

## RESEARCH ARTICLE

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# A note on contributions concerning nonseparable spaces with respect to signal processing within Bayesian frameworks

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In this paper, we discuss the study of some signal processing problems within Bayesian frameworks and semigroups theory, in the case where the Banach space under consideration may be nonseparable. For applications, the suggested approach may be of interest in situations where approximation in the norm of the space is not possible. We describe the idea for the case of the abstract Cauchy problem for the evolution equation and provide more detailed example of the diffusion equation with the initial data in the nonseparable Morrey space.

## KEYWORDS

Bayesian approach, diffusion equation, equations in function spaces, evolution equation, Morrey spaces, nonseparable Banach space, operator semigroup, parabolic equation, stochastic partial differential equation

## MSC CLASSIFICATION

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## 1 | INTRODUCTION

Bayesian methods have proven to be the most rigorous probabilistic framework to identify target variables and evaluate the corresponding uncertainties using the available information. As known, the Bayesian interface is a probabilistic method of inference that allows to form probabilistic estimates of certain parameters from a given series of observations. We refer, for instance, to the survey paper<sup>1</sup> for the Bayesian approach to signal processing problems in which the signal is a solution of a stochastic partial differential equation (SPDE).

In the survey,<sup>1</sup> a focus there was to show that the idea to work with probability measures on a suitable function space is a key idea, which leads to the notion of a well-posed signal-processing problem. This approach helps to avoid dealing with the algorithms to which the use of known methods based on application of the standard Markov chain Monte Carlo (MCMC) method after a discretization leads. One of the disadvantages of such algorithms is that they perform poorly under refinement of the discretization.

The use of a proper mathematical formulation of the problems on domain space while working with probability measures on function spaces leads to efficient sampling techniques, defined on path-space as the domain space, and therefore robust under the introduction of discretization. A wide variety of signal processing problems is overviewed in Hairer et al.<sup>1</sup> These problems lead to a posterior probability measure on a separable Banach space. The separability of a Banach space provides a possibility for approximation in the space. However, when looking at the problems tackled in Bayesian

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frameworks, it is also of interest to explore what potential “gains” and/or “losses” might be in the case where the space is not necessarily separable. Such kind of questions sometimes are discussed also at some online forums. In this note, we touch some questions concerning such issues.

As known, the theory of Markov stochastic processes is a natural source of some parabolic equations. When passing to a macro description of such processes with respect to the densities of transition probabilities, under special conditions there appear partial differential equations. It is worth mentioning that A.P. Kolmogorov yet in 1934 (see Kolmogoroff<sup>2</sup>), proceeding from the problems of the theory of probability, defined and studied an “interesting” equation, which he called the equation of diffusion with inertia. The properties of solutions of such equations and the methods of their study, for natural reasons, are very closely related to the properties of solutions of the heat equation and diffusion equations. Parabolic equations of diffusion-type and their various generalizations are known to be widely studied and may be found in various books, in particular in the Hörmander books.<sup>3</sup>

Study of such equations under special conditions and in various function spaces could make possible a more precise observing the results and the possibilities of their further applications.

The theory of parabolic equations has deep connections with functional analysis, especially with the theory of evolution equations with unbounded operators in Banach spaces and the theory of semigroups.

The theory of stochastic processes, especially the theory of Markov processes and stochastic differential equations, very closely interacts with the theory of parabolic equations.

In this paper, we suggest an approach for the study of properties of solutions to such equations in the case of nonseparable function spaces. In case of problems formulated as filtering problems in addition to smoothing problems, an approach based on study in the frameworks of weighted nonstandard function spaces could also be useful.

One of the main approaches to evolution equations relates to the theory of semigroups of operators. This approach is under discussion in our paper with respect to the Cauchy problem for an abstract evolution equation.

## 2 | SOLUTION OF CAUCHY PROBLEM VIA SEMIGROUPS

One of the approaches to the study of evolution equations is based on the use of semigroups of operators. This approach has a long history for about 80 years. Already in the paper<sup>4</sup> of 1954, the reader can find a comparison of different settings of the Cauchy problem for an abstract evolution equation, for which the operator semigroups approach is applicable. The presentation of this approach may be found in a big variety of books and surveys. We refer, for instance, to previous works<sup>5–13</sup> and Yosida<sup>14</sup> on semigroups of operators and their applications to partial differential equations.

