A DISSERTATION FOR THE DEGREE OF DOCTOR SCIENTIARUM

Wave Damping and Momentum Transfer

Manifestation of Momentum Transfer in case of Ocean Surface Waves being Damped by an Elastic Film or a Viscous Layer

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To Bestemamma

PREFACE

This project was initiated already in 1987 and has in the later years appeared as to become "everlasting". However, now my thesis finally is completed, which is a great relief.

It has been a long journey from the beginning until today. There were times when I was close to giving up, but support from my supervisor, professor Kristian B. Dysthe, collegues, family and friends encouraged me to carry on. Unfortunate circumstances made the original plans difficult to carry through, but my supervisor helped design an adjusted project. The new project incorporated part of the original project and so prevented the already invested efforts from being wasted.

By the end of 1992 I accepted a position at the University Library – quite confident that my thesis would be completed within short time. But full-time job and family occupied my days and my energy, so it certainly took me a while … !

Having reached my goal at last, I want to thank my friends and collegues at the University, especially at the Physics Department, for their support during these years. In different ways they have all helped me carry through the project. I want, in particular, to mention my master degree supervisor, Inge Røeggen, for encouraging talks when "everything" was going wrong, Noralv Bjørnå and Truls Lynne Hansen for their support in general and in obtaining funding, Åshild Fredriksen for all our pleasant lunch-hours and Liv Larssen for her assistance in drawing illustrations and also for typing my initial manuscript. Further, I want to thank Einar Mjølhus, whose help was essential in obtaining plots, Kirsten Nymann Petersen for her encouragement and Stein Høydalsvik for scanning figures and photos. Finally, I want to thank my superiors at the University Library, Helge Salvesen and Sigmund Nesset, for permitting periods of full-time work on my thesis, that was of vital importance for me managing to complete the work.

However, my most sincere thanks go to my supervisor, professor Kristian B. Dysthe. His support and patience have been of great importance. He always encouraged me and urged me to keep on. His insight into the topic and his broad scientific experience have been most valuable. Without his guidance this thesis had never become a reality.

To my family – my husband Tom, my son Lars (16) and my daughter Marte (12) – I shall say: Thank you, for being so patient! You have been fabulous! Without your support I would have had to give up this project years ago.

The work has been financed partly by the Norwegian Council of Research (1987-88 and 1992), and partly by the University of Tromsø (1989-91).

I dedicate this thesis to the memory of my late grandmother (Bestemamma), who always had such great belief in me and was so proud of her granddaughter. I am only sad I didn't manage to complete this work while she was still amongst us. She would have appreciated this day just as much as I do.

Tromsø, December 2000

Marianne Foss

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I N T R O D U C T I O N

1. Motivation

The dynamics of the upper ocean has been the object of experimental and theoretical studies for centuries. The interest in the field has varied, but has increased in the last few decades. The developing of the off-shore industry and the intensified focus upon environmental questions have strongly effected the enhanced efforts. In the important field of ocean environmental surveillance thorough knowledge of the ocean surface dynamics is a necessity, and it is crucial for correct interpretation of satellite radar data. Many topics and mechanisms in this complex field still remain to be fully understood, however. By this thesis we hope to add a contribution to the understanding of *some* aspects of the subject.

Observations have revealed that accumulations of oil spill, biological material, sea ice etc cover great parts of the ocean surface. If not maintained by some kind of energy input, short surface waves are heavily damped in such areas. Our attention has been attracted to the momentum transfer associated with such decaying waves, and we shall present a theoretical study of how this momentum transfer may manifest itself.

Based upon a general dynamic model for fluid motion the topic will be approached by performing a detailed analysis in two selected cases of wave damping surface layers: surface-active film and a layer of very viscid fluid. The film case applies, for instance, in cases of oil spill or biological surface material, whereas viscid fluid is an appropriate representation of sea ice in the commonly existent condition of grease ice. These cases represent qualitatively different dynamic situations.

The dynamics of *pure* ocean surface waves is quite well explained and understood today. Linear models appear to be sufficient in many cases. Momentum transfer and mass transport are, however, among the non-linear features which need more sophisticated treatment. In this work some non-linear dynamic effects of surface waves propagating into a wave damping surface layer will be studied. As introduction, we shall briefly summarize the established theories and findings being relevant to the present work.

2. Established theory

The famous scientist G.G. Stokes did as early as in 1847 [Stokes, 1847] prove the oscillatory motion of pure water surface waves to be accompanied by a mean mass transport of second order, the now well-known *Stokes drift*. This drift was predicted on the assumption of an inviscid fluid. Later experiments have shown, however, that for a real fluid the mass transport velocity may differ considerably from the Stokes drift.

Works by Longuet-Higgins [1953] and others have established that viscous damping results in a mean flow in addition to the inviscid Stokes drift. The viscous damping generates a vorticity creating a stress gradient in the surface layer.

Although the viscous *stresses* in the surface layer are important, the layer itself makes a negligible contribution to the overall rate of energy *dissipation*, since the rates of strain there are no larger than in the irrotational flow and the layer thickness is small. The energy losses arise almost entirely from the straining of the horizontal motion.

The discrepancy between the Stokes drift and the observed mass transport velocity in a real fluid led M.S. Longuet-Higgins [1953] to develope a new and more general model for mass transport in water waves, taking also the viscosity of the water into account. An important feature of Longuet-Higgins' model is the partitioning of the fluid into an interior part and boundary layers. Inviscid theory applies in the interior, but in the thin boundary layers the effect of viscosity must be considered. By letting the solutions at the interior boundaries of the boundary layers replace the surface and bottom boundary conditions of the interior problem, Longuet-Higgins obtained a model which predicted results for the mass transport in agreement with observation. Longuet-Higgins considered *permanent* waves, implying some kind of energy input to the wave field to compensate the viscid energy dissipation. It was later shown [Huang, 1970; Ünlüata and Mei, 1970] that Longuet-Higgins' model led to infinite surface drift for an ocean of infinite depth. Ünlüata and Mei discussed this seemingly unphysical result. They argued that for finite depth steady state will finally always be reached, but claimed that for infinite depth no steady state is to be expected. Consequently the analysis was not likely to apply in that case. For viscous *attenuated* waves Liu and Davis [1977] obtained a finite mean surface velocity also in the case of infinite depth – using the same model.

In a paper published in 1969 Longuet-Higgins [1969] studied a slowly decaying wave train. He showed that a mean, second-order vorticity is induced in the viscid surface layer. Although a non-zero viscosity was crucial for this vorticity to *exist*, the vorticity itself turned out to be independent of the viscosity and of the boundary layer thickness. This induced vorticity will diffuse into the interior of the fluid, producing a mean Eulerian drift which adds to the Stokes drift and is of equal order. The total drift obtained was in agreement with experimental data. Longuet-Higgins showed that the induced vorticity may be associated by an equivalent "virtual tangential wave stress" applied to the surface, being proportional to the mean Eulerian velocity gradient. It was also shown that the decaying waves eventually would be transferring *all* their momentum to the boundary layer at the free surface, this momentum then being transferred to the mean flow by viscous diffusion, or - as it may equivalently be considered – by the virtual surface stress. Hence, the final distribution of momentum would be very different from the initial distribution.

In 1970 Ümit Ünlüata and Chiang C. Mei [1970] published their paper "Mass Transport in Water Waves". Their approach followed Longuet-Higgins in the sense of dividing the fluid into an irrotational interior region surrounded by viscid boundary layers, but they applied a

Lagrangian description of fluid motion and a perturbation approach, as outlined by Pierson [1962]. In Lagrangian fluid description the physical properties are regarded as being attached to marked fluid particles, rather than to fixed positions in space as in the Eulerian description applied by Longuet-Higgins. When applying the Lagrangian approach the kinematic surface boundary condition becomes *linear*, which is very convenient. Ünlüata and Mei addressed permanent waves, assuming free surface and solid bottom as did Longuet-Higgins in his 1953-paper. A second-order analysis was carried out under the assumption that a steady, second-order field of vorticity had been established throughout the depth of the fluid, being diffused inwards from the boundary layers. In contrast to Longuet-Higgins' Eulerian approach, in which the boundary layers needed special treatment, the Lagrangian approach yielded an overall valid solution, also giving the mean mass transport in a straightforward manner. The results were in agreement with those of Longuet-Higgins [1953], but were obtained much more straightforward.

J.E. Weber has studied the mean mass transport induced by wave motion in several papers, applying the Lagrangian approach along with the "Pierson method", as did Ünlüata and Mei. In a work from 1983 Weber [1983a] studied mean drift currents due to spatially periodic surface waves in a viscid, rotating ocean. No energy input was assumed, leaving the wave field to be attenuated by viscous dissipation. It was shown that the mean drift became finite everywhere for these attenuated waves, also without rotation. A not negligible net surface mean mass transport was shown to take place even for small viscosity. The largest net displacements were shown as most likely to appear for intermediate viscosities – stronger viscous damping also more effectively limited the growth of the current, while weaker damping led to smaller velocities. Weber displayed results only for the mean drift at the surface, but the expressions indicate that the mean flow decreases exponentially with depth. Both the primary waves and the induced mean flow were seen to be attenuated in time due to the viscosity.

Weber pointed out that the viscid wave motion problem was qualitatively different from the inviscid problem. The works of Longuet-Higgins, Ünlüata and Mei and others had already clearly shown viscosity – whatever small – to alter the mass transport drastically, not only in the boundary layers, but also in the interior of the fluid. Longuet-Higgins [1953, 1960] and Phillips [1977] had pointed out that the equations of motion have a singular character with respect to viscosity – the viscous mass transport solutions do not generally approach the inviscid solutions in the limit of vanishing viscosity. Weber's analysis revealed that in the inviscid case the wave equations of motion are independent of the mass transport equations of motion, whereas these sets of equations are coupled in the viscid case.

In another paper published that year, Weber [1983 b] considered *permanent* surface waves in a deep viscid, rotating ocean. In contrast to Longuet-Higgins [1953] and Ünlüata and Mei [1970] who assumed a free surface and hence no surface stress, Weber allowed a varying external surface stress due to wind – adjusted as to provide the wave field with an energy input exactly compensating the energy loss due to viscous dissipation. Hence permanent waves were achieved. In the limit of a non-rotating, inviscid ocean the surface waves would induce a mean mass transport, the familiar Stokes drift, if no stress was applied to the surface. Taking viscosity into account introduced only a small correction in the mass transport itself, but the vorticity and hence the dynamic surface boundary condition were greatly affected.

Weber adopted a physical interpretation of the Lagrangian mean flow based upon considerations by Andrews and McIntyre [1978], which through a center-of-mass approach concluded that the Lagrangian mean flow is a direct measure of the net mass transport associated with the waves.

Weber [1987] also studied the wave-induced drift occuring when gravity waves in a slightly viscid, rotating ocean are being damped by a very thin layer of highly concentrated brash-like ice. He determined the wave-induced drift below the ice layer, finding a jet-like mean flow just below the ice. Also the mean viscous stress exerted on the ice was determined, showing that the viscous stress on the ice from the waves might be comparable with the size of the wind stress induced by moderate winds.

