB22] that such inequalities can be used to produce examples of uniform nonnegative polynomials which are not sums of squares for sufficiently large number of variables. Such a polynomial is given by $\frac{1}{18}p_{(2^5)} + 3p_{(8,2)} + 6p_{(6,4)} - 3p_{(6,2^2)}$ (see Proposition II.7.26).

## 1.4 Summary of Papers

**Paper I** analysis the description of the sets of sums of squares of forms which are invariant under the action of a finite group. In contrast to the general situation investigated in [GP04], we focus on the action of a reflection group. Using representation theory we are able to use the symmetry inherent in the convex cones to give more efficient descriptions. In the cases of $A_n, B_n$ and $D_n$ we use higher Specht polynomials [MY98] to prove complexity bounds and uniform descriptions for a fixed degree. Namely, we prove stabilization of the isotypic decomposition of $H_{n,d}$ for a fixed degree and increasing number of variables (Theorem I.3.21). Such a stabilization was already known for the symmetric group ([Rie+13, Theorem 5.5]) but proved with different methods. As a consequence we obtain a uniform representation of the invariant sums of squares for a fixed degree (Corollary I.3.22) and we present explicit calculations as applications. We give an elementary proof of Harris' result on ternary octics ([Har99, Theorem 4.1.]) (see Corollary I.4.2) and classify the nonnegativity versus sums of squares question for the group $D_n$ (Theorem I.4.26). Finally, we study smallish cones of invariant nonnegative forms. In general, the set of nonnegative forms cannot be represented as a projection of a higher dimensional spectrahedron whenever the set does not equal the cone of sums of squares ([Sch18, Corollary 4.25]). We show that there are cases such that the set of invariant nonnegative forms can be represented as a projection of a higher dimensional spectrahedron, although the set does not equal the cone of invariant sums of squares (see Theorem I.4.28).

**Paper II** discusses various aspects of symmetric nonnegative and sums of squares forms uniformly in the number of variables and for countably infinitely many variables. We describe the boundary of the even Vandermonde map (Theorems II.3.6 and II.3.7 and Corollary II.3.10) which provides analogous results to [Arn86] in finitely many variables and [Kos04; Kos07] at infinity. We analyse the convex hull of the map of elementary symmetrics on the probability simplex (Theorem II.4.1 and Corollary II.4.3) and explain how the convex hull relates to test sets (Corollary II.4.8). This generalizes and geometrically explains the test set in [CLR87] for even symmetric sextics. Additionally, we discuss the combinatorial properties of the boundary, classify the nonnegativity versus sums of squares question for (even) symmetric homogeneous functions (Theorem II.5.1) and provide explicit examples (see Theorem II.5.5). Moreover, we prove that determining validity of nonnegativity for multisymmetric homogeneous functions on copies of the probability simplex is undecidable (Theorem II.6.1) which can be followed from [HN11] on undecidability of linear inequalities in graph homomorphism densities. Finally, we present another approach to study the set of nonnegative even symmetric homogeneous functions using tropicalization. The tropicalization of the image of the even Vandermonde map has a simple description (see Theorem II.7.2) which allows us to compare the tropicalizations of the dual cones to even symmetric homogeneous nonnegative and sums of squares functions. Therefore, we provide several technical statements about tropical convex sets and sets with Hadamard property.

**Paper III** studies the Specht ideals of the hyperoctahedral group $\mathcal{B}_n$. The Paper generalizes [MRV21] and shows that analogous statements to the combinatorial and complexity theoretical statements on the symmetric group and its Specht ideals exist for the hyperoctahedral group, as well. The bipartitions of $n$ naturally encode the irreducible representations of $\mathcal{B}_n$. We present a partial order on bipartitions of $n$ (see Definition III.4.1) which captures the poset structure of Specht ideal and variety inclusion (Theorem III.5.1). We classify all the covering relations in the poset of bipartitions (Theorem III.4.3) as in ([Bry73, Proposition 2.3]) for the poset of partitions, and investigate other combinatorial properties which are known for the poset of partitions with respect to dominance order. We introduce the notation of a $\mathcal{B}_n$-orbit of an element (Definition III.6.2) and present a set decomposition of the Specht varieties based on the partial order and the orbit type (Theorem III.6.6). Finally, we present applications to $\mathcal{B}_n$-invariant ideals. We bound the dimension of the coordinate ring (Theorem III.7.2) and the structure of elements in the affine variety based on a sparsity condition (Corollary III.7.3).

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# Papers

Paper I

# Reflection groups and cones of sums of squares

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I

**Abstract**

We consider cones of real forms which are sums of squares and invariant under a (finite) reflection group. Using the representation theory of these groups we are able to use the symmetry inherent in these cones to give more efficient descriptions. We focus especially on the $A_n$, $B_n$, and $D_n$ case where we use so-called higher Specht polynomials to give a uniform description of these cones. These descriptions allow us, to deduce that the description of the cones of sums of squares of fixed degree $2d$ stabilizes with $n > 2d$. Furthermore, in cases of small degree, we are able to analyze these cones more explicitly and compare them to the cones of nonnegative forms.

## Contents



## I.1 Introduction

A real form (homogeneous polynomial) $f \in \mathbb{R}[X_1, \ldots, X_n]$ is called a *sum of squares* if it admits a representation in the form $f = f_1^2 + \ldots + f_m^2$ for some real forms $f_1, \ldots, f_m \in \mathbb{R}[X_1, \ldots, X_n]$ and it is called *positive semidefinite* or *nonnegative* if it assumes only nonnegative values on $\mathbb{R}^n$. We will denote by $\Sigma_{n,2d}$ the cone of sums of squares forms in $n$ variables of degree $2d$ and by $\mathcal{P}_{n,2d}$

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the corresponding cone of nonnegative forms. Clearly, every sum of squares is also nonnegative, and we therefore have the inclusion $\Sigma_{n,2d} \subset \mathcal{P}_{n,2d}$. Hilbert [Hil88] addressed and solved the question to characterize the cases, when the two cones coincide. As it turns out this only seldom happens, namely only in the case of bivariate forms ($n = 2$), quadratic forms ($2d = 2$), and ternary quartics ($n = 3, 2d = 4$). Sums of squares play a fundamental role in real algebraic geometry and have in the last two decades become also a very important tool for polynomial optimisation (see for example [Sch09]). Several authors have considered situations in which one supposes that the forms are invariant under the action of a group: For a group $G \subset \mathrm{Gl}_n(\mathbb{R})$ we denote by $\mathcal{P}^G_{n,2d}$ and $\Sigma^G_{n,2d}$ the invariant forms in the respective cones. Since this additional requirement can shrink the dimensions of the cones, their study may become more tractable. Furthermore, as presented in [GP04], representation theory of groups can be particularly used to simplify the sums of squares decomposition. Building on this, it was found in [Rie+13; Rie11] that sums of squares invariant under the symmetric group are highly structured, and the complexity of a sum of squares decomposition in this case stabilizes with $n \geq 2d$. Furthermore, symmetric sums of squares appear quite naturally in various contexts (for example [Ray+18]). This makes these cones an interesting object of study. Choi and Lam [CL77] initiated a systematic study of Hilbert's classification restricted to the case of symmetric forms, and in a collaboration with Reznick they further provided a complete study of the cone of even symmetric sextics [CLR87]. Whereas they could show that in the sextic case there exists a form which is nonnegative but not a sum of squares Harris [Har99], who studied the case of even symmetric octics, was able to show that the cones of even symmetric ternary octics that are sums of squares coincides with the nonnegative cone. Recently, Goel, Kuhlmann and Reznick [GKR17] constructed even symmetric polynomials of every degree $2d > 8$ and every number of variables $n \geq 3$ which are nonnegative but not a sum of squares, so for even symmetric forms Harris' example and the quartics in any number of variables remain the only exceptional cases compared to Hilbert's classification. Despite the classical case analysis done by Hilbert, it can also be interesting to study the quantitative comparison of sums of squares on nonnegative polynomials in an asymptotic situation, i.e., when the number of variables grows to infinity. In contrary to the general situation, where for large numbers of variables almost every nonnegative form is not a sum of squares (see [Ble06]), a detailed analysis of the symmetric sum of squares cone and symmetric nonnegative cone in [BR21] showed that this is not the case in the symmetric case and that in particular in the quartic case the two cones coincide in the limit.

In this article, we study further the previously mentioned lines of research by focusing on the situation of sums of squares invariant under some families of finite real reflection groups $G \subset \mathrm{Gl}_n(\mathbb{R})$. Such groups are generated by a set of orthogonal reflections across hyperplanes passing through the origin. The invariant theory of these groups is well understood and generalizes the theory of symmetric polynomials. Therefore, our setup provides a natural unification and extension to the previously mentioned works on symmetric and even symmetric forms.

**Outline of the article and contributions:** The beginning of the next section gives a short general introduction to the machinery of symmetry reduction for sums of squares based on linear representation theory. In the case of finite reflection groups these techniques combined with results from invariant theory, and in particular the coinvariant algebra and harmonic polynomials, allow for a concrete description of the qudratic module of invariant sums of squares in Theorem I.2.23. The results we give in this second section are similar to previous works, notably [BR21; DGV+17; GP04; Val09].

Section I.3 then turns to the special situation of the three infinite families $A_n$, $B_n$ and $D_n$ of irreducible reflection groups for which we can integrate the notion of the higher Specht polynomials [ATY97; MY98] with the previously mentioned techniques. These polynomial allow for a convenient way to combinatorially describe an isotypic decomposition of the coinvariant algebra in the case of finite reflection groups whose irreducible components fall to the classes $A_n, B_n, D_n$ (see Theorem I.3.7). As we show in Theorem I.3.10 this combinatorial description then in turn implies a concrete characterization of the cone of invariant sums of squares. In particular, we show in Theorem I.3.21 that if the degree $2d$ is fixed and the number of variables $n$ is growing, a stabilization of the isotypic decomposition and a resulting combinatorial stabilization of the structure of the cone of invariant sums of squares is happening in the case of all three families.

Building on these general results we study the cone of even symmetric (i.e., $B_n$-invariant) forms of degree 8 in more detail in Subsection I.4.1. In Theorem I.4.1 we obtain an explicit description of the dual cone of even symmetric ternary octics. As one application of this result we are able to revisit the remarkable finding of Harris, which follow immediately from our description. Furthermore, we provide a complete description of the cone of even symmetric octic sums of squares for all number of variables in Theorem I.4.15. Following our discussion of even symmetric forms we turn to forms that are $D_n$-invariant in Subsection I.4.2. We first show in addition to the case of even symmetric ternary quartics also all ternary quartics invariant by the slightly smaller group $D_3$ are positive semidefinite if and only if they can be written as a sum of squares (see Theorem I.4.18). We then examine the dual cone of $D_4$-invariant quartic sums of squares in Theorem I.4.22, which turns out to be simplicial. Similarly to our approach in the even symmetric case this yields in particular that every $D_4$-invariant quarternary quartic nonnegative form is a sum of squares. These results allow us to conclude a complete charaterization of the cases in which for $D_n$-invariant forms we have an equality between the cones of sums of squares and nonnegative forms (see Theorem I.4.26). To conclude our considerations, we highlight some connections to nonnegativity testing of forms with the help of semidefinite programming in the last subsection. It follows from recent work of Scheiderer [Sch18] that the cone of nonnegative forms in general is not a so called spectrahedral shadow, i.e., it can in general not be represented as a feasibility set of semidefinite programming. In contrast to this result, we observe that additionally to the cases where the cone of invariant sums of squares coincides with the corresponding cone of nonnegative forms, there are cases where we can represent the cone of nonnegative forms by projections of sets defined by linear matrix inequalities.

## I.2 Invariant sums of squares

### I.2.1 General symmetry reduction

Let $X = (X_1, \ldots, X_n)$ always denote a tuple of variables and write $\mathbb{R}[X] = \mathbb{R}[X_1, \ldots, X_n] = \bigoplus_{d \in \mathbb{N}_0} H_{n,d}$ for the polynomial ring in these variables, where $H_{n,d}$ denotes the subspace of forms of degree $d$. Let $G \subset \mathrm{Gl}_n(\mathbb{R})$ be a finite group acting linearly on $\mathbb{R}^n$. This action then naturally gives rise to an action of $G$ on the polynomial ring $\mathbb{R}[X_1, \ldots X_n]$ and thus we can view this $\mathbb{R}$-vector space as a $G$-module. It follows from Maschke's theorem that this $G$-module is completely reducible, and thus for any degree $d$ there exists an isotypic decomposition, i.e., the $G$-module $H_{n,d}$ decomposes into a direct sum of the form

$$H_{n,d} \;=\; V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(h)} \tag{I.1}$$

with $V^{(j)} = \theta_1^{(j)} \oplus \cdots \oplus \theta_{\eta_j}^{(j)}$ and $\vartheta_j := \dim \theta_i^{(j)}$, where $\theta_{i_1}^{(u)}, \theta_{i_2}^{(v)}$ are $G$-isomorphic if and only if $u = v$ i.e., we denote by $\eta_j$ the *multiplicity* of an irreducible $G$-module and by $\vartheta_j$ its *dimension*. Here, the $\theta_i^{(j)}$ are the *irreducible components* and the $V^{(j)}$ are the *isotypic components*, i.e., the direct sum of isomorphic irreducible components. The component with respect to the trivial irreducible representation in $\mathbb{R}[X]$ is the invariant ring $\mathbb{R}[X]^G$. In general, an irreducible representation $\theta_i^{(j)}$ will occur with infinite multiplicity in $\mathbb{R}[X]$. Any irreducible representation $\theta$ occurs $\dim \theta$ many times in the regular representation $\mathbb{R}[G]$ of $G$, i.e., $\vartheta = \eta$ for a representation $\theta$ in $\mathbb{R}[G]$. For $f \in \mathbb{R}[X]$ we write $\langle f \rangle_G$ for the $G$-module which is the linear span of $\{\sigma f : \sigma \in G\}$.

It is classically known that $\mathbb{R}[X]^G$ is a finitely generated $\mathbb{R}$-algebra, and furthermore each isotypic component in $\mathbb{R}[X]$ is a finitely generated $\mathbb{R}[X]^G$-module (see [Sta79, Theorem 1.3]). These properties follow for finite groups from the existence of a linear projection onto $\mathbb{R}[X]^G$, called the Reynolds-Operator.

**Definition I.2.1.** For a finite group $G$ the linear map

$$\begin{array}{rcl} \mathcal{R}_G : H_{n,d} & \longrightarrow & H_{n,d}^G \\ f & \longmapsto & \frac{1}{|G|} \sum_{\sigma \in G} \sigma(f) \end{array}$$

is called the *Reynolds operator* of $G$.

*Remark* I.2.2. Although we restrict to finite groups, most of the theory presented in this section can be directly translated to the more general setup of reductive groups.

An important tool for the study of invariant sums of squares is Schur's lemma, which we include for the convenience of the reader.

**Lemma I.2.3** (Schur's lemma)**.** *Let $\mathbb{K}$ be a field which is algebraically closed and $V$ be a $G$-module defined over $\mathbb{K}$. Further, let $\mathcal{V}, \mathcal{W}$ denote two irreducible $G$-submodules of $V$. Then the $G$-module* $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ *of $G$-homomorphism between*

*$\mathcal{V}$ and $\mathcal{W}$ satisfies* $\mathrm{Hom}_G(\mathcal{V},\mathcal{W}) \simeq \mathbb{K}$ *if and only if $\mathcal{V}$ and $\mathcal{W}$ are $G$-isomorphic. Otherwise, we have* $\mathrm{Hom}_G(\mathcal{V},\mathcal{W}) = 0$.

*Remark* I.2.4. In the sequel, we will mostly work with $G$-modules defined over the real numbers. In this setup, one devotes some care to the fact that irreducible representations defined over the reals may be reducible over the complex numbers. This additional difficulty is in fact not hard to overcome and, in particular, in the case of real reflection groups, which are the main focus of this work, all complexifications of real irreducible $G$-modules remain irreducible [Hum90].

Let $\mathcal{V} = \langle f_1 \rangle_G$ be irreducible. As a consequence of Schur's lemma, we obtain that any $G$-homomorphism $\phi \in \mathrm{Hom}_G(\mathcal{V},\mathcal{W})$ is uniquely defined by $f_2 := \phi(f_1)$. If further $\phi \neq 0$ then for any $\psi \in \mathrm{Hom}_G(\mathcal{V},\mathcal{W})$ we have $\psi = \lambda\phi$ for a scalar $\lambda \in \mathbb{K}$. This motivates the following:

**Definition I.2.5.** Let $V$ be a finite dimensional $G$-module with isotypic decomposition

$$V = \bigoplus_{j=1}^{l} \bigoplus_{i=1}^{\eta_j} \theta_i^{(j)}$$

and $f_{ji} \in \theta_i^{(j)}$ be such that for every $j$ each $f_{ji}$ is the image of $f_{j1}$ under a $G$-isomorphism. Then $(f_{11}, \ldots, f_{1\eta_1}, f_{21}, \ldots, f_{l\eta_l})$ is called a *symmetry adapted basis* of $V$.

We point out that while a symmetry adapted basis of a $G$-module is usually not a vector space basis, a system of linear generators is given by its $G$-orbit.
For a $\mathbb{R}$-vector space $W$ we write $\sum W^2$ for the sums of squares of elements in $W$. Note, an invariant polynomial which can be expressed as a sum of squares in the ring $\mathbb{R}[X]$ will not necessarily have a sum of squares decomposition in invariant polynomials, i.e.,

$$\mathbb{R}[X]^G \bigcap \sum \mathbb{R}[X]^2 \neq \sum (\mathbb{R}[X]^G)^2.$$

For instance, the symmetric polynomial $X_1^2 + X_2^2$ cannot be a sum of squares of symmetric polynomials of degree 1.
By integrating the idea of a symmetry adapted basis together with Schur's lemma, one arrives at the following observation more or less directly (see also [BR21; CKS09; GP04; Rie+13] for more details on the following statement).

**Theorem I.2.6.** *Let $\{f_{11}, f_{12}, \ldots, f_{l\eta_l}\}$ be a symmetry adapted basis of the $G$-module $H_{n,d}$ of forms of degree $d$. Then any $G$-invariant sum of squares form in $H_{n,2d}^G$ is contained in the set*

$$\sum_{j=1}^{l} \mathcal{R}_G \left( \langle f_{j1}, \ldots, f_{j\eta_j} \rangle_{\mathbb{R}}^2 \right)$$

In some situations, it is convenient to formulate Theorem I.2.6 in terms of matrix polynomials, i.e., matrices with polynomial entries. For two $k \times k$

symmetric matrices $A$ and $B$ we define their inner product as $\langle A, B\rangle = \mathrm{Tr}(AB)$. We define a block-diagonal symmetric matrix $B$ with $j$ blocks $B^{(1)}, \ldots, B^{(j)}$ and

$$B^{(j)} = (\mathcal{R}_G(f_{ju} \cdot f_{jv}))_{u,v}. \tag{I.2}$$

Then Theorem I.2.6 is equivalent to the following statement:

**Corollary I.2.7.** *$g \in \Sigma^G_{n,2d}$ if and only if $g = \langle A_1, B^{(1)}\rangle + \ldots + \langle A_l, B^{(l)}\rangle$ for some $A_j \in \mathbb{R}^{\eta_j \times \eta_j}$ symmetric and positive semidefinite matrices.*

### I.2.2 Representation theory of finite reflection groups

The aim of this subsection is to provide an introduction to the representation theory of finite real reflection groups and how their symmetry can be exploited to reduce complexity in calculations. The presented material is mainly based on work in [BR21; DGV+17; GP04; Rie+13].

**Definition I.2.8.** A *real reflection group* is a pair $(G, \rho)$, where $G$ is a finite group and $\rho : G \to \mathrm{Gl}_n$ a linear representation of $G$ such that $\rho(G)$ is generated by a set of reflections. A reflection group is *essential* if $\mathbb{R}^n$ does not contain a non-trivial $G$-submodule.

Usually, we just say that a group $G$ is a *reflection group* and the relevant linear map $\rho$ should be understood from the context.

**Example I.2.9.**

(i) The symmetric group $\mathfrak{S}_n$ on $n$ letters is a reflection group acting via coordinate permutation on $\mathbb{R}^n$. The action of $\mathfrak{S}_n$ on $\mathbb{R}^n$ is not essential, as the linear subspace $\mathbb{R} \cdot (1, \ldots, 1)$ is fixed point wise. The induced action of $\mathfrak{S}_n$ on $\mathbb{R}^n/\mathbb{R} \cdot (1, \ldots, 1)$ is known as the reflection group of type $A_{n-1}$ and is essential.

(ii) The symmetry group of the regular $m$-gon is a reflection group denoted by $I_2(m)$ and called *dihedral group*.

*Remark* I.2.10. Any real reflection group can be identified with a direct product of essential reflection groups. The essential real reflection groups have been classified and are precisely the infinite series $A_{n-1}, B_n, D_n, I_2(m)$ and the six exceptional reflection groups $E_6, E_7, E_8, F_4, H_3, H_4$ (see e.g. [Hum90]).

The reflection group of type $B_n$ can be identified with the hyperoctahedral group $\mathfrak{S}_2 \wr \mathfrak{S}_n$ acting on $\mathbb{R}^n$ via sign changing and permutation of coordinates. Then $B_n$ is generated by the reflections at $\{x_i = \pm x_j\}$, for $1 \leq i \leq j \leq n$. Furthermore, $D_n$ can be identified with the subgroup of $B_n$ of index 2, generated by the reflections at $\{x_i = \pm x_j\}$, for $1 \leq i < j \leq n$. $D_n$ is the group of "even sign changes".

**Theorem I.2.11** (Chevalley-Shephard-Todd)**.** *Let $G$ be a finite group and let $G$ act linearly on $\mathbb{R}^n$. Then the invariant ring $\mathbb{R}[X]^G$ is as $\mathbb{R}$-algebra isomorphic to a polynomial ring if and only if $G$ is a real reflection group. Moreover, in this* *case* $\mathbb{R}[X]^G$ *is generated by* $n$ *algebraically independent forms* $\psi_1, \ldots, \psi_n$*, i.e.,* $\mathbb{R}[X]^G = \mathbb{R}[\psi_1, \ldots, \psi_n]$.

While the generators are not unique but well explored (e.g., the elementary symmetric or the power sum polynomials are generators of $\mathbb{R}[X]^{\mathcal{S}_n}$), the multisets of their degrees $\{d_1, \ldots, d_n\}$ are unique and $\prod_i d_i = |G|$ (see e.g. [Hum90] for further details).

**Definition I.2.12.** Let $G$ be a reflection group which acts linearly on $\mathbb{R}^n$ and $\mathbb{R}[X]^G = \mathbb{R}[\psi_1, \ldots, \psi_n]$. The forms $\psi_1, \ldots, \psi_n$ are the *fundamental invariants* of $G$. Let $(d_1, \ldots, d_n)$ be the ordered *sequence of degrees* of the fundamental invariants. We define

$$N_G(k) := |\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \,:\, \alpha_1 d_1 + \ldots + \alpha_n d_n = k\}|.$$

With this definition the following is a direct consequence of Theorem I.2.11.

**Corollary I.2.13.** *Let* $G$ *be a finite reflection group. The dimension of the vector space of* $G$*-invariant forms of degree* $d$ *equals* $N_G(d)$*, i.e.,* $\dim H_{n,d}^G = N_G(d)$.

**Example I.2.14.**

(i) $\mathbb{R}[X]^{\mathfrak{S}_n} = \mathbb{R}[e_1, e_2, \ldots, e_n] = \mathbb{R}[p_1, p_2, \ldots, p_n]$, where $e_j(X) := \sum_{I \subset [n] : |I| = j} \prod_{i \in I} X_i$ are the elementary symmetric and $p_j(X) := \sum_{i=1}^n X_i^j$ are the power sum polynomials.

(ii) $\mathbb{R}[X]^{B_n} = \mathbb{R}[e_1(X^2), e_2(X^2), \ldots, e_n(X^2)] = \mathbb{R}[p_2, p_4, \ldots, p_{2n}]$, where $X^2 := (X_1^2, \ldots, X_n^2)$.

(iii) $\mathbb{R}[X]^{D_n} = \mathbb{R}[p_2, p_4, \ldots, p_{2n-2}, e_n]$.

(iv) $\mathbb{R}[X]^{I_2(m)} = \mathbb{R}[X_1^2 + X_2^2, (X_1 + \sqrt{-1}X_2)^m + (X_1 - \sqrt{-1}X_2)^m]$.

*Remark* I.2.15. For $\lambda := (\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l$ we write $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_l}$ for the $l$ products of the power sums $p_{\lambda_i}$ and analogously $e_\lambda$ for the products of elementary symmetrics.

From a computational perspective, invariant theory as outlined above can be used to reduce computations for polynomials in $\mathbb{R}[X]$ to the smaller ring $\mathbb{R}[X]^G$. Since $\mathbb{R}[X]$ is in general a finite $\mathbb{R}[X]^G$-module, the quadratic module $\mathbb{R}[X]^G \bigcap \sum \mathbb{R}[X]^2$ can be described conveniently. We outline this in the case of reflection groups below by using the coinvariant algebra and a theorem of Chevalley.

**Definition I.2.16.** The quotient algebra of the polynomial ring modulo the ideal generated by the non-constant elements of the invariant ring is called the *coinvariant algebra* of $G$ and is denoted by $\mathbb{R}[X]_G$, i.e.,

$$\mathbb{R}[X]_G = \mathbb{R}[X] / \left(\psi_1, \ldots, \psi_n\right)_{\mathbb{R}[X]}.$$

Note, by definition the coinvariant algebra of $G$ has the structure of a $G$-module.

**Theorem I.2.17** ([Che55]). *Let $G$ be a real reflection group acting linearly on $\mathbb{R}^n$. Then the coinvariant algebra $\mathbb{R}[X]_G$ is as $G$-module isomorphic to the regular representation and*

$$\mathbb{R}[X] \simeq \mathbb{R}[X]^G \otimes_{\mathbb{R}} \mathbb{R}[X]_G$$

*as graded $\mathbb{R}$-algebras.*

**Corollary I.2.18.** *Let $\mathbb{R}[X]^G = \mathbb{R}[\psi_1, \ldots, \psi_n]$ be a polynomial ring in the fundamental invariants $\psi_1, \ldots, \psi_n$ and let $\mathbb{R}[X]_G = \bigoplus_{j=1}^{l} \eta_j \theta^{(j)}$ be the isotypic decomposition of the coinvariant algebra. Then there exists a symmetry adapted basis $f_{11}, \ldots, f_{l\eta_l} \in \mathbb{R}[X]$ of $\mathbb{R}[X]_G$ and any $f \in \mathbb{R}[X]$ can be written as a sum of polynomials of the form*

$$\sum_{j=1}^{l} \sum_{i=1}^{\eta_j} \sum_{\sigma \in G} g_{ji,\sigma} \sigma f_{ji}$$

*for some $g_{ji,\sigma} \in \mathbb{R}[X]^G$.*

*Proof.* The existence of the symmetry adapted basis $(f_{11}, \ldots, f_{l\eta_l})$ of $\mathbb{R}[X]_G$ follows by Schur's lemma I.2.3. Further, by definition, the $G$-orbit of $(f_{11}, \ldots, f_{l\eta_l})$ spans the coinvariant algebra. The claim follows from the graded tensor decomposition in Theorem I.2.17, since the basic tensors of $\mathbb{R}[X]$ are elements described above. ∎

The second summation in the representation of a polynomial in Corollary I.2.18 goes up to $\eta_j$. We recall that the multiplicity $\eta_j$ of an irreducible representation $\theta^{(j)}$ in the coinvariant algebra equals its dimension $\vartheta_j$.

*Remark* I.2.19. The calculation of a symmetry adapted basis of the coinvariant algebra allows easily the computation of the isotypic decomposition of the $G$-module $H_{n,d}$ for any degree. As a rough general procedure, one needs to compute the products of elements from the symmetry adapted basis with fundamental invariants of $G$, such that the degree of the obtained forms equal $d$.

**Definition I.2.20.** Let $S := \{s_1, \ldots, s_{|G|}\}$ be a basis of $\mathbb{R}[X]_G$. Then we define the matrix polynomial $H^S(\psi_1, \ldots, \psi_n) \in \mathbb{R}[\psi_1, \ldots, \psi_n]^{|G| \times |G|}$ entry wise

$$H^S_{u,v} := \mathcal{R}_G(s_u \cdot s_v),$$

and each entry $\mathcal{R}_G(s_u \cdot s_v)$ is expressed as a polynomial in the fundamental invariants $\psi_1, \ldots, \psi_n$.

**Lemma I.2.21.** *Let $f \in \mathbb{R}[X]$ be $G$-invariant and let $\gamma \in \mathbb{R}[\psi_1, \ldots, \psi_n]$ with $\gamma(\psi_1, \ldots, \psi_n) = f$ then $f$ is a sum of squares if and only if $\gamma(\psi_1, \ldots, \psi_n)$ admits a representation of the form*

$$\gamma = \langle T, H^S \rangle$$

*where $T$ is a sum of squares matrix polynomial, i.e., $T = L^T L$ for a matrix $L \in \mathbb{R}[\psi_1, \ldots, \psi_n]^{n \times m}$ and an integer $1 \leq m \leq n$.*

*Proof.* This follows from the decomposition $\mathbb{R}[X] \simeq \mathbb{R}[X]^G \otimes \mathbb{R}[X]_G$ in Theorem I.2.17. ■

**Definition I.2.22.** For every irreducible representation $\theta^{(j)}$ of $G$ we construct a matrix polynomial $H^{\vartheta_j} \in \mathbb{R}[\psi_1, \ldots, \psi_n]^{\eta_j \times \eta_j}$ in the following way: Let $\mathbb{R}[X]_G = \bigoplus_{j=1}^{l} \mathbb{R}[X]_G^{\vartheta_j}$ be the isotypic decomposition of the coinvariant algebra and $\{s_{1,1}, \ldots, s_{1,\eta_1}, s_{2,1}, \ldots, s_{l,\eta_l}\}$ be a symmetry adapted basis of $\mathbb{R}[X]_G$. Then we define

$$H^{\vartheta_j}_{u,v} = R_G(s_{j,u} \cdot s_{j,v}).$$

Combining above definition and lemma, and the results from Schur's lemma we immediately obtain

**Theorem I.2.23.** *Let $G$ be a finite reflection group with $\mathbb{R}[X]^G = \mathbb{R}[\psi_1, \ldots, \psi_n]$. Then,*

$$\Sigma\mathbb{R}[X]^2 \cap \mathbb{R}[X]^G = \left\{ g \in \mathbb{R}[\psi_1, \ldots, \psi_n] \,:\, g = \sum_{j=1}^{l} \langle H^{\vartheta_j}, A_j \rangle \right\},$$

*where $A_j \in \mathbb{R}[\psi_1, \ldots, \psi_n]^{\eta_j \times \eta_j}$ is a sum of squares matrix polynomial.*

**Example I.2.24.** Let $f \in \mathbb{R}[X_1, X_2]$ be a form of degree $2d$ which is invariant under the dihedral group $I_2(k)$. The dihedral group $I_2(k)$ has only irreducible representations of dimension 1 or 2. In fact, if $k$ is odd (resp. even), then 2 (resp. 4) representations of dimension one and $\frac{k-1}{2}$ (resp. $\frac{k-2}{2}$) representations of dimension two. By block-diagonalisation we end up with $H^S(z)$ having 2 (resp. 4) $1 \times 1$ blocks $H^{\theta_1}, H^{\theta_2}$ (resp. $H^{\theta_1}, \ldots, H^{\theta_4}$) and $\frac{k-1}{2}$ (resp. $\frac{k-2}{2}$) $2 \times 2$ blocks $H^{\theta_3}, \ldots, H^{\theta_{\frac{k+3}{2}}}$ (resp. $H^{\theta_5}, \ldots, H^{\theta_{\frac{k+6}{2}}}$). Then for $n$ odd (resp. even) $f$ nonnegative if and only if there exist sums of squares matrix polynomials $A_j \in \mathbb{R}[X_1^2 + X_2^2, (X_1 + \sqrt{-1}X_2)^k + (X_1 - \sqrt{-1}X_2)^k]^{\dim\theta_j \times \dim\theta_j}$ such that

$$f = \sum_{j=1}^{m} \langle H^{\theta_j}, A_j \rangle$$

and $m = \frac{k+3}{2}$ (resp. $m = \frac{k+6}{2}$).
For $k = 3$ the coinvariant algebra $\mathbb{R}[x, y]_{I_2(3)}$ decomposes into the direct sum of

$$\theta^{(1)} = \langle 1 \rangle, \theta^{(2)} = \langle -x^3 + 3xy^2 \rangle, \theta_1^{(3)} = \langle x \rangle_{I_2(3)}, \theta_2^{(3)} = \langle xy \rangle_{I_2(3)},$$

where $\theta_1^{(3)}$ and $\theta_2^{(3)}$ are $I_2(3)$-isomorphic through $x \mapsto xy$. Then $H^{\theta^{(1)}} = (1)$, $H^{\theta^{(2)}} = \left(\mathcal{R}_{I_2(3)}(3xy^2 - x^3)^2\right)$ and $H^{\theta^{(3)}} = \begin{pmatrix} \mathcal{R}_{I_2(3)}(x^2) & \mathcal{R}_{I_2(3)}(x^2 y) \\ \mathcal{R}_{I_2(3)}(x^2 y) & \mathcal{R}_{I_2(3)}(x^2 y^2) \end{pmatrix}$.

**Definition I.2.25.** Let $G$ be a finite reflection group and $\theta$ be an irreducible representation. We write $h_k^\vartheta$ for the multiplicity of $\theta$ in $(\mathbb{R}[X]_G^\theta)_k$.

I.e., $h_k^\vartheta$ equals the multiplicity of $\theta$ in the isotypic decomposition of the subspace of the coinvariant algebra of forms of degree $k$. We recall that $N_G(d)$ denotes the vector space dimension of $G$-invariant forms of degree $d$ (see I.2.13).

**Corollary I.2.26.** *Let $G$ be a finite reflection group and $\theta$ be an irreducible representation. Then the multiplicity of the corresponding irreducible representation in the $G$-module $H_{n,d}$ equals $\sum_{k=0}^d N_G(d-k)\cdot h_k^\vartheta$.*

### I.2.3 G-harmonic polynomials

In this subsection we present a specific basis of the coinvariant algebra for reflection groups which can be simply computed.

**Definition I.2.27.** For a polynomial $f(X)=\sum_\alpha c_\alpha X^\alpha\in\mathbb{R}[X]$ we define $f(\partial)$ as the linear operator

$$\begin{array}{rcl} f(\partial):\mathbb{R}[X] & \longrightarrow & \mathbb{R}[X] \\ g & \longmapsto & \sum_\alpha c_\alpha \frac{\partial^\alpha}{(\partial X)^\alpha} g \end{array}.$$

I.e., $f(\partial)$ is a linear map which is a formal sum of scaled partial derivatives.

**Example I.2.28.** Let $f(X)=X_1^2+X_1X_2\in\mathbb{R}[X_1,X_2,X_3]$, then $f(\partial)=\frac{\partial^2}{\partial X_1\partial X_1}+\frac{\partial^2}{\partial X_1\partial X_2}$ and $f(\partial)\left(X_1^2+X_2^2+X_3^2+X_1X_2X_3\right)=2+X_3$.

**Definition I.2.29.** Let $G$ be a reflection group and $\mathbb{R}[X]^G=\mathbb{R}[\psi_1,\psi_2,\ldots,\psi_n]$. The $\mathbb{R}$-vector space of *harmonic polynomials* is defined as $\mathcal{H}_G:=\left(\mathbb{R}[X]^G\right)^\perp$ with respect to the inner product

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle:\mathbb{R}[X]\times\mathbb{R}[X] & \longrightarrow & \mathbb{R}[X] \\ (f,g) & \longmapsto & \mathrm{ev}_{(0,\ldots,0)}\left(f(\partial)g(X)\right). \end{array}$$

**Theorem I.2.30** ([Ber09])**.** *Let $G$ be a real reflection group and $\Delta:=\prod L_i$, be the product of a minimal system of linear polynomials defining the reflection hyperplanes. Then, the vector space of $G$-harmonic polynomials $\mathcal{H}_G$ is generated by all partial derivatives of $\Delta$, i.e., $\mathcal{H}_G=\langle\frac{\partial^\alpha}{(\partial x)^\alpha}\Delta:\alpha\in\mathbb{N}_0^n\rangle_\mathbb{R}$. Furthermore, $\mathcal{H}_G$ is as $G$-module isomorphic to the regular representation of $G$ and $\mathbb{R}[X]=\mathbb{R}[X]^G\otimes_\mathbb{R}\mathcal{H}_G$.*

Note, $\Delta$ is only defined up to scalar multiplication by $c\in\mathbb{R}\setminus\{0\}$.

*Remark* I.2.31. Let $G$ be a reflection group, $\psi_1,\ldots,\psi_n$ be the fundamental invariants and consider the map

$$\begin{array}{rcl} \Psi:\mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ X & \longmapsto & (\psi_1(X),\ldots,\psi_n(X)). \end{array}$$

Then, thanks to a statement of Steinberg [Ste60] we have

$$\Delta = \det(\mathrm{Jac}\Psi)$$

where $\mathrm{Jac}\Psi$ denotes the Jacobian of $\Psi$. The choice of fundamental invariants $\psi_1, \ldots, \psi_n$ does not matter.

**Example I.2.32.** For $\mathfrak{S}_n$ the symmetric group acting on $\mathbb{R}^n$ via coordinate permutation and $\psi_i = \sum_{j=1}^{n} X_j^i$ the power sums, we obtain $\Delta = \prod_{i<j}(x_i - x_j)$ equals the determinant of the Vandermonde matrix, which is precisely the product over all reflections of $\mathfrak{S}_n$.

*Remark* I.2.33. Computing a basis of the coinvariant algebra $\mathbb{R}[X]_G = \mathbb{R}[X]/\mathbb{R}[X]_{>0}^G$, which is defined as a quotient space, is challenging and involves the calculation of a Gröbner basis. However, the approach using harmonic polynomials is more efficient. It is based on linear algebra for given fundamental invariants and the fundamental invariants of real reflection groups are well-known, so one can simply calculate the polynomial $\Delta$ and all its partial derivatives.

### I.2.4 Convex geometric properties of $\Sigma^G$ and $\mathcal{P}^G$

The convex cones of sums of squares and nonnegative forms, and their dual cones have been studied intensively in the research on nonnegativity versus sums of squares (see e.g. [Ble12] on Hilbert's inequality cases or [BPT12]). In this subsection, we present known and adapted knowledge on the convex geometrical properties of $\Sigma_{n,2d}^G$ and $\mathcal{P}_{n,2d}^G$. We refer to ([BR21, Subsection 4.5]) for more details.

The sets $\Sigma_{n,2d}^G$ and $\mathcal{P}_{n,2d}^G$ are *convex cones*, i.e., they are convex sets which are closed under scalar multiplication by nonnegative scalars. Moreover, these sets are closed and *pointed*, i.e., they do not contain a non-trivial linear subspace. We refer to [Ble06] for details.
The dual cone of a set $K \subset \mathbb{R}^N$ is denoted by $K^*$ and is defined as $K^* = \left\{\ell \in \mathbb{R}^{N,*} : \ell(K) \subseteq \mathbb{R}_{\geq 0}\right\}$.

*Remark* I.2.34. To study the set $\Sigma_{n,2d}^{G,*}$ we associate the elements in $\Sigma_{n,2d}^{G,*}$ with positive semidefinite quadratic forms. We associate a linear functional $\ell \in H_{n,2d}^{G,*}$ with a $G$-invariant quadratic form $Q_\ell$ defined as

$$\begin{array}{rcl} Q_\ell : H_{n,d} & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \ell\left(\mathcal{R}_G(f^2)\right). \end{array}$$

Note, although $\ell$ is defined on the space of invariant forms, the quadratic form $Q_\ell$ is defined on the space of all forms.

Since our considered polynomials are homogeneous we have the following description of the dual cone of invariant nonnegative forms. For $a \in \mathbb{R}^n$ we write $\mathrm{ev}_a$ for the *point-evaluation* of $a$, i.e.,

$$\begin{array}{rcl} \mathrm{ev}_a : \mathbb{R}[X] & \longrightarrow & \mathbb{R} \\ f(X) & \longmapsto & f(a). \end{array}$$

**Proposition I.2.35** ([Ble06]). *The dual cone of the nonnegative invariant forms is the convex cone that is generated by all point-evaluations, i.e.,*

$$\mathcal{P}_{n,2d}^{G,*} = \operatorname{cone}\{\mathrm{ev}_a : a \in \mathbb{S}^{n-1}\}.$$

By duality any $f \in \mathcal{P}_{n,2d}^G$ contained in the boundary of $\mathcal{P}_{n,2d}^G$ has a real zero. We formulate the dual version of Theorem I.2.6.

**Lemma I.2.36.** *Let $\ell \in H_{n,2d}^{G,*}$ and $\{f_{11}, \ldots, f_{1\eta_1}, f_{21}, \ldots, f_{l\eta_l}\}$ be a symmetry adapted basis of $H_{n,d}$ and $B^{(j)} = (\mathcal{R}_G(f_{ju} \cdot f_{jv}))_{u,v}$. Then $\ell \in \Sigma_{n,2d}^{G,*}$ if and only if $\ell(B_j)$ is positive semidefinite for all $j = 1, \ldots, l$.*

The following lemma enables the characterisation of extremal elements through their kernels.

**Lemma I.2.37** ([Ble12], Lemma 2.2). *Let $V$ be a $\mathbb{R}$-vector space, $\mathcal{A}$ the vector space of quadratic forms on $V$ and $\mathcal{A}^+ \subset \mathcal{A}$ the cone of positive semidefinite quadratic forms. Let $L$ be a linear subspace of $\mathcal{A}$ and $K$ be the section of $\mathcal{A}^+$ with $L$, i.e., $K = \mathcal{A}^+ \cap L$. Then a quadratic form $Q \in K$ spans an extreme ray of $K$ if and only if its kernel is maximal among all kernels of quadratic forms in $L$, i.e., if $\ker Q \subseteq \ker P$ for a $P \in L$, it is $P = \lambda Q$ for some $\lambda \in \mathbb{R}$.*

In order to examine the kernels of invariant quadratic forms, we use the following construction. For a linear subspace $W \subset H_{n,d}$, we define its *quadratic symmetrization* with respect to $G$ as

$$W^{<2>} := \left\{ h \in H_{n,2d}^G \,:\, h = \mathcal{R}_G\left(\sum f_i g_i\right) \text{ for } f_i \in W \text{ and } g_i \in H_{n,d} \right\}.$$

In order to characterize the extreme rays of $\Sigma_{n,2d}^{G,*}$ we use Lemma I.2.36 to identify $\Sigma_{n,2d}^{G,*}$ with a linear section of the cone of positive semidefinite quadratic forms on $H_{n,d}$ with the subspace of $G$-invariant quadratic forms on $H_{n,d}$.

**Proposition I.2.38** ([BR21]). *An element $\ell \in \Sigma_{n,2d}^{G,*}$ is extremal if and only if $\ker Q_\ell$ is maximal among all kernels of $G$-invariant quadratic forms on $H_{n,d}$. Let $W := \ker Q_\ell$, then $W^{<2>}$ is equal to the kernel of $\ell$. Moreover, if $(f_{11}, \ldots, f_{l\eta_l})$ is a symmetry adapted basis of $H_{n,d}$ and $\left(g_{11}, \ldots, g_{l\eta_l'}\right)$ is a symmetry adapted basis of $W$ such that $\langle g_{ji_1} \rangle_G \simeq_G \langle f_{ji_2} \rangle_G$ and $g_{ji_1} \mapsto f_{ji_2}$ define the unique $G$-isomorphism, then*

$$W^{\langle 2 \rangle} = \langle \mathcal{R}_G(g_{ji_1} \cdot f_{ji_2}) : 1 \le j \le l, 1 \le i_1 \le \eta_j', 1 \le i_2 \le \eta_j \rangle_{\mathbb{R}}.$$

*Proof.* The first claim follows from Lemma I.2.37. The second claim follows from the positive semidefiniteness of the quadratic form $Q_\ell$. The complexity reduction gives the above description of $W^{\langle 2 \rangle}$ according to the use of a symmetry adapted basis and by applying Schur's lemma. ∎

To prove equality or inequality between the sets $\Sigma^G_{n,2d}$ and $\mathcal{P}^G_{n,2d}$ we can use the dual approach.

**Corollary I.2.39.** *Suppose the convex cones $\Sigma^G_{n,2d}, \mathcal{P}^G_{n,2d}$ are full dimensional. Then $\Sigma^G_{n,2d} = \mathcal{P}^G_{n,2d}$ if and only if any extremal ray in $\Sigma^{G,*}_{n,2d}$ is generated by a point-evaluation.*

*Proof.* The primal cones $\mathcal{P}^G_{n,2d}$ and $\Sigma^G_{n,2d}$ are equal if and only if the dual cones are equal. By Minkowski's theorem, any $\ell \in \Sigma^{G,*}_{n,2d}$ can be written as a sum of extremal elements. If any extremal ray in $\Sigma^{G,*}_{n,2d}$ is generated by a point-evaluation, then there exists a set $M \subset \mathbb{R}^n$ such that

$$\mathcal{P}^{G,*}_{n,2d} \subseteq \Sigma^{G,*}_{n,2d} = \operatorname{cone}\{\operatorname{ev}_a : a \in M\} \subset \operatorname{cone}\{\operatorname{ev}_a : a \in \mathbb{S}^{n-1}\} = \mathcal{P}^{G,*}_{n,2d}$$

where the last equality follows by Proposition I.2.35.
Conversely, if $\Sigma^G_{n,2d} = \mathcal{P}^G_{n,2d}$ then also the dual cones are equal. However, $\mathcal{P}^{G,*}_{n,2d}$ is the convex cone that is generated by all point-evaluations. Hence, any extremal ray in $\Sigma^{G,*}_{n,2d}$ is generated by a point-evaluation. ∎

## I.3 Sums of squares invariant under $A_n$, $B_n$, and $D_n$

In this section we present an algorithmic approach for calculating a symmetry adapted basis of the coinvariant algebra for reflection groups of type $A_{n-1}, B_n$ or $D_n$ which was introduced by Morita and Yamada [MY98]. We prove stabilization of the isotypic decomposition for a fixed degree and large enough number of variables, for those series of essential reflection groups.

### I.3.1 Higher Specht polynomials

A well known classical construction of the irreducible $\mathfrak{S}_n$-modules in the real polynomial ring is due to Specht [Spe37]. The $\mathfrak{S}_n$-generators of these representations are called *Specht polynomials*. However, we are interested in the decomposition of the coinvariant algebra. An elegant combinatorial algorithm to decompose the coinvariant algebra into all irreducible submodules for all pseudoreflection groups of type $G(r,p,n)$ was introduced in [MY98]. In the following, we briefly present their work.
We begin with recalling some basic definitions from combinatorics.

**Definition I.3.1.** A non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_l)$ is called a *partition* and $l$ is the *length* of $\lambda$. We denote by $|\lambda| = \sum_{i=1}^l \lambda_i = n$ the *value* of $\lambda$ and say that $\lambda$ is a partition of $n$, which we denote by $\lambda \vdash n$, if $|\lambda| = n$. For partitions $\lambda^1$ and $\lambda^2$ we call the pair $\Lambda = (\lambda^1, \lambda^2)$ a *bipartition* and allow $\lambda^1 = \emptyset$ or $\lambda^2 = \emptyset$. We say that $|\Lambda| = |\lambda^1| + |\lambda^2| = n$ is the value of $\Lambda$ and write $\Lambda \vdash n$ when $\Lambda$ is bipartition of $n$.

We always denote bipartitions by capital letters and partitions by small letters. However, sometimes we write $(\lambda, \emptyset)$ instead of $\lambda$ for a partition $\lambda$. I.e., we write also $\Lambda$ instead of $(\lambda, \emptyset)$ sometimes.

**Definition I.3.2.** The *Young diagram* associated to a partition $\lambda \vdash n$ is a sequence of ordered boxes starting from the left which $i$-th line contains $\lambda_i$ boxes. If one fills the boxes with all the integers in $[n]$, one calls the obtained object a *Young tableau* or *tableau* of shape $\lambda$. If the numbers in all columns and rows are increasing we call the tableau *standard*.
Bipartitions are associated with their pairs of Young diagrams. A *Young bitableau* or *bitableau* is a filling of both diagrams with all the numbers in $[n]$ and we call it standard if both diagrams are standard.
We denote by $\mathrm{YT}(\Lambda)$ the set of (bi-)tableaux of shape $\Lambda$ and by $\mathrm{SYT}(\Lambda)$ the subset of standard (bi-)tableaux.

In the following, we will denote an irreducible representation indexed by a (bi-)partition $\Lambda$ by $\mathbb{S}^\Lambda$, i.e., $\mathbb{S}^\Lambda$ is a *Specht module*. The underlying group should be clear from the context.
The famous Robinson-Schensted correspondence gives a bijection between the standard tableaux of shape $\lambda$ and the elements in the conjugacy class of $\mathfrak{S}_n$ which are labelled by $\lambda$. Hence, this number equals the multiplicity of the Specht module $\mathbb{S}^\lambda$ in the coinvariant algebra. The correspondence has been adapted to pseudoreflection groups of type $G(r,p,n)$ and in particular for the contained series of reflection groups of types $B_n = G(2,1,n)$ and $D_n = G(2,2,n)$, e.g., see ([Cas11, Section 10]).

Following [ATY97; MY98] we construct a symmetry adapted basis of the coinvariant algebra. The group $\mathfrak{S}_n$ acts naturally on a tableau by replacing the entry $i$ with $\sigma(i)$ for $\sigma \in \mathfrak{S}_n$.

**Definition I.3.3.** Let $T$ be a Young tableau of shape $\lambda \vdash n$. The $\mathfrak{S}_n$-subgroups

$$\begin{aligned}\mathcal{C}_T &:= \{\sigma \in \mathfrak{S}_n : \sigma T \text{ is obtained by permutation of the columns of } T\}\\ \mathcal{R}_T &:= \{\sigma \in \mathfrak{S}_n : \sigma T \text{ is obtained by permutation of the rows of } T\}\end{aligned}$$

are the *column* and *row stabilizer* of $T$. We define the formal linear combination

$$\epsilon_T := \frac{f^\lambda}{n!} \sum_{\sigma\in\mathcal{C}_T,\tau\in\mathcal{R}_T} \mathrm{sgn}(\sigma)\sigma\tau \in \mathbb{R}[\mathfrak{S}_n],$$

where $f^\lambda$ is the number of standard tableau of shape $\lambda$. For a bitableau $T = (T^1, T^2)$ we define $\epsilon_{T^1}, \epsilon_{T^2} \in \mathbb{R}[\mathfrak{S}_n]$ analogously and set $\epsilon_T := \epsilon_{T^1} \cdot \epsilon_{T^2}$.

We associate (bi-)tableau with sequences, monomials and polynomials:

**Definition I.3.4.** Let $T = (T^1, T^2) \in \mathrm{YT}(\Lambda)$ be a (bi-)tableau. The *word* of $T$ is the sequence $w(T) \in \mathbb{N}^{|\lambda|}$ where we read and notate each column of the tableau $T^1$ from the bottom to the top, starting from the left. We continue with this procedure for the tableau $T^2$.
We define the *index* $i(T) \in \mathbb{N}^{|\Lambda|}$ of $T$ as follows. The number 1 in the word $w(T)$ has index 0. If $k$ in the word has index $p$, then $k+1$ has index $p$ or $p+1$ according as it lies to the right or the left of $k$. We call the sum of the entries of

$i(T)$ the *charge* of $T$ and write $\mathrm{ch}(T)$.
We associate to a pair of (bi-)tableaux $(T,S) \in \mathrm{YT}(\Lambda) \times \mathrm{YT}(\Lambda)$ a monomial in $n$ variables $X_T^S := X_{w(T)_1}^{i(w(S))_1} \cdots X_{w(T)_{|\Lambda|}}^{i(w(S))_{|\Lambda|}}$. Moreover, we define polynomials associated to $(T,S)$

$$F_T^S := \epsilon_T \cdot X_T^S \in \mathbb{R}[X] \text{ and } \widehat{F}_T^S := F_T^S(X^2) \cdot \prod_{j \in T^2} X_j$$

where $X^2 := (X_1^2, \ldots, X_n^2)$.

We note, associating tableaux with words is a standard technique in the combinatorics of tableaux (see e.g. [FF97]).

**Example I.3.5.** Let $\Lambda = ((2,1),(1)) \vdash 4$ be a bipartition and $S = \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \boxed{3}\right)$, $T = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \boxed{3}\right) \in \mathrm{SYT}(\Lambda)$. The word of $S$ is $w(S) = (2,1,4,3)$ and the word of $T$ is $w(T) = (4,1,2,3)$. We calculate the indices $i(S) = (1,0,2,1)$ and $i(T) = (1,0,0,0)$ and compute $X_T^S = X_4^1 X_1^0 X_2^2 X_3^1 = X_2^2 X_3 X_4$, $F_T^S = X_1^2 X_3 X_4 + X_2^2 X_3 X_4 - X_1 X_2^2 X_3 - X_1 X_3 X_4^2$.

The authors in [MY98] introduced the following polynomials in analogy to Specht's polynomial representation of the irreducible $\mathfrak{S}_n$-modules.

**Definition I.3.6.** Let $n \in \mathbb{N}$ and let $\mathcal{L} := \{(\lambda, \mu) \vdash n : \lambda \neq \mu, |\lambda| \geq |\mu|\}$.

I.3.6.1. For $A_{n-1}$ the *higher Specht polynomials* are the polynomials $\left\{F_T^S : (T,S) \in \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)\right\}$.

I.3.6.2. For $B_n$ the *higher Specht polynomials* are the polynomials $\left\{\widehat{F}_T^S : (T,S) \in \bigcup_{\Lambda \vdash n} \mathrm{SYT}(\Lambda) \times \mathrm{SYT}(\Lambda)\right\}$.

I.3.6.3. For $D_n$ the *higher Specht polynomials* are the polynomials

$$\left\{\widehat{F}_T^S : (T,S) \in \bigcup_{\Lambda \in \mathcal{L}} \mathrm{SYT}(\Lambda) \times \mathrm{SYT}(\Lambda)\right\}, \text{ and}$$

$$\left\{\widehat{F}_{(T^1,T^2)}^S \pm \widehat{F}_{(T^2,T^1)}^S : ((T^1,T^2),S) \in \bigcup_{\lambda \vdash \frac{n}{2}} \mathrm{SYT}((\lambda,\lambda)) \times \mathrm{SYT}((\lambda,\lambda))\right\}.$$

If $n$ is odd there are no partitions of $\frac{n}{2}$. Thus, the $D_n$ higher Specht polynomials differ in their structure when $n$ is even or odd.

**Theorem I.3.7** ([MY98], Theorem 3)**.** *For the reflection groups of type $A_{n-1}, B_n$ or $D_n$ the higher Specht polynomials form a vector space basis of the coinvariant algebra. For $(P,Q),(P',Q') \in \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$ and $(T,S),(T',S') \in \mathrm{SYT}(\Lambda) \times \mathrm{SYT}(\Lambda)$ we have*

$$\mathbb{S}^\lambda \simeq_{A_{n-1}} \langle F_{(P,Q)}^{(P',Q')} \rangle_{A_{n-1}} = \langle F_{(P'',Q'')}^{(P',Q')} : (P'',Q'') \in \mathrm{SYT}(\lambda) \rangle_{\mathbb{R}}$$
$$\mathbb{S}^\Lambda \simeq_{B_n} \langle \widehat{F}_{(T,S)}^{(T',S')} \rangle_{B_n} = \langle \widehat{F}_{(T'',S'')}^{(T',S')} : (T'',S'') \in \mathrm{SYT}(\Lambda) \rangle_{\mathbb{R}}$$

*Furthermore, for $\lambda \neq \mu$ the associated irreducible $B_n$-representations $(\lambda, \mu)$ and $(\mu, \lambda)$ remain $D_n$-irreducible, but are $D_n$-isomorphic. For a pair $((T^1, T^2), S)$ of standard bitableaux of shape $(\lambda, \lambda) \vdash n$ we have*

$$\langle \widehat{F}_T^S \rangle_{D_n} = \langle \widehat{F}_T^S + \widehat{F}_{(T^2,T^1)}^S \rangle_{D_n} \oplus \langle \widehat{F}_T^S - \widehat{F}_{(T^2,T^1)}^S \rangle_{D_n} \simeq_{D_n}: \mathbb{S}_+^{(\lambda,\lambda)} \oplus \mathbb{S}_-^{(\lambda,\lambda)}$$

*and the $D_n$-modules $\mathbb{S}_+^{(\lambda,\lambda)}, \mathbb{S}_-^{(\lambda,\lambda)}$ are $D_n$-irreducible and non-isomorphic.*

Moreover, we find the following as a consequence of Schur's lemma I.2.3 and the statements in [MY98]: For the groups $A_{n-1}, B_n$ and $D_n$ and standard (bi-)tableaux $T = (T^1, T^2), S_1, S_2$ of shape $\Lambda$ (resp. $\lambda$) the maps

$$F_T^{S_1} \mapsto F_T^{S_2} \text{ for } A_{n-1} \text{ and } \widehat{F}_T^{S_1} \mapsto \widehat{F}_T^{S_2} \text{ for } B_n, D_n$$

define the (up to scalar multiplication) unique $G$-module isomorphisms. If $\Lambda = (\lambda, \lambda)$, then the unique $D_n$-isomorphisms are

$$\widehat{F}_{(T^1,T^2)}^{S_1} \pm \widehat{F}_{(T^2,T^1)}^{S_1} \mapsto \widehat{F}_{(T^1,T^2)}^{S_2} \pm \widehat{F}_{(T^2,T^1)}^{S_2} .$$

**Definition I.3.8.** Let $G \in \{A_{n-1}, B_n, D_n\}$ and $\Lambda \vdash n$ be a (bi-)partition. We write $q_d^\Lambda$ for the multiplicity of the $G$-module $\mathbb{S}^\Lambda$ in $H_{n,d}$.