Below, we provide necessary standard definitions concerning operator semigroups.

A family of continuous linear operators  $T_t$ ,  $t > 0$ , in a Banach space  $E$  is called a semigroup of operators if

$$T_t T_s = T_{t+s}, \quad t > 0, s > 0 \quad \text{and} \quad T_0 = I.$$

A semigroup of operators is called strongly continuous if

$$T_t f \rightarrow T_s f \quad \text{in } E \quad \text{as } t \rightarrow s, \quad s > 0, \quad \text{for all } f \in E.$$

A semigroup  $T_t$  is said to be of class  $C_0$  if

$$\lim_{t \rightarrow 0} \|T_t f - f\|_E = 0.$$

The operator

$$A f := \lim_{t \rightarrow 0} \frac{1}{t} (T_t f - f),$$

defined for  $f \in E$  whenever this limit exists, is called the infinitesimal operator of the semigroup  $T_t$ .

If the operator  $A$  admits an extension to a closed operator  $\bar{A}$ , then the operator  $\bar{A}$  is referred to as the generator of the semigroup  $T_t$ .

More information on semigroups of operators may be found, for example, in previous books<sup>6,15,16</sup> and Yosida.<sup>14</sup>

To describe the principal idea of the operator semigroup approach, we consider the standard Cauchy problem

$$\begin{cases} u_t = Au, \\ u|_{t=0} = u_0 \end{cases}, \quad (2.1)$$

where  $u$  is the unknown function and  $A$  is a linear operator. It may be a function  $u(x, t)$ , where  $x \in \Omega \subseteq \mathbb{R}^n, t > 0$ , and  $A = \Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$  being the classical Laplace operator corresponding to the heat or diffusion processes.

More generally, it may be in abstract form, that is,  $u$  is an element of a Banach space  $E$ , depending on parameter  $t > 0$ , with the derivative  $u_t = \frac{\partial u}{\partial t}$  interpreted in a proper sense and  $A$  is a linear operator with the domain  $D(A) \subset E$  (now we present a formal procedure, precise assumptions and formulations being considered later).

Formally solving the equation  $\frac{\partial u}{\partial t} = Au$  as an ordinary differential equation and taking the initial condition into account, we get

$$u = e^{tA}u_0,$$

assuming that  $e^{tA}$  is well defined. Formally again  $\{e^{tA}\}_{t>0}$  is a semigroup. Thus, we say that the semigroup  $T_t = e^{tA}$  solves the Cauchy problem (2.1).

Certainly, this formal procedure needs justification. Such a justification needs certain assumptions on the space  $E$  and the operator  $A$  and is well known in the literature. These assumptions usually include the notion of nonpositivity or resolvent set of the operator  $A$  and the condition

$$D(A) = E. \quad (2.2)$$

For our goals in the sequel, we underline that in applications the condition (2.2) is in fact the assumption that the space  $E$  is separable. In this paper, we discuss a possibility to adjust the above operator semigroup procedure for non-necessarily separable spaces.

Concretely for the abstract Cauchy problem (2.1), the semigroup approach under various assumptions is dispersed in the literature and goes back to the paper.<sup>4</sup>

The proof of the following theorem was given in Phillips,<sup>4</sup> Theorem 3.1; see also Richtmyer.<sup>12</sup>

**Theorem 2.1.** *Let  $E$  be a Banach space. Consider a closed linear operator  $A$  with the domain  $D(A)$  dense in  $E$  and nonempty resolvent set. Suppose that the Cauchy problem (2.1) is uniquely solvable for every  $u_0 \in D(A)$ . Then there exists a semigroup  $T_t$  of the class  $C_0$  which solves the Cauchy problem (2.1).*

*Remark 2.2.* There are known sufficient conditions for the unique solvability of the Cauchy problem supposed in Theorem 2.1 given in terms of the generator of the semigroup  $T_t$ ; see, for example, Phillips,<sup>4</sup> Theorem 3.3