The effect of an insoluble and inextensible surface film on the drift velocity of capillary-gravity waves has been studied by Weber and Førland [1989]. Their results show the induced mean drift to depend crucially upon how the surface viscous stress is prescribed. The calculations also demonstrate that the induced mean flow exists on a much longer time scale than does the primary wave field. In another paper [Weber and Førland, 1990] the surface film was replaced by air. Similar results were achieved, but the effects were weaker because of the weaker attenuation. Also the effect of the air on the drift velocity for *spatially* damped waves was investigated. In this case the virtual wave stress became independent of time, and it was argued that the Coriolis force was needed to balance the wave stress in order to avoid infinitely large drift velocities as time approached infinity. In agreement with the result of Longuet-Higgins, also Weber and Førland's method showed *all* the original wave momentum being finally transferred to the mean current. Both Weber and Førland [Weber, 1983a; Weber and Førland, 1989] have discussed the relation between the mean Eulerian drift caused by the wave-induced mean vorticity and the mean Lagrangian drift. In the Lagrangian approach, a homogeneous solution had to be added to the particular solution for the mean drift, to meet the boundary conditions. Weber and Førland showed that the particular solution may be interpreted as the (generalized) Stokes drift, whereas the homogeneous solution should be interpreted as the mean Eulerian drift.

Weber and Sætra [1995] studied the effect of an *elastic* surface film on wave-induced drift. They assumed no surface stress. Again the Lagrangian approach was adopted. For waves attenuated in time the non-linear drift was obtained. Provided sufficiently high values of the film elasticity, the drift was shown to have its maximum just *below* the surface. It was also shown that the effect of the film was to enhance the surface drift at short times, whereas at larger times the stronger attenuation of the wave field resulted in smaller drift currents. This was analogous to the result of Weber and Førland [1989] for the special case of inextensible film. Weber and Sætra showed that the elastic film strongly effected the strength of the virtual wave stress at the surface, which basically is responsible for the induced mean, Eulerian drift of the fluid. Maximum damping was shown to occur when the frequency of the Marangoni-waves (longitudinal, elastic waves in the film) *nearly* coincided with the frequency of the transverse capillary-gravity waves – in agreement with a result obtained by Dysthe and Rabin [1986]. Lucassen [1968] originally stated that this happened when the *wavelengths* of the two wave types coincided.

Together with R.W. Stewart had Longuet-Higgins earlier introduced the concept of *radiation* *stress* (cf electromagnetic radiation pressure) in the context of ocean surface waves [Longuet-Higgins and Stewart, 1960-64]. The radiation stress is defined as the *excess* momentum flux of the fluid due to the wave motion. Longuet-Higgins and Stewart demonstrated that several dynamic phenomena may be explained rather intuitively by taking the radiation stress into account, hence the radiation stress concept may be a convenient tool in analysing complex wave phenomena. Despite the promising results obtained by Longuet-Higgins and Stewart using this concept, the idea seems to have attracted limited attention.

3. Aim of this work

As the above outline shows, the topic has been approached in different ways over the years. Mostly have special cases been studied. Despite considerable progress, the picture is still far from complete, though. The dynamics associated with surface waves propagating through a single viscid fluid with free surface is now well understood. But the picture becomes more complicated as another fluid is involved. This may alter the boundary conditions and by that possibly change the dynamic picture. How the boundary conditions are altered in different cases, and the effect of this, is not fully explored. A complete theory does not exist, and because of the complexity it is still necessary to study special cases to make further progress. It is to hope that a unified theory eventually will result.

Following the line of making progress through studying special cases, we chose to theoretically investigate two selected cases,

- capillary-gravity waves propagating into a surfacefilm-covered area
- short gravity waves propagating into an area with a very viscid surface layer.

In both cases heavy damping is assumed to occur. Most earlier efforts had been considering waves attenuated in time – we chose to concentrate on spatially damped waves (see photo at next page, fig. I.1). We aimed at especially focusing upon the mean surface stress in the first case and the mean flow in the other. The solutions were to be obtained analytically.

It had been demonstrated by several authors that Lagrangian fluid description was to prefer when studying this kind of problems. By the perturbation method of Pierson [1962] very good results were achieved. The involved expressions became quite lengthy as soon as the second-order problem was to be considered, however. One of our objects was to develope this model further, intending to obtain a more compact representation.

Weber and Førland had theoretically determined the mean drift caused by waves attenuated in time by an inelastic film. Together with Sætra, Weber extended this work to the case of an elastic film. In this work we shall study waves attenuated in *space* by an elastic film. Instead of prescribing the mean tangential stress, we want to *determine* this stress – under the assumption of mean, steady state.

The viscid-layer case is to be treated by a two-layer model. The viscosity of the upper layer is assumed to be much greater than the normal viscosity of (sea-) water. In a laboratory experiment S. Martin and P. Kauffman [1981] had studied wave damping by grease ice, and had observed a wave induced mean mass circulation within the viscid grease ice layer. Weber [1987] had theoretically treated a similar case, but he determined the drift *below* a very thin layer of viscid brash-like ice. We wanted to determine theoretically the mean drift *within* the upper, viscid layer, in order to explain the experimental results of Martin and Kauffman.

Fig. I.1. Capillary-gravity waves damped in space. Photo from Kaldfjord, Kvaløya (outside Tromsø).

The radiation stress concept introduced by Longuet-Higgins and Stewart [1960] had not been given very much attention by other authors, despite the promising results. To test its convenience in our context, we investigated our two selected cases also within this concept.

4. Structure of thesis

The structure of this thesis is that after this introductory chapter, the basics of fluid dynamics and the necessary theory of surface waves are presented in chapter I, along with the terminology to be used throughout the thesis. An introduction to Lagrangian fluid description is given, together with an outline of the governing equations within this approach.

Chapter II is dedicated to the establishing of a hydrodynamic model for surface waves within the Lagrangian approach. Combining the Jacoby determinant approach of Chang [1969] with the perturbation expansion introduced by Pierson [1962], a very compact and symmetrical – and hence easy interpretable – model is developed. The stream function plays an important role. We present a thorough discussion of how the Lagrangian solutions for the fluid motion may be implemented in an Eulerian description of the fluid particle paths. Special care has to be taken, since the solutions are valid only for small perturbations. The implementation is not trivial, and has – in our opinion – been given too little attention by other authors.

The special case of surface film is treated in chapter III. The mean tangential surface stress in case of steady state is determined. The chapter includes a historical review of how the theory of wave-damping by oil developed. In chapter IV is referred how sea ice in the condition of grease ice forms, and the physical properties of grease ice as a very viscid fluid are described. The special case of a viscid surface layer is investigated, and the mean, horizontal, second-order flow within this layer is determined.

The results from chapters III and IV are briefly discussed in a qualitative manner within the concept of radiation stress in chapter V, where this concept is also given a thorough introduction.

In the final chapter VI we summarize our work, discuss the results and draw conclusions.

CHAPTER I

BASIC HYDRODYNAMIC THEORY OF WATER WAVES

In chapter I a short outline of the established mathematical theory of water waves is given. The terminology and concepts to be used throughout this thesis is introduced.

1. The governing equations

In the common Eulerian description of fluid motion the basic dynamic equations for a Newtonian fluid are the Navier-Stokes equation which expresses Newton's second law and the continuity equation for conservation of mass. Exposed to gravity as the only body force, the equations may be written

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot \left(\rho \underline{v}\,\underline{v} + \underline{\underline{P}}\right) = \rho \underline{g} \tag{1.1}$$

and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0\,. \tag{1.2}$$

In accordance with common notation ρ denotes the density of mass, $\underline{v}$ the velocity of the fluid, $\underline{\underline{P}}$ is the stress tensor and $\underline{g}$ the acceleration of gravity. Since the Eulerian description of fluid motion is a *field* description, all quantities are functions of time and position in a fixed coordinate system.

At an interface between water and air the (surface) kinematic and dynamic boundary conditions will be, respectively,

$$\left.\begin{aligned}&\frac{\partial \eta}{\partial t}+\underline{v}\cdot\nabla\eta=\underline{e}_z\cdot\underline{v}\\&\underline{\underline{P}}\cdot\underline{n}=\kappa_s\sigma\underline{n}-\left(\underline{\underline{I}}-\underline{n}\,\underline{n}\right)\cdot\nabla\sigma\end{aligned}\right\}\quad z=\eta \tag{1.3}$$

if the influence of the air is neglected. As illustrated in fig.1.1 below, η denotes the vertical displacement of the surface with respect to an equilibrium position $z = 0$ (the vertical z-axis pointing upwards), $\underline{e}_z$ is a unit vector in the positive z-direction and $\underline{n}$ is a unit vector

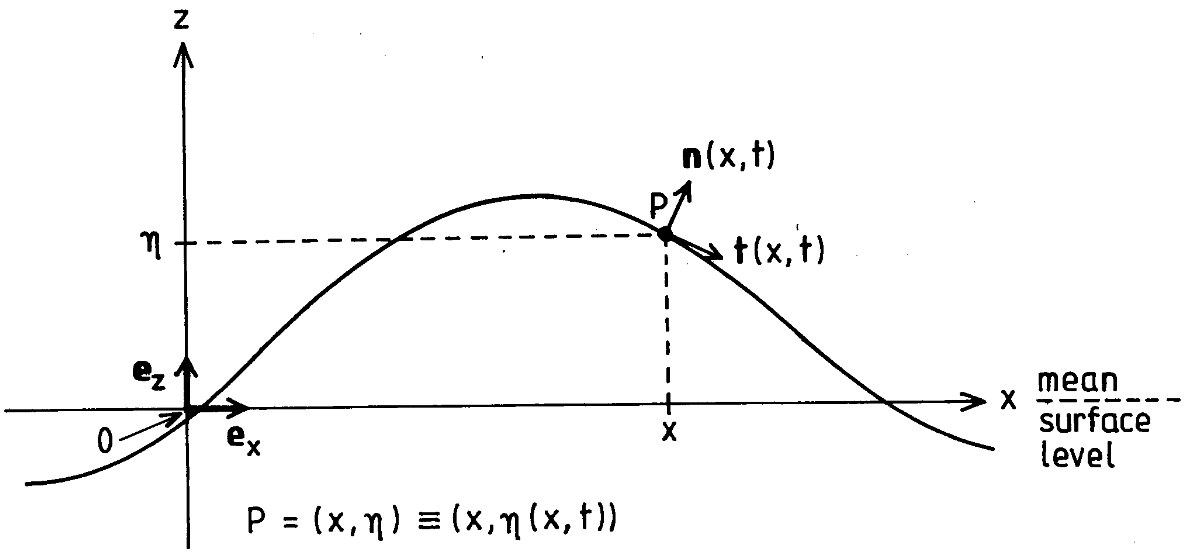

Fig. 1.1. Illustration of the parameters.

normal to the surface (pointing outwards). κ_s is the mean curvature of the surface, σ is the surface tension and $\left(\underline{\underline{I}}-\underline{n}\,\underline{n}\right)\cdot\nabla\sigma$, with $\underline{\underline{I}}$ being the unit tensor, is the gradient of the surface tension along the surface. If the surface tension is uniform along the surface, as in the case of a pure water surface, this gradient vanishes. A solid bottom at $z = -h$ gives the additional kinematic boundary condition

$$\left(\nabla h+\underline{e}_z\right)\cdot\underline{v}=0,\qquad z=-h. \tag{1.4}$$

Since any fluid pressure would be balanced by a solid bottom, it is not possible to define a dynamic boundary condition similar to the one in eq. (1.3) at the bottom.

We will be concentrating on incompressible fluids only, in which case the continuity equation (1.2) is reduced to

$$\nabla\cdot\underline{v}=0. \tag{1.5}$$

The stress tensor $\underline{\underline{P}}$ in a Newtonian, viscid fluid is, as shown by H. Lamb [1932], given by

$$\underline{\underline{P}} = p\underline{\underline{I}} - \mu\left(\nabla\underline{\mathrm{v}} + (\nabla\underline{\mathrm{v}})^T \right), \tag{1.6}$$

where p is the dynamic pressure and μ the coefficient of viscosity. (The supscript T means transposed.) For an incompressible fluid the Navier-Stokes equation hence becomes

$$\frac{\partial\underline{\mathrm{v}}}{\partial t} + \underline{\mathrm{v}}\cdot\nabla\underline{\mathrm{v}} + \frac{1}{\rho}\nabla p - \nu\nabla^2\underline{\mathrm{v}} - \underline{g} = \underline{0}, \tag{1.7}$$

where ν=μ/ρ is the kinematic coefficient of viscosity.