*Remark* I.3.9. From Theorem I.3.7 we obtain a combinatorial description of $h_k^\theta$, i.e., of the multiplicity of an irreducible representation $\theta$ in the subspace of the coinvariant algebra of forms of degree $k$. Namely, in the case of $A_{n-1}$ $\theta$ is labelled by a partition $\lambda \vdash n$ and

$$h_k^\lambda = |\{T \in \mathrm{SYT}(\lambda) : \mathrm{ch}(T) = k\}|.$$

While for $B_n$ and $D_n$ $\theta$ is labelled by a bipartition $\Lambda = (\lambda, \mu) \vdash n$ and

$$h_k^\Lambda = |\{(T, S) \in \mathrm{SYT}(\Lambda) : 2\,\mathrm{ch}(T, S) + |\mu| = k\}|.$$

In particular, the multiplicity of $\mathbb{S}^\Lambda$ in $H_{n,d}$ can be described combinatorially through the number of standard (bi-)tableaux and the degrees of $G$

$$q_d^\Lambda = \sum_{k=0}^{d} N_G(d-k) \cdot h_k^\Lambda.$$

By integrating the above presented construction with the general setup, the degrees of the considered reflection groups and the number of standard (bi-)tableaux combinatorially encode the following information about the invariant sums of squares.

**Theorem I.3.10.** *Let $G \in \{A_{n-1}, B_n\}$.*

(1) *The isotypic decomposition of $H_{n,d}$ is*

$$\bigoplus_{\Lambda \vdash n} q_d^\Lambda \cdot \mathbb{S}^\Lambda,$$

*where $\Lambda$ ranges over partitions for $A_{n-1}$ and bipartitions for $B_n$.*

(2) *There exists a symmetry adapted basis of the coinvariant algebra* $\mathbb{R}[X]_G$ *consisting of higher Specht polynomials* $(s_1^\Lambda, \ldots, s_{\vartheta_\Lambda}^\Lambda)_{\Lambda \vdash n}$*, where* $\vartheta_\Lambda$ *denotes the dimension of* $\mathbb{S}^\Lambda$*. By defining symmetric matrix polynomials* $(H_{v,u}^\Lambda) = \left(\mathcal{R}_G(s_v^\Lambda \cdot s_u^\Lambda)\right) \in \mathbb{R}[X]^{\vartheta_\Lambda \times \vartheta_\Lambda}$ *we have*

$$\Sigma\mathbb{R}[X]^2 \cap \mathbb{R}[X]^G = \left\{ g \in \mathbb{R}[\psi_1, \ldots, \psi_n] \, : \, g = \sum_{\Lambda \vdash n} \langle H^{\vartheta_j}, A_\Lambda \rangle \right\},$$

*where* $A_\Lambda \in \mathbb{R}[\psi_1, \ldots, \psi_n]^{\vartheta_\Lambda \times \vartheta_\Lambda}$ *are sums of squares matrix polynomial.*

(3) *There exists a symmetry adapted basis of* $H_{n,d} = \bigoplus_{\Lambda \vdash n} q_d^\Lambda \cdot \mathbb{S}^\Lambda$ *such that its elements* $\left(s_1^\Lambda, \ldots, s_{q_d^\Lambda}^\Lambda\right)$ *which belong to the isotypic component* $q_d^\Lambda \cdot \mathbb{S}^\Lambda$ *are products each of one higher Specht polynomial and a monomial in* $\psi_1, \ldots, \psi_n$*. By defining matrix polynomials* $B^\Lambda = \left(\mathcal{R}_G(s_v^\Lambda \cdot s_u^\Lambda)\right)_{v,u} \in \left(\mathbb{R}[X]^G\right)^{q_d^\Lambda \times q_d^\Lambda}$ *a form* $f \in H_{n,2d}^G$ *is a sum of squares if and only if*

$$f = \sum_{\Lambda \vdash n} \langle B^\Lambda, A_\Lambda \rangle$$

*for some positive semidefinite matrices* $A_\Lambda \in \mathbb{R}^{q_d^\Lambda \times q_d^\Lambda}$*.*

*Proof.* The isotypic decomposition of $H_{n,d}$ can be realized through multiplying the higher Specht polynomials of $G$ of degree $\leq d$ with products of fundamental invariants by Theorems I.3.7 and I.2.17. For every $k$ the multiplicity of $G$-modules $G$-isomorphic to $\mathbb{S}^\Lambda$ in the subspace of the coinvariant algebra of degree $k$ is precisely $h_k^\Lambda$, while $N_G(d-k)$ gives the dimension of $H_{n,d-k}^G$. Now, (2) and (3) follow from Theorem I.2.23 and Corollary I.2.7. ∎

*Remark* I.3.11. For $D_n$ one can provide analogous assertions. The isotypic decomposition in (1) and the sizes of the matrices in (2) and (3) differ slightly, since then the $D_n$-module $\mathbb{S}^{(\lambda,\lambda)}$ decomposes into two irreducible $D_n$-modules, and since $\mathbb{S}^{(\lambda,\mu)}$ is $D_n$-isomorphic to $\mathbb{S}^{(\mu,\lambda)}$.

**Example I.3.12.** The $D_4$ fundamental invariants are the following:

$$\begin{aligned} p_2 &= X_1^2 + X_2^2 + X_3^2 + X_4^2, \;\; p_4 = X_1^4 + X_2^4 + X_3^4 + X_4^4, \\ p_6 &= X_1^6 + X_2^6 + X_3^6 + X_4^6, \;\; e_4 = X_1 X_2 X_3 X_4, \end{aligned}$$

i.e., we have $\mathbb{R}[X]^{D_4} = \mathbb{R}[p_2, p_4, p_6, e_4]$. By Corollary I.2.18 and Theorem I.3.10 the symmetry adapted basis of $H_{4,2}$ can be obtained by multiplying fundamental invariants with higher Specht polynomials such that the degree equals 2.
We apply Theorem I.3.7 to calculate the $D_4$ higher Specht polynomials. For a bipartition $\Lambda \vdash 4$ the minimal degree of a higher Specht polynomial associated with $\Lambda$ is given by the smallest integer in $\{2\,\mathrm{ch}(T) + |\lambda^2| : T \in \mathrm{SYT}(\Lambda)\}$.
Since the degrees of the fundamental invariants are at least 2, we need to compute all higher Specht polynomials of degree 0 and 2. Therefore, we

only need to consider bipartitions $(\lambda^1, \lambda^2) \vdash 4$ with $\lambda^2 \vdash m \in \{0, 2\}$ as otherwise the degree is odd. In the case $\lambda^2 \vdash 2$ it must be $\mathrm{ch}(T) = 0$. This can only occur for $w(T) = (1, 2, 3, 4)$ which forces $\Lambda = ((2), (2))$. The possible remaining cases are $\Lambda^1 = ((4), \emptyset), \Lambda^2 = ((3, 1), \emptyset), \Lambda^3 = ((2, 2), \emptyset), \Lambda^4 = ((2, 1, 1), \emptyset), \Lambda^5 = ((1, 1, 1, 1), \emptyset)$. We are looking for a standard bitableau $T$ of shape $\Lambda^j, j \in \{1, 2, 3, 4, 5\}$, such that $\mathrm{ch}(T) \in \{0, 1\}$. The case that the charge is 0 is only possible for $\Lambda^1$. In the remaining cases, $\mathrm{ch}(T) = 1$ if and only if $T = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \emptyset\right)$. Then, the $D_4$-module $\mathbb{S}^{((2),(2))}$ decomposes by Theorem I.3.7 into two irreducible, non-isomorphic modules $\mathbb{S}_+^{((2),(2))}$ and $\mathbb{S}_-^{((2),(2))}$. Hence, the $D_4$-module $H_{4,2}$ has the isotypic decomposition

$$H_{4,2} = \mathbb{S}^{((4),\emptyset)} \oplus \mathbb{S}^{((3,1),\emptyset)} \oplus \mathbb{S}_+^{((2),(2))} \oplus \mathbb{S}_-^{((2),(2))}.$$

The relevant higher Specht polynomials are 1 for $\mathbb{S}^{((4),\emptyset)}$, $X_4^2 - X_1^2$ for $\mathbb{S}^{((3,1),\emptyset)}$ and $X_1X_2 \pm X_3X_4$ for $\mathbb{S}_+^{((2),(2))}$ and $\mathbb{S}_-^{((2),(2))}$.

### I.3.2 Stabilization of the isotypic decompositions

In the following, we prove a stabilization of the isotypic decompositions of the $Z_n$-modules $H_{n,d}$ for large $n$ and $(Z_n)_n \in \{(A_{n-1})_n, (B_n)_n, (D_n)_n\}$.

**Definition I.3.13.** For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ we write $\lambda + 1$ for the partition of $n+1$ obtained from $\lambda$ by replacing $\lambda_1$ with $\lambda_1 + 1$. For a bipartition $\Lambda \vdash n$ we define $\Lambda + 1$ as the bipartition $(\lambda + 1, \mu) \vdash n + 1$.

Note, $\lambda + 1 = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_l)$. We use the combinatorial description of the degrees of a symmetry adapted basis of $H_{n,d}$ from Remark I.3.9. For $A_{n-1}$ and a standard tableau $T$ we have $\deg F_T^T = \mathrm{ch}(T)$, while for $B_n$ and $(T, S)$ we have $\deg \widehat{F}_{(T,S)}^{(T,S)} = 2\,\mathrm{ch}(T, S) + |\mu|$. Our aim is to identify the relevant standard (bi-)tableaux whose associated higher Specht polynomials occur in $H_{n,d}$.

**Lemma I.3.14.** *Let $k \geq 1$ be an integer and $\lambda \vdash n = d + k$ be a partition. In the case that the first row of a tableau $T \in \mathrm{SYT}(\lambda)$ does not begin with $1, 2, \ldots, k$ we have $\deg F_T^T > d$.*

*Proof.* We assume that a standard tableau $T$ of shape $\lambda$ does not contain $1, 2, \ldots, k$ in the first row. Let $\tilde{k}$ be the first entry of $T$ in the second row. It must be $\tilde{k} \leq k$ and $i(T)$ does contain at least $n - \tilde{k} + 1$ entries which are larger than or equal to 1. Therefore,

$$\deg F_T^T = \mathrm{ch}(T) \geq n - \tilde{k} + 1 \geq n - k + 1 = d + 1.$$

∎

We formulate Lemma I.3.14 for bipartitions.

**Lemma I.3.15.** *Let $(\lambda, \mu) \vdash n$ be a bipartition, with $|\mu| \leq d$ and $|\lambda| \geq \frac{d-1}{2} + j$ for an integer $j \geq 1$. Let $(T, S)$ be a standard bitableau of shape $(\lambda, \mu)$ where*

*$\alpha_1 < \ldots < \alpha_{|\lambda|}$ are all the entries in $T$. Suppose the first row of $T$ does not begin with $\alpha_1, \ldots, \alpha_j$ then $\deg \widehat{F}^{(T,S)}_{(T,S)} > d$.*

*Proof.* We suppose that for some $i \leq j$ the $i$-th entry in the first row of $T$ is not $\alpha_i$ and let $i$ be minimal with this property. Then $\alpha_i$ must be the first entry in the second row and $|\lambda| - i + 1$ entries in $i(T,S)$ are at least 1. Hence

$$\deg \widehat{F}^{(T,S)}_{(T,S)} = 2\operatorname{ch}(T,S) + |\mu| \geq 2(|\lambda| - i + 1) \geq 2\left(\frac{d-1}{2} + j - j + 1\right) \geq d + 1.$$

■

We write $T = (\alpha_{ij})$ for a standard tableau $T$ of shape $\lambda$ and $\alpha_{ij}$ denotes the entry in the i-th row and j-th coloumn of $T$, counted from the left to the right and the top to the bottom. Analogously, we write $(T,S) = ((\alpha_{ij}), (\beta_{ij}))$ for a standard bitableau.

**Definition I.3.16.** For a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n = d + k$ we define

$$\Pi_k^\lambda := \{(\alpha_{ij}) \in \operatorname{SYT}(\lambda) : \alpha_{1j} = j, 1 \leq j \leq k\}.$$

For a bipartition $\Lambda \vdash n = d + k$ we define $\Pi_k^\Lambda$ as the set

$$\{((\alpha_{ij}), (\beta_{ij})) \in \operatorname{SYT}(\Lambda) : (\alpha_{1j}) \text{ starts with the } k \text{ smallest integers in } \{\alpha_{ij}\}\}$$

and $(\alpha_{1j})$ denotes the first row of $T$.

**Example I.3.17.**

$$\Pi_3^{((3,1),(1))} = \left\{ \left(\ytableaushort{234,5}, \ytableaushort{1}\right), \left(\ytableaushort{134,5}, \ytableaushort{2}\right), \left(\ytableaushort{124,5}, \ytableaushort{3}\right), \left(\ytableaushort{123,5}, \ytableaushort{4}\right), \left(\ytableaushort{123,4}, \ytableaushort{5}\right) \right\}$$

**Lemma I.3.18.** *Let $n = d + k$, $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ be a partition and*

$$\begin{array}{rccc} \rho^\lambda_{n,n+1} : & \Pi_k^\lambda & \longrightarrow & \Pi_{k+1}^{\lambda+1} \\ & S = (\alpha_{ij}) & \longmapsto & \widetilde{S} = (\widetilde{\alpha}_{ij}) \end{array}$$

*where $\widetilde{\alpha}_{1j} = j$ for $1 \leq j \leq \alpha_{21}$. Further, $\widetilde{\alpha}_{1j} = \alpha_{1j-1} + 1$ for $j \geq \alpha_{21} + 1$ and $\widetilde{\alpha}_{ij} = \alpha_{ij} + 1$ for all $i \geq 2$ and $j \geq 1$.*
*The map $\rho^\lambda_{n,n+1}$ is injective and $i(S), i(\widetilde{S})$ differ only by a 0, i.e., any non-zero entry in $i(S)$ occurs with the same multiplicity in $i(\widetilde{S})$, while 0 occurs once more. Furthermore, if $k > d-1$ then for any $\widetilde{S} \in \Pi_{k+1}^{\lambda+1} \setminus \rho^\lambda_{n,n+1}(\Pi_k^\lambda)$ we have $\operatorname{ch}(\widetilde{S}) > d$.*

*Proof.* Since $S \in \Pi_k^\lambda$ is standard, we observe that $\alpha_{21}$ is the smallest integer $t$ for which $\alpha_{1t} \neq t$, if such a $t$ exists, and otherwise $\alpha_{21} = \max_j\{\alpha_{1j}\} + 1$. For $S \in \Pi_k^\lambda$ the tableau $\widetilde{S}$ of shape $\lambda + 1$ is indeed standard: $\widetilde{S}$ is filled with $1, \ldots, n+1$. Increasing rows and columns are inherited from $S$, as $\alpha_{1\alpha_{21}} > \alpha_{21}$, if $\alpha_{21} < \max_j\{\alpha_{1j}\}$. $\widetilde{S}$ is clearly increasing in any column from the second row onward. But also from the first row to the second. For $1 \leq j \leq \alpha_{21}$ this is clear from $S$. For $j \geq \alpha_{21} + 1$ this follows because

$\widetilde{\alpha}_{1j} = \alpha_{1,j-1} + 1 < \alpha_{2,j-1} + 1 < \alpha_{2,j} + 1 = \widetilde{\alpha}_{2j}$.
The smallest $p$ which is written left of $p-1$ in $w(S)$ (resp. $w(\widetilde{S})$) is $\alpha_{21}$ if $\alpha_{21} < \max_j\{\alpha_{1j}\}$ and otherwise $\min\{\alpha_{22}, \alpha_{31}\}$. From there any $p > \alpha_{21}$ is left of $p-1$ in $w(S)$ if and only if $p+1$ is left of $p$ in $w(\widetilde{S})$. Hence, $i(S)$ and $i(\widetilde{S})$ differ only by a 0.
Consider $\psi^{\lambda+1}_{n+1,n} : \Pi^{\lambda+1}_{k+1} \to \mathrm{YT}(\lambda)$ which maps a standard tableau $\widetilde{S}$ to a tableau $S$ by removing the box of the first entry $\widetilde{\alpha}_{1j}$ in the first row of $\widetilde{S}$, that is strictly smaller than $\widetilde{\alpha}_{1j+1} - 1$, and if such an entry does not exist then the last entry. The boxes to the right are shifted to the left such that one obtains a diagram. Any entry that was to the right of $\widetilde{\alpha}_{1j}$ or in a row below is decreased by one. If $\psi^{\lambda+1}_{n+1,n}(\widetilde{S}) =: S$ is again standard, then $\psi^{\lambda+1}_{n+1,n} \circ \rho^{\lambda}_{n,n+1}(S) = S$. This shows the injectivity of $\rho^{\lambda}_{n,n+1}$.
If $S$ is not standard, then one entry in the first column must be smaller than the entry below. Assume that this happens at $S$'s entry $\alpha_{1j}$. By assumption $j > k$, but this means $\lambda_2 \geq j > k$ and we observe

$$\mathrm{ch}(\widetilde{S}) \geq \lambda_2 + 1 \geq k + 2 \geq d + 1.$$

■

We present the analogous assertion for bipartitions.

**Lemma I.3.19.** *Let $n = d + k$, $\Lambda = (\lambda, \mu) = ((\lambda_1, \dots, \lambda_l), \mu) \vdash n$ be a bipartition and*

$$\begin{array}{rccc} \rho^{\Lambda}_{n,n+1} : & \Pi^{\Lambda}_{k} & \longrightarrow & \Pi^{\Lambda+1}_{k+1} \\ & (T,S) = ((\alpha_{ij}), (\beta_{ij})) & \longmapsto & (\widetilde{T}, \widetilde{S}) = ((\widetilde{\alpha}_{ij}), (\widetilde{\beta}_{ij})) \end{array},$$

*where $(\widetilde{T}, \widetilde{S})$ is defined by: Let $t$ be minimal with $\alpha_{1t} \neq t$, then $\widetilde{\alpha}_{1j} = j$, $1 \leq j \leq t$ and $\widetilde{\alpha}_{1j} = \alpha_{1j-1} + 1$, for $t + 1 \leq j \leq \lambda_1 + 1$, $\widetilde{\alpha}_{ij} = \alpha_{ij} + 1$, when $i \geq 2$, and $\widetilde{\beta}_{ij} = \beta_{ij} + 1$ for all $i, j$. If such a $t$ does not exist, then $\widetilde{\alpha}_{1j} = j$ for all $1 \leq j \leq \lambda_1 + 1$ and $\widetilde{\alpha}_{ij} = \alpha_{ij} + 1$ if $i \geq 2, j \geq 1$ and $\widetilde{\beta}_{ij} = \beta_{ij} + 1$ for all $i, j$. Then the map $\rho^{\Lambda}_{n,n+1}$ is injective and $i(S,T), i(\widetilde{S}, \widetilde{T})$ differ only by a 0, i.e., any non-zero entry in $i(S,T)$ occurs with the same multiplicity in $i(\widetilde{S}, \widetilde{T})$ and 0 occurs once more. Furthermore, if $k > \frac{d}{2} - 2$ then for any $(\widetilde{T}, \widetilde{S}) \in \Pi^{\Lambda+1}_{k+1} \setminus \rho^{\Lambda}_{n,n+1}(\Pi^{\Lambda}_{k})$ it must be $2\,\mathrm{ch}(\widetilde{T}, \widetilde{S}) > d$.*

*Proof.* For $(T, S) \in \Pi^{\Lambda}_{k}$ we note $(\widetilde{T}, \widetilde{S})$ is indeed a standard bitableau of shape $\Lambda + 1$, since increasing entries in all rows and columns are inherited from $(T, S)$. An integer $p$ occurs left of $p - 1$ in $w(T, S)$ if and only if $p + 1$ occurs left of $p$ in $w(\widetilde{T}, \widetilde{S})$. In particular, $i(T, S)$ and $i(\widetilde{T}, \widetilde{S})$ differ only by an additional 0 entry and hence their charges are equal.
Consider $f : \Pi^{\Lambda+1}_{k+1} \to \mathrm{YT}(\Lambda)$ which maps an element $(\widetilde{T}, \widetilde{S}) \in \Pi^{\Lambda+1}_{k+1}$ to a bitableau of shape $\Lambda$ by removing $\widetilde{\alpha}_{11}$, if $\widetilde{\alpha}_{11} \neq 1$. Otherwise, we remove the box containing the largest entry in the first row of $\widetilde{T}$ that is no longer the predecessor of the following number, and subtract 1 from any larger entry $\widetilde{\alpha}_{ij}, \widetilde{\beta}_{ij}$. Then $f$

is an inverse of $\rho^{\Lambda}_{n,n+1}$ and therefore $\rho^{\Lambda}_{n,n+1}$ is injective.
If $f(\widetilde{T},\widetilde{S})$ is not standard, we have $\lambda_2 \geq k+1$. For $k > \frac{d}{2}-2$ we have

$$2\operatorname{ch}(\widetilde{T},\widetilde{S}) \geq 2(k+2) > d.$$

■

**Definition I.3.20.** For $m > n \geq d$ and (bi-)partitions $\Lambda, \lambda \vdash n$ we define $\rho^{\lambda}_{n,m} := \rho^{\lambda+m-n-1}_{m-1,m} \circ \cdots \circ \rho^{\lambda}_{n,n+1}$ and $\rho^{\Lambda}_{n,m} := \rho^{\Lambda+m-n-1}_{m-1,m} \circ \cdots \circ \rho^{\Lambda}_{n,n+1}$.

Now, we can prove the stabilization of the isotypic decomposition, which was already proven in [Rie+13; Rie11] for the symmetric group with different methods.

**Theorem I.3.21.** *Let $n \in \mathbb{N}$, $\Lambda \vdash n$ and $Z_n \in \{A_{n-1}, B_n, D_n\}$. For large enough $n$ the $Z_n$- and $Z_{n+1}$-isotypic decompositions are equal in the sense that $\mathbb{S}^{(\lambda,\mu)}$ occurs with the same multiplicity in $H_{n,d}$ as $\mathbb{S}^{(\lambda+1,\mu)}$ in $H_{n+1,d}$. The stabilization of the isotypic decomposition of $H_{n,d}$ occurs from $n = 2d$ for $A_{n-1}$, $n = d$ for $B_n$ and $n = 2d+1$ for $D_n$.*

*Proof.* We restrict us to the cases $A_{n-1}$ with $n \geq 2d$, and $B_n$ with $n \geq d$. For $n > d$ the relevant fundamental invariants of degree $\leq d$ are equal for $B_n$ and $D_n$. Thus, for $D_n$ and $n > 2d$ the same argument as for $B_n$ applies, since no bipartition of $n$ can be of the form $(\lambda, \lambda)$. By iteration, it is sufficient to compare the isotypic decompositions of $H_{n,d}$ and $H_{n+1,d}$.

Let $n \geq d$ and $\Lambda = (\lambda, \mu) \vdash n$ be a bipartition with $|\mu| \leq d$ and $\kappa \vdash n \geq 2d$ be a partition. Further, be $f_1, \ldots, f_m$ a symmetry adapted basis of the isotypic compoment $\bigoplus_{i=1}^{m} \mathbb{S}^{\Lambda}$ (resp. $\bigoplus_{i=1}^{m} \mathbb{S}^{\kappa}$) from Theorem I.3.7. We suppose there exist $m$ standard (bi-)tableaux $T := T_1$ and $T_2 \ldots, T_m$ of shape $\Lambda$ (resp. $\kappa$) with $f_j = \pi \widehat{F}^{T_j}_{T}$ (resp. $f_j = \pi F^{T_j}_{T}$), for some $\pi \in \mathbb{R}[X]^{Z_n}$. $\pi$ can be chosen as a product of fundamental invariants of $Z_n$ by a change of basis, since $\pi f_j$ must be homogeneous. The degree of a polynomial $f_j$ is determined by the degrees of fundamental invariants $d_1, \ldots, d_n$, the charge of the standard (bi-)tableau $T_j$ and $|\mu|$.

The degrees $\leq d$ of fundamental invariants are equal for $n$ and $n+1$. By Lemma I.3.14 we have $T_1, \ldots, T_m \in \Pi^{\kappa}_{n-d}$ and by Lemma I.3.18 for all $i$ the tableau $\rho^{\kappa}_{n,n+1}(T_i)$ is standard with same charge as $T_i$. Furthermore, the map $\rho^{\kappa}_{n,n+1}$ is injective and any standard tableau that is not contained in the image has too large charge. The claim follows since only the standard tableaux in $\rho^{\kappa}_{k,k+1}(\Pi^{\kappa}_{k})$ are possible options for higher Specht polynomials in $H_{n+1,d}$.
By the Lemmas I.3.15 and I.3.19 the standard bitableaux $(T,S)$ of shape $\Lambda$ with $2\operatorname{ch}(T,S) \leq d$ are in bijection with the standard bitableaux $(\widetilde{T},\widetilde{S})$ of shape $\Lambda+1$ with $2\operatorname{ch}(\widetilde{T},\widetilde{S}) \leq d$ and the bijection preserves the charge. Furthermore, our bijection adds a 0 to the index of the image bitableau and preserves the other entries.
Finally, note that $\mathbb{S}^{(d,d)} \subset H_{2d,d}$, since the tableau $\begin{array}{|c|c|c|c|}\hline 1 & 2 & \ldots & d \\ \hline v & w & \ldots & 2d \\ \hline \end{array}$ with $v = d+1$ and $w = d+2$ has charge $d$, but $(d-1,d)$ is not a partition of $2d-1$. Similarly,

$\mathbb{S}^{(\emptyset,(d))} \subset H_{d,d}$ for the bitableau $(\emptyset\ ,\ \boxed{1|2|\ldots|d})$ with charge 0. For $D_n$ we observe $\mathbb{S}^{((d),(d))} \subset H_{2d,d}$ but the $D_{2d}$-module $\mathbb{S}^{((d),(d))}$ is special since it decomposes which does not happen for $\mathbb{S}^{((d-1),(d))}$. ■

We note the following for the group $D_n$. If $n = d$ then an additional fundamental invariant of degree $d$ occurs, which does not occur for $n > d$ anymore. Thus, at least the trivial representation occurs with larger multiplicity in $H_{d,d}$ than in $H_{d+1,d}$.

**Corollary I.3.22.** *For a fixed degree $d$ and a sequence $(Z_n)_n$ of reflection groups $(A_{n-1})_n$, $(B_n)_n$ or $(D_n)_n$ the symmetry adapted description of the set $\Sigma_{n,2d}^{Z_n}$ are equal up to the map $\rho_{n,m}^{\Lambda}$, for $n \geq 2d$, $n \geq d$ or $n > 2d$ respectively.*

The corollary says that up to $\rho_{n,m}^{\Lambda}$ and $\rho_{n,m}^{\lambda}$ the same matrix polynomials can be used in a sum of squares representation.

*Proof.* This follows from Theorem I.3.21 and Lemmas I.3.14, I.3.15, I.3.18, I.3.19. ■

The case $n = 2d$ is the last where $\Lambda \vdash n$ can be of the form $\Lambda = (\lambda, \lambda)$, i.e., the $B_{2d}$-module $\mathbb{S}^{\Lambda}$ is not $D_{2d}$-irreducible in $H_{n,d}$ but the $B_{2d+1}$-module $\mathbb{S}^{\Lambda+1}$ is $D_{2d+1}$-irreducible in $H_{n+1,d}$ (see Theorem I.3.7). Nevertheless, the multiplicities of $\mathbb{S}^{\Lambda}$ in $H_{2d,d}$ and $H_{n,d}$ are equal for $n \geq 2d$. Moreover, whenever $n \geq d$ for $B_n$, or $n > d$ in case of $D_n$ one can use that if $\mathbb{S}^{\Lambda} \subset H_{n,d}$, for $\Lambda = (\lambda, \mu) \vdash n$, and $d$ even (odd), then $|\mu|$ must also be even (odd).

## I.4 Concrete examples and applications

We apply the results from the preceding Section I.3 to solve nonnegativity versus sums of squares questions. In contrast to the non-equivariant case, the $B_n$-invariant forms have a non-trivial equality between the sets of even symmetric sums of squares and nonnegative forms in 3 variables and degree 8. This was proven by Harris [Har99]. In fact, it turns out that this case and quartics are the only non-trivial equality cases [GKR17]. We will present a characterization of the dual and primal cones of $B_3$-invariant sums of squares ternary octics and obtain a new elementary proof of Harris' theorem. Moreover, we study $D_n$-invariant forms, prove that $\mathcal{P}_{4,4}^{D_4}$ is a simplicial cone and answer the nonnegativity versus sums of squares question there.
In general, testing nonnegativity of a polynomial in more than two variables is already for quartics an NP-hard problem (see e.g., [Blu+98] or [MK85]). In equivariant situations, it is therefore of interest to exploit the symmetry of invariant polynomials to reduce this complexity. The works in [AV16; FRS18; Har99; MRV21; Rie12; Rie16; Tim03] focus on providing test sets for verification of nonnegativity of invariant polynomials. We also examine test sets for $B_n$ and $D_n$ invariant forms and small degrees.
We remark that each group in the infinite series $I_2(m)$ of dihedral groups acts on $\mathbb{R}^2$. In particular, any $I_2(m)$ invariant nonnegative form is a sum of squares.

### I.4.1 Even symmetric octics

One of the well known and rare cases of equality between sums of squares and nonnegative forms in equivariant situations was proven by Harris [Har99]. Harris' proof is quite analytical. In this subsection we derive a lower dimensional test set for nonnegativity of even symmetric ternary octics and as a byproduct we give a new proof of equality. Furthermore, we present a uniform description of the cones of $n$-ary even symmetric sums of squares octics.

**Theorem I.4.1.** *The dual cone of even symmetric ternary octic sums of squares has the following description*

$$\Sigma_{3,8}^{B_3,*} = \left\{ \mathrm{ev}_{\left(a,\sqrt{1-a^2},0\right)}, \mathrm{ev}_{(b,c,c)} : \frac{1}{2} \leq a \leq 1, 0 \leq b \leq 1, c = \sqrt{\frac{(1-b^2)}{2}} \right\}.$$

As a consequence of Theorem I.4.1 we can give a new proof for Harris' result.

**Corollary I.4.2** ([Har99], Theorem 4.1)**.** *The sets of nonnegative and sums of squares even symmetric ternary octics are equal, i.e.,* $\Sigma_{3,8}^{B_3} = \mathcal{P}_{3,8}^{B_3}$.

*Proof.* By Theorem I.4.1 the cone $\Sigma_{3,8}^{B_3,*}$ is generated by point-evaluations. Thus, the claim follows from Corollary I.2.39. ∎

We elaborate a study of the even symmetric sums of squares ternary octics.

**Lemma I.4.3.** *The $B_3$-module $H_{3,4}$ has the isotpyic decomposition*

$$H_{3,4} = 2 \cdot \mathbb{S}^{((3),\emptyset)} \oplus 2 \cdot \mathbb{S}^{((2,1),\emptyset)} \oplus 2 \cdot \mathbb{S}^{((1),(2))} \oplus \mathbb{S}^{((1),(1,1))}.$$

*A symmetry adapted basis of $H_{3,4}$ realising the $B_3$-isotypic decomposition is given by the following polynomials:*

$$\mathbb{S}^{((3),\emptyset)} : \left\{ e_1(X^2)^2, e_2(X^2) \right\}, \qquad \mathbb{S}^{((2,1),\emptyset)} : \left\{ e_1(X^2)(X_3^2 - X_1^2), X_2^2X_3^2 - X_1^2X_2^2 \right\},$$
$$\mathbb{S}^{((1),(2))} : \left\{ e_1(X^2)X_2X_3, X_1^2X_2X_3 \right\}, \quad \mathbb{S}^{((1),(1,1))} : \left\{ (X_3^2 - X_2^2)X_2X_3 \right\}.$$

*Proof.* We need to determine the multiplicity of the irreducible $B_3$-modules $\mathbb{S}^{(\lambda,\mu)}$ in $H_{3,4}$ for all bipartitions $(\lambda, \mu) \vdash 3$. We can immediately exclude some of them. Since we only need higher Specht polynomials of degree $0, 2$ or $4$ by Theorem I.3.10, the degree - which equals 2 times the charge of a standard bitableau of shape $(\lambda, \mu)$ plus $|\mu|$ - must be $0, 2$ or $4$. However, this implies that only bipartitions with $\mu \in \{\emptyset, (2), (1,1)\}$ are feasible to obtain an even degree. By going through all the remaining cases one obtains precisely the following higher Specht polynomials of degree $0, 2$ and $4$:

$$\{1, X_3^2 - X_1^2, X_2^2X_3^2 - X_1^2X_2^2, X_2X_3, X_1^2X_2X_3, (X_3^2 - X_2^2)X_2X_3.\}$$

Multiplying by the invariants $1, e_1(X^2)^2$ and $e_2(X^2)$ results accordingly in the above mentioned symmetry adapted basis. ∎

**Corollary I.4.4.** *A form $f \in H_{3,8}^{B_3}$ is a sum of squares if and only if there exist positive semidefinite matrices $A^{(1)}, A^{(2)}, A^{(3)} \in \mathbb{R}^{2\times 2}$ and $A^{(4)} \in \mathbb{R}^{1\times 1}$ such that*

$$f = \langle A^{(1)}, B^{(1)}\rangle + \langle A^{(2)}, B^{(2)}\rangle + \langle A^{(3)}, B^{(3)}\rangle + \langle A^{(4)}, B^{(4)}\rangle$$

*where the $B^{(j)}$'s are the following matrix polynomials corresponding to the $B_3$-modules in $H_{3,4}$*

$$B^{(1)} := \begin{pmatrix} e_1(X^2)^4 & e_1(X^2)^2e_2(X^2) \\ e_1(X^2)^2e_2(X^2) & e_2(X^2)^2 \end{pmatrix},$$

$$B^{(2)} := \begin{pmatrix} \frac{2}{3}e_1(X^2)^4 - 2e_1(X^2)^2e_2(X^2) & -3e_1(X^2)e_3(X^2) + \frac{1}{3}e_1(X^2)^2e_2(X^2) \\ -3e_1(X^2)e_3(X^2) + \frac{1}{3}e_1(X^2)^2e_2(X^2) & \frac{2}{3}e_2(X^2)^2 - 2e_1(X^2)e_3(X^2) \end{pmatrix},$$

$$B^{(3)} := \begin{pmatrix} \frac{1}{3}e_1(X^2)^2e_2(X^2) & e_1(X^2)e_3(X^2) \\ e_1(X^2)e_3(X^2) & \frac{1}{3}e_1(X^2)e_3(X^2) \end{pmatrix},$$

$$B^{(4)} := \left(e_1(X^2)e_3(X^2) - \frac{4}{3}e_2(X^2)^2 + \frac{1}{3}e_1(X^2)^2e_2(X^2)\right).$$

*Proof.* The matrices $B^{(1)}, \ldots, B^{(4)}$ are the symmetrizations of the products of the symmetry adapted basis from Lemma I.4.3. By Theorem I.2.6 any invariant sum of squares form has such a representation. ∎

**Corollary I.4.5.** *A linear form $\ell \in H_{3,8}^{B_3,*}$ is contained in $\Sigma_{3,8}^{B_3,*}$ if and only if the following four matrices are positive semidefinite*

$$\begin{pmatrix} m_{(1^4)} & m_{(2,1^2)} \\ m_{(2,1^2)} & m_{(2^2)} \end{pmatrix}, \quad \begin{pmatrix} \frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)} & \frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)} \\ \frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)} & \frac{2}{3}m_{(2^2)} - 2m_{(3,1)} \end{pmatrix},$$
$$\begin{pmatrix} \frac{1}{3}m_{(2,1^2)} & m_{(3,1)} \\ m_{(3,1)} & \frac{1}{3}m_{(3,1)} \end{pmatrix}, \quad \left(\ \frac{1}{3}m_{(2,1^2)} - \frac{4}{3}m_{(2^2)} + m_{(3,1)}\ \right),$$

*where we write $m_{(1^4)} := \ell(e_1(X^2)^4), m_{(3,1)} := \ell(e_1(X^2)e_3(X^2)), m_{(2,1^2)} := \ell(e_1(X^2)^2e_2(X^2))$ and $m_{(2^2)} := \ell(e_2(X^2)^2)$.*

*Proof.* By Lemma I.2.36 this is precisely the dual statement to Corollary I.4.4. ∎

*Remark* I.4.6. We observe

$$\begin{aligned} H_{3,8}^{B_3} &= \langle p_2^4, p_2^2p_4, p_2p_6, p_4^2\rangle_{\mathbb{R}} \\ &= \langle e_1(X^2)^4, e_1(X^2)e_3(X^2), e_1(X^2)^2e_2(X^2), e_2(X^2)^2\rangle_{\mathbb{R}} \end{aligned}$$

is a 4-dimensional $\mathbb{R}$-vector space. We choose as fundamental invariants the elementary symmetric polynomials evaluated in $X^2 = (X_1^2, X_2^2, X_3^2)$ and work with the $\mathbb{R}$-basis

$$(e_1(X^2)^4, e_1(X^2)e_3(X^2), e_1(X^2)^2e_2(X^2), e_2(X^2)^2)$$

of $H_{3,8}^{B_3}$. We study the extremal rays in $\Sigma_{3,8}^{B_3,*}$ and show that all of them are spanned by point-evaluations. This is then used to prove Theorem I.4.1.
In the remaining part of the subsection we always use the following notation for

an extremal element $\ell \in \Sigma_{3,8}^{B_3,*}$. $\mathcal{Q}_\ell$ denotes the associated $B_3$-invariant quadratic form on $H_{3,4}$, $W_\ell := \ker \mathcal{Q}_\ell$ its kernel and

$$W_\ell^{\langle 2\rangle} := \ker \ell = \left\{\mathcal{R}_G\left(\sum f_i g_i\right) \in H_{n,2d}^G \,:\, f_i \in W, g_i \in H_{n,d}\right\}$$

(see Proposition I.2.38). A hyperplane in $H_{3,8}^{B_3}$ is of dimension 3. Thus, we must have $\dim W_\ell^{\langle 2\rangle} = 3$. By Lemma I.4.3 the isotypic decomposition of the $B_3$-submodule $W_\ell$ of $H_{3,4}$ has the form

$$W_\ell = \ker \mathcal{Q}_\ell = \alpha \cdot \mathbb{S}^{((3),\emptyset)} \oplus \beta \cdot \mathbb{S}^{((2,1),\emptyset)} \oplus \gamma \cdot \mathbb{S}^{((1),(2))} \oplus \delta \cdot \mathbb{S}^{((1),(1,1))},$$

where $\alpha, \beta, \gamma \in \{0,1,2\}$ and $\delta \in \{0,1\}$.
Frequently, we use that $\ker \ell$ is maximal among all kernels of elements in $\Sigma_{3,8}^{G,*}$, i.e., when $\ker \ell$ contains a non-trivial zero then $\ell$ must be a scalar of the point-evaluation at this point (see Lemma I.2.37).

In the following lemmas we analyse possible combinations of the integers $\alpha, \beta, \gamma$ and $\delta$ through a case distinction to obtain a classification of all extremal elements in $\Sigma_{3,8}^{B_3,*}$.

**Lemma I.4.7.** *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element. Then $\alpha < 2$, i.e., the multipilcity of the trivial representation in $W_\ell$ is smaller than 2.*

*Proof.* If $\alpha = 2$ then $e_1(X^2)^2 \in W_\ell$ and hence $e_1(X^2)^4 \in W_\ell^{\langle 2\rangle} = \ker \ell$. However, any monomial of degree 8 that is a square occurs with positive coefficients in $e_1(X^2)^4$, which implies $\ell = 0$ must be the 0 map. ∎

**Lemma I.4.8.** *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element and $\alpha = 0$. Then $\ell$ is a scalar of the point-evaluation $\mathrm{ev}_z$, where $z \in \{(1,1,1),(1,0,0),(1,1,0)\}$.*

*Proof.* In the case $\beta = 2$ we know by dimension reasons on $W_\ell^{\langle 2\rangle}$ that any other $B_3$-module occurring in $W_\ell$ must already be contained in $2 \cdot \mathbb{S}^{((2,1),\emptyset)}$. However, the forms in the module $2 \cdot \mathbb{S}^{((2,1),\emptyset)}$ have the common zero $(1,1,1)$.
If $\beta = 1$, then it must be $\gamma \geq 1$ or $\delta = 1$ such that $W_\ell^{\langle 2\rangle}$ is a hyperplane. For $\delta = 1$ the elements in $W_\ell$ have the common root $(1,1,1)$. Now, we consider the case $\beta = 1, \gamma \geq 1$. Thus for some pairs $(a,b),(c,d) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$ae_1(X^2)(X_3^2 - X_1^2) + b(X_2^2X_3^2 - X_1^2X_2^2), ce_1(X^2)X_2X_3 + dX_1^2X_2X_3 \in W_\ell,$$

and the symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2\rangle}$, i.e.,

$$\begin{aligned}
0 =& a\left(\frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)}\right) + b\left(\frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)}\right),\\
0 =& a\left(\frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)}\right) + b\left(\frac{2}{3}m_{(2^2)} - 2m_{(3,1)}\right),\\
0 =& \frac{c}{3}m_{(2,1^2)} + dm_{(3,1)},\\
0 =& cm_{(3,1)} + \frac{d}{3}m_{(3,1)}.
\end{aligned}$$

Now, we distinguish between $m_{(3,1)}$ equals or not equals 0:

i) In the case that $m_{(3,1)} \neq 0$ we have that $c + \frac{d}{3} = 0$. Since $W_\ell$ is a linear space we can set $c = 1$ and $d = -3$. However, then the $B_3$-module $W_\ell$ has the common zero $(1,1,1)$. Thus $\ell$ is a scalar of the point-evaluation $\mathrm{ev}_{(1,1,1)}$.

ii) Let $m_{(3,1)} = 0$. We first assume that $c \neq 0$. Then $m_{(2,1^2)} = 0$ and since $m_{(1^4)} > 0$ we have $a = 0$. Hence, $b \neq 0$ and $m_{(2^2)} = 0$ which implies that the elements in $W_\ell$ all vanish at $(1,0,0)$ and $\ell$ is a scalar of $\mathrm{ev}_{(1,0,0)}$.
If $c = 0$ we have

$$\begin{aligned} 0 =& a\left(\frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)}\right) + b\left(\frac{1}{3}m_{(2,1^2)}\right), \\ 0 =& a\left(\frac{1}{3}m_{(2,1^2)}\right) + b\left(\frac{2}{3}m_{(2^2)}\right). \end{aligned}$$

If $a = 0$ then $\ell$ is a scalar of $\mathrm{ev}_{(1,0,0)}$, since any form in $W_\ell^{\langle 2\rangle}$ has the zero $(1,0,0)$. Otherwise, we may assume that $a = 1$ since $W_\ell^{\langle 2\rangle}$ is a linear space. It is

$$\begin{aligned} 0 =& \frac{2}{3}m_{(1^4)} + (-2 + \frac{b}{3})m_{(2,1^2)}, \\ 0 =& \frac{1}{3}m_{(2,1^2)} + \frac{2b}{3}m_{(2^2)}. \end{aligned}$$

Through scaling of $\ell$ and $m_{(1^4)} > 0$, we can assume that $m_{(1^4)} = 1$. If $b = 0$, then $0 = m_{(1^4)} = 1$ which cannot be true. So $b \neq 0$ and $m_{(2,1^2)} = \frac{2}{6-b}, m_{(2^2)} = \frac{1}{-6b+b^2}$, for a non zero $b \neq 6$. From the positive semidefiniteness conditions in Corollary I.4.5 we obtain from the first matrix

$$\det\begin{pmatrix} 1 & m_{(2,1^2)} \\ m_{(2,1^2)} & m_{(2^2)} \end{pmatrix} \geq 0,$$

which implies that $-2 \leq b < 0$. And the positive semidefiniteness of the last matrix in I.4.5

$$\frac{1}{3}m_{(2,1^2)} - \frac{4}{3}m_{(2^2)} + m_{(3,1)} \geq 0$$

implies that $b \leq -2$ or $0 < b < 6$. Thus $b = -2$ and $\ell$ is the point-evaluation $\mathrm{ev}_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)}$.

Finally, if $\gamma \geq 1$, then $\beta = 1$ or $\delta = 1$. However, we have already examinied the case $\beta = 1$. For $\delta = 1$ the elements in $W_\ell$ have the common zero $(1,0,0)$. Thus $\ell$ is a scalar of $\mathrm{ev}_{(1,0,0)}$. ∎

Therefore we proceed with the cases where $\alpha = 1$, which implies that $ae_1(X^2)^2 + e_2(X^2) \in W_\ell$ for an $a \in \mathbb{R}$, since $e_1(X^2)^4 \notin W_\ell$. This means for $\ker \ell$

$$am_{(1^4)} + m_{(2,1^2)} = 0,$$

$$am_{(2,1^2)} + m_{(2^2)} = 0.$$

Moreover, since $m_{(1^4)} > 0$ and since $\ell$ is a linear form, without loss of generality we can suppose $m_{(1^4)} = 1$, as $\ell$ is then just a positive scalar. The positive semidefinitness conditions with the reductions $m_{(2,1^2)} = -am_{(1^4)}, m_{(2^2)} = a^2 m_{(1^4)}$ and $m_{(1^4)} = 1$ are that the following four matrices must be positive semidefinite.

$$\begin{pmatrix} 1 & -a \\ -a & a^2 \end{pmatrix}, \begin{pmatrix} \frac{2}{3}+2a & -\frac{1}{3}a-3m_{(3,1)} \\ -\frac{1}{3}a-3m_{(3,1)} & \frac{2}{3}a^2-2m_{(3,1)} \end{pmatrix}, \begin{pmatrix} -\frac{1}{3}a & m_{(3,1)} \\ m_{(3,1)} & \frac{1}{3}m_{(3,1)} \end{pmatrix}, \begin{pmatrix} \frac{-a}{3}-\frac{4a^2}{3}+m_{(3,1)} \end{pmatrix} \quad \text{(I.3)}$$

From the positive semidefiniteness of the second matrix and $-a = m_{(2,1^2)} \geq 0$ we obtain $a \in [-\frac{1}{3}, 0]$.

We now proceed with a case distinction on the paramaters $\beta, \gamma, \delta$:

**Lemma I.4.9.** *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element. If $\alpha = \delta = 1$, then $\ell$ is a scalar of a point-evaluation in $(1,1,0)$.*

*Proof.* $\delta = 1$ means that $\mathbb{S}^{((1),(1,1))} \subset W_\ell$ which implies $(X_3^2 - X_2^2)X_2X_3 \in W_\ell$ and

$$-\frac{a}{3} - \frac{4a^2}{3} + m_{(3,1)} = 0.$$

Positiveness yields $0 \leq m_{(3,1)} = \frac{1}{3}(a + 4a^2)$ and therefore that $a \leq -\frac{1}{4}$. We use that the determinant of the second matrix in (I.3) is nonnegative, i.e.,

$$0 \leq \left(\frac{2}{3} + 2a\right)\left(\frac{2}{3}a^2 - 2m_{(3,1)}\right) - \left(-\frac{1}{3}a - 3m_{(3,1)}\right)^2 = -\frac{4}{9}a(1+3a)^2(1+4a).$$

This is not satisfied for $a < -\frac{1}{4}$. Hence $a = -\frac{1}{4}, m_{(1^4)} = 1, m_{(3,1)} = 0, m_{(2,1^2)} = \frac{1}{4}, m_{(2^2)} = \frac{1}{16}$ and $\ell$ is a scalar of $\mathrm{ev}_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}$. ■

**Lemma I.4.10.** *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element. If $\alpha = 1, \gamma \geq 1$, then $\ell$ is a scalar of a point-evaluation in $(1,0,0), (1,1,1)$ or $\left(\sqrt{\frac{1}{2} + \sqrt{a + \frac{1}{4}}}, \sqrt{\frac{1}{2} - \sqrt{a + \frac{1}{4}}}, 0\right)$, for $-\frac{1}{4} \leq a \leq 0$.*

*Proof.* We have $\mathbb{S}^{((1),(2))} \subset W_\ell$, i.e., for a pair $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$be_1(X^2)X_2X_3 + cX_1^2X_2X_3 \in W_\ell$$

and the symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2 \rangle}$, i.e.,

$$\begin{aligned} 0 =& b\frac{-a}{3} + cm_{(3,1)}, \\ 0 =& bm_{(3,1)} + \frac{c}{3}m_{(3,1)}. \end{aligned}$$

Inserting $\frac{ab}{3} = cm_{(3,1)}$ in the second equation gives $b\left(\frac{a}{9} + m_{(3,1)}\right) = 0$.

a) We first assume that $b \neq 0$. Then $m_{(3,1)} = -\frac{a}{9}$. In this case we obtain from the positive semidefiniteness of the second matrix in (I.3) that

$$0 \leq \frac{2}{3}a^2 - 2m_{(3,1)} = \frac{2}{3}a(a + \frac{1}{3}).$$

Thus $a \in \{0, -\frac{1}{3}\}$. If $a = 0$ then $m_{(3,1)} = m_{(2,1^2)} = m_{(2^2)} = 0$ and $\ell = \mathrm{ev}_{(1,0,0)}$. For $a = -\frac{1}{3}$ it is $m_{(3,1)} = \frac{1}{27}, m_{(2,1^2)} = \frac{1}{3}, m_{(2^2)} = \frac{1}{9}$ and $\ell = \mathrm{ev}_{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}$.

b) In the remaining case $b = 0$ we can assume by linearity of $W_\ell$ that $c = 1$, which implies $m_{(3,1)} = 0$. By the nonnegativity of the last $1 \times 1$ matrix in (I.3), i.e.,

$$0 \leq -\frac{a}{3} - \frac{4a^2}{3} + m_{(3,1)}$$

we obtain $-\frac{1}{4} \leq a \leq 0$. However, for any such $-\frac{1}{4} \leq a \leq 0$ it is $m_{(1^4)} = 1, m_{(3,1)} = -a, m_{(2,1^2)} = a^2, m_{(2^2)} = 0$ and $\ell = \mathrm{ev}_{\left(\sqrt{\frac{1}{2}+\sqrt{a+\frac{1}{4}}},\sqrt{\frac{1}{2}-\sqrt{a+\frac{1}{4}}},0\right)}$.

∎

**Lemma I.4.11.** *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element. If $\alpha = \beta = 1$, then $\ell$ is a scalar of a point-evaluation in $\left(\sqrt{\frac{1+2\sqrt{1+3a}}{3}}, \sqrt{\frac{1-\sqrt{1+3a}}{3}}, \sqrt{\frac{1-\sqrt{1+3a}}{3}}\right)$, for $-\frac{1}{3} \leq a \leq 0$, or at $\left(\sqrt{\frac{1-2\sqrt{1+3b}}{3}}, \frac{\sqrt{1+\sqrt{1+3b}}}{3}, \frac{\sqrt{1+\sqrt{1+3b}}}{3}\right)$, for $-\frac{1}{3} \leq b \leq -\frac{1}{4}$.*

*Proof.* If $\beta = 1$ then $\mathbb{S}^{((2,1),\emptyset)} \subset W_\ell$, i.e., for a pair $(b, c) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$be_1(X^2)(X_3^2 - X_1^2) + c(X_2^2X_3^2 - X_1^2X_2^2) \in W_\ell$$

and the symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2\rangle}$, i.e.,

$$\begin{aligned} 0 =& b\left(\frac{2}{3} + 2a\right) + c\left(-\frac{1}{3}a - 3m_{(3,1)}\right), \\ 0 =& b\left(-\frac{1}{3}a - 3m_{(3,1)}\right) + c\left(\frac{2}{3}a^2 - 2m_{(3,1)}\right). \end{aligned}$$

We distinguish two cases:

i) If $b = 0, c = 1$ or if $b = 1, c = 0$ then $-\frac{1}{3} = a, m_{(3,1)} = \frac{1}{27}$ and $\ell = \mathrm{ev}_{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}$.

ii) We continue with the remaining case $b \neq 0$ and $c \neq 0$. Since $W_\ell$ is a vector space we assume without loss of generality that $b = 1$ and obtain $m_{(3,1)} = \frac{2}{9c} + \frac{2a}{3c} - \frac{a}{9}$ and $\frac{2(1+3a)(-3-2c+ac^2)}{9c} = 0$. Hence $a = \frac{-1}{3}$ (then

$\ell = \mathrm{ev}_{(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})}$ or $-3-2c+ac^2=0$. If $a=0$ then $c=-\frac{3}{2}$ and $m_{(3,1)}=-\frac{4}{27}$ which does not satisfy the positive semidefiniteness conditions. If $-\frac{1}{3}<a<0$ then either $c=\frac{1}{a}-\sqrt{\frac{1+3a}{a^2}}$ or $c=\frac{1}{a}+\sqrt{\frac{1+3a}{a^2}}$.

In the first case it is $m_{(1^4)}=1, m_{(3,1)}=\frac{a\left(1+a\left(6+\sqrt{\frac{1+3a}{a^2}}\right)\right)}{9-9a\sqrt{\frac{1+3a}{a^2}}}, m_{(2,1^2)}=-a, m_{(2^2)}=a^2$. For any $-\frac{1}{3}<a<0$ $\ell$ is the point-evaluation at $\left(\sqrt{\frac{1+2\sqrt{1+3a}}{3}},\sqrt{\frac{1-\sqrt{1+3a}}{3}},\sqrt{\frac{1-\sqrt{1+3a}}{3}}\right)$.

In the second case it is $m_{(1^4)}=1, m_{(3,1)}=\frac{a\left(1-a\left(-6+\sqrt{\frac{1+3a}{a^2}}\right)\right)}{9+9a\sqrt{\frac{1+3a}{a^2}}}, m_{(2,1^2)}=-a, m_{(2^2)}=a^2$. However, $m_{(3,1)}\geq 0$ is equivalent to $-\frac{1}{3}<a\leq -\frac{1}{4}$. For any $-\frac{1}{3}<a\leq-\frac{1}{4}$ $\ell$ is the point-evaluation at $\left(\sqrt{\frac{1-2\sqrt{1+3a}}{3}},\frac{\sqrt{1+\sqrt{1+3a}}}{3},\frac{\sqrt{1+\sqrt{1+3a}}}{3}\right)$.

∎

*Proof of Theorem I.4.1.* In Lemmas I.4.7, I.4.8, I.4.9, I.4.10 and I.4.11 we have seen that the extremal rays in $\Sigma_{3,8}^{B_3,*}$ are all generated by point-evaluations. Those generators are the point-evaluations at elements in the set

$$\left\{\left(a,\sqrt{1-a^2},0\right),(b,c,c):\frac{1}{2}\leq a\leq 1, 0\leq b\leq 1, c=\frac{1}{\sqrt{2}}\sqrt{(1-b^2)}\right\}.$$

∎

**Corollary I.4.12.** *$\mathcal{P}_{3,8}^{B_3}$ is the convex cone generated by the following six forms*

$e_1(X^2)^4-3e_1(X^2)^2e_2(X^2), -9e_1(X^2)e_3(X^2)+e_1(X^2)^2e_2(X^2), e_2(X^2)^2-3e_1(X^2)e_3(X^2),$
$e_1(X^2)^2e_2(X^2), e_1(X^2)e_3(X^2), 3e_1(X^2)e_3(X^2)-4e_2(X^2)^2+e_1(X^2)^2e_2(X^2)$

*and the following two families of forms*

$$\left(\ ae_1(X^2)^4+e_1(X^2)e_2(X^2), ae_1(X^2)e_2(X^2)+e_2(X^2)^2:-\tfrac{1}{3}\leq a\leq 0\ \right)$$

*Proof.* These forms are precisely the sums of squares contained in the kernels of extremal rays of $\Sigma_{3,8}^{B_3,*}$. Since $\Sigma_{3,8}^{B_3}=\mathcal{P}_{3,8}^{B_3}$ by Corollary I.4.2, these forms are also the extremal elements in the pointed convex cone $\mathcal{P}_{3,8}^{B_3}$. The claim follows from Minkowski's theorem. ∎

*Remark* I.4.13. Harris showed that $\Omega:=\{(a,a,b),(0,a,b):a,b\in\mathbb{R}_{\geq 0}\}$ is a test set for even symmetric ternary octics and used this as main ingredient in his proof of equality [Har99]. In fact, our description in Theorem I.4.1 provides the subset of $\Omega$ consisting of all points of norm 1, which we derived from describing $\Sigma_{3,8}^{B_3,*}$.