### 3 | WHAT CAN BE SAVED IF THE BANACH SPACE IS NONSEPARABLE?

Note that the assumption that the domain of the operator is dense in the considered Banach space, roughly speaking, presupposes that the space is separable. In this section, we provide some arguments which allow us to include nonseparable spaces into the operator semigroup approach. We explain these arguments for the model case of the Cauchy problem (2.1). Let  $X$  be a nonseparable space. Consider any Banach space  $E$  presumably separable, such that  $X \hookrightarrow E$  and the operator  $A$  obeys Theorem 2.1 in the space  $E$ . By  $D_X(A)$  and  $D_E(A)$ , we denote the domains of the operator  $A$  in the spaces  $X$  and  $E$ , respectively. Then clearly for  $f \in D_X(A) \subseteq D_E(A)$  by Theorem 2.1, we have the solution  $u(t) = T_t u_0 \in E$  for all  $t > 0$ .

However, our interest is to know that  $u(t) \in X, t > 0$  and  $u(t) \rightarrow u_0$  in  $X$  as  $t \rightarrow 0$ . The requirement that  $u(t) \in X, t > 0$ , is easily covered by the assumption that the operators  $T_t$  are bounded in  $X$  for  $t > 0$ . However, we cannot assume that  $T_t$  is of class  $C_0$  in  $X$ , since the latter in general does not hold in nonseparable spaces. Anyway, we have a weaker convergence  $\|u(t) - u_0\|_E \rightarrow 0$  as  $t \rightarrow 0$ .

Thus, dealing with a smaller space  $X \hookrightarrow E$ , from the known solvability in the space  $E$ , we can gain the information that  $u(t) \in X, t > 0$ , but have to keep a weaker  $E$ -convergence of  $u(t)$  to  $u_0$ .

We summarize this in the form of the following theorem:

**Theorem 3.1.** *Let the Banach space  $E$  and the operator  $A$  satisfy all the assumptions of Theorem 2.1, and let  $T_t$  be the semi-group provided by that theorem. Let also  $X$  be an arbitrary Banach space, non-necessarily separable, such that  $X \hookrightarrow E$ . Then  $u(t) \in X, t > 0$  for all  $u_0 \in D_X(A)$ , if the operator  $T_t, t > 0$  is bounded in  $X$ . The convergence  $T_t u_0 \rightarrow u_0$  holds for  $u_0 \in D_X(A)$ , but in a weaker norm of the space  $E$ .*

In the above scheme,  $X$  could be an arbitrary space. There is known a variety of nonseparable function spaces. Apart of the well known space  $L^\infty$  of essentially bounded functions in analysis and PDEs, there are also known such nonseparable spaces as Hölder spaces, Morrey spaces, and recently developed grand Lebesgue spaces. Observe that grand

Lebesgue spaces proved to be very important in some applications to PDEs; see for instance the papers D'Onofrio et al.<sup>17</sup> and Sbordone.<sup>18</sup>

In the example below, we take as  $X$  the Morrey space  $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ . Note that Morrey spaces are very popular both in analysis and applications to PDEs; see for instance the books of Giaquinta<sup>19</sup> and Sawano et al.<sup>20</sup>

Let us consider the diffusion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = k(\Delta u)(x,t), & x \in \mathbb{R}^n \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (3.1)$$

where  $k > 0$ . We take  $E = L^q_\beta(\mathbb{R}^n)$ ,  $1 < q < \infty$ ,  $\beta \in \mathbb{R}$ , where  $L^q_\beta(\mathbb{R}^n)$  is the weighted Lebesgue space defined by the norm

$$\|f\|_{L^q_\beta(\mathbb{R}^n)} = \left[ \int_{\mathbb{R}^n} \left| \frac{f(x)}{(1+|x|)^\beta} \right|^q dx \right]^{\frac{1}{q}}$$

and choose  $X = \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ ,  $0 < p < \infty$ ,  $0 < \lambda < n$ , where  $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$  is the Morrey space defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \sup_{r > 0} \left( \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

It is a nonseparable space. For more details on Morrey spaces, we refer to the book of Pick et al.<sup>21</sup> The subspace of functions  $f \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ , such that

$$\lim_{h \rightarrow 0} \|f(\cdot + h) - f(\cdot)\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = 0$$

is referred to as the Zorko space; see Zorko.<sup>22</sup> The Zorko space is a proper subspace of the Morrey space, and it is known that this is the maximal subspace of the Morrey space where the heat semigroup is of the class  $C_0$ ; see Kato,<sup>23</sup> Lemma 3.1