It is well established that an ideal (i.e. inviscid) fluid which at some initial time is irrotational (∇x$\underline{\mathrm{v}}$=0), will stay irrotational. The velocity may then be expressed as the gradient of a velocity potential φ, $\underline{\mathrm{v}} = \nabla\varphi$. In this case the Navier-Stokes equation may be written

$$\nabla\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{p}{\rho} + g\,z\right) = 0 \tag{1.8}$$

and the incompressibility condition (1.5) becomes

$$\nabla^2\varphi = 0\,. \tag{1.9}$$

Eq. (1.8) is fulfilled if the integrated Navier-Stokes equation,

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{p}{\rho} + g\,z = 0\,, \tag{1.10}$$

known as the Bernoulli equation, is fulfilled.

The boundary conditions (1.3)-(1.4) are in this case, if also assuming a pure surface and a horizontal bottom, reduced to

$$\left.\begin{array}{ll}\left.\begin{array}{l}\dfrac{\partial\eta}{\partial t}+\nabla\varphi\cdot\nabla\eta=\dfrac{\partial\varphi}{\partial z}\\ \dfrac{\partial\varphi}{\partial t}+\dfrac{1}{2}(\nabla\varphi)^2+g\eta=-\dfrac{1}{\rho}\kappa_s\,\sigma\end{array}\right\} & z=\eta\\ \dfrac{\partial\varphi}{\partial z}=0 & z=-h\end{array}\right\}. \tag{1.11}$$

2. Surface waves

In the case of two-dimensional irrotational flow a plane-wave solution for progressive deep-water (i.e. $h\to\infty$) surface waves is obtained from the linearized form of eqs. (1.9)-(1.11). The linearized equations become in this case

$$\left.\begin{array}{ll}\dfrac{\partial\varphi}{\partial t}+\dfrac{p}{\rho}+g\,z=0 & \\ \nabla^2\varphi=0 & \\ \left.\begin{array}{l}\dfrac{\partial\eta}{\partial t}=\dfrac{\partial\varphi}{\partial z}\\ \dfrac{\partial\varphi}{\partial t}+g\eta=\dfrac{\sigma}{\rho}\nabla^2\eta\end{array}\right\} & z=0\\ \dfrac{\partial\varphi}{\partial z}=0 & z=-\infty\end{array}\right\}, \tag{1.12}$$

where the surface curvature κ_s is expressed in terms of the surface elevation η.

In a real (=viscid) fluid the motion is not irrotational, however. The standard method of establishing a wave solution in a viscid fluid is to add a rotational part to the irrotational solution. In two-dimensional motion the rotational part may be expressed as the curl of a vector potential normal to the plane of motion. Defining the *(x.z)*-plane as the plane of motion, the total velocity $\underline{v}$ may then be written as

$$\underline{v}=\nabla\varphi-\nabla\times\underline{e}_y\psi\,, \tag{1.13}$$

where $\underline{e}_y$ is a unit vector in the y-direction. (It is known from general vector analysis that a vector field *always* may be expressed as the sum of the gradient of a scalar potential and the curl of a vector potential.)

The linear Navier-Stokes equation becomes in this case

$$\nabla\left(\frac{\partial\varphi}{\partial t}+\frac{p}{\rho}+gz\right)-\nabla\times\underline{e}_y\left(\frac{\partial\psi}{\partial t}-\nu\nabla^2\psi\right)=0\,. \tag{1.14}$$

Eq. (1.14) is fulfilled if the scalar potential φ fulfils the two first equations of eq. (1.12), and the vector potential ψ fulfils

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\psi=0\,. \tag{1.15}$$

The boundary conditions must also be adjusted, and the vorticity (=∇x$\underline{v}$) becomes in this case $\nabla^2\psi$ ($\neq 0$).

In the inviscid case a possible wave solution of eqs. (1.12) is given by a velocity potential φ having the form

$$\varphi=\frac{a\omega}{k}\,e^{kz}\sin\left(kx-\omega t\right). \tag{1.16}$$

Here is *a* the amplitude of the wave, *k* is the wave number and ω the angular frequency of the wave motion. The parameters *x* and *t* refer to position at the *x*-axis and to time, respectively, and the solution describes a wave propagating along the positive *x*-axis. In order to fulfil the boundary conditions we must have

$$\eta=a\cos\left(kx-\omega t\right) \tag{1.17}$$

and

$$\omega^2=gk+\frac{\sigma}{\rho}k^3\,. \tag{1.18}$$

The solution (1.16) is correct to the lowest order in the wave steepness (*ak*) for an ideal fluid.

The relation between wave number and angular frequency given in eq.(1.18) constitutes the *dispersion relation* of the wave. Depending on the ratio $\rho g/\sigma k^2$ the waves are characterized as gravity waves ($\rho g \gg \sigma k^2$) or capillary waves ($\rho g \ll \sigma k^2$). In the former case surface tension may be neglected, in the latter gravity. In clean water capillary waves have wavelengths less than about 1 cm, whereas gravity waves have wavelengths from about 10 cm and upwards. In the middle area ($\rho g \sim \sigma k^2$) neither of the terms may be neglected - such waves are known as capillary-gravity waves.

A common approximation in the viscid case is to assume the fluid to be irrotational in the interior of the fluid [Longuet-Higgins, 1953], taking the vorticity $(\nabla x \underline{v})$ into account only in the boundary layers. The above solution will then apply in the interior, while a solution in the boundary layers must include contributions from the function ψ. ψ may be given by

$$\psi = \mathrm{Re}\left\{be^{mz}e^{i(kx-\omega t)}\right\}, \tag{1.19}$$

where the coefficient b is a constant to be determined by the boundary conditions and the coefficient m in the exponent must fulfil (cf eq. (1.15))

$$m^2 = k^2 - i\frac{\omega}{\nu}. \tag{1.20}$$

The characteristic thickness of the boundary layers is δ, given by

$$\delta = \sqrt{2\nu/\omega}. \tag{1.21}$$

(It should be mentioned that Longuet-Higgins [1953] has shown that vorticity slowly diffuses into the total volume of the fluid, so that the irrotational approximation supposed in the interior will be invalid after some time.)

3. Mass transport

The simple solution (1.16)-(1.18) gives zero mean velocity when averaged over one wave period, $\overline{\nabla\varphi} = 0$. (An overbar will generally be denoting average over one wave period or one wave length throughout the thesis.) In fact it may be shown – at least for gravity waves – that eqs. (1.16)-(1.18) constitute a solution even to the third order in the wave steepness *(ak)*. Hence, there is no second-order mean Eulerian velocity in this case of irrotational motion.

Following a *marked* fluid particle, however, the situation becomes different. To the first order in (*ak*) the fluid particles follow circular orbits (still assuming deep-water waves), and there is no mean particle motion or mass transport. But, as already pointed out by Stokes [1847], to the second order in (*ak*) there will be a mean horizontal drift of the fluid in the direction of wave propagation – the well known Stokes drift v_s –

$$\mathrm{v}_\mathrm{S} \equiv a^2\omega\, k\, e^{2kz} = (ak)^2 \frac{\omega}{k} e^{2kz} . \tag{1.22}$$

(See, for instance, Phillips [1977], p. 43-44.) The profile of the Stokes drift is sketched in fig.1.2.

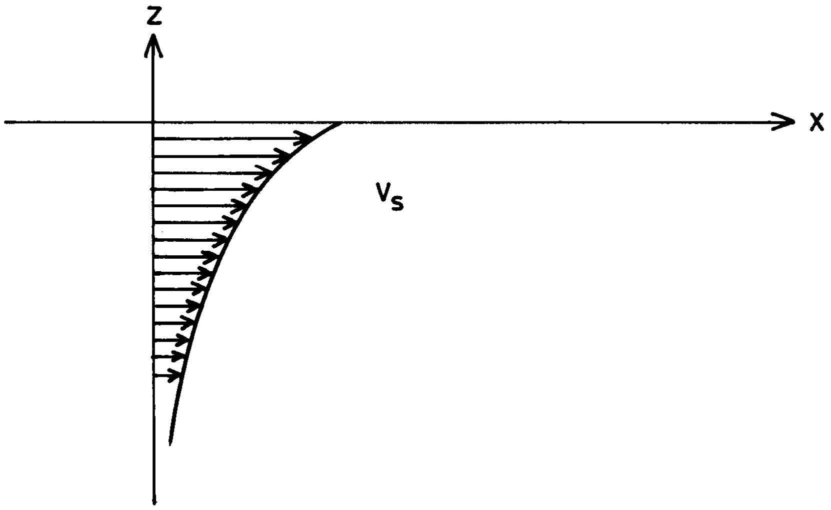

Fig. 1.2. Profile of the standard Stokes drift.

Surprisingly one finds - if calculating the mean horizontal momentum $\overline{M}$ of the wave per unit surface area to the second order in (*ak*) – that no matter which of the two approaches adopted, the same result is obtained for $\overline{M}$ (v_E denotes the horizontal Eulerian velocity $\partial\varphi/\partial x$):

$$\left.\begin{aligned} \overline{M_E} &\equiv \rho \overline{\int_{-\infty}^{\eta} \mathrm{v}_\mathrm{E}\, dz} = \rho \overline{\int_{-\infty}^{\eta} \frac{\partial\varphi}{\partial x}\, dz} = \rho \frac{a\,\omega}{k} \overline{\cos(kx-\omega t)\, e^{k\eta}} = \frac{1}{2}\rho\, a^2\omega \\ \overline{M_L} &\equiv \rho \overline{\int_{-\infty}^{\eta} \mathrm{v}_\mathrm{S}\, dz} = \rho \int_{-\infty}^{0} \mathrm{v}_\mathrm{S}\, dz = \frac{1}{2}\rho\, a^2\omega \\ \Rightarrow \qquad & \overline{M_E} = \overline{M_L} = \overline{M}. \end{aligned}\right\} \tag{1.23}$$

This shows that the mean horizontal momentum $\overline{M}$ also is equal to the mass flux associated with the mean particle drift. Both approaches give zero vertical mean drift to second order. A qualitative difference in the two views should be noted, however: In the first integral only a thin surface layer contributes, in the latter there is a contribution from the total depth of the fluid:

$$\overline{M_E} = \rho \overline{\int_{-\infty}^{\eta} \mathrm{v_E}\, dz} \approx \rho \overline{\int_{0}^{\eta} \mathrm{v_E}\, dz}, \qquad \overline{M_L} = \rho \overline{\int_{-\infty}^{\eta} \mathrm{v_S}\, dz} \approx \rho \overline{\int_{-\infty}^{0} \mathrm{v_S}\, dz}\,. \tag{1.24}$$

In the solution above it was understood that the wave parameters a, ω and k were constants. Allowing for a slow variation – with time or position – in one or more of these wave parameters, a mean *Eulerian* velocity of order $(ak)^2$ is also possible. Locally – to the first order – the wave parameters may be considered as constants, and hence the first-order solution will still have vanishing average. To the second order we must take the variation of the parameters into account, and we will in general get a non-zero mean Eulerian velocity of order $(ak)^2$. The mean Eulerian velocity will give an additional contribution to the mean mass transport. Expressing the solution as

$$\varphi = \varphi_1 + \varphi_2, \qquad \eta = \eta_1 + \eta_2, \tag{1.25}$$

where (φ_1,η_1) is the first-order solution given in eqs. (1.16)-(1.18) and (φ_2,η_2) is of second order, one finds – using the general equations (1.9)-(1.11) combined with the fact that the first-order solution fulfils eqs. (1.12) – that the mean, second-order functions $\overline{\varphi_2}$ and $\overline{\eta_2}$ must fulfil

$$\left.\begin{aligned} &\nabla^2 \overline{\varphi_2} = 0 \\ &\left.\begin{aligned} &\frac{\partial \overline{\varphi_2}}{\partial t} + g\,\overline{\eta_2} - \frac{\sigma}{\rho}\frac{\partial^2 \overline{\eta_2}}{\partial x^2} = 0 \\ &\frac{\partial \overline{\eta_2}}{\partial t} - \frac{\partial \overline{\varphi_2}}{\partial z} = 0 \end{aligned}\right\} \quad z = 0 \\ &\frac{\partial \overline{\varphi_2}}{\partial z} = 0, \qquad\qquad z = -\infty \end{aligned}\right\}. \tag{1.26}$$

Also other physical situations may produce mean Eulerian velocities of second order, although having vanishing first order mean. An example is strongly decaying waves, which we shall be investigating in chapter III and IV.