Note, Harris result does not follow from Hilbert's equality case $\Sigma_{3,4}^{\mathfrak{S}_3} = \mathcal{P}_{3,4}^{\mathfrak{S}_3}$ under canonical identification through the $\mathfrak{S}_3$-isomorphism

$$\begin{array}{cccc}\Phi: & H_{3,8}^{B_3} & \longrightarrow & H_{3,4}^{\mathfrak{S}_3} \\ & \sum_{\alpha\in 2\mathbb{N}_0^3} c_\alpha X^\alpha & \longmapsto & \sum_{\alpha\in 2\mathbb{N}_0^3} c_\alpha X^{\frac{1}{2}\alpha}\end{array}.$$

For $g \in H_{3,4}^{\mathfrak{S}_3}$ we have $\Phi^{-1}(g) = g(X_1^2, X_2^2, X_3^2)$. Then $g$ is nonnegative on $\mathbb{R}_{\geq 0}^3$ if and only if $\Phi^{-1}(g)$ is nonnegative. However, the example

$$f(X) := e_1(X^2)e_3(X^2) = (X_1^2 + X_2^2 + X_3^2)(X_1^2X_2^2X_3^2) \in \mathcal{P}_{3,8}^{B_3}$$

with $\Phi(f)(-1,-1,1) = -1 < 0$ shows $\mathcal{P}_{3,4}^{\mathfrak{S}_3} \subsetneq \Phi(\mathcal{P}_{3,8}^{B_3})$.

We demonstrate Theorem I.3.21, i.e., the stabilization of $B_n$-Specht modules in $H_{n,d}$ for large enough number of variables for $d = 4$. This allows a uniform description of the sets $\Sigma_{n,8}^{B_n}$, as observed in Corollary I.3.22.
We work with *power means* $p_i^{(n)}(X) := \frac{1}{n}\sum_{j=1}^n X_j^i \in \mathbb{R}[X]^{\mathfrak{S}_n}$. The upper index $n$ denotes that $p_i^{(n)}$ is a power mean in $n$ variables. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ we write $p_\lambda^{(n)} := p_{\lambda_1}^{(n)} \cdot \ldots \cdot p_{\lambda_l}^{(n)}$. A reason for working with power means is that they are weighted, i.e., for all $i$ and all $n$ we have $p_i^{(n)}(1,1,\ldots,1) = 1$ and $p_i^{(n)}(1,0,\ldots,0) = \frac{1}{n}$.

**Lemma I.4.14.** *The $B_n$-isotypic decomposition of $H_{n,4}$ for $n \geq 4$ is*

$$2\cdot\mathbb{S}^{((n),\emptyset)} \oplus 2\cdot\mathbb{S}^{((n-1,1),\emptyset)} \oplus \mathbb{S}^{((n-2,2),\emptyset)} \oplus 2\cdot\mathbb{S}^{((n-2),(2))} \oplus \mathbb{S}^{((n-2),(1,1))} \oplus \mathbb{S}^{((n-3,1),(2))} \oplus \mathbb{S}^{((n-4),(4))}.$$

*A symmetry adapted basis of $H_{n,4}$ realising the $B_n$-isotypic decomposition is given by the following seven sets of polynomials*

$$\mathbb{S}^{((n),\emptyset)}: \left\{p_{(4)}^{(n)}, p_{(2^2)}^{(n)}\right\}, \qquad \mathbb{S}^{((n-1,1),\emptyset)}: \left\{(X_n^2 - X_1^2)p_{(2)}^{(n)}, X_n^4 - X_1^4\right\},$$

$$\mathbb{S}^{((n-2,2),\emptyset)}: \left\{(X_1^2 - X_3^2)(X_2^2 - X_4^2)\right\}, \qquad \mathbb{S}^{((n-2),(1,1))}: \left\{(X_n^2 - X_{n-1}^2)X_{n-1}X_n\right\},$$

$$\mathbb{S}^{((n-4),(4))}: \{X_1X_2X_3X_4\}, \qquad \mathbb{S}^{((n-3,1),(2))}: \left\{(X_n^2 - X_1^2)X_{n-2}X_{n-1}\right\}$$

$$\mathbb{S}^{((n-2),(2))}: \left\{X_{n-1}X_np_{(2)}^{(n)}, (X_{n-1}^2 + X_n^2)X_{n-1}X_n\right\}.$$

*Proof.* We determine the multiplicity of an irreducible $B_n$-module $\mathbb{S}^{(\lambda,\mu)}$ in $H_{n,4}$ for bipartitions $(\lambda,\mu) \vdash n$ using Theorem I.3.7. We can immediately exclude some bipartitions. The fundamental invariants of degree $\leq 4$ are of degree 2 and 4. Only $(\lambda,\mu)$ such that $\mu \vdash n_2$, with $n_2 \leq 4$ can occur, since a corresponding higher Specht polynomial has as a factor the monomial consisting of all products of the $X_i$'s, where $i$ ranges over the entries of the second bitableau. Furthermore, we only need to consider bipartitions $(\lambda,\mu)$ such that $|\mu|$ is even because a factor of the higher Specht polynomial is of degree $|\mu|$, while the additional factor has even degree. We can restrict us to bipartitions $(\lambda,\mu)$ such that there exist

$(T,S) \in \mathrm{SYT}(\lambda,\mu)$ with $2\,\mathrm{ch}(T,S)+|\mu| \leq 4$. Therefore a charge $\leq 2$ is necessary. We calculated all relevant higher Specht polynomials for $n \geq 4$:

$$\mathbb{S}^{((n),\emptyset)} : \{1\}, \qquad \mathbb{S}^{((n-1,1),\emptyset)} : \left\{ X_n^2 - X_1^2, \frac{1}{n}\sum_{i=2}^{n-1} X_i^2(X_n^2 - X_1^2) \right\},$$

$$\mathbb{S}^{((n-2,2),\emptyset)} : \left\{(X_1^2 - X_3^2)(X_2^2 - X_4^2)\right\}, \qquad \mathbb{S}^{((n-2),(1,1))} : \left\{(X_n^2 - X_{n-1}^2)X_{n-1}X_n\right\},$$

$$\mathbb{S}^{((n-4),(4))} : \{X_1X_2X_3X_4\}, \qquad \mathbb{S}^{((n-3,1),(2))} : \left\{(X_n^2 - X_1^2)X_{n-2}X_{n-1}\right\}.$$

and $\mathbb{S}^{((n-2),(2))} : \left\{X_{n-1}X_n, \frac{1}{n-2}(X_1^2 + \ldots + X_{n-2}^2)X_{n-1}X_n\right\}$. Multiplying them with power means gives a $B_n$-symmetry adapted basis of $H_{n,4}$. However, since

$$X_n^4 - X_1^4 \in \langle p_2^{(n)}(X_n^2 - X_1^2), \frac{1}{n}\sum_{i=2}^{n-1} X_i^2(X_n^2 - X_1^2)\rangle_{\mathbb{R}},$$

$$(X_{n-1}^2 + X_n^2)X_{n-1}X_n \in \langle p_2^{(n)}X_{n-1}X_n, \frac{1}{n-2}(X_1^2 + \ldots + X_{n-2}^2)X_{n-1}X_n\rangle_{\mathbb{R}},$$

we can work with the above mentioned symmetry adapted basis. ■

**Theorem I.4.15.** *For $n \geq 4$, $f \in H_{n,8}^{B_n}$ is a sum of squares if and only if there exist positive semidefinite matrices $A^{((n),\emptyset)}, A^{((n-1,1),\emptyset)}, A^{((n-2,2),\emptyset)}, A^{((n-2),(2))} \in \mathbb{R}^{2\times 2}$ and $A^{((n-2),(1,1))}, A^{((n-4),(4))}, A^{((n-3,1),(2))} \in \mathbb{R}^{1\times 1}_{\geq 0}$ such that*

$$\begin{aligned} \mathfrak{f} = {} & \langle A^{((n),\emptyset)}, B^{((n),\emptyset)}\rangle + \langle A^{((n-1,1),\emptyset)}, B^{((n-1,1),\emptyset)}\rangle + \langle A^{((n-2,2),\emptyset)}, B^{((n-2,2),\emptyset)}\rangle \\ & + \langle A^{((n-2),(2))}, B^{((n-2),(2))}\rangle + A^{((n-2),(1,1))}B^{((n-2),(1,1))} \\ & + A^{((n-4),(4))}B^{((n-4),(4))} + A^{((n-3,1),(2))}B^{((n-3,1),(2))} \end{aligned}$$

*where*

$$B^{((n),\emptyset)} := \begin{pmatrix} p^{(n)}_{(4^2)} & p^{(n)}_{(4,2^2)} \\ p^{(n)}_{(4,2^2)} & p^{(n)}_{(2^4)} \end{pmatrix},$$

$$B^{((n-1,1),\emptyset)} := \begin{pmatrix} p^{(n)}_{(4,2^2)} - p^{(n)}_{(2^4)} & p^{(n)}_{(6,2)} - p^{(n)}_{(4,2^2)} \\ p^{(n)}_{(6,2)} - p^{(n)}_{(4,2^2)} & p^{(n)}_{(8)} - p^{(n)}_{(4^2)} \end{pmatrix},$$

$$B^{((n-2,2),\emptyset)} := \left( \frac{-n+1}{n^2} p^{(n)}_{(8)} + \frac{4n-4}{n^2} p^{(n)}_{(6,2)} + \frac{n^2-3n+3}{n^2} p^{(n)}_{(4^2)} - 2p^{(n)}_{(4,2^2)} + p^{(n)}_{(2^4)} \right),$$

$$B^{((n-2),(2))} := \begin{pmatrix} p^{(n)}_{(2^4)} - \frac{1}{n}p^{(n)}_{(4,2^2)} & 2p^{(n)}_{(4,2^2)} - \frac{2}{n}p^{(n)}_{(6,2)} \\ 2p^{(n)}_{(4,2^2)} - \frac{2}{n}p^{(n)}_{(6,2)} & 2p^{(n)}_{(6,2)} + 2p^{(n)}_{(4^2)} - \frac{4}{n}p^{(n)}_{8} \end{pmatrix},$$

$$B^{((n-2),(1,1))} := \left( p^{(n)}_{(6,2)} - p^{(n)}_{(4^2)} \right),$$

$$B^{((n-4),(4))} := \left( p^{(n)}_{(2^4)} - \frac{6}{n}p^{(n)}_{(4,2^2)} + \frac{3}{n^2}p^{(n)}_{(4^2)} + \frac{8}{n^2}p^{(n)}_{(6,2)} - \frac{6}{n^3}p^{(n)}_{(8)}) \right),$$

$$B^{((n-3,1),(2))} := \left( \frac{2}{n^2}p^{(n)}_{(8)} - \frac{2n+2}{n^2}p^{(n)}_{(6,2)} - \frac{1}{n}p^{(n)}_{(4^2)} + \frac{n+3}{n}p^{(n)}_{(4,2^2)} - p^{(n)}_{(2^4)} \right).$$

*Proof.* The matrices $B^{(i)}$ are the matrices which contain the symmetrized products of the symmetry adapted basis of the $B_n$-module $H_{n,4}$ from Lemma I.4.14. By Theorem I.2.6 any invariant sums of squares form has such a representation. ∎

We observe that for $n \geq 4$ the $\mathbb{R}$-vector spaces

$$H_{n,8}^{B_n} = \langle p_{(2^4)}^{(n)}, p_{(4,2^2)}^{(n)}, p_{(4^2)}^{(n)}, p_{(4,2)}^{(n)}, p_{(6,2)}^{(n)}, p_8^{(n)} \rangle_{\mathbb{R}}$$

have the same dimension. We identify the vector spaces with respect to the isomorphism

$$p_\lambda^{(n)} \mapsto p_\lambda^{(m)}$$

for $n, m \in \mathbb{N}_{\geq 4}$. Blekherman and the second author studied symmetric quartic forms [BR21] and defined a limit set as the linear span of all $\mathfrak{p}_\lambda := \lim_{n\to\infty} p_\lambda^{(n)}$. They showed that for symmetric quartics the limits of the cones of sums of squares and nonnegative forms are equal. As a first step towards a similar result in the $B_n$ case we provide a classification of the limit of the cones of even symmetric octics which are sums of squares.

*Remark* I.4.16. The matrices in Theorem I.4.15 have the following limits for $n \to \infty$

$$\begin{aligned}
\mathcal{B}^{((n),\emptyset)} &:= \begin{pmatrix} \mathfrak{p}_{(4^2)} & \mathfrak{p}_{(4,2^2)} \\ \mathfrak{p}_{(4,2^2)} & \mathfrak{p}_{(2^4)} \end{pmatrix} \\
\mathcal{B}^{((n-1,1),\emptyset)} &:= \begin{pmatrix} \mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)} & \mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)} \\ \mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)} & \mathfrak{p}_{(8)} - \mathfrak{p}_{(4^2)} \end{pmatrix}, \\
\mathcal{B}^{((n-2,2),\emptyset)} &:= \left( \mathfrak{p}_{(4^2)} - 2\mathfrak{p}_{(4,2^2)} + \mathfrak{p}_{(2^4)} \right), \\
\mathcal{B}^{((n-2),(2))} &:= \begin{pmatrix} \mathfrak{p}_{(2^4)} & 2\mathfrak{p}_{(4,2^2)} \\ 2\mathfrak{p}_{(4,2^2)} & 2\mathfrak{p}_{(6,2)} + 2\mathfrak{p}_{(4^2)} \end{pmatrix}, \\
\mathcal{B}^{((n-2),(1,1))} &:= \left( \mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)} \right), \\
\mathcal{B}^{((n-4),(4))} &:= \left( \mathfrak{p}_{(2^4)} \right), \\
\mathcal{B}^{((n-3,1),(2))} &:= \left( \mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)} \right).
\end{aligned}$$

**Corollary I.4.17.** *An even symmetric homogeneous octic limit sum of squares inequality $\mathfrak{f}$ has the form*

$$\begin{aligned}
\mathfrak{f} = {} & \alpha_1 \mathfrak{p}_{(4^2)} + 2\alpha_2 \mathfrak{p}_{(4,2^2)} + \alpha_3 \mathfrak{p}_{(2^4)} \\
& + \beta_1(\mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)}) + 2\beta_2(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)}) + \beta_3(\mathfrak{p}_{(8)} - \mathfrak{p}_{(4^2)}) \\
& + \delta(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)})
\end{aligned}$$

*where* $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{pmatrix}, (\delta)$ *are positive semidefinite real matrices.*

*Proof.* We observe that an invariant limit sum of squares coming from the irreducible representation $\mathbb{S}^{((n-2,2),\emptyset)}$, i.e., $\mathfrak{p}_{(4^2)} - 2\mathfrak{p}_{(4,2^2)} + \mathfrak{p}_{(2^4)}$, is contained in the first line. The limit sum of squares $\mathfrak{p}_{(2^4)}$ from $\mathbb{S}^{((n-4),(4))}$ is also contained in the first line, while the limit form from $\mathbb{S}^{((n-3,1),(2))}$, i.e., $\mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)}$, is contained in the second line for $\beta_1 = 1$. Furthermore,

$$(\alpha,\beta)\begin{pmatrix} \mathfrak{p}_{(2^4)} & 2\mathfrak{p}_{(4,2^2)} \\ 2\mathfrak{p}_{(4,2^2)} & 2\mathfrak{p}_{(6,2)} + 2\mathfrak{p}_{(4^2)} \end{pmatrix}(\alpha,\beta)^T$$
$$= 2\beta^2(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)}) + \langle \begin{pmatrix} 4\beta^2 & 2\alpha\beta \\ 2\alpha\beta & \alpha^2 \end{pmatrix}, \begin{pmatrix} \mathfrak{p}_{(4^2)} & \mathfrak{p}_{(4,2^2)} \\ \mathfrak{p}_{(4,2^2)} & \mathfrak{p}_{(2^4)} \end{pmatrix} \rangle.$$

■

It is a question for further studies to determine the relation between the limit cones of even symmetric sums of squares and nonnegatives octics.

### I.4.2 Forms invariant under $D_n$

It is natural to wonder, to what extend Harris' result on ternary forms invariant under $B_3$ carries over to the slightly smaller group $D_3$. As is shown in the following theorem we obtain equality between the sets $\Sigma_{3,8}^{D_3}$ and $\mathcal{P}_{3,8}^{D_3}$. Furthermore, we prove that $\mathcal{P}_{4,4}^{D_4}$ is a simplicial cone which gives a test set for nonnegativity consisting of three points. We prove that for quaternary quartics invariant under $D_4$ we also have that nonnegativity implies having a sums of squares representation. We conclude with a full characterization of the nonnegativity versus sums of squares question for forms invariant under $D_n$.

**Theorem I.4.18.** *The sets of nonnegative and sums of squares ternary octics invariant under $D_3$ are equal, i.e., $\Sigma_{3,8}^{D_3} = \mathcal{P}_{3,8}^{D_3}$.*

*Proof.* The invariant ring $\mathbb{R}[X_1, X_2, X_3]^{D_3} = \mathbb{R}[p_2, e_3, p_4]$ is a polynomial ring in the symmetric polynomials $p_2, e_3$ and $p_4$. A vector space basis of $H_{3,8}^{D_3}$ is given by $\left(p_{(2^4)}, p_{(4,2^2)}, p_{(4^2)}, p_2e_3^2\right)$. In Remark I.4.6 we have seen that $H_{3,8}^{B_3} = \langle p_{(2^4)}, p_{(4,2^2)}, p_{(4^2)}, p_{(6,2)}\rangle_{\mathbb{R}}$. The functions $p_6$ and $e_3^2$ occur linearly in the following identity for symmetric functions in three variables

$$p_{(2^3)} - 3p_{(4,2)} + 2p_6 - 6e_3^2 = 0.$$

Hence we deduce that $H_{3,8}^{D_3} = H_{3,8}^{B_3}$. The claim follows by Corollary I.4.2. ■

*Remark* I.4.19. We have the same conical generators and test set for nonnegative ternary octics invariant under $D_3$ as for $B_3$, i.e., a form $f \in H_{3,8}^{D_3}$ is nonnegative if and only if $f(y) \geq 0$ for all $y \in \{(a,a,b), (0,a,b) : a, b \in \mathbb{R}_{\geq 0}\}$.

In the following we study quaternary quartics invariant under $D_4$.

**Lemma I.4.20.** *The $D_4$-module $H_{4,2}$ has the isotypic decomposition*

$$H_{4,2} = \mathbb{S}^{((4),\emptyset)} \oplus \mathbb{S}^{((3,1),\emptyset)} \oplus \mathbb{S}_+^{((2),(2))} \oplus \mathbb{S}_-^{((2),(2))}.$$

*A symmetry adapted basis which realizes the $D_4$-module decomposition of $H_{4,2}$ is the following:*

$$\mathbb{S}^{((4),\emptyset)} : \{p_{(2)}\}, \qquad \mathbb{S}^{((3,1),\emptyset)} : \{X_4^2 - X_1^2\},$$
$$\mathbb{S}_+^{((2),(2))} : \{X_1X_2 + X_3X_4\}, \qquad \mathbb{S}_-^{((2),(2))} : \{X_1X_2 - X_3X_4\}.$$

*Proof.* By Theorem I.3.7 we have to determine the multiplicity of the irreducible $D_4$-modules labelled by bipartitions $(\lambda,\mu) \vdash 4$ of the form $|\lambda| \geq |\mu|$. We are just interested in higher Specht polynomials of degree 0 or 2, since the only $D_4$ fundamental invariant of degree $\leq 2$ is $p_2$. Thus, it must be $|\mu| \in \{0,2\}$. If $\mu = \emptyset$, then both bipartitions $((4),\emptyset),((3,1),\emptyset)$ have exactly one standard bitableau whose charge is at most 1, i.e., they occur precisely once in $H_{4,2}$. Any occurring module labelled by $(\lambda,\mu)$ with $|\mu| = 2$ must have a standard bitableau with index $(0,0,0,0)$. This can only occur if the word equals $(1,2,3,4)$. Thus, only the bipartition $((2),(2))$ has a standard bitableau with charge 0. By Theorem I.3.7 the module $\mathbb{S}^{((2),(2))}$ decomposes into two irreducible $D_4$-modules $\mathbb{S}_+^{((2),(2))}$ and $\mathbb{S}_-^{((2),(2))}$. We calculated the relevant higher Specht polynomials according to Theorem I.3.7

$$\{1,\, X_4^2 - X_1^2,\, X_1X_2 + X_3X_4,\, X_1X_2 - X_3X_4\},$$

and find accordingly the polynomials above. ∎

**Corollary I.4.21.** *A $D_4$-invariant quaternary quartic $f \in H_{4,4}^{D_4}$ is a sum of squares if and only if there exist positive numbers $A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)} \in \mathbb{R}_{\geq 0}$ such that $f = A^{(1)}B^{(1)} + A^{(2)}B^{(2)} + A^{(3)}B^{(3)} + A^{(4)}B^{(4)}$, where*

$$B^{((4),\emptyset)} := \left(p_{(2^2)}\right), \qquad B^{((3,1),\emptyset)} := \left(\frac{2}{3}p_{(4)} - \frac{1}{6}p_{(2^2)}\right),$$
$$B_+^{((2),(2))} := \left(\frac{1}{6}p_{(2^2)} - \frac{1}{6}p_{(4)} + 2e_4\right), \quad B_-^{((2),(2))} := \left(\frac{1}{6}p_{(2^2)} - \frac{1}{6}p_{(4)} - 2e_4\right).$$

*Proof.* The matrices $B^{(i)}$ are obtained by calculating the Reynolds operator evaluated at squares of the symmetry adapted basis of the irreducible $D_4$-modules from Lemma I.4.20. By Theorem I.2.6 any invariant sum of squares form has such a representation. ∎

**Theorem I.4.22.** *$\Sigma_{4,4}^{D_4,*}$ is a simplicial cone with the following description $\Sigma_{4,4}^{D_4,*} = \operatorname{cone}\left\{\mathrm{ev}_{(1,0,0,0)}, \mathrm{ev}_{(1,1,1,-1)}, \mathrm{ev}_{(1,1,1,1)}\right\}$.*

*Proof.* Let $\ell \in \Sigma_{4,4}^{D_4,*}$ denote an extremal element. Let

$$W_\ell := \alpha \cdot \mathbb{S}^{((4),\emptyset)} \oplus \beta \cdot \mathbb{S}^{((3,1),\emptyset)} \oplus \gamma \cdot \mathbb{S}_+^{((2),(2))} \oplus \delta \cdot \mathbb{S}_-^{((2),(2))}$$

be the $D_4$-submodule of $H_{4,2}$ which is the kernel of the associated quadratic form, for $\alpha,\beta,\gamma,\delta \in \{0,1\}$. Now, we show that $\ell$ must be a scalar of one of

the three point-evaluations above, respectively that $W_\ell^{\langle 2\rangle}$ must have one of the points as a zero.
Since $p_{(2^2)}$ is not contained in the boundary of $\Sigma_{4,4}^{D_4}$ it must be $\alpha = 0$. Furthermore, $\dim_{\mathbb{R}} W_\ell^{\langle 2\rangle} = 2$ and therefore we have that precisely two of the parameters are non-zero, because the symmetrized squares of the symmetry adapted basis elements belonging to the $D_4$-modules $\mathbb{S}^{((3,1),\emptyset)}$, $\mathbb{S}_+^{((2),(2))}$ and $\mathbb{S}_-^{((2),(2))}$ are linearly independent.

i) We start by examining the case $\gamma = \delta = 1$. Then $\ell(e_4) = 0, \ell(p_{(2^2)}) = \ell(p_{(4)})$ and

$$W_\ell^{\langle 2\rangle} = \langle e_4, p_{(2^2)} - p_{(4)}\rangle_{\mathbb{R}}.$$

$W_\ell^{\langle 2\rangle}$ has the root $(1,0,0,0)$.

We proceed with the cases $\gamma = \beta = 1$ or $\beta = \delta = 1$.

ii) We notice that if $\gamma = \beta = 1$ then

$$W_\ell = \langle X_4^2 - X_1^2, X_1X_2 + X_3X_4\rangle_{D_4},$$

but all elements in $W_\ell$ have the common root $(1,1,1,-1)$.

iii) If $\beta = \delta = 1$ then

$$W_\ell = \langle X_4^2 - X_1^2, X_1X_2 - X_3X_4\rangle_{D_4}$$

with the common root $(1,1,1,1)$.

■

**Corollary I.4.23.** *The set of nonnegative and sums of squares quaternary quartics invariant under $D_4$ are equal, i.e., $\Sigma_{4,4}^{D_4} = \mathcal{P}_{4,4}^{D_4}$.*

The corollary does not already follow from $\Sigma_{4,4}^{B_4} = \mathcal{P}_{4,4}^{B_4}$ [Har99], because $H_{4,4}^{D_4} \setminus H_{4,4}^{B_4} \neq \emptyset$.

*Proof.* By Theorem I.4.22 the cone $\Sigma_{4,4}^{D_4,*}$ is generated by point-evaluations. Hence any extremal ray in $\Sigma_{4,4}^{D_4,*}$ is spanned by a point-evaluation and the claim follows from Corollary I.2.39. ■

By reformulating Theorem I.4.22 we obtain the following very simple test set for $D_4$-quartics:

**Corollary I.4.24.** *A form $f(X) = a(X_1^2 + X_2^2 + X_3^2 + X_4^2)^2 + b(X_1^4 + X_2^4 + X_3^4 + X_4^4) + cX_1X_2X_3X_4$, with $a, b, c \in \mathbb{R}$, is nonnegative if and only if $f(z) \geq 0$ for all $z \in \{(1,0,0,0), (1,1,1,-1), (1,1,1,1)\}$.*

*Proof.* An invariant form $f \in H_{4,4}^{D_4}$ is nonnegative if and only if $\ell(f) \geq 0$ for all $\ell \in \mathcal{P}_{4,4}^{D_4,*}$. By Corollary I.4.23 we have $\mathcal{P}_{4,4}^{D_4,*} = \Sigma_{4,4}^{D_4,*}$. But we saw in Theorem I.4.22 that $\Sigma_{4,4}^{D_4,*}$ has those 3 extreme rays. ■

**Corollary I.4.25.** *The convex cone $\mathcal{P}_{4,4}^{D_4}$ of nonnegative $D_4$-quartics is a simplicial cone generated by*

$$4p_{(4)} - p_{(2^2)}, \quad p_{(2^2)} - p_{(4)} + 12e_4, \quad p_{(2^2)} - p_{(4)} - 12e_4.$$

*Proof.* The sets $\mathcal{P}_{4,4}^{D_4}$ and $\Sigma_{4,4}^{D_4}$ are equal by Corollary I.4.23. The boundary of $\Sigma_{4,4}^{D_4}$ is equal to the union of all kernels of extremal elements in $\Sigma_{4,4}^{D_4,*}$ intersected with $\Sigma_{4,4}^{D_4}$. The claimed forms are precisely the invariant sums of squares contained in the kernels of the three extremal rays in Theorem I.4.22. ∎

The results from the previous two subsections allow to conclude the following classification of the equivariant nonnegativity versus sums of squares question for the reflection group $D_n$.

**Theorem I.4.26.** *The sets $\Sigma_{n,2d}^{D_n}$ and $\mathcal{P}_{n,2d}^{D_n}$ are equal if and only if $(n, 2d) \in \{(2, 2d), (n, 2), (n, 4), (3, 8)\}$.*

*Proof.* Suppose that there exists $f \in \mathcal{P}_{n,2d}^{B_n} \setminus \Sigma_{n,2d}^{B_n}$. This implies $f \in \mathcal{P}_{n,2d}^{D_n} \setminus \Sigma_{n,2d}^{D_n}$. Therefore, we can directly rely on the classification carried out in [GKR17] and we only need to consider those cases specifically, where all even symmetric positive semidefinite forms are sums of squares. These are only the following non-trivial cases: $(n, 2d) \in \{(3, 8), (n, 4)\}$. But we have shown in Theorem I.4.18 that in the case $(3, 8)$ the equality does survive, and while following Corollary I.4.23 it does also for $(4, 4)$. Furthermore, if $n > 4$ then the invariant quartics with respect to $B_n$ are precisely the invariant quartics with respect to $D_n$ as $H_{n,4}^{B_n} = \langle p_{(2^2)}, p_{(4)} \rangle_{\mathbb{R}} = H_{n,4}^{D_n}$ for $n \geq 5$, which completes the proof. ∎

### I.4.3 LMIs and nonnegativity testing

In general testing nonnegativity of a polynomial in more than two variables is already for quartics an NP-hard problem (see e.g. [Blu+98] or [MK85]). On the other hand, certifying that a given polynomial is a sum of squares can be done with so called semidefinite programming. Although the complexity status of this procedure in the Turing or in the real numbers model is not yet known (see [Ram97]) SDPs can be solved numerically in polynomial time to a given accuracy through the ellipsoid algorithm and this approach generally provides a tractable way to certify that a polynomial is nonnegative, if it is a sum of squares. For real symmetric matrices $A, B \in \mathbb{R}^{n \times n}$ we write $A \succeq B$ if $A - B$ is positive semidefinite. The feasible region of a semidefinite program is given by the projection of a set defined by a *linear matrix inequality* (LMI), i.e., an inequality of the form $A_0 + x_1A_1 + x_2A_2 + \ldots + x_nA_n \succeq 0$, where $A_0, \ldots, A_n$ are real symmetric matrices all of the same size and $x_1, \ldots, x_n$ are supposed to be real scalars. The set of all $x \in \mathbb{R}^n$ satisfying a given LMI is called a *spectrahedron*. For every $f \in H_{n,2d}$ one can construct an LMI ([PW98]) which possesses a solution if and only if $f$ is a sum of squares. The corresponding spectrahedron is called the *Gram spectrahedron* of $f$ [Chu+16], and it represents in fact all possible ways to decompose $f$ into sums of squares. Accordingly, it is non-empty if and only if $f$ is a sum of squares. The results presented in the article can be directly transferred into the setup of symmetry adapted Gram-spectrahedra, which were, for example, recently studied by [HHS21].

**Theorem I.4.27.** *Let $G$ be a finite reflection group and consider $f \in H_{n,2d}^G$ and $\theta_1, \ldots, \theta_l$ be all non $G$-isomorphic irreducible representations. Then the Gram spectrahedron of $f$ can be defined by a block diagonal matrix, consisting of $l$ blocks $B_1, \ldots, B_l$ and the size of the block $B_i$ equals*

$$\sum_{k=0}^{d} N(d-k) \cdot h_k^{\vartheta_i}.$$

*In particular, in the case $G \in \{A_{n-1}, B_n, D_n\}$ the size of the matrix is independent of $n$, for large $n$*

*Proof.* This follows from choosing a symmetry adapted basis of $H_{n,d}$ and Corollary I.2.26. When $G \in \{A_{n-1}, B_n, D_n\}$ the stabilization follows from Corollary I.3.22 ∎

A convex set which can be obtained as the projection of a higher dimensional spectrahedron is called *spectrahedral shadow*. Following a question by Nemirovski, which convex sets can be represented as projections of spectrahedra, Scheiderer [Sch18] showed that the cones of nonnegative forms in general are not spectrahedral shadows. In the next theorem we give some examples of invariant nonnegative forms, which form spectrahedral shadows.

**Theorem I.4.28.** *For all $n$ the families of cones $\mathcal{P}_{n,4}^{\mathfrak{S}_n}$, $\mathcal{P}_{n,6}^{B_n}$, $\mathcal{P}_{n,8}^{B_n}$ and $\mathcal{P}_{n,10}^{B_n}$ are spectrahedral shadows. Moreover, for forms in any of these families, there exists an LMI of size $O(n^3)$ certifying the nonnegativity.*

*Proof.* For $n \leq 2$ this is trivial, and in the case $n = 3$ this follows either from Hilbert's Theorem in the $\mathfrak{S}_3$ case or from Harris' result I.4.2 in the $B_3$ case. So we assume $n \geq 4$. By the half-degree principle, an element $f \in H_{n,4}^{\mathfrak{S}_n}$ is nonnegative on $\mathbb{R}^n$ if and only if for any partition $\lambda \vdash n$ of length 2 the form $f^\lambda \in H_{2,4}$ is nonnegative on $\mathbb{R}^2$, where $f^\lambda(x,y) := f(x,\ldots,x,y,\ldots,y)$ and $x$ occurs precisely $\lambda_1$ times and $y$ $\lambda_2$ times. Notice that each $f^\lambda$ is nonnegative if and only if it is a sum of squares, i.e., if we have $f^\lambda \in \Sigma_{2,4}$. If we denote by $\Phi^\lambda$ the linear map $f \mapsto \tilde{f}^\lambda(x,y)$ and if $\lambda^1, \ldots, \lambda^m$ are all partitions of $n$ with length 2 then

$$\mathcal{P}_{n,4}^{\mathfrak{S}_n} = \bigcap_{i=1}^{m} \left(\Phi^{\lambda^i}\right)^{-1} (\Sigma_{2,4})$$

which proves the claim in the $\mathfrak{S}_n$ case. Using the half-degree principle [Rie16, Theorem 3.1] for $B_n$ and considering instead of $f(X) \in \mathbb{R}[X]^{B_n}$ the form $f(\sqrt{X_1}, \ldots, \sqrt{X_n}) \in \mathbb{R}[X]^{\mathfrak{S}_n}$, one can argue analogously with slight modifications. ∎

*Remark* I.4.29. In the case of symmetric polynomials, the above statement was implicitly already stated in [Rie+13, Theorem 5.5] for symmetric quartic forms, albeit without mentioning of the term spectrahedral shadow.

The core of the proof above is the reduction to bivariate forms through test sets.

**Theorem I.4.30.** *For the families of cones $\mathcal{P}_{n,6}^{\mathfrak{S}_n}$, $\mathcal{P}_{n,12}^{B_n}$ and $\mathcal{P}_{n,14}^{B_n}$ membership can be decided with $O(n^3)$ many LMIs, each of which has size bounded independent of $n$.*

*Proof.* Using the half-degree principle [Rie16, Theorem 3.1] one finds that membership in each of the above mentioned cones can be decided by reducing to $O(n^3)$ many ternary forms, similarly to the proof above. For each of these ternary forms, one can decide nonnegativity individually. De Klerk and Pasechnik [KP04] provided a construction to decide nonnegativity of a ternary form of degree $2d$ by means of $d/4$ LMIs each of which is polynomial in $d$. Combining their construction with the arguments above thus yields an LMI of the announced size. ∎

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Paper II

# At the limit of symmetric nonnegative forms

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### Abstract

We study nonnegative symmetric and even symmetric forms uniformly in the number of variables for a fixed degree and investigate their limits which are known as symmetric functions. We relate our study to the Vandermonde map at infinity, test sets for nonnegativity, classify all cases for which the sets of nonnegative and sums of squares symmetric functions are equal, and prove undecidability of verifiability of nonnegativity for multisymmetric functions. Finally, we present an alternative approach to prove strict inclusion between the sets of (even) symmetric homogeneous functions which are nonnegative and sums of squares respectively based on tropicalization. Tropicalization also provides quantitative information on the difference between these sets.

## Contents



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## II.1 Introduction

A real polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is called nonnegative if the associated function assumes only nonnegative values and it is called a *sum of squares* if there exists a decomposition $f = q_1^2 + \ldots + q_k^2$ for real polynomials $q_i \in \mathbb{R}[X_1, \ldots, X_n]$. Clearly, in the latter case $f$ is also nonnegative and moreover the algebraic identity of a sum of squares decomposition certifies this nonnegativity. In particular, also the fact that sums of squares decompositions can be found rather efficiently with the help of *semidefinite programming* has revived the interest in the question, which polynomials are sums of squares, which originated from the work of Hilbert. In this article we are focusing on symmetric and even symmetric polynomials. Such polynomials are characterized through the property of being invariant by all permutations of the variables. The natural restriction from symmetric polynomials in $n+1$ variables to the symmetric polynomials in $n$ variables defines an inverse system with a limit which is sometimes also referred to as the ring of *symmetric functions* [Mac98], and we study the limit objects corresponding to the symmetric nonnegative polynomials and symmetric sums of squares respectively. The elements of these limit can naturally be seen as functions defined on the *image at infinity* of the so called *Vandermonde map*, which has been studied extensively already by Arnold, Givental, and Kostov in finitely many variables [Arn86; Giv87; Kos04; Kos89; Kos99] and at infinity [Kos04; Kos07]. Sometimes the image is defined via the first elementary symmetric polynomials. This different viewpoint amounts to study the Vandermonde map composed with a polynomial diffeomorphism induced by *Newton's identities*. We study this set in detail in the even symmetric setup, where we provide a description of the boundary in the finite case (Theorem II.3.6) and show that the image at infinity is not semialgebraic (see Corollary II.3.12) expanding on earlier work of Kostov. We then turn to the cone of nonnegative symmetric homogeneous functions which also is shown to be not semialgebraic. Moreover, we show that the elements of these limit cones are, *multisymmetric* setup, not computationally traceable, i.e., the membership problem for this cone is undecidable (see Theorem II.6.1). This is in sharp contrast to the case of finitely many variables, where it follows from Artin's solution to Hilbert's 17th problem that verifying nonnegativity of any polynomial is decidable. Recently, *tropicalization* has been applied beyond classical algebraic geometry in extremal combinatorics [Ble+22b; BR22] and applied real algebraic geometry [AB22; Ble+22a]. Using tropicalization in the sense of log-limits we study the combinatorial shadows of the dual cones of the nonnegative and sums of squares homogeneous even symmetric functions. The tropicalization of the image of the Vandermonde map at infinity is a polyhedral convex cone (see Theorem II.7.2) and we provide via tropicalization a quantification of the difference between the sets of nonnegative and sums of squares even symmetric homogeneous functions. The limit behavior between the sets of sums of squares and nonnegative polynomials was investigated earlier. The second author proved that for a fixed degree there are significantly more nonnegative polynomials than sums of squares [Ble06] by comparing the ratio of the volumes of conical compact bases of these sets in different number of variables. It is shown that

the ratio grows in the number of variables and asymptotic bounds are given. Limit behavior of the corresponding sets for symmetric forms has also been investigated before but this was done using a different notion of limit.
Every symmetric polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ can be uniquely written as a polynomial in the power sums $f = g(p_1^{(n)}, \dots, p_n^{(n)})$. *Normalized limits* of symmetric polynomials have the form $\lim_{n\to\infty} g(p_1^{(n)}/n, \dots, p_n^{(n)}/n)$. The second and fourth author proved that normalized limits of symmetric nonnegative and sums of squares forms are equal in degree 4 and conjectured that equality is true in any degree [BR21]. However, the first and second author showed in [AB22] that equality for normalized limits does not survive to any higher degree and strict inclusion occurs from degree 10 onward for the normalized limits of even symmetrics forms but not for smaller degrees.

The paper is structured as follows: Section II.2 overviews the situation of symmetric polynomials and functions, and introduces the limits of the cones of invariant nonnegative and sums of squares forms. We initiate the study of the even Vandermonde map in Section II.3 with the description of its boundary and explicit parametrization in the plane. In Section II.4 we study the convex hull of the image of the map whose coordinates are the elementary symmetrics and analyse its combinatorial properties. Following this, we classify the cases of equality between the limit cones of invariant nonnegative and sums of squares forms and provide explicit examples in Section II.5. In Section II.6 we prove that determining validity of nonnegativity is undecidable for multisymmetric functions. Finally, we present a different approach to the study of nonnegativity versus sums of squares using tropicalization in Section II.7, before concluding the paper with closing remarks and open questions in Section II.8.

## II.2 Symmetric polynomials and functions

### II.2.1 Symmetric polynomials and partitions

We call a polynomial *symmetric* if it is invariant with respect to the action of the symmetric group $\mathcal{S}_n$. The algebra of symmetric polynomials $\mathbb{R}[X_1, \dots, X_n]^{\mathcal{S}_n}$ is isomorphic to a polynomial ring. Our prototype of symmetric polynomials which we often use are the *power sum polynomials* and *elementary symmetric polynomials* $p_k^{(n)} := \sum_{i=1}^n X_i^k$ and $e_k^{(n)} := \sum_{I\subset[n],|I|=k} \prod_{i\in I} X_i$ . It is classically known that the polynomials $p_1^{(n)}, \dots, p_n^{(n)}$ and $e_1^{(n)}, \dots, e_n^{(n)}$ are algebraically independent, and the ring of symmetric polynomials is isomorphic to a polynomial ring in the power sums and in the elementary symmetrics, i.e.,

$$\mathbb{R}[X]^{\mathcal{S}_n} = \mathbb{R}[p_1^{(n)}, \dots, p_n^{(n)}] = \mathbb{R}[e_1^{(n)}, \dots, e_n^{(n)}] \ .$$

Thus, any symmetric polynomial $f \in \mathbb{R}[X_1, \dots, X_n]^{\mathcal{S}_n}$ can be written as $f = g(p_1^{(n)}, \dots, p_n^{(n)})$ for a unique $n$-variate real polynomial $g$.
We write $\mathcal{B}_n$ for the hyperoctahedral group acting on $\mathbb{R}[X_1, \dots, X_n]$ via permutation of variables and switching of signs and note $\mathcal{B}_n = \{\pm 1\} \wr \mathcal{S}_n$.

A $\mathcal{B}_n$-invariant polynomial is called *even symmetric*. The ring of even symmetric polynomials is isomorphic to a polynomial ring in the even power sums $p_{2i}^{(n)}(X) = p_i^{(n)}(X^2)$ and in the elementary symmetrics evaluated in $X^2 := (X_1^2, \ldots, X_n^2)$, i.e.,

$$\mathbb{R}[X]^{\mathcal{B}_n} = \mathbb{R}[p_2^{(n)}, \ldots, p_{2n}^{(n)}] = \mathbb{R}[e_1^{(n)}(X^2), \ldots, e_n^{(n)}(X^2)] \ .$$

A priori, the power sums and the elementary symmetrics provide equally good bases to work with the vector space of (even) symmetric forms of a fixed degree. We frequently interchange between elementary symmetrics and power sums since we observe that both bases have advantages and disadvantages in the study of invariant nonnegative and sums of squares forms.

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$ is a sequence of non-increasing, positive integers whose *value* $|\lambda| := \sum_{i=1}^{l} \lambda_i$ equals $n$. We say that $\emptyset$ is the unique partition of 0 and write $\lambda \vdash n$ for $\lambda$ being a partition of $n$. A *bipartition* of $n$ is a pair of partitions $(\lambda, \mu)$ satisfying $|\lambda| + |\mu| = n$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash d$ we write $p_\lambda^{(n)} := p_{\lambda_1}^{(n)} p_{\lambda_2}^{(n)} \cdots p_{\lambda_l}^{(n)}$ and $e_\lambda^{(n)} := e_{\lambda_1}^{(n)} e_{\lambda_2}^{(n)} \cdots e_{\lambda_l}^{(n)}$ .

Let $H_{n,d}^{\mathcal{S}_n}$ ($H_{n,d}^{\mathcal{B}_n}$) denote the vector space of (even) symmetric forms in $n$ variables of degree $d$. The vector space $H_{n,d}^{\mathcal{S}_n}$ has the linear bases $\left(p_\lambda^{(n)} : \lambda \vdash d\right)$ and $\left(e_\lambda^{(n)} : \lambda \vdash d\right)$. Analogously, vector space bases of $H_{n,2d}^{\mathcal{B}_n}$ are given by $\left(p_{2\lambda}^{(n)} : \lambda \vdash d\right)$ and $\left(e_\lambda^{(n)}(X^2) : \lambda \vdash d\right)$, where $2\lambda = (2\lambda_1, 2\lambda_2, \ldots)$. Note, $\dim H_{n,d}^{\mathcal{S}_n} = \dim H_{n,2d}^{\mathcal{B}_n} = \pi(d)$ the number of partitions of $d$ for all $n \geq d$.

### II.2.2 Symmetric and even symmetric sums of squares

The problem of verifying nonnegativity occurs naturally in applications, e.g., in polynomial optimization. It is known to be an NP-hard problem already for polynomials of degree 4 [Blu+98; MK85].

A real polynomial $f(X_1, \ldots, X_n)$ is *nonnegative* if the polynomial attains only nonnegative values, i.e., if $f(a) \geq 0$ for all $a \in \mathbb{R}^n$. When a real polynomial $p \in \mathbb{R}[X_1, \ldots, X_n]$ can be written as a sum of squares of real polynomial, i.e., if $p = q_1^2 + \ldots + q_m^2$ for some polynomials $q_1, \ldots, q_m \in \mathbb{R}[X_1, \ldots, X_n]$, then $p$ is called a *sum of squares*. It turns out that a polynomial is nonnegative (a sum of squares) if and only if its homogenization is nonnegative (a sum of squares). A form of degree $2d$ can only be a sum of squares of forms of degree $d$. Sums of squares are obviously nonnegative but the converse statement is not true in general. In 1888 Hilbert proved that any nonnegative form in $n$ variables and degree $2d$ is a sum of squares if and only if $(n, 2d) \in \{(2, 2d), (n, 2), (3, 4)\}$. Nevertheless, it is regarded as a non-trivial task to provide nonnegative forms that are not sums of squares [Sch09]. We write $\Sigma_{n,2d}^{\mathcal{S}_n}$ and $\mathcal{P}_{n,2d}^{\mathcal{S}_n}$ for the sets of $n$-variate symmetric forms of degree $2d$ which are sums of squares and nonnegative respectively. Analogously, we write $\Sigma_{n,2d}^{\mathcal{B}_n}$ and $\mathcal{P}_{n,2d}^{\mathcal{B}_n}$ for the corresponding sets of even symmetric forms.

Several authors investigated the nonnegativity versus sums of squares question for (even) symmetric forms [BR21; Cho75; CL77a; CL77b; CLR87; GKR16;

GKR17; Har99]. Using representation theory one can describe the invariant sums of squares under $\mathcal{S}_n$ and $\mathcal{B}_n$ more efficiently and algorithmically [GP04].

It is well known that the irreducible representations of $\mathcal{S}_n$ correspond to partitions of $n$, while the irreducible representations of $\mathcal{B}_n$ correspond to bipartitions of $n$. The irreducible representations are called *Specht modules* and are denoted by $\mathbb{S}^\lambda$, resp. $\mathbb{S}^{(\lambda,\mu)}$ (see e.g. [Sag01] for background information). We recall useful properties of isoytpic decompositions with respect to $(\mathcal{S}_n)_n$ or $(\mathcal{B}_n)_n$ and a fixed degree.

**Proposition II.2.1** ([DR20; Rie+13]).

(i) *The $\mathcal{S}_n$-isotypic decomposition of $H_{n,d}$ stabilizes for $n \geq 2d$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash 2d$ the multiplicity of its irreducible representation in $H_{2d,d}$ equals the multiplicity of $(\lambda_1 + k, \lambda_2, \ldots)$ in $H_{2d+k,d}$ and all irreducible representations in $H_{2d+k,d}$ are of this form.*

(ii) *The $\mathcal{B}_n$-isotypic decomposition of $H_{n,d}$ stabilizes for $n \geq d$. For a bipartition $(\lambda, \mu) = ((\lambda_1, \lambda_2, \ldots), \mu)$ of $d$ the multiplicity of its irreducible representation in $H_{2d,d}$ equals the multiplicity of $((\lambda_1 + k, \lambda_2, \ldots), \mu)$ in $H_{d+k,d}$ and all irreducible representations in $H_{d+k,d}$ are of this form.*

**Proposition II.2.2.** *For $n \geq 2d$ there exist symmetric matrices $A_n^{(1)} \in \left(H_{n,2d}^{\mathcal{S}_n}\right)^{n_1 \times n_1}, \ldots, A_n^{(l)} \in \left(H_{n,2d}^{\mathcal{S}_n}\right)^{n_l \times n_l}$ such that any symmetric sum of squares $f \in H_{n,2d}^{\mathcal{S}_n}$ can be written as*

$$f = \sum_{i=1}^{l} \operatorname{Tr}(A_n^{(i)} B^{(i)})$$

*for some real symmetric matrices $B^{(i)}$.*

The same is true verbatim for $\mathcal{B}_n$ and $n \geq d$. Symmetry reduction and higher Specht polynomials allow a uniform representation of the invariant sums of squares in sufficiently large number of variables. Higher Specht polynomials [MY98] allow the calculation of a so called symmetry adapted basis for $H_{n,d}$. This basis gives immediately the isotypic decomposition of $H_{n,d}$. Wee refer to ([BR21, § 4]) for details.

*Remark* II.2.3. The following applies for both groups $\mathcal{S}_n$ and $\mathcal{B}_n$. The matrices $A_n^{(i)}$ in Proposition II.2.2 correspond to the isotypic components, i.e., to the Specht modules, and the size of a matrix $A_n^{(i)}$ equals the multiplicity of its associated irreducible representation in $H_{n,d}$. We have

$$A_n^{(i)} = \left( \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \sigma \cdot (f_{i,j} f_{i,k}) \right)_{j,k}$$

where $(f_{i,j})$ forms a symmetry adapted basis of $H_{n,d}$. Actually, the $f_{ij}$'s can be chosen uniformly for $n \geq 2d$, resp. $n \geq d$, if we embed $H_{n,d} \hookrightarrow H_{n+k,d}$ and

substitute $p_i^{(n)}$ by $p_i^{(n+k)}$. Then, all entries of $A_n^{(i)}$ are linear combinations in the $p_\lambda^{(n)}$'s with $\lambda \vdash 2d$, whose coefficients are rational functions in $n$ with highest degree 0.
We can also consider the limit for $n \to \infty$, by multiplication of the rows with powers of $n$ such that all rational functions in $n$ on the diagonal of $A_n^{(i)}$ have the same degree. In this way we obtain a characterization of the limit set of symmetric sums of squares which is introduced in the next subsection.

### II.2.3 Symmetric functions

In algebraic combinatorics the ring of symmetric functions is usually considered in countably infinitely many variables and constructed as a specific limit of the rings of symmetric polynomials in an increasing number of variables. Newton's identities and other identities of symmetric polynomials which do not depend on the number of variables hold in this limit ring.

Let $\mathcal{S}_\infty := \bigcup_{n\geq 1} \mathcal{S}_n$ be the permutation group of all finite subsets of $\mathbb{N}$. A *symmetric function* $f$ is a formal power series $f(X_1, X_2, \ldots)$ in countably infinitely many variables which is invariant with respect to the action of $\mathcal{S}_\infty$ and the set of degrees of the monomials in $f$ is finite. A homogeneous symmetric function is called a *limit form*. We denote the ring of symmetric functions by

$$\mathbb{R}[X_1, X_2, \ldots]^{\mathcal{S}_\infty}.$$

Our prototypes of symmetric functions are the formal power series analogous of the power sums and elementary symmetrics, i.e., the *power sum functions* $\mathfrak{p}_k = \sum_{i\in\mathbb{N}} X_i^k$ and *elementary symmetric functions* $\mathfrak{e}_k = \sum_{I\subset\mathbb{N},|I|=k} \prod_{i\in I} X_i$ . We observe

$$\mathbb{R}[X_1, X_2, \ldots]^{\mathcal{S}_\infty} = \mathbb{R}[\mathfrak{p}_1, \mathfrak{p}_2, \ldots] = \mathbb{R}[\mathfrak{e}_1, \mathfrak{e}_2, \ldots] \ ,$$

i.e., the ring of symmetric functions is isomorphic to the ring of polynomials in the the power sum functions and elementary symmetric functions. Newton's identities, i.e.,

$$\mathfrak{p}_k = (-1)^{k-1} k \mathfrak{e}_k + \sum_{i=1}^{k-1} (-1)^{k-1+i} \mathfrak{e}_{k-i} \mathfrak{p}_i \tag{II.1}$$

for all $k \in \mathbb{N}$, remain true if the number of variables $n \geq k$ is finite. We refer to [SF99] for background information.

Following ([Mac98, §I.2 Remark 1]) one can introduce the ring of symmetric functions as inverse limit of the rings of symmetric polynomials. We observe that if $f \in \mathbb{R}[X_1, \ldots, X_{n+1}]^{\mathcal{S}_{n+1}}$ then $f(X_1, \ldots, X_n, 0)$ which we consider as a $n$-variate polynomial is $\mathcal{S}_n$-invariant. For $m \geq n$ we write

$$\rho_{m,n} : \mathbb{R}[X]^{\mathcal{S}_m} \to \mathbb{R}[X]^{\mathcal{S}_n}$$

for the forgetful function that maps a symmetric polynomial in $m$ variables to the symmetric polynomial in $n$ variables obtained from setting the last $m-n$ variables equal to 0 and write $\rho_n := \rho_{n+1,n}$. For example, we have $\rho_{m,n}(p_k^{(m)}) = p_k^{(n)}$. Thus a power sum in $m$ variables is mapped to the "same" power sum in $n$ variables. When we fix the degree we obtain from the algebraic independence of power sums that the transition maps $\rho_{n+k,n}$ induce isomorphisms

$$\mathbb{R}[X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+k}]_{\leq n}^{\mathcal{S}_{n+k}} \simeq \mathbb{R}[X_1, \ldots, X_n]_{\leq n}^{\mathcal{S}_n} \tag{II.2}$$

for every $k \geq 0$. The ring of symmetric functions is then the inverse limit of $(\mathbb{R}[X]^{\mathcal{S}_n})_n$ with respect to the transition maps $\rho_n$ in the category of graded rings. Moreover, the homogeneous part of degree $\leq d$ of the ring of symmetric functions is again isomorphic to $\mathbb{R}[X_1, \ldots, X_n]_{\leq d}^{\mathcal{S}_n}$ for all $n \geq d$, i.e.,

$$\mathbb{R}[X_1, X_2, \ldots]_{\leq d}^{\mathcal{S}_\infty} \simeq \mathbb{R}[X_1, \ldots, X_n]_{\leq d}^{\mathcal{S}_n} \text{ with respect to } \mathfrak{p}_k \mapsto p_k^{(n)} \ .$$

When we compare sets of symmetric forms in different number of variables we do implicitly identify them using the transition maps $\rho_{m,n}$ but usually do not write the isomorphism explicitly. The sequences

$$\Sigma_{n+k,d}^{\mathcal{S}_{n+k}} \subset \Sigma_{n,d}^{\mathcal{S}_n}, \quad \mathcal{P}_{n+k,d}^{\mathcal{S}_{n+k}} \subset \mathcal{P}_{n,d}^{\mathcal{S}_n} \tag{II.3}$$

are nested for all $k \geq 1$. These inclusions follow, since if $f \in \mathbb{R}[X_1, \ldots, X_{n+k}]^{\mathcal{S}_{n+k}}$ is sum of squares/nonnegative, then $\rho_{n+k,n}(f)$ is also sum of squares/nonnegative.

**Definition II.2.4.** The *limit sets* of sums of squares and nonnegative symmetric forms in $n$ variables of degree $2d$ are defined as

$$\begin{aligned} \mathfrak{S}_{2d}^{\mathcal{S}} &:= \bigcap_{n \geq 2d} \Sigma_{n,2d}^{\mathcal{S}_n} \text{ and} \\ \mathfrak{P}_{2d}^{\mathcal{S}} &:= \bigcap_{n \geq 2d} \mathcal{P}_{n,2d}^{\mathcal{S}_n} \ . \end{aligned}$$

We also call $\mathfrak{S}_{2d}^{\mathcal{S}}$ and $\mathfrak{P}_{2d}^{\mathcal{S}}$ the sets of sums of squares and nonnegative homogeneous symmetric functions.