The embedding  $X \hookrightarrow E$  in this case holds under the conditions

$$q < p \text{ and } \beta > \frac{\lambda}{p} + n \left( \frac{1}{q} - \frac{1}{p} \right), \quad (3.2)$$

as derived from Samko.<sup>24</sup> Corollary 3.14

As is well known, the solution of the problem (3.1) is given by the semigroup

$$u(x,t) = T_t u_0(x), \quad T_t u_0(x) := \int_{\mathbb{R}^n} \mathcal{K}_t(x-y) u_0(y) dy, \quad (3.3)$$

where

$$\mathcal{K}_t(x) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4kt}}.$$

The fact that this semigroup is of class  $C_0$  in the space  $L^q(\mathbb{R}^n)$  is well known. In the lemma below, we justify that the same holds for the weighted Lebesgue space  $L^q_\beta(\mathbb{R}^n)$ .

**Lemma 3.2.** *Let  $q > 1$ , and  $\frac{1}{q} + \frac{1}{q'} = 1$ . Then the semigroup (3.3) is of class  $C_0$  in the space  $L^q_\beta(\mathbb{R}^n)$  if  $-\frac{n}{q'} < \beta < \frac{n}{q}$ .*

*Proof.* The kernel  $\mathcal{K}_t(x)$  of the convolution operator (3.3) has the form

$$\mathcal{K}_t(x) = \frac{1}{(\sqrt{t})^n} \mathcal{K}_1\left(\frac{x}{\sqrt{t}}\right).$$

Convolution operators with such a dilation kernel are uniformly in  $t$  dominated by the maximal operator, provided that  $\mathcal{K}_1(x)$  is radial, integrable on  $\mathbb{R}^n$  and as a radial function is increasing on  $\mathbb{R}_+$ ; see Stein.<sup>25</sup> Hence,

$$|T_t u_0(x)| \leq c M u_0(x), \quad M f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y-x|<r} |u_0(y)| dy. \quad (3.4)$$

The maximal operator  $M$  is bounded in the weighted space  $L^q_\beta(\mathbb{R}^n)$  if  $-\beta$  lies in the ‘‘Muckenhoupt interval’’  $(-\frac{n}{q}, \frac{n}{q'})$ ; see Dyn'kin and Osilenker.<sup>26</sup> Thus, the operators  $T_t$ ,  $t \in \mathbb{R}_+$ , are even uniformly in  $t$  bounded in  $L^q_\beta(\mathbb{R}^n)$ .

It remains to verify that

$$\lim_{t \rightarrow 0} \|T_t u_0 - u_0\|_{L^q_\beta(\mathbb{R}^n)} = 0.$$

This can be checked by the standard procedure via approximation of  $u_0 \in L^q_\beta(\mathbb{R}^n)$  by  $C_0^\infty$ -functions in  $L^q_\beta(\mathbb{R}^n)$  and using the uniform boundedness of  $T_t$ . The proof is complete.  $\square$

Finally, we present the following statement for the nonseparable Morrey space  $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$  :

**Theorem 3.3.** *Let  $u_0 \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ ,  $1 < p < \infty$ ,  $0 < \lambda < n$ . Then the unique solution  $u(x, t)$  of the problem (3.1) has the property  $u(\cdot, t) \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$  uniformly in  $t \in \mathbb{R}_+$ . The convergence*

$$\|T_t u_0 - u_0\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \rightarrow 0 \text{ as } t \rightarrow 0 \quad (3.5)$$

holds if  $u_0$  is in the Zorko subspace  $Z^{p,\lambda}(\mathbb{R}^n)$  of  $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ .

*Proof.* We embed the space  $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$  into the space  $L^q_\beta$  with some  $q$  and  $\beta$ , which is possible under the conditions (3.2). Choose also  $\beta < \frac{n}{q}$  and observe that the interval  $(\frac{\lambda}{p} + \frac{n}{q} - \frac{n}{p}, \frac{n}{q})$  is nonempty. Then in view of this embedding and Lemma 3.2 for the solution  $u(x, t)$ , we have the representation (3.3). Then the uniformness

$$\sup_{t>0} \|u(\cdot, t)\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{t>0} \|T_t u_0\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \leq \infty$$

of the inclusion  $u(t) \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$  follows from the uniform point-wise domination of the semigroup  $T_t$  by the maximal operator and the boundedness of the maximal operator in Morrey spaces. The latter was proved in Chiarenza and Frasca<sup>27</sup>; see also the proof for more general class of sublinear operators in Samko.<sup>28</sup>