4. Lagrangian fluid description

Fluid motion may be described in two fundamentally different manners. Eulerian description, which is applied in the previous sections, is most commonly applied. In Eulerian description are velocity, pressure and all other features characterizing the dynamic state of the fluid expressed as functions of position in space, referred to a fixed coordinate system, and of time. The Eulerian description is hence a *field* description. Such a description is well suited in many cases – in the first-order perturbation study of surface wave motion, for instance. Other phenomena are not so easily handled in Eulerian description; among those are mean mass transport phenomena associated with surface wave motion. These phenomena will be of second order in the perturbations [Longuet-Higgins, 1953], and for a proper treatment of the boundary conditions at the surface one may not anylonger neglect the motion of the surface itself. In Eulerian description this will demand the use of curvilinear coodinates, which will make the treatment quite laborious.

The preferable alternative is to use *Lagrangian* description of fluid motion [Lamb, 1932]. Within this description the idea is to keep track of each individual fluid particle, and then describe the time evolution of the fluid in terms of the individual fluid particles. The two types of description are illustrated in fig.1.3:

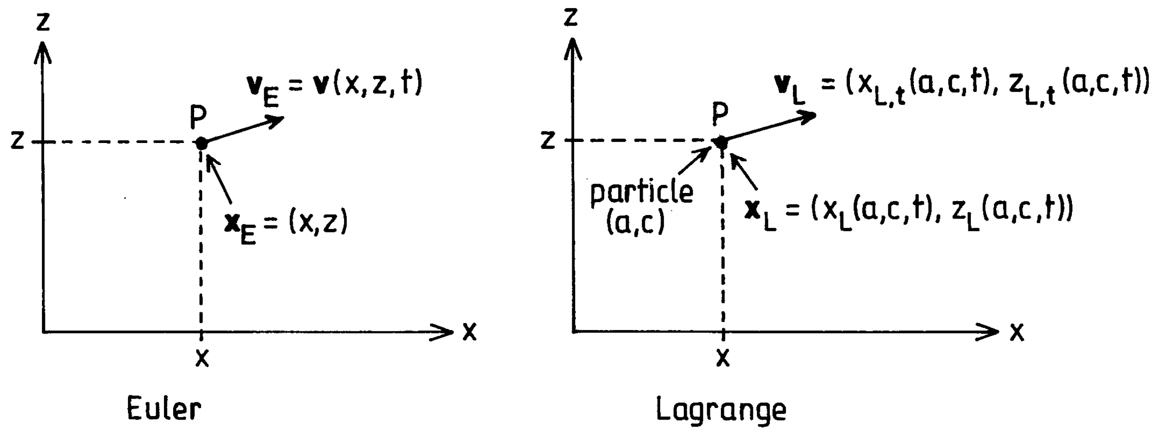

Fig. 1.3. Eulerian versus Lagrangian fluid description.

When describing the mean mass transport itself, the benefit of using Lagrangian description is obvious. Also – within Lagrangian description one may avoid the troublesome curvilinear coordinates when establishing the boundary conditions at the surface.

To keep track of the individual fluid particles a unique labelling is needed. This may be done in different ways, the only demand is that for a system in three dimensions, three independent variables are necessary. Let *a,b* and *c* be 3 independent and continuous variables, chosen so that there is a one-to-one correspondence between the set *{(a,b,c)}* and the fluid particles, and so that the (lot of) fluid particles with labels in a small volume surrounding (a_0,b_0,c_0) in the space of variables constitute all the fluid within a small volume surrounding the fluid particle labelled (a_0,b_0,c_0). All features of the fluid may then be related to the individual fluid particles and hence be expressed as continuous functions of the labels *(a,b,c)* and of time. Having established this set of variables, we are in the position to find the Lagrangian form of the basic equations of fluid motion.

Let us by

$$\underline{x} \equiv (x,y,z) = \underline{\tilde{x}}(a,b,c,t) \tag{1.27}$$

denote the position of the fluid particle *(a,b,c)* at the time *t*. $\underline{x}$ is the Eulerian point in space, whereas $\underline{\tilde{x}}$ is a function of *a,b,c* and *t*. The velocity and acceleration of the fluid particle will then, in Lagrangian description, be the first and second partial derivative of $\underline{\tilde{x}}\,(a,b,c,t)$ with respect to time, denoted by $\underline{\tilde{x}}_t$ and $\underline{\tilde{x}}_{tt}$, respectively.

Following Chang [1969] we state that provided the transform between the sets (*x,y,z,t*) and (*a,b,c,t*) of variables is invertible, there must exist – for any function $F = F(x,y,z,t)$ – a function $f = f(a,b,c,t)$ so that

$$F(x,y,z,t) = f(a,b,c,t)\,, \tag{1.28}$$

where *a,b* and *c* all are functions of *x,y,z* and *t* and vice versa. By standard theory for transformations between different sets of variables the derivatives of *F* may be expressed as

$$\left.\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial(f,\tilde{y},\tilde{z})/\partial(a,b,c)}{\partial(\tilde{x},\tilde{y},\tilde{z})/\partial(a,b,c)} \\
\frac{\partial F}{\partial y} &= \frac{\partial(\tilde{x},f,\tilde{z})/\partial(a,b,c)}{\partial(\tilde{x},\tilde{y},\tilde{z})/\partial(a,b,c)} \\
\frac{\partial F}{\partial z} &= \frac{\partial(\tilde{x},\tilde{y},f)/\partial(a,b,c)}{\partial(\tilde{x},\tilde{y},\tilde{z})/\partial(a,b,c)}
\end{aligned}\right\}, \tag{1.29}$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ denote *x, y* and *z* written as functions of *a,b,c* and *t*, and the expression

$$\frac{\partial(f,\tilde{y},\tilde{z})}{\partial(a,b,c)} \equiv \det \begin{Bmatrix} f_a & f_b & f_c \\ \tilde{y}_a & \tilde{y}_b & \tilde{y}_c \\ \tilde{z}_a & \tilde{z}_b & \tilde{z}_c \end{Bmatrix} \tag{1.30}$$

is the Jacoby determinant of $(f, \tilde{y}, \tilde{z})$ with respect to (a,b,c). (Similarly for the other equations in (1.29).) The subscripts mean partial differentiation with respect to the indicated variable.

We proceed by finding the Lagrangian form of our basic equations (1.1) and (1.2), which in Eulerian form alternatively may be written

$$\frac{D\underline{v}}{Dt} = -\frac{1}{\rho}\nabla p + \underline{g} + \nu\left(\nabla^2\underline{v} + \nabla(\nabla\cdot\underline{v})\right), \tag{1.31}$$

and

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\underline{v} = 0\,. \tag{1.32}$$

The operator $D/Dt \equiv \partial/\partial t + \underline{v}\cdot\nabla$ is the substantial derivative with respect to time; that is, it represents differentiation with respect to time while following the motion (or the fluid particles). The velocity $\underline{v}$, pressure p and denstity ρ are all functions of x,y,z and t. We will denote the corresponding functions of a,b,c and t (cf eq.(1.28)) by $\underline{\tilde{x}}_t$, $\tilde{p}$ and $\tilde{\rho}$, that is

$$\left.\begin{aligned} \underline{v}(x,y,z,t) &= \tilde{x}_t(a,b,c,t) \\ p(x,y,z,t) &= \tilde{p}(a,b,c,t) \\ \rho(x,y,z,t) &= \tilde{\rho}(a,b,c,t) \end{aligned}\right\}. \tag{1.33}$$

Hence, eq.(1.32) becomes, by eq.(1.29),

$$\frac{\partial\tilde{\rho}}{\partial t} + \tilde{\rho}\,\frac{\left\{\dfrac{\partial(\tilde{x}_t,\tilde{y},\tilde{z})}{\partial(a,b,c)} + \dfrac{\partial(\tilde{x},\tilde{y}_t,\tilde{z})}{\partial(a,b,c)} + \dfrac{\partial(\tilde{x},\tilde{y},\tilde{z}_t)}{\partial(a,b,c)}\right\}}{\dfrac{\partial(\tilde{x},\tilde{y},\tilde{z})}{\partial(a,b,c)}} \quad = \quad 0 \tag{1.34}$$

or

$$\frac{\partial}{\partial t}\left\{\tilde{\rho}\,\frac{\partial(\tilde{x},\tilde{y},\tilde{z})}{\partial(a,b,c)}\right\}=0\,. \tag{1.35}$$

The volume of the (small) mass or fluid particle (a_0,b_0,c_0) limited by the planes $a=a_0$, $a=a_0+\delta a$, $b=b_0$, $b=b_0+\delta b$, $c=c_0$, $c=c_0+\delta c$ in the "label" space occupies the volume

$$\Delta V=\frac{\partial(\tilde{x},\tilde{y},\tilde{z})}{\partial(a,b,c)}\,\Delta a\,\Delta b\,\Delta c \tag{1.36}$$

in physical space. Hence, eq.(1.35) states that the mass of a fluid particle is invariant with respect to time. So – if $(\tilde{x}_0,\tilde{y}_0,\tilde{z}_0)$ is the position and $\tilde{\rho}_0$ the density of the fluid particle at time t_0, for any other time t we must have

$$\tilde{\rho}\,\frac{\partial(\tilde{x},\tilde{y},\tilde{z})}{\partial(a,b,c)}=\tilde{\rho}_0\,\frac{\partial(\tilde{x}_0,\tilde{y}_0,\tilde{z}_0)}{\partial(a,b,c)}\,, \tag{1.37}$$

which is the general continuity equation in the Lagrangian description.