In the Kuratowski convergence definition for sequences of convex sets [Kur14] we have:

**Theorem II.2.5.** *The sets $\mathfrak{S}_{2d}^{\mathcal{S}}$ and $\mathfrak{P}_{2d}^{\mathcal{S}}$ are full dimensional pointed closed convex cones and*

$$\mathfrak{S}_{2d}^{\mathcal{S}} = \lim_{n \to \infty} \Sigma_{n,2d}^{\mathcal{S}_n} \ , \quad \mathfrak{P}_{2d}^{\mathcal{S}} = \lim_{n \to \infty} \mathcal{P}_{n,2d}^{\mathcal{S}_n} \ .$$

*Proof.* To prove that the limit sets defined in II.2.4 are indeed the limits of the sequences of convex sets $(\Sigma_{n,2d}^{\mathcal{S}_n})_n$ and $(\mathcal{P}_{n,2d}^{\mathcal{S}_n})_n$ we need to prove

$$\begin{aligned} &\limsup_n \Sigma_{n,2d}^{\mathcal{S}_n} \subset \mathfrak{S}_{2d}^{\mathcal{S}} \subset \liminf_n \Sigma_{n,2d}^{\mathcal{S}_n} \text{ and} \\ &\limsup_n \mathcal{P}_{n,2d}^{\mathcal{S}_n} \subset \mathfrak{P}_{2d}^{\mathcal{S}} \subset \liminf_n \mathcal{P}_{n,2d}^{\mathcal{S}_n} \end{aligned}$$

which follow from the nestedness of the sequences (II.3).
By definition the limit sets are closed pointed convex cones as intersections of closed pointed convex cones. We have to show that they are also full dimensional. However, this follows from ([BR18, Lemma 6.2.]) when we make sure that their equation (6.3) holds also in our notion of limit. The second and fourth author proved that the homogeneous part of $\mathbb{R}[X_1, X_2, \ldots]^{\mathcal{S}_\infty}$ of degree $d$ is spanned by

$$\{\mathfrak{p}_\lambda \mathfrak{p}_\mu : \lambda, \mu \vdash d\} \cup \{\mathfrak{p}_{a+b}\mathfrak{p}_\lambda \mathfrak{p}_\mu : 1 \leq a, b \leq d, \lambda \vdash d-a, \mu \vdash d-b\}.$$

This follows, since they show that one can divide the partitions of $2d$ into those which have up to reordering the form $(\lambda, \mu)$ for $\lambda, \mu \vdash d$, or the form $(a+b, \lambda, \mu)$ for $1 \leq a, b \leq d$ and $\lambda \vdash d-a, \mu \vdash d-b$. Using symmetry reduction we note, for $\lambda, \mu \vdash d$ the forms $p_\lambda^{(n)}, p_\mu^{(n)} \in H_{n,d}$ are equivariants of the Specht module $\mathbb{S}^{(n)}$ which implies

$$\mathfrak{p}_{(\lambda,\lambda)} + \mathfrak{p}_{(\mu,\mu)} - 2\mathfrak{p}_{(\lambda,\mu)} = \lim_{n\to\infty} p^{(n)}_{(\lambda,\lambda)} + p^{(n)}_{(\mu,\mu)} - 2p^{(n)}_{(\lambda,\mu)} \in \mathfrak{S}^{\mathcal{S}}_{2d} .$$

Analogously, for some integers $1 \leq a, b \leq d$ and partitions $\lambda \vdash d-a, \mu \vdash d-b$ we have $(X_1^a - X_2^a)p_\lambda^{(n)}, (X_1^b - X_2^b)p_\mu^{(n)} \in H_{n,d}$ are equivariants of the Specht module $\mathbb{S}^{(n-1,1)}$ and

$$\mathfrak{p}_{a+b}\mathfrak{p}_\lambda\mathfrak{p}_\mu = \lim_{n\to\infty} \frac{n}{2} \sum_{\sigma \in \mathcal{S}_n} \frac{1}{n!} \sigma \cdot \left( (x_1^a - x_2^a)(x_1^b - x_2^b) p_\lambda^{(n)} p_\mu^{(n)} \right)$$

which shows that there exists a symmetric homogeneous function that is a sum of squares and contains $\mathfrak{p}_{a+b}\mathfrak{p}_\lambda\mathfrak{p}_\mu$ linearly. Thus, the convex cones $\mathfrak{S}^{\mathcal{S}}_{2d} \subset \mathfrak{P}^{\mathcal{S}}_{2d}$ contain a full dimensional subcone and are therefore full dimensional themselves. ∎

*Remark* II.2.6. Analogously, we can define transition maps for the hyperoctahedral group $\mathcal{B}_n$ to construct the *ring of even symmetric functions* denoted by $\mathbb{R}[X_1, X_2, \ldots]^{\mathcal{B}}$ which can be considered as a subring of the ring of symmetric functions. The *limit sets* of even symmetric sums of squares and nonnegative forms are defined analogously and denoted by $\mathfrak{S}^{\mathcal{B}}_{2d}, \mathfrak{P}^{\mathcal{B}}_{2d}$. Again, the sets $\mathfrak{S}^{\mathcal{B}}_{2d}$ and $\mathfrak{P}^{\mathcal{B}}_{2d}$ are full dimensional pointed convex cones.

It can be seen more directly that the limit cones of even symmetric functions are full dimensional. Namely, every even power sum is already a sum of squares. Therefore, it follows immediately that the convex cone $\mathfrak{S}^{\mathcal{B}}_{2d}$ contains a full dimensional subcone.

**Example II.2.7.** We calculate a description of $\mathfrak{S}^{\mathcal{B}}_{6}$. First, an isotypic decomposition of $H_{n,3}$ with respect to the group $\mathcal{B}_n$ has to be calculated. Using the higher Specht polynomial approach we obtain for $n \geq 3$

$$H_{n,3} \simeq_{\mathcal{B}_n} 2 \cdot \mathbb{S}^{((n-1),(1))} \oplus \mathbb{S}^{((n-3),(3))} \oplus \mathbb{S}^{((n-2,1),(1))}$$

and a system of equivariants is given by

$$\left\{X_n, X_n^3\right\}, \quad \left\{X_1X_2X_3\right\}, \quad \left\{(X_n^2 - X_1^2)X_{n-1}\right\}.$$

Then the representing matrices for the cones of sums of squares are

$$A_n^{(1)} = \begin{pmatrix} p_{(2^3)}^{(n)} & p_{(4,2)}^{(n)} \\ p_{(4,2)}^{(n)} & p_{(6)}^{(n)} \end{pmatrix}, \quad A_n^{(2)} = \left(\frac{1}{6}p_{(2^3)}^{(n)} - \frac{1}{2}p_{(4,2)}^{(n)} + \frac{1}{3}p_{(6)}^{(n)}\right)$$

$$A_n^{(3)} = \left(\frac{-1}{n(n-1)(n-2)}p_{(2^3)}^{(n)} + \frac{n+1}{n(n-1)(n-2)}p_{(4,2)}^{(n)} + \frac{-1}{(n-1)(n-2)}p_{(6)}^{(n)}\right)$$

which converge to the limit matrices

$$A^{(1)} = \begin{pmatrix} \mathfrak{p}_{(2^3)} & \mathfrak{p}_{(4,2)} \\ \mathfrak{p}_{(4,2)} & \mathfrak{p}_{(6)} \end{pmatrix}, \quad A^{(2)} = \left(\frac{1}{6}\mathfrak{p}_{(2^3)} - \frac{1}{2}\mathfrak{p}_{(4,2)} + \frac{1}{3}\mathfrak{p}_{(6)}\right),$$

$$A^{(3)} = \left(\mathfrak{p}_{(4,2)} - \mathfrak{p}_{(6)}\right).$$

Thus,

$$\mathfrak{S}_6^{\mathcal{B}} = \{\mathrm{Tr}(A^{(1)}B^{(1)}) + A^{(2)}B^{(2)} + A^{(3)}B^{(3)} : B^{(i)} \text{ positive semidefinite}\}.$$

*Remark* II.2.8. The ring of symmetric functions can also be defined as a direct limit. For positive integers $m = n + k \geq n$ let $\phi_{m,n}$ denote the inverse of the restriction of the map $\rho_{m,n}$ in (II.2). Then, $\phi_{m,n}(p_k^{(n)}) = p_k^{(m)}$ for all $k \leq n$ and thus we obtain injections $\mathbb{R}[X]^{\mathcal{S}_n} \hookrightarrow \mathbb{R}[X]^{\mathcal{S}_m}$. The ring of symmetric functions can be constructed as the direct limit with respect to the transition maps $\phi_{m,n}$ which is just the union of the sets $\mathbb{R}[X]_{\leq n}^{\mathcal{S}_n}$.

We conclude the section with one more motivation for the identification of (even) symmetrics in different number of variables. Instead of identifying power sums in arbitrary many variables using the transition maps $\rho_{m,n}$ we could use as basis elements *normalized power sums* $\hat{p}_k^{(n)}(X_1, \ldots, X_n) := \frac{1}{n}\sum_{i=1}^n X_i^k$. Analogously, we define transition maps $\hat{\rho}_{m,n}$ which map a normalized power sum to the "same" normalized power sum in a different number of variables. This results in the *normalized limit* of symmetric polynomials. The nonnegativity versus sums of squares question in this setting was first investigated in [BR21]. The second and fourth author show that the normalized limit cones of symmetric quartics are equal and conjectured that equality of normalized limits is true for any degree. This was recently disproved in [AB22]. The first and second author proved that the sets are different already in degree 6 and for even symmetrics from degree 10 onward.

## II.3 The Vandermonde map

The *Vandermonde map* of degree $d$ in $n$ variables is the function

$$\begin{aligned} \nu_{n,d} : \mathbb{R}^n &\longrightarrow \mathbb{R}^d \\ x &\longmapsto (x_1 + \cdots + x_n, x_1^2 + \cdots + x_n^2, \ldots, x_1^d + \cdots + x_n^d) \end{aligned}.$$

The Vandermonde map and the analogous function in the first $d$ elementary symmetrics has been studied by various authors, for example in [Arn86; Giv87;

[Kos89; Kos99; Meg92]. We write $\mathcal{M}_{n,d} = \nu_{n,d}(\mathbb{R}^n)$ for the image of the Vandermonde map. In our work the Vandermonde map appears naturally in the study of nonnegative symmetric forms. We observe that the images form an increasing chain in the number of variables for a fixed degree $d$, i.e., we have $\mathcal{M}_{n,d} \subseteq \mathcal{M}_{n+1,d}$ for all $n \geq 1$. We are particularly interested in the image of the *Vandermonde map at infinity*, i.e., in the closure of the union of all images in an increasing number of variables

$$\mathcal{M}_d := \mathrm{cl}\left(\bigcup_{n=1}^{\infty} \mathcal{M}_{n,d}\right) ,$$

in the *even Vandermonde map*

$$\begin{aligned} \nu_{n,d}^{e} : \mathbb{R}^n &\longrightarrow \mathbb{R}^d \\ x &\longmapsto (x_1^2 + \cdots + x_n^2, x_1^4 + \cdots + x_n^4, \ldots, x_1^{2d} + \cdots + x_n^{2d}) \end{aligned},$$

its image $\mathcal{N}_{n,d} = \nu_{n,d}^{e}(\mathbb{R}^n)$, and its *image at infinity* $\mathcal{N}_d := \mathrm{cl}\left(\bigcup_{n=1}^{\infty} \mathcal{N}_{n,d}\right)$ .
Kostov studied in [Kos04; Kos07] polynomial images of the set $\mathcal{M}_d$. He worked with elementary symmetric polynomials instead of power sums and focused on degree 4. Although $\mathfrak{p}_4$ is non linearly in $\mathfrak{e}_1$ and $\mathfrak{e}_2$ we see that $d = 4$ is also the largest degree $d$ for which we can recover a description of $\mathcal{M}_d$ as a linear image of the corresponding map in the elementary symmetrics.
We have

$$\mathfrak{P}_{2d}^{\mathcal{S},*} = \mathrm{cone}\left(\nu_d\left(\mathcal{M}_d\right)\right) \text{ and } \mathfrak{P}_{2d}^{\mathcal{B},*} = \mathrm{cone}\left(\nu_d\left(\mathcal{N}_d\right)\right), \tag{II.4}$$

where $\nu_d : \mathbb{R}^d \to \mathbb{R}^{\pi(d)}$ is the monomial map $(x_1, \ldots, x_d) \mapsto (x_1^d, x_1^{d-2}x_2, \ldots, x_d)$ and $\mathrm{cone}(\cdot)$ denotes the *convex conical hull*. We observe that the set $\mathcal{N}_{n,d}$ equals the image of $\nu_{n,d}$ restricted to the nonnegative orthant.

### II.3.1 Properties of the Vandermonde map

We collect properties of Vandermonde maps which will be relevant in the following sections. In Section II.5 we analyse $\mathcal{M}_4$ to prove $\mathfrak{S}_4^{\mathcal{S}} \subsetneq \mathfrak{P}_4^{\mathcal{S}}$ and the properties of $\mathcal{N}_d$ will be essential in Section II.7 where we tropicalize $\mathcal{N}_d$.

**Lemma II.3.1.** *The sets $\mathcal{M}_{n,d}, \mathcal{M}_d$ are weighted homogeneous with respect to all $\lambda \in \mathbb{R}$, and the sets $\mathcal{N}_{n,d}, \mathcal{N}_d$ are weighted homogeneous with respect to all $\lambda \in \mathbb{R}_{\geq 0}$, i.e., if $(a_1, \ldots, a_d)$ is contained then also $(\lambda a_1, \ldots, \lambda^d a_d)$ for all $\lambda \in \mathbb{R}$ (resp. $\lambda \in \mathbb{R}_{\geq 0}$).*

*Proof.* We only prove the claim for $\mathcal{M}_{n,d}$ and $\mathcal{M}_d$ since the proof works completely analogously for $\mathcal{N}_{n,d}$ and $\mathcal{N}_d$.
Let $a = (a_1, \ldots, a_d) \in \mathcal{M}_{n,d}$. By definition we have $a = (p_1^{(n)}(x), \ldots, p_d^{(n)}(x))$ for some $x \in \mathbb{R}^n$ and hence

$$(\lambda a_1, \lambda^2 a_2, \ldots, \lambda^d a_d) = (p_1^{(n)}(\lambda x), \ldots, p_d^{(n)}(\lambda x)) \in \mathcal{M}_{n,d}.$$

It follows by continuity that $\mathcal{M}_d$ is weighted homogeneous. ■

**Definition II.3.2.** A set $S \subset \mathbb{R}^n$ has *Hadamard property* if $a \cdot b := (a_1 b_1, \ldots, a_n b_n) \in S$ for all $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in S$.

**Proposition II.3.3.** *The sets $\mathcal{M}_d$ and $\mathcal{N}_d$ have Hadamard property and are closed under addition.*

*Proof.* We only prove the properties for $\mathcal{M}_d$ since the arguments work analogously for $\mathcal{N}_d$.
The point $((x_1 + \cdots + x_m) \cdot (y_1 + \cdots + y_n), \ldots, (x_1^d + \cdots + x_m^d) \cdot (y_1^d + \cdots + y_n^d))$ is the image of $(x_1 y_1, \ldots, x_m y_n)$ under $\nu_{mn,d}$. Thus, for $x, y \in \mathcal{M}_d$ and sequences $(x^{(n)})_n, (y^{(n)})_n \in \bigcup_{n=1}^{\infty} \nu_{n,d}(\mathbb{R}^n)$ which converge to $x$ and $y$, we observe that the Hadamard product of $x^{(n)}$ and $y^{(n)}$ gives a sequence whose limit is the Hadamard product of $x$ and $y$. So $\mathcal{M}_d$ has Hadamard property.
If $u, v \in \bigcup_{n=1}^{\infty} \nu_{n,d}(\mathbb{R}^n)_d$ then $u = \nu_{m,d}(x)$ and $v = \nu_{n,d}(y)$ for some $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Hence $u + v = \nu_{m+n,d}(z) \in \mathcal{M}_d$ for $z = (x, y) \in \mathbb{R}^{m+n}$ being the concatenation of $x$ and $y$. So $\mathcal{M}_d$ is closed under addition. ∎

Proposition II.3.3 may fail for the finite cases $\mathcal{M}_{n,d}$ and $\mathcal{N}_{n,d}$. The following observation was already made in [Kos04; Kos07].

**Proposition II.3.4.** *$\mathcal{M}_d$ is a prism with respect to the first coordinate, i.e., $(a_1, a_2, \ldots, a_d) \in \mathcal{M}_d$ if and only if $(0, a_2, \ldots, a_d) \in \mathcal{M}_d$.*

*Proof.* Let $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ be a point and $x_n = (\frac{-a_1}{n}, \frac{-a_1}{n}, \ldots, \frac{-a_1}{n}) \in \mathbb{R}^n$ be a sequence. Then

$$\mathcal{M}_d \ni \lim_{n \to \infty} (p_1^{(n)}, \ldots, p_d^{(n)})(x_n) = \lim_{n \to \infty} (-a_1, \frac{a_1^2}{n}, \ldots, \frac{(-1)^d a_1^d}{n^{d-1}}) = (-a_1, 0, \ldots, 0).$$

Therefore, we have

$$a \in \mathcal{M}_d \text{ if and only if } (0, a_2, \ldots, a_d) = a + (-a_1, 0, \ldots, 0) \in \mathcal{M}_d$$

since the set $\mathcal{M}_d$ is closed under addition by Proposition II.3.3. ∎

**Lemma II.3.5.** *The points $(1, t^2, \ldots, t^2)$ and $(0, 1, \frac{1}{t}, \frac{1}{t^2}, \ldots, \frac{1}{t^{d-2}})$ belong to $\mathcal{M}_d$ for all positive integers $t^2$.*

*Proof.* Let $q = (1, \ldots, 1) \in \mathbb{R}^k$, so $\nu_{k,d}(q) = (k, \ldots, k) \in \mathcal{M}_d$ for all positive integers $k$. Now, since $\mathcal{M}_d$ is a prism with respect to the first coordinate by Proposition II.3.4, setting $k = t^2$ we obtain $(0, t^2, \ldots, t^2), (1, t^2, \ldots, t^2) \in \mathcal{M}_d$ and $v_{n,d}(\frac{1}{t}, 0, \ldots, 0) = (\frac{1}{t}, \frac{1}{t^2}, \ldots, \frac{1}{t^d}) \in \mathcal{M}_d$. Thus, we have $(0, t^2, \ldots, t^2) \cdot (\frac{1}{t}, \frac{1}{t^2}, \ldots, \frac{1}{t^d}) = (0, 1, \frac{1}{t}, \frac{1}{t^2}, \ldots, \frac{1}{t^{d-2}}) \in \mathcal{M}_d$ since $\mathcal{M}_d$ has Hadamard property II.3.3. ∎

### II.3.2 The boundary of the even Vandermonde map

The boundary of $\mathcal{M}_{n,d}$ has been described in ([Kos89, Theorem 1.14]). In Theorem II.3.6 we present the description of the boundary of $\mathcal{N}_{n,d}$ which we prove in Subsection II.3.3. We provide a parametrization of the boundary of $\mathcal{N}_{n,3}$, generalizations to projections of boundaries of higher degrees, and a parametrization of the boundary of $\mathcal{N}_3$. We conclude with showing that $\mathcal{N}_d$ is not semialgebraic for all $d \geq 3$.

We write $\widetilde{\mathcal{N}}_{n,d} := (e_1^{(n)}, \dots, e_d^{(n)})(\mathbb{R}^n_{\geq 0})$ for the image of the first $d$ elementary symmetrics in $n$ variables on the nonnegative orthant. It follows from Newton's identities (II.1) that $\mathcal{N}_d$ and $\widetilde{\mathcal{N}}_d$ are the same up to a polynomial diffeomorphism for $n \geq d$. For $a = (a_1, \dots, a_{d-1}) \in \mathbb{R}^{d-1}$ we define the affine variety

$$V(a) = \{x \in \mathbb{R}^n : p_{2i}^{(n)}(x) = a_i, 1 \leq i \leq d-1\}.$$

Note that the variety $V(a)$ is either empty or compact for all choices of $a$ and $n \geq d \geq 2$.

**Theorem II.3.6.** *For $n \geq d$, the boundary of the sets $\mathcal{N}_{n,d}$ and $\widetilde{\mathcal{N}}_{n,d}$ is given by the closure of the set of evaluations at all points whose coordinates consist of $0 < x_1 < x_2 < \ldots < x_{d-1}$ and are of the following types:*

(1e) *$d$ even, $(x_1, \dots, x_1, x_2, x_3, \dots, x_{d-1})$ with $x_{2k-1}$ has multiplicity $\geq 1$, while $x_{2k}$ has multiplicity $1$ for all $k$;*

(1o) *$d$ odd, $(0, \dots, 0, x_1, x_2, \dots, x_2, x_3, \dots, x_{d-1})$ with $x_{2k}$ has multiplicity $\geq 1$ and $0$ has arbitrary multiplicity, while $x_{2k-1}$ has multiplicity $1$ for all $k$;*

(2e) *$d$ even, $(0, \dots, 0, x_1, x_2, \dots, x_2, x_3, \dots, x_{d-1})$ with $x_{2k}$ has multiplicity $\geq 1$ and $0$ has arbitrary multiplicity, while $x_{2k-1}$ has multiplicity $1$ for all $k$;*

(2o) *$d$ odd, $(x_1, \dots, x_1, x_2, x_3, \dots, x_{d-1})$ with $x_{2k-1}$ has multiplicity $\geq 1$, while $x_{2k}$ has multiplicity $1$ for all $k$.*

*Moreover, for $x \in \mathbb{R}^n$ of type (1e),(1o)/(2e),(2o) $p_{2d}^{(n)}$ attains a minimum/maximum at*

$$V(p_2^{(n)}(x), \dots, p_{2d-2}^{(n)}(x)) \ .$$

*If $d$ is odd then $e_d^{(n)}(X^2)$ attains a minima/maxima at $x$ of type (1e),(1o)/(2e),(2o) and otherwise a maxima/minima.*

For a fixed $a \in \mathbb{R}^{d-1}$ and $n \geq d$ there exists a scalar $c_a \in \mathbb{R}$ such that

$$e_d^{(n)}(x) = (-1)^{d-1} p_d^{(n)}(x) + c_a$$

for all $x \in V(a)$ which shows that the last claim on $e_d^{(n)}(X^2)$ follows from the classification of the minima and maxima of $p_{2d}^{(n)}$.

The theorem above allows univariate parametrization of the boundaries of $\mathcal{N}_{n,3}$ in a finite number of variables and at infinity.

**Theorem II.3.7.** *For $n \geq 3$, a parametrization of the boundary of the set*

$$\{(p_4^{(n)}(x), p_6^{(n)}(x)) : x \in \mathbb{R}^n,\ p_2^{(n)}(x) = 1\}$$

*is given by the following $n$ univariate parametrizations. The upper part of the boundary is parametrized by*

$$\left(\frac{(1-t)^2}{n-1} + t^2, \frac{(1-t)^3}{(n-1)^2} + t^3\right) : \frac{1}{n} \leq t \leq 1 \tag{II.5}$$

*while the lower part is parametrized by the families*

$$\left(\left(\frac{(1-t)^2}{n-k-1} + t^2, \frac{(1-t)^3}{(n-k-1)^2} + t^3\right) : 0 \leq t \leq \frac{1}{n-k}\right)_{0 \leq k \leq n-2} . \tag{II.6}$$

*Proof.* We note

$$\{(p_4^{(n)}(x), p_6^{(n)}(x)) : x \in \mathbb{R}^n_{\geq 0},\ p_2^{(n)}(x) = 1\} = \mathcal{N}_{n,3} \cap \{p_2^{(n)} = 1\}$$

and thus we can apply Theorem II.3.6 to determine the boundary. Since we restrict to the nonnegative orthant we can consider instead the alternative representation

$$\{(p_2^{(n)}(x), p_3^{(n)}(x)) : x \in \mathbb{R}^n_{\geq 0}, \sum_{i=1}^{n} x_i = 1\}$$

and II.3.6 still applies. Therefore, the boundary consists of the closure of the set of all point evaluations in $(p_2^{(n)}, p_3^{(n)})$ at all points $(0, \ldots, 0, x_1, \ldots, x_1, x_2, \ldots, x_2) \in \mathbb{R}^n$ of the form $0 < x_1 < x_2$ of type (1o) or (2o) and whose sum equals 1.
Note that any point of type (2o) must be of the form $(a, \ldots, a, b)$ with $0 < a < b$ and $(n-1)a + b = 1$. Thus, $a = \frac{1-b}{n-1}$, $\frac{1}{n} < b < 1$ and we observe that the upper part of the boundary is indeed parametrized by the curve in (II.5).
We note that there are essentially $n-1$ points of type (1o). Namely, points of the form

$$(\underbrace{0, \ldots, 0}_{\#=k}, a, \underbrace{b, \ldots, b}_{\#=n-k-1})$$

for $0 \leq k \leq n-2$ satisfying $b = \frac{1-a}{n-k-1}, a \leq \frac{1}{n-k}$. We obtain precisely the parametrizations (II.6) of the lower part of the boundary. ∎

We obtain parametrizations of the boundary of $\widetilde{N}_{n,3}$ by applying the polynomial diffeomorphism induced by Newton's identities. See Figure II.2 for a visualisation of these boundaries.

*Remark* II.3.8. Theorem II.3.7 generalizes to a parametrization of the boundary of the set

$$\{(p_{2k}^{(n)}(x), p_{2m}^{(n)}(x)) : x \in \mathbb{R}^n, p_2^{(n)}(x) = 1\}$$

for $2 \leq k \leq m$. However, we observe that the upper part of the boundary cannot be described by just one smooth parametrizitation. This is since there are essentially more points of type (2e) resp. (2o) than for $k = 2$ and $m = 3$, where there is only 1. However, a careful analysis can lead to a description of the boundary.

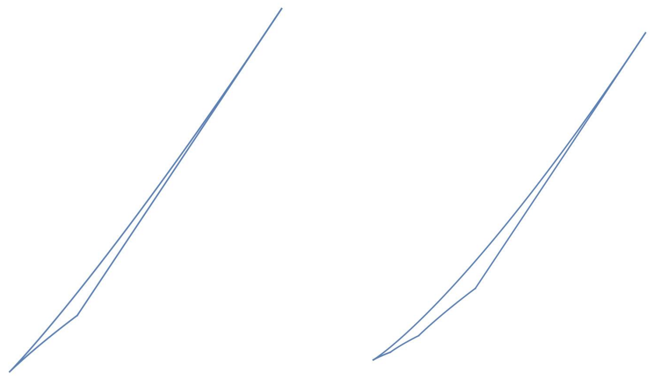

Figure II.1: The boundary of the sets $\mathcal{N}_{n,3} \cap \{p_2^{(n)} = 1\}$ for $n = 3$ (left) and $n = 5$ (right)

Figure II.2: The boundary of the sets $\widetilde{\mathcal{N}}_{n,3} \cap \{p_2^{(n)} = 1\}$ for $n = 3$ (left) and $n = 5$ (right)

**Example II.3.9.** For $2 \leq k \leq 3$, the lower part of the boundary of the set

$$\{(p_{2k}^{(4)}(x), p_8^{(4)}(x)) : x \in \mathbb{R}^4, p_2^{(4)}(x) = 1\}$$

is the union of the images of the following two parametrizations

$$\left(2s^k + t^k + (1-2s-t)^k, 2s^4 + t^4 + (1-2s-t)^4\right) : 0 \leq s \leq t \leq \frac{1}{2} - s \ , \tag{II.7}$$

$$\left(s^k + t^k + \frac{1}{2^{k-1}}(1-s-t)^k, s^4 + t^4 + \frac{1}{8}(1-s-t)^4\right) : 0 \leq s \leq t < \frac{1}{3} - \frac{1}{3}s \ . \tag{II.8}$$

Parametrization (II.7) comes from the points with multiplicity vector $(x_1, x_1, x_2, x_3)$ and (II.8) from the points $(x_1, x_2, x_3, x_3)$ and in both cases $0 < x_1 < x_2 < x_3$.
The upper part of the boundary is the union of the images of the following two parametrizations

$$\left(s^k + 2t^k + (1-s-2t)^k, s^4 + 2t^4 + (1-s-2t)^4\right) : 0 \leq s \leq t \leq \frac{1}{3} - \frac{1}{3}s \ , \tag{II.9}$$

$$\left(s^k + t^k + (1-s-t)^k, s^4 + t^4 + (1-s-t)^4\right)\right) : 0 \le s \le t \le \frac{1}{2} - \frac{1}{2}t \ . \tag{II.10}$$

Parametrization (II.9) comes from the points with multiplicity vector $(x_1, x_2, x_2, x_3)$ and (II.10) from those with $(0, x_1, x_2, x_3)$ and in both cases $0 < x_1 < x_2 < x_3$.

We note that by going from $\mathcal{N}_{n,3} \cap \{p_2^{(n)} = 1\}$ to $\mathcal{N}_{n+1,3} \cap \{p_2^{(n+1)} = 1\}$ in Theorem II.3.7 the upper part of the boundary grows slowly and converges. Its limit has the parametrization $(t, t^{3/2}), 0 \le t \le 1$. Moreover, every point on the lower part of the boundary for $n$ remains on the boundary for $n+1$, but a single new smooth curve is added. Namely, the smooth curve with parametrization

$$\left(\frac{(1-t)^2}{n} + t^2, \frac{(1-t)^3}{n^2} + t^3\right) \ , \ 0 \le t \le \frac{1}{n+1}$$

appears additionally.

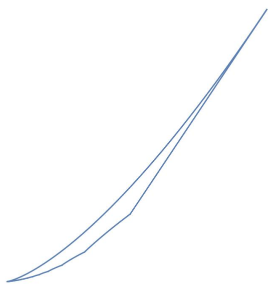

Figure II.3: The boundary of the set $\mathcal{N}_{20,3} \cap \{p_2^{(20)} = 1\}$

We can immediately give a description of the boundary of $\mathcal{N}_3 \cap \{\mathfrak{p}_2 = 1\}$.

**Corollary II.3.10.** *The boundary of the set $\mathcal{N}_3 \cap \{\mathfrak{p}_2 = 1\}$ equals*

$$\left\{(t, t^{3/2}) : 0 \le t \le 1\right\} \cup \bigcup_{k \in \mathbb{N}_{>1}} \left\{\left(\frac{(1-t)^2}{k} + t^2, \frac{(1-t)^3}{k^2} + t^3\right) : 0 \le t \le \frac{1}{k+1}\right\}.$$

We note that two different parametrizations of the lower part of the boundary

$$\begin{aligned} &\left(\frac{(1-t)^2}{k} + t^2, \frac{(1-t)^3}{k^2} + t^3\right) \ : \ 0 \le t \le \frac{1}{k+1} \text{ and} \\ &\left(\frac{(1-s)^2}{l} + s^2, \frac{(1-s)^3}{l^2} + s^3\right) \ : \ 0 \le s \le \frac{1}{l+1} \end{aligned}$$

intersect if and only if $k = l-1$ or $k = l+1$ which can be verified using a computer algebra system. Without loss of generality be $k = l-1$. The intersection has cardinality 1 and the curves meet at $\left(\frac{1}{l}, \frac{1}{l^2}\right)$ for $t = \frac{1}{l}$ and $s = 0$. Moreover, the gradients $(0, 0)$ and $\left(\frac{-2}{l}, \frac{-3}{l}\right)$ differ at this point which shows that $\left(\frac{1}{l}, \frac{1}{l^2}\right)$ is a singular point of the boundary.

**Corollary II.3.11.** *The compact set $\mathcal{N}_3 \cap \{\mathfrak{p}_2 = 1\}$ has countably infinite isolated singular points which are the points of the form*

$$\left(\frac{1}{k}, \frac{1}{k^2}\right), \; k \in \mathbb{N}_{>0} \textit{ and } (0, 0) \; .$$

*Proof.* It follows from the discussion above that only neighboring parametrizations of the lower part of the boundary intersect and their intersection point is a singular point of the boundary. The intersection points are all of the form $\left(\frac{1}{k+1}, \frac{1}{(k+1)^2}\right)$ for all $k \in \mathbb{N}$. However, $(1, 1)$ is an intersection of the parametriztation $(t, t^{3/2}), 0 \leq t \leq 1$ of the upper part of the boundary and $((1-s)^2 + s^2, (1-s)^3 + s^3) : 0 \leq s \leq 1/2$ of the lower part. For $t = 1$ and $s = 0$, but again the gradients are different which shows that $(1, 1)$ is a singular point. Moreover, any singular point must be an intersection of two parametrizations. But the intersection points are precisely the points of the claimed form and the limit point $(0, 0)$.
Since all the singular points lie in the rational moment curve $(t, t^2)$ the points are indeed isolated (see e.g. ([Bar02, Chapter II.9.])). ∎

**Corollary II.3.12.** *The sets $\mathcal{N}_d$ and $\widetilde{\mathcal{N}}_d$ are not semialgebraic for all $d \geq 3$.*

*Proof.* We show that $\mathcal{N}_3$ is not semialgebraic. The general case follows, since for $d \geq 3$ we observe $\mathcal{N}_3 = \pi(\mathcal{N}_d)$, where $\pi : \mathbb{R}^d \to \mathbb{R}^3$ denotes the projection onto the first 3 coordinates. Moreover, $\widetilde{\mathcal{N}}_d$ is a polynomial image of the set $\mathcal{N}_d$ which must then also be non semialgebraic.
We suppose that the set $\mathcal{N}_3$ is semialgebraic. Then the intersection $K$ of $\mathcal{N}_3$ with the hypersurface $\{\mathfrak{p}_2 = 1\}$ must also be semialgberaic. However, by Corollary II.3.11 the semialgebraic set $K$ has countably infinite isolated singular points. Let $T$ denote the union of all the singular points. The union of all singular points of a semialgebraic set is again semialgebraic since this condition can be formalized as the vanishing and non-vanishing of certain polynomial equalities. Thus, $T$ is semialgebraic. By ([BCR13, Theorem 2.4.4]) every semialgebraic set is the disjoint union of a finite number of semialgebraically connected semialgebraic sets. However, there are countably infinite isolated points in $T$ which contradicts $T$ being semialgebraic. In particular, $\mathcal{N}_3$ cannot be semialgebraic. ∎

### II.3.3 Proof of Theorem II.3.6

We provide a proof of Theorem II.3.6. Our proof is an adaption of the work in [Arn86; Kos89], which can also be found with more details in [Meg92; Rai04].

**Lemma II.3.13.** *Let $a \in \mathbb{R}^{d-1}$. Then a point $x \in V(a)$ with at least $d-1$ distinct non-zero absolute values of coordinates is a smooth point.*

*Proof.* This follows by considering the Jacobian of $(p_2^{(n)}, \dots, p_{2d-2}^{(n)})$ evaluated at $x$. Indeed, for the rank of this matrix to be strictly less than $d-1$, $x$ cannot have $d-1$ distinct non-zero coordinates up to absolute value. ∎

**Lemma II.3.14.** *Let $a \in \mathbb{R}^{d-1}$. The critical points of $p_{2d}^{(n)}$ on the regular part of $V(a)$ are exactly the points with precisely $d-1$ distinct absolute values of non-zero coordinates.*

*Proof.* This follows by considering the Jacobian of $(p_2^{(n)}, \dots, p_{2d-2}^{(n)})$. We find that this matrix has rank $d-1$ at a point $x$ if and only if $x$ has exactly $d-1$ distinct non-zero squares of coordinates. ∎

I.e., up to $\mathcal{B}_n$-action a critical point is of the form

$$(x_1, x_2, \dots, x_{d-1}, y_1, \dots, y_{n-(d-1)}) \in \mathbb{R}^n_{\geq 0},$$

with $\#\{| x_1 |, | x_2 |, \dots, | x_{d-1} |\} = d-1$, $y_i \in \{x_1, x_2, \dots, x_{d-1}, 0\}$, and $x_i \neq 0$ for all $i$.

The following proposition is an adaption of ([Arn86, § 5 Corollary]) from power sums to even power sums for which a proof can be found in ([Meg92, Proposition 7]) and ([Rai04, Proposition 3.2.5.]).

**Proposition II.3.15.** *Let $a \in \mathbb{R}^{d-1}$. For a critical point $x$ of $p_{2d}^{(n)}$ on $V(a)$ let $m_0$ denote the number of times $0$ appears, and $m_i$ denote the number of times the $i$-th smallest positive coordinate appears. Further set $r_i = m_i - 1$. Then, if $d$ is odd (even), the Hessian of $p_{2d}^{(n)}$ on $V(a)$ at the point $x$ is the sum of a negative (positive) definite quadratic form on $\mathbb{R}^a$ and a positive (negative) definite form on $\mathbb{R}^b$, where $a = \sum_{i<d, i\notin 2\mathbb{N}} r_i$ and $b = m_0 + \sum_{i<d, i\in 2\mathbb{N}} r_i$.*

*Proof.* Let $x$ be a critical point which we assume without loss of generality to have only nonnegative coordinates. By Lemma II.3.14 we can assume

$$x = (\underbrace{0, \dots, 0}_{m_0}, \underbrace{x_1, \dots, x_1}_{r_1}, \dots, \underbrace{x_{d-1}, \dots, x_{d-1}}_{r_{d-1}}, x_1, \dots, x_{d-1})$$

for some positive pairwise distinct $x_i$'s. Let

$$\tilde{x} = (\underbrace{0, \dots, 0}_{m_0}, \underbrace{x_1, \dots, x_1}_{r_1}, \dots, \underbrace{x_{d-1}, \dots, x_{d-1}}_{r_{d-1}}) \in \mathbb{R}^{n-d+1}$$

denote the point consisting of the first $n-d+1$ coordinates of $x$. The first $n-d+1$ coordinates can be used as a system of local coordinates for $V(a)$ in a neighborhood of $x$ by II.3.14.
We note that there exist Lagrange multipliers $\lambda_1^*, \dots, \lambda_{d-1}^* \in \mathbb{R}$ such that all partial derivatives of the Lagrangian function

$$L(X) = p_{2d}^{(n)}(X) - \sum_{i=1}^{d-1} \lambda_i^* (p_{2i}^{(n)}(X) - a_i)$$

vanish at $x$. Thus, there exists an univariate polynomial

$$g(t) := 2dt^{2d-1} - \sum_{i=1}^{d-1} 2i\lambda_i^* t^{2i-1} = t(2dt^{2d-2} - \sum_{i=1}^{d-1} 2i\lambda_i^* t^{2i-2})$$

such that any coordinate of $x$, and in particular $\tilde{x}$, is a root of $g$ which shows that the zeros of $g$ are contained in $\{0, \pm x_1, \ldots, \pm x_{d-1}\}$. By the intermediate value theorem we note $g'(t)$ has $d-1$ positive zeros $v_1, \ldots, v_{d-1}$ satisfying $0 < v_1 < x_1 < v_2 < x_2 < \ldots < x_{d-2} < v_{d-1} < x_{d-1}$. Moreover, since the leading coefficient of $g'(t)$ is positive we can observe

$$g'(x_{d-1}) > 0, g'(x_{d-2}) < 0, g'(x_{d-3}) > 0, \ldots, (-1)^q g'(x_1) < 0, (-1)^q g'(0) > 0$$

with $q \in \mathbb{N}_{>0}$, and $q$ even if and only if $d$ is odd.
Then, the Hessian of the Lagrangian function satisfies

$$\begin{aligned}(h_1, \ldots, h_{n-d+1}) d^2 L(\tilde{x})(h_1, \ldots, h_{n-d+1})^T =& (h_1^2 + \ldots + h_{m_0}^2) g'(0) + (h_{m_0+1}^2 + \ldots + h_{m_0+r_1+1}^2) g'(x_1) \\ &+ \ldots + (h_{m_0+r_1+\ldots+r_{d-2}+1}^2 + \ldots + h_{n-d+1}^2) g'(x_{d-1})\end{aligned}$$

since $\frac{\partial^2 L}{\partial X_i \partial X_j} = 0$ for $i \neq j$, and $\frac{\partial^2 L}{\partial X_i \partial X_i} = g'(x_i)$. This shows that the Hessian of $p_{2d}^{(n)}$ on $V(a)$ at $x$ has indeed the claimed form. ■

*Remark* II.3.16 ([Rai04], Proposition 3.2.5). The last proposition implies that $p_{2d}^{(n)}$ is a Morse function on $V(a)$ for $a \in \mathbb{R}^{d-1}$.

Thus, we immediately obtain:

**Corollary II.3.17.** *Let $x$ be a critical point of $p_{2d}^{(n)}$ on $V(a)$. Then $x$ is a strict local minimum/maximum if $x$ is of type* (1e),(1o)/(2e),(2o) *(depending on d even or odd).*

The following was proven by Kostov for the Vandermonde map ([Kos89], Lemma 2.2]).

**Lemma II.3.18.** *Let $d \geq 2$. The image of the function $p_{2d}^{(n)} : \mathbb{R}^n \to \mathbb{R}$ on the set $\{x \in \mathbb{R}^n : (p_2^{(n)}, \ldots, p_{2d-2}^{(n)})(x) = a\}$ is either empty or an interval for all $a \in \mathbb{R}^{d-1}$.*

*Proof.* Givental proves that the set

$$\{x \in \mathbb{R}^n_{\geq 0} : x_i \leq x_{i+1}, \forall\ i\} \cap \{x \in \mathbb{R}^n : (p_2^{(n)}, \ldots, p_{2d-2}^{(n)})(x) = a\}$$

is generically either contractible or empty [Giv87]. Kostov showed that generically implies globally ([Kos89], Lemma 2.6]). Thus, the image of $p_{2d}^{(n)}$ on the restriction is connected and compact. Therefore, the non-empty image of $p_{2d}^{(n)}$ on the set $\{x \in \mathbb{R}^n : (p_2^{(n)}, \ldots, p_{2d-2}^{(n)})(x) = a\}$ is an interval. ■

Finally, we can present a proof of Theorem II.3.6.

*Proof of Theorem II.3.6.* For $n \geq d$, the even Vandermonde map $(p_2^{(n)}, \ldots, p_{2d}^{(n)})$ maps elements in the interior of the cone $0 \leq x_1 \leq \ldots \leq x_n$ to points in the interior of the image $\mathcal{N}_d$, and points from the boundary to the boundary [Giv87]. Thus, any point of type (1e),(1o) or (2e),(2o) is indeed mapped to the boundary. Now, we assume that $(p_2^{(n)}, \ldots, p_{2d}^{(n)})(x)$ is contained in the boundary of the set $\mathcal{N}_{n,d}$ and is non-singular. Then, since the set

$$p_{2d}^{(n)}\left(V((p_2^{(n)}, \ldots, p_{2d-2}^{(n)})(x))\right)$$

is an interval by Lemma II.3.18, we observe that $p_{2d}^{(n)}$ is either minimized or maximized at $x$ on the interval $V((p_2^{(n)}, \ldots, p_{2d-2}^{(n)})(x))$. We can apply Corollary II.3.17 and obtain that $x$ must be of type (1e),(1o) or (2e),(2o). If $(p_2^{(n)}, \ldots, p_{2d}^{(n)})(x)$ is a singular point then $x$ can be obtained as the limit of a sequence of such points. ∎

## II.4 The convex hull for elementary symmetrics and test sets for nonnegativity

Similarly to our work in Section II.3 we analyze the convex hull of the images of elementary symmetric polynomials and power sums on the nonnegative orthant in a fixed number of variables and at infinity. Although, the boundary of the image described in Theorem II.3.6 is the same for elementary symmetrics and power sums up to diffeomorphism, we show that the convex hull of the image of elementary symmetrics satisfies useful properties which are not shared by the convex hull of the image of power sums. The descriptions of the vertices of the convex hulls can be reformulated in terms of test sets to verify nonnegativity of specific even symmetric (limit) forms. The test sets are a generalization of the degree 6 case investigated by Choi, Lam and Reznick [CLR87]. Moreover, we use Gale's evenness condition to describe the facets of the convex hull.

Let $\Delta_n := \{x \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i = 1\}$ denote the $n-1$ dimensional *probability simplex* and $\Delta := \lim_{n\to\infty} \Delta_n = \operatorname{cl}\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right)$ the *infinite probability simplex*. For $n \geq d$ we provide a description of the convex sets $\mathcal{E}_{n,d} := \operatorname{conv}((e_2^{(n)}, \ldots, e_d^{(n)})(\Delta_n))$, and $\mathcal{E}_d := \operatorname{cl}\left(\bigcup_{n\geq d} \mathcal{E}_{n,d}\right)$, which turn out to be a polytope with $n$ vertices in the first case and respectively the closure of a union of nested polytopes. We note $\mathcal{E}_{n,d} = \operatorname{conv}(\pi(\widetilde{\mathcal{N}}_{n,d} \cap \{p_2^{(n)} = 1\}))$ and $\mathcal{E}_d = \operatorname{conv}(\pi(\widetilde{\mathcal{N}}_d \cap \{\mathfrak{p}_2 = 1\}))$, where $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ denotes the projection onto the last $d-1$ coordinates.

The following theorem was already known before in different contexts. For instance, it can be found in [For87; KKR12]. The result appeared even earlier in the context of extremal combinatorics and was proven by Bollobás in the plane [Bol76] to give a description of the convex hull of the range of edge versus triangle densities, and the result was extended to larger dimensions shortly afterwards.

**Theorem II.4.1.** *The set $\mathcal{E}_{n,d}$ is equal to the polytope*

$$\operatorname{conv}\left\{\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) : k \in [n]\right\},$$

*where $\binom{k}{j} = 0$ if $k < j$. For $d \geq 3$ the polytope $\mathcal{E}_{n,d}$ has $n$ vertices.*

We recall that for a convex set $S \subset \mathbb{R}^n$ a point $a \in S$ is *extremal* if the set $S \setminus \{a\}$ is again convex. We follow ([Zha22, Lemma 5.4.3]) for a proof of II.4.1.

*Proof.* By the Krein-Milman theorem every compact and convex set is equal to the convex hull of its extremal elements. The extremal elements of $\mathcal{E}_{n,d}$ are precisely the minima of affine linear maps on $\mathcal{E}_{n,d}$.
Let $n \geq d$ and $\phi(e_2,\ldots,e_d) = c_1 + c_2e_2 + \ldots + c_ne_d : \mathbb{R}^n \to \mathbb{R}$ be an affine non-constant linear map on $\mathcal{E}_{n,d}$ and let $x^*$ be a mininizer of $\phi^* = \phi(e_2^{(n)},\ldots,e_d^{(n)})$ on $\Delta_n$. We show that $x^* = (1/k,\ldots,1/k,0,\ldots,0)$ up to permutation for a $1 \leq k \leq n$. We assume that $x_1, x_2 > 0$ and write $\phi^*(x) = x_1A + x_2B + x_1x_2C + D$, where $A, B, C, D$ are functions in $x_3,\ldots,x_n$. Then, since $\phi^*$ is symmetric $A = B$ and by fixing $x_1 + x_2 = x_1^* + x_2^*$ we obtain $\phi^*(x) = x_1x_2C + D'$. We have, if $C(x^*) \geq 0$ we set either $x_1 = 0$ or $x_2 = 0$ with holding $x_1 + x_2 = x_1^* + x_2^*$ fixed and obtain that $x^*$ was not a minimum. If $C(x^*) < 0$ we obtain $\phi^*(x^*)$ is minimized at $x_1^* = x_2^*$. Iteratively, we observe $x^* = (1/k,\ldots,1/k,0,\ldots,0)$.
It follows from Corollary II.4.5 that all the claimed points are indeed vertices of $\mathcal{E}_{n,d}$. ■

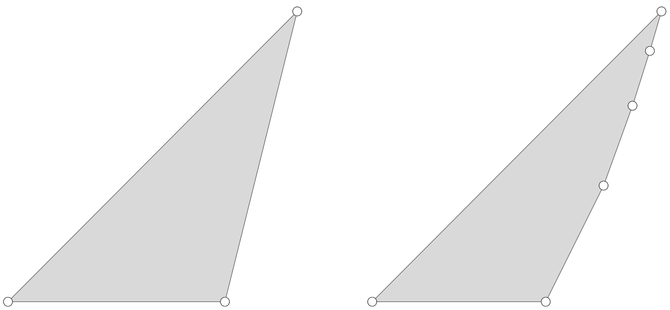

Figure II.4: The sets $\mathcal{E}_{3,2}$ (left) and $\mathcal{E}_{6,2}$ (right)

For $\mathcal{E}_{n,3}$ Theorem II.4.1 is equivalent to the following theorem by Choi, Lam and Reznick.

**Theorem II.4.2** ([CLR87], Theorem 3.7)**.** *Let $f(p_2^{(n)}, p_4^{(n)}, p_6^{(n)})$ be an even symmetric sextic in $n \geq 3$ variables. Then, $f$ is nonnegative if and only if $f\left(1, \frac{1}{k}, \frac{1}{k^2}\right)$ is nonnegative for all $k \in [n]$.*

For $n \geq 3$ we observe from Newton's identities

$$p_2^{(n)}(X) = 1\ ,p_4^{(n)}(X) = 1 - 2e_2^{(n)}(X^2)\ ,p_6^{(n)}(X) = 1 - 3e_2^{(n)}(X^2) + 3e_3^{(n)}(X^2)\ .$$

Thus, the power sums $p_4^{(n)}$ and $p_6^{(n)}$ are linear in $e_2^{(n)}(X^2)$ and $e_3^{(n)}(X^2)$ on $\{x : p_2^{(n)}(x) = 1\}$. An even symmetric sextic has the form $f = c_1 p_{(2^3)}^{(n)} + c_2 p_{(4,2)}^{(n)} + c_3 p_{(6)}^{(n)}$. The only non linearly occurring power sum is $p_{(2)}^{(n)} = e_1^{(n)}(X^2)$. However, for testing nonnegativity we can restrict to $p_{(2)}^{(n)} = 1$ since $f$ is a form. Thus, $f$ is nonnegative if and only if $f(1, p_4, p_6)$ is nonnegative on the set $\operatorname{conv}\{(p_4^{(n)}(x), p_6^{(n)}(x)) : x \in \Delta_n\}$ which we show to be a linear transformation of $\mathcal{E}_{n,3}$ in Proposition II.4.4.

Since

$$\lim_{n\to\infty}\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) = \left(\frac{1}{2!},\ldots,\frac{1}{d!}\right)$$

we immediately obtain a description of the limit.

**Corollary II.4.3.** $\mathcal{E}_d = \operatorname{conv}\left\{\left\{\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) : k \in \mathbb{N}\right\} \uplus \left\{\left(\frac{1}{2!},\frac{1}{3!},\ldots,\frac{1}{d!}\right)\right\}\right\}$.

*Proof.* We observe that the set $\bigcup_{n\geq d}\mathcal{E}_{n,d}$ is convex: if $v, w \in \bigcup_{n\geq d}\mathcal{E}_{n,d}$ then for some integer $N$ we have $v, w$ are contained in the convex set $\mathcal{E}_{n,d}$, because the sets $\mathcal{E}_{n,d}$ are nested. Thus, $\mathcal{E}_d$ is convex as the closure of the convex set $\bigcup_{n\geq d}\mathcal{E}_{n,d}$. We note

$$\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right), \left(\frac{1}{2!},\frac{1}{3!},\ldots,\frac{1}{d!}\right) \in \mathcal{E}_d$$

per definition and since $\mathcal{E}_d$ is closed. Thus, the set on the right hand side is contained in $\mathcal{E}_d$. Moreover, we have

$$\mathcal{E}_{n,d} \subset \operatorname{conv}\left\{\left\{\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) : k \in \mathbb{N}\right\} \uplus \left\{\left(\frac{1}{2!},\frac{1}{3!},\ldots,\frac{1}{d!}\right)\right\}\right\}$$

for any $n \geq d$ and thus

$$\operatorname{cl}\left(\bigcup_{n\geq d}\mathcal{E}_{n,d}\right) \subset \operatorname{conv}\left\{\left(\frac{1}{2!},\frac{1}{3!},\ldots,\frac{1}{d!}\right), \left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) : k \in \mathbb{N}\right\}$$

since the set on the right hand side is closed. ∎

Figure II.5 displays $\mathcal{E}_{20,2}$ and visualizes how the additional vertices eventually accumulate around the point $\left(\frac{1}{2!},\frac{1}{3!},\ldots,\frac{1}{d!}\right)$.

We deduce the following proposition from Newton's identities and by observing

$$\left(\frac{1}{k},\frac{1}{k^2},\ldots,\frac{1}{k^{d-1}}\right) = \left(p_i^{(n)}\left(\frac{1}{k},\ldots,\frac{1}{k},0,\ldots,0\right)\right)_{2\leq i\leq d}.$$

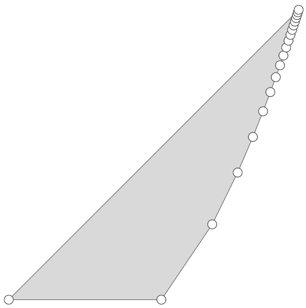

Figure II.5: The set $\mathcal{E}_{20,2}$

**Proposition II.4.4.** *For $n \geq d$, the map*

$$\begin{array}{rcl}\Phi_d : \left\{\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) : k \in [n]\right\} & \longrightarrow & \left\{\left(\frac{1}{k},\frac{1}{k^2},\ldots,\frac{1}{k^{d-1}}\right) : k \in [n]\right\} \\ \left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) & \longmapsto & \left(\frac{1}{k},\frac{1}{k^2},\ldots,\frac{1}{k^{d-1}}\right)\end{array}$$

*induces an affine isomorphism on the vector spaces $\mathbb{R}^{d-1}$.*

*Proof.* Let $n \geq m \geq 2$ and $n \geq k$ be nonnegative integers. Let $z_m := (1, 2, \ldots, m-1, 0, \ldots, 0) \in \mathbb{R}^n$. Then by Vieta's formula we have

$$\begin{aligned}\binom{k}{m}\frac{1}{k^m} &= \frac{\prod_{i=1}^{m-1}(k-i)}{m! \cdot k^{m-1}} \\ &= \frac{1}{m! \cdot k^{m-1}}(k^{m-1} - e_1^{(n)}(z_m)k^{m-2} \pm \cdots + (-1)^{m-1}(e_{m-1}^{(n)}(z_m)) \\ &= \frac{1}{m!} - \frac{1}{2(m-2)!}\frac{1}{k} + \cdots + \frac{(-1)^{m-1}}{m}\frac{1}{k^{m-1}}\end{aligned}$$

which shows that for any $k \in [n]$ the same affine linear relation of the $m$-th coordinates of points in the sets $\left\{\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) : k \in [n]\right\}$ and $\left\{\left(\frac{1}{k},\frac{1}{k^2},\ldots,\frac{1}{k^{d-1}}\right) : k \in [n]\right\}$ is satisfied. ∎

Thus, for all degrees $d$ there exists indeed a bijective linear map, even though Newton's identities provide only polynomial transition maps between $\widetilde{\mathcal{N}}_d$ and $\mathcal{N}_d$. For instance, already the power sum $p_4^{(n)}$ is quadratic in $e_2^{(n)}$ for $n \geq 4$.

**Corollary II.4.5.** *For $n \geq d \geq 2$, the set* $\operatorname{conv}\left\{\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) : k \in [n]\right\}$ *does not contain the point* $\left(\binom{n+1}{2}\frac{1}{(n+1)^2},\ldots,\binom{n+1}{d}\frac{1}{(n+1)^d}\right)$.

*Proof.* This follows immediately from Proposition II.4.4 since it is true for the points on the moment map (see e.g. ([Bar02, Chapter II.9.])). ∎

We observed in Corollary II.3.11 that the projection of the set $\mathcal{N}_{n,3} \cap \{p_2^{(n)} = 1\}$ onto its last two coordinates is contained in $\operatorname{conv}\{(\frac{1}{k}, \frac{1}{k^2}) : k \in [n]\}$. So it seems naturally to ask whether an analogous result to Theorem II.4.1 in the power sums generalizes to higher degrees. We provide a negative answer in terms of the convex hull of points of the form $(1/k, \ldots, 1/k^{d-1})$ on the moment curve.

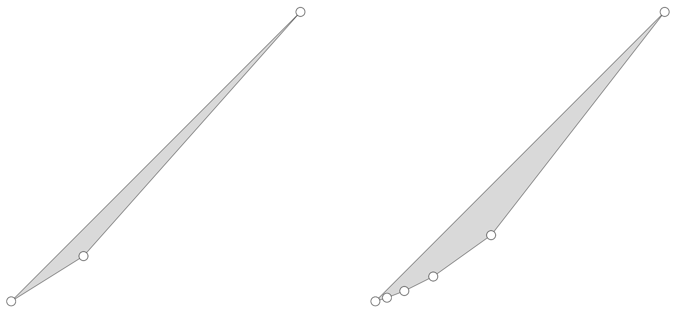

Figure II.6: The polytopes with vertex sets $\{(1/k, 1/k^2) : k \in [n]\}$ for $n = 3$ (left) and $n = 6$ (right)

**Proposition II.4.6.** *Let $d \geq 4$. Then, for sufficiently large $n$ (and in the limit) the set*

$$\operatorname{conv}\left\{\left(\frac{1}{k}, \frac{1}{k^2}, \cdots, \frac{1}{k^{d-1}}\right) : k \in [n]\right\}$$

*does not contain the set $(p_2^{(n)}, p_3^{(n)}, \ldots, p_d^{(n)})(\Delta_n)$.*

*Proof.* We consider $f = 2p_4^{(n)} - 3p_{(3,1)}^{(n)} + p_{(2,1^2)}^{(n)}$. The form $f$ is nonnegative on the convex hull of the rational points on the moment curve of the form $(1/k, 1/k^2, 1/k^3)$, since

$$f(1/k, 1/k^2, 1/k^3) = \frac{(k - 3/2)^2 - 1/4}{k^3} \geq 0$$

for all $k \in \mathbb{N}$. However, for $n = m + 1$ we have

$$f(a, \underbrace{1, \ldots, 1}_{\#1's = m}) = -ma^3 + a^2m^2 + a^2m + 2am^2 - 3am + m^3 - 3m^2 + 2m =: g_m(a)$$

and thus for fixed $m$ we observe $g_m(a)$ has a negative leading coefficient. For sufficiently large $a > 0$ we must have $f(a, 1, \ldots, 1) < 0$. Therefore, $f$ cannot be globally nonnegative and since $f$ is homogeneous $f$ cannot be nonnegative on the restriction to $p_2^{(n)} = 1$, i.e., $f$ is not nonnegative on the probability simplex $\Delta_n$. This shows the existence of a point in the set $(p_2^{(n)}, p_3^{(n)}, \ldots, p_d^{(n)})(\Delta_n) \setminus \operatorname{conv}\left\{\left(\frac{1}{k}, \frac{1}{k^2}, \cdots, \frac{1}{k^{d-1}}\right) : k \in [n]\right\}$ and the claim follows. ∎

We reformulate the descriptions of the vertex sets of $\mathcal{E}_{n,d}$ and $\mathcal{E}_d$ to provide test sets for nonnegativity of certain homogeneous even symmetric polynomials and functions.

