



An Analogy of the Carleson–Hunt Theorem with Respect to Vilenkin Systems

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Abstract

In this paper we discuss and prove an analogy of the Carleson–Hunt theorem with respect to Vilenkin systems. In particular, we use the theory of martingales and give a new and shorter proof of the almost everywhere convergence of Vilenkin–Fourier series of $f \in L_p(G_m)$ for $p > 1$ in case the Vilenkin system is bounded. Moreover, we also prove sharpness by stating an analogy of the Kolmogorov theorem for $p = 1$ and construct a function $f \in L_1(G_m)$ such that the partial sums with respect to Vilenkin systems diverge everywhere.

Keywords Fourier analysis · Vilenkin system · Vilenkin group · Vilenkin–Fourier series · Almost everywhere convergence · Carleson–Hunt theorem · Kolmogorov theorem

Mathematics Subject Classification 42C10 · 42B25

1 Introduction

In 1947 Vilenkin [61–63] investigated a group G_m , which is a direct product of the additive groups $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ of integers modulo m_k , where $m :=$

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(m_0, m_1, \dots) are positive integers not less than 2, and introduced the Vilenkin systems $\{\psi_j\}_{j=0}^\infty$. These systems include as a special case the Walsh system.

The classical theory of Hilbert spaces (for details see e.g. the books [58, 60]) says that if we consider the partial sums $S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k$, with respect to Vilenkin systems, then

$$\|S_n f\|_2 \leq \|f\|_2.$$

In the same year Schipp [45], Simon [51] and Young [67] (see also the book [49]) generalized this inequality for $1 < p < \infty$: there exists an absolute constant c_p , depending only on p , such that

$$\|S_n f\|_p \leq c_p \|f\|_p, \text{ when } f \in L_p(G_m).$$

From this it follows that for every $f \in L_p(G_m)$ with $1 < p \leq \infty$,

$$\|S_n f - f\|_p \rightarrow 0, \text{ as } n \rightarrow \infty.$$

The boundedness does not hold for $p = 1$, but Watari [64] (see also Gosselin [23] and Young [67]) proved that there exists an absolute constant c such that, for $n = 1, 2, \dots$, the weak type estimate holds:

$$y \mu \{|S_n f| > y\} \leq c \|f\|_1, \quad f \in L_1(G_m), \quad y > 0.$$

The almost-everywhere convergence of Fourier series for L_2 functions was postulated by Luzin [39] in 1915 and the problem was known as Luzin's conjecture. Carleson's theorem is a fundamental result in mathematical analysis establishing the pointwise (Lebesgue) almost everywhere convergence of Fourier series of L_2 functions, proved by Carleson [10] in 1966. The name is also often used to refer to the extension of the result by Hunt [26] which was given in 1968 to L_p functions for $p \in (1, \infty)$ (also known as the Carleson–Hunt theorem).

Carleson's original proof is exceptionally hard to read, and although several authors have simplified the arguments there are still no easy proofs of his theorem. Expositions of the original Carleson's paper were published by Kahane [28], Mozzochi [40], Jorsboe and Mejlbø [27] and Arias de Reyna [43]. Moreover, Fefferman [16] published a new proof of Hunt's extension, which was done by bounding a maximal operator of partial sums

$$S^* f := \sup_{n \in \mathbb{N}} |S_n f|.$$

This, in turn, inspired a much simplified proof of the L_2 result by Lacey and Thiele [35], explained in more detail in Lacey [33]. The books Fremlin [17] and Grafakos [24] also give proofs of Carleson's theorem. An interesting extension of Carleson–Hunt result much more closer to L_1 space than L_p for any $p > 1$ was done by Carleson's student Sjölin [56] and later on, by Antonov [2]. Already in 1923, Kolmogorov [31] showed

that the analogue of Carleson's result for L_1 is false by finding such a function whose Fourier series diverges almost everywhere (improved slightly in 1926 to diverging everywhere). This result indeed inspired many authors after Carleson proved positive results in 1966. In 2000, Kolmogorov's result was improved by Konyagin [32], by finding functions with everywhere-divergent Fourier series in a space smaller than L_1 , but the candidate for such a space that is consistent with the results of Antonov and Konyagin is still an open problem.

The famous Carleson theorem was very important and surprising when it was proved in 1966. Since then this interest has remained and a lot of related research has been done. In fact, in recent years this interest has even been increased because of the close connections to e.g. scattering theory [41], ergodic theory [14, 15], the theory of directional singular integrals in the plane [4, 11, 13, 34] and the theory of operators with quadratic modulations [36]. We refer to [33] for a more detailed description of this fact. These connections have been discovered from various new arguments and results related to Carleson's theorem, which have been found and discussed in the literature. We mean that these arguments share some similarities, but each of them has also a distinct new idea behind, which can be further developed and applied. It is also interesting to note that, for almost every specific application of Carleson's theorem in the aforementioned fields, mainly only one of these new arguments was used.

The analogue of Carleson's theorem for Walsh system was proved by Billard [5] for $p = 2$ and by Sjölin [55] for $1 < p < \infty$, while for bounded Vilenkin systems by Gosselin [23]. Schipp [46, 47, 49] investigated the so called tree martingales, i.e., martingales with respect to a stochastic basis indexed by a tree, and generalized the results about maximal function, quadratic variation and martingale transforms to these martingales (see also [48, 65]). Using these results, he gave a proof of Carleson's theorem for Walsh–Fourier series. A similar proof for bounded Vilenkin systems can be found in Schipp and Weisz [48, 65]. In each proof, they show that the maximal operator of the partial sums is bounded on $L_p(G_m)$, i.e., there exists an absolute constant c_p such that

$$\|S^* f\|_p \leq c_p \|f\|_p, \quad \text{as } f \in L_p(G_m), \quad 1 < p < \infty.$$

Recent proof of almost everywhere convergence of Walsh–Fourier series was given by Demeter [12] in 2015. By using some methods of martingale Hardy spaces, almost everywhere convergence of subsequences of Vilenkin–Fourier series was considered in [8]. Antonov [3] proved that for $f \in L_1(\log^+ L)(\log^+ \log^+ L)(G_m)$ its Walsh–Fourier series converges a.e. Similar result for the bounded Vilenkin systems was proved by Oniani [42]. However, there exists a function from $L_1(\log^+ \log^+ L)(G_m)$ whose Vilenkin–Fourier series diverges everywhere, where in this result G_m is a general (not necessary “bounded”) Vilenkin group (see Tarkaev [59]).

Stein [57] constructed an integrable function whose Walsh–Fourier series diverges almost everywhere. Later Schipp [44, 49] proved that there exists an integrable function whose Walsh–Fourier series diverges everywhere. Kheladze [29, 30] proved that for any set of measure zero there exists a function in $f \in L_p(G_m)$ ($1 < p < \infty$) whose Vilenkin–Fourier series diverges on the set, while the result for continuous or bounded function was proved by Harris [25] or Bitsadze [6]. Moreover, Simon

[53] constructed an integrable function such that its Vilenkin–Fourier series diverges everywhere. Bochkarev [9] considered rearrangements of Vilenkin–Fourier series of bounded type.

It is not known whether Carleson’s theorem holds for unbounded Vilenkin systems. However, some theorems were proved for unbounded Vilenkin systems by Gát [18–21], Simon [52, 53] and Tarkaev [59].

In this paper, we use the theory of martingales and give a new an shorter proof of the almost everywhere convergence of Vilenkin–Fourier series of $f \in L_p(G_m)$ for $p > 1$ in the case the Vilenkin system is bounded. The positive results of this paper are derived in Sect. 3. In Theorem 2 we prove the boundedness of the maximal operator on L_p ($1 < p < \infty$) spaces. By using this result, we derive the L_p norm convergence of the partial sums of Vilenkin–Fourier series (Theorem 3) as well as the analogue of the Carleson–Hunt theorem, i.e., the almost everywhere convergence of the partial sums of $f \in L_p$ (Theorem 4), when $1 < p < \infty$. The proof is built up by proving some new lemmas of independent interest. The corresponding sharpness and almost everywhere divergence are stated and proved in Sect. 4, see Theorems 5 and 6. Especially Theorem 6 is the Kolmogorov type result and also here the proof is built up by proving some lemmas of independent interest. In order not to disturb our discussion later, some necessary preliminaries are presented in Sect. 2.