In incompressible fluids the density of each fluid particle - and hence the occupied volume - remains constant in time, and so $\tilde{\rho}=\tilde{\rho}_0$ in eq.(1.37). This is equivalent to the Eulerian condition $\nabla\cdot\underline{v}=0$ (eq.(1.5)). In a similar way as with the continuity equation (eq.(1.32)), the Lagrangian form of the components of the Navier-Stokes equation for an incompressible fluid is found to be (the curly marks on the Lagrangian functions from now on being left out for the sake of simplicity),

$$\left.\begin{aligned} x_{tt} &= -\frac{1}{\rho}\,\frac{\partial(p,y,z)/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)}+\nu\left(\nabla^2 u\right)_L \\ y_{tt} &= -\frac{1}{\rho}\,\frac{\partial(x,p,z)/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)}+\nu\left(\nabla^2 \mathrm{v}\right)_L \\ z_{tt}+g &= -\frac{1}{\rho}\,\frac{\partial(x,y,p)/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)}+\nu\left(\nabla^2 \mathrm{w}\right)_L \end{aligned}\right\}. \tag{1.38}$$

Here u,v and w are the components of the Eulerian velocity; the notation $(\nabla^2 u)_L$ denoting the Lagrangian function corresponding to the Eulerian function $\nabla^2 u$, that is (cf eqs.(1.28) and (1.29))

$$\left(\nabla^2 u\right)_L = \left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right\}_L$$

$$= \frac{\partial\left(\dfrac{\partial(x_t,y,z)/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)},\, y,\, z\right)\Big/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)}$$

$$+\frac{\partial\left(x,\,\dfrac{\partial(x,x_t,z)/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)},\, z\right)\Big/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)}$$

$$+\frac{\partial\left(x,\, y,\,\dfrac{\partial(x,y,x_t)/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)}\right)\Big/\partial(a,b,c)}{\partial(x,y,z)/\partial(a,b,c)} \quad , \tag{1.39}$$

which is seen to be a sum of Jacobians of Jacobians. By $(\nabla^2\mathrm{v})_L$ and $(\nabla^2 w)_L$ are meant similar expressions.

In a two-dimensional problem, where nothing is to change in the y-direction, two independent variables will suffice to identify the fluid particles. Choosing a and c to be the x- and z-coordinates, respectively, of the position of the fluid particle (a,c) at the time t_0, that is, $(a,c)\Leftrightarrow(x_0,z_0)$, the basic equations for the two-dimensional, incompressible problem are reduced to

$$x_{tt} = -\frac{1}{\rho}\frac{\partial(p,z)}{\partial(a,c)} + \nu\left(\nabla^2 u\right)_L , \tag{1.40}$$

$$z_{tt} + g = -\frac{1}{\rho}\frac{\partial(x,p)}{\partial(a,c)} + \nu\left(\nabla^2 \mathrm{w}\right)_L \tag{1.41}$$

and

$$\partial(x,z)/\partial(a,c) = 1 , \tag{1.42}$$

where

$$\left(\nabla^2 u\right)_L = \frac{\partial\left(\partial\left(x_t, z\right)/\partial(a,c),\, z\right)}{\partial(a,c)} + \frac{\partial\left(x,\, \partial\left(x, x_t\right)/\partial(a,c)\right)}{\partial(a,c)} \tag{1.43}$$

and

$$\left(\nabla^2 \mathrm{w}\right)_L = \frac{\partial\left(\partial\left(z_t, z\right)/\partial(a,c),\, z\right)}{\partial(a,c)} + \frac{\partial\left(x,\, \partial\left(x, z_t\right)/\partial(a,c)\right)}{\partial(a,c)}. \tag{1.44}$$

The model we develope in chapter II, and our further investigations in chapter III and IV, will all be based upon the Lagrangian description of fluid motion. Eqs.(1.40)-(1.42) will be the governing equations of motion for the analysis and calculations.

CHAPTER II

A MATHEMATICAL MODEL FOR DAMPED OCEAN SURFACE WAVES

In this chapter the mathematical model to be applied in our calculations is established. The model is based on Lagrangian description of fluid motion and perturbation theory. Formal equations for the second-order stream function are obtained. The equations appear in a much more symmetrical and compact form than earlier reported. This form greatly simplifies the task of interpretation.

1. A perturbation approach

In the well-established linear model for surface waves the fluid particles of an incompressible fluid move in closed orbits as illustrated in fig. 2.1. Closer inspection [Stokes, 1847] has

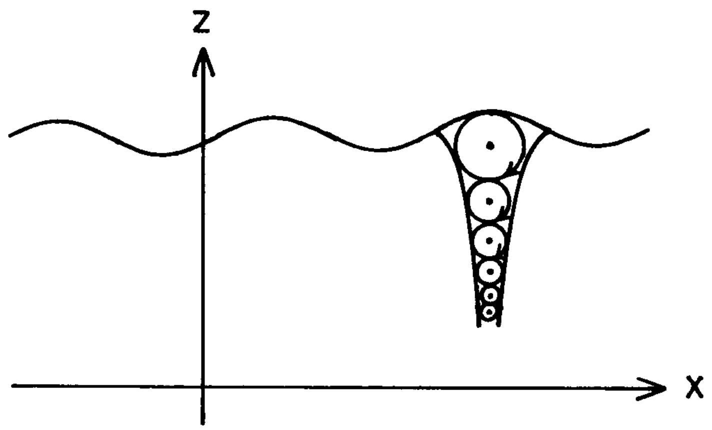

Fig. 2.1. Fluid particle motion in surface waves, given incompressible fluid and a linear model.

shown, however, that the orbits are not quite closed, rather there will be a weak, mean drift of the fluid particles, cf also section 3 of chapter I. This drift is typically of second order, compared to the first-order velocity in the linear model. This is also verified experimentally. The velocity profile of this Stokes drift was sketched in fig. 1.2.

When wanting to describe the fluid motion within the Lagrangian approach – i.e. by following each fluid particle – a natural choice is therefore to use a perturbation model where the actual positions of the fluid particles are considered as (small) perturbations from their equilibrium positions. This was first done by W.J. Pierson [1962], and we shall in principle follow his method. But instead of writing out the second-order equations explicitly, we shall keep the Jacoby determinant notation used by Chang [1969, p.1523], cf section 4 of chapter I, throughout the developement of our model. By keeping the determinant notation the equations become quite symmetrical and neat, and hence easier to handle and to interpret. This will be seen to have great advantages, both when analysing the model and when performing the actual calculations. As far as we have seen, this has not been reported before. Even Chang used the determinant notation only when establishing the initial equations.

The actual position of a fluid particle will be expressed in terms of the deviation from its equilibrium position. A reasonable assumption seems to be that the mean deviation will be small compared to the deviation in the linear model, as long as a moderate time interval (a few periods) is considered.[1)] Taking the equilibrium positions to be the reference positions (labels) of the individual fluid particles, the horizontal and vertical positions of fluid particle *(a,c)* at the (moderate) time *t* may be written [Pierson,1962]

$$\left.\begin{aligned} x(a,c,t) &= a + \varepsilon\, x^{(1)}(a,c,t) + \varepsilon^2 x^{(2)}(a,c,t) + \cdots \\ z(a,c,t) &= c + \varepsilon\, z^{(1)}(a,c,t) + \varepsilon^2 z^{(2)}(a,c,t) + \cdots , \end{aligned}\right\} \tag{2.1}$$

where ε is a small dimensionless number (typically the characteristic wave steepness),

$$0 < \varepsilon << 1 , \tag{2.2}$$

and the functions $x^{(1)}, x^{(2)},\ldots,z^{(1)}, z^{(2)},\ldots$ are all of the same order.

The components of the particle velocity and acceleration will be the first and second partial derivatives of eqs. (2.1) with respect to time. The fluid pressure is expressed in a similar manner, in terms of the deviation from the hydrostatic pressure at the equilibrium position of the fluid particle, that is

$$p(a,c,t) = -\rho g c + \varepsilon\, p^{(1)}(a,c,t) + \varepsilon^2 p^{(2)}(a,c,t) + \cdots . \tag{2.3}$$

(The constant atmospheric pressure at the surface at hydrostatic equilibrium is put equal to zero.)

---

1) In the long run the mean drift may cause the fluid particles to move far away from their original equilibrium positions. This will be discussed in section 7 for the case of steady state.

The governing equations of motion for this Lagrangian description of fluid were established in chapter I, eqs. (1.40)-(1.42). Approximate solutions may be obtained by inserting the expressions (2.1) and (2.3), then solving the equations to a certain order of ε; the higher order, the more accurate solutions.

For convenience, we define a new operator D by

$$D \equiv \frac{\partial}{\partial t} - \nu \nabla_L^2, \tag{2.4}$$

where

$$\nabla_L^2 \equiv \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial c^2}, \tag{2.5}$$

that is, ∇_L^2 is the Laplacian in the variables a and c. To the first order in ε the governing equations (1.40)-(1.42) then become

$$D x_t^{(1)} + g z_a^{(1)} + \frac{1}{\rho} p_a^{(1)} = 0 \tag{2.6}$$

$$D z_t^{(1)} - g x_a^{(1)} + \frac{1}{\rho} p_c^{(1)} = 0 \tag{2.7}$$

$$x_a^{(1)} + z_c^{(1)} = 0, \tag{2.8}$$

whereas the second-order governing equations become

$$\begin{aligned} D x_t^{(2)} + g z_a^{(2)} + \frac{1}{\rho} p_a^{(2)} &= -\frac{1}{\rho} \frac{\partial\left(p^{(1)}, z^{(1)}\right)}{\partial(a,c)} \\ &+ \nu \left\{ \frac{\partial\left(x_t^{(1)}, z_a^{(1)} - x_c^{(1)}\right)}{\partial(a,c)} + 2\frac{\partial\left(x_{ta}^{(1)}, z^{(1)}\right)}{\partial(a,c)} - 2\frac{\partial\left(x_{tc}^{(1)}, x^{(1)}\right)}{\partial(a,c)} \right\}, \end{aligned} \tag{2.9}$$

$$D z_t^{(2)} - g x_a^{(2)} + \frac{1}{\rho} p_c^{(2)} = -\frac{1}{\rho}\frac{\partial\left(x^{(1)}, p^{(1)}\right)}{\partial(a,c)}$$
$$+\nu\left\{\frac{\partial\left(z_t^{(1)}, z_a^{(1)} - x_c^{(1)}\right)}{\partial(a,c)} + 2\frac{\partial\left(z_{ta}^{(1)}, z^{(1)}\right)}{\partial(a,c)} - 2\frac{\partial\left(z_{tc}^{(1)}, x^{(1)}\right)}{\partial(a,c)}\right\}, \tag{2.10}$$

$$x_a^{(2)} + z_c^{(2)} = -\frac{\partial\left(x^{(1)}, z^{(1)}\right)}{\partial(a,c)}. \tag{2.11}$$

The equations (2.6)-(2.11) agree with the equivalent equations derived by, for instance, Pierson [1962] and Weber [1983a]. The strong symmetry between the horizontal and the vertical components is, however, hidden in the form these equations are reported before.