**Theorem II.4.7.** *Let $f = f(e_1^{(n)}(X^2), e_2^{(n)}(X^2), \ldots, e_d^{(n)}(X^2))$ be an even symmetric form in $n \geq d$ variables in which only $e_1^{(n)}(X^2)$ occurs non linearly. Then, $f$ is nonnegative if and only if*

$$f\left(1, \binom{k}{2}\frac{1}{k^2}, \ldots, \binom{k}{d}\frac{1}{k^d}\right)$$

*is nonnegative for any $k \in [n]$.*

*Proof.* Since $f(e_1^{(n)}(X^2), e_2^{(n)}(X^2), \ldots, e_d^{(n)}(X^2))$ is homogenous we can restrict to the case $p_2^{(n)}(X) = e_1^{(n)}(X^2) = 1$. As $f$ is linear in the remaining elementary symmetrics we observe $f$ is nonnegative if and only if $f$ is nonnegative on the convex hull of the image of the elementary symmetrics on $\Delta_n$. By Theorem II.4.1 the convex hull equals

$$\mathcal{E}_{n,d} = \operatorname{conv}\left(\left(\binom{k}{2}\frac{1}{k^2}, \ldots, \binom{k}{d}\frac{1}{k^d} : 1 \leq k \leq n\right)\right).$$

In particular, $f$ is nonnegative if and only if $f(1, y_2, \ldots, y_d)$ is nonnegative on the vertices of $\mathcal{E}_{n,d}$, which are precisely the claimed points as observed in II.4.5. ■

**Corollary II.4.8.** *Let $f(\mathfrak{e}_1(X^2), \mathfrak{e}_2(X^2), \ldots, \mathfrak{e}_d(X^2))$ be an even symmetric limit form in which only $\mathfrak{e}_1(X^2)$ occurs non linearly. Then, $f$ is nonnegative if and only if $f$ is nonnegative on the discrete set*

$$\left\{\left(1, \binom{k}{2}\frac{1}{k^2}, \ldots, \binom{k}{d}\frac{1}{k^d}\right) : k \in \mathbb{N}\right\} \cup \left\{\left(1, \frac{1}{2!}, \ldots, \frac{1}{d!}\right)\right\}.$$

Actually, we note that the test set for even symmetric forms in Theorem II.4.7 and the description of $\mathcal{E}_{n,d}$ in Theorem II.4.1 are reformulations and thus equivalent. The same is true for the description of the limit $\mathcal{E}_d$ and the limit test set in Corollaries II.4.3 and II.4.8.

*Remark* II.4.9. The result of Choi-Lam-Reznick, i.e., Theorem II.4.2, can be recovered from Theorem II.4.7. Newton's identities provide a linear transformation between $(e_2^{(n)}, e_3^{(n)})$ and $(p_2^{(n)}, p_3^{(n)})$ for $e_1^{(n)} = p_1^{(n)} = 1$. However, a generalization of test sets to higher degrees cannot be given in the power sum basis which follows from Proposition II.4.6 where we considered degree 4.

For $d \geq 3$ we have seen that the sets $\widetilde{\mathcal{N}}_d$ and $\mathcal{N}_d$ are not semialgebraic in Corollary II.3.12. Thus, the set $\mathcal{E}_d$ cannot be semialgebraic since it is the convex hull of $\widetilde{\mathcal{N}}_d \cap \{\mathfrak{p}_2 = 1\}$.

*Remark* II.4.10. A proof by contradiction of Corollary II.3.12 may be done by assuming the set $\mathcal{E}_d$ is semialgebraic. We do know that $\mathcal{E}_d$ has countably infinite vertices by Corollary II.4.5. The union of all vertices of a semialgebraic set is again semialgebraic ([Sin15, Remark 2.16 (a)]), but this set would then consist of infinitely semialgebraically connected components which cannot be satisfied by any semialgebraic set ([BCR13, Theorem 2.4.4]). From knowing that $\mathcal{E}_d$ is not semialgberaic we can deduce that the sets $\widetilde{\mathcal{N}}_d$ and $\mathcal{N}_d$ are not semialgebraic since slices of the convex hull of these sets with $\{\mathfrak{p}_2 = 1\}$ are polynomial images of $\mathcal{E}_d$.

In the remaining part of the section we show that the facets of $\mathcal{E}_{n,d}$ can be described by Gale's evenness condition. For every $n \geq d \geq 3$ we note that the convex polytope

$$\operatorname{conv}\left\{\left(\frac{1}{k}, \frac{1}{k^2}, \ldots, \frac{1}{k^{d-1}}\right) : k \in [n]\right\}$$

is a *cyclic polytope* $C(n, d-1)$ by definition (see e.g. ([Zie12, Section 0])). We immediately obtain

**Corollary II.4.11.** *The set $\mathcal{E}_{n,d}$ is a cyclic polytope.*

*Proof.* It follows from Proposition II.4.4 that $\mathcal{E}_{n,d}$ is a cyclic polytope as an affine transformation of the cyclic polytope $C(n, d-1)$. ∎

The facets of $C(n, d-1)$ (and thus of $\mathcal{E}_{n,d}$) are fully characterized by Gale's evenness condition. We write $\overline{k} = \left(\frac{1}{k}, \frac{1}{k^2}, \ldots, \frac{1}{k^{d-1}}\right)$.

**Theorem II.4.12** ([Gal63]). *For $n > d$, the facets of $C(n, d)$ are given by all $\{\overline{k} : k \in S\}$, where $S \subset [n]$ is any set of size $d$ satisfying*

- *If $d$ is even, then $S$ is either a disjoint union of consecutive pairs $\{i, i+1\}$, or a disjoint union of consecutive pairs $\{i, i+1\}$ and $\{1, n\}$.*
- *If $d$ is odd, then $S$ is a disjoint union of consecutive pairs $\{i, i+1\}$ and either the singleton $\{1\}$ or $\{n\}$.*

For $d \geq 3$ the cyclic polytope $\mathcal{E}_{n,d}$ has $2\binom{n-e}{e-1}$ facets if $d = 2e$ even, and $\frac{n}{n-e}\binom{n-e}{e}$ facets if $d = 2e+1$ is odd ([Zie12, Exercise 0.9]). The vertices $\Phi_{n,d}^{-1}(\overline{k})$ of facets of $\mathcal{E}_{n,d}$ come from those index sets $S \subset [n]$ described in Theorem II.4.12 for all $k \in S$.

We follow ([Zie12, Page 14]) where Ziegler describes the H-representation of $C(n, d)$. For $S = \{k_1, \ldots, k_{d-1}\} \subset [n]$ with $|S| = d-1$ let $\tilde{\ell}_S : \mathbb{R}^{d-1} \to \mathbb{R}$ be the linear map

$$\det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ X_1 & k_1 & \ldots & k_{d-1} \\ X_2 & k_1^2 & \ldots & k_{d-1}^2 \\ & & \ddots & \\ \ldots & \ldots & & \ldots \\ X_{d-1} & k_1^{d-1} & \ldots & k_{d-1}^{d-1} \end{pmatrix}.$$

It follows from properties of the Vandermonde determinant, that $\tilde{\ell}_S(k, k^2, \ldots, k^{d-1}) = 0$ if and only if $k \in S$. Thus, $\tilde{\ell}_S$ defines a linear map which kernel is generated by all $\overline{k} \in S$. The $H$-representation of $C(n, d)$ is then given

by inequalities of the form $\pm\tilde{\ell}_S(X) \leq r_S$ for all facet defining sets $S \subset [n]$ and some $r_S \in \mathbb{R}$. We write $\ell_S$ for $\tilde{\ell}_S$ multiplied by $-1$ to the correct power.
We have $\mathcal{E}_{n,d} = \Phi_d^{-1}(\mathrm{conv}\{(1/k, \dots, 1/k^{d-1}) : k \in [n]\})$ by Proposition II.4.4. Thus, we can formulate the $H$-representation of $\mathcal{E}_{n,d}$.

**Proposition II.4.13.** *Let $n \geq d$ be nonnegative integers. Let $\mathcal{C}_{d-1}$ denote the collection of facet defining sets of $C(n, d-1)$ in Theorem II.4.12. Then the $H$-representation of $\mathcal{E}_{n,d}$ is given by the inequalities*

$$\{\ell_S \circ \Phi_d(X) \leq r_S \ : \ S \in \mathcal{C}_{d-1}\}.$$

*Proof.* The claim follows from the discussion above and since

$$\begin{aligned}\mathcal{E}_{n,d} &= \Phi_d^{-1}(\mathrm{conv}\{(1/k, \dots, 1/k^{d-1}) : k \in [n]\}) \\ &= \Phi_d^{-1}(\{x \in \mathbb{R}^{d-1} : \ell_S(x) \leq r_S, S \in \mathcal{C}_{d-1}\})\end{aligned}$$

by Proposition II.4.4, we have

$$\mathcal{E}_{n,d} = \{x \in \mathbb{R}^{d-1} : \ell_S \circ \Phi_d(x) \leq r_S, S \in \mathcal{C}_{d-1}\}.$$

■

*Remark* II.4.14. The boundary of the diffeomorphic sets

$$(p_2^{(n)}, \dots, p_d^{(n)})(\Delta_n) \text{ and } (e_2^{(n)}, \dots, e_d^{(n)})(\Delta_n)$$

has the combinatorial structure of a cyclic polytope in the sense that the boundary can be considered as a glueing of patches. Each patch is a hypersurface and contains $d$ cusps. The cusps are precisely the elements of the families of $d$-subsets of $[n]$ above satisfying Gale's evenness condition.

**Example II.4.15.** Using Sage ([Ste07]) we calculate the defining inequalities of $\mathcal{E}_{n,2}$ for $3 \leq n \leq 6$ and obtain:

$$\begin{aligned}\mathcal{E}_{3,2} &= \{x \in \mathbb{R}^2 : x_2 \geq 0,\ x_1 - 9x_2 \geq 0,\ -4x_1 + 9x_2 \geq -1\}, \\ \mathcal{E}_{4,2} &= \{x \in \mathbb{R}^2 : x_2 \geq 0,\ x_1 - 6x_2 \geq 0,\ -11x_1 + 18x_3 \geq -3,\ -4x_1 + 9x_2 \geq -1\}, \\ \mathcal{E}_{5,2} &= \{x \in \mathbb{R}^2 : x_2 \geq 0,\ -4x_1 + 9x_2 \geq -1,\ -11x_1 + 18x_2 \geq -3, \\ &\qquad -7x_1 + 10x_2 \geq -2,\ x_1 - 5x_2 \geq 0\}, \\ \mathcal{E}_{6,2} &= \{x \in \mathbb{R}^2 : x_2 \geq 0,\ -4x_1 + 9x_2 \geq -1,\ -11x_1 + 18x_2 \geq -3, \\ &\qquad -7x_1 + 10x_2 \geq -2, -34x_1 + 45x_2 \geq -10,\ 2x_1 - 9x_2 \geq 0\}.\end{aligned}$$

The code can be found in Appendix B.

In the sequel we analyse what happens with facets of $\mathcal{E}_{n,d}$ for increasing $n$. For a set $S \subset [n]$ containing $n$ we write $S_m := S \setminus \{n\} \uplus \{m\}$ for any integer $m \geq n$ or $m = \infty$.

Let $S \subset [n]$ be a facet defining set of $\mathcal{E}_{n,d}$.
First, suppose $d$ is odd. If $S = \biguplus\{i, i+1\}$ then $S$ is also a facet defining set of

$\mathcal{E}_{m,d}$ for any $m \geq n$. If $S$ does not have such a form then $S_m$ is a facet defining set of $\mathcal{E}_{m,d}$.
Second, suppose $d$ is even. If $S = \biguplus\{i, i+1\} \uplus \{1\}$ then $S$ is also a facet defining set of $\mathcal{E}_{m,d}$ for any $m \geq n$. If $S$ does not have such a form then $S_m$ is also a facet defining set of $\mathcal{E}_{m,d}$.

We observe that for a facet $S_n$ of $\mathcal{E}_{n,d}$, which depends on $n \in S_n$, the sequence of facets $(S_m)_{m \geq n}$ of the cyclic polytopes $\mathcal{E}_{m,d}$ converges to a "limit facet". More precisely:

**Proposition II.4.16.** *Let $n \geq d \geq 2$ and let $S \subset [n]$ be a facet defining set of indices of $\mathcal{E}_{n,d}$. If $d$ is odd we assume $S = \biguplus\{i, i+1\} \uplus \{1, n\}$. If $d$ is even we assume $S = \biguplus\{i, i+1\} \uplus \{n\}$. Then, for all $m \geq n$ the inequalities $\ell_{S_m} \circ \Phi_d \leq r_{S_m}$ corresponding to a facet of $\mathcal{E}_{m,d}$ converge to an inequality $\ell_{S_\infty} \circ \Phi_d \leq r_{S_\infty}$ defining a facet of $\mathcal{E}_d$.*

*Proof.* Because all but one of the vertices of the facets corresponding to $S_m$ are equal, the remaining sequence of changing vertices converges to the limit vertex

$$\left(\binom{m}{2}\frac{1}{m^2}, \ldots, \binom{m}{d}\frac{1}{m^d}\right) \to \left(\frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{d!}\right) \in \mathcal{E}_d, \ m \to \infty.$$

Thus, the facets corresponding to $S_m$ in $\mathcal{E}_{m,d}$ converge to the facet $S_\infty$ in $\mathcal{E}_d$. By continuity the defining linear inequalities must also converge which was to show. ∎

We can show that $\mathcal{E}_d$ is contained in a set defined as an intersection of countably infinite halfspaces.

**Proposition II.4.17.** *Let $d \geq 2$ and $\mathcal{C}_e := \{S \subset \mathbb{N} \cup \{\infty\} : |S| = d, S = \biguplus\{i, i+1\} \uplus \{1, \infty\}\}$ and $\mathcal{C}_o := \{S \subset \mathbb{N} \cup \{\infty\} : |S| = d, S = \{\kappa\} \uplus \biguplus\{i, i+1\},\ \kappa \in \{1, \infty\}\}$. Then*

$$\mathcal{E}_d = \left\{ x \in \mathbb{R}^{d-1} \ : \ \ell_S \circ \Phi_d(x) \leq r_S : \begin{array}{ll} S \in \mathcal{C}_e, & \textit{if } d-1 \textit{ is even;} \\ S \in \mathcal{C}_o, & \textit{if } d-1 \textit{ is odd} \end{array} \right\}$$

*Proof.* We restrict us to the case where $d-1$ is even, since the odd case follows analogously. Let $\mathcal{Z}$ denote the set on the right hand side. $\mathcal{Z}$ is a closed convex set as the intersection of closed convex sets.
We have

$$a_m := \left(\binom{n}{2}\frac{1}{n^2}, \ldots, \binom{n}{d}\frac{1}{n^d}\right) \in \mathcal{E}_{m,d}$$

for any $m \geq n$. Thus, the facet defining inequalities of $\mathcal{E}_{m,d}$ are valid on $a_m$. Furthermore, the limit inequalities are valid on $a_m$ by continuity (see Proposition II.4.16). We have $a_m \in \mathcal{Z}$, because all the countably infinite inequalities defining $\mathcal{Z}$ are valid at $a_m$. Since $\mathcal{Z}$ is convex and closed we have $\mathcal{E}_d = \operatorname{conv}\{(1/2!, \ldots, 1/d!), a_m : m \in \mathbb{N}\} \subset \mathcal{Z}$. ∎

## II.5 The limits of the positivity cones $\mathfrak{P}^{\mathcal{S}}_{2d}, \mathfrak{P}^{\mathcal{B}}_{2d}$

We prove that for any non-trivial case we have strict inclusion between the limit sets of sums of squares and nonnegative limit forms, i.e., we show

**Theorem II.5.1.** *$\mathfrak{S}^{\mathcal{S}}_{2d} \subsetneq \mathfrak{P}^{\mathcal{S}}_{2d}$ for all $2d \geq 4$ and $\mathfrak{S}^{\mathcal{B}}_{2d} \subsetneq \mathfrak{P}^{\mathcal{B}}_{2d}$ for all $2d \geq 6$.*

For any other degree we have equality by Hilbert's famous theorem from 1888 and since $\Sigma^{\mathcal{B}_n}_{n,4} = \mathcal{P}^{\mathcal{B}_n}_{n,4}$ for all $n$ [Har99]. Our proof of the theorem is divided into two parts. The first part considers symmetric quartic functions, while the second part treats the cases of (even) symmetric functions of degree $\geq 6$ simultaneously and uses the non semialgebraicness of the set $\mathcal{N}_d$ for $d \geq 3$ proven in Section II.3.

### II.5.1 Symmetric quartics

Besides answering the nonnegativity versus sums of squares question for symmetric quartics, we present test sets for verifying nonnegativity and being a sum of squares. This leads to the construction of a uniform nonnegative but not sum of squares symmetric function of degree 4 which turns out to be never a sum of squares.

We begin with presenting all the limit symmetric quartic sums of squares. Therefore, we state the isotypic decomposition

$$H_{n,2} \simeq_{\mathcal{S}_n} 2 \cdot \mathbb{S}^{(n)} \oplus 2 \cdot \mathbb{S}^{(n-1,1)} \oplus \mathbb{S}^{(n-2,2)}$$

with the representing matrices for the limit

$$\begin{pmatrix} \mathfrak{p}_{(2^2)} & \mathfrak{p}_{(2,1^2)} \\ \mathfrak{p}_{(2,1^2)} & \mathfrak{p}_{(1^4)} \end{pmatrix}, \begin{pmatrix} \mathfrak{p}_{(2,1^2)} & \mathfrak{p}_{(3,1)} \\ \mathfrak{p}_{(3,1)} & \mathfrak{p}_{(4)} \end{pmatrix}, \left(\mathfrak{p}_{(2^2)} - \mathfrak{p}_{(4)}\right).$$

The dual cone to sums of squares quartics at infinity, $\mathfrak{S}^{\mathcal{S},*}_4$, is the spectrahedron containing precisely the $(a,b,c,d,e) \in \mathbb{R}^5$ such that $X = A \oplus B \oplus C \succeq 0$ where $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $B = \begin{pmatrix} b & d \\ d & e \end{pmatrix}$, and $C = c - e$. Thus, the primal cone $\mathfrak{S}^{\mathcal{S}}_4$ is described with $a = \mathfrak{p}_{(1^4)}$, $b = \mathfrak{p}_{(2,1^2)}$, $c = \mathfrak{p}_{(2^2)}$, $d = \mathfrak{p}_{(3,1)}$, $e = \mathfrak{p}_{(4)}$.
It is well known that the dual of the cone of sums of squares can be identified with a set of positive semidefinite quadratic forms. The second and fourth author show in [BR21] that an analysis of the extremal rays in the symmetric case can be done similarly to the general case [Ble12; BS17]. The extremal rays of $\mathfrak{S}^{\mathcal{S},*}_4$ correspond to positive semidefinite quadratic forms with maximal kernel, i.e., to those vectors $(a,b,c,d,e) \in \mathbb{R}^5$ for which $X$ is positive semidefinite and the quadratic form has maximal kernel ([BS17, Proposition 4.20]).

**Proposition II.5.2.** *Every point in the set*

$$S = \{(1, t^2, t^4, st^2, s^2t^2) \,|\, t \geq 0,\ s \in [-t, t]\}$$

*spans an extreme ray of $\mathfrak{S}^{\mathcal{S},*}_4$. Moreover, every extreme ray of $\mathfrak{S}^{\mathcal{S},*}_4$ is spanned by a point in $S$ or by $(0,0,1,0,0)$ or $(0,0,1,0,1)$.*

We present a case analysis on the kernel of the quadratic form analogously to ([BR21, Lemma 5.2]).

*Proof.* Suppose $(a, b, c, d, e)$ spans a extreme ray of $\mathfrak{S}_4^*$. We begin by a case distinction on the rank of $X$.

Case $\operatorname{rank} A = 1$ and $\operatorname{rank} B = 0$. If $\operatorname{rank} B = 0$ then $b = d = e = 0$. Hence, since $\operatorname{rank} A = 1$ then either $a = 0$ or $c = 0$. If $c = 0$ we have a extreme ray spanned by $(1, 0, 0, 0, 0)$, and if $a = 0$ we have a extreme ray spanned by $(0, 0, 1, 0, 0)$.

Case $\operatorname{rank} A = 1$ and $\operatorname{rank} B = 1$. If $b = 0$ then from $B$ we get $d = 0$, and from $A$ either $a = 0$ or $c = 0$. In the first case then $c \neq 0$ and $c = e$ will give maximal kernel, so $(0, 0, 1, 0, 1)$ spans a extreme ray. In the second case then from $C$ we get $e = 0$, so $\operatorname{rank} B = 0$, impossible. If $a = 0$ then from $A$ we get $b = 0$ and the same follows. So we can assume from now on that $a$ and $b$ are nonzero.

From $\det A = \det B = 0$ we obtain $c = b^2/a$ and $e = d^2/b$. So $(a, b, c, d, e) = (a, b, \frac{b^2}{a}, d, \frac{d^2}{b})$, and dividing by $a$ we obtain $(1, t, t^2, \frac{d}{b}t, \frac{d^2}{b^2}t)$ where $t = \frac{b}{a}$. Setting $s = \frac{d}{b}$ we obtain $(1, t, t^2, st, s^2t)$. If $C = 0$ then $c = e$ and so $b^3 = ad^2$ or $\frac{b}{a} = \frac{d^2}{b^2}$, i.e., $t = s^2$, and so we obtain the family of extreme rays $(1, s^2, s^4, s^3, s^4)$ for $s \in \mathbb{R} \setminus \{0\}$. If $C > 0$ then $c > e$ or $t > s^2$, and we get the family of extreme rays $(1, t, t^2, st, s^2t)$ with $t > 0$ and $s \in (-t^{1/2}, t^{1/2})$.

The cases $\operatorname{rank} A = 2$ or $\operatorname{rank} B = 2$ cannot occur due to the maximality of the kernel of the positive semidefinite quadratic form. ■

Kostov describes in [Kos07] the set $\mathcal{M}_4$ but in the coordinates of elementary symmetrics. He writes $\Pi_d(\infty)$ to denote the closure of the set $\bigcup_{n \geq 1}(e_1^{(n)}, e_2^{(n)}, \ldots, e_d^{(n)})(\mathbb{R}^n)$ which he calls the set of *stably hyperbolic* polynomials of degree $d$. Kostov motivates the study of $\Pi_d(\infty)$ through explaining that the set is the closure of the set of all monic hyperbolic polynomials whose first $d+1$ coefficients are contained in $\{1\} \times \bigcup_{n \geq 1}(e_1^{(n)}, e_2^{(n)}, \ldots, e_d^{(n)})(\mathbb{R}^n)$. The paper focuses on degree 4 and the parametrization of the boundary of $\Pi_4(\infty) \cap \{x \in \mathbb{R}^4 : x_1 = 0, x_2 = -1\}$. Since $\mathcal{M}_4$ and $\Pi_4(\infty)$ are weighted homogeneous by Lemma II.3.1 and prisms with respect to the first coordinate by Proposition II.3.4 this is not a restriction of the general case. Newton's identity provides an isomorphism

$$\mathcal{M}_4 \cap \{x \in \mathbb{R}^4 : x_1 = 0, x_2 = 1\} \simeq \Pi_4(\infty) \cap \{x \in \mathbb{R}^4 : x_1 = 0, x_2 = -1\}.$$

Combining Kostov's results on $\mathcal{M}_4$ with the description of the dual cone $\mathfrak{S}_4^{\mathcal{S},*}$ allows us to prove the following theorem.

**Theorem II.5.3.** $\mathfrak{S}_4^{\mathcal{S}} \subsetneq \mathfrak{P}_4^{\mathcal{S}}$.

*Proof.* Setting $s = 1$ in Proposition II.5.2 we see that, for all $t \geq 1$, $(1, t^2, t^4, t^2, t^2)$ spans a extreme ray of $\mathfrak{S}_4^*$. Any such extreme ray comes from a point evaluation if and only if $(1, t^2, t^2, t^2) \in \mathcal{M}_4$ by (II.4), i.e., if the point is contained in the image of the Vandermonde map at the limit. The proof of Lemma II.3.5 shows that $(1, t^2, t^2, t^2) \in \mathcal{M}_4$ implies $(0, 1, \frac{1}{t}, \frac{1}{t^2}) \in \mathcal{M}_4$. However $(0, 1, \frac{1}{t}, \frac{1}{t^2}) \notin \mathcal{M}_4$ when $t^2$ is not an integer which was proven by Kostov ([Kos07]) but formulated for the image of elementary symmetric functions. For example, for $t = \frac{2}{\sqrt{3}}$ we claim that $(0, 1, \frac{\sqrt{3}}{2}, \frac{3}{4}) \notin \mathcal{M}_4$. To prove this we transform to elementary symmetric coordinates using Newton's identities, and rescale appropriately so that the first two coordinates are $(0, -1)$, giving the last two coordinates $(\frac{\sqrt{6}}{3}, -\frac{1}{4})$. We see that this point lies strictly above the arc $B_1$ (since $1 < t = \frac{2}{\sqrt{3}} < 2$) by using the parametrization given in ([Kos07, p.102]), i.e., we verify $(0, -1, \frac{\sqrt{6}}{3}, -\frac{1}{4}) \notin \Pi_4(\infty)$. Namely, we prove that if $\frac{2\sqrt{2}}{3}\cos^3 t + \frac{2\sqrt{2}}{3}\sin^3 t = \frac{\sqrt{6}}{3}$ then $\frac{1}{2} - \cos^4 t - \sin^4 t < -\frac{1}{4}$. Equivalently

$$\begin{aligned} \cos^3 t + \sin^3 t &= \frac{\sqrt{3}}{2} \\ \Rightarrow \cos^4 t + \sin^4 t &> \frac{3}{4}. \end{aligned} \tag{II.11}$$

For this purpose observe that squaring the supposed equality (II.11) implies

$$\begin{aligned} \frac{3}{4} &= (\cos^3 t + \sin^3 t)^2 \\ &= (\cos^4 t + \sin^4 t)(\cos^2 t + \sin^2 t) + 2\cos^3 t \sin^3 t - \cos^2 t \sin^2 t \end{aligned}$$

Therefore, using the Pythagorean identity and $2 \sin t \cos t = \sin t$ we obtain

$$\cos^4 t + \sin^4 t = \frac{3}{4} + \cos^2 t \sin^2 t(1 - \sin 2t) \geq \frac{3}{4}.$$

It only remains to prove that equality is impossible, which happens if and only if $\cos t = 0$ or $\sin t = 0$ or $t = \frac{\pi}{4}$ or $t = \frac{5\pi}{4}$ but neither of these satisfy all the above equations simultaneously. ∎

Based on Kostov's description of the boundary of $\Pi_4(\infty) \cap \{x \in \mathbb{R}^4 : x_1 = 0, x_2 = -1\}$ and Proposition II.5.2 we present test sets for quartic limits to be nonnegative or a sum of squares.

Let $\mathcal{K}$ be *Kostov's leaf* ([Kos07, Fig. 2]), i.e., the image of the parametrization of the boundary of $\Pi_4(\infty) \cap \{x \in \mathbb{R}^4 : x_1 = 0, x_2 = -1\}$. Then the extreme points of $\operatorname{conv}(\mathcal{K})$ are precisely the cusps of the arcs of $\mathcal{K}$. Since $\mathcal{K}$ has countably infinite cusps the set $\operatorname{conv}(\mathcal{K})$ has countably infinite vertices. Let $\mathcal{K}'$ be the projection of $\mathcal{M}_4 \cap \{\mathfrak{p}_2 = 1\}$ onto the last two coordinates.
We observe, $\mathcal{K}'$ is the image of $\mathcal{K}$ under a linear invertible map coming from Newton's identities and so the extreme points of $\operatorname{conv}(\mathcal{K}')$ are the images of

the extreme points of conv($\mathcal{K}$) under the same map. By ([Kos07, p. 102]) the extreme points of conv($\mathcal{K}$) are

$$\left\{ \left( \pm\frac{2}{3}\sqrt{\frac{2}{s}}, \frac{1}{2} - \frac{1}{s} \right) : s \in \mathbb{N}_{>0} \right\}.$$

Thus, we obtain that the set $\mathcal{T}$ of extreme points of conv($\mathcal{K}'$) equals

$$\mathcal{T} = \{(\pm n^{-1/2}, n^{-1}) : n \in \mathbb{N}_{>0}\}.$$

This observation suffices to provide a test set for nonnegativity. Since the first coordinate of $\mathcal{M}_d$ is free by Lemma II.3.5 and we can restrict to $\mathfrak{p}_2 = 1$ by homogeneity, then a limit symmetric quartic

$$f := c_1\mathfrak{p}_1^4 + c_2\mathfrak{p}_2\mathfrak{p}_1^2 + c_3\mathfrak{p}_3\mathfrak{p}_1 + c_4\mathfrak{p}_2^2 + c_5\mathfrak{p}_4$$

is nonnegative if and only if each univariate polynomial in the family

$$\mathscr{F}' := \{c_1x^4 + c_2x^2 + c_3p_3x + c_4 + c_5p_4 \mid (p_3, p_4) \in \mathcal{K}'\}$$

is nonnegative (if $\mathfrak{p}_2 = 0$ then $\mathfrak{p}_4 = 0$ because $\mathfrak{p}_2^2 \geq \mathfrak{p}_4 \geq 0$, and $\mathfrak{p}_3 = 0$ because $\mathfrak{p}_2\mathfrak{p}_4 \geq \mathfrak{p}_3^2$ and in such case $c_1 \geq 0$). Now, since any convex combination of nonnegative polynomials is nonnegative then $f$ is nonnegative if and only if each univariate polynomial in the family conv($\mathscr{F}'$) is nonnegative. Moreover, since the coefficients of the polynomials in this family depend linearly on $p_3$ and $p_4$ then $f$ is nonnegative if and only if each univariate polynomial in the family

$$\mathscr{F} = \{c_1x^4 + c_2x^2 + c_3p_3x + c_4 + c_5p_4 \mid (p_3, p_4) \in \mathcal{T}, p_3 > 0\}$$

is nonnegative. We note that it is sufficient to restrict to the test set $\{(n^{-1/2}, n^{-1}) : n \in \mathbb{N}_{>0}\}$.

**Theorem II.5.4.** *Let $f(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4)$ be a limit symmetric quartic. Then, $f$ is nonnegative if and only if $f$ is nonnegative on the discrete set of parallel lines*

$$\{(x, 1, n^{-1/2}, n^{-1}) \mid x \in \mathbb{R},\ n \in \mathbb{N}_{>0}\}.$$

*Moreover, $f$ is a sum of squares if and only $f$ it is nonnegative on*

$$\{(x, 1, u, u^2) \mid x \in \mathbb{R},\ 0 \leq u \leq 1\}.$$

*Proof.* The first claim follows from the discussion above while the second follows from setting $\mathfrak{p}_2 = 1$ in the description of $\mathfrak{S}_4^{\mathcal{S},*}$ in Proposition II.5.2. ∎

To the best of our knowledge the following is the first uniform sequence of symmetric nonnegatives but not sum of squares polynomials, i.e., the coefficients are independently from the number of variables.

**Theorem II.5.5.** *The limit symmetric quartic $f := 4\mathfrak{p}_1^4 - 5\mathfrak{p}_2\mathfrak{p}_1^2 - \frac{139}{20}\mathfrak{p}_3\mathfrak{p}_1 + 4\mathfrak{p}_2^2 + 4\mathfrak{p}_4$ belongs to the set $\mathfrak{P}_4^{\mathcal{S}} \setminus \mathfrak{S}_4^{\mathcal{S}}$. The corresponding forms are nonnegative in any number of variables. Moreover, these forms are never sums of squares for any number of variables $n \geq 4$.*

*Proof.* We write $f_n$ for $f$ considered as quartic form in $n$ variables. For $x \in \mathbb{R}$ we define the auxiliary quadratic polynomial

$$g_x(u) := 4x^4 - 5x^2 - 139/20xu + 4 + 4u^2.$$

Then, $f \in \mathfrak{P}_4^{\mathcal{S}}$ if and only if $g_x(u) \geq 0$ for all $u \in S := \{\frac{1}{\sqrt{n}} : n \in \mathbb{N}_{>0}\} \cup \{0\}$ and $x \in \mathbb{R}$, while $f \in \mathfrak{S}_4^{\mathcal{S}}$ if and only if $g_x(u) \geq 0$ for all $0 \leq u \leq 1$ and $x \in \mathbb{R}$ using the test set II.5.4.
We have $g_1(0.85) \approx -0.0175 < 0$ and so $f \notin \mathfrak{S}_4^{\mathcal{S}}$. Thus, $f_N \notin \Sigma_{N,4}^{\mathcal{S}}$ for some $N \geq 4$, but we have already $f \notin \Sigma_{4,4}^{\mathcal{S}}$ and verified this using the SumsOfSquares package ([CKP20]) in Macaulay2 ([Eis+01]).
We still have to prove $f \in \mathfrak{P}_4^{\mathcal{S}}$. We claim that for each $x \in \mathbb{R}$, $g_x(u)$ has roots either in $[\frac{1}{\sqrt{2}}, 1]$ or $[-1, -\frac{1}{\sqrt{2}}]$, or has no real roots. Since the coefficient of $xu$ in $g_x(u)$ is negative, it suffices that for all $x \geq 0$, $g_x(u)$ has roots in $[\frac{1}{\sqrt{2}}, 1]$ or has no real roots. This follows since for $x < 0$ everything gets reflected over the $y$-axis because the only coefficient of $g_x(u)$ that changes when $x \to -x$ is the coefficient of $u$. Considering the discriminant and $x \geq 0$, we have $g_x(u)$ has only roots in $[\frac{1}{\sqrt{2}}, 1]$ if and only if

$$4\sqrt{2} \leq +6.95x \pm \sqrt{-64x^4 + 128.3025x^2 - 64} \leq 8$$

for any $x \geq 0$ such that $-64x^4 + 128.3025x^2 - 64 \geq 0$. Otherwise, $g_x(u)$ has no real roots. The set of all points in $\mathbb{R}$ for which $-64x^4 + 128.3025x^2 - 64 \geq 0$ is an interval $[a, b] \subset [0.96, 1.04] \subset \mathbb{R}_{\geq 0}$. We observe $4\sqrt{2} - 6.95x \leq 0$ and $8 - 6.95x \geq 0$ on $[0.96, 1.04]$. Now, the claim follows from the global inequalities

$$\begin{aligned} 64x^4 - 80x^2 - \frac{278\sqrt{2}}{5}x + 96 &\geq 0 \\ 64x^4 - 80x^2 - \frac{556}{5}x + 128 &\geq 0. \end{aligned}$$

■

**Example II.5.6.** The limit form $4\mathfrak{p}_1^4 - 5\mathfrak{p}_2\mathfrak{p}_1^2 - 4\sqrt{3}\mathfrak{p}_3\mathfrak{p}_1 + 4\mathfrak{p}_2^2 + 4\mathfrak{p}_4$ is contained in the boundary of $\mathfrak{S}_4^{\mathcal{S}}$ but not in the boundary of $\mathfrak{P}_4^{\mathcal{S}}$. To see this we consider

$$\begin{aligned} g_x(u) &= 4u^2 - 4\sqrt{3}xu + 4x^4 - 5x^2 + 4 \\ &= (2u - \sqrt{3}x)^2 + 4(x^2 - 1)^2 \end{aligned}$$

and observe $g_x(u) = 0$ if and only if $x = \pm 1$ and $u = \pm\frac{\sqrt{3}}{2}$. However, this shows that $g_x(v)$ is strictly positive for any $v \in S$, but attains 0 at $u = \frac{\sqrt{3}}{2} \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

### II.5.2 The remaining cases

By the Tarski-Seidenberg transfer principle the sets $\mathcal{P}_{n,2d}^{\mathcal{S}_n}$ and $\mathcal{P}_{n,2d}^{\mathcal{B}_n}$ are semialgebraic. We show that for any even degree $2d \geq 6$ the limit set of nonnegative (even) symmetric forms is not semialgebraic. This implies strict inclusion between the limit sets of (even) symmetric sums of squares and positive semidefinite forms of degree $2d$. It also demonstrates that the limit case has a higher level of complexity even though we work with symmetric functions.

**Lemma II.5.7.** *The dual cone of a semialgebraic set $S \subset \mathbb{R}^n$ is also semialgebraic, i.e., the set $S^* = \{a \in \mathbb{R}^n : \sum_{i=1}^n a_i w_i \geq 0, \forall w \in S\}$ is semialgebraic.*

*Proof.* We observe

$$\begin{aligned} S^* &= \{a \in \mathbb{R}^n : \exists w \in S, \sum_{i=1}^n a_i w_i < 0\}^c \\ &= \pi\left(\{(a,w) \in \mathbb{R}^n \times \mathbb{R}^n : w \in S, \sum_{i=1}^n a_i w_i < 0\}\right)^c, \end{aligned}$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the projection onto the first $n$ coordinates. Thus, $S^*$ is semialgebraic. ∎

**Theorem II.5.8.** *For any even degree $2d \geq 6$ the cones $\mathfrak{P}_{2d}^{\mathcal{S}}$ and $\mathfrak{P}_{2d}^{\mathcal{B}}$ are not semialgebraic.*

*Proof.* We fix the products of power sum functions as a basis of the vector space of homogeneous symmetric functions of degree $2d$. We observed in (II.4) that the dual cone to the even symmetric nonnegative limit forms is the convex conical hull of a polynomial image of the image of the even Vandermonde map at infinity, i.e.,

$$\mathfrak{P}_{2d}^{\mathcal{B},*} = \operatorname{cone}(\nu_d(\mathcal{N}_d)).$$

However, for $n \geq 3$ the set $\mathcal{N}_d$ is known to be non semialgebraic by Corollary II.3.12. Thus, the set $\nu_d(\mathcal{N}_d)$ must also be non semialgebraic as a polynomial image of $\mathcal{N}_d$. Since the convex conical hull of a non semialgebraic set is also not semialgebraic we obtain that $\mathfrak{P}_{2d}^{\mathcal{B},*}$ is not semialgebraic for all $d \geq 3$. In particular, the set $\mathfrak{P}_{2d}^{\mathcal{B}}$ is not semialgebraic by Lemma II.5.7.
Let $d \geq 3$. We suppose that the set

$$\mathfrak{P}_{2d}^{\mathcal{S}} \simeq \left\{ (c_\lambda)_{\lambda \vdash \pi(2d)} : \sum_{\lambda \vdash \pi(2d)} c_\lambda \mathfrak{p}_\lambda \text{ is psd} \right\}.$$

is semialgebraic and consider the linear subspace

$$\mathcal{H}_{2d} := \{(c_\lambda)_{\lambda \vdash \pi(2d)} : c_\mu = 0,\ \forall \mu \notin \bigcup_{m \geq 0} (2\mathbb{N})^k\}$$

containing the subspace of even symmetric forms of degree $2d$. Then, the intersection $K_{2d} := \mathfrak{P}_{2d}^{\mathcal{S}} \cap \mathcal{H}_{2d}$ is semialgebraic. However, $K_{2d} \simeq \mathfrak{P}_{2d}^{\mathcal{B}}$ which we already know to be not semialgebraic. This is a contradiction and therefore the set $\mathfrak{P}_{2d}^{\mathcal{B}}$ cannot be semialgebraic. ∎

We obtain strict inclusion between (even) symmetric sums of squares and nonnegative limit forms for degree $2d \geq 6$.

**Corollary II.5.9.** *Let $2d \geq 6$. Then, the set of all (even) symmetric sums of squares limit forms of degree $2d$ is strictly contained in the set of all (even) symmetric nonnegative limit forms of degree $2d$, i.e.,*

$$\mathfrak{S}_{2d}^{\mathcal{S}} \subsetneq \mathfrak{P}_{2d}^{\mathcal{S}}, \text{ and } \mathfrak{S}_{2d}^{\mathcal{B}} \subsetneq \mathfrak{P}_{2d}^{\mathcal{B}}.$$

*Proof.* For any degree $2d$ the sets $\mathfrak{S}_{2d}^{\mathcal{S},*}$ and $\mathfrak{S}_{2d}^{\mathcal{B},*}$ have spectrahedral representations and are therefore semialgebraic. Thus, their duals are semialgebraic by Lemma II.5.7 which shows that the set of nonnegatives cannot be equal to the set of sums of squares. ∎

## II.6 Undecidability of nonnegativity for multisymmetric functions

We prove that the problem of verifying nonnegativity of multisymmetric functions is undecidable using the vertex description of the convex set $\mathcal{E}_d$ in Section II.4. We follow work on undecidability in graph homomorphism densities [BRW22; HN11].

We consider the diagonal action of the group $\mathcal{S}_n^k = \prod_{i=1}^k \mathcal{S}_n$ via permutation of $k$ groups of variables $(X_{i,j})_{1\leq i\leq k, 1\leq j\leq n}$ on $\mathbb{R}[X^k] := \mathbb{R}[X_{1,1},\ldots,X_{1,n},\ldots,X_{k,1},\ldots,X_{k,n}]$. Then, $\mathcal{S}_n^k$ acts as a reflection group on $\mathbb{R}[X^k]$ and the invariant ring is again a polynomial ring in elementary symmetrics, i.e.,

$$\mathbb{R}[X^k]^{\mathcal{S}_n^k} = \mathbb{R}[e_{1,(1)}^{(n)},\ldots,e_{n,(1)}^{(n)},\ldots,e_{1,(k)}^{(n)},\ldots,e_{n,(k)}^{(n)}],$$

where $e_{l,(i)}^{(n)}$ denotes the $l$-th elementary symmetric function in $X_{i,1},\ldots,X_{i,n}$. Analogously, we define the invariant ring under the diagonal action of $\mathcal{B}_n^k$, i.e., $k$ copies of the signed symmetric group. A form is called *(even) multisymmetric* or *(even) $k$-symmetric* if it is invariant under $\mathcal{S}_n^k$ ($\mathcal{B}_n^k$). In particular, any (even) symmetric form is (even) 1-symmetric. Analogous to the definition of (even) symmetric limit forms in Subsection II.2.3, we define (even) multisymmetric limit forms. Let

$$\Delta^k := \left\{ (x_{1,1},\ldots,x_{k,1},x_{1,2},\ldots,x_{k,2},\ldots) : \sum_{j\geq 1} x_{i,j} = 1, x_{i,j} \geq 0, \forall i,j \right\}$$

denote $k$-copies of the infinite probability simplex. Now, we prove undecidability of the determination of validity of nonnegativity of even $k$-symmetric limit forms on copies of the probability simplex.

**Theorem II.6.1.** *The following problem is undecidable.*

Instance: *A positive integer $k$ and a $k$-symmetric limit form $f$.*

Question: *Does the inequality $f(x) \geq 0$ hold for all $x = (x_{1,1}, \ldots, x_{k,1}, x_{1,2}, \ldots, x_{k,2}, \ldots) \in \Delta^k$?*

In the proof we follow ([HN11, § 5]) and use their notation. Hatami and Norin's work concerns undecidability of determining the validity of polynomial inequalities between graph homomorphism densities for graphons. Their proof answers negative a question of Lovász ([Lov08, Problem 17]). By adapting only very few parts of Hatami and Norin's proof we show that an undecidable problem can be embedded into the problem of deciding nonnegativity for even multisymmetric limit forms. Namely, by Matiyasevich's solution to Hilbert's tenth problem ([Mat70]) deciding nonnegativity of multivariate polynomials on the natural numbers is undecidable. We relate the nonnegativity of a multivariate polynomial on $\mathbb{N}^k$ to the nonnegativity of a $k$-symmetric function on $\Delta^k$.

*Proof of Theorem II.6.1.* By ([HN11, Lemma 5.1]) it follows from Matiyasevich's solution to Hilbert's tenth problem that the following validity problem is undecidable:

Instance: A positive integer $k$ and a polynomial $p \in \mathbb{Z}[Y_1, \ldots, Y_k]$.

Question: Do there exist $x_1, \ldots, x_k \in \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ with $p(x_1, \ldots, x_k) < 0$?

From Corollary II.4.3 we know that the convex hull of the image of the scaled elementary symmetric functions $2\mathfrak{e}_2, 6\mathfrak{e}_3$ on the infinite probability simplex equals

$$C := \operatorname{conv}\left\{(1,1), \left(1 - \frac{1}{n}, \frac{(n-1)(n-2)}{n^2}\right) : n \in \mathbb{N}\right\}.$$

Let $g(x) := 2x^2 - x$ and define the piecewise linear function

$$L(x) := \frac{3t^2 - t - 2}{t(t+1)}x - \frac{2(t-1)}{t+1}$$

on the interval $[0,1]$, where $t \in [0,1)$ is chosen such that $x \in \left[1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right]$ for some $t \in \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$, and $L(1) := 1$. The piecewise linear function $L$ takes the same value as $g$ on all the endpoints of the intervals $\left[1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right]$ and $L(x) \geq g(x)$. Further, let $R := \{(x,y) \in [0,1]^2 : y \geq L(x)\}$. The images of each piecewise linear part of $L$ on $[0,1]$ are precisely the facets of the lower part of the boundary of $C$.
Let $p \in \mathbb{R}[Y_1, \ldots, Y_k]$ be a polynomial and let $M$ be the sum of the absolute values of its coefficients multiplied by $100 \deg(p)$. Consider the real auxiliary polynomial $q(Y_1, \ldots, Y_k, Z_1, \ldots, Z_k)$ defined as

$$q := p \prod_{i=1}^{k} (1 - Y_i)^6 + M \left(\sum_{i=1}^{k} Z_i - g(Y_i)\right).$$

Then, by ([HN11, Lemma 5.4]) and the observation $(1,1) \in R$ the following are equivalent:

(i) $q(x_1,\ldots,x_k,y_1,\ldots,y_k) < 0$ for some $x_1,\ldots,x_k,y_1,\ldots,y_k$ with $(x_i,y_i) \in R$ for all $1 \le i \le k$;

(ii) $p(x_1,\ldots,x_k) < 0$ for some $x_1,\ldots,x_k \in \{1, 1-\frac{1}{n} : n \in \mathbb{N}\}$.

Now, we consider the map

$$\begin{aligned} \tau : \mathbb{R}[Y_1,\ldots,Y_k,Z_1,\ldots,Z_k] &\longrightarrow \mathbb{R}[X^k]^{\mathcal{S}^k} \\ f(Y_1,\ldots,Y_k,Z_1,\ldots,Z_k) &\longmapsto \prod_{i=1}^k \mathfrak{e}_{1,(i)}^{3\deg f} \cdot f\left(\frac{\mathfrak{e}_{2,(1)}}{\mathfrak{e}_{1,(1)}^2},\ldots,\frac{\mathfrak{e}_{2,(k)}}{\mathfrak{e}_{1,(k)}^2},\frac{\mathfrak{e}_{3,(1)}}{\mathfrak{e}_{1,(1)}^3},\ldots,\frac{\mathfrak{e}_{3,(k)}}{\mathfrak{e}_{1,(k)}^3}\right). \end{aligned}$$

For $f \in \mathbb{R}[Y_1,\ldots,Y_k,Z_1,\ldots,Z_k]$ the rational function $\tau(f)$ is actually an even $k$-multisymmetric limit form. This is, since $\mathfrak{e}_{2,(i)}$ and $\mathfrak{e}_{1,(i)}^2$ (resp. $\mathfrak{e}_{3,(i)}$ and $\mathfrak{e}_{1,(i)}^3$) have degree 2 (resp. 3) and thus every monomial in the rational multisymmetric function $f\left(\frac{\mathfrak{e}_{2,(1)}}{\mathfrak{e}_{1,(1)}^2},\ldots,\frac{\mathfrak{e}_{2,(k)}}{\mathfrak{e}_{1,(k)}^2},\frac{\mathfrak{e}_{3,(1)}}{\mathfrak{e}_{1,(1)}^3},\ldots,\frac{\mathfrak{e}_{3,(k)}}{\mathfrak{e}_{1,(k)}^3}\right)$ has degree 0. Note that multiplying by $\mathfrak{e}_{1,(i)}^{3\deg f}$ ensures that $\tau(f)$ has always nonnegative exponent in $\mathfrak{e}_{1,(i)}$.

We claim that the following assertions are equivalent

(a) $q(x_1,\ldots,x_k,y_1,\ldots,y_k) < 0$ for some $x_1,\ldots,x_k,y_1,\ldots,y_k$ with $(x_i,y_i) \in R$ for all $1 \le i \le k$;

(b) $\tau(q)$ attains a negative value on $\Delta^k$.

We suppose (a). Hatami and Norin show in the proof of ([HN11, Lemma 5.4]) that
$q(x_1,\ldots,x_k,y_1,\ldots,y_k) < 0$ for $x_1,\ldots,x_k,y_1,\ldots,y_k$ with $(x_i,y_i) \in R$ for all $1 \le i \le k$ then the $x_i$'s can be chosen as $x_1,\ldots,x_k \in \{1, 1-\frac{1}{n} : n \in \mathbb{N}\}$, and $y_i = L(x_i)$. Thus, $\tau(q)$ is negative on $\Delta^k$ by Corollary II.4.3. More precisely, $\mathfrak{e}_{1,(i)} = 1, \mathfrak{e}_{2,(i)} = x_i$ and $\mathfrak{e}_{3,(i)} = y_i$ is feasible and thus $\tau(q)$ is not nonnegative. Suppose $q(x_1,\ldots,x_k,y_1,\ldots,y_k) \ge 0$ for any $x_i, y_i$ with $(x_i,y_i) \in R$ for all $1 \le i \le k$, then $\tau(q)$ is nonnegative on $\Delta^k$, since $\mathcal{E}_3^k \subset R^k$.

So the assertions (ii) and (b) are equivalent. This proves the Theorem since we have embedded an undecidable problem into the problem which we claimed to be undecidable. ∎

*Remark* II.6.2. We deduce from Theorem II.6.1 that there cannot exist a unified algorithm or effective certificate to determine the validity of polynomial inequalities of multisymmetric functions on copies of the probability simplex. Note that for a finite number of variables it follows by Artin's solution to Hilbert's 17th problem [Art27] that validity of polynomial inequalities on semialgebraic sets is decidable.

## II.7 Tropicalization

Another approach to study the limit cones of nonnegative and sums of squares forms uses tropicalization. This approach is developed independently from our previous studies and provides quantitative information on how $\mathfrak{P}_{2d}^{\mathcal{B}_n}$ and $\mathfrak{S}_{2d}^{\mathcal{B}_n}$ differ. Tropicalization is often used to study real or complex algebraic varieties. In [Ble+22b; BR22] the authors introduced and studied the tropicalization of graph profiles and densities to provide applications of tropicalization in extremal combinatorics. Moreover, tropicalization was recently applied in real algebra to study the sets of nonnegative and sums of squares polynomials, and their duals. The work in [Ble+22a] concerns the study of truncated moments and pseudomoments on semialgebraic sets and provides new insights into limitations of sums of squares approximations. The first and second author apply tropicalization to study the nonnegativity versus sums of squares question for normalized limits of even symmetric forms [AB22].

Let

$$\begin{array}{cccc} \log_a : & \mathbb{R}^s_{>0} & \longrightarrow & \mathbb{R}^s \\ & (x_1, \ldots, x_s) & \longmapsto & (\log_a(x_1), \ldots, \log_a(x_s)) \end{array}$$

denote the *logarithm map* for a positive $a > 0$. Further denote $\log := \log_e$ the logarithm map with respect to $e$ and $\operatorname{trop} : \mathbb{R}^s_{>0} \to \mathbb{R}^s$ denotes the *tropicalization* which is defined as $\lim_{t\to\infty} \log_t$. For a set $\mathcal{S} \subset \mathbb{R}^s_{\geq 0}$ we write $\log(\mathcal{S}) := \log(\mathcal{S} \cap \mathbb{R}^s_{>0})$ and $\operatorname{trop}(S) := \operatorname{trop}(S \cap \mathbb{R}^s_{>0})$.

By ([Ale13, Proposition 2.2]) $\operatorname{trop}(\mathcal{S})$ is a closed cone for any set $\mathcal{S} \subset \mathbb{R}^s_{\geq 0}$. Large parts of this section are technical. We list the main results for which no further notation is needed.

**Theorem II.7.1.** *The minimal degree $2d$ for which $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*}) \subsetneq \operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ is $2d = 10$.*

We present in Lemma II.7.3 how $\operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ can be computed. Actually, the lemma provides only an inclusion but we will see that our spectrahedra $\mathfrak{S}_{2d}^{\mathcal{B},*}$ satisfy additional structure such that the lemma can be applied. The tropicalization of $\mathfrak{P}_{2d}^{\mathcal{B},*}$ is more challenging. We take a detour and start with tropicalizing $\mathcal{N}_d$. The polyhedral cone $\operatorname{trop}(\mathcal{N}_d)$ turns out to have a simple description, although it is challenging to understand the set $\mathcal{N}_d$.

**Theorem II.7.2.** *The tropicalization of the image of the even Vandermonde map has the following characterization:*

$$\operatorname{trop}(\mathcal{N}_d) = \left\{ (y_1, \ldots, y_d) \in \mathbb{R}^d : \begin{cases} y_k + y_{k+2} \geq 2y_{k+1}\,, & k = 1, \ldots, d-2 \\ dy_{d-1} \geq (d-1)y_d\,. & \end{cases} \right\}$$

We combine Theorem II.7.2 with the technical Proposition II.7.13 to obtain a description of $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*})$ in Proposition II.7.23.

The guideline to this section is as follows. In Subsection II.7.1 we present how one can tropicalize the dual to the sums of squares. In II.7.2 we investigate properties of max-closed sets. II.7.3 deals about tropicalization of sets having Hadamard property. In Subsection II.7.4 we use the knowledge about max-closed sets to prove Theorem II.7.2 and present the tropicalization of the dual to the nonnegative functions. Finally, we present explicit examples in Subsection II.7.5 and how those can be applied to study nonnegativity versus sums of squares for even symmetric homogeneous functions.

### II.7.1 Tropicalization of $\mathfrak{S}_{2d}^{\mathcal{B},*}$

Tropicalizations of spectrahedra have been investigated in ([AGS20, Theorem 5.17]) where some assumptions on the tropicalization are made. In principal, we may use their results to understand $\operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$. However, our spectahedra have a specific structure which makes their analysis simpler. Lemmas 4.1 and 4.3 in [Ble+22b] describe the tropicalization of spectrahedra $\{x \in \mathbb{R}^s_{\geq 0} : A(x) \succeq 0\}$ defined by a symmetric matrix $A(X)$ whose entries are monomials in $X = (X_1, \ldots, X_s)$. In general, the spectrahedra $\mathfrak{S}_{2d}^{\mathcal{B},*}$ may not be of this form.

We work with products of even power sums as a vector space basis of the even symmetric limit forms. The dual cone to the sums of squares is contained in the nonnegative orthant, i.e., $\mathfrak{S}_{2d}^{\mathcal{B},*} \subset \mathbb{R}^{\pi(d)}_{\geq 0}$, since any basis element is a sum of squares.

Although the technical Lemma II.7.3 provides only one inclusion for the tropicalization of spectrahedra, it provides the correct answer for the spectrahedra $\mathfrak{S}_{2d}^{\mathcal{B},*}$ which we compute in Subsection II.7.5 for $2d \leq 10$.
We use the following notation: $\max\{a \in \emptyset\} := -\infty$ and if $L = (L_{ij}(X))_{ij}$ is a symmetric matrix, whose entries $L_{ij}(X) = L_{ji}(X) = \sum_{k=1}^{s} a_{ijk} X_k$ are real linear forms on $\mathbb{R}^s$, we write $\ell_{ij}^{+} := \{k : a_{ijk} > 0\}, \ell_{ij}^{-} := \{k : a_{ijk} < 0\}$ and $\ell_{ij} := \{k : a_{ijk} \neq 0\}$.