2 Preliminaries

Denote by \mathbb{N}_+ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k .

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of Z_{m_j} ’s. The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$. In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots), \quad (x_j \in Z_{m_j}).$$

It is easy to give a base for the neighborhood of G_m :

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\},$$

where $x \in G_m, n \in \mathbb{N}$. Denote $I_n := I_n(0)$ for $n \in \mathbb{N}_+$, and $\overline{I_n} := G_m \setminus I_n$.

If we define the so-called generalized number system based on m by

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ ($j \in \mathbb{N}_+$) and only a finite number of n_j 's differ from zero. For two natural numbers $n = \sum_{j=1}^{\infty} n_j M_j$ and $k = \sum_{j=1}^{\infty} k_j M_j$, we define that

$$n \oplus k := \sum_{i=0}^{\infty} ((n_i + k_i) \pmod{m_i}) M_i, \quad n_i, k_i \in Z_{m_i}.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. First, we define the complex-valued function $r_k(x) : G_m \rightarrow \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k), \quad (i^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system when $m \equiv 2$. The norms (or quasi-norms) of the spaces $L_p(G_m)$ ($0 < p < \infty$) is defined by

$$\|f\|_p^p := \int_{G_m} |f|^p d\mu.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (for details see e.g. the books [1, 49]).

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$, we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\begin{aligned} \widehat{f}(n) &:= \int_{G_m} f \overline{\psi_n} d\mu, \quad (n \in \mathbb{N}) \\ S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \quad \text{and} \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+) \end{aligned}$$

respectively. Recall that (see e.g. Simon [50, 54] and Golubov et al. [22])

$$\sum_{s=0}^{m_k-1} r_k^s(x) = \begin{cases} m_k, & \text{if } x_k = 0, \\ 0, & \text{if } x_k \neq 0. \end{cases} \tag{1}$$

and

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases} \tag{2}$$

It is known that (for the details see e.g. [1, 7, 37, 38]) there exist absolute constants C_1 and C_2 such that

$$C_1 n \leq \|D_{q_n}\|_1 \leq C_2 n, \quad \text{for } q_n = M_{2n} + M_{2n-2} + M_2 + M_0. \tag{3}$$

A function P is called Vilenkin polynomial if $P = \sum_{k=0}^n c_k \psi_k$. The spectra of the Vilenkin polynomial P is defined by

$$sp(P) = \{n \in \mathbb{N} : \widehat{P}(n) \neq 0.\}$$

3 Martingale Inequalities

We will also need some martingale inequalities. The σ -algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by \mathcal{F}_n ($n \in \mathbb{N}$). If \mathcal{F} denotes the set of Haar measurable subsets of G_m , then obviously $\mathcal{F}_n \subset \mathcal{F}$. By a Vilenkin interval we mean one of the form $I_n(x)$, $n \in \mathbb{N}$, $x \in G_m$. The conditional expectation operators relative to \mathcal{F}_n are denoted by E_n . An integrable sequence $f = (f_n)_{n \in \mathbb{N}}$ is said to be a martingale if f_n is \mathcal{F}_n -measurable for all $n \in \mathbb{N}$ and $E_n f_m = f_n$ in the case $n \leq m$. We can see that if $f \in L_1(G_m)$, then $(E_n f)_{n \in \mathbb{N}}$ is a martingale. Martingales with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ are called Vilenkin martingales. It is easy to show (see e.g. Weisz [65, p. 11]) that the sequence $(\mathcal{F}_n, n \in \mathbb{N})$ is regular, i.e.,

$$f_n \leq R f_{n-1} \quad (n \in \mathbb{N}) \tag{4}$$

for all non-negative Vilenkin martingales (f_n) , where $R := \max_{n \in \mathbb{N}} m_n$.

Using (2), we can show that $E_n f = S_{M_n} f$ for all $f \in L_p(G_m)$ with $1 \leq p \leq \infty$ (see e.g. [65, Sect. 1.2]). By the well known martingale theorems, this implies that

$$\|S_{M_n} f\|_p \leq c_p \|f\|_p, \quad \text{for all } f \in L_p(G_m) \text{ when } 1 \leq p < \infty$$

and

$$\|S_{M_n} f - f\|_p \rightarrow 0, \quad \text{as } n \rightarrow \infty \text{ for all } f \in L_p(G_m) \text{ when } p \geq 1. \tag{5}$$

For a Vilenkin martingale $f = (f_n)_{n \in \mathbb{N}}$, the maximal function is defined by

$$f^* := \sup_{n \in \mathbb{N}} |f_n|.$$

For a martingale $f = (f_n)_{n \geq 0}$ let

$$d_n f = f_n - f_{n-1} \quad (n \geq 0)$$

denote the martingale differences, where $f_{-1} := 0$. The square function and the conditional square function of f are defined by

$$S(f) = \left(\sum_{n=0}^{\infty} |d_n f|^2 \right)^{1/2}, \quad s(f) = \left(|d_0 f|^2 + \sum_{n=0}^{\infty} E_n |d_{n+1} f|^2 \right)^{1/2}.$$

We have shown the following theorem in [65].

Theorem 1 *If $0 < p < \infty$, then*

$$\|f^*\|_p \sim \|S(f)\|_p \sim \|s(f)\|_p.$$

If in addition $1 < p \leq \infty$, then

$$\|f^*\|_p \sim \|f\|_p.$$

We will use the following convexity, concavity theorem proved in [65].

Proposition 1 *Let \mathbb{T} be a countable index set and $(\mathcal{A}_t, t \in \mathbb{T})$ be an arbitrary (not necessarily monotone) sequence of sub- σ -algebras of \mathcal{F} . Suppose that for all $h \in L_p(G_m)$ and all $1 < p < \infty$ Doob's inequality*

$$\left\| \sup_{t \in \mathbb{T}} |E_t h| \right\|_p \leq C_p \|h\|_p$$

holds where E_t denotes the conditional expectation operator relative to \mathcal{A}_t . If $(f_t, t \in \mathbb{T})$ is a sequence of non-negative measurable functions, then for all $1 \leq p < \infty$,

$$\int_{G_m} \left(\sum_{t \in \mathbb{T}} E_t f_t \right)^p d\mu \leq C_p \int_{G_m} \left(\sum_{t \in \mathbb{T}} f_t \right)^p d\mu$$

and for all $0 < q \leq 1$,

$$\int_{G_m} \left(\sum_{t \in \mathbb{T}} f_t \right)^q d\mu \leq C_q \int_{G_m} \left(\sum_{t \in \mathbb{T}} E_t f_t \right)^q d\mu.$$

4 A.E. Convergence of Vilenkin–Fourier Series

We introduce some notations. For $j, k \in \mathbb{N}$ we define the following subsets of \mathbb{N} :

$$I_{jM_k}^k := [jM_k, jM_k + M_k) \cap \mathbb{N}$$

and

$$\mathcal{J} := \left\{ I_{jM_k}^k : j, k \in \mathbb{N} \right\}.$$

We introduce also the partial sums taken in these intervals:

$$s_{I_{jM_k}^k} f := \sum_{i \in I_{jM_k}^k} \widehat{f}(i) \psi_i.$$

For simplicity, we suppose that $\widehat{f}(0) = 0$. The last author has proved in [66] that, for an arbitrary $n \in I_{jM_k}^k$,

$$s_{I_{jM_k}^k} f = \psi_n E_k (f \overline{\psi}_n). \tag{6}$$

For

$$n = \sum_{j=0}^{\infty} n_j M_j \quad (0 \leq n_j < m_j),$$

we introduce

$$n(k) := \sum_{j=k}^{\infty} n_j M_j, \quad I_{n(k)}^k = [n(k), n(k) + M_k) \quad (n \in \mathbb{N}). \tag{7}$$

For $I = I_{n(k)}^k$, let

$$T^I f := T^{I_{n(k)}^k} f := \sum_{\substack{[n(k+1), n(k)] \supset J \in \mathcal{J} \\ |J|=M_k}} s_J f. \tag{8}$$

Since $I_{n(k)}^k = I_{\tilde{n}(k)}^k$ implies $n(k + 1) = \tilde{n}(k + 1)$, the operators T^I ($I \in \mathcal{J}$) are well defined. Note that there are n_k summands in (8).