Making use of the continuity equations (2.8) and (2.11), the first- and second-order equations of motion, eqs. (2.6)-(2.7) and (2.9)-(2.10), respectively, may be written even more compact, as the vector equations

$$D\left(x_t^{(1)}, z_t^{(1)}\right) + \nabla_L\left(g z^{(1)} + \frac{1}{\rho} p^{(1)}\right) = \underline{0}, \tag{2.12}$$

$$D\left(x_t^{(2)}, z_t^{(2)}\right) + \nabla_L\left(g z^{(2)} + \frac{1}{\rho} p^{(2)}\right) = (Q_1, Q_2) \equiv \underline{Q}. \tag{2.13}$$

In equations (2.12) and (2.13) the operator ∇_L is defined by

$$\nabla_L \equiv \left(\frac{\partial}{\partial a}, \frac{\partial}{\partial c}\right), \tag{2.14}$$

Q_1 is the right-hand side of the second-order equation for the horizontal motion (eq.(2.9)), and Q_2 is similar for the vertical motion, except for the first term, that is

$$Q_2 \equiv -g\frac{\partial\left(x^{(1)}, z^{(1)}\right)}{\partial(a,c)} + \begin{pmatrix} right - hand\ side \\ of\ eq.\,(2.10) \end{pmatrix}. \tag{2.15}$$

By taking the curl of the first- and second-order vector equations of motion, eqs. (2.12)-(2.13), another convenient set of equations is obtained,

$$D\left(x_{tc}^{(1)} - z_{ta}^{(1)}\right) = 0 \tag{2.16}$$

and

$$D\left(x_{tc}^{(2)} - z_{ta}^{(2)}\right) = Q_{1c} - Q_{2a}\,. \tag{2.17}$$

The term *(Q_{1c} - Q_{2a})* may be expressed as

$$\begin{aligned} Q_{1c} - Q_{2a} = &-\frac{\partial\left(x^{(1)}, D\,x_t^{(1)}\right)}{\partial(a,c)} - \frac{\partial\left(z^{(1)}, D\,z_t^{(1)}\right)}{\partial(a,c)} \\ &+\nu\left\{4\frac{\partial\left(x_a^{(1)}, x_{at}^{(1)}\right)}{\partial(a,c)} + 2\frac{\partial\left(x_c^{(1)}, x_{ct}^{(1)}\right)}{\partial(a,c)} + 2\frac{\partial\left(z_a^{(1)}, z_{at}^{(1)}\right)}{\partial(a,c)} + 2\frac{\partial\left(x^{(1)}, \nabla_L^2 x_t^{(1)}\right)}{\partial(a,c)}\right. \\ &\qquad + 2\frac{\partial\left(z^{(1)}, \nabla_L^2 z_t^{(1)}\right)}{\partial(a,c)} + \frac{\partial\left(\nabla_L^2 x^{(1)}, x_t^{(1)}\right)}{\partial(a,c)} + \frac{\partial\left(\nabla_L^2 z^{(1)}, z_t^{(1)}\right)}{\partial(a,c)} \\ &\qquad\left. + \frac{\partial\left(x_c^{(1)} - z_a^{(1)}, x_{ct}^{(1)} - z_{at}^{(1)}\right)}{\partial(a,c)}\right\} \end{aligned} \tag{2.18}$$

or

$$\begin{aligned} Q_{1c} - Q_{2a} = &-\frac{\partial\left(x^{(1)}, x_{tt}^{(1)}\right)}{\partial(a,c)} - \frac{\partial\left(z^{(1)}, z_{tt}^{(1)}\right)}{\partial(a,c)} \\ &+\nu\left\{4\frac{\partial\left(x_a^{(1)}, x_{at}^{(1)}\right)}{\partial(a,c)} + 2\frac{\partial\left(x_c^{(1)}, x_{ct}^{(1)}\right)}{\partial(a,c)} + 2\frac{\partial\left(z_a^{(1)}, z_{at}^{(1)}\right)}{\partial(a,c)} + 3\frac{\partial\left(x^{(1)}, \nabla_L^2 x_t^{(1)}\right)}{\partial(a,c)}\right. \\ &\qquad + 3\frac{\partial\left(z^{(1)}, \nabla_L^2 z_t^{(1)}\right)}{\partial(a,c)} + \frac{\partial\left(\nabla_L^2 x^{(1)}, x_t^{(1)}\right)}{\partial(a,c)} + \frac{\partial\left(\nabla_L^2 z^{(1)}, z_t^{(1)}\right)}{\partial(a,c)} \\ &\qquad\left. + \frac{\partial\left(x_c^{(1)} - z_a^{(1)}, x_{ct}^{(1)} - z_{at}^{(1)}\right)}{\partial(a,c)}\right\}. \end{aligned} \tag{2.19}$$

(To obtain eqs. (2.18)-(2.19) we have used the first-order governing equations (2.6)-(2.7) to find expressions for the pressure functions $p_a^{(1)}$ and $p_c^{(1)}$ in terms of $x^{(1)}$ and $z^{(1)}$. A term proportional to g (acceleration of gravity) became identically zero by the second-order continuity equation, eq.(2.11).)

2. The surface boundary conditions in Lagrangian description

We shall adopt the standard hydrodynamic assumption that the surface consists of the same fluid particles at all times. Then, letting the equilibrium position of the surface particles have the vertical coordinate (Euler) $z = 0$, the surface will at all times be identified by putting $c = 0$ according to the notation introduced in the previous section. This will replace the Eulerian kinematic boundary condition at the surface, given in eqs.(1.3) of chapter I.

The dynamic boundary conditions at the surface for a two-dimensional problem become, from eqs.(1.3) of chapter I with the expression (1.6) inserted for the stress tensor,

$$p-\frac{2\mu}{\left(1+\eta_x^2\right)}\left\{\mathrm{w}_z-\left(u_z+\mathrm{w}_x\right)\eta_x+u_x\eta_x^2\right\}=-\frac{\sigma\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}} \tag{2.20}$$

and

$$\frac{\mu}{\left(1+\eta_x^2\right)}\left\{u_z+\mathrm{w}_x-2\left(u_x-\mathrm{w}_z\right)\eta_x-\left(u_z+\mathrm{w}_x\right)\eta_x^2\right\}=\frac{\partial\sigma}{\partial s}, \tag{2.21}$$

both to be satisfied at $z = \eta$. Eqs.(2.20) and (2.21) give the conditions normal and tangential to the surface, respectively.

To find the Lagrangian form of these boundary conditions we use the correspondences

$$\left.\begin{array}{l}(u,\mathrm{w})\leftrightarrow(x_t,z_t)\\ \quad\eta\quad\leftrightarrow z\,(c=0)\,.\end{array}\right\} \tag{2.22}$$

The differentiating rules from eqs.(1.28) and (1.29) then give us

$$\begin{aligned}p-\frac{2\mu}{\left(1+z_c^2z_a^2\right)}&\left\{\frac{\partial(x,z_t)}{\partial(a,c)}-\left[\frac{\partial(x,x_t)}{\partial(a,c)}+\frac{\partial(z_t,z)}{\partial(a,c)}\right]z_cz_a+\frac{\partial(x_t,z)}{\partial(a,c)}z_c^2z_a^2\right\}\\ &=-\frac{\sigma}{\left(1+z_c^2z_a^2\right)^{3/2}}\left[\frac{\partial(z_c,z)}{\partial(a,c)}z_a+z_c^2z_{aa}\right],\ c=0\end{aligned} \tag{2.23}$$

and

$$\frac{\mu}{\left(1+z_c^2 z_a^2\right)}\left\{\left[\frac{\partial(x,x_t)}{\partial(a,c)}+\frac{\partial(z_t,z)}{\partial(a,c)}\right]\left(1-z_c^2 z_a^2\right)-2\left[\frac{\partial(x_t,z)}{\partial(a,c)}-\frac{\partial(x,z_t)}{\partial(a,c)}\right]z_c z_a\right\}$$

$$=\frac{\partial\sigma}{\partial s}, \qquad c=0\,. \tag{2.24}$$

Eqs. (2.23)-(2.24) constitute the dynamic surface boundary conditions in Lagrangian description. (It is understood that by p and σ here are meant the corresponding *Lagrangian* functions for pressure and surface tension.)

When substituting the expansions (2.1) and (2.3) for *x, z* and *p* into the boundary conditions (2.23) and (2.24), again collecting terms of equal order in ε, we get the first-order dynamic boundary conditions

$$p^{(1)}-2\mu\, z_{tc}^{(1)}=-\sigma\, z_{aa}^{(1)}, \quad c=0 \tag{2.25}$$

$$\mu(x_{ct}^{(1)}+z_{at}^{(1)})=\left[\frac{\partial\sigma}{\partial s}\right]^{(1)}, \quad c=0\,, \tag{2.26}$$

and the second-order dynamic boundary conditions

$$p^{(2)}-2\mu\left\{z_{ct}^{(2)}+\frac{\partial(x^{(1)},z_t^{(1)})}{\partial(a,c)}-\left(x_{ct}^{(1)}+z_{at}^{(1)}\right)z_a^{(1)}\right\}$$

$$=-\sigma\left\{z_{aa}^{(2)}+z_{ac}^{(1)}z_a^{(1)}+2z_c^{(1)}z_{aa}^{(1)}\right\}, \quad c=0 \tag{2.27}$$

$$\mu\left\{x_{ct}^{(2)}+z_{at}^{(2)}+\frac{\partial(x^{(1)},x_t^{(1)})}{\partial(a,c)}+\frac{\partial(z_t^{(1)},z^{(1)})}{\partial(a,c)}-2\left(x_{at}^{(1)}-z_{ct}^{(1)}\right)z_a^{(1)}\right\}=\left[\frac{\partial\sigma}{\partial s}\right]^{(2)} \quad c=0\,. \tag{2.28}$$

There is a discrepancy between this form of the second-order boundary conditions and the form given by, for instance, Weber [1983 b]. This is because Weber is considering the horizontal-vertical pair of components, whereas we consider the tangential-normal pair. The sets of equations are, of course, equivalent.

3. The first-order wave solution

The linear version of the Navier-Stokes equation (1.7) for an incompressible fluid is

$$\frac{\partial \underline{v}}{\partial t} + \frac{1}{\rho}\nabla p - \nu\,\nabla^2\,\underline{v} - \underline{g} = \underline{0}\,. \tag{2.29}$$

Eq. (2.29) may also be written

$$\left(\frac{\partial}{\partial t} - \nu\,\nabla^2\right)\underline{v} + \nabla\left(g\,z + \frac{1}{\rho}\,p\right) = \underline{0}\,, \tag{2.30}$$

which is formally identical to the first-order Lagrangian equation of motion (eq. (2.12)) if we make the substitutions

$$\left.\begin{array}{lcl} & \nabla \to \nabla_L & \\ \underline{v} \to (x_t^{(1)}, z_t^{(1)}) & & p \to \widetilde{p} \\ z \to z^{(1)} & & \rho \to \widetilde{\rho}\quad . \end{array}\right\} \tag{2.31}$$

Hence, the two equations may be assumed to have formally similar solutions.

For convenience, we shall be using complex notation all over. The physical solutions are, of course, given by the real parts of the complex functions. Writing the first-order velocity $(x_t^{(1)},z_t^{(1)})$ in the form (cf eq. (1.13))

$$(x_t^{(1)}, z_t^{(1)}) = \nabla_L\varphi - \nabla_L \times \underline{e}_y\widetilde{\psi}\,, \tag{2.32}$$

we therefore let the functions φ and $\widetilde{\psi}$ be

$$\varphi = \left(A\,e^{kc} + B\,e^{-kc}\right)e^{i(k\,a-\omega\,t)} \tag{2.33}$$

and

$$\tilde{\psi} = \left(C\, e^{mc} + D\, e^{-mc}\right) e^{i(k\,a - \omega t)}\,, \tag{2.34}$$

cf the Eulerian solution, eqs. (1.16) and (1.19). (In the deep-water solutions (1.16) and (1.19) the coefficients corresponding to *B* and *D* in eqs. (2.33)-(2.34) were equal to zero to assure physically valid solution in the deep-water limit.) The curly mark on the vector potential $\tilde{\psi}$ is to avoid confusion with the stream function ψ to be introduced in section 5.

As in chapter I, the function φ must fulfil a Bernoulli equation -

$$\varphi_t + g\, z^{(1)} + \frac{1}{\rho}\, p^{(1)} = 0\,, \tag{2.35}$$

whereas the function $\tilde{\psi}$ must fulfil

$$D\tilde{\psi} \equiv \frac{\partial \tilde{\psi}}{\partial t} - \nu\, \nabla_L^2 \tilde{\psi} = 0\,. \tag{2.36}$$

Eq. (2.36) leads to the condition

$$m^2 = k^2 - i\,\frac{\omega}{\nu}\,, \tag{2.37}$$

and the parameter *m* is taken to be the root of eq. (2.37) having positive real part. The parameter *k* is also supposed to have positive real part. The coefficients *A, B, C* and *D* are to be determined by the appropriate boundary conditions.