**Lemma II.7.3.** *Let $L = (L_{ij}(X))_{i,j} = (\sum_{k=1}^{s} a_{ijk} X_k)_{i,j}$ be a symmetric $N \times N$ matrix whose entries are real linear forms on $\mathbb{R}^s$. Let $K := \{x \in \mathbb{R}^s_{\geq 0} : L(x) \succeq 0\}$ and let $T \subset \mathbb{R}^s$ be the set of all points $x \in \mathbb{R}^s$ which satisfy the following two conditions*

(1) $\max\{x_k : k \in \ell_{ii}^{+}\} \geq \max\{x_k : k \in \ell_{ii}^{-}\}$, *for any* $1 \leq i \leq N$.

(2) $\max\{x_k : k \in \ell_{ii}^{+}\} + \max\{x_k : k \in \ell_{jj}^{+}\} \geq 2 \max\{x_k : k \in \ell_{ij}\}$, *for any* $1 \leq i < j \leq N$.

*If for all $v \in \operatorname{int}(T)$ the inequalities in* (1) *and* (2) *are strict then* $\operatorname{cl}(\operatorname{int}(T)) \subset \operatorname{trop}(K) \subset T$.

*Proof.* First, we show $\operatorname{trop}(K) \subset T$. The conditions (1) and (2) follow for every element $v \in \operatorname{trop}(K)$ from the positive semidefiniteness of the $1 \times 1$ and $2 \times 2$ principal minors of the defining matrix $L$.
Next, we prove $\operatorname{cl}(\operatorname{int}(T)) \subset \operatorname{trop}(K)$. It suffices to show that any $v \in \operatorname{int}(T)$ is contained in $\operatorname{trop}(K)$, since $\operatorname{trop}(K)$ is a closed set by ([Ale13, Proposition 2.2]).

Let $v \in \mathrm{int}(T)$. We show that $t^v := (t^{v_1}, \ldots, t^{v_s}) \in K$ for all $t > 0$ sufficiently large. By assumption, all the inequalities in (1) and (2) are strict at $v$. We consider the diagonal entries of $L$ at $t^v$. Let $q \in [n]$ be an index at which the maximum over all $x_j$ with $j \in \ell_{ii}^+$ is attained. Then $v_q > 0$ and $v_k - v_q < 0$ for all $k$ with $a_{iik} < 0$. Thus, for all $k \in \ell_{ii}^-$ we have $t^{v_k - v_q} \to 0$ if $t \to \infty$. Hence, for sufficiently large $t$

$$\frac{L_{ii}(t^v)}{t^{v_q}} = \sum_{k \in \ell_{ii}} a_{iik} t^{v_k - v_q} = \sum_{k \in \ell_{ii}^+} a_{iik} t^{v_k - v_q} \geq a_{iiq} > 0.$$

Now, let $k \geq 2$ and consider the Leibniz formula of the leading $k \times k$-principal minor of $L(t^v)$ given by

$$\sum_{\sigma \in \mathcal{S}_k} \mathrm{sgn}(\sigma) \prod_{i=1}^{k} L_{i\sigma(i)}(t^v).$$

We treat the product of linear forms in $t^v$ as univariate exponential polynomials in $t$ and claim

$$\deg_t \left( \prod_{i=1}^{k} L_{ii}(t^v) \right) > \deg_t \left( \prod_{i=1}^{k} L_{i\sigma(i)}(t^v) \right) \tag{II.12}$$

for any $\sigma \in \mathcal{S}_k \setminus \{\mathrm{id}\}$. Equivalently to (II.12), since $L$ is symmetric

$$\deg_t \left( \prod_{i=1}^{k} L_{ii}^2(t^v) \right) > \deg_t \left( \prod_{i=1}^{k} L_{i\sigma(i)}(t^v) L_{\sigma(i)i}(t^v) \right).$$

The leading coefficient of the univariate exponential polynomial $\prod_{i=1}^{k} L_{ii}^2(t^v)$ equals a product of positive coefficients of each $L_{ii}$ by assumption (1). Combining (1) and (2) we obtain at $t^v$

$$\begin{aligned} \deg_t \left( L_{ii} L_{\sigma(i)\sigma(i)} \right) &= \deg_t \left( L_{ii} \right) + \deg_t \left( L_{\sigma(i)\sigma(i)} \right) \\ &> 2 \deg_t \left( L_{i\sigma(i)} \right) = \deg_t \left( L_{i\sigma(i)} L_{\sigma(i)i} \right). \end{aligned}$$

Therefore, $\sum_{\sigma \in \mathcal{S}_k} \mathrm{sgn}(\sigma) \prod_{i=1}^{k} L_{i\sigma(i)}(t^v) > 0$ for all sufficiently large $t$. Analogously, for any $k \times k$ principal minor of $L(t^v)$ the product of the diagonal entries has degree larger than the product of any other generalized diagonal obtained from a permutation of the indices. Thus, for all $t$ sufficiently large we have $t^v \in K$. ∎

*Remark* II.7.4. In general, the set $T$ in Lemma II.7.3 is not necessarily convex, since the inequalities over the max do not need to split into a finite sum of linear inequalities. However, $T$ is a polyhedral fan, i.e., a polyhedral complex in which every polyhedron is a cone from the origin [Ale13]. We showed that all interior points of the fan are guaranteed to lie in the tropicalization.

**Proposition II.7.5.**

$$\operatorname{trop}(\mathfrak{S}_6^{\mathcal{B},*}) = \left\{(y_{(2^3)}, y_{(4,2)}, y_{(6)}) \in \mathbb{R}^3 : y_{(2^3)} + y_{(6)} \geq 2y_{(4,2)}, y_{(2^3)} \geq y_{(4,2)} \geq y_{(6)}\right\}.$$

*Proof.* By Example II.2.7 the set $\mathfrak{S}_6^{\mathcal{B}}$ has the description:

$$\begin{pmatrix} \mathfrak{p}_{(2^3)} & \mathfrak{p}_{(4,2)} \\ \mathfrak{p}_{(4,2)} & \mathfrak{p}_{(6)} \end{pmatrix}, \left(\frac{1}{6}\mathfrak{p}_{(2^3)} - \frac{1}{2}\mathfrak{p}_{(4,2)} + \frac{1}{3}\mathfrak{p}_{(6)}\right), \left(\mathfrak{p}_{(4,2)} - \mathfrak{p}_{(6)}\right). \tag{II.13}$$

We observe that the set

$$S := \left\{ (y_{(2^3)}, y_{(4,2)}, y_{(6)}) \in \mathbb{R}^3 \; : \; \begin{array}{c} y_{(2^3)} + y_{(6)} \geq 2y_{(4,2)}, \\ \max\{y_{(2^3)}, y_{(6)}\} \geq y_{(4,2)}, \\ y_{(4,2)} \geq y_{(6)} \end{array} \right\}$$

equals the defined auxiliary set $T$ in Lemma II.7.3 and note $\max\{y_{(2^3)}, y_{(6)}\} = y_{(6)}$ implies $y_{(2^3)} = y_{(4,2)} = y_{(6)} \in S$. Thus, the max-inequality can be replaced by a linear inequality. The set $S$ is a full dimensional closed convex polyhedral cone. Therefore we have $S = \operatorname{cl}(\operatorname{int} S) \subset \operatorname{trop}(\mathfrak{S}_6^{\mathcal{B},*}) \subset S$ by Lemma II.7.3 which shows that we can apply Lemma II.7.3 to determine $\operatorname{trop}(\mathfrak{S}_6^{\mathcal{B},*})$ and $\operatorname{trop}(\mathfrak{S}_6^{\mathcal{B},*})$ has indeed the claimed form. ■

The following example shows that the condition of all inequalities in (1) and (2) being strict for all points in the interior of the auxiliary set $T$ cannot be omitted.

**Example II.7.6.** Note that by an orthogonal change of basis with $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ the $2 \times 2$ matrix in (II.13) equals

$$\begin{pmatrix} \mathfrak{p}_{(2^3)} & \mathfrak{p}_{(2^3)} - \mathfrak{p}_{(4,2)} \\ \mathfrak{p}_{(2^3)} - \mathfrak{p}_{(4,2)} & \mathfrak{p}_{(2^3)} + \mathfrak{p}_{(6)} - 2\mathfrak{p}_{(4,2)} \end{pmatrix}.$$

The new $2 \times 2$ matrix and the two $1 \times 1$ matrices provide again a description of $\mathfrak{S}_6^{\mathcal{B}}$. The auxiliary set in Lemma II.7.3 applied to the new matrices becomes $T = \{(y_{(2^3)}, y_{(4,2)}, y_{(6)} \in \mathbb{R}^3 : y_{(2^3)} \geq y_{(4,2)} \geq y_{(6)}\}$. However, condition (2) in II.7.3 on the new $2 \times 2$ matrix requires the inequality $2\mathfrak{p}_{(2^3)} \geq 2\mathfrak{p}_{(2^3)}$ to be strict for all points in the interior of $T$ which can certainly not be true. Moreover, we saw in Proposition II.7.5 $\operatorname{trop}\left(\mathfrak{S}_6^{\mathcal{B},*}\right) \subsetneq T$.

## II.7.2 Properties of max-closed sets

We prove some general properties of max-closed sets and in particular examine the structure of extremal rays of convex cones which are max-closed. These results will be used to present the defining linear inequalities of the convex cone $\operatorname{trop}(\mathcal{N}_d)$ in Subsection II.7.4.

Let $\oplus$ denote *tropical addition* with respect to taking the coordinate wise maximum, i.e., for $x, y \in \mathbb{R}^n$ we have

$$x \oplus y := (\max(x_1, y_1), \ldots, \max(x_n, y_n)).$$

**Definition II.7.7.** Let $M \subset \mathbb{R}^n$ be a set. Then $M$ is called *max-closed* if $x \oplus y \in M$ for all $x, y \in M$. The *max-closure* $\overline{M}$ of $M$ is the smallest max-closed set containing $M$. For $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$ we write $a \odot x = (a + x_1, \dots, a + x_n)$ to denote tropical scalar multiplication.

We observe that the max-closure of a set $M \subset \mathbb{R}^n$ is the intersection of all max-closed subsets of $\mathbb{R}^n$ containing $M$. Therefore, the max-closure is well defined since the intersection is non-empty as $\mathbb{R}^n$ is a max-closed set which contain $M$.

In the following lemma we provide a description of the max-closure of sets $M \subset \mathbb{R}^n$ which contain a vector $v \in \mathbb{R}^n_{>0}$ in their *linearity space*, i.e., $x + \lambda v \in M$ for all $\lambda \in \mathbb{R}$ and all $x \in M$. We denote by $\mathcal{Q}_i \subset \mathbb{R}^n$ the orthant where the $i$-th coordinate is non-positive and any other nonnegative, i.e.,

$$\mathcal{Q}_i := \{x \in \mathbb{R}^n \,:\, x_i \leq 0, x_j \geq 0 \;\; \forall j \in [n] \setminus \{i\}\}.$$

**Lemma II.7.8.** *Let $M \subset \mathbb{R}^n$ and let $v \in \mathbb{R}^n_{>0}$ be contained in the linearity space of $M$. Then, the max-closure of $M$ equals $\bigcap_{i=1}^n M + \mathcal{Q}_i$.*

*Proof.* We write $\widehat{M} := \bigcap_{i=1}^n M + \mathcal{Q}_i$.
First, we prove that any $x \in \widehat{M}$ is the tropical sum of elements in $M$. Let $x \in \widehat{M}$. We have to show the existence of $y_1, \dots, y_m \in M$ with $x = y_1 \oplus y_2 \oplus \dots \oplus y_m$. Since $x \in \widehat{M}$ there exist $y_i \in M$ and $g_i = (g_{i1}, \dots, g_{in}) \in \mathbb{Q}_i$ such that $x = y_i + g_i$ for all $i$. Since $g_{ii} \leq 0, g_{ij} \geq 0$ for $j \neq i$ we have $y_1 \oplus \dots \oplus y_n = (y_{11}, \dots, y_{nn})$ and $x_i \leq y_{ii}$. Assume that for some $i$ actually $y_{ii} > x_i$. Then let $\lambda \in \mathbb{R}_{>0}$ such that $x_i = y_{ii} - \lambda v_i$. By assumption $\widetilde{y}_i := y_i - \lambda v \in M$ and $x = \widetilde{y}_i + g_i + \lambda v \in M + \mathbb{Q}_i$ since $g_{ii} + \lambda v_i = 0$ and any other coordinate is positive. Thus, there exist $y_1, \dots, y_n \in M$ with $x = y_1 \oplus \dots \oplus y_n$.
Second, we prove that any tropical sum of elements in $M$ is contained in $\widehat{M}$. Let $y_1, \dots, y_k \in M$ and $x := y_1 \oplus \dots \oplus y_k$. For any $j \in [n]$ let $i_j \in [k]$ be such that $x_j = y_{i_j j}$. Since $x_k \geq y_{i_j k}$ for any $k \in [n]$ it is $x \in y_{i_j} + \mathbb{Q}_j$. In particular, $x \in \bigcap_i M + \mathbb{Q}_i$.
Finally, we show that $\widehat{M}$ is max-closed. For $a, b \in \widehat{M}$ there exist finite sequences of elements in $M$ such that $a$ and $b$ are their tropical sums. However, $a \oplus b$ is the tropical sum of all those elements and we have already seen that $\widehat{M}$ is closed under tropical summation with elements in $M$.
Thus, $\widehat{M}$ is max-closed and the elements in $\widehat{M}$ are precisely the tropical sums of elements in $M$ which proves the claim. ∎

**Corollary II.7.9.** *Let $S \subset \mathbb{R}^n$ be a convex cone containing $v \in \mathbb{R}^n_{>0}$ in its linearity space. Then, the max closure of $S$ is a convex cone.*

*Proof.* The max-closure of $S$ is the intersection of convex cones by Lemma II.7.8. Thus the max-closure is again a convex cone. ∎

We denote the all one vector $(1, \dots, 1)$ by $\mathbf{1}$.

**Definition II.7.10.** Let $M \subset \mathbb{R}^n$ be a set. $M$ is called *tropical convex* if the set $M$ contains all tropical convex combinations, i.e., all points of the form $a \odot x \oplus b \odot y$ for all $x, y \in M$ and all $a, b \in \mathbb{R}$ with $a \oplus b = 0$. We define the *tropical convex hull* $\operatorname{tconv}(M)$ of $M$ as the smallest set containing all tropical convex combinations of $M + \mathbb{R} \cdot \mathbf{1}$, i.e.,

$$\operatorname{tconv}(M) = \{a_1 \odot x_1 \oplus \ldots \oplus a_l \odot x_l : l \in \mathbb{N}, a_1, \ldots, a_l \in \mathbb{R}, x_1, \ldots, x_m \in M\}.$$

We refer to ([DS03, Proposition 4]) for a proof of the set theoretical description of $\operatorname{tconv}(M)$ in Definition II.7.10. Note that $M + \mathbb{R} \cdot \mathbf{1}$ contains $\mathbf{1}$ in its linearity space. Thus, the set $\operatorname{tconv}(M)$ is the max-closure of $M + \mathbb{R} \cdot \mathbf{1}$. We observe

$$\operatorname{tconv}(M) = \bigcap_{j=1}^{n} (M + \mathbb{R} \cdot \mathbf{1} + \mathbb{Q}_j) \tag{II.14}$$

which follows from Lemma II.7.8.

**Corollary II.7.11.** *Let $M \subset \mathbb{R}^n$ contain an element in $\mathbb{R}^n_{>0}$ in its linearity space. Then, the linear inequalities characterizing the max-closure of $M$ are the linear inequalities on $M$ with exactly one non-positive coordinate.*

*Proof.* By Lemma II.7.8 we have

$$\overline{M}^* = \left( \bigcap_{i=1}^{n} (M + \mathbb{Q}_i) \right)^* = \bigoplus_{i=1}^{n} M^* \cap \mathbb{Q}_i^*.$$

The claim follows now from $\mathbb{Q}_i^* = \mathbb{Q}_i$. ∎

We apply Corollary II.7.11 to show that all extremal rays of $\operatorname{trop}(\mathcal{N}_d)^*$ have exactly one nonnegative coefficient.

*Remark* II.7.12. Analogous results for the min-closure and the tropical convex hull can be obtained by using min convention instead of defining tropical addition as coordinate wise maximum.

### II.7.3 Properties of sets having Hadamard property

We prove technical properties about tropicalizations of sets which have Hadamard property. Although the results presented here are theoretical, we apply them in Subsection II.7.4 to compute $\operatorname{trop}(\mathfrak{P}_{2d}^{B,*})$.
For a given set $S \subset \mathbb{R}^n_{\geq 0}$ and a polynomial map $f$ computing $\operatorname{trop}(\operatorname{cone}(f(S)))$ may be way more difficult than computing $\operatorname{trop}(f(S))$. Subsection II.7.4 focuses on the non semialgebraic set $S = \mathcal{N}_d$. The following useful proposition is known for sets $S$ which are semialgebraic ([AGS19, Lemma 8]). We prove that it remains valid when $S$ has Hadamard property, although it may fail in general.

**Proposition II.7.13.** *Let $S \subset \mathbb{R}^n_{\geq 0}$ and assume the set $S$ has Hadamard property. Then,*

$$\operatorname{trop}(\operatorname{cone}(S)) = \operatorname{tconv}(\operatorname{trop}(S)).$$

Even though we could immediately prove the reverse inclusion, we present some lemmas which will allow us to prove the primal inclusion.

**Lemma II.7.14** ([BR22], Lemma 2.2.)**.** *Let $S \subset \mathbb{R}^n_{\geq 0}$ be a convex cone and assume that $S$ has Hadamard property. Then,* $\operatorname{trop}(\overline{S})$ *is a max-closed convex cone.*

The lemma is actually proven in a more general context for semirings closed under coordinate wise addition and Hadamard multiplication.

The following lemma from [AB22] will be useful to prove one of the inclusions, since certain linear inequalities valid on $\operatorname{trop}\left(\mathfrak{P}^{\mathcal{B},*}_{2d}\right)$ transfer to binomial inequalities in power sums. We provide a proof for completeness.

**Lemma II.7.15** ([AB22])**.** *Let $I \subsetneq [n]$ and $m \in [n] \setminus I$. Then, the set of all points $x \in \mathbb{R}^n_{\geq 0}$ satisfying the binomial inequality*

$$\prod_{i \in I} x_i^{\alpha_i} \geq x_m^d,$$

*where $\alpha_i, d \in \mathbb{N}_{>0}$ and $d = \sum_{i \in I} \alpha_i$ is a convex cone.*

*Proof.* It suffices to prove that the inequality holds for conical combinations $\alpha x + \beta x'$ of points $x$ and $x'$ which satisfy it. By considering more than $n$ variables if necessary we prove the following, where $J$ is a set of indices of size $d$. The claim follows then from replacing $J$ by $I$ and allowing natural numbers as exponents. We claim

$$\prod_{j \in J} (\alpha x_j + \beta x'_j)^{\frac{1}{d}} \geq \alpha \left( \prod_{j \in J} x_j \right)^{\frac{1}{d}} + \beta \left( \prod_{j \in J} x'_j \right)^{\frac{1}{d}} \geq \alpha x_m + \beta x'_m.$$

The second inequality follows by hypothesis, and the first inequality is known as Mahler's inequality which is a direct consequence of the following inequality: If $a_i, b_i \in \mathbb{R}_{\geq 0}$ then

$$\prod_{i=1}^{d} (a_i + b_i)^{\frac{1}{d}} \geq \left( \prod_{i=1}^{d} a_i \right)^{\frac{1}{d}} + \left( \prod_{i=1}^{d} b_i \right)^{\frac{1}{d}}.$$

Mahler's inequality is trivial if one of the $a_i, b_i$'s is 0. Otherwise, if all $a_i$ and $b_i$ are positive it follows after adding the AM-GM inequalities below:

$$\frac{1}{d} \sum_{i=1}^{d} \frac{a_i}{a_i + b_i} \geq \frac{\left(\prod a_i\right)^{\frac{1}{d}}}{\prod (a_i + b_i)^{\frac{1}{d}}},$$

$$\frac{1}{d} \sum_{i=1}^{d} \frac{b_i}{a_i + b_i} \geq \frac{\left(\prod b_i\right)^{\frac{1}{d}}}{\prod (a_i + b_i)^{\frac{1}{d}}}.$$

■

*Remark* II.7.16. Not all binomial inequalities necessarily transfer to a convex cone or even to Minkowski sums. For example, $x^\alpha \geq x^\beta$ does not transfer if $\alpha = (1,1,1)$ and $\beta = (2,1,0)$. For instance, we observe $x = (1,0,0)$ and $y = (0,1,0)$ satisfy the inequality but their sum $x + y = (1,1,0)$ does not.

We obtain a generalization of Lemma II.7.15 to real exponents.

**Corollary II.7.17.** *Let $I \subsetneq [n]$ and $m \in [n] \setminus I$. Then, the set of all points $x \in \mathbb{R}^n_{\geq 0}$ satisfying the inequality*

$$\prod_{i \in I} x_i^{\alpha_i} \geq x_m^d,$$

*where $\alpha_i, d \in \mathbb{R}_{>0}$ and $d = \sum_{i \in I} \alpha_i$ is a convex cone.*

*Proof.* First we observe that Lemma II.7.15 generalizes to homogeneous binomial inequalities, where $d, \alpha_i \in \mathbb{Q}_{>0}$ and $d = \sum_{i \in I} \alpha_i$. This can be seen from taking the exponential on both sides of the inequality with respect to the least common denominator of all reduced fractions $d, \alpha_i$.
Now, we notice that a real homogeneous binomial inequality can be approximated by a sequence of rational binomial inequalities. Note that the intersection of convex cones is again a convex cone. Thus, II.7.15 remains also valid when $\alpha_i \in \mathbb{R}_{>0}$. ∎

We are prepared to present a proof of the main result Proposition II.7.13.

*Proof of Proposition II.7.13.* We suppose $z \in \operatorname{tconv}(\operatorname{trop}(S))$, i.e., we have $z = a_1 \odot v_1 \oplus \ldots \oplus a_k \odot v_k$ for some $a_i \in \mathbb{R}$ and $v_i \in \operatorname{trop}(S)$. First, we show that $a_i \odot v_i \in \operatorname{trop}(\operatorname{cone}(S))$ for all $1 \leq i \leq k$. Since $v_i \in \operatorname{trop}(S)$ we have

$$v_i = \lim_{m \to \infty} \log_{\frac{1}{\tau_m}} (w_i^{(m)})$$

for a sequence $(w_i^{(m)})_m \subset S \cap \mathbb{R}^s_{>0}$ and a sequence $(\tau_m)_m \subset (0, \epsilon)$ converging to 0 ([Ale13, Proposition 2.1]). For all $m$ we have $\frac{1}{\tau_m^{a_i}} w_i^{(m)} \in \operatorname{cone}(S)$ and

$$\begin{aligned} a_i \odot v_i &= a_i \odot \lim_{m \to \infty} \log_{\frac{1}{\tau_m}} (w_i^{(m)}) \\ &= \lim_{m \to \infty} \log_{\frac{1}{\tau_m}} \left( \frac{1}{\tau_m^{a_i}} \right) \odot \log_{\frac{1}{\tau_m}} (w_i^{(m)}) \\ &= \lim_{m \to \infty} \log_{\frac{1}{\tau_m}} \left( \frac{1}{\tau_m^{a_i}} w_i^{(m)} \right) \end{aligned}$$

which shows $a_i \odot v_i \in \operatorname{trop}(\operatorname{cone}(S))$. Since $\operatorname{cone}(S)$ has Hadamard property, we observe that the set $\operatorname{trop}(\operatorname{cone}(S))$ is max-closed by Lemma II.7.14. Thus, $z \in \operatorname{trop}(\operatorname{cone}(S)))$ because $z$ is a tropical sum of the $a_i \odot v_i$'s.
To prove the remaining inclusion we first note that $\operatorname{trop}(\operatorname{cone}(S))$ and $\operatorname{tconv}(\operatorname{trop}(S))$ are closed convex cones in $\mathbb{R}^n$ by ([Ble+22b, Lemma 2.2 (2)]). More precisely, this follows since $\operatorname{cone}(S)$ has Hadamard property and because

trop($S$) is a closed convex cone the set tconv(trop($S$)) is the intersection of closed convex cones by (II.14). We show instead the equivalent formulation

$$\operatorname{tconv}(\operatorname{trop}(S))^* \subseteq \operatorname{trop}(\operatorname{cone}(S))^*.$$

Corollary II.7.11 implies

$$\begin{aligned}\operatorname{tconv}(\operatorname{trop}(S))^* &= \left(\bigcap_{k=1}^{n} \operatorname{trop}(S) + \mathbb{R}\cdot\mathbf{1} + Q_k\right)^* \\ &= \bigoplus_{k=1}^{n} \operatorname{trop}(S)^* \cap \mathbb{R}\cdot\mathbf{1}^* \cap Q_k^* \\ &= \bigoplus_{k=1}^{n} \operatorname{trop}(S)^* \cap \{z\in\mathbb{R}^n : z_i \geq 0, i\neq k, z_k \leq 0, \sum_{i\neq k} z_i = -z_k\}.\end{aligned}$$

Therefore, any extremal ray in tconv(trop($S$))$^*$ has precisely one negative coefficient and the sum over all positive coefficients equals the absolute value of the negative coefficient. By ([BR22, Proposition 2.4]) any $\alpha = \alpha_+ - \alpha_- \in \operatorname{trop}(S)^*$ with $\alpha_+, \alpha_- \in \mathbb{R}^n_{\geq 0}$ and $\alpha_- \neq 0$ transfers to a valid binomial inequality $x^{\alpha_+} \geq x^{\alpha_-}$ on $S$. Thus, any extremal ray in tconv(trop($S$))$^*$ gives rise to a valid homogenenous binomial inequality $x^\alpha \geq x^\beta$, and $\beta$ has precisely one non-zero entry and $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$. We saw in Corollary II.7.17 that the set of all solutions in $\mathbb{R}^s_{\geq 0}$ of the real homogeneous binomial inequality forms a convex cone containing $S$. Thus, it is a valid binomial inequality on the set cone($S$). We deduce that any extremal ray in tconv(trop($S$))$^*$ is contained in trop(cone($S$))$^*$. ∎

### II.7.4 Tropicalization of $\mathcal{N}_d$ and $\mathfrak{P}_{2d}^{\mathcal{B},*}$

We prove in Theorem II.7.2 a uniform description of trop($\mathcal{N}_d$) as a polyhedral cone and present a motivation for the defining linear inequalities. The proof uses methods from ([BR22, § 2.1]). Then, we determine trop($\mathfrak{P}_{2d}^{\mathcal{B},*}$) via decomposing the tropicalization using the results in Subsection II.7.3. The decomposition technique was already applied by the first and second author in their study of normalized limits [AB22].

We begin with presenting some properties of $\mathcal{N}_d$ which allow us to apply the results in Subsections II.7.2 and II.7.3.

**Lemma II.7.18.** *The set* trop($\mathcal{N}_d$) *is a max-closed closed convex cone containing the line spanned by* $(1, 2, 3, \ldots, d)$ *and the ray spanned by the all one vector* $\mathbf{1}$*.*

*Proof.* The set $\mathcal{N}_d$ has Hadamard property and is closed under addition by Proposition II.3.3. Since $\mathcal{N}_d \subset \mathbb{R}^d_{\geq 0}$ has Hadamard property, trop($\mathcal{N}_d$) is a closed convex cone by ([Ble+22b, Lemma 2.2 (2)]). Moreover, since $\mathcal{N}_d$ is closed under addition the set trop($\mathcal{N}_d$) is max-closed by ([BR22, Lemma 2.2]). For $\lambda \in \mathbb{R}$ we have

$$2\lambda(1,2,\ldots,d) = \lim_{m\to\infty} \log_{\frac{1}{\tau_m}} \left( \left(\frac{1}{\tau_m^\lambda}\right)^2, \left(\frac{1}{\tau_m^\lambda}\right)^4, \ldots, \left(\frac{1}{\tau_m^\lambda}\right)^{2d} \right)$$

for any sequence $(\tau_m)_m \subset \mathbb{R}_{>0}$ converging to 0, and

$$\mathbf{1} = \log_m(m,\ldots,m) = \log_m((1^2+\ldots+1^2),(1^4+\ldots+1^4),\ldots,(1^{2d}+\ldots+1^{2d}))$$

for all $m \in \mathbb{N}$. Thus, we have inclusions $\mathbb{R}\cdot(1,2,3,\ldots,d), \mathbb{R}_{\geq 0}\cdot\mathbf{1} \subset \operatorname{trop}(\mathcal{N}_d)$. ■

The following corollary will be useful in our proof of the characterization of $\operatorname{trop}(\mathcal{N}_d)$.

**Corollary II.7.19.** *An extreme ray of* $\operatorname{trop}(\mathcal{N}_d)^*$ *is spanned by a vector with at most one negative coordinate.*

*Proof.* This follows directly from Corollary II.7.11. The set $\operatorname{trop}(\mathcal{N}_d)$ is a max-closed convex cone which contains the line $\mathbb{R}\cdot(1,2,\ldots,d)$ by Lemma II.7.18. Since $\operatorname{trop}(\mathcal{N}_d)$ is max-closed it contains $(1,2,\ldots,d)$ in its linearity space. ■

**Corollary II.7.20.** *The set* $\nu_d(\mathcal{N}_d)$ *is a cone and has Hadamard property. Moreover, the set* $\mathfrak{P}_{2d}^{\mathcal{B},*}$ *is a convex cone which has Hadamard property.*

*Proof.* Since any $z \in \mathcal{N}_d$ is the limit of a sequence in $\bigcup_{n=1}^\infty \nu_{d,n}^e(\mathbb{R}^n)$ we can use that the map $\nu_d \circ (p_2^{(n)},\ldots,p_{2d}^{(n)})$ is homogeneous and obtain that $\nu_d(\mathcal{N}_d)$ is a cone. For $x = \nu_d(a), y = \nu_d(b) \in \nu_d(\mathcal{N}_d)$ we have

$$(x_1y_1,\ldots,x_sy_s) = (a_1^d b_1^d, a_1^{d-2}a_2 b_1^{d_2} b_2, \ldots, a_d b_d) \in \nu_d(\mathcal{N}_d)$$

since $\mathcal{N}_d$ has Hadamard property by Proposition II.3.3.
The set $\mathfrak{P}_{2d}^{\mathcal{B},*} = \operatorname{cone}(\nu_d(\mathcal{N}_d))$ has Hadamard property because Hadamard multiplication of convex combinations of elements in $\nu_d(\mathcal{N}_d)$ gives again a convex combination of elements in $\nu_d(\mathcal{N}_d)$. ■

We make one additional definition before presenting the description of $\operatorname{trop}(\mathcal{N}_d)$.

**Definition II.7.21.** Let $S \subset \mathbb{R}^n$ be a cone containing a point $v \in \mathbb{R}^n_{>0}$ in its linearity space. Then we define the *double hull* of $S$ as the max-closure of the convex hull of $S$. We write $\operatorname{dh}(S)$.

Now, we can prove Theorem II.7.2 i.e., the description of the polyhedral cone $\operatorname{trop}(\mathcal{N}_d)$. We show that the following families of power sum binomial inequalities transfer to a linear characterization of $\operatorname{trop}(\mathcal{N}_d)$.

1. $\mathfrak{p}_{2k}^{k+1} \geq \mathfrak{p}_{2k+2}^{k}$ for all positive integers $k$.
2. $\mathfrak{p}_{2k}\cdot\mathfrak{p}_{2k+4} \geq \mathfrak{p}_{2k+2}^2$ for all positive integers $k$.

I.e., we show $\operatorname{trop}(\mathcal{N}_d)$ is characterized by the linearizations of those inequalities. The first family of inequalities comes from the well-known monotonicity of $p$-norms and the second family follows from Muirhead's inequality [Mui02], since

$$p_{2k}^{(n)} \cdot p_{2k+4}^{(n)} - p_{2k+2}^{(n)^2} = \sum_{1 \leq i \neq j \leq n} X_i^{2k} X_j^{2k+4} - X_i^{2k+2} X_j^{2k+2}$$

for any $n$ and $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2k \\ 2k+4 \end{pmatrix} = \begin{pmatrix} 2k+2 \\ 2k+2 \end{pmatrix}$.

The case distinction in the proof goes analogously to ([BR22, Theorem 2.15]) which proves the tropicalization of graph profiles of sets of even cycles.

*Proof of Theorem II.7.2.* Let $\mathcal{Q}$ denote the closed convex cone on the right hand side.
**Claim 1:** $\operatorname{trop}(\mathcal{N}_d) \subset \mathcal{Q}$.
By ([Ble+22b, Lemma 2.2 (2)]) $\operatorname{trop}(\mathcal{N}_d) = \operatorname{cl}(\operatorname{conv}(\log(\mathcal{N}_d)))$ since $\mathcal{N}_d$ has Hadamard property. Thus, by taking logarithm the families of binomial inequalities in power sums from the discussion above transfer to valid linear inequalities on $\log(\mathcal{N}_d)$ which are preserved under taking the convex hull. Moreover, those inequalities are precisely the linear inequalities defining $\mathcal{Q}$.
**Claim 2:** The rays $\mathbf{1} = (1, \ldots, 1), (1, 2, \ldots, d), (-1, -2, \ldots, -d)$ are contained in $\operatorname{trop}(\mathcal{N}_d)$.
This was shown in Lemma II.7.18.
**Claim 3:** We have $\mathcal{Q} \subset \operatorname{dh}(\operatorname{cone}(\mathbf{1}, (1, 2, \ldots, d), (-1, -2, \ldots, -d)))$, and so $\mathcal{Q} \subset \operatorname{trop}(\mathcal{N}_d)$.
Let $\mathcal{S} := \operatorname{cone}(\mathbf{1}, (1, 2, \ldots, d), (-1, -2, \ldots, -d))$ and let $\mathcal{D} := \operatorname{dh}(\mathcal{S})$ denote its double hull. From Corollary II.7.11 we know that $\mathcal{D}^* = \bigoplus_{i=1}^d \mathcal{S}^* \cap Q_i$, where $Q_i$ is the orthant of $\mathbb{R}^d$ in which the $i$-th coordinate is non-positive and any other coordinate is nonnegative. Therefore, the set of extreme rays of $\mathcal{D}^*$ is contained in the union of extreme rays of $\mathcal{S}^* \cap Q_i$ for $i \in [d]$. We point out that the extreme rays of $\mathcal{D}^*$ correspond to linear inequalities valid on $\mathcal{D}$.
For all $i$ the closed convex cone $\mathcal{S}^* \cap Q_i$ is defined by $d+3$-many inequalities, since

$$\mathcal{S}^* = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : a_1 + \ldots + a_d \geq 0, a_1 + 2a_2 + \ldots + da_d = 0\}.$$

We observe that $\mathcal{S}^*$ is not a full dimensional convex cone. At least $d-1$ many inequalities of $\mathcal{S}^* \cap Q_i$ must be tight to form an extreme ray. In particular, $d-2$ of the inequalities in $\mathcal{A}_i := \{a_1 + \ldots + a_d \geq 0, a_j \geq 0, a_i \leq 0, j \neq i\}$ are tight for some $i \in [d]$. In the following we examine the various combinations for which $d-2$ of these inequalities are tight.
We consider a ray $\mathbf{r} := (r_1, \ldots, r_d)$ of $\mathcal{S}^* \cap Q_i$ with $d-2$ many tight inequalities from $\mathcal{A}_i$. Thus, there are at least $d-3$ many tight inequalities of $Q_i$.

(1) Let $r_\rho = 0$ be the tight inequalities for $\rho \in [d] \setminus \{k, l\}$ together with $kr_k + lr_l = 0$. Furthermore, we assume $r_k + r_l > 0$ and $r_k, r_l > 0$. This gives a contradiction.

(2) Let $r_\rho = 0$, $\rho \in [d] \setminus \{k, l\}$ and $kr_k + lr_l = 0$ be the tight inequalities and $r_k + r_l > 0, r_k > 0, r_l < 0$. Without loss of generality be $r_l = -k$ then $r_k = l$ and $0 < r_k + r_l = l - k$ implies that $l > k$. Thus $\mathbf{r} = (0, \ldots, 0, l, 0, \ldots, 0, -k, 0, \ldots, 0)$ where $l$ is the $k$-th and $-k$ the $l$-th coordinate.

(3) Let $r_1 + \ldots + r_d = 0, r_1 + 2r_2 + \ldots + d \cdot r_d = 0, r_\rho = 0, \rho \in [d] \setminus \{k, l, m\}$ be the tight inequalities and $r_k, r_l, r_m > 0$. This gives a contradiction.

(4) Let $r_1 + \ldots + r_d = 0$, $r_1 + \ldots + d \cdot r_d = 0, r_\rho = 0$, for $\rho \in [d] \setminus \{k, l, m\}$ be the tight inequalities and $r_k, r_l > 0, r_m < 0$. Without loss of generality be $0 > r_m = k - l$, i.e., $l > k$. Then $0 \leq r_k = l - m$, $0 < a_l = m - k$ and $l > m > k$. Thus $\mathbf{r} = (0, \ldots, l - m, 0, \ldots, k - l, \ldots, m - k, 0, \ldots, 0)$, where $l - m$ is the $k$-th, $k - l$ the $m$-th and $m - k$ the $l$-th coordinate.

The proof of Claim 3 in ([BR22, Theorem 2.15]) shows that many of the above inequalities are redundant since they are conic, convex combinations of the inequalities defining $\mathcal{Q}$.
Thus $\mathcal{Q} \subset \mathcal{D}$ and we know $\mathcal{D} \subset \operatorname{trop}(\mathcal{N}_d)$ because $\operatorname{trop}(\mathcal{N}_d)$ is a convex cone by Lemma II.7.18 containing the line $\mathbb{R} \cdot (1, 2, \ldots, d)$ and the ray $\mathbb{R}_{\geq 0} \cdot \mathbf{1}$ by Claim 1, and since $\operatorname{trop}(\mathcal{N}_d)$ is a max-closed set by definition. Thus, we obtain $\mathcal{Q} \subset \operatorname{trop}(\mathcal{N}_d)$. ∎

We present an example for $3 \leq d \leq 5$.

**Example II.7.22.**

(i) $\operatorname{trop}(\mathcal{N}_3) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 + y_3 \geq 2y_2, 3y_2 \geq 2y_3\}$;

(ii) $\operatorname{trop}(\mathcal{N}_4) = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : y_1 + y_3 \geq 2y_2, y_2 + y_4 \geq 2y_3, 4y_3 \geq 3y_4\}$;

(iii) $\operatorname{trop}(\mathcal{N}_5) = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5 : y_1 + y_3 \geq 2y_2, y_2 + y_4 \geq 2y_3, y_3 + y5 \geq 2y_4, 5y_4 \geq 4y_5\}$.

Instead of following the upper line in the diagram (II.15) we first tropicalize $\mathcal{N}_d$ which we already understand.

$$\begin{tikzcd} \mathbb{R}^d_{\geq 0} \arrow[r, "\nu_d"] \arrow[d, "\mathrm{trop}"] & \mathbb{R}^{\pi(d)}_{\geq 0} \arrow[r, "\mathrm{conv}"] & \mathbb{R}^{\pi(d)}_{\geq 0} \arrow[d, "\mathrm{trop}"] \\ \mathbb{R}^d \arrow[r, "\tilde{\nu}_d"] & \mathbb{R}^{\pi(d)} \arrow[r, "\mathrm{tconv}"] & \mathbb{R}^{\pi(d)} \end{tikzcd} \tag{II.15}$$

where

$$\begin{array}{rccc} \tilde{\nu}_d : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^{\pi(d)} \\ & (X_1, \ldots, X_d) & \longmapsto & (dX_1, (d-2)X_1 + X_2, \ldots, X_d) \end{array}$$

denotes the tropicalization of the monomial map $\nu_d$ whose coordinates are the expressions of the form $\sum_{i=1}^d \alpha_i X_i$ where $\alpha_i \in \mathbb{N}_0$ and $\sum_{i=1}^d i\alpha_i = d$. Our goal is to show an analogous to the result in [AB22] on normalized limits.

**Proposition II.7.23.** $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*}) = \operatorname{tconv}(\tilde{\nu}_d(\operatorname{trop}(\mathcal{N}_d)))$.

Claim 1 transfers verbally from [AB22] and we present the proof for completeness.

*Proof.* We have $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*}) = \operatorname{trop}(\operatorname{cone}(\nu_d(\mathcal{N}_d)))$. We divide the proof into two parts.
**Claim 1:** $\tilde{\nu}_d(\operatorname{trop}(\mathcal{N}_d)) = \operatorname{trop}(\nu_d(\mathcal{N}_d))$.
The sets $\mathcal{N}_d$ and $\nu_d(\mathcal{N}_d)$ contain the all one vector $\mathbf{1}$ and satisfy the Hadamard property by Proposition II.3.3 and Corollary II.7.20. Thus, by ([Ble+22b, Lemma 2.2 (2)]) Claim 1 is equivalent to

$$\tilde{\nu}_d(\operatorname{cl}(\operatorname{conv}(\log(\mathcal{N}_d)))) = \operatorname{cl}(\operatorname{conv}(\log(\nu_d(\mathcal{N}_d)))).$$

However,

$$\operatorname{cl}(\operatorname{conv}(\log(\nu_d(\mathcal{N}_d)))) = \operatorname{cl}(\operatorname{conv}(\tilde{\nu}_d(\log(\mathcal{N}_d)))) = \operatorname{cl}(\tilde{\nu}_d(\operatorname{conv}(\log(\mathcal{N}_d)))),$$

where the first equality follows from the definition of log and the second equality follows because taking convex hull and applying a linear map commute. Moreover, since $\tilde{\nu}_d$ is injective and linear we have $\tilde{\nu}_d(\operatorname{cl}(A)) = \operatorname{cl}(\tilde{\nu}_d(A))$ for all sets $A \subset \mathbb{R}^{\pi(d)}$. Therefore, Claim 1 follows.
**Claim 2:** $\operatorname{trop}(\operatorname{cone}(S)) = \operatorname{tconv}(\operatorname{trop}(S))$ for $S = \nu_d(\mathcal{N}_d)$.
This follows from Proposition II.7.13 since the set $S$ has Hadamard property by Corollary II.7.20.

Therefore,

$$\operatorname{trop}(\mathcal{P}_{2d}^{\mathcal{B},*}) = \operatorname{trop}(\operatorname{cone}(\nu_d(\mathcal{N}_d))) = \operatorname{tconv}(\operatorname{trop}(\nu_d\mathcal{N}_d)) = \operatorname{tconv}(\tilde{\nu}_d(\operatorname{trop}(\mathcal{N}_d)))$$

which was to prove. ∎

### II.7.5 Applications and examples

We present the tropicalization of $\mathfrak{P}_{2d}^{\mathcal{B},*}$ and $\mathfrak{S}_{2d}^{\mathcal{B},*}$ for degrees $6, 8$ and $10$. Since $\mathfrak{S}_4^{\mathcal{B}} = \mathfrak{P}_4^{\mathcal{B}}$, the tropicalization of the dual cones of quartics must be equal. In Theorem II.7.1 we show that strict inclusion between the tropicalizations of the dual cones occurs for the first time in degree 10, even though we have $\mathfrak{S}_6^{\mathcal{B}} \subsetneq \mathfrak{P}_6^{\mathcal{B}}$ and $\mathfrak{S}_8^{\mathcal{B}} \subsetneq \mathfrak{P}_8^{\mathcal{B}}$ by Corollary II.5.9. We use a linear inequality valid on $\operatorname{trop}(\mathfrak{P}_{10}^{\mathcal{B},*})$ but not on $\operatorname{trop}(\mathfrak{S}_{10}^{\mathcal{B},*})$ to provide an example of a form in $\mathfrak{P}_{10}^{\mathcal{B},*} \setminus \mathfrak{S}_{10}^{\mathcal{B},*}$.

We define systems of linear equations in $3, 5$ and $7$ variables whose coordinates are indexed by the partitions of $6, 8$ and $10$ which contain only even entries. $\mathcal{L}_1 := \left\{ y_{(2^3)} + y_{(6)} \geq 2y_{(4,2)},\ y_{(4,2)} \geq y_{(6)} \right\}$, $\mathcal{L}_2 := \left\{ \begin{matrix} y_{(2^4)} + y_{(4^2)} \geq 2y_{(4,2^2)}, & y_{(4,2^2)} + y_{(8)} \geq 2y_{(6,2)}, \\ y_{(4^2)} \geq y_{(8)}, & y_{(6,2)} \geq y_{(4^2)}. \end{matrix} \right\}$ and $\mathcal{L}_3$ equals

$$\left\{ \begin{matrix} y_{(6,2^2)} \geq y_{(4,2^2)}, & y_{(8,2)} \geq y_{(6,4)}, & y_{(6,4)} \geq y_{(10)}, \\ y_{(6,2^2)} + y_{(10)} \geq 2y_{(8,2)}, & y_{(4^2,2)} + y_{(10)} \geq 2y_{(6,4)}, & y_{(4^2,2)} \geq y_{(8,2)}, \\ y_{(4,2^3)} + y_{(6,4)} \geq 2y_{(4^2,2)}, & y_{(4,2^3)} + y_{(8,2)} \geq 2y_{(6,2^2)}, & y_{(2^5)} + y_{(4^2,2)} \geq 2y_{(4,2^3)} \end{matrix} \right\}.$$

We write $\mathcal{L}_i(y)$ if $y$ satisfies all inequalities in $\mathcal{L}_i$.

**Proposition II.7.24.** *The sets* $\operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ *are convex polyhedral cones for* $2d \in \{6, 8, 10\}$. *Moreover,*

$$\begin{aligned}
\operatorname{trop}(\mathfrak{S}_6^{\mathcal{B},*}) &= \{(y_{(2^3)}, y_{(4,2)}, y_{(6)}) \in \mathbb{R}^3 : \mathcal{L}_1(y)\};\\
\operatorname{trop}(\mathfrak{S}_8^{\mathcal{B},*}) &= \left\{(y_{(2^4)}, y_{(4,2^2)}, y_{(4^2)}, y_{(6,2)}, y_{(8)}) \in \mathbb{R}^5 : \mathcal{L}_2(y)\right\};\\
\operatorname{trop}(\mathfrak{S}_{10}^{\mathcal{B},*}) &= \{(y_{(2^5)}, y_{(4,2^3)}, y_{(4^2,2)}, y_{(6,2^2)}, y_{(6,4)}, y_{(8,2)}, y_{(10)}) \in \mathbb{R}^7 : \mathcal{L}_3(y)\}.
\end{aligned}$$

*Proof.* We already proved the description of $\operatorname{trop}(\mathfrak{S}_6^{\mathcal{B},*})$ in Proposition II.7.5. We omit here the calculations of the sets $\mathfrak{S}_8^{\mathcal{B},*}, \mathfrak{S}_{10}^{\mathcal{B},*}$ which can be calculated using symmetry reduction and higher Specht polynomials (see Remark II.2.3 and ([DR20, Theorem 4.15.]) provides a description for octics expressed in elementary symmetrics). The naive tropicalization from Lemma II.7.3 gives also in degrees 8 and 10 a full dimensional closed polyhedral convex cone which then completes the proof of the Proposition since the auxiliary sets $T$ satisfy $T = \operatorname{cl}(\operatorname{int}(T)) \subset \operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*}) \subset T$. ■

The representations given above are actually H-representations of the polyhedral cones. We present a proof of Theorem II.7.1 saying that the minimal degree for which $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*}) \subsetneq \operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ is $2d = 10$.

*Proof of Theorem II.7.1.* In Example II.7.22 we have calculated $\operatorname{trop}(\mathcal{N}_d)$ for $3 \leq d \leq 5$. We can apply Proposition II.7.23 to obtain $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*})$, i.e., we calculate a vertex representation of the polyhedral cone $\operatorname{trop}(\mathcal{N}_d)$ and apply $\tilde{\nu}_d$. Then we calculate the tropical convex hull and compare the cones $\operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ calculated in II.7.24 with $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*})$. The computations were done using Sage and the code can be found in Appendix A. ■

The convex polyhedral cones in degree 10 differ in the following sense. Every linear inequality in the $H$-representation of $\operatorname{trop}(\mathfrak{S}_{10}^{\mathcal{B},*})$ is also in the $H$-representation of $\operatorname{trop}(\mathfrak{P}_{10}^{\mathcal{B},*})$, but there exists precisely one additional linear inequality

$$y_{(2^5)} + y_{(6,4)} + y_{(8,2)} \geq 3y_{(6,2^2)}$$

in the $H$-representation of $\operatorname{trop}(\mathfrak{P}_{10}^{\mathcal{B},*})$.
We can use this linear inequality to produce an example of a limit nonnegative but not sum of squares even symmetric function of degree 10. This was already done verbatim for an analogous inequality for normalized limits and degree 6 in [AB22].

**Lemma II.7.25** ([AB22])**.** *Let* $a_1, a_2, a_3 \in \mathbb{R}_{>0}$ *such that* $a_1a_2a_3 = 1$. *Then the even symmetric limit form*

$$a_1\mathfrak{p}_{(2^5)} + a_2\mathfrak{p}_{(6,4)} + a_3\mathfrak{p}_{(8,2)} - 3\mathfrak{p}_{(6,2^2)}$$

*is nonnegative, i.e., the form is nonnegative in any number of variables.*

*Proof.* By ([Gan79, p. 203]) we have $\mathfrak{p}_{(2k,2k)} \leq \mathfrak{p}_{(2k+2,2k-2)}$ for any $k \geq 1$. In particular, we have $\mathfrak{p}_6^3 \leq \mathfrak{p}_8 \cdot \mathfrak{p}_6 \cdot \mathfrak{p}_4$. Therefore

$$\mathfrak{p}_{(6,2^2)} = \sqrt[3]{\mathfrak{p}_6^3\mathfrak{p}_2^6} \leq \sqrt[3]{\mathfrak{p}_8\mathfrak{p}_6\mathfrak{p}_4\mathfrak{p}_2^6}$$

and we obtain by the claim from the arithmetic and geometric mean inequality

$$\sqrt[3]{\mathfrak{p}_8\mathfrak{p}_6\mathfrak{p}_4\mathfrak{p}_2^6} = \sqrt[3]{(a_1\mathfrak{p}_{(2^5)})(a_2\mathfrak{p}_{(6,4)})(a_3\mathfrak{p}_{(8,2)})} \leq \frac{a_1\mathfrak{p}_{(2^5)} + a_2\mathfrak{p}_{(6,4)} + a_3\mathfrak{p}_{(8,2)}}{3}.$$

■

**Proposition II.7.26.** *The even symmetric limit form*

$$\frac{1}{18}\mathfrak{p}_{(2^5)} + 3\mathfrak{p}_{(8,2)} + 6\mathfrak{p}_{(6,4)} - 3\mathfrak{p}_{(6,2^2)}$$

*is nonnegative but not a sum of squares.*

*Proof.* The nonnegativity of the limit form follows from Lemma II.7.25. The dual spectrahedron $\mathfrak{S}_{10}^{\mathcal{S},*}$ is the set of all $(a,b,c,d,e,f,g) \in \mathbb{R}^7$ such that the following 7 matrices are positive semidefinite:

$$\begin{pmatrix} a & b & b & c \\ b & d & d & f \\ b & d & c & e \\ c & f & e & g \end{pmatrix}, \begin{pmatrix} b-c & c-e & d-e \\ c-e & e-g & f-g \\ d-e & f-g & f-g \end{pmatrix}, (b-3d-3c+5f+6e-6g), (c-d).$$

$$\begin{pmatrix} a-3b+2c & 3b-3d-6c+6e \\ 3b-3d-6c+6e & 6d+3c-15f-12e+18g \end{pmatrix}, (a-10b+15d+20c-20f-30e+24g),$$
$$(d-2f-e+2g).$$

Note that the linear map $\ell = (450228, 75326, 24986, 12656, 8325, 4159, 2803) : \mathbb{R}^7 \to \mathbb{R}$ is contained in $\mathfrak{S}_{10}^{\mathcal{S},*}$ and satisfies $\ell(\frac{1}{18}p_{(2^5)} + 3p_{(8,2)} + 6p_{(6,4)} - 3p_{(6,2^2)}) = -49/3$. Thus, the limit form cannot be a sum of squares. ■

Note, replacing $a = \mathfrak{p}_{(2^5)}, b = \mathfrak{p}_{(4,2^3)}, c = \mathfrak{p}_{(6,2^2)}, d = \mathfrak{p}_{(4^2,2)}, e = \mathfrak{p}_{(8,2)}, f = \mathfrak{p}_{(6,4)}, g = \mathfrak{p}_{(10)}$ in the proof above gives the matrices that define the set $\mathfrak{S}_{10}^{\mathcal{S}}$.

## II.8 Conclusion and open questions

In this article we studied the sets of nonnegative and sums of squares (even) symmetric functions. Our results provide new insights into the discrepancy of the cones of nonnegative and sums of squares of symmetric polynomials in an increasing number of variables. Although the cones $\Sigma_{n,2d}^{\mathcal{S}_n}$ and $\mathcal{P}_{n,2d}^{\mathcal{S}_n}$ are shrinking and approach limits, those limits are still different. We observed that the limit has even a higher complexity in some sense since the set of nonnegative symmetric functions is no longer semialgebraic and testing nonnegativity of multisymmetric functions is no longer decidable. We observed that working with power sums turns out to be useful to describe the image of the Vandermonde map, while elementary symmetrics provide more information on the convex hull.

Although the analytical description of the image of the even Vandermonde map is a difficult task it has very nice combinatorial properties. The boundary has the combinatorial structure of a cyclic polytope. Moreover, we described the combinatorial shadow of the image of the even Vandermonde map at infinity, i.e., its tropicalizations, through a simple set of linear inequalities arising from two families of binomial inequalities in power sum functions. Our presented proof of $\mathfrak{S}_{2d}^{\mathcal{S}} \subsetneq \mathfrak{P}_{2d}^{\mathcal{S}}$ for all $2d \geq 4$ and $\mathfrak{S}_{2d}^{\mathcal{B}} \subsetneq \mathfrak{P}_{2d}^{\mathcal{B}}$ for all $2d \geq 6$ is unsatisfactory in a certain way. We show that the sets of nonnegative limit forms in those degrees are not semialgebraic, while the sets of sums of squares are semialgebraic. The proof does not provide quantitative information on the difference of the sets $\mathfrak{S}_{2d}^{\mathcal{S}}$ and $\mathfrak{P}_{2d}^{\mathcal{S}}$. However, we used tropicalization to give quantitative information in the sense that we compared the H-representations of the cones $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*})$ and $\operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ for degree $2d = 10$ and used a linear inequality in $\operatorname{trop}(\mathfrak{P}_{10}^{\mathcal{B},*})^* \setminus \operatorname{trop}(\mathfrak{S}_{10}^{\mathcal{B},*})^*$ to find a form in $\mathfrak{P}_{10}^{\mathcal{B}} \setminus \mathfrak{S}_{10}^{\mathcal{B}}$. The computations for degree 10 can be similarly done in higher degrees.

In Section II.4 we presented the countably infinite vertices of the convex set $\mathcal{E}_d$. Moreover, we showed that $\mathcal{E}_d$ is the union of nested cyclic polytopes. We conjecture that the defining linear inequalities of the set $\mathcal{E}_d$ can also be deduced from Gale's evenness condition. Then the set $\mathcal{E}_d$ would be an intersection of countably infinite halfspaces and Proposition II.4.17 would actually be an equality.

**Conjecture II.8.1.** *Let $d \geq 2$, $\mathcal{C}_e := \{S \subset \mathbb{N} \cup \{\infty\} : |S| = d, S = \biguplus \{i, i+1\} \uplus \{1, \infty\}\}$ and $\mathcal{C}_o := \{S \subset \mathbb{N} \cup \{\infty\} : |S| = d, S = \{\kappa\} \uplus \biguplus \{i, i+1\},\ \kappa \in \{1, \infty\}\}$. Then in the notation from Section II.4*

$$\mathcal{E}_d = \left\{ x \in \mathbb{R}^{d-1} \ : \ \ell_S \circ \Phi_d(x) \leq r_S : \begin{array}{ll} S \in \mathcal{C}_e, & \text{if } d-1 \text{ is even;} \\ S \in \mathcal{C}_o, & \text{if } d-1 \text{ is odd} \end{array} \right\}$$

In Theorem II.6.1 we showed that the problem of determining validity of nonnegativity of multisymmetric functions on copies of the probability simplex is undecidable by embedding the problem of deciding nonnegativity of multivariate polynomials on copies of the natural numbers into our problem. Since validity of nonnegativity of univariate polynomials on the natural numbers is decidable we conjecture that Theorem II.6.1 is true for symmetric functions.