Lemma 1 *For all $n \in \mathbb{N}$, we have*

$$S_n f = \sum_{k=0}^{\infty} T^{I_{n(k)}^k} f = \psi_n \sum_{k=0}^{\infty} \sum_{l=0}^{n_k-1} \bar{r}_k^{n_k-l} E_k \left(d_{k+1} (f \overline{\psi}_n) r_k^{n_k-l} \right),$$

where $I_{n(k)}^k$ is defined in (7).

Proof We sketch the proof, only. It is proved in [66] that

$$\begin{aligned} T^{I_{n(k)}^k} f &= \sum_{j \in [n(k+1), n(k)]} \widehat{f}(j) \psi_j \\ &= \psi_n \sum_{l=0}^{n_k-1} \bar{r}_k^{n_k-l} E_k \left(d_{k+1} (f \bar{\psi}_n) r_k^{n_k-l} \right). \end{aligned} \tag{9}$$

Moreover, n is contained in $I_{n(k)}^k$ and $I_{n(k)}^k \subset I_{n(k+1)}^{k+1}$. Since

$$[0, n) = \bigcup_{k=0}^{\infty} [n(k+1), n(k)),$$

we get that

$$S_n f = \sum_{k=0}^{\infty} T^{I_{n(k)}^k} f.$$

This finishes the proof of Lemma 1. □

Lemma 2 For all $k, n \in \mathbb{N}$, we have

$$\left| T^{I_{n(k)}^k} f \right| \leq R E_k \left(\left| s_{I_{n(k+1)}^{k+1}} f - s_{I_{n(k)}^k} f \right| \right),$$

where $R := \max(m_n, n \in \mathbb{N})$.

Proof Equalities (9) and (6) imply

$$\begin{aligned} |T^{I_{n(k)}^k} f| &\leq m_k E_k \left(|d_{k+1} (f \bar{\psi}_n)| \right) \\ &\leq R E_k \left(|\psi_n E_{k+1} (f \bar{\psi}_n) - \psi_n E_k (f \bar{\psi}_n)| \right) \\ &= R E_k \left(\left| s_{I_{n(k+1)}^{k+1}} f - s_{I_{n(k)}^k} f \right| \right), \end{aligned} \tag{10}$$

which shows the lemma. □

Lemma 3 For all $n \in \mathbb{N}$, $(\bar{\psi}_n T^{I_{n(k)}^k} f)_{k \in \mathbb{N}}$ is a martingale difference sequence with respect to $(\mathcal{F}_{k+1})_{k \in \mathbb{N}}$.

Proof First, $\bar{\psi}_n T^{I_{n(k)}^k} f$ is \mathcal{F}_{k+1} measurable because of (9) and the fact that r_k is \mathcal{F}_{k+1} measurable. Since $E_k(r_k^i) = 0$ for $i = 1, \dots, m_n - 1$, we can see that

$$E_k \left(\bar{\psi}_n T^{I_{n(k)}^k} f \right) = E_k \left(\sum_{l=0}^{n_k-1} \bar{r}_k^{n_k-l} E_k \left(d_{k+1} (f \bar{\psi}_n) r_k^{n_k-l} \right) \right) = 0,$$

hence

$$\left(\sum_{l=0}^{n_k-1} \bar{r}_k^{n_k-l} E_k \left(d_{k+1}(f \bar{\psi}_n) r_k^{n_k-l} \right) \right)_{k \in \mathbb{N}}$$

is a martingale difference sequence. □

Before proving our main theorem, we need some further notations and lemmas. In what follows, I, J, K denote some elements of \mathcal{J} . Let

$$\mathcal{F}_K := \mathcal{F}_n \quad \text{and} \quad E_K := E_n \quad \text{if} \quad |K| = M_n.$$

Assume that $\epsilon = (\epsilon_K, K \in \mathcal{J})$ is a sequence of functions such that ϵ_K is \mathcal{F}_K measurable. Set

$$T_{\epsilon, I, J} f := \sum_{I \subset K \subsetneq J} \epsilon_K T^K f$$

and

$$T_{\epsilon; I}^* f := \sup_{I \subset J} |T_{\epsilon; I, J} f|, \quad T_{\epsilon}^* f := \sup_{I \in \mathcal{J}} |T_{\epsilon; I}^* f|.$$

If $\epsilon_K(t) = 1$ for all $K \in \mathcal{J}$ and $t \in G_m$, then we omit the notation ϵ and we write simply $T_{I, J} f, T_I^* f$ and $T^* f$.

For $I \in \mathcal{J}$ with $|I| = M_n$, let $I^+ \in \mathcal{J}$ such that $I \subset I^+$ and $|I^+| = M_{n+1}$. Moreover, let $I^- \in \mathcal{J}$ denote one of the sets $I^- \subset I$ with $|I^-| = M_{n-1}$. Note that $\mathcal{F}_{I^-} = \mathcal{F}_{n-1}$ and $E_{I^-} = E_{n-1}$ are well defined. We introduce the maximal functions

$$s_I^* f := \sup_{K \subset I} E_{K^-} |s_K f|$$

and

$$s^* f := \sup_{I \in \mathcal{J}} s_I^* f.$$

Since $|s_{I^+} f|$ is \mathcal{F}_{I^+} measurable, by the regularity condition (4),

$$|s_{I^+} f| \leq R E_I |s_{I^+} f| \leq R s_{I^+}^* f. \tag{11}$$

Lemma 4 *For any real number $x > 0$ and $K \in \mathcal{J}$, let*

$$\epsilon_K := \chi_{\{t \in G_m : x < s_{K^+}^* f(t) \leq 2x\}}$$

and

$$\alpha_K := \chi_{\{t \in G_m : s_K^* f(t) > x, s_I^* f(t) \leq x, I \subsetneq K\}}.$$

Then

$$T_{\epsilon}^* f \leq 2 \sup_{K \in \mathcal{J}} \alpha_K T_{\epsilon;K}^* f + 4R^2 x \chi_{\{t \in G_m : s^* f(t) > x\}}. \tag{12}$$

Proof Let us fix $I \subsetneq J$ in \mathcal{J} and t in G_m . Set

$$\tau_K := \chi_{\{t \in G_m : s_K^* f(t) > x\}} \quad (K \in \mathcal{J}).$$

Therefore $\epsilon_K = \tau_K + \epsilon_K$. Consequently, if the set

$$\{K \in \mathcal{J} : I \subset K \subsetneq J, \tau_K(t) = 1\}$$

is empty then $T_{\epsilon;I,J} f(t) = 0$ or else let K_1 be its minimum element. Moreover, denote by K_0 one of the minimum elements of the set

$$\{K \in \mathcal{J} : K \subset K_1^+, \tau_K(t) = 1\}.$$

This means that if $L \subsetneq K_0$, then $\tau_L(t) = 0$. Thus $\alpha_{K_0}(t) = 1$ and

$$\begin{aligned} T_{\epsilon;I,J} f(t) &= T_{\epsilon;K_1,J} f(t) = \epsilon_{K_1} T^{K_1} f(t) + T_{\epsilon;K_1^+,J} f(t) \\ &= \epsilon_{K_1} T^{K_1} f(t) + \alpha_{K_0}(t) \left(T_{\epsilon;K_0,J} f(t) - T_{\epsilon;K_0,K_1^+} f(t) \right). \end{aligned}$$

By Lemma 2 and (11),

$$\begin{aligned} \epsilon_{K_1} |T^{K_1} f| &\leq R \epsilon_{K_1} E_{K_1} \left(|s_{K_1^+} f - s_{K_1} f| \right) \\ &\leq 2R^2 \epsilon_{K_1} E_{K_1} \left(\epsilon_{K_1} s_{K_1^+}^* f \right) \leq 4R^2 x \chi_{\{t \in G_m : s^* f(t) > x\}}. \end{aligned}$$