As for the convergence  $\|T_t u_0 - u_0\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \rightarrow 0$  as  $t \rightarrow 0$ , it certainly does not hold for all  $u_0 \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ ; see counterexample in Kato.<sup>23</sup> Example 3.4

The convergence (3.5) with  $u_0 \in Z^{p,\lambda}(\mathbb{R}^n)$  for the heat semigroup was proved in Kato.<sup>23</sup> Lemma 3.1

Observe that in any case, we have such a convergence for all  $u_0 \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$  in a weaker norm  $\|\cdot\|_{L^q_\beta(\mathbb{R}^n)}$ , for  $\beta > \frac{\lambda}{p}$  and arbitrarily close to  $\frac{\lambda}{p}$ , since  $q$  in (3.2) may be chosen arbitrarily close to  $p$ . Convergence in  $L^q_\beta(\mathbb{R}^n)$ -norm becomes ‘‘less weak’’ when  $\beta \rightarrow \frac{\lambda}{p}$ .  $\square$

*Remark 3.4.* Theorem 3.3 may be extended to the so called generalized Morrey spaces (see their definition, e.g., in Rafeiro et al.<sup>29</sup>) via the use of embedding between generalized weighted Morrey and Lebesgue spaces obtained in Samko.<sup>24</sup>

Since generalized Morrey spaces provide more flexible characterisation of how behave the averages  $\frac{1}{|B(x,r)|} \int_{B(x,r)} |u_0(y)|^p dy$ , this may give more possibilities for realization of the approach of Theorem 3.1, in particular, the ‘‘gap’’ between the convergences in  $X$  and  $E$  may be made more narrow in comparison with Theorem 3.3.

Various operators of harmonic analysis, important for applications in PDEs were studied in such spaces. We refer, for instance, to Lukkassen et al<sup>30</sup> for Singular Integral operators and their applications and to Lundberg and Samko<sup>31</sup> for anisotropic Hardy operators and their applications, and to the book of Sawano et al<sup>20</sup> for more operators and their applications.

## 4 | CONCLUSION

There are analyzed relations between some signal processing problems within Bayesian frameworks and semigroups theory, in the case where the Banach space under consideration may be nonseparable. For applications the suggested approach is of interest in situations where approximation in the norm of the space is not possible. Its realization in the general form is presented in the case of the abstract Cauchy problem for the evolution equation. An explicit application of this approach is given for the diffusion equation with the initial data in the nonseparable Morrey space. The approach may be also developed for other nonseparable spaces and other signal processing problems.

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## CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

## DECLARATIONS

The authors equally contributed to the research in this paper. The authors read and approved the final manuscript. The authors have no financial or proprietary interests in any material discussed in this article.