**Conjecture II.8.2.** *The following determination of validity problem is decidable.*

Instance: *A symmetric limit form $f$.*

Question: *Is $f(X) < 0$ for some $X \in \Delta$?*

We noticed in Section II.7 that $\operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ is a rational polyhedral cone for degree $2d \leq 10$. However, we only proved that $\operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ is a polyhedral fan in general (see Remark II.7.4). On the other hand, we showed $\operatorname{trop}(\mathfrak{P}_{2d}^{\mathcal{B},*})$ is always a rational polyhedral cone. Thus, an open question is whether $\operatorname{trop}(\mathfrak{S}_{2d}^{\mathcal{B},*})$ is a rational polyhedral cone for all degrees. The open question would have a positive answer if it was possible to define the spectrahedra $\mathfrak{S}_{2d}^{\mathcal{B},*}$ through matrices having only monomials as coefficients. Then $\mathfrak{S}_{2d}^{\mathcal{B},*}$ has Hadamard property which shows that $\operatorname{trop}(\mathcal{S}_{2d}^{\mathcal{B},*})$ is a convex cone by ([Ble+22b, Lemma 2.2]).

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## Appendix A

In the following we list the code used for calculating the tropicalizations in Sage [Ste07]. The code to calculate the max-closure was provided by Josephine Yu.

Listing II.1: Code to calculate max-closure

```
    # Computes tropical convex hull
def tropConv(P):
    d = P.ambient_dim();
    I = matrix.identity(d);
    E = I.rows();
    sectors = [Polyhedron(rays=[E[j] for j in range(d) if j != i],
                   lines=[Sequence([1 for k in range(d)])]) for i in range(d)];
    MinkSums = [P.minkowski_sum(S) for S in sectors];
    tconv = MinkSums[0];
    for i in range(1,d):
        tconv = tconv.intersection(MinkSums[i])
    return tconv

# non-normalized even symmetric sextics:
print('nn even sym sextics:')

# The first 0 is for >=0, coordinates are in lexicographic order p2,p4,p6
emoment6=Polyhedron(ieqs=[[0,2,-1,0],[0,0,3,-2],[0,1,-2,1]]);

# extreme rays for k
emoment6.Vrepresentation()

# trop of monomial partition map -- from p2,p4,p6,p8 to p2222,p422,p44,
# p6p2,p8  && image of extreme rays above
M6=matrix([[3,0,0],[1,1,0],[0,0,1]]);
M6*vector([1,2,3]);
M6*vector([0,-1,-2]);
M6*vector([0,-2,-3]);

# cone spanned by the image of the extreme rays above
im_emoment6=Polyhedron(rays=[[1,1,1],[-1,-1,-1],[0,-1,-2],[0,-2,-3]]);

#tropical convex hull of im_k (trop of nnevenPSD_6*)
epsd6=tropConv(im_emoment6);

# print('ieqs for trop(nneP_6*):')
epsd6.Hrepresentation()

print('sos6:')
sos6=Polyhedron(ieqs = [[0,1,-2,1],[0,1,-1,0],[0,0,1,-1]]);
```

```
sos6.Hrepresentation()

# we compare the cones
print('trop psd* subset trop sos*:')
epsd6._is_subpolyhedron(sos6)
print('trop sos* subset trop psd*:')
sos6._is_subpolyhedron(epsd6)

## non-normalized even symmetric octics:
print('nn even sym octics:')


# The first 0 is for >=0, coordinates are in lexicographic order p2,p4,
# p6,p8
emoment8=Polyhedron(ieqs=[[0,2,-1,0,0],[0,0,3,-2,0],[0,0,0,4,-3],
                        [0,1,-2,1,0],[0,0,1,-2,1]]);

# extreme rays for k
emoment8.Vrepresentation()

# trop of monomial partition map -- from p2,p4,p6,p8 to p2222,p422,p44,
# p6p2,p8 && image of extreme rays above
M8=matrix([[4,0,0,0],[2,1,0,0],[0,2,0,0],[1,0,1,0],[0,0,0,1]]);
M8*vector([1,2,3,4]);
M8*vector([0,-2,-3,-4]);
M8*vector([0,-1,-2,-3]);
M8*vector([0,-3,-6,-8]);

# cone spanned by the image of the extreme rays above
im_emoment8=Polyhedron(rays=[[1,1,1,1,1],[-1,-1,-1,-1,-1],[0,-2,-4,-3,-4],
                            [0,-1,-2,-2,-3],[0,-3,-6,-6,-8]]);

# tropical convex hull of im_k (trop of nnevenPSD_8*)
epsd8=tropConv(im_emoment8)

print('ieqs for trop(nneP_8*):')
epsd8.Hrepresentation()

print('sos8:')
sos8=Polyhedron(ieqs = [[0,1,-1,0,0,0],[0,0,0,1,0,-1],[0,0,0,-1,1,0],
                    [0,0,1,0,-1,0],[0,0,1,0,-2,1],[0,1,-2,1,0,0]]);
sos8.Hrepresentation()

# we compare the cones
print('trop psd* subset trop sos*:')
epsd8._is_subpolyhedron(sos8)
print('trop sos* subset trop psd*:')
sos8._is_subpolyhedron(epsd8)


## Non normalized even symmetric psd decics:
print('nn even sym decics:')

# The first 0 is for >=0, coordinates are in lexicographic order: p2,p4,p6,
# p8,p10
emoment10=Polyhedron(ieqs = [[0,2,-1,0,0,0],[0,0,3,-2,0,0],[0,0,0,4,-3,0],
    [0,0,0,0,5,-4],[0,1,-2,1,0,0],[0,0,1,-2,1,0],[0,0,0,1,-2,1]]);
emoment10.Vrepresentation()

```

```
# trop of monomial partition map -- from p2,p4,p6,p8,p10 to
# p22222,p4222,p442,p622,p64,p82,p10 && image of extreme rays of emoment10
M10=matrix([[5,0,0,0,0],[3,1,0,0,0],[1,2,0,0,0],[2,0,1,0,0],
            [0,1,1,0,0],[1,0,0,1,0],[0,0,0,0,1]]);
M10*vector([1,2,3,4,5]);
M10*vector([0,-2,-3,-4,-5]);
M10*vector([0,-3,-6,-8,-10]);
M10*vector([0,-1,-2,-3,-4]);
M10*vector([0,-4,-8,-12,-15]);

# cone spanned by the image of the extreme rays above
im_emoment10=Polyhedron(rays=[[1,1,1,1,1,1,1],[0,-2,-4,-3,-5,-4,-5],
[0,-3,-6,-6,-9,-8,-10],[0,-1,-2,-2,-3,-3,-4],[0,-4,-8,-8,-12,-12,-15]]);

# tropical convex hull of im_d (trop of evenPSD_10*)
epsd10=tropConv(im_emoment10);

print('ieqs for trop(nneP10*):')
epsd10.Hrepresentation()

print('sos10:')
sos10=Polyhedron(ieqs = [[0,1,-1,0,0,0,0,0],[0,0,1,0,-1,0,0,0],[0,0,0,-1,
1,0,0,0],[0,0,0,1,0,0,-1,0],[0,0,0,0,0,-1,1,0],[0,0,0,0,0,1,0,-1],[0,1,-2,
1,0,0,0,0],[0,1,0,0,-2,0,0,1],[0,0,1,-2,0,1,0,0],[0,0,1,0,-2,0,1,0],[0,0,
0,1,0,-2,0,1],[0,0,0,0,1,0,-2,1]]);
sos10.Hrepresentation()

# we compare the cones
print('trop psd* subset trop sos*:')
epsd10._is_subpolyhedron(sos10)
print('trop sos* subset trop psd*:')
sos10._is_subpolyhedron(epsd10)
```

# Appendix B

Listing II.2: Sage code used to calculate inequalities defining $\mathcal{E}_{n,2}$

```
print("d=2 ,n=3")
ele32 = Polyhedron(vertices=[(0,0),(1/2^2,0),(3/(3^2),1/3^3)])
ele32.Hrepresentation()

print("d=2 ,n=4")
ele42 = Polyhedron(vertices=[(0,0),(1/2^2,0),(3/(3^2),1/3^3),
        (6/(4^2),4/(4^3))])
ele42.Hrepresentation()

print("d=2 ,n=5")
ele52 = Polyhedron(vertices=[(0,0),(1/2^2,0),(3/(3^2),1/3^3),
        (6/(4^2),4/(4^3)),
        (2/5, 2/25)])
ele52.Hrepresentation()

print("d=2 ,n=6")
ele62 = Polyhedron(vertices=[(0,0),(1/2^2,0),(3/(3^2),1/3^3),
        (6/(4^2),4/(4^3)),(2/5, 2/25),(5/12, 5/54)])
ele62.Hrepresentation()
```

Paper III

# The poset of Specht ideals for hyperoctahedral groups

**Sebastian Debus, Philippe Moustrou, Cordian Riener, Hugues Verdure**

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**Abstract**

Specht polynomials classically realize the irreducible representations of the symmetric group. The ideals defined by these polynomials provide a strong connection with the combinatorics of Young tableaux and have been intensively studied by several authors. We initiate similar investigations for the ideals defined by the Specht polynomials associated to the hyperoctahedral group $\mathcal{B}_n$. We introduce a bidominance order on bipartitions which describes the poset of inclusions of these ideals and study algebraic consequences on general $\mathcal{B}_n$-invariant ideals and varieties, which can lead to computational simplifications.

## Contents



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## III.1 Introduction

Symmetries provide beautiful connections between algebra, geometry and efficient computations: on the one hand, the symmetries of geometrical objects can be described with the algebraic language of group theory, while on the other hand algebraic problems affording additional structure can be solved more efficiently once symmetry is appropriately taken into consideration. A particular incarnation of these phenomena occurs when studying algebraic systems of polynomial equations whose solution set is invariant under a group action. In this set-up, when looking at the corresponding polynomial ideal, the machinery of invariant and representation theory can be employed to gain information about the solutions of the initial system, and to simplify its resolution.

This kind of questions have been extensively studied in the literature for the *symmetric group* $\mathcal{S}_n$, acting on the polynomial ring $\mathbb{K}[X_1, \ldots, X_n]$ over a field $\mathbb{K}$ by permuting variables. In particular, it has been observed in different computational tasks that the understanding of this action can lead to substantial algorithmic improvements (see for example [BR21a; BR21b; Fau+20; HS21; KLM20; Mou+22; Rie+13; RS18]. These improvements mostly build on the fact that in this situation, both representation and invariant theory are classically understood, and are closely related to the combinatorics of partitions and Young Tableaux. More precisely, the irreducible representations of $\mathcal{S}_n$ are in bijection with the partitions of $n$, through a construction due to Specht: for every partition, one can define a polynomial whose $\mathcal{S}_n$-orbit spans an irreducible $\mathcal{S}_n$-module, called *Specht module* [Spe37b]. This motivates the study of *Specht ideals*, the ideals generated by such modules, since they can be seen as building blocks of the action of the symmetric group on a polynomial ring. The study of these objects has shown to be fruitful from various aspects, and the connection between these ideals and the combinatorics of partitions turns out to be deeper: not only there is a bijection between $\mathcal{S}_n$-Specht ideals and partitions of $n$, but this correspondence respects the poset structures. First results were proven by Haiman and Woo (see Woo's doctoral thesis [Woo05]), and then independently revisited and extended for algorithmic purposes in [MRV21]. In turn, this combinatorial understanding also provides information on these ideals from the point of view of commutative algebra: for instance they all are radical [MOY21], and the partitions for which they are Cohen-Macaulay are understood [Yan21]. The study of these ideals has also paved algorithmic ways to simplify calculations for $\mathcal{S}_n$-closed ideals and their corresponding varieties. They allow to understand the symmetry of the coordinates of the points in the variety, which in turn gives information on their dimension. This information can then be used to design more efficient algorithms by reducing the number of variables.

In this article, we initiate a similar study for the action of the *hyperoctahedral group* $\mathcal{B}_n$ on a polynomial ring $\mathbb{K}[X_1, \ldots, X_n]$. The field $\mathbb{K}$ is assumed to be infinite, although many results remain valid for finite fields. In this representation, this group can be seen as the group generated by permutations of variables and sign switches of variables, namely maps sending $X_i$ to $-X_i$. The group is isomorphic to the Weyl group of type $B$ and appears in several different areas, as hyperplane arrangements ([AM17, Section 6.7],[Abe+20]), representation theory [Che93; CS93; Mus93], and has applications in the study of non negative even symmetric polynomials [CLR87; Har99] and optimization [DGV+17]. Similarly to the case of permutations, this situation is profoundly connected to combinatorics. In this case, instead of partitions, the irreducible representations of $\mathcal{B}_n$ are in bijection with *bipartitions* of $n$. Furthermore, polynomial generators of the irreducible $\mathcal{B}_n$-modules can be constructed in a similar way [Spe37a]. We aim at a first investigation of the corresponding ideals with the goal to extend the connections between algebra and combinatorics as far as possible. In contrast to the $\mathcal{S}_n$-case where there is a natural order on partitions, several orders are possible on bipartitions [AMP81; DJM95]. However, while in the $\mathcal{S}_n$ case the poset of the standard order on partitions reflects the corresponding poset of ideal inclusions, none of the previously studied orders of bipartitions satisfies this property. Therefore, we define another order on bipartitions. After studying the basic properties of this order, we are able to show that it indeed translates well to the ideal inclusion. Similarly to the case of $\mathcal{S}_n$, this combinatorial connection finds consequences for the corresponding varieties. In addition to the inclusion of varieties we are able to give a complete characterization in terms of orbit types of the points in these varieties. Further, this gives information on the possible orbit types of points in general $\mathcal{B}_n$-invariant varieties, allowing for complexity reduction in the resolution of $\mathcal{B}_n$-closed polynomial systems.

The paper is structured as follows: Section III.2 overviews the situation of $\mathcal{S}_n$-Specht ideals. We initiate the study of $\mathcal{B}_n$-Specht ideals in Section III.3 with definitions and natural connections to the $\mathcal{S}_n$ case. In Section III.4 we define our order for bipartitions and study its combinatorial properties. Following this, we show equivalence between our poset of bipartitions and the posets of Specht ideals and varieties in Section III.5. In Section III.6, study possible decompositions of Specht varieties in terms of orbit types. Finally, we extend our study to general $\mathcal{B}_n$-invariant ideals in Section III.7, before concluding the paper with closing remarks and open questions in Section III.8.

## III.2 $\mathcal{S}_n$-Specht ideals

### III.2.1 Definitions

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$ is a sequence of non-increasing non negative integers such that $\sum_{i\geq 1} \lambda_i = n$. We write $\lambda \vdash n$ when $\lambda$ is a partition of $n$ and say that $\emptyset$ is the unique partition of 0. The *size* of a partition $\lambda$ is $\mid \lambda \mid = \sum_{i\geq 1} \lambda_i$. The *length* of a non-empty partition $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ is the maximal $l \in \mathbb{N}_0$ with $\lambda_l > 0$, while the length of $\emptyset$ is 0. We denote the length by $\operatorname{len}(\lambda)$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ we use the convention that $\lambda_s = 0$ for every $s \geqslant l + 1$.

Let $\lambda, \mu \vdash n$ be partitions of the same size. Then $\lambda$ *dominates* $\mu$ if and only if $\sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} \mu_j$ for any $k$. We denote domination by $\mu \trianglelefteq \lambda$. A partition $\lambda$

can be represented via its *(Young) diagram*, i.e., the ordered sequence of boxes from the left to the right and the top to the bottom, where the $i$-th line contains $\lambda_i$ many boxes. We say that the associated diagram has *shape* $\lambda$. A *tableau* of shape $\lambda$ is a filling of a diagram of shape $\lambda$ with all the numbers $[n] = \{1, \ldots, n\}$. Then, we write $\mathrm{sh}(T) = \lambda$ if $T$ is a tableau of shape $\lambda$. For instance, $S = \begin{array}{|c|c|c|c|}\hline 9 & 3 & 6 & 4 \\ \hline 2 & 1 & 8 \\ \cline{1-3} 5 & 7 \\ \cline{1-2}\end{array}$ is a tableau of shape $(4, 3, 2)$. A *generalized* tableau is a filling of a diagram with elements in $\mathbb{K}$. The *conjugate partition* $\lambda^\vee$ of a partition $\lambda$ is the partition whose diagram is the one obtained from the diagram of $\lambda$ by interchanging the rows and columns.

For a sequence $(i_1, \ldots, i_m)$ of natural numbers, we define the associated *Vandermonde polynomial* in the variables $X_{i_1}, \ldots, X_{i_m}$ as

$$\Delta_{(i_1,\ldots,i_m)}(X) = \prod_{j<k\in[m]} (X_{i_j} - X_{i_k}),$$

while $\Delta_{(i)} = \prod_\emptyset (X_{i_j} - X_{i_k}) = 1$.

**Definition III.2.1.** Let $T$ be a tableau of shape $\lambda \vdash n$ with $m$ columns and let $T_i$ be the sequence of natural numbers containing the entries of the $i$-th column of $T$ from above to below. Then, the associated $\mathcal{S}_n$ *Specht polynomial* $\mathrm{sp}_T(X)$ is the product of all the column Vandermonde polynomials of the columns, i.e.,

$$\mathrm{sp}_T(X) = \prod_{j=1}^m \Delta_{T_j}.$$

For the tableau $S$ of shape $(4, 3, 2)$ above, we have $\mathrm{sp}_S(X)$ equals

$$\begin{aligned} &\Delta_{(9,2,5)}(X)\Delta_{(3,1,7)}(X)\Delta_{(6,8)}(X)\Delta_{(4)}(X) \\ &= (X_9 - X_2)(X_9 - X_5)(X_2 - X_5)(X_3 - X_1)(X_3 - X_7)(X_1 - X_7)(X_6 - X_8). \end{aligned}$$

**Definition III.2.2.** Let $\lambda$ be a partition of $n$. We define the $\mathcal{S}_n$*-Specht ideal*

$$I_\lambda = \langle \mathrm{sp}_T(X) : T \text{ is a tableau of shape } \lambda \rangle \subset \mathbb{K}[X_1, \ldots, X_n]$$

and the $\mathcal{S}_n$*-Specht variety*

$$V_\lambda = \{a \in \mathbb{K}^n : f(a) = 0 \text{ for all } f \in I_\lambda\} \subset \mathbb{K}^n$$

associated to $\lambda$.

The group $\mathcal{S}_n$ acts transitively on the set of tableaux of shape $\lambda$, where an element $\sigma \in \mathcal{S}_n$ acts on a tableau $T$ by replacing every entry $i$ in a box by $\sigma(i)$. Thus, the $\mathcal{S}_n$-Specht ideal $I_\lambda$ is the ideal generated by the $\mathcal{S}_n$-orbit of a Specht polynomial of a tableau of shape $\lambda$.

**Definition III.2.3.** For a partition $\lambda \vdash n$ we write $\mathcal{S}_\lambda = \mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \ldots \subset \mathcal{S}_n$ and define the $\mathcal{S}_n$*-orbit set* $H_\lambda = \{z \in \mathbb{K}^n : \mathrm{Stab}_{\mathcal{S}_n}(z) \simeq \mathcal{S}_\lambda\}$. If $z \in H_\lambda$ we call $\lambda$ the $\mathcal{S}_n$*-orbit type* of $z$.

The orbit set of any partition is non-empty and the $H_\lambda$'s define a set partition of $\mathbb{K}^n$. For instance, $H_{(3,2,2,1)}$ is the $\mathcal{S}_n$ orbit of the set

$$\{(a_1, a_1, a_1, a_2, a_2, a_3, a_3, a_4) \in \mathbb{K}^n : a_i \neq a_j,\ \forall i \neq j\}.$$

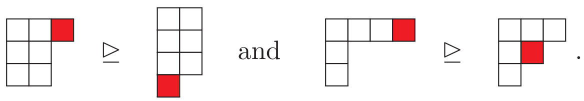
### III.2.2 Inclusions and applications

The dominance order for integer partitions is well studied and understood. We recall that if $(P, \preccurlyeq)$ is a poset and $p, q \in P$ then $p$ *covers* $q$ if and only if $p \neq q$, $q \preccurlyeq p$, and for any $r \in P$, $q \preccurlyeq r \preccurlyeq p$ implies $r \in \{p, q\}$. Brylawski studied the lattice of integer partitions of $n$ with respect to the dominance order and classified the covering relations ([Bry73, Proposition 2.3]). Let $\lambda, \mu \vdash n$ be partitions. Then, $\mu \trianglelefteq \lambda$ is a covering if and only if $\lambda$ is of the form

$$\lambda = (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \ldots, \mu_l),$$

and either $j = i + 1$ or $\mu_i = \mu_{j-1}$ (and $\mu_{i-1} > \mu_i$ and $\mu_j > \mu_{j+1}$ to ensure that $\mu$ is a partition). In particular, the diagram of shape $\mu$ can be obtained from the diagram of shape $\lambda$ via moving one box from the end of row $i$ to row $j$.

**Example III.2.4.** The following are two coverings of partitions displayed by their diagrams.




The following theorem shows the equivalences of the posets of partitions with respect to dominance order, and the posets of $\mathcal{S}_n$-Specht ideals and varieties with respect to inclusion.

**Theorem III.2.5** ([MRV21], Theorem 1)**.** *Let $\lambda$ and $\mu$ be partitions of $n$. Let $I_\lambda, I_\mu$ denote their associated $\mathcal{S}_n$-Specht ideals and $V_\lambda, V_\mu$ their associated $\mathcal{S}_n$-Specht varieties. Then, the following assertions are equivalent:*

III.2.5.1. *The partition $\lambda$ dominates $\mu$, i.e. $\lambda \trianglerighteq \mu$;*

III.2.5.2. *The $\mathcal{S}_n$-Specht ideal $I_\lambda$ contains the $\mathcal{S}_n$-Specht ideal $I_\mu$, i.e. $I_\lambda \supset I_\mu$;*

III.2.5.3. *The $\mathcal{S}_n$-Specht variety $V_\lambda$ is contained in the $\mathcal{S}_n$-Specht variety $V_\mu$, i.e. $V_\lambda \subset V_\mu$.*

The $\mathcal{S}_n$-Specht varieties can be decomposed using $\mathcal{S}_n$ orbit sets.

**Theorem III.2.6** ([MRV21], Corollary 1)**.** *Let $\mu \vdash n$ be a partition. Then, the associated $\mathcal{S}_n$-Specht variety is*

$$V_\mu = \left( \bigcup_{\lambda \trianglelefteq \mu} H_\lambda \right)^c = \bigcup_{\lambda \ntrianglelefteq \mu} H_\lambda.$$

This characterization already shows that in general $\mathbb{K}[X_1,\dots,X_n]/I_\lambda$ is not Cohen-Macaulay for a $\mathcal{S}_n$-Specht ideal $I_\lambda$, since the varieties are not equidimensional. Yanagawa classified the few cases when a Specht ideal is Cohen-Macaulay.

**Theorem III.2.7** ([Yan21], Corollary 4.4)**.** *The ring $\mathbb{K}[X_1,\dots,X_n]/I_\lambda$ is Cohen-Macaulay if and only if $\lambda$ is one of the following form*

III.2.7.1. $\lambda = (n-d,1,\dots,1)$;

III.2.7.2. $\lambda = (n-d,d)$;

III.2.7.3. $\lambda = (a,a,1)$.

The authors in [MOY21; Woo05] prove that a $\mathcal{S}_n$-Specht ideal is radical. Their proof uses Theorem III.2.6, i.e., that $\mathcal{S}_n$-Specht varieties can be written as disjoint unions of $\mathcal{S}_n$ orbit sets, and the non-emptyness of any orbit set $H_\lambda$.

## III.3 Definition and first properties of $\mathcal{B}_n$-Specht ideals

### III.3.1 Definitions

A *bipartition* of $n$ is a pair $(\lambda,\mu)$, where $\lambda \vdash n_1$, $\mu \vdash n_2$ are partitions and $n_1+n_2=n$. We denote the set of all bipartitions of $n$ by $\mathrm{BP}_n$. A *(Young) bidiagram* of a bipartition $(\lambda,\mu)$ is the pair of diagrams of shape $\lambda$ and $\mu$. A *bitableau* is a filling of a bidiagram with all the numbers in $[n]$. We write $\mathrm{sh}(T,S)=(\lambda,\mu)$ if $(T,S)$ is a bitableau of shape $(\lambda,\mu)$. For example, $(T',S') = \left(\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & \\ \cline{1-1} 5 & \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 6 \\ \hline 1 \\ \hline\end{array}\right)$ is a bitableau of shape $((2,1,1),(1,1))$. When we consider representatives of $\mathcal{B}_n$-orbits of points, we do not need to distinguish between the signs of coordinates. Thus, we write $X^2=(X_1^2,\dots,X_n^2)$ and analogously $z^2=(z_1^2,\dots,z_n^2)$ for points $z\in\mathbb{K}^n$. A *generalized* bitableau is a filling of a bidiagram with elements in $\mathbb{K}$.

**Definition III.3.1.** Let $(T,S)$ be a bitableau and let $T_i,\mathcal{S}_i$ be the sequences of natural numbers containing the entries of the $i$-th column of $T$ and $S$ from above to below. Then, the associated $\mathcal{B}_n$ *Specht polynomial* is

$$\mathrm{sp}_{(T,S)}(X) \;=\; \mathrm{sp}_T(X^2)\,\mathrm{sp}_S(X^2)\prod_{k\in S}X_k \;=\; \prod_{i\geq 1}\Delta_{T_i}(X^2)\prod_{j\geq 1}\Delta_{\mathcal{S}_j}(X^2)\prod_{k\in S}X_k$$

where the notation $\mathrm{sp}_T$ is naturally adapted in this context to a Tableau $T$ which is not necessarily filled with the integers $1,\dots,k$. For the bitableau $(T',S')$ of shape $((2,1,1),(1,1))$ above, we have

$$\begin{aligned}\mathrm{sp}_{(T,S)}(X) &= \Delta_{(4,2,5)}(X^2)\cdot\Delta_{(3)}(X^2)\cdot\Delta_{(6,1)}(X^2)\cdot X_6X_1\\ &= (X_4^2-X_2^2)(X_4^2-X_5^2)(X_2^2-X_5^2)(X_6^2-X_1^2)X_1X_6.\end{aligned}$$

From now on, if not specified, Specht polynomials will stand for $\mathcal{B}_n$-Specht polynomials.

**Definition III.3.2.** Let $(\lambda, \mu)$ be a bipartition of $n$. We define the $\mathcal{B}_n$-*Specht ideal*

$$I_{(\lambda,\mu)} = \langle \mathrm{sp}_{(T,S)}(X) : (T, S) \text{ is a bitableau of shape } (\lambda, \mu)\rangle \subset \mathbb{K}[X_1, \ldots, X_n]$$

and the $\mathcal{B}_n$-*Specht variety*

$$V_{(\lambda,\mu)} = \{z \in \mathbb{K}^n : f(z) = 0 \text{ for all } f \in I_{(\lambda,\mu)}\} \subset \mathbb{K}^n$$

associated to $(\lambda, \mu)$.

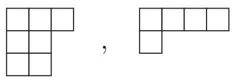
Again, the $\mathcal{B}_n$-Specht ideal $I_{(\lambda,\mu)}$ is the ideal generated by the $\mathcal{S}_n$ orbit of a Specht polynomial of a bitableau of shape $(\lambda, \mu)$. We observe that switching of signs of variables in a Specht polynomial $\mathrm{sp}_{(T,S)}$ returns $\pm\,\mathrm{sp}_{(T,S)}$. Thus, $I_{(\lambda,\mu)}$ also equals the $\mathcal{B}_n$ orbit of $\mathrm{sp}_{(T,S)}$.

### III.3.2 Relations between $\mathcal{B}_n$ and $\mathcal{S}_n$ Specht polynomials

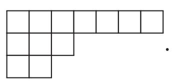
**Definition III.3.3.** Let $\lambda \vdash n_1, \mu \vdash n_2$ be partitions. Then, the *glueing* of $\lambda$ and $\mu$ is the partition $\lambda \uplus \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots) \vdash n_1 + n_2$. The *concatenation* $\lambda \vee \mu \vdash n_1 + n_2$ is the partition obtained by rearranging $(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t)$ in decreasing order.

The glueing $\lambda \uplus \mu$ defines indeed again a partition. Since $\lambda_i \geq \lambda_{i+1}$ and $\mu_i \geq \mu_{i+1}$ we have $\lambda_i + \mu_i \geq \lambda_{i+1} + \mu_{i+1}$ for any $i$.

**Example III.3.4.** The glueing of the partitions $(3, 2, 2), (4, 1)$ with diagrams



is the partition $(7, 3, 2)$ with diagram



**Lemma III.3.5.** *Let $\lambda, \mu$ be two partitions. Then, we have*

$$(\lambda \uplus \mu)^{\perp} = \lambda^{\perp} \vee \mu^{\perp}.$$

*As a consequence, the columns of the diagram of shape $\lambda \uplus \mu$ are in bijection with the columns in the bidiagram of shape $(\lambda, \mu)$.*

*Proof.* Let $\rho = \left(\lambda^{\perp} \vee \mu^{\perp}\right)^{\perp}$. Then

$$\begin{aligned} \rho_k = \mid \{j,\ (\lambda^{\perp} \vee \mu^{\perp})_j \geqslant k\} \mid \quad &= \quad \mid \{j,\ (\lambda^{\perp})_j \geqslant k\} + \#\{j,\ (\mu^{\perp})_j \geqslant k\} \mid \\ &= \quad ((\lambda^{\perp})^{\perp})_k + ((\mu)^{\perp})^{\perp})_k \\ &= \quad \lambda_k + \mu_k \\ &= \quad (\lambda \uplus \mu)_k. \end{aligned}$$

The last claim of the statement follows, since the columns of $\lambda \uplus \mu$ are the rows of its conjugation. ∎

The previous Lemma provides a natural connection between bitableaux of shape $(\lambda, \mu)$ and Tableaux of shape $\lambda \uplus \mu$. Concretely, let $(T, S)$ be a bitableau of shape $(\lambda, \mu)$. Then, we can consider the tableau $T \uplus S$ of shape $\lambda \uplus \mu$, where the columns of $T \uplus S$ are filled like the columns of $T$ and $S$. When two columns in $\lambda \uplus \mu$ have the same length, they are ordered by their occurrence in the bitableau $(T, S)$ from the left to the right. For instance, for $(T^*, S^*) = \left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & 10 & 9 \\ \hline 4 & 8 & 7 & \\ \cline{1-3} 6 & & & \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline \end{array} \right)$ we have $T^* \uplus S^* = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 10 & 3 & 9 \\ \hline 4 & 8 & 7 & 5 & \\ \cline{1-4} 6 & & & & \\ \cline{1-1} \end{array}$. Since this map is invertible we get:

**Proposition III.3.6.** *The tableaux of shape $\lambda \uplus \mu$ are in 1:1 correspondence with the bitableaux of shape $(\lambda, \mu)$. A bijection is given by $(T, S) \mapsto T \uplus S$.*

The lemma below describes the connection between $\mathcal{S}_n$ and $\mathcal{B}_n$-Specht polynomials. In particular, it motivates the definition of the same operations on the bidiagram of shape $(\lambda, \mu)$ and on the diagram of its glueing $\lambda \uplus \mu$ via moving some of the boxes in a diagram in section III.4.

**Lemma III.3.7.** *Let $(\lambda, \mu) \in \mathrm{BP}_n$ be a bipartition and let $(T, S)$ be a bitableau of shape $(\lambda, \mu)$. Then*

$$\mathrm{sp}_{(T,S)}(X_1, \dots X_n) = \mathrm{sp}_{T \uplus S}(X_1^2, \dots, X_n^2) \prod_{j \in S} X_j.$$

*Proof.* It is an immediate consequence of Proposition III.3.6, since the Specht polynomials are defined as product of Vandermonde polynomials on the columns of the glued partition. ■

### III.3.3 Existing orders on bipartitions

As mentioned in the introduction, partial orders on the set of bipartitions of $n$ have been studied by several authors. Let $(\lambda, \mu), (\lambda', \mu') \in \mathrm{BP}_n$ be bipartitions. For instance the following statements define partial orders:

$$\begin{cases} \sum_{j \leq k} \lambda'_j \leq \sum_{j \leq k} \lambda_j, & \text{for all } k, \text{ and} \\ \mid \lambda' \mid + \sum_{j \leq k} \mu'_j \leq \mid \lambda \mid + \sum_{j \leq k} \mu_j, & \text{for all } k. \end{cases} \tag{III.1}$$

was introduced in [DJM95] to study Hecke algebras of type $\mathcal{B}_n$, and was recently proven to occur naturally in the field of spin group theory [Xia18]. Ariki generalized their order to multipartitions to study Hecke algebras of type $G(m, 1, n)$ [Ari01]. The partial order

$$\begin{cases} \mid \lambda' \mid < \mid \lambda \mid, & \text{or} \\ \mid \lambda' \mid = \mid \lambda \mid, & \text{and } \lambda' \trianglelefteq \lambda, \mu' \trianglelefteq \mu. \end{cases} \tag{III.2}$$

was formalized in [AMP81] to construct $\mathcal{B}_n$-irreducible representations based on a more general procedure valid for finite groups. The partial orders (III.1) and (III.2) are not equivalent. For instance, in (III.1) the bipartitions $((2), (1))$ and

$((1,1,1),\emptyset)$ are not comparable, while the latter is larger than the first in (III.2).

Moreover, these orders do not capture inclusions of ideals and varieties. Namely, for both orders, we have the following ordering of bipartitions of $n = 2$:

$$((2),\emptyset) \succ ((1,1),\emptyset) \succ ((1),(1)),$$

while the corresponding ideals are

$$I_{((1,1),\emptyset)} =< X_1^2 - X_2^2 >\subsetneq I_{((1),(1))} =< X_1, X_2 >\subsetneq I_{(2),\emptyset} =< 1 > .$$

In the next section, we introduce a new order on bipartitions that will capture inclusion of Specht ideals.

## III.4 The poset of bipartitions

In this section we introduce our new order for bipartitions:

**Definition III.4.1.** Let $(\lambda,\mu),(\lambda',\mu') \in \mathrm{BP}_n$ be biparitions of $n$. We say that $(\lambda,\mu)$ *bidominates* $(\lambda',\mu')$ if and only if

$$\begin{aligned}
\sum_{j=1}^{k}(\lambda'_j+\mu'_j) &\leq \sum_{j=1}^{k}(\lambda_j+\mu_j), \quad \text{and} \\
\sum_{j=1}^{k-1}(\lambda'_j+\mu'_j)+\lambda'_k &\leq \sum_{j=1}^{k-1}(\lambda_j+\mu_j)+\lambda_k
\end{aligned}$$

for all positive integer $k$. If $(\lambda,\mu)$ bidominates $(\lambda',\mu')$ we write $(\lambda',\mu') \trianglelefteq (\lambda,\mu)$. We call $\trianglelefteq$ the *bidominance order*.

We point out that the first condition is just a condition on the glueing of the bipartitions, i.e.,

$$\lambda' \uplus \mu' \trianglelefteq \lambda \uplus \mu.$$

**Example III.4.2.** The following bipartitions of 8 are comparable: $((2,1,1),(3,1)) \trianglelefteq ((3,2),(2,1))$, since

$$2 \leq 3, \quad 5 \leq 5, \quad 6 \leq 7, \quad 7 \leq 8, \quad 8 \leq 8.$$

However, the bipartitions $((2),(1,1))$ and $(\emptyset,(4))$ are not comparable, since $2 > 0$ but $3 < 4$.

Although we use the same symbol for dominance and bidominance, this should not create any confusion, as they are defined on sets with empty intersection. We identify bipartitions with their associated bidiagrams and speak about *boxes* in a bipartition.
It follows from the definition that our bidominance order is a partial order on $\mathrm{BP}_n$, and the previous example shows that it is not a total order.

Before proving our main theorem in the next section, we need a better understanding of our poset of bipartitions.

The smallest element in $(\mathrm{BP}_n, \trianglelefteq)$ is $(\emptyset, (1, \ldots, 1))$, while the largest element is $((n), \emptyset)$. The following theorem characterizes the covering relations in the poset $(\mathrm{BP}_n, \trianglelefteq)$. It turns out that there are four different cases, that are illustrated in Example III.4.4.

**Theorem III.4.3.** *Let $(\lambda, \mu), (\lambda', \mu') \in \mathrm{BP}_n$ be bipartitions and let $i = \min\{j \in [n] : (\lambda_j, \mu_j) \neq (\lambda'_j, \mu'_j)\}$. Then, $(\lambda, \mu)$ covers $(\lambda', \mu')$ if and only if one of the following statements is true:*

III.4.3.1. *$\mu = \mu'$, $\lambda$ covers $\lambda'$ with respect to the dominance order on partitions with $\lambda'_i = \lambda_i - 1$, and for $k$ such that $\lambda'_k = \lambda_k + 1$, we have $\mu_{i-1} = \mu_i = \ldots = \mu_k$;*

III.4.3.2. *$\lambda = \lambda'$, $\mu$ covers $\mu'$ with respect to the dominance order on partitions, with $\mu'_i = \mu_i - 1$, and for $k$ such that $\mu'_k = \mu_k + 1$, we have $\lambda_i = \lambda_{i+1} = \ldots = \lambda_{k+1}$;*

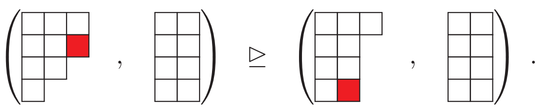
III.4.3.3. *$\lambda \neq \lambda'$, $\mu \neq \mu'$ and $\lambda_i > \lambda'_i$. If $k$ is maximal with $\lambda_i = \lambda_k$, then $\mu_i = \mu_k$, $(\lambda'_j, \mu'_j) = (\lambda_j - 1, \mu_j + 1)$ for any integer $i \leq j \leq k$, and $(\lambda'_j, \mu'_j) = (\lambda_j, \mu_j)$ otherwise;*

III.4.3.4. *$\lambda \neq \lambda'$, $\mu \neq \mu'$, $\lambda_i = \lambda'_i$ (and therefore $\mu_i > \mu'_i$). If $k$ is maximal with $\mu_i = \mu_k$, then $\lambda_{i+1} = \lambda_{k+1}$, $(\mu'_j, \lambda'_{j+1}) = (\mu_j - 1, \lambda_{j+1} + 1)$ for any integer $i \leqslant j \leqslant k$ and there is equality otherwise.*

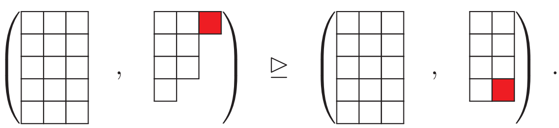
The example below shows instances for all the covering cases of bipartitions. The boxes that are moved are colored in red.

**Example III.4.4.**

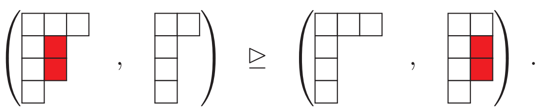
- An example of a covering of type (1), where $i = 2$ and $k = 4$:



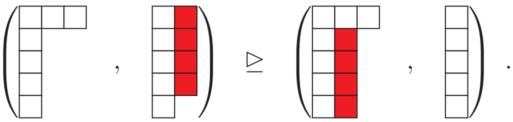
- An example of a covering of type (2), where $i = 1$ and $k = 4$:



- An example of a covering of type (3), where $i = 2$:



- An example of a covering of type (4), where $i = 1$:



One can think of the cases (3) and (4) as moving a *partial column* from the left diagram in the bidiagram $(\lambda, \mu)$ to the right side and staying in the same row, or from moving a partial column from the right diagram to the left and going down one row.

Now we present a proof of the theorem.

*Proof of Theorem III.4.3.* We start the proof by showing that operations (1)−(4) define covering relations. Suppose that $(\lambda', \mu')$ is obtained from $(\lambda, \mu)$ by one of these operations. We need to show that $(\lambda, \mu)$ covers $(\lambda', \mu')$, that is, if $(\lambda^*, \mu^*)$ is such that $(\lambda, \mu) \trianglerighteq (\lambda^*, \mu^*) \trianglerighteq (\lambda', \mu')$, then either $(\lambda^*, \mu^*) = (\lambda, \mu)$ or $(\lambda^*, \mu^*) = (\lambda', \mu')$. Since the proofs for operations (2) and (4) are respectively similar to (1) and (3), we will focus on these two operations.

(A) Suppose that $(\lambda', \mu')$ is obtained from $(\lambda, \mu)$ by operation (1). In particular, $\mu = \mu'$, and there exists $1 < i < k$ such that $\lambda'_i = \lambda_i - 1$, $\lambda'_k = \lambda_k + 1$, $\lambda'_j = \lambda_j$ for $j \neq i, k$ and $\mu_{i-1} = \ldots = \mu_k$. It is not difficult to show, by taking the difference between two consecutive partial sums, that

$$\forall j \in \{1, \ldots i-1, k+1, \ldots\},\ \lambda_j = \lambda^*_j = \lambda'_j \text{ and } \mu_j = \mu^*_j = \mu'_j,$$

as well as $\mu^*_k = \mu_k = \mu'_k$. Since $\mu^*_{i-1} = \mu_{i-1} = \mu_k = \mu^*_k$, this implies that $\mu^*_{i-1} = \ldots = \mu^*_k$ as well. In turn, this means that

$$\overline{\lambda} = (\lambda_i, \ldots, \lambda_k) \trianglerighteq \overline{\lambda^*} = (\lambda^*_i, \ldots, \lambda^*_k) \trianglerighteq (\lambda'_i, \ldots, \lambda'_k) = \overline{\lambda'}.$$

By hypothesis, $\overline{\lambda}$ covers $\overline{\lambda'}$, so that either $\overline{\lambda^*} = \overline{\lambda}$ or $\overline{\lambda^*} = \overline{\lambda'}$. In turn, this shows that $(\lambda, \mu)$ covers $(\lambda', \mu')$.

(B) Now, we assume that $(\lambda', \mu')$ is obtained from $(\lambda, \mu)$ by operation (3). This means, that there exists $i \leqslant k$ such that $\lambda_i = \lambda_k > \lambda_{k+1}$, $\mu_{i-1} > \mu_i = \mu_k$ and $\lambda'_j = \lambda_j - 1$, $\mu'_j = \mu_j + 1$, for $i \leq j \leq k$ and otherwise $\lambda'_j = \lambda_j$ and $\mu'_j = \mu_j$.
As above we observe that

$$\forall j < i \text{ or } j > k, \lambda_j = \lambda^*_j = \lambda'_j \text{ and } \mu_j = \mu^*_j = \mu'_j.$$

In the same way, it is easy to show that

$$\forall i \leqslant j \leqslant k,\ \lambda_j + \mu_j \geqslant \lambda^*_j + \mu^*_j \geqslant \lambda'_j + \mu'_j = \lambda_j + \mu_j,$$

and

$$\forall i \leqslant j \leqslant k,\ \lambda_j \geqslant \lambda^*_j \geqslant \lambda'_j = \lambda_j - 1.$$

Together, this implies that

$$\forall i \leqslant j \leqslant k,\ \lambda^*_j \in \{\lambda_j, \lambda_j - 1\} \text{ and } \mu^*_j \in \{\mu_j, \mu_j + 1\}.$$

a) Assume first that $\lambda_i^* = \lambda_i - 1$. Then for $i \leqslant j \leqslant k$,

$$\lambda_j - 1 \leqslant \lambda_j^* \leqslant \lambda_i^* = \lambda_i - 1 = \lambda_j - 1$$

which implies $\lambda_j^* = \lambda_j - 1$ and $\mu_j^* = \mu_j + 1$, that is, $(\lambda^*, \mu^*) = (\lambda', \mu')$.

b) On the other hand, if $\lambda_i^* = \lambda_i$ or equivalently $\mu_i^* = \mu_i$, then for $i \leqslant j \leqslant k$, we have

$$\mu_j \leqslant \mu_j^* \leqslant \mu_i^* = \mu_i = \mu_j$$

which implies that $\mu_j^* = \mu_j =$ and $\lambda_j^* = \lambda_j$, that is $(\lambda^*, \mu^*) = (\lambda, \mu)$.

Thus, they describe a covering relation in the poset $(\mathrm{BP}_n, \trianglelefteq)$.

Now, we prove the converse. Let $(\lambda, \mu)$ and $(\lambda', \mu')$ be two different bipartitions of $n$ and assume that $(\lambda', \mu') \trianglelefteq (\lambda, \mu)$. We show that there exists a bipartition $(\lambda^*, \mu^*)$ of $n$ that can be obtained from $(\lambda, \mu)$ through one of the cases (1)-(4), and $(\lambda', \mu') \trianglelefteq (\lambda^*, \mu^*) \trianglelefteq (\lambda, \mu)$. Let $i \in \mathbb{N}$ be minimal with $(\lambda_i, \mu_i) \neq (\lambda_i', \mu_i')$.

(A) We consider first the case where $\lambda_i > \lambda_i'$ and show that we can obtain $(\lambda^*, \mu^*)$ using one of the operations (1),(3), or (4). Let $k \in \mathbb{N}$ be maximal with $\lambda_i = \lambda_k$.

We begin our analysis with distinguishing between the following two cases. Either there exists a $p \in \mathbb{N}$ such that $i \leq p \leq k$ and $\mu_p < \mu_{p-1}$ or not, with the convention that $\mu_1 < \mu_0$.

a) First, we assume that there exists such a $p$ and we fix the minimal $p$ with this property. Then $\mu_p$ is the first place after $\mu_{i-1}$ where we can put a box to still obtain a partition. Let $q \in \mathbb{N}$ be minimal such that $p \leq q \leq k$ and $\mu_q = \ldots = \mu_k$. We define $(\lambda^*, \mu^*)$ as the bipartition of $n$ with $(\lambda_j^*, \mu_j^*) = (\lambda_j - 1, \mu_j + 1)$ for every $q \leq j \leq k$ and otherwise $(\lambda_j^*, \mu_j^*) = (\lambda_j, \mu_j)$. We observe easily that $(\lambda^*, \mu^*) \trianglelefteq (\lambda, \mu)$ and $\lambda^* \uplus \mu^* = \lambda \uplus \mu \trianglerighteq \lambda' \uplus \mu'$, and we are just left with verifying

$$\lambda_t^* + \sum_{j=1}^{t-1} (\lambda_j^* + \mu_j^*) \geq \lambda_t' + \sum_{j=1}^{t-1} (\lambda_j' + \mu_j'),$$

for any $t \in \mathbb{N}$. However, this is clear for any $t < q$ and $t > k$. If $q \leqslant t \leqslant k$, we have

$$\lambda_t' \leqslant \lambda_i' \leqslant \lambda_i - 1 = \lambda_t - 1 = \lambda_t^*$$

and

$$\sum_{j=1}^{t-1} (\lambda_j^* + \mu_j^*) = \sum_{j=1}^{t-1} (\lambda_j + \mu_j) \geq \sum_{j=1}^{t-1} (\lambda_j' + \mu_j')$$

so that we also have

$$\lambda_t^* + \sum_{j=1}^{t-1}(\lambda_j^* + \mu_j^*) \geq \lambda_t' + \sum_{j=1}^{t-1}(\lambda_j' + \mu_j').$$

This is operation (3).

b) Next, we assume that no such $p$ exists. In particular, $i > 1$ and $\mu_{i-1} = \ldots = \mu_k$. We consider the closest possible free place in the bidiagram $(\lambda, \mu)$, namely we take $r > k$ to be the minimal integer with $\lambda_r < \lambda_k - 1$ or ($\lambda_r = \lambda_k - 1$ and $\mu_r < \mu_k$). Such a $r$ always exists, since we allow ourselves to extend the partitions with empty rows. If it did not exist, that would mean that $\lambda_k = 1$ and $\mu_k = 0$. By definition of $k$ and $i$, this would mean that $\lambda_j' = \lambda_j$ for $1 \leqslant j < i$, $\mu_j' = \mu_j$ for $1 \leqslant j < i$, that $1 = \lambda_k = \ldots \lambda_i > \lambda_i' = 0$. Also, we have $0 = \mu_k = \cdots = \mu_{i-1} = \mu_{i-1}'$. That means that the sizes of the bipartitions $(\lambda, \mu)$ and $(\lambda', \mu')$ are different, which is absurd. We proceed again with a case distinction.

i. Let us start with assuming that $\lambda_r = \lambda_k - 1$ and $\mu_r < \mu_k$. We define $\mu_j^* = \mu_j - 1$ and $\lambda_{j+1}^* = \lambda_{j+1} + 1$ for all $k \leq j \leq r-1$, while $\mu_j^* = \mu_j$ and $\lambda_{j+1}^* = \lambda_{j+1}$ for any other $j \in \mathbb{N}_0$. Since $\lambda_{k+1} < \lambda_k$ and $\mu_{r-1} = \mu_k > \mu_r$, $(\lambda^*, \mu^*)$ is a bipartition. Clearly $(\lambda^*, \mu^*) \trianglelefteq (\lambda, \mu)$. By construction we have

$$\lambda_t^* + \sum_{j=1}^{t-1}(\lambda_j^* + \mu_j^*) = \lambda_t + \sum_{j=1}^{t-1}(\lambda_j + \mu_j) \geq \lambda_t' + \sum_{j=1}^{t-1}(\lambda_j' + \mu_j')$$

for any $t \in \mathbb{N}$. Also, for $t < k$ or $t \geqslant r$, we have

$$\sum_{j=1}^{t}(\lambda_j + \mu_j) = \sum_{j=1}^{t}(\lambda_j^* + \mu_j^*)$$

Thus, it remains to show this inequality holds for $k \leqslant t < r$. The following inequalities follow from the definitions of $k$, $i$ and $r$:

$$\begin{array}{ll} \lambda_k^* = \lambda_k = \lambda_i \geqslant \lambda_i' + 1 \geqslant \lambda_k' + 1, & \\ \lambda_t^* = \lambda_t + 1 = \lambda_k - 1 + 1 \geqslant \lambda_k' + 1 \geqslant \lambda_t' + 1 & \text{for } k < t \leqslant r \\ \mu_t^* + 1 = \mu_t = \mu_k = \mu_{i-1} = \mu_{i-1}' \geqslant \mu_t' & \text{for } k \leqslant t < r. \end{array}$$

This shows that $\lambda^* \uplus \mu^* \trianglerighteq \lambda' \uplus \mu'$, and we obtain $(\lambda^*, \mu^*)$ from $(\lambda, \mu)$ by operation (4).

ii. Finally, we assume that $\lambda_r < \lambda_k - 1$. In particular, $\mu_{i-1} = \ldots = \mu_{r-1}$ and $\lambda_{r-1} = \ldots = \lambda_{k+1} = \lambda_k - 1 = \ldots = \lambda_i - 1$. We distinguish between two cases.

- First, assume that $\mu_{r-1} > \mu_r$. We define $\lambda^*_r = \lambda_r + 1 \leqslant \lambda_{r-1}$ and $\mu^*_{r-1} = \mu_{r-1} - 1 \geqslant \mu_r$, while $\mu^*_j = \mu_j$ and $\lambda^*_j = \lambda_j$ otherwise. Then $(\lambda^*, \mu^*)$ is a well-defined bipartition, and as usual, $(\lambda^*, \mu^*) \trianglelefteq (\lambda, \mu)$. Also, proving that $(\lambda', \mu') \trianglelefteq (\lambda^*, \mu^*)$ is straightforward, except maybe proving that
$$\sum_{j=1}^{r-1}(\lambda^*_j + \mu^*_j) \geqslant \sum_{j=1}^{r-1}(\lambda'_j + \mu'_j).$$
But as previously, we have:
$$\begin{array}{ll} \lambda^*_j = \lambda_j = \lambda'_j & \text{for } 1 \leqslant j < i \\ \mu^*_j = \mu_j = \mu'_j & \text{for } 1 \leqslant j < i \\ \lambda^*_j = \lambda_j = \lambda_i \geqslant \lambda'_i + 1 \geqslant \lambda'_j + 1 & \text{for } i \leqslant j \leqslant k \\ \lambda^*_j = \lambda_j = \lambda_k - 1 = \lambda_i - 1 \geqslant \lambda'_i \geqslant \lambda'_j & \text{for } k < j \leqslant r-1 \\ \mu^*_j = \mu_j = \mu_{i-1} = \mu'_{i-1} \geqslant \mu'_j & \text{for } i \leqslant j \leqslant r-2 \\ \mu^*_{r-1} = \mu_{i-1} - 1 = \mu'_{i-1} - 1 \geqslant \mu'_{r-1} - 1 & \end{array}$$
All together, this gives
$$\sum_{j=1}^{r-1}(\lambda^*_j + \mu^*_j) \geqslant \sum_{j=1}^{r-1}(\lambda'_j + \mu'_j) + (k-i) \geqslant \sum_{j=1}^{r-1}(\lambda'_j + \mu'_j)$$
as wanted. This also means that we obtain $(\lambda^*, \mu^*)$ from $(\lambda, \mu)$ by operation (4).
- We are left with the case $\mu_r = \mu_{r-1}$. By a previous remark, this means that $\mu_{i-1} = \ldots = \mu_r$. We set $\mu^* = \mu$ and $\lambda^*_k = \lambda_k - 1, \lambda^*_r = \lambda_r + 1$, and otherwise $\lambda^*_j = \lambda_j$. Then, by assumption $(\lambda^*, \mu^*)$ is a bipartition and $(\lambda^*, \mu^*) \trianglelefteq (\lambda, \mu)$. It is also straightforward to show that $(\lambda^*, \mu^*) \trianglerighteq (\lambda', \mu')$ except maybe the partial sums inequalities in rows $k$ to $r$. Since $\lambda^*_k + \lambda^*_r = \lambda_k + \lambda_r$, we only need to look at rows $k$ to $r-1$. To this purpose, we remark that:
$$\begin{array}{ll} \lambda^*_j = \lambda_j = \lambda'_j & \text{for } 1 \leqslant j < i \\ \mu^*_j = \mu_j = \mu'_j & \text{for every } j \\ \lambda^*_j = \lambda_j = \lambda_i \geqslant \lambda'_i + 1 \geqslant \lambda'_j + 1 & \text{for } i \leqslant j < k \\ \lambda^*_k = \lambda_k - 1 = \lambda_i - 1 \geqslant \lambda'_i \geqslant \lambda'_k & \\ \lambda^*_j = \lambda_j = \lambda_k - 1 = \lambda_i - 1 \geqslant \lambda'_i \geqslant \lambda'_j & \text{for } k < j \leqslant r-1 \end{array}$$
Then for any $k \leqslant j < r$, we have
$$\sum_{t=1}^{j}(\lambda^*_t + \mu^*_t) \geqslant \sum_{t=1}^{j}(\lambda'_t + \mu'_t) + (k-i) \geqslant \sum_{t=1}^{j}(\lambda'_t + \mu'_t)$$
and in the same way
$$\lambda^*_j + \sum_{t=1}^{j-1}(\lambda^*_t + \mu^*_t) \geqslant \lambda'_j + \sum_{t=1}^{j-1}(\lambda'_t + \mu'_t).$$

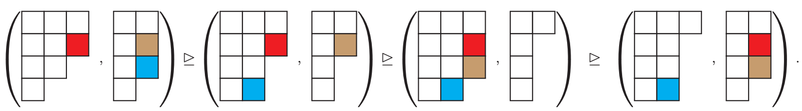
Thus we obtain $(\lambda^*, \mu^*)$ from $(\lambda, \mu)$ by operation (1).

(B) It remains to deal with the case $\lambda_i = \lambda'_i$, and $\mu_i > \mu'_i$. It can easily be deduced from the previous case, by noticing the following: let $\rho = (\mu_1, \mu_1, \mu_2, \ldots)$ and $\rho' = (\mu_1, \mu'_1, \mu'_2, \ldots)$, then $(\rho, \lambda)$ and $(\rho', \lambda')$ are bipartitions of $n + \mu_1$ such that $(\rho, \lambda) \trianglerighteq (\rho', \lambda')$. If $j$ is minimal such that $(\rho_j, \lambda_j) \neq (\rho'_j, \lambda'_j)$, then $j = i + 1$ and $\rho_j = \mu_i > \mu'_i = \rho'_j$. From what we have just seen, there exists $(\rho^*, \lambda^*)$, obtained from $(\rho, \lambda)$ by operations (1), (3) and (4), such that $(\rho, \lambda) \trianglerighteq (\rho^*, \lambda^*) \trianglerighteq (\rho', \lambda')$. It is then clear that $\rho^*_1 = \rho_1 = \rho'_1 = \mu_1$. Let $\mu^* = (\rho^*_2, \rho^*_3, \ldots)$. It is clear that $(\lambda^*, \rho^*)$ is a bipartition and obviously $(\lambda, \mu) \trianglerighteq (\lambda^*, \mu^*) \trianglerighteq (\lambda', \mu')$. Moreover if we obtained $(\mu^*, \rho^*)$ from $(\mu, \rho)$ by operations (1), (3) or (4) respectively, we obtain $(\lambda^*, \mu^*)$ from $(\lambda, \mu)$ by operations (2), (4) and (3) respectively.

■

It is in general not true that if $\lambda$ is a partition covering $\lambda'$, then $(\lambda, \mu)$ covers $(\lambda', \mu)$, as the following example shows:

**Example III.4.5.** Consider $(\lambda, \mu) = ((3,3,2,1),(2,2,2,1))$ and $(\lambda', \mu') = ((3,2,2,2),(2,2,2,1))$. Indeed, it is $\mu_1 = \mu_2 = \mu_3 > \mu_4$. Thus, there exist bipartitions which lie in between.