On the other hand,

$$\left| T_{\epsilon;K_0,J} f(t) - T_{\epsilon;K_0,K_1^+} f(t) \right| \leq 2T_{\epsilon;K_0}^* f(t).$$

Taking the supremum over all $I \subsetneq J$, we get (12). □

Now we introduce the quasi-norm $\| \cdot \|_{p,q}$ ($0 < p, q < \infty$) by

$$\|f\|_{p,q} := \sup_{x>0} x \left(\int_{G_m} \left(\sum_{I \in \mathcal{J}} \alpha_I \right)^{p/q} d\mu \right)^{1/p},$$

where α_I is defined in Lemma 4. Observe that α_I can be rewritten as

$$\alpha_I := \chi_{\{t \in G_m : E_{I-} |s_I f(t)| > x, E_{J-} |s_J f(t)| \leq x, J \subsetneq I\}}. \tag{13}$$

Denote by $P^{p,q}$ the set of functions $f \in L_1$ which satisfy $\|f\|_{p,q} < \infty$. For $q = \infty$, define

$$\|f\|_{p,\infty} := \sup_{x>0} x \left(\int_{\tilde{G}_m} \left(\sup_{I \in \mathcal{J}} \alpha_I \right)^p d\mu \right)^{1/p} \quad (0 < p < \infty).$$

It is easy to see that

$$\|f\|_{p,\infty} \leq \|f\|_{p,q} \quad (0 < q < \infty)$$

and

$$\|f\|_{p,\infty} = \sup_{x>0} x \mu(s^* f > x)^{1/p}.$$

Lemma 5 *Let $\max(1, p) < q < \infty$, $f \in P^{p,q}$ and $x, z > 0$. Then*

$$\mu \left(\sup_{I \in \mathcal{J}} \alpha_I T_{\epsilon;I}^* f > zx \right) \leq C_{p,q} z^{-q} x^{-p} \|f\|_{p,q}^p,$$

where α_I is defined in Lemma 4.

Proof Equality (9) implies that

$$\xi T^K(f) = T^K(f\xi)$$

for any \mathcal{F}_K measurable function ξ . By Lemma 3, for a suitable $n \in I$, $(\bar{\psi}_n T^K f)_{I \subset K}$ is a martingale difference sequence relative to $(\mathcal{F}_{K^+})_{I \subset K}$. We have

$$T_{\epsilon;I}^* f = \sup_{I \subset J} \left| \sum_{I \subset K \subsetneq J} \epsilon_K \bar{\psi}_n T^K f \right| = \sup_{I \subset J} \left| \sum_{I \subset K \subsetneq J} \bar{\psi}_n T^K (f \epsilon_K) \right|.$$

Using Burkholder–Gundy’s inequality (see Theorem 1) together with (10), we obtain

$$\begin{aligned} E_I (|T_{\epsilon;I}^* f|^{p_0}) &\leq C_{p_0} E_I \left(\sum_{I \subset K} |\bar{\psi}_n T^K (f \epsilon_K)|^2 \right)^{p_0/2} \\ &\leq C_{p_0} E_I \left(\sum_{I \subset K} E_K |d_{K^+}(f \epsilon_K \bar{\psi}_n)|^2 \right)^{p_0/2}, \end{aligned}$$

where $p_0 > 1$. Applying again Theorem 1, one can establish that

$$\begin{aligned} E_I (|T_{\epsilon;I}^* f|^{p_0}) &\leq C_{p_0} E_I \left| \sum_{I \subset K} d_{K^+}(f \epsilon_K \bar{\psi}_n) \right|^{p_0} \\ &= C_{p_0} E_I \left| \sum_{I \subset K} \epsilon_K d_{K^+}(f \bar{\psi}_n) \right|^{p_0}. \end{aligned}$$

For fixed I and $t \in G_m$ let us denote by $K_0(t) \in \mathcal{J}$ (resp. $K_1(t) \in \mathcal{J}$) the smallest (resp. largest) interval $K \supset I$ for which $\epsilon_{K_0(t)} = 1$ (resp. $\epsilon_{K_1(t)} = 1$). Then

$$\begin{aligned} \psi_n(t) \sum_{I \subset K} \epsilon_K(t) d_{K^+}(f \bar{\psi}_n)(t) &= \psi_n(t) \sum_{K_0(t) \subset K \subset K_1(t)} \epsilon_K(t) d_{K^+}(f \bar{\psi}_n)(t) \\ &= \epsilon_{K_1(t)}(t) (\psi_n(t) E_{K_1(t)^+}(f \bar{\psi}_n)(t) - \psi_n(t) E_{K_0(t)}(f \bar{\psi}_n)(t)) \\ &= \epsilon_{K_1(t)}(t) (s_{K_1(t)^+} f(t) - s_{K_0(t)} f(t)). \end{aligned}$$

By (11) and by the definition of ϵ_K ,

$$\begin{aligned} \epsilon_{K_1(t)}(t) |s_{K_1(t)^+} f(t) - s_{K_0(t)} f(t)| &\leq R \epsilon_{K_1(t)}(t) (s_{K_1(t)^+}^* f(t) + s_{K_0(t)}^* f(t)) \\ &\leq 2R \epsilon_{K_1(t)}(t) s_{K_1(t)^+}^* f(t) \leq 4R x. \end{aligned} \tag{14}$$

Hence

$$E_I (|T_{\epsilon;I}^* f|^{p_0}) \leq C_{p_0} x^{p_0}.$$

By Tsebishev’s inequality and the concavity theorem (see Proposition 1), for $p_0 \geq q > 1$, one can see that

$$\begin{aligned} \mu \left(\sup_{I \in \mathcal{J}} \alpha_I T_{\epsilon;I}^* f > zx \right) &\leq (zx)^{-q} \int_{\tilde{G}_m} \left(\sup_{I \in \mathcal{J}} \alpha_I T_{\epsilon;I}^* f \right)^q d\mu \\ &\leq (zx)^{-q} \int_{\tilde{G}_m} \left(\sum_{I \in \mathcal{J}} \alpha_I T_{\epsilon;I}^* f^{p_0} \right)^{q/p_0} d\mu \\ &\leq C_{p_0,q} (zx)^{-q} \int_{\tilde{G}_m} \left(\sum_{I \in \mathcal{J}} \alpha_I E_I (T_{\epsilon;I}^* f^{p_0}) \right)^{q/p_0} d\mu \\ &\leq C_{p_0,q} z^{-q} \int_{\tilde{G}_m} \left(\sum_{I \in \mathcal{J}} \alpha_I \right)^{q/p_0} d\mu. \end{aligned}$$

Set $p_0 := q^2/p \geq q > 1$ and observe that

$$\begin{aligned} \mu \left(\sup_{I \in \mathcal{J}} \alpha_I T_{\epsilon; I}^* f > zx \right) &\leq C_{p,q} z^{-q} \int_{G_m} \left(\sum_{I \in \mathcal{J}} \alpha_I \right)^{p/q} d\mu \\ &\leq C_{p,q} z^{-q} x^{-p} \|f\|_{p,q}^p, \end{aligned}$$

which shows the lemma. □

Lemma 6 *Let $\max(1, p) < q < \infty$ and $f \in P^{p,q}$. Then*

$$\sup_{y>0} y^p \mu \left(T^* f > (2 + 8R^2)y \right) \leq C_{p,q} \|f\|_{p,q}.$$

Proof First we define a decomposition generated by the sequences $\epsilon^k = (\epsilon_K^k, K \in \mathcal{J})$, where

$$\epsilon_K^k := \chi_{\{t \in G_m : 2^k < s_{K^+}^* f(t) \leq 2^{k+1}\}} \quad (k \in \mathbb{Z}).$$

Notice that (10) and (14) imply

$$\chi_{\{t \in G_m : s^* f(t) = 0\}} T^K f = \chi_{\{t \in G_m : s^* f(t) = 0\}} \chi_{\{t \in G_m : s_{K^+}^* f(t) = 0\}} T^K f = 0.$$