In terms of the functions φ and $\tilde{\psi}$ the boundary conditions (eqs. (2.25)-(2.26)) become

$$\left.\begin{aligned} &\varphi_t + 2\nu\varphi_{cc} - 2\nu\tilde{\psi}_{ac} = \frac{\sigma}{\rho}\, z_{aa}^{(1)} - g\, z^{(1)} \\ &\mu\,(2\,\varphi_{ac} + \tilde{\psi}_{cc} - \tilde{\psi}_{aa}) = \left[\frac{\partial \sigma}{\partial s}\right]^{(1)} \end{aligned}\right\} \quad c = 0\,, \tag{2.38}$$

where the Bernoulli equation (2.35) has been used to eliminate the pressure from eq. (2.38).

For an ideal fluid of infinite depth we have a solution – as in the Eulerian picture –

$$(x_t^{(1)}, z_t^{(1)}) = \nabla_L \varphi\,, \tag{2.39}$$

with the coefficient B in the function φ (eq. (2.33)) equal to zero, hence the linear velocity components become

$$\left.\begin{aligned} x_t^{(1)} &= i\,k\,A e^{kc} e^{i(k\,a-\omega t)} \\ z_t^{(1)} &= \;k\,A e^{kc} e^{i(k\,a-\omega t)} \end{aligned}\right\}. \tag{2.40}$$

The linear solution for the Eulerian velocity v=*(u,w)* is in this case given in terms of the velocity potential in eq. (1.16). In complex form this velocity potential may be written

$$\varphi' = A' e^{k'z} e^{i(k'x-\omega' t)}\,, \tag{2.41}$$

where the marks on the parameters serve to separate them from the Lagrangian parameters in eq. (2.40). Hence, the Eulerian velocity components become

$$\left.\begin{aligned} u &= i\,k'\,A' e^{k'z} e^{i(k'x-\omega' t)} \\ \mathrm{w} &= k'\,A' e^{k'z} e^{i(k'x-\omega' t)} \end{aligned}\right\}. \tag{2.42}$$

Comparing the expressions (2.40) and (2.42), having in mind the restriction that the Lagrangian solution is valid only for small deviations from the equilibrium positions of the fluid particles, we find that the parameters k and ω correspond to the wavenumber k' and the angular frequency ω′, respectively, in the Eulerian description. Hence we may - in the Lagrangian solutions - adopt the interpretation that the parameters k and ω represent the wave number and angular frequency (possibly complex) of an "Eulerian wave". The dispersion relation will be determined by the boundary conditions.

If the first-order velocities are assumed to be periodic in time, the frequency ω must be real, and if they are assumed to be periodic in space, i.e. in the variable a, the wave number k must be real.

4. Time-averaged second-order equations

If the first-order velocities $x_t^{(1)}$ and $z_t^{(1)}$ from section 3 are periodic in time, they will have vanishing mean,

$$\overline{x_t^{(1)}} = 0, \qquad \overline{z_t^{(1)}} = 0 , \tag{2.43}$$

the bar indicating average in time over one period. Two arbitrary functions *f* and *g*, being periodic in time, will always fulfil

$$\overline{f_t\, g} = -\overline{f\, g_t}\ . \tag{2.44}$$

This property will be frequently exploited to manipulate the functions as to obtain convenient equations.

The time derivative of the second-order continuity equation (eq.(2.11)) is given by

$$x_{at}^{(2)} + z_{at}^{(2)} + \frac{\partial(x_t^{(1)}, z^{(1)})}{\partial(a,c)} + \frac{\partial(x^{(1)}, z_t^{(1)})}{\partial(a,c)} = 0\,. \tag{2.45}$$

Averaging this equation in time, taking advantage of the property (2.44) for periodic functions, we obtain a very simple continuity equation for the second-order time-averaged motion,

$$\overline{x_{at}^{(2)}} + \overline{z_{ct}^{(2)}} = 0\,. \tag{2.46}$$

By means of this continuity equation the time-averaged version of the second-order Navier-Stokes vector equation (2.13) may be manipulated to become

$$\frac{\partial}{\partial t}\left(\overline{x_t^{(2)}}, \overline{z_t^{(2)}}\right) + \nu\,\nabla_L \times \underline{e}_y\left(\overline{x_{ct}^{(2)}} - \overline{z_{at}^{(2)}}\right) + \nabla_L\left(g\,\overline{z^{(2)}} + \frac{1}{\rho}\overline{p^{(2)}}\right) = \underline{\overline{Q}}\,. \tag{2.47}$$

We shall assume the mean, second-order acceleration to be negligible compared to the second-order viscous term in eq. (2.47):

$$\left.\begin{aligned} \left|\overline{x_{tt}^{(2)}}\right| &<< \nu\left|\nabla_L^2 \overline{x_t^{(2)}}\right| = \nu\left|\frac{\partial}{\partial c}\left(\overline{x_{ct}^{(2)}} - \overline{z_{at}^{(2)}}\right)\right| \\ \left|\overline{z_{tt}^{(2)}}\right| &<< \nu\left|\nabla_L^2 \overline{z_t^{(2)}}\right| = \nu\left|\frac{\partial}{\partial a}\left(\overline{x_{ct}^{(2)}} - \overline{z_{at}^{(2)}}\right)\right|. \end{aligned}\right\} \qquad (2.48)$$

The equalities are valid because of the continuity equation (2.46). (The validity of the assumption (2.48) will be discussed in chapter VI.) The second-order time-averaged equation of motion, eq.(2.47), is by this assumption reduced to

$$\nu\nabla_L \times \underline{e}_y\left(\overline{x_{ct}^{(2)}} - \overline{z_{at}^{(2)}}\right) + \nabla_L\left(g\,\overline{z^{(2)}} + \frac{1}{\rho}\overline{p^{(2)}}\right) = \underline{\overline{Q}}. \qquad (2.49)$$

Once more using the common property that any vector field may be written in terms of the gradient of a scalar potential and the curl of a vector potential, we express the right-hand side, $\underline{\overline{Q}}$, of the equation of motion as

$$\underline{\overline{Q}} \equiv \nabla_L G + \nabla_L \times\left(\underline{e}_y \nu H\right), \qquad (2.50)$$

where G is the scalar potential and $\nu H\underline{e}_y$ the vector potential. (The kinematic viscosity ν is written explicitly only for convenience.) Taking the divergence and curl, respectively, of eq. (2.50), gives us the following equations for the scalar potential G and the vector potential $\nu H\underline{e}_y$:

$$\nabla_L^2 G = \nabla_L \cdot \underline{\overline{Q}} = \overline{Q_{1a}} + \overline{Q_{2c}} \qquad (2.51)$$

and

$$\underline{e}_y \nabla_L^2(\nu H) = -\nabla_L \times \underline{\overline{Q}} = \underline{e}_y\left(\overline{Q}_{2a} - \overline{Q}_{1c}\right), \qquad (2.52)$$

where

$$\overline{Q}_{1a} + \overline{Q}_{2c} = \overline{z_t^{(1)} \nabla_L^2 z_t^{(1)}} + \overline{x_t^{(1)} \nabla_L^2 x_t^{(1)}} - 2\overline{\frac{\partial(x_t^{(1)}, z_t^{(1)})}{\partial(a,c)}}$$
$$+ 2\nu \left\{ \overline{\frac{\partial(z^{(1)}, \nabla_L^2 x_t^{(1)})}{\partial(a,c)}} - \overline{\frac{\partial(x^{(1)}, \nabla_L^2 z_t^{(1)})}{\partial(a,c)}} \right\}, \tag{2.53}$$

$$\overline{Q}_{2a} - \overline{Q}_{1c} = -2\nu \left\{ 4\overline{x_{aa}^{(1)} x_{act}^{(1)}} + 2\overline{x_{ac}^{(1)} x_{cct}^{(1)}} + 2\overline{z_{aa}^{(1)} z_{act}^{(1)}} + 2\overline{\frac{\partial(x^{(1)}, \nabla_L^2 x_t^{(1)})}{\partial(a,c)}} \right.$$
$$\left. + 2\overline{\frac{\partial(z^{(1)}, \nabla_L^2 z_t^{(1)})}{\partial(a,c)}} + \overline{\left(\nabla_L^2 x^{(1)}\right)\left(\nabla_L^2 z_t^{(1)}\right)} \right\}. \tag{2.54}$$

Substituting $\underline{\overline{Q}}$ by the expression (2.50), the (reduced) equation of motion, eq.(2.49), may be written

$$\nu \nabla_L \times \underline{e}_y \left(\overline{x_{ct}^{(2)}} - \overline{z_{at}^{(2)}} - H \right) + \nabla_L \left(g\, \overline{z^{(2)}} + \frac{1}{\rho} \overline{p^{(2)}} - G \right) = \underline{0}\,. \tag{2.55}$$

A possible solution is that each of the two parts of eq. (2.55) equals zero. This is fulfilled if the functions in the parentheses are constant,

$$\overline{x_{ct}^{(2)}} - \overline{z_{at}^{(2)}} = H + constant\,, \tag{2.56}$$

$$g\,\overline{z^{(2)}} + \frac{1}{\rho}\overline{p^{(2)}} = G + constant\,. \tag{2.57}$$

By taking the curl of the equation of motion in the form (2.49), the alternative equation

$$\nabla_L^2 \left(\overline{x_{ct}^{(2)}} - \overline{z_{at}^{(2)}} \right) = \frac{1}{\nu} \left(\overline{Q}_{2a} - \overline{Q}_{1c} \right) \tag{2.58}$$

for the mean, second-order velocity components is obtained.

5. The stream function

In two-dimensional flow the (Eulerian) velocity field $\underline{v}$ = *(u,w)* of an incompressible fluid may always be given in terms of a *stream function* Ψ, through the equations

$$u = \partial\Psi/\partial z, \qquad w = -\partial\Psi/\partial x . \tag{2.59}$$

As an Eulerian function the stream function Ψ must have a corresponding function ψ in Lagrangian description, so that

$$\Psi(x,z,t) = \psi(a,c,t) . \tag{2.60}$$

(This Lagrangian stream function ψ must not be mistaken to be the function ψ in eq. (1.13).)

Applying the concept of stream function will be of great advantage in the following calculations.