Since the poset of partitions for the standard order is a lattice, it is natural to ask whether this holds for our order on bipartitions. However, this is not the case already for $n = 4$. Consider $a = ((2),(1,1))$ and $b = ((2,2),\emptyset)$. Now take $c = ((2,1,1),\emptyset)$. It is covered by both $a$ and $b$, so if $a$ and $b$ have a meet, that is a greatest lower bound, it has to be $c$. However, for $d = (\emptyset,(2,2))$, $d < a$ and $d < b$, but $c$ and $d$ are not comparable. Similarly, one can ask if the poset $(\mathrm{BP}_n, \trianglelefteq)$ is graded,i.e., any maximal chain has equal length. However, already for $n = 3$ the poset is non-graded since there exist maximal chains of length 6 and 7.

## III.5 The posets of Specht ideals and varieties

In this section, we state and prove our main theorem:

**Theorem III.5.1.** *Let $(\lambda, \mu)$ and $(\vartheta, \omega)$ be bipartitions of $n$. Let $I_{(\lambda,\mu)}, I_{(\vartheta,\omega)}$ denote their associated Specht ideals and $V_{(\lambda,\mu)}, V_{(\vartheta,\omega)}$ their associated Specht varieties. Then, the following assertions are equivalent:*

III.5.1.1. *The bipartition $(\lambda, \mu)$ bidominates $(\vartheta, \omega)$, i.e. $(\lambda, \mu) \trianglerighteq (\vartheta, \omega)$;*

III.5.1.2. *The $\mathcal{B}_n$-Specht ideal $I_{(\lambda,\mu)}$ contains the $\mathcal{B}_n$-Specht ideal $I_{(\vartheta,\omega)}$, i.e. $I_{(\lambda,\mu)} \supset I_{(\vartheta,\omega)}$;*

III.5.1.3. *The $\mathcal{B}_n$-Specht variety $V_{(\lambda,\mu)}$ is contained in the $\mathcal{B}_n$-Specht variety $V_{(\vartheta,\omega)}$, i.e. $V_{(\lambda,\mu)} \subset V_{(\vartheta,\omega)}$.*

We start with the first implication, namely that a dominance of bipartitions implies the containment of the corresponding $\mathcal{B}_n$-Specht ideals.

**Proposition III.5.2.** *Let $(\lambda,\mu),(\lambda',\mu') \in \mathrm{BP}_n$ be bipartitions of $n$ and let $(\lambda',\mu') \trianglelefteq (\lambda,\mu)$. Then, $I_{(\lambda',\mu')} \subset I_{(\lambda,\mu)}$.*

*Proof.* It is sufficient to prove the theorem in the four covering cases in III.4.3.

In cases (1) and (2), we have in particular ($\lambda' \trianglelefteq \lambda$ and $\mu' = \mu$) or ($\mu' \trianglelefteq \mu$ and $\lambda' = \lambda$), and the result follows from the proof of ([MRV21, Theorem 1]), combined with the definition of $\mathcal{B}_n$-Specht polynomials.

Now, we consider case (3). In this case, to go from $(\lambda',\mu')$ to $(\lambda,\mu)$, we remove a number $a$ of boxes from a column $U_1$ in $\mu'$, that will be added to a column $U_2$ in $\lambda'$. We can restrict our attention to these two columns. Let $b = \mid U_2 \mid$, we then have $\mid U_1 \mid = a + \mid U_2 \mid = a+b$. Let $A = \{1,\dots,a\}$, $B_1 = \{a+1,\dots,a+b\}$ and $B_2 = \{a+b+1,\dots,a+2b\}$. Up to permutation, it suffices to show that the polynomial

$$P(X) = \Delta_{B_2}(X^2)\Delta_{A\cup B_1}(X^2) \prod_{i\in A\cup B_1} X_i$$

is in the ideal generated by polynomials of the form $\Delta_S(X^2)\Delta_{\overline{S}}(X^2)\prod_{i\in\overline{S}} X_i$, where $\{1,\dots,a+2b\}$ is the disjoint union of $S$ and $\overline{S}$, and $\mid S \mid = a+b$. Let us consider

$$Q(X) = \Delta_{A\cup B_2}(X^2)\Delta_{B_1}(X^2) \prod_{i\in B_1} X_i,$$

which is a polynomial of the expected form, and

$$\begin{aligned}\tilde{Q}(X) &= Q(X)\prod_{i\in A} X_i \\ &= \Delta_{A\cup B_2}(X^2)\Delta_{B_1}(X^2) \prod_{i\in A\cup B_1} X_i,\end{aligned}$$

Note that we have:

$$\deg(P) = b(b-1) + (a+b)(a+b-1) + a + b = 2b^2 + a^2 + b(2a-1)$$

and

$$\deg(Q) = (a+b)(a+b-1) + b(b-1) + b = 2b^2 + a^2 + b(2a-1) - a$$

so that $P$ and $\tilde{Q}$ have the same degree. We are going to show that $P$ is a combination of $\epsilon(\sigma)\sigma\cdot\tilde{Q}$ for $\sigma$'s in $G = \mathcal{S}_{A\cup B_1}$. Note that we can rewrite

$$\Delta_{A\cup B_2}(X^2) = \Delta_A(X^2)\Delta_{B_2}(X^2)\prod_{i\in A} R(X_i^2)$$

where

$$R(y) = \prod_{j \in B_2} (y - X_j^2).$$

Then, since $\prod_{i \in A \cup B_1} X_i$ and $\Delta_{B_2}(X^2)$ are invariant by $G$, we can factor them out and focus on the remaining terms, then looking at

$$P^*(X) = \Delta_{A \cup B_1}(X^2)$$

and

$$Q^*(X) = \Delta_A(X^2)\Delta_{B_1}(X^2) \prod_{i \in A} R(X_i^2).$$

Also consider the subgroup $H = \mathcal{S}_A \times \mathcal{S}_{B_1}$ of $G$. Then, for $\tau_1 \in \mathcal{S}_A$, $\tau_2 \in \mathcal{S}_{B_1}$, we have $\tau_1\tau_2(\Delta_A(X^2)) = \epsilon(\tau_1)(\Delta_A(X^2))$ and $\tau_1\tau_2(\Delta_{B_1}(X^2)) = \epsilon(\tau_2)(\Delta_{B_1}(X^2))$, and because $\prod_{i \in A} R(X_i^2)$ is $H$-invariant, we get

$$\epsilon(\tau_1\tau_2)\tau_1\tau_2 Q^* = Q^*,$$

allowing us to consider the sum

$$\overline{Q} = \sum_{\sigma \in G/H} \epsilon(\sigma)\sigma Q^*,$$

and we claim that $P^* = \overline{Q}$.

First, we show that $P^*$ divides $\overline{Q}$, namely that for every $i \neq j \in A \cup B_1$, $X_i^2 - X_j^2$ divides $\overline{Q}$. Since for every $\sigma \in G$, $\sigma\overline{Q} = \pm\overline{Q}$, and $G$ acts transitively on pairs $(i, j)$, it is enough to check that $X_1^2 - X_2^2$ divides $\overline{Q}$. We hence have to show that $\overline{Q}$ vanishes when imposing $X_1^2 = X_2^2$. To see this, first observe that the terms in the sum are in correspondence with set partitions $K \cup \overline{K}$ of $A \cup B_1$, where $\mid K \mid= a$. Indeed, up to permutation by elements of $H$, we only need to choose where to send the subset $A = \{1, \ldots, a\}$. Now, if $\sigma$ sends 1 and 2 in the same subset, the corresponding Vandermonde determinant in $\epsilon(\sigma)\sigma Q^*$ vanishes whenever $X_1^2 = X_2^2$. We then only need to focus on partitions where 1 and 2 are not in the same subset. There are two kinds of such partitions: those with $1 \in K$ and $2 \in \overline{K}$, and those $2 \in K$ and $1 \in \overline{K}$. The transposition $(12)$ naturally induces a bijection between these sets of partitions. If $\sigma \in G$ is a representative for a partition of the first kind, then $(12)\sigma$ is a representative for the corresponding partition of the second kind. When $X_1^2 = X_2^2$, we have $(12)\sigma Q^*(x) = \sigma Q^*(x)$, and because $\epsilon((12)\sigma) = -\epsilon(\sigma)$, the two corresponding terms cancel out.

Then, we need to check that $P^*$ and $\overline{Q}$ have the same leading term with respect to the lexicographical ordering. The leading term of $P^*$ is $X_1^{2(a+b-1)}X_2^{2(a+b-2)} \cdots X_{a+b-1}^2$. Then, for $\sigma \in G$ sending $\{1, \ldots, k\}$ onto $K$, the partial degree of $\sigma Q^*$ in $X_1$ is

$$\begin{cases} 2(a-1) + 2b & \text{if } 1 \in K \\ 2(b-1) & \text{if } 1 \notin K \end{cases}$$

and therefore $\sigma Q^*$ can give a contribution to the leading term of $\overline{Q}$ only if $1 \in K$. By the same argument, 2 has to be in $K$, and in the end, $K = \{1, \ldots, a\}$: Indeed, assume there is a minimal $i \leqslant a$ with $i \notin K$. Then, the leading term of $\sigma Q^*$ is of the form $X_1^{2(a+b-1)} X_2^{2(a+b-2)} \ldots X_{j-1}^{2(a+b-j+1)} X_j^{2(b-1)} m$ where $m$ is a monomial in the variables $X_{j+1}, \ldots, X_{a+b}$. Since, $j \leq a$, then $2(b-1) < 2(a+b-j)$, and therefore the leading monomial of $\sigma Q^*$ is strictly lower than that of $Q^*$. Thus, the leading term of $\overline{Q}$ is exactly the leading term of $Q^*$, which is $X_1^{2(a+b-1)} X_2^{2(a+b-2)} \cdots X_{a+b-1}^2$, as expected. This concludes the covering case (3).

The proof for the covering case (4) is very similar. In this situation, to go from $(\lambda', \mu')$ to $(\lambda, \mu)$, we remove $a$ boxes from a column $U_1$ in $\lambda'$, before adding them to a column $U_2$ in $\mu'$, with $\mid U_2 \mid = b$ and $\mid U_1 \mid = a+b+1$. We can apply the previous argument, where this time $A = \{1, \ldots, a\}$, $B_1 = \{a+1, \ldots, a+b+1\}$, $B_2 = \{a+b+2, \ldots, a+2b+1\}$,

$$P = \Delta_{A \cup B_1}(X^2) \Delta_{B_2}(X^2) \prod_{i \in B_2} X_i$$

$$Q = \Delta_{B_1}(X^2) \Delta_{A \cup B_2}(X^2) \prod_{i \in A \cup B_2} X_i$$

and

$$\begin{aligned} \tilde{Q} &= \Big(\prod_{j \in A} X_j\Big) Q \\ &= \Delta_{B_1}(X^2) \Delta_{A \cup B_2}(X^2) \prod_{i \in A} X_i^2 \prod_{i \in B_2} X_i. \end{aligned}$$

■

The second implication of III.5.1 is clear, it remains to prove that (3) implies (1):

**Proposition III.5.3.** *Let $(\lambda, \mu), (\vartheta, \omega)$ be bipartitions and $V_{(\lambda,\mu)} \subset V_{(\vartheta,\omega)}$. Then, $(\lambda, \mu) \trianglerighteq (\vartheta, \omega)$.*

To prove this implication, we will consider two types of points in $\mathbb{K}^n$:

**Lemma III.5.4.** *Let $(\vartheta, \omega) \in \mathrm{BP}_n$ and $\Lambda = \vartheta \uplus \omega$ be a partition of $n$. Consider the point*

$$z = (\underbrace{a_1, \ldots, a_1}_{\Lambda_1}, \underbrace{a_2, \ldots, a_2}_{\Lambda_2}, \ldots, \underbrace{a_m, \ldots, a_m}_{\Lambda_m})$$

*with $a_i^2 \neq a_j^2$ if $i \neq j$ and $a_i \neq 0$ if $i \leqslant \mathrm{len}(\omega)$.*

i) $z \notin V_{(\vartheta,\omega)}$.

ii) *If $(\lambda, \mu) \in \mathrm{BP}_n$ is a bipartition such that $z \notin V_{(\lambda,\mu)}$, then $\lambda \uplus \mu \trianglerighteq \Lambda$.*

*Proof.*

i) Let $(T, S)$ be the generalized bitableau of shape $(\vartheta, \omega)$ which has the filling

$$\left( \begin{array}{llll} \boxed{a_1} & \boxed{a_1} & \cdots & \boxed{a_1} \\ \boxed{a_2} & \boxed{a_2} & \cdots & \\ \vdots & & & \\ \vdots & & & \\ \boxed{a_h} & \cdots & & \end{array} \quad , \quad \begin{array}{lllll} \boxed{a_1} & \boxed{a_1} & \cdots & \cdots & \boxed{a_1} \\ \boxed{a_2} & \boxed{a_2} & \cdots & \cdots & \\ \vdots & & & & \\ \boxed{a_l} & \cdots & & & \\ & & & & \end{array} \right).$$

i.e., the $i$-th row of both $T$ and $S$ contains only $a_i$'s. The assumption $a_i \neq 0$ for $i \leqslant \operatorname{len}(\omega)$ ensures that $S$ contains no 0 entry, and by construction the squares of column entries are pairwise different. Thus, $z \in V^c_{(\vartheta,\omega)}$.

ii) By assumption, there is a bitableau $(T, S)$ of shape $(\lambda, \mu)$ such that $\operatorname{sp}_{(T,S)}(z) \neq 0$. Then, according to Lemma III.3.7, there is a tableau $U = T \uplus S$ of shape $\lambda \uplus \mu$ such that $\operatorname{sp}_U(z^2) \neq 0$. Therefore $z^2$ does not belong to the $\mathcal{S}_n$-Specht variety $V_{\lambda \uplus \mu}$, and ([MRV21, Prop 1.ii)]) gives

$$\Lambda = \Lambda(z) = \Lambda(z^2) \trianglelefteq \lambda \uplus \mu,$$

which proves the Lemma.

■

**Lemma III.5.5.** *Let $(\vartheta, \omega) \in \mathrm{BP}_n$ be a bipartition of $n$. Consider the point*

$$z = (\underbrace{0, \dots, 0}_{\vartheta_1}, \underbrace{a_1, \dots, a_1}_{\omega_1 + \vartheta_2}, \underbrace{a_2 \dots, a_2}_{\omega_2 + \vartheta_3}, \dots, \underbrace{a_m, \dots, a_m}_{\omega_m + \vartheta_{m+1}})$$

*with $a_i^2 \neq a_j^2$ if $i \neq j$ and $a_i \neq 0$. Then:*

i) $z \notin V_{(\vartheta,\omega)}$.

ii) *If $(\lambda, \mu) \in \mathrm{BP}_n$ is a bipartition such that $z \notin V_{(\lambda,\mu)}$, then*

$$\sum_{j=1}^{k-1} (\lambda_j + \mu_j) + \lambda_k \geq \sum_{j=1}^{k-1} (\vartheta_j + \omega_j) + \vartheta_k$$

*for any integer $k \geq 1$.*

*Proof.*

i) Let $(T, S)$ be the generalized bitableau of shape $(\vartheta, \omega)$ which has the filling

$$\left( \begin{array}{llll} \boxed{0} & \boxed{0} & \cdots & \boxed{0} \\ \boxed{a_1} & \boxed{a_1} & \cdots & \\ \vdots & & & \\ \boxed{a_h} & \cdots & & \\ & & & \end{array} \quad , \quad \begin{array}{llll} \boxed{a_1} & \boxed{a_1} & \cdots & \boxed{a_1} \\ \boxed{a_2} & \boxed{a_2} & \cdots & \\ \vdots & & & \\ \vdots & & & \\ \boxed{a_l} & \cdots & & \end{array} \right)$$

i.e., $\vartheta_i$ contains only $a_{i-1}$'s and $\omega_i$ contains only $a_i$'s, where $a_0 = 0$. We observe that no entry in $S$ equals 0 and the squares of column entries are pairwise different. Thus, we have $z \in V^c_{(\vartheta,\omega)}$.

ii) By assumption, there exists a bitableau $(T, S)$ of shape $(\lambda, \mu)$ such that

$$0 \neq \mathrm{sp}_{(T,S)}(z) = \mathrm{sp}_T(z^2)\,\mathrm{sp}_S(z^2) \cdot \prod_{j \in S} z_j.$$

Let $(T^*, S^*)$ be the generalized bitableau obtained from $(T, S)$ by replacing $i$ with $a_i$ in any box. This means that the zeros of $z$ are written in $T^*$ and no column in $T^*$ or $S^*$ contains entries with equal squares. Since permutation of the column entries can only change the sign of $\mathrm{sp}_{(T,S)}(z)$, we can assume that the entries in every column in $(T^*, S^*)$ are sorted increasingly by the indices of the $a_i$'s from above to below, and with $a_0 = 0$.
We obtain that all the 0's must be written in the first row of $T^*$ which implies $\lambda_1 \geq \vartheta_1$. Now, for an integer $k \geq 1$ the $a_k$'s in $(T^*, S^*)$ must be written in different columns in the generalized bitableau $(T^*, S^*)$. Since the entries in $(T^*, S^*)$ are written with increasing indices in each column from the top to the bottom, we know that the $a_j$'s with $0 \leqslant j \leqslant k$ must be written within the first $k$ rows of $S$ and the first $k+1$-rows in $T$. Thus, by the pigeon hole principle, we have

$$\sum_{j=1}^{k} (\lambda_j + \mu_j) + \lambda_{k+1} \geq \sum_{j=1}^{k} (\vartheta_j + \omega_j) + \vartheta_{k+1}.$$

■

Now, we can prove Proposition III.5.3:

*Proof of Proposition III.5.3.* The assumption is equivalent to $V^c_{(\vartheta,\omega)} \subset V^c_{(\lambda,\mu)}$. We have to prove that $\lambda \uplus \mu \trianglerighteq \vartheta \uplus \omega$ and that $\sum_{j=1}^{k-1} (\lambda_j + \mu_j) + \lambda_k \geq \sum_{j=1}^{k-1} (\vartheta_j + \omega_j) + \vartheta_k$ for every integer $k \geq 1$.

For the first claim, consider the point

$$z = (\underbrace{a_1, \dots, a_1}_{\vartheta_1+\omega_1}, \underbrace{a_2, \dots, a_2}_{\vartheta_2+\omega_2}, \dots, \underbrace{a_m, \dots, a_m}_{\vartheta_m+\omega_m})$$

with $a_i^2 \neq a_j^2$ for $i \neq j$, and $a_i \neq 0$ if $i \leq \mathrm{len}(\omega)$. According to i) in Lemma III.5.4, $z \in V^c_{(\vartheta,\omega)}$. By assumption, we then have $z \in V^c_{(\lambda,\mu)}$, and ii) in Lemma III.5.4 gives

$$\lambda \uplus \mu \trianglerighteq \Lambda(z) = \vartheta \uplus \omega.$$

For the second claim, consider the point

$$z = (\underbrace{0, \dots, 0}_{\vartheta_1}, \underbrace{a_1, \dots, a_1}_{\omega_1+\vartheta_2}, \underbrace{a_2 \dots, a_2}_{\omega_2+\vartheta_3}, \dots, \underbrace{a_m, \dots, a_m}_{\omega_m+\vartheta_{m+1}})$$

with $a_i^2 \neq a_j^2$ for $i \neq j$, and $a_i \neq 0$. According to i) in Lemma III.5.5, $z \in V^c_{(\vartheta,\omega)}$. By assumption, we then have $z \in V^c_{(\lambda,\mu)}$, and ii) in Lemma III.5.5 gives

$$\sum_{j=1}^{k-1} (\lambda_j + \mu_j) + \lambda_k \geq \sum_{j=1}^{k-1} (\vartheta_j + \omega_j) + \vartheta_k$$

for any integer $k \geq 1$. ■

## III.6 Orbit types

In this section we define orbit types of elements in $\mathbb{K}^n$ with respect to the action of the hyperoctahedral group. Compared with the $\mathcal{S}_n$-orbit types, they allow a finer set decomposition of $\mathbb{K}^n$ since one distinguishes whether coordinates are 0 or not. This leads to a set partition of the $\mathcal{B}_n$-Specht varieties based on the combinatorics of the poset $(\mathrm{BP}_n, \trianglelefteq)$.

Recall that if $z = (a_1, \dots, a_n) \in \mathbb{K}^n$, then the $\mathcal{S}_n$-orbit type of $z$ is the unique partition $\Lambda(z) = (\Lambda_1, \dots, \Lambda_l) \vdash n$ such that $\mathrm{Stab}_{\mathcal{S}_n}(z) \simeq \mathbb{Z}/\Lambda_1\mathbb{Z} \times \dots \times \mathbb{Z}/\Lambda_l\mathbb{Z}$, or equivalently, there exists $b_1, \dots, b_l$ pairwise distinct such that $z \in \mathcal{S}_n \cdot (\underbrace{b_1, \dots, b_1}_{\Lambda_1}, \dots, \underbrace{b_l, \dots, b_l}_{\Lambda_l})$.

**Definition III.6.1.** Let $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ be a partition and $t \in \mathbb{N}_0$. Let $j = \min\{i : \lambda_i < t\}$ with the convention that $j = m+1$ if $t = 0$. Then the *$t$-cut* of $\lambda$ is the bipartition $(\rho, \sigma)$ defined as $\rho = (t, \cdots, t, \lambda_j, \cdots, \lambda_m)$ and $\sigma = (\lambda_1 - t, \cdots, \lambda_{j-1} - t)$. We denote it by $\mathrm{cut}(\lambda, t)$.

We have $\mathrm{cut}(\lambda, 0) = (\emptyset, \lambda)$ while $\mathrm{cut}(\lambda, t) = (\lambda, \emptyset)$ for any $t \geqslant \lambda_1$. We observe that if $\mathrm{cut}(\lambda, t) = (\sigma, \rho)$, then $\sigma \uplus \rho = \lambda$.

We are now ready to define the $\mathcal{B}_n$-orbit type of a point in $\mathbb{K}^n$ and the notion of $\mathcal{B}_n$-orbit set:

**Definition III.6.2.** Let $z \in \mathbb{K}^n$. The $\mathcal{B}_n$*-orbit type* of $z$ is

$$\Omega(z) = \operatorname{cut}(\Lambda(z^2), t_z)$$

where $t_z$ is the number of 0 coordinates of $z$.

For $(\lambda, \mu) \in \mathrm{BP}_n$, the $\mathcal{B}_n$*-orbit set* associated to $(\lambda, \mu)$ is then

$$H_{(\lambda,\mu)} = \{z \in \mathbb{K}^n : \ \Omega(z) = (\lambda, \mu)\}.$$

We observe that $\mathcal{B}_n$-orbits sets might be empty. The non-empty $\mathcal{B}_n$-orbits sets correspond to bipartitions $(\lambda, \mu)$ such that $\lambda_1 = \ldots = \lambda_{\mathrm{len}(\mu)+1}$. Moreover, if $(\lambda, \mu)$ is such that $H_{(\lambda,\mu)} \neq \emptyset$, then any point $z \in H_{(\lambda,\mu)}$ is of the following form: let $m = \mathrm{len}(\mu)$. Then there exists non-zero elements $a_1, \ldots, a_l \in \mathbb{K}$, with distinct squares such that

$$z = \sigma \cdot (\underbrace{a_1, \ldots, a_1}_{\lambda_1+\mu_1}, \ldots, \underbrace{a_m, \ldots, a_m}_{\lambda_1+\mu_m}, \underbrace{0, \ldots, 0}_{\lambda_1}, \underbrace{a_{m+1}, \ldots, a_{m+1}}_{\lambda_{m+2}}, \ldots, \underbrace{a_l, \ldots, a_l}_{\lambda_{l+1}})$$

for some $\sigma \in \mathcal{B}_n$. It is straightforward that $l = m$ if $\mathrm{len}(\lambda) \leqslant m$ (which implies $\lambda = \emptyset$) and $l = \mathrm{len}(\lambda) - 1$ if $\mathrm{len}(\lambda) > m$.

**Example III.6.3.** We present the orbit types of $\mathbb{K}^3$. Let $a, b, c \in \mathbb{K}^*$ be such that they have distinct squares.

| $z$ | $\Omega(z)$ |
|---|---|
| $(0,0,0)$ | $((3), \emptyset)$ |
| $(\pm a, 0, 0)$ | $((2,1), \emptyset)$ |
| $(\pm a, \pm b, 0)$ | $((1,1,1), \emptyset)$ |
| $(\pm a, \pm a, 0)$ | $((1,1), (1))$ |
| $(\pm a, \pm a, \pm a)$ | $(\emptyset, (3))$ |
| $(\pm a, \pm a, \pm b)$ | $(\emptyset, (2,1))$ |
| $(\pm a, \pm b, \pm c)$ | $(\emptyset, (1,1,1))$ |

The remaining bipartitions $((2),(1)), ((1),(2)), ((1),(1,1))$ of 3 have an empty orbit set.

The following proposition follows from the previous definitions and comments:

**Proposition III.6.4.** *Let $z \in \mathbb{K}^n$ and $(\lambda, \mu) = \Omega(z)$. Then $\Lambda(z^2) = \lambda \uplus \mu$. Moreover, the $\mathcal{B}_n$-orbit sets define a set partition of $\mathbb{K}^n$, namely $\mathbb{K}^n = \biguplus_{(\lambda,\mu)\in \mathrm{BP}_n} H_{(\lambda,\mu)}$.*

**Proposition III.6.5.** *Let $z \in \mathbb{K}^n$, and $(\lambda, \mu) \in \mathrm{BP}_n$. Then:*

III.6.5.1. $z \notin V_{\Omega(z)}$,

III.6.5.2. $z \notin V_{(\lambda,\mu)} \Rightarrow (\lambda, \mu) \trianglerighteq \Omega(z)$.

*Proof.* Let $(\vartheta, \omega) = \Omega(z)$ and $m = \mathrm{len}(\omega)$. Then there exists $\sigma \in \mathcal{B}_n$ and $a_1, \ldots, a_l \in \mathbb{K}^*$ with distinct squares such that $z = \sigma z'$ with

$$z' = (\underbrace{a_1, \ldots, a_1}_{\vartheta_1+\omega_1}, \ldots, \underbrace{a_m, \ldots, a_m}_{\vartheta_1+\omega_m}, \underbrace{0, \ldots, 0}_{\lambda_1}, \underbrace{a_{m+1}, \ldots, a_{m+1}}_{\vartheta_{m+2}}, \ldots, \underbrace{a_l, \ldots, a_l}_{\vartheta_{l+1}}).$$

Since the Specht varieties are invariant under the action of $\mathcal{B}_n$, we can assume that $z = z'$. With this shape, we can apply Lemma III.5.4 to $z$ which gives immediately (1), and partly (2): if $z \notin V(\lambda, \mu)$, then $\lambda \uplus \mu \trianglerighteq \vartheta \uplus \omega$. It remains to prove that if $z \notin V(\lambda, \mu)$, then $\sum_{j=1}^{k-1}(\lambda_j + \mu_j) + \lambda_k \geq \sum_{j=1}^{k-1}(\vartheta_j + \omega_j) + \vartheta_k$ for any integer $k \geq 1$. To do so, it is enough to observe that since $\vartheta_1 = \ldots = \vartheta_{m+1}$, the point

$$z'' = (\underbrace{0, \ldots, 0}_{\vartheta_1}, \underbrace{a_1, \ldots, a_1}_{\omega_1+\vartheta_2}, \underbrace{a_2 \ldots, a_2}_{\omega_2+\vartheta_3}, \ldots, \underbrace{a_m, \ldots, a_m}_{\omega_m+\vartheta_{m+1}})$$

is in the same orbit, and we can apply Lemma III.5.5 to conclude the proof. ∎

As a consequence of our previous results, we get a decomposition of $\mathcal{B}_n$-Specht varieties in terms of orbit sets:

**Theorem III.6.6.**

$$V_{(\lambda,\mu)} = \left( \bigcup_{(\vartheta,\omega)\in \mathrm{BP}_n, (\vartheta,\omega)\trianglelefteq(\lambda,\mu)} H_{(\vartheta,\omega)} \right)^c = \bigcup_{(\vartheta,\omega)\in \mathrm{BP}_n, (\vartheta,\omega)\ntrianglelefteq(\lambda,\mu)} H_{(\vartheta,\omega)}.$$

*Proof.* The collection $\{H_{(\vartheta,\omega)} : (\vartheta, \omega) \in \mathrm{BP}_n\}$ defines a set partition of $\mathbb{K}^n$ by definition, which explains the second equality.
In order to prove the first one, we first assume that $z \notin V_{(\lambda,\mu)}$. By part (2) in Proposition III.6.5 we obtain that $(\lambda, \mu) \trianglerighteq \Omega(z)$. Thus $z \in \bigcup_{(\vartheta,\omega)\in \mathrm{BP}_n, (\vartheta,\omega)\trianglelefteq(\lambda,\mu)} H_{(\vartheta,\omega)}$.
Conversely, let $z \in \bigcup_{(\vartheta,\omega)\trianglelefteq(\lambda,\mu)} H_{(\vartheta,\omega)}$. In other words, $\Omega(z) \trianglelefteq (\lambda, \mu)$. Then, part (1) in Proposition III.6.5 implies $z \notin V_{\Omega(z)}$. On the other hand, by Theorem III.5.1, $V_{(\lambda,\mu)} \subset V_{\Omega(z)}$. Therefore $z \notin V_{(\lambda,\mu)}$. ∎

**Example III.6.7.** We calculate the Specht variety corresponding with the bipartition $((1, 1), (2))$ using Theorem III.6.6. The bipartitions $(\vartheta, \omega)$ encoding non-empty orbit sets such that $(\vartheta, \omega) \ntrianglelefteq ((1, 1), (2))$ are the bipartitions in

$$\Omega = \{((2, 2), \emptyset), ((2, 1, 1), \emptyset), (\emptyset, (4)), ((3, 1), \emptyset), ((4), \emptyset)\} \subset \mathrm{BP}_4 .$$

Then,

$$\begin{aligned} V_{((1,1),(2))} &= \bigcup_{(\lambda,\mu)\in\Omega} H_{(\lambda,\mu)} \\ &= H_{((2,2),\emptyset)} \cup H_{((2,1,1),\emptyset)} \cup H_{(\emptyset,(4))} \cup H_{((3,1),\emptyset)} \cup H_{((4),\emptyset)}, \end{aligned}$$

which means

$$V_{((1,1),(2))} = \mathcal{B}_4 \cdot \{(0, 0, a, a), (0, 0, a, b), (a, a, a, a), (0, 0, 0, a), (0, 0, 0, 0) : a, b \in \mathbb{K}_{>0}\}.$$

One might look for a more natural orbit-type, only involving the number of zeroes of a point, and the $\mathcal{S}_n$-orbit-type of the remaining non-zero squared coordinates. Indeed, our previous decomposition can be reformulated in such a way, and it can be obtained either using $S_n$-invariance and results of [MRV21], or as a consequence of our previous results on bipartitions. We just briefly describe here the latter approach because such a point of view, even if it gives a natural decomposition, does not give information on inclusions of $\mathcal{B}_n$-Specht varieties, which will be needed for our applications in the next section.

If $z = (a_1, \ldots, a_n) \in \mathbb{K}^n$ is a point, then

$$\mathrm{Stab}_{\mathcal{S}_n}(a_1^2, \ldots, a_n^2) \simeq \mathcal{S}_{\Lambda_1} \times \ldots \times \mathcal{S}_{\Lambda_l} \times \mathcal{S}_t$$

where $\Lambda_1 \geqslant \ldots \geqslant \Lambda_l$ and $t$ is the number of zero coordinates of $z$. We could have defined the orbit type of $z$ as

$$\Lambda(z) = (t, (\Lambda_1, \ldots, \Lambda_l)).$$

Then, there is a bijection $\varphi$ between the set of pairs $(t, \Lambda)$ where $n \geqslant t \geqslant 0$ and $\Lambda \vdash n - t$ and the set of bipartitions $(\lambda, \mu)$ such that $H_{(\lambda,\mu)} \neq \emptyset$ given by

$$(t, \Lambda) \mapsto \mathrm{cut}(\overline{(\Lambda_1, \Lambda_2, \ldots, \Lambda_l, t)}, t)$$

where $\overline{(\Lambda_1, \Lambda_2, \ldots, \Lambda_l, t)}$ denotes the partition obtained by rearranging $(\Lambda_1, \Lambda_2, \ldots, \Lambda_l, t)$ in non increasing order.

Now, for $t \leqslant 0$ and $\Lambda \vdash n-t$, if we denote by $G_{t,\Lambda} = \{z \in \mathbb{K}^n;\ \Lambda(z) = (t, \Lambda)\}$, we have by construction

$$G_{t,\Lambda} = H_{\phi(t,\Lambda)}.$$

Moreover, $\phi$ preserves the orders in the following sense: for $t$ fixed, $\phi(t, \Lambda) \trianglelefteq \phi(t, \Lambda')$ in our poset of bipartitions if and only if $\Lambda \trianglelefteq \Lambda'$ in the poset of partitions. Also, it is obvious that if $t > \lambda_1$, then $z \in V_{(\lambda,\mu)}$.

As a consequence, our decomposition in Theorem III.6.6 becomes in this context

$$\left(V_{(\lambda,\mu)}\right)^c = \bigcup_{t=1}^{\lambda_1} \bigcup_{\substack{\Lambda, \\ \phi(t,\Lambda) \trianglelefteq (\lambda,\mu)}} G_{t,\Lambda}.$$

Actually, if one fixes $0 \leqslant t \leqslant \lambda_1$, one can prove that

$$\phi(t, \Lambda) \trianglelefteq (\lambda, \mu) \Leftrightarrow \Lambda \trianglelefteq \lambda^{(t)} \uplus \mu,$$

where $\lambda^{(t)}$ is defined as follows: if $s = \max\{i;\ \lambda_i \geqslant k\}$, then

$$\lambda^{(t)} = (\lambda_1, \ldots, \lambda_{s-1}, \lambda_s + \lambda_{s+1} - k, \lambda_{s+2}, \ldots, \lambda_l).$$

Finally, we can reformulate the decomposition as:

$$\left(V_{(\lambda,\mu)}\right)^c = \bigcup_{t=1}^{\lambda_1} \bigcup_{\Lambda \trianglelefteq \lambda^{(t)} \uplus \mu} G_{(t,\Lambda)}.$$

# III.7 Applications to $\mathcal{B}_n$-invariant ideals

## III.7.1 Specht ideals in $\mathcal{B}_n$-invariant ideals

The main result in the article [MRV21] studies the $\mathcal{S}_n$-Specht ideals contained in an ideal $I$ which is globally invariant under the action of $\mathcal{S}_n$. We give here the main ideas of this statement, before extending it to the case of $\mathcal{B}_n$.

For $P$ a polynomial in an $\mathcal{S}_n$-invariant ideal $I \subset \mathbb{K}[X_1, \ldots, X_n]$ of degree $d$, we denote by $\mathrm{wt}(P_d)$ the number of variables appearing in the highest component $P_d$ of $P$. Moreover, for a monomial $m$ of degree $d$ in $P$, the partial degrees of $m$ induce a partition $(k_1, \ldots, k_\ell)$, and under the assumption $\mathrm{wt}(P_d) + d \leq n$, we have in particular $\mathrm{wt}(P_d) + \ell \leq n$, and therefore we can define the partition

$$\mu(m) = (k_1 + 1, k_2 + 1, \ldots, k_\ell + 1, \underbrace{1, \ldots, 1}_{n-d-\ell})$$

of $n$. It is then proved ([MRV21, Theorem 4]) that for every monomial $m \in \mathrm{Mon}(P_d)$, the ideal $I$ contains every $\mathrm{sp}_T$ for which $\mathrm{sh}(T) \trianglelefteq \mu(m)^\perp$.

The proof works as follows: up to permutation of the variables, we may assume that

$$m = X_1^{k_1} X_2^{k_2} \cdots X_\ell^{k_\ell}$$

and since $\mathrm{wt}(P_d) + d \leq n$, there exists $d = k_1 + \ldots + k_\ell$ many variables in $\{X_1, \ldots X_n\}$ that do not appear in $P_d$. More precisely, we can take $I_1, \ldots, I_\ell$, disjoint subsets of $\{1, \ldots, n\}$ such that for any $1 \leq i \leq \ell$, there are $k_i$ elements in $I_i$, and none of them appears in $P_d$. Then, if for $1 \leq i \leq \ell$, $J_i = \{i\} \cup I_i$, we can prove

$$\Delta_{J_1} \cdots \Delta_{J_\ell} = \frac{k}{k_1! \cdots k_\ell!} \sum_{\sigma \in \mathrm{Sym}_{J_1} \times \cdots \times \mathrm{Sym}_{J_\ell}} \epsilon(\sigma) \sigma(\Delta_{I_1} \cdots \Delta_{I_\ell} P) \tag{III.3}$$

where $\Delta_I$ is the Vandermonde polynomial of the ordered set $I$ and $k \neq 0$.

Now, we want to generalize this result to $\mathcal{B}_n$-invariant ideals. First, we need to associate a bipartition to a given monomial:

**Definition III.7.1.** Let $m$ be a monomial in $\mathbb{K}[X_1, \ldots, X_n]$. There exist unique sets $I_1$ and $I_2$ such that we can write $m$ as

$$m = \prod_{i \in I_1} X_i^{2k_i} \prod_{i \in I_2} X_i^{2r_i} \prod_{i \in I_2} X_i$$

and $k_i \neq 0$. Denote $\ell = \mid I_1 \mid$, $d_1 = \sum_{i \in I_1} k_i$, $s = \mid I_2 \mid$, and $d_2 = \sum_{i \in I_2} r_i$. The sets $\{k_i, i \in I_1\}$ and $\{r_i, i \in I_2\}$ respectively induce partitions $(\lambda_1, \ldots, \lambda_\ell)$ of $d_1$ and $(\mu_1, \ldots, \mu_s)$ of $d_2$. If moreover we assume that $\ell + s + d_1 + d_2 \leq n$, we can define a bipartition $\Gamma(m)$ of $n$ by

$$\Gamma(m) = (\tilde{\lambda}, \tilde{\mu}) = ((\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_\ell + 1, \underbrace{1, \ldots, 1}_{n-(\ell+s+d_1+d_2)}), (\mu_1 + 1, \mu_2 + 1, \ldots, \mu_s + 1)).$$

Finally, we define

$$\Gamma^*(m) = (\tilde{\lambda}^\perp, \tilde{\mu}^\perp) = (n - (s + d_1 + d_2), \ell, \tilde{\lambda}_3^\perp, \ldots), (s, \tilde{\mu}_2^\perp, \ldots)),$$

which is a bipartition of $n$ as well.

With this notion, we get, for $\mathcal{B}_n$-invariant ideals:

**Theorem III.7.2.** *Let $I \subset \mathbb{K}[X_1, \ldots, X_n]$ be a $\mathcal{B}_n$-invariant ideal, and let $P \in I$. Assume that $m$ is a monomial in the homogeneous component of highest degree of $P$. Using the notation of Definition III.7.1, assume that $\mathrm{wt}(P_d) + d_1 + d_2 \leqslant n$. Then, we have the ideal inclusion*

$$I_{\Gamma^*(m)} \subset I.$$

*Proof.* Up to permutation, we may assume that

$$m = \prod_{i=1}^{\ell} X_i^{2k_i} \prod_{i=\ell+1}^{\ell+s} X_i^{2r_i} \prod_{i=\ell+1}^{\ell+s} X_i$$

and that the coefficient of $m$ in $P$ is 1. Let $\epsilon_i \in \mathcal{B}_n$ the map changing $X_i$ in $-X_i$. Then, the polynomial

$$\frac{P - \epsilon_i P}{2}$$

is a polynomial in $I$ whose terms are exactly the terms of $P$ having an odd degree in $X_i$, and therefore divisible by $X_i$. After applying this transformation for every $i \in \{\ell + 1, \ldots, \ell + s\}$, we may substitute $P$ with a polynomial of the form

$$\tilde{P}(X) \prod_{i=\ell+1}^{\ell+s} X_i,$$

containing $m$ in its leading term, and where every term in $\tilde{P}$ has even degree in $X_i$, for every $i \in \{\ell + 1, \ldots, \ell + s\}$. Further, for $i \notin \{\ell + 1, \ldots, \ell + s\}$, we can apply the transformation

$$\frac{\tilde{P}(X) + \epsilon_i \tilde{P}(X)}{2}$$

to get a polynomial which is still in $I$, but its terms are exactly the terms of $P$ having an even degree in $X_i$. In the end, we may assume that $P$ is of the form:

$$P(X) = Q(X^2) \prod_{i=\ell+1}^{\ell+s} X_i,$$

where $m$ is still a monomial of the leading term.

Now we can apply a strategy similar to the one described for $\mathcal{S}_n$-invariant ideals. Since $\ell + s + d_1 + d_2 \leq \mathrm{wt}(P_d) + d_1 + d_2 \leq n$, there exists $d_1 + d_2 = k_1 + \ldots + k_\ell + r_1 + \ldots + r_s$ many variables in $\{X_1, \ldots X_n\}$ that do not appear in $P_d$, and we can take $I_1, \ldots, I_\ell$, $I_{\ell+1}, \ldots, I_{\ell+s}$, disjoint subsets of $\{1, \ldots, n\}$ such that for any $1 \le i \le \ell$, there are $k_i$ elements in $I_i$, for any $\ell+1 \le i \le \ell+s$, there are $r_i$ elements in $I_i$, and none of them appears in $P_d$. Then, for $1 \le i \le \ell+s$, denote

$$J_i = \{i\} \cup I_i,$$

and $\tilde{\Delta}_J(X) = \Delta_J(X^2)$ the Vandermonde polynomial associated with the variables $X_i^2$ for $i \in J$. Consider

$$R(X) = P(X)\tilde{\Delta}_{I_1} \cdots \tilde{\Delta}_{I_{\ell+s}} \prod_{i \in I_{\ell+1} \times \cdots \times I_{\ell+s}} X_i$$

We then have for $\mathcal{T} := \mathrm{Sym}_{J_1} \times \cdots \times \mathrm{Sym}_{J_{\ell+s}}$

$$\begin{aligned}
\sum_{\sigma \in \mathcal{T}} \epsilon(\sigma)\sigma(R(X)) &= \sum_{\sigma \in \mathcal{T}} \epsilon(\sigma)\sigma(P(X)\tilde{\Delta}_{I_1} \cdots \tilde{\Delta}_{I_{\ell+s}} \prod_{i \in I_{\ell+1} \times \cdots \times I_{\ell+s}} X_i) \\
&= \sum_{\sigma \in \mathcal{T}} \epsilon(\sigma)\sigma \left( Q(X^2)\tilde{\Delta}_{I_1} \cdots \tilde{\Delta}_{I_{\ell+s}} \prod_{i \in J_{\ell+1} \times \cdots \times J_{\ell+s}} X_i \right) \\
&= \prod_{i \in J_{\ell+1} \times \cdots \times J_{\ell+s}} X_i \sum_{\sigma \in \mathrm{Sym}_{J_1} \times \cdots \times \mathrm{Sym}_{J_\ell}} \epsilon(\sigma)\sigma \left( Q(X^2)\tilde{\Delta}_{I_1} \cdots \tilde{\Delta}_{I_{\ell+s}} \right) \\
&= k_1! \cdots k_\ell! \cdot r_{\ell+1}! \cdots r_{\ell+s}! \tilde{\Delta}_{J_1} \cdots \tilde{\Delta}_{J_\ell} \tilde{\Delta}_{J_{\ell+1}} \cdots \tilde{\Delta}_{J_{\ell+s}} \prod_{i \in J_{\ell+1} \times \cdots \times J_{\ell+s}} X_i
\end{aligned}$$

where the last equality follows from (III.3). Therefore, $I$ contains the polynomial

$$\tilde{\Delta}_{J_1} \cdots \tilde{\Delta}_{J_\ell} \tilde{\Delta}_{J_{\ell+1}} \cdots \tilde{\Delta}_{J_{\ell+s}} \prod_{i \in J_{\ell+1} \times \cdots \times J_{\ell+s}} X_i$$

which is one of the generators of $I_{\Gamma^*(m)}$. By symmetry, we get the theorem. ∎

**Corollary III.7.3.** *Let $I \subset \mathbb{K}[X_1, \ldots, X_n]$ be a $\mathcal{B}_n$-invariant ideal, and let $P \in I$. Assume that $m$ is a monomial in the leading term of $P$. Using the notation of Definition III.7.1, assume that $\mathrm{wt}(P_d) + d_1 + d_2 \leqslant n$. Then, we have the inclusion of varieties*

$$V(I) \subset V(I_{\Gamma^*(m)})$$

*and $V(I) \cap H_{(\vartheta,\omega)} = \emptyset$ for any bipartition $(\vartheta, \omega) \in \mathrm{BP}_n$ bidominated by $\Gamma^*(m)$.*

*Proof.* The inclusion of varieties follows immediately from the set inclusion in Theorem III.7.2, while the second claim follows from the set partition of Specht varieties in Theorem III.6.6. ∎

We illustrate how this can be applied:

**Example III.7.4.** Let $P = X_2X_3(X_1^2 - 1) \in \mathbb{K}[X_1, \ldots, X_4]$. The polynomial $P$ contains a unique monomial $m = X_1^2X_2X_3$ of highest degree. Using the notation of Definition III.7.1 we obtain $\mathrm{wt}(P_4) = 3, l = 1, s = 2, d_1 = 1, d_2 = 0,$

$\Gamma(m) = ((2),(1,1))$, and $\Gamma^*(m) = ((1,1),(2))$. Let $I$ denote the ideal that is generated by the $\mathcal{B}_n$ orbit of $P$. Then, by Corollary III.7.3 it must be $V(I) \subset V_{((1,1),(2))}$. Thus, we have

$$V(I) \subset \mathcal{B}_4 \cdot \{(0,0,a,a),(0,0,a,b),(a,a,a,a),(0,0,0,a),(0,0,0,0) : a,b \in \mathbb{K}_{>0}\}.$$

We observe that $V(I) = \{(1,1,1,1),(0,0,0,a),(0,0,0,0) : a \in \mathbb{K}\}$. Thus $V(I)$ contains already points of three of the five possible orbit types.

### III.7.2 Connections with Representation Theory

We assume that $\mathbb{K} = \mathbb{C}$, or $\mathbb{K} = \mathbb{R}$ if $G$ is a real reflection group. Let $G$ be a finite group acting linearly on the polynomial ring $\mathbb{K}[X_1,\ldots,X_n]$. The polynomials fixed by this action form a finitely generated subalgebra $\mathbb{K}[X_1,\ldots,X_n]^G$. Moreover, each finite group admits - up to isomorphism - a finite number of irreducible $\mathbb{K}[G]$-modules and the action on $\mathbb{K}[X_1,\ldots,X_n]$ can be decomposed into isotypic components, i.e., we have a decomposition of the form

$$\mathbb{K}[X_1,\ldots,X_n] = \bigoplus_{\chi} \mathbb{K}[X_1,\ldots,X_n]_\chi,$$

where $\chi$ runs over the pairwise non-isomorphic representations and each isotypic component $\mathbb{K}[X_1,\ldots,X_n]_\chi$ contains only pairwise isomorphic $\mathbb{K}[G]$-submodules. Notice, that with this notion the invariant polynomials $\mathbb{K}[X_1,\ldots,X_n]^G$ correspond to the trivial representation. Clearly, $\mathbb{K}[X_1,\ldots,X_n]$ has the structure of a $\mathbb{K}[X_1,\ldots,X_n]^G$-module. Finally, let $J_+ \subset \mathbb{K}[X_1,\ldots,X_n]$ be the ideal generated by invariant polynomials of positive degree. Then the algebra $\mathbb{K}[X_1,\ldots,X_n]_G := \mathbb{K}[X_1,\ldots,X_n]/J_+$ is called the *coinvariant algebra*. Whereas these algebras can be defined and studied for all finite groups, it was shown by Chevalley ([Che55, Theorem (B)]) and Sephard-Todd ([ST54]) that for finite reflection groups, these algebras are very regular and moreover that they charaterize finite reflection groups by what is now known as Chevalley–Shephard–Todd theorem:

**Theorem III.7.5** ([Che55; ST54]). *Let $G$ be a finite group. Then, the following are equivalent*

III.7.5.1. *$G$ is a group generated by reflections.*

III.7.5.2. *The algebra of polynomial invariants $\mathbb{K}[X_1,\ldots,X_n]^G$ is a (free) polynomial algebra.*

III.7.5.3. *The algebra $\mathbb{K}[X_1,\ldots,X_n]$ is a free module over $\mathbb{K}[X_1,\ldots,X_n]^G$*

III.7.5.4. *$\mathbb{K}[X_1,\ldots,X_n]_G$ affords the regular representation of $G$, i.e.,*

$$\mathbb{K}[X_1,\ldots,X_n]_G \simeq \bigoplus_{\chi} \dim(\chi)\chi,$$

*where $\chi$ runs over the pairwise non-isomorphic representations of $G$.*

Let $(\lambda,\mu)\in \mathrm{BP}_n$ be a bipartition, denote by $\mathbb{K}[X_1,\dots,X_n]_{(\lambda,\mu)}$ the isotypic component corresponding to $(\lambda,\mu)$, and by $I_{(\lambda,\mu)}$ the associated Specht ideal. Notice that by Theorem III.7.5, $\mathbb{K}[X_1,\dots,X_n]_{(\lambda,\mu)}$ is also a finitely generated $\mathbb{K}[X_1,\dots,X_n]^G$ module, generated by $s_{\lambda,\mu}$ many elements, where $s_{\lambda,\mu}$ denotes the dimension of the corresponding irreducible representation. It follows from ([MY98, Theorem 1 (2)]) that $s_{\lambda,\mu}$ is in fact equal to the number of standard bitableaux of shape $\lambda,\mu$. We note the following Proposition:

**Proposition III.7.6.** *Let $d$ be minimal with*

$$\mathcal{V}:=\mathbb{K}[X_1,\dots,X_n]_{(\lambda,\mu)}\cap\mathbb{K}[X_1,\dots,X_n]_{\le d}\neq\emptyset.$$

*Then, the multiplicity of an irreducible representation of type $(\lambda,\mu)$ in $\mathcal{V}$ is 1. This unique irreducible representation is given by the $\mathcal{B}_n$-Specht polynomials of shape $(\lambda,\mu)$. Moreover, $\mathbb{K}[X_1,\dots,X_n]_{(\lambda,\mu)}$ is contained in the ideal generated by this unique irreducible representation, i.e.,*

$$\mathbb{K}[X_1,\dots,X_n]_{(\lambda,\mu)}\subset I_{(\lambda,\mu)}.$$

*Proof.* Since $\mathbb{K}[X_1,\dots,X_n]_{(\lambda,\mu)}$ is a direct sum of irreducible $\mathcal{B}_n$-modules isomorphic to the standard $\mathcal{B}_n$-Specht module, $W_{(\lambda,\mu)}$, it is enough to show that for every Specht polynomial $Q=\mathrm{sp}_{(T,S)}$ with $\mathrm{sh}(T,S)=(\lambda,\mu)$, and every $\mathcal{B}_n$-isomorphism $\phi$, the polynomial

$$P=\phi(Q)$$

is divisible by $Q$. First, for every $i\in\{1,\dots,n\}$, if $\epsilon_i$ is the map replacing $X_i$ with $-X_i$, since $\phi$ respects the action of $\mathcal{B}_n$, we must have

$$\epsilon_i P=\begin{cases}-P & \text{if } i\in S\\ P & \text{if } i\notin S\end{cases},$$

which implies that $P$ is of the form

$$P(X)=\tilde{P}(X^2)\prod_{i\in S}X_i.$$

Then, for every $\tau$ switching two elements in a same column of $T$ or $S$, we must have $\tau\tilde{P}=-\tilde{P}$, so that $P$ is divisible by $Q$. ∎

*Remark* III.7.7. We remark that the statement about multiplicity 1 of an irreducible representation in Proposition III.7.6 does not apply in general. Consider the real reflection group $D_n\subset\mathcal{B}_n$ which is generated by all permutations and those maps that switch an even number of signs. Then the $\mathcal{B}_n$-irreducible representation of type $(\lambda,\mu)$ and $(\mu,\lambda)$ remain $D_n$-irreducible if $\lambda\neq\mu$, but are $D_n$-isomorphic. By Theorem III.5.1 we have

$$I_{((2),(1,1))}\not\subset I_{((1,1),(2))}\quad\text{and}\quad I_{((1,1),(2))}\not\subset I_{((2),(1,1))}.$$

Thus, no polynomial in the $\mathcal{B}_n$-orbit of $(X_3^2 - X_4^2)X_3X_4$ divides the polynomial $(X_1^2 - X_2^2)X_3X_4$ although

$$\langle \mathrm{sp}_{(T,S)} : \mathrm{sh}(T,S) = ((2),(1,1))\rangle_{\mathbb{K}} \simeq_{D_n} \langle \mathrm{sp}_{(T,S)} : \mathrm{sh}(T,S) = ((1,1),(2))\rangle_{\mathbb{K}}.$$

The $D_n$-irreducible representation $((2),(1,1))$ occurs for the first time in $\mathbb{K}[X_1,\ldots,X_4]_{\leq 4}$ but with multiplicity 2.

Combining Corollary III.7.3 with Proposition III.7.6 we obtain the following.

**Theorem III.7.8.** *Let $I \subset \mathbb{K}[X_1,\ldots,X_n]$ a $\mathcal{B}_n$-invariant ideal satisfying the conditions of III.7.3. Consider the the associated coordinate ring $R_I = \mathbb{K}[X_1,\ldots,X_n]/I$. Then, viewed as a $\mathbb{K}[\mathcal{B}_n]$-module, $R_I$ does not contain any irreducible $\mathbb{K}[\mathcal{B}_n]$-submodule which is isomophic to $W_{\lambda,\mu}$. Moreover, $R_I$ is a finite $\mathbb{K}[X_1,\ldots,X_n]^{\mathcal{B}_n}$ module of rank bounded by $\sum_{(\vartheta,\omega)\trianglelefteq(\lambda,\mu)} s^2_{(\vartheta,\omega)}$.*

## III.8 Conclusion and open questions

We initiated in this article the investigation of a class of polynomial ideals which are naturally linked to the action of a group on a polynomial ring. Our results provide an analogue of the relation of the combinatorics of integer partitions and $\mathcal{S}_n$-Specht ideals to bipartitions and $\mathcal{B}_n$-Specht ideals. The present work shows that it is indeed possible to derive an analogous connection between combinatorics and algebra for the case of the hyperoctahedral group as was observed in the case of the symmetric group. Both groups are finite reflection groups, and they thus share important similarities from a view point of invariant theory and representation theory. Our results here lead to the natural question, if similar relations between integer (bi)-partitions and ideals can be derived for other (pseudo)-reflection groups. Indeed, in [MY98] a similar basis of the coinvariant algebra is provided for complex reflection groups of type $G(r,p,n)$, where $r,p,n \in \mathbb{Z}_{\geq 1}$ and $p \mid n$. We recover $G(1,1,n) \simeq \mathcal{S}_n$, $G(2,1,n) \simeq \mathcal{B}_n$, and $G(2,2,n) \simeq D_n$. It seems plausible to envision similar results to the ones presented here in these cases as well. More precisely, that there is a partial order on $r$-multipartitions, which are linked to the irreducible representations of the complex reflection group $G(r,1,n)$, which transfers to the inclusion of the $G(r,1,n)$-Specht ideals and their corresponding varieties. Furthermore, it remains to investigate if the $\mathcal{B}_n$-Specht ideals also have similarly nice algebraic properties as their $S_n$ counter parts. Indeed, it is known that the $\mathcal{S}_n$-Specht ideals are radical (see [MOY21, Theorem 1.1] and [Woo05, Proposition 4]). Both proofs rely on the understanding of the $\mathcal{S}_n$-Specht varieties in terms of orbit sets, i.e., Theorem III.6.6, and crucially depend on the property that any $\mathcal{S}_n$-orbit set is non-empty which is not true for $\mathcal{B}_n$-Specht varieties. Nevertheless, computational evidence for small number of variables motivates the conjecture, similar to the $S_n$ situation ([Lie21; MOY21; Rie21; Woo05]).

**Conjecture III.8.1.** *The $\mathcal{B}_n$-Specht ideals are radical. Moreover, for a bipartition $(\lambda,\mu) \in \mathrm{BP}_n$ the $\mathcal{B}_n$-Specht polynomials $\{\mathrm{sp}_{(T,S)}$ :*

*$(T,S)$ is a bitableau of shape $(\vartheta,\omega) \trianglelefteq (\lambda,\mu)\}$ form a universal Gröbner basis of $I_{(\lambda,\mu)}$.*

Finally, Yanagawa [Yan21] classified the partitions for which the associated $S_n$-Specht ideals are Cohen-Macaulay and it would be interesting to derive a similar characterization of bipartitions $(\lambda,\mu)$ for which the corresponding $\mathcal{B}_n$-Specht ideals have this property.

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