Henceforth

$$T^K f = \chi_{\{t \in G_m : s^* f(t) > 0\}} T^K f = \sum_{k \in \mathbb{Z}} \epsilon_K^k T^K f$$

and

$$T^* f \leq \sum_{k \in \mathbb{Z}} T_{\epsilon^k}^* f.$$

Let us apply Lemma 4 to ϵ^k and $x = 2^k$ to write

$$T_{\epsilon^k}^* f \leq 2 \sup_{K \in \mathcal{J}} \alpha_K^k T_{\epsilon^k; K}^* f + 2^{k+2} R^2 \chi_{\{t \in G_m : s^* f(t) > 2^k\}},$$

where

$$\alpha_K^k := \chi_{\{t \in G_m : s^* f(t) > 2^k, s_j^* f(t) \leq 2^k, I \subsetneq K\}} \quad (K \in \mathcal{J}).$$

Choosing $j \in \mathbb{Z}$ such that $2^j < y \leq 2^{j+1}$, we get that

$$\begin{aligned} \chi_{\{t \in G_m : s^* f(t) \leq y\}} T^* f &\leq 2 \sum_{k \leq j} \sup_{K \in \mathcal{J}} \alpha_K^k T_{\epsilon^k; K}^* f + \sum_{k \leq j} 2^{k+2} R^2 \chi_{\{t \in G_m : s^* f(t) > 2^k\}} \\ &\leq 2 \sum_{k \leq j} \sup_{K \in \mathcal{J}} \alpha_K^k T_{\epsilon^k; K}^* f + 8R^2 y. \end{aligned}$$

By Lemma 5, for any $k \in \mathbb{Z}$ and $z_k > 0$, we have

$$\mu \left(\sup_{K \in \mathcal{J}} \alpha_K^k T_{\epsilon^k; K}^* f > z_k 2^k \right) \leq C_{p,q} z_k^{-q} 2^{-pk} \|f\|_{p,q}^p. \tag{15}$$

Consequently,

$$\begin{aligned} y^p \mu \left(T^* f > (2 + 8R^2)y \right) &\leq y^p \mu(s^* f > y) + y^p \mu \left(T^* f > (2 + 8R^2)y, s^* f \leq y \right) \\ &\leq \|f\|_{p,\infty}^p + y^p \mu \left(\sum_{k \leq j} \sup_{K \in \mathcal{J}} \alpha_K^k T_{\epsilon^k; K}^* f > y \right) \\ &\leq \|f\|_{p,q}^p + y^p \mu \left(\sum_{k \leq j} \sup_{K \in \mathcal{J}} \alpha_K^k T_{\epsilon^k; K}^* f > 2^j \right). \end{aligned}$$

To use (15), observe that $c_\beta \sum_{k \leq j} 2^{\beta(k-j)} = 1$ if $\beta > 0$ and $c_\beta = 1 - 2^{-\beta}$. Set

$$2^j c_\beta 2^{\beta(k-j)} = c_\beta 2^{(\beta-1)(k-j)} 2^k =: z_k 2^k.$$

Then for $\beta = (q - p)/(2q)$, we get

$$z_k^{-q} 2^{-pk} \leq C_{p,q} 2^{-pj} 2^{p(j-k)+q(\beta-1)(j-k)} \leq C_{p,q} y^{-p} 2^{(q-p)(k-j)/2}.$$

Thus, by (15),

$$\begin{aligned} \mu \left(\sum_{k \leq j} \sup_{K \in \mathcal{J}} \alpha_K^k T_{\epsilon^k; K}^* f > 2^j \right) &\leq \sum_{k \leq j} \mu \left(\sup_{K \in \mathcal{J}} \alpha_K^k T_{\epsilon^k; K}^* f > z_k 2^k \right) \\ &\leq C_{p,q} \sum_{k \leq j} z_k^{-q} 2^{-pk} \|f\|_{p,q}^p \\ &\leq C_{p,q} y^{-p} \|f\|_{p,q}^p \sum_{k \leq j} 2^{(q-p)(k-j)/2} \\ &\leq C_{p,q} y^{-p} \|f\|_{p,q}^p, \end{aligned}$$

so the lemma is proved. □

Let Δ denote the closure of the triangle in \mathbb{R}^2 with vertices $(0, 0)$, $(1/2, 1/2)$ and $(1, 0)$ except the points $(x, 1 - x)$, $1/2 < x \leq 1$.

Lemma 7 *Suppose that $1 < p, q < \infty$ satisfy $(1/p, 1/q) \in \Delta$. Then, for all $f \in L_p$,*

$$\|f\|_{p,q} \leq C_{p,q} \|f\|_p.$$

Proof For an arbitrary $x > 0$, let us use the definition of α_I given in (13). Then α_I is \mathcal{F}_{I^-} -measurable and, obviously,

$$\alpha_I \alpha_J = 0 \text{ if } I \not\subseteq J \text{ or } J \not\subseteq I.$$

For all $I \in \mathcal{J}$, introduce the projections $F_I := \alpha_I s_I$ and observe that $s_I \circ s_J = 0$ for every incomparable I and J . Therefore, we get for every $g \in L_1$ and $I, J \in \mathcal{J}$ that

$$F_I(F_J g) = \alpha_I s_I(s_J(\alpha_J g)) = s_I(\alpha_I \alpha_J s_J g) = \delta_{I,J} F_I g,$$

where $\delta_{I,J}$ is the Kronecker symbol. Thus the projections F_I are orthogonal and Bessel's inequality implies for any $g \in L_2$ that

$$\|(F_I g, I \in \mathcal{J})\|_{L_2(l_2)}^2 = \sum_{I \in \mathcal{J}} \|F_I g\|_2^2 \leq \|g\|_2^2.$$

Let us introduce the operators

$$G_I g := E_{I^-}(\eta_I F_I g) \quad (g \in L_1, I \in \mathcal{J}),$$

where $(\eta_I, I \in \mathcal{J})$ is a fixed sequence of functions satisfying $\|\eta_I\|_\infty \leq 1$ for each $I \in \mathcal{J}$. Then

$$\begin{aligned} \|(G_I g, I \in \mathcal{J})\|_{L_2(l_2)}^2 &\leq \int_{G_m} \sum_{I \in \mathcal{J}} E_{I^-} |F_I g|^2 d\mu \\ &= \int_{G_m} \sum_{I \in \mathcal{J}} |F_I g|^2 d\mu \leq \|g\|_2^2. \end{aligned}$$

Furthermore, by Doob's inequality,

$$\|(G_I g, I \in \mathcal{J})\|_{L_s(l_\infty)} \leq \left\| \sup_{I \in \mathcal{J}} E_{I^-} |g| \right\|_s \leq C_s \|g\|_s$$

for any $1 < s \leq \infty$ and $g \in L_s$. It follows by interpolation that

$$\|(G_I g, t \in \mathcal{J})\|_{L_p(l_q)} \leq C_{p,q} \|g\|_p \quad (g \in L_p)$$

where $1/p = (1-t)/2 + t/s$ and $1/q = (1-t)/2$ for any $0 \leq t \leq 1$. Setting $g := f$ and $\eta_t := \text{sign } s_t f$, we have

$$\left(\int_{G_m} \left(\sum_{I \in \mathcal{J}} (\alpha_I E_{I^-} |s_t f|)^q \right)^{p/q} d\mu \right)^{1/p} \leq C_{p,q} \|f\|_p.$$

Using the fact that

$$\alpha_I E_{I-|S_I f|} > x \alpha_I,$$

we can see that

$$x \left(\int_{G_m} \left(\sum_{I \in \mathcal{J}} \alpha_I \right)^{p/q} d\mu \right)^{1/p} \leq C_{p,q} \|f\|_p,$$

which finishes the proof. \square

Now we are ready to formulate our first main result.

Theorem 2 *Let $f \in L_p(G_m)$, where $1 < p < \infty$. Then*

$$\|S^* f\|_p \leq c_p \|f\|_p,$$

where

$$S^* f := \sup_{n \in \mathbb{N}} |S_n f|.$$

Proof It is easy to see that Lemma 1 implies $S^* f \leq T^* f$. It follows from Lemmas 6 and 7 that

$$\sup_{y>0} y^p \mu(S^* f > y) \leq C_p \|f\|_p$$

for $1 < p < \infty$. Now the proof of the theorem follows by the Marcinkiewicz interpolation theorem. \square

The next norm convergence result in L_p spaces for $1 < p < \infty$ follow from the density of the Vilenkin polynomials in $L_p(G_m)$ and from Theorem 2.