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## REFERENCES

- Hairer M, Stuart A, Voss J. Signal processing problems on function space: Bayesian formulation, stochastic PDEs and effective MCMC methods. *The Oxford Handbook of Nonlinear Filtering*. Oxford: Oxford Univ. Press; 2011:833-873.
- Kolmogoroff A. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann of Math*. 1934;35(1):116-117.
- Hörmander L. *The Analysis of Linear Partial Differential Operators. I-IV*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag. Vol. 256-257 (1983); Vol. 274-275 (1985).
- Phillips RS. A note on the abstract Cauchy problem. *Proc Nat Acad Sci USA*. 1954;40:244-248.
- Applebaum D. *Semigroups of Linear Operators, Volume 93 of London Mathematical Society Student Texts. With Applications to Analysis, Probability and Physics*. Cambridge: Cambridge University Press; 2019.
- Butzer PL, Berens H. *Semi-groups of Operators and Approximation*, Die Grundlehren Der Mathematischen Wissenschaften Band, vol. 145. New York: Springer-Verlag, New York Inc.; 1967.
- Egorov YV, Shubin MA. Linear partial differential equations. Foundations of the classical theory. *Partial differential equations, I*, Encyclopaedia Math. Sci., vol. 30. Berlin: Springer; 1992:1-259.
- Egorov YV, Shubin MA. Linear partial differential equations. Elements of the modern theory. *In Partial differential equations, II*, Encyclopaedia Math. Sci., vol. 31. Berlin: Springer; 1994:1-120.
- Komech AI. Linear partial differential equations with constant coefficients. *Partial differential equations, II*, Encyclopaedia Math Sci, vol. 31. Berlin: Springer; 1994:121-255.
- Lorenzi L, Rhandi A. *Semigroups of Bounded Operators and Second-Order Elliptic and Parabolic Partial Differential Equations*, Monographs and Research Notes in Mathematics. Boca Raton, FL: CRC Press; 2021.
- Reed M, Simon B. *Methods of Modern Mathematical Physics I-IV*. New York: Academic Press, Inc. [Harcourt Brace Jovanovich Publishers]; 1981. 1978, 1979, 1980.
- Richtmyer RD. *Principles of Advanced Mathematical Physics. Vol. I-II*, Texts and Monographs in Physics. New York-Heidelberg: Springer-Verlag; 1978. 1981.
- Rudnicki R. Stochastic operators and semigroups and their applications in physics and biology. In: *Evolutionary equations with applications in natural sciences, Lecture Notes in Math., 2126*, Springer; 2015:255-318.
- Yosida K. *Functional Analysis*, Die Grundlehren Der mathematischen Wissenschaften, Band 123. Academic Press, Inc., New York. Berlin: Springer-Verlag; 1965.
- McBride AC. *Semigroups of Linear Operators: An Introduction*, Pitman Research Notes in Mathematics Series, vol. 156. New York: Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc.; 1987.



16. Pazy A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44. New York: Springer-Verlag; 1983.
17. D'Onofrio L, Sbordone C, Schiattarella R. Grand Sobolev spaces and their applications in geometric function theory and PDEs. *J Fixed Point Theory Appl*. 2013;13(2):309-340.
18. Sbordone C. New estimates for div-curl products and very weak solutions of PDEs. *Ann Scuola Norm Sup Pisa Cl Sci*. 1998;25(3-4):739-756. 1997. Dedicated to Ennio De Giorgi.
19. Giaquinta M. *Multiple Integrals in the Calculus of Variations and Non-linear Elliptic Systems*: Princeton Univ. Press; 1983.
20. Sawano Y, Di Fazio G, Hakim DI. *Morrey Spaces. Introduction and Applications to Integral Operators and PDE's, I and II*: CRC Press, Taylor and Francis Group; 2020.
21. Pick L, Kufner A, John O, Fučík S. *Function Spaces. Vol. 1*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 14. Berlin: Walter de Gruyter & Co.; 2013.
22. Zorko CT. Morrey space. *Proc Amer Math Soc*. 1986;98(4):586-592.
23. Kato T. Strong solutions of the Navier-Stokes equation in Morrey spaces. *Bol Soc Brasil Mat (NS)*. 1992;22(2):127-155.
24. Samko N. Embeddings of weighted generalized Morrey spaces into Lebesgue spaces on fractal sets. *Fract Calc Appl Anal*. 2019;22(5):1203-1224.
25. Stein EM. *Singular Integrals and Differentiability Properties of Functions*: Princeton University Press; 1970.
26. Dyn'kin EM, Osilenker BB. Weighted norm estimates for singular integrals and their applications. *J Sov Math*. 1985;30:2094-2154.
27. Chiarenza F, Frasca M. Morrey spaces and Hardy-Littlewood maximal function. *Rend Math*. 1987;7:273-279.
28. Samko N. Maximal, Potential and singular operators in vanishing generalized Morrey spaces. *J Glob Optim*. 2013;57(4):1385-1399.
29. Rafeiro H, Samko N, Samko S. *Morrey-Campanato Spaces: An Overview*, Operator Theory: Advances and Applications, vol. 228: Birkhäuser; 2013.
30. Lukkassen D, Meidell A, Persson L-E, Samko N. Hardy and singular operators in weighted generalized Morrey spaces with applications to singular integral equations. *Math Methods Appl Sci*. 2012;35(11):1300-1311.
31. Lundberg S, Samko N. On some hyperbolic type equations and weighted anisotropic Hardy operators. *Math Methods Appl Sci*. 2017;40(5):1414-1421.

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