Theorem 3 *Let $f \in L_p(G_m)$, where $1 < p < \infty$. Then*

$$\|S_n f - f\|_p \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Our announced Carleson–Hunt type theorem reads:

Theorem 4 *Let $f \in L_p(G_m)$, where $p > 1$. Then*

$$S_n f \rightarrow f, \text{ a.e., as } n \rightarrow \infty.$$

The proof follows directly by using Theorem 2 and the fact that the Vilenkin polynomials are dense in L_p .

5 Almost Everywhere Divergence of Vilenkin–Fourier Series

A set $E \subset G_m$ is called a set of divergence for $L_p(G_m)$ if there exists a function $f \in L_p(G_m)$ whose Vilenkin–Fourier series diverges on E .

Lemma 8 *If E is a set of divergence for $L_1(G_m)$, then there is a function $f \in L_1(G_m)$ such that $S^* f = \infty$ on E .*

Proof We claim that given any $g \in L_1(G_m)$, there is an unbounded monotone increasing sequence $\lambda = (\lambda_j, j \in \mathbb{N})$ of positive real numbers and a function $f \in L_1(G_m)$ such that

$$\hat{f}(j) = \lambda_j \hat{g}(j) \quad (j \in \mathbb{N}). \tag{16}$$

To prove this claim use (5) for $p = 1$ to choose a strictly increasing sequence of positive integer n_1, n_2, \dots such that

$$\|S_{M_{n_k}} g - g\|_1 < M_k^{-1} \quad (k \in \mathbb{N}_+). \tag{17}$$

Consider the function f defined by

$$f := g + \sum_{k=1}^{\infty} (g - S_{M_{n_k}} g).$$

By (17), the series converges in the norm of $L_1(G_m)$. In particular, f belongs to $L_1(G_m)$ and

$$\hat{f}(j) = \hat{g}(j) + \sum_{k=1}^{\infty} \int_{G_m} (g - S_{M_{n_k}} g) \overline{\psi}_j d\mu$$

for $j \in \mathbb{N}$. Therefore, the claim follows from orthogonality if we set

$$\lambda_j := 1 + \sum_{k \in \mathbb{N}_+ : M_{n_k} \leq j} 1 \quad (j \in \mathbb{N}).$$

To prove the theorem, suppose that $g \in L_1(G_m)$ is a function whose Vilenkin–Fourier series diverges on E . Use the claim to choose a monotone increasing, unbounded sequence λ which satisfies (16). By Abel’s transformation,

$$\begin{aligned} S_n g - S_m g &= \sum_{j=m}^{n-1} (S_{j+1} f - S_j f) \frac{1}{\lambda_j} \\ &= \frac{S_n f}{\lambda_{n-1}} - \frac{S_m f}{\lambda_m} + \sum_{j=m+1}^{n-1} \left(\frac{1}{\lambda_{j-1}} - \frac{1}{\lambda_j} \right) S_j f \end{aligned}$$

for any integers $n, m \in \mathbb{N}$ with $n > m$. Since λ is increasing, it follows that

$$|S_n g - S_m g| \leq \frac{2}{\lambda_m} S^* f \quad (n, m \in \mathbb{N}, n > m).$$

Since λ is unbounded, it follows that $S_n g$ converge at x when $S^* f(x)$ is finite. In particular, $(S^* f)(x) = \infty$ for all $x \in E$. □

Lemma 9 *A set $E \subseteq G_m$ is a set of divergence for $L_1(G_m)$ if and only if there exist Vilenkin polynomials P_1, P_2, \dots such that*

$$\sum_{j=1}^{\infty} \|P_j\|_1 < \infty \tag{18}$$

and

$$\sup_{j \in \mathbb{N}_+} S^* P_j(x) = \infty \quad (x \in E). \tag{19}$$

Proof Suppose first that E is a set of divergence for $L_1(G_m)$. Let $g \in L_1(G_m)$ be a function whose Vilenkin–Fourier series diverges on E . By repeating the proof of Lemma 8, we can choose an unbounded, monotone increasing positive sequence $(\lambda_j, j \in \mathbb{N})$ and a function $f \in L_1(G_m)$ such that

$$S_n g - S_m g = \frac{S_n f}{\lambda_{n-1}} - \frac{S_m f}{\lambda_m} + \sum_{j=m+1}^{n-1} \left(\frac{1}{\lambda_{j-1}} - \frac{1}{\lambda_j} \right) S_j f$$

for all integers $n, m \in \mathbb{N}, m < n$.

Let $(\omega_j, j \in \mathbb{N})$ be an unbounded sequence of positive, increasing numbers which satisfy

$$\sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j} - \frac{1}{\lambda_{j+1}} \right) \omega_j < \infty.$$

For example, let

$$\omega_j := \frac{1}{\frac{1}{\sqrt{\lambda_j}} + \frac{1}{\sqrt{\lambda_{j+1}}}}.$$

Indeed, then

$$\sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j} - \frac{1}{\lambda_{j+1}} \right) \omega_j \leq \sum_{j=1}^{\infty} \left(\frac{1}{\sqrt{\lambda_j}} - \frac{1}{\sqrt{\lambda_{j+1}}} \right) = \frac{1}{\sqrt{\lambda_1}}.$$

Fix $x \in E$. If

$$|S_j f(x)| = O(\omega_j), \quad \text{as } j \rightarrow \infty,$$

then

$$|S_n g(x) - S_m g(x)| \rightarrow 0, \text{ as } n, m \rightarrow \infty$$

and we get that $S_n g(x)$ is a convergent series for any $x \in E$, which is contradiction. Consequently, the inequality

$$|S_n f(x)| > \omega_n \tag{20}$$

holds for infinitely many integers $n \in \mathbb{N}$.

Use (5) for $p = 1$ to choose strictly increasing sequences of positive integers $(n_j, j \in \mathbb{N})$ and $(\alpha_j, j \in \mathbb{N})$ which satisfy $n_j < \alpha_j + 1$,

$$\|f - S_{M_{n_j}} f\|_1 < M_j^{-1} \tag{21}$$

and

$$\|S^*(S_{M_{n_j}} f)\|_\infty < \frac{\omega_{\alpha_j}}{2} \quad (j \in \mathbb{N}). \tag{22}$$

Consider the functions defined by

$$P_j := S_{M_{\alpha_{j+1}}}(f - S_{M_{n_j}} f) \quad (j \in \mathbb{N}_+).$$

Clearly, these functions are Vilenkin polynomials. We will show that they satisfy (18) and (19). Since $\|S_{M_n} h\|_1 \leq \|h\|_1$, for $n \in \mathbb{N}$ and $h \in L_1(G_m)$, (18) is a direct consequence of (21). To verify (19), fix $x \in E$ and choose an $n \in \mathbb{N}$ satisfying (20) which is large enough so that $\alpha_j < n \leq \alpha_{j+1}$ for some $j \in \mathbb{N}_+$. Since the definition of P_j implies

$$S_n P_j = S_n f - S_n(S_{M_{n_j}} f),$$

we have by (20) and (22) that

$$|(S_n P_j)(x)| \geq |(S_n f)(x)| - \frac{\omega_{\alpha_j}}{2} \geq \frac{1}{2} \omega_{\alpha_j}.$$

Hence (19) follows from the fact that $\omega_n \rightarrow \infty$ as $n \rightarrow \infty$.

Conversely, suppose that

$$P_j := \sum_{k=0}^{M_{\alpha_{j-1}}} c_k^{(j)} \psi_k \quad (\alpha_j \in \mathbb{N}, j \in \mathbb{N}_+)$$

is a sequence of Vilenkin polynomials which satisfies (18) and (19). Let $n_1 := \alpha_1$ and for $j > 1$ set $n_j := 1 + \max\{n_{j-1}, \alpha_j\}$. Then $(n_j, j \in \mathbb{N})$ is a strictly increasing sequence of integers and it is easy to see that

$$M_{n_{j+1}} \oplus k_1 > M_{n_j} \oplus k_0 \tag{23}$$

for any choice of integers k_0 and k_1 which satisfy $0 \leq k_0 \leq M_{\alpha_j}, 0 \leq k_1 \leq M_{\alpha_{j+1}}$ and $j \in \mathbb{N}$. Let

$$f := \sum_{j=1}^{\infty} \psi_{M_{n_j}} P_j$$

and observe by (18) that $f \in L_1(G_m)$. It is clear that the series defining f converges in $L_1(G_m)$ norm. Consequently, this series is the Vilenkin–Fourier series of f . Moreover, (23) can be used to see that

$$S_{M_{n_j}+k} f - S_{M_{n_j}} f = \psi_{M_{n_j}} S_k P_j$$

for $0 \leq k < M_{n_{j+1}} - M_{n_j}, j \in \mathbb{N}_+$. In particular, (19) implies the Vilenkin–Fourier series of f diverges at each $x \in E$. □

Corollary 1 *If E_1, E_2, \dots are sets of divergence for $L_1(G_m)$, then*

$$E := \cup_{n=1}^{\infty} E_n$$

is also a set of divergence for $L_1(G_m)$.

Proof Apply Lemma 9 to choose Vilenkin polynomials $P_1^{(n)}, P_2^{(n)}, \dots$ such that

$$\sum_{j=1}^{\infty} \|P_j^{(n)}\|_1 < \infty$$

and

$$\sup_{j \in \mathbb{N}_+} (S^* P_j^{(n)})(x) = \infty \quad (x \in E_n, n \in \mathbb{N}_+). \tag{24}$$

Thus there exist integers $\alpha_1 < \alpha_2 < \dots$ such that

$$\sum_{j=\alpha_n}^{\infty} \|P_j^{(n)}\|_1 < \frac{1}{M_n} \quad (n \in \mathbb{N}_+).$$

Let $(Q_j, j \in \mathbb{N}_+)$ be any enumeration of the polynomials

$$\left\{ P_j^{(n)} : j \geq \alpha_n, n = 1, 2, \dots \right\}.$$

e.g.,

$$\begin{aligned} Q_1 &:= P_{\alpha_1}^{(1)}, \\ Q_2 &:= P_{\alpha_2}^{(2)}, \quad Q_3 := P_{\alpha_1+1}^{(1)}, \\ Q_4 &:= P_{\alpha_1+2}^{(1)}, \quad Q_5 := P_{\alpha_2+1}^{(2)}, \quad Q_6 := P_{\alpha_3}^{(3)}, \end{aligned}$$

$$Q_7 := P_{\alpha_4}^{(4)}, \quad Q_8 := P_{\alpha_3+1}^{(3)}, \quad Q_9 := P_{\alpha_2+2}^{(2)}, \quad Q_{10} := P_{\alpha_1+3}^{(1)}$$

$$Q_{11} := P_{\alpha_1+4}^{(1)}, \quad Q_{12} := P_{\alpha_2+3}^{(2)}, \quad Q_{13} := P_{\alpha_3+2}^{(3)}, \quad Q_{14} := P_{\alpha_4+1}^{(4)}, \quad Q_{15} := P_{\alpha_5}^{(5)} \dots$$

Each Q_j is a Vilenkin polynomial and

$$\sum_{j=1}^{\infty} \|Q_j\|_1 < \sum_{n=1}^{\infty} \frac{1}{M_n} < \infty.$$

In particular, by Lemma 9 it suffices to show that

$$\sup_{j \in \mathbb{N}_+} (S^* Q_j)(x) = \infty, \quad \text{for } x \in E.$$

But this follows from the construction and from (24) since every $x \in E$ necessarily belongs to some E_n . □

Theorem 5 *If $1 \leq p < \infty$ and $E \subseteq G_m$ is a set of Haar measure zero, then E is a set of divergence for $L_p(G_m)$.*

Proof We begin with a general remark. If $A \subseteq G_m$ is a finite union of intervals I_1, I_2, \dots, I_n for some $n \in \mathbb{N}_+$ and if N is any non-negative integer, then there exists a Vilenkin polynomial P such that, for some $i \geq N$,

$$P = \sum_{k=M_N}^{M_i-1} c_k \psi_k,$$

which satisfies

$$|P(x)| = 1, \quad (x \in A) \quad \text{and} \quad \int_{G_m} |P|^p \, d\mu = \mu(A).$$

Indeed, if $i := \max\{M_N, 1/\mu(I_j) : 1 \leq j \leq n\}$, then, in view of (2), we find that $P := \chi(A)\psi_i$ is such a polynomial.

To prove the theorem, suppose $E \subseteq G_m$ satisfies $\mu(E) = 0$. Cover E with intervals $(I_k, k \in \mathbb{N})$ such that

$$\sum_{k=0}^{\infty} \mu(I_k) < 1$$

and each $x \in E$ belongs to infinitely many of the sets I_k . Set $n_0 := 0$ and choose integers $n_0 < n_1 < n_2 \dots$ such that

$$\sum_{k=n_j}^{\infty} \mu(I_k) < M_j^{-1} \quad (j \in \mathbb{N}).$$

Apply the general remark above successively to the sets

$$A_j := \bigcup_{k=n_j}^{n_{j+1}-1} I_k \quad (j \in \mathbb{N})$$

to generate integers $\alpha_0 := 0 < \alpha_1 < \alpha_2 < \dots$ and Vilenkin polynomials P_0, P_1, \dots such that $sp(P_j) \subset [M_{\alpha_j}, M_{\alpha_{j+1}})$:

$$P_j = \sum_{k=M_{\alpha_j}}^{M_{\alpha_{j+1}}-1} c_k \psi_k,$$

$$\|P_j\|_p^p = \mu(A_j) \leq M_j^{-1} \tag{25}$$

and

$$|P_j(x)| = 1 \quad x \in A_j, \quad \text{for } j \in \mathbb{N}. \tag{26}$$

Setting

$$f := \sum_{j=1}^{\infty} P_j,$$

we observe by (25) that this series converges in $L_p(G_m)$ norm. Hence $f \in L_p(G_m)$ and this series is the Vilenkin–Fourier series of f . Moreover, since the spectra of the polynomials P_j are pairwise disjoint, we have

$$S_{M_{\alpha_{j+1}}} f - S_{M_{\alpha_j}} f = P_j \quad (j \in \mathbb{N}_+).$$

Since every $x \in E$ belongs to infinitely many of the sets A_j , it follows from (26) that the Vilenkin–Fourier series of f diverges at every point $x \in E$. □

This theorem cannot be improved for $1 < p < \infty$ and measurable sets with non-zero measure. Indeed, in this case the Vilenkin–Fourier series of an $f \in L_p(G_m)$ converges a.e. (see Theorem 4). However, it can be improved considerably for $p = 1$.

Theorem 6 *There is a function $f \in L_1(G_m)$ whose Vilenkin–Fourier series diverges everywhere.*

Proof Fix $\alpha_n \in [M_{n-1}, M_n)$, where $n \in \mathbb{N}_+$ is odd. By using the lower estimate for the Lebesgue constant in (3), we can conclude that there exists an absolute constant $C > 0$, which does not depend on n , such that

$$\left\| \sum_{k=0}^{\alpha_n-1} \psi_k \right\|_1 = \|D_{\alpha_n}\|_1 > Cn.$$

Consider the function

$$g_n(x) = \begin{cases} \frac{D_{\alpha_n}(x)}{|D_{\alpha_n}(x)|}, & \text{if } D_{\alpha_n}(x) \neq 0, \\ 0, & \text{if } D_{\alpha_n}(x) = 0. \end{cases}$$

It is constant on any set of the form $I_n(x)$, $x \in G_m$. Hence g_n is a Vilenkin polynomial of order at most M_n . Moreover, since

$$\psi_k(x - t) = \psi_k(x)\overline{\psi}_k(t) = \overline{\psi}_k(t), \quad 0 \leq k < M_n, \quad x \in I_n(0).$$

we get that

$$D_{\alpha_n}(x - t) = \overline{D}_{\alpha_n}(t), \quad x \in I_n(0).$$

Hence, by the choice of $\alpha_n \in [M_{n-1}, M_n)$ we have

$$(S_{\alpha_n}g_n)(x) = \int_{G_m} g_n(t)D_{\alpha_n}(x - t) = \|D_{\alpha_n}\|_1 > Cn, \quad (x \in I_n(0)).$$

For $k = \sum_{s=0}^{n-1} k_s M_s$, ($k_s \in Z_{m_s}$), let us define the points $x_k^{(n)} \in G_m$, ($0 \leq k < M_n$) by $x_k^{(n)} := (k_0, k_1, \dots, k_{n-1}, 0, 0, \dots)$ and set

$$Q_n := \prod_{k=0}^{M_n-1} \left(1 - \tau_{x_k^{(n)}} g_n \frac{\sum_{s=1}^{m_{n+k}-1} r_{n+k}^s}{m_{n+k} - 1} \right), \quad \text{where } \tau_{x_k^{(n)}} g_n(x) = g_n(x - x_k^{(n)}).$$

By using (1) we find that

$$\frac{\sum_{s=1}^{m_{n+k}-1} r_{n+k}^s(x)}{m_{n+k} - 1} = \begin{cases} 1, & \text{if } x_{n+k} = 0, \\ \frac{-1}{m_{n+k}-1}, & \text{if } x_{n+k} \neq 0 \end{cases} \tag{27}$$

and

$$1 - \frac{\sum_{s=1}^{m_{n+k}-1} r_{n+k}^s(x)}{m_{n+k} - 1} = \begin{cases} 0, & \text{if } x_{n+k} = 0, \\ 1 + \frac{1}{m_{n+k}-1}, & \text{if } x_{n+k} \neq 0. \end{cases} \tag{28}$$

It is easy to show that for any $x \in G_m$ there exists $x_j^{(n)} := (j_0, j_1, \dots, j_{n-1}, 0, 0, \dots)$ such that $I_n(x) = I_n(x_j^{(n)})$, that is

$$x_0 = j_0, \quad x_1 = j_1, \quad x_{n-1} = j_{n-1}.$$

Consider the j -th term of the expression of Q_n and let $x \in I_n(x_j^{(n)})$. Since $\tau_{x_j^{(n)}} g_n(x_j^{(n)}) = g_n(0) = 1$, according to (28) we can conclude that

$$1 - \tau_{x_j^{(n)}} g_n(x) \frac{\sum_{s=1}^{m_{n+j}-1} r_{n+j}^s(x)}{m_{n+j} - 1} = 1 - \frac{\sum_{s=1}^{m_{n+j}-1} r_{n+j}^s(x)}{m_{n+j} - 1} = 0 \text{ if } x_{n+j} = 0.$$

Since $x \in I_n(x_j^{(n)})$ for some $0 \leq k \leq M_n - 1$,

$$Q_n(x) = \prod_{k=0}^{M_n-1} \left(1 - \tau_{x_k^{(n)}} g_n(x) \frac{\sum_{s=1}^{m_{n+k}-1} r_{n+k}^s(x)}{m_{n+k} - 1} \right) = 0 \text{ if } x_{n+k} = 0.$$

On the other hand, (27) and $|\tau_{x_k^{(n)}} g_n(x)| \leq 1$ imply that if $x_{n+k} \neq 0$, we get that

$$\begin{aligned} \left| 1 - \tau_{x_k^{(n)}} g_n \frac{\sum_{s=1}^{m_{n+k}-1} r_{n+k}^s}{m_{n+k} - 1} \right| &\leq 1 + \left| \tau_{x_k^{(n)}} g_n \frac{\sum_{s=1}^{m_{n+k}-1} r_{n+k}^s}{m_{n+k} - 1} \right| \\ &\leq 1 + \frac{1}{m_{n+k} - 1} = \frac{m_{n+k}}{m_{n+k} - 1}. \end{aligned}$$

It follows that

$$|Q_n(x)| \leq \prod_{k=0}^{M_n-1} \frac{m_{n+k}}{m_{n+k} - 1}, \text{ if } x \in I_n(x) \text{ and } x_{n+k} \neq 0, \text{ for all } 0 \leq k \leq M_n - 1.$$

Hence, we can conclude that $Q_n \in L_1(G_m)$. Indeed,

$$\begin{aligned} &\int_{G_m} |Q_n| d\mu \\ &\leq \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_{n-1}=0}^{m_{n-1}-1} \sum_{x_n=0}^{m_n-1} \cdots \sum_{x_{n+M_n-1}=1}^{m_{n+M_n-1}-1} \int_{I_{n+M_n}(x)} \left(\prod_{k=0}^{M_n-1} \frac{m_{n+k}}{m_{n+k} - 1} \right) d\mu \\ &= \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_{n-1}=0}^{m_{n-1}-1} \sum_{x_n=0}^{m_n-1} \cdots \sum_{x_{n+M_n-1}=1}^{m_{n+M_n-1}-1} \frac{1}{M_{n+M_n}} \left(\prod_{k=0}^{M_n-1} \frac{m_{n+k}}{m_{n+k} - 1} \right) \\ &= \frac{1}{M_n \prod_{k=0}^{M_n-1} m_{n+k}} \left(\prod_{k=0}^{M_n-1} \frac{m_{n+k}}{m_{n+k} - 1} \right) \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_{n-1}=0}^{m_{n-1}-1} \sum_{x_n=0}^{m_n-1} \cdots \sum_{x_{n+M_n-1}=1}^{m_{n+M_n-1}-1} 1 \\ &= \frac{1}{M_n \prod_{k=0}^{M_n-1} m_{n+k}} \left(\prod_{k=0}^{M_n-1} \frac{m_{n+k}}{m_{n+k} - 1} \right) M_n \prod_{k=0}^{M_n-1} (m_{n+k} - 1) = 1. \end{aligned}$$

Clearly, Q_n is a Vilenkin polynomial. Moreover, since the terms of the expanded product have pairwise disjoint spectra, by expanding the product used to define Q_n , it is easy to see for $k = 0, 1, \dots, M_n - 1$ that

$$S_{M_{n+k}+\alpha_n} Q_n - S_{M_{n+k}} Q_n = \frac{\pm 1}{m_{n+k} - 1} r_{n+k} S_{\alpha_n}(\tau_{x_k^{(n)}} g_n),$$

where $+$ sign is if $M_n - 1$ is even number and $-$ sign is if $M_n - 1$ is odd.

Since

$$|(S_{M_{n+k}} Q_n)(x)| \leq \|D_{M_{n+k}}\|_1 \|Q_n\|_1 \leq 1,$$

choice of the integers α_n therefore, for sufficiently large n imply

$$\begin{aligned} |(S_{M_{n+k}+\alpha_n} Q_n)(x)| &> Cn - |(S_{M_{n+k}} Q_n)(x)| \\ &> \frac{C}{2}n, \quad (x \in I_n(x_k^{(n)})). \end{aligned} \tag{29}$$

Let $n_1 < n_2 < \dots$ be positive integers chosen so that

$$\sum_{j=1}^{\infty} \frac{1}{\sqrt{n_j}} < \infty \quad \text{and set} \quad j := \frac{Q_{n_j}}{\sqrt{n_j}} \quad (j \in \mathbb{N}_+).$$

It is evident that

$$\sum_{j=1}^{\infty} \|P_j\|_1 < \infty.$$

Moreover, for a fixed $x \in G_m$ it is possible to choose integers $0 \leq k_{(j)} < 2^{n_j}$ such that $x \in I_{n_j}(x_{k_{(j)}}^{(n_j)})$ ($j \in \mathbb{N}_+$). Hence, (29) implies

$$(S^* P_j)(x) \geq \frac{C}{2} \sqrt{n_j}$$

for $j \in \mathbb{N}_+$ and $x \in G_m$. Consequently, G_m is a set of divergence for $L_1(G_m)$ by Lemma 9. The proof is complete. \square

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