The *Lagrangian* (or fluid particle) velocity components x_t and z_t are given – in terms of the stream function ψ – as

$$x_t = \psi_c , \qquad z_t = -\psi_a . \tag{2.61}$$

Expanding also the Lagrangian stream function ψ by powers of ε,

$$\psi = \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \cdots , \tag{2.62}$$

we have

$$\left.\begin{array}{ll} x_t^{(1)} = \psi_c^{(1)}, & z_t^{(1)} = -\psi_a^{(1)} \\ x_t^{(2)} = \psi_c^{(2)}, & z_t^{(2)} = -\psi_a^{(2)} \\ \vdots & \vdots \end{array}\right\} . \tag{2.63}$$

In terms of the stream function the first- and second-order equations of motion, eqs. (2.12) and (2.13), may be expressed as

$$D\left(\psi_c^{(1)},-\psi_a^{(1)}\right)+\nabla_L\left(g\,z^{(1)}+\frac{1}{\rho}p^{(1)}\right)=\underline{0} \tag{2.64}$$

$$D\left(\psi_c^{(2)},-\psi_a^{(2)}\right)+\nabla_L\left(g\,z^{(2)}+\frac{1}{\rho}p^{(2)}\right)=\underline{Q}\,, \tag{2.65}$$

the differential operator D still being defined by eq. (2.4). By taking the curl of eqs. (2.64)-(2.65) we obtain very simple equations for the first- and second-order stream functions,

$$D\nabla_L^2\,\psi^{(1)}=0 \tag{2.66}$$

and

$$D\nabla_L^2\,\psi^{(2)}=Q_{1c}-Q_{2a}\,. \tag{2.67}$$

When the wave solutions to the first order, $x_t^{(1)}$ and $z_t^{(1)}$, are periodic as in section 3, so are the derivatives of the first-order stream function $\psi^{(1)}$ (cf eq. (2.63)). Hence, they have vanishing mean,

$$\overline{\psi_a^{(1)}}=0,\qquad \overline{\psi_c^{(1)}}=0\,. \tag{2.68}$$

But then the first-order stream function itself must have vanishing mean,

$$\overline{\psi^{(1)}}=0\,. \tag{2.69}$$

An equation for the time-averaged, second-order stream function $\overline{\psi^{(2)}}$ is obtained by averaging eq. (2.67), giving

$$D\nabla_L^2 \overline{\psi^{(2)}} = \overline{Q_{1c}} - \overline{Q_{2a}} \,. \qquad (2.70)$$

Again neglecting the mean, second-order acceleration (cf eq. (2.48)), eq. (2.70) is reduced to

$$\nabla_L^4 \overline{\psi^{(2)}} = -\frac{1}{\nu}\left(\overline{Q_{1c}} - \overline{Q_{2a}}\right). \qquad (2.71)$$

By the scheme of section 4 we shall have a possible solution for the second-order time-averaged stream function $\overline{\psi^{(2)}}$ if also the equation (cf eq.(2.56))

$$\nabla_L^2 \overline{\psi^{(2)}} = H + constant \qquad (2.72)$$

is fulfilled. The function H is defined in eq.(2.50).

By solving eqs. (2.71) and (2.72) together with the appropriate boundary conditions, the Lagrangian mean, second-order stream function $\overline{\psi^{(2)}}$ may be determined.

6. Interpretation of the Lagrangian solution

The curves determined by the Eulerian stream function Ψ = *constant* (*t* fixed) will constitute the field lines of the Eulerian velocity field, known as the stream lines of the fluid, at this certain moment. In case of steady state the stream lines are known to be identical to the actual paths of the fluid particles.

The stream lines must, at a certain moment, be equally well determined by claiming the corresponding *Lagrangian* stream function ψ to be constant (when *t* fixed), but in an indirect manner – by picking out the individual fluid particles constituting the particular stream line at the time.

The components *u* and w of the Eulerian velocity become – in terms of the Lagrangian stream function ψ and the Lagrangian position functions *x* and *z* –

$$u = \frac{\partial \Psi}{\partial z} = \frac{\partial(\tilde{x},\psi)}{\partial(a,c)}, \qquad \mathrm{w} = -\frac{\partial \Psi}{\partial x} = \frac{\partial(\psi,\tilde{z})}{\partial(a,c)}, \qquad (2.73)$$

cf the definition (2.59) of the stream function Ψ and the differentiation rules (1.28)-(1.29), the label *(a,c)* again referring to a position. The Jacobians in eq. (2.73) may be expanded by powers of ε, by inserting the expressions (2.61) for ψ and (2.1) for the Lagrangian functions x and z (the curly marks are omitted):

$$\left.\begin{aligned}\frac{\partial(x,\psi)}{\partial(a,c)} &= \varepsilon\psi_c^{(1)} + \varepsilon^2\left\{\psi_c^{(2)} + \frac{\partial(x^{(1)},\psi^{(1)})}{\partial(a,c)}\right\} + o(\varepsilon^3)\\ \frac{\partial(\psi,z)}{\partial(a,c)} &= \varepsilon\psi_a^{(1)} + \varepsilon^2\left\{\psi_a^{(2)} + \frac{\partial(\psi^{(1)},z^{(1)})}{\partial(a,c)}\right\} + o(\varepsilon^3)\,.\end{aligned}\right\} \tag{2.74}$$

By combining eqs. (2.63) and (2.74) we may state that the Eulerian velocity to the first order (the linear approximation) equals the first-order Lagrangian velocity:

$$\left.\begin{aligned}u^{(1)} &= \varepsilon\psi_c^{(1)} = \varepsilon\, x_t^{(1)}\\ \mathrm{w}^{(1)} &= -\varepsilon\psi_a^{(1)} = \varepsilon\, z_t^{(1)}\,.\end{aligned}\right\} \tag{2.75}$$

To the second order there is a discrepancy between the Eulerian velocity and the fluid particle velocity:

$$\left.\begin{aligned}u^{(2)} &= \varepsilon^2\left\{\psi_c^{(2)} + \frac{\partial(x^{(1)},\psi^{(1)})}{\partial(a,c)}\right\} = \varepsilon^2 x_t^{(2)} + \varepsilon^2\frac{\partial(x^{(1)},\psi^{(1)})}{\partial(a,c)}\\ \mathrm{w}^{(2)} &= -\varepsilon^2\left\{\psi_a^{(2)} + \frac{\partial(\psi^{(1)},z^{(1)})}{\partial(a,c)}\right\} = \varepsilon^2 z_t^{(2)} + \varepsilon^2\frac{\partial(z^{(1)},\psi^{(1)})}{\partial(a,c)}\,.\end{aligned}\right\} \tag{2.76}$$

This discrepancy may be interpreted as a kind of generalized Stokes drift, as may be seen by the following analysis: Assuming the linear solution (2.40), the real parts of the functions are (with A, k and ω real)

$$\left.\begin{aligned}x_t^{(1)} &= -k\,A e^{kc}\sin(ka-\omega t)\\ z_t^{(1)} &= k\,A e^{kc}\cos(ka-\omega t)\,.\end{aligned}\right\} \tag{2.77}$$

By integrating eqs. (2.77) with respect to time we obtain

$$\left.\begin{aligned} x^{(1)} &= -\frac{k\,A}{\omega} e^{kc} \cos(ka-\omega t) \\ z^{(1)} &= -\frac{k\,A}{\omega} e^{kc} \sin(ka-\omega t)\,. \end{aligned}\right\} \tag{2.78}$$

The surface elevation is given by

$$\varepsilon\, z^{(1)}(c=0) = -\varepsilon\, \frac{k\,A}{\omega} \sin(ka-\omega t)\,, \tag{2.79}$$

hence, the wave amplitude is $A_0 = \varepsilon kA/\omega$. Inserting the solutions (2.77)-(2.79), the Jacobian in the first of eqs. (2.76) becomes

$$\begin{aligned} \varepsilon^2 \frac{\partial(x^{(1)},\psi^{(1)})}{\partial(a,c)} &= \varepsilon^2 \left(x_a^{(1)}\psi_c^{(1)} - x_c^{(1)}\psi_a^{(1)} \right) \\ &= -\varepsilon^2 \frac{k^3 A^2}{\omega} e^{2kc} \left(\sin^2(ka-\omega t) + \cos^2(ka-\omega t) \right) \\ &= -\varepsilon^2 \frac{k^3 A^2}{\omega} e^{2kc} \\ &= -\omega k\, A_0^2 e^{2kc}\;. \end{aligned} \tag{2.80}$$

By comparing with the expression (1.22) for the Stokes drift, we find that eq. (2.80) gives the (horizontal) Stokes drift with opposite sign (if the variable c is substituted by the variable z).

For the vertical velocity we obtain, in a similar manner,

$$\varepsilon^2 \frac{\partial(z^{(1)},\psi^{(1)})}{\partial(a,c)} = 0\,, \tag{2.81}$$

which agrees with (minus) the vertical Stokes drift.

Hence, we may interpret the Jacobians in eqs. (2.76) as minus the Stokes drift $\underline{v}_s$, and so the relation between the Eulerian and the Lagrangian velocities is given by

$$\left.\begin{aligned} \underline{v}_E^{(1)} &= \underline{v}_L^{(1)} \\ \underline{v}_E^{(2)} &= \underline{v}_L^{(2)} - \underline{v}_S \end{aligned}\right\}. \tag{2.82}$$

See also Phillips [1977, p. 43].

From eq. (2.82) it is clear that in absence of second-order Eulerian drift, the fluid drift velocity will equal the (generalized) Stokes drift as would be expected.

7. Steady mean motion

The terms "steady state" and "steady motion" used in describing a fluid, serve to characterize a state for which the fluid velocity – at any fixed point in space – is constant in time. Consequently, the term "steady *mean* motion" characterizes a state for which this is the case for the *mean* velocity instead of the instantaneous velocity.

In the case of steady state the stream lines of a fluid are identical to the paths of the fluid particles and are curves fixed in space (independent of time). Hence, having steady state, a stream line will be constituted by the same fluid particles at all times. Describing, in time, the motion of a particular fluid particle will consequently give exactly the same information as if the motion is described, at a fixed moment of time, for all the fluid particles constituting that particular stream line.

The fluid is *not* in steady state in the case of surface waves, according to the definition above. To the first order in the small parameter ε the fluid particles have a periodic motion, but to the second order there is also a non-periodic drift. A fruitful way of describing this kind of motion is to regard the fluid particle as orbiting around a *guiding centre* moving with the much slower mean velocity. This is illustrated in fig. (2.2) below.

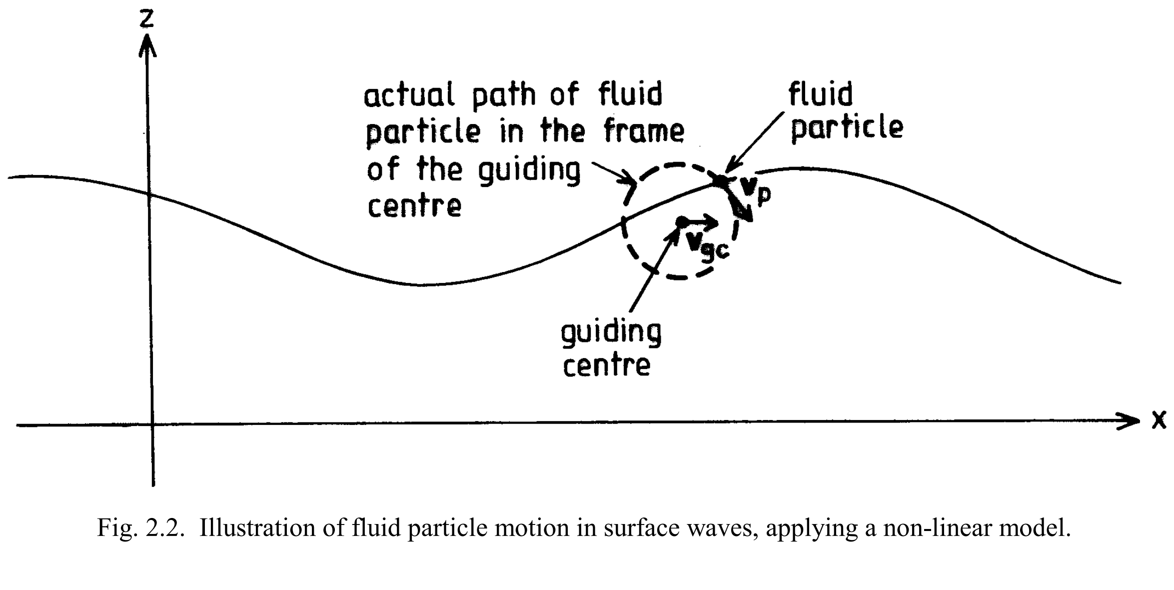

Fig. 2.2. Illustration of fluid particle motion in surface waves, applying a non-linear model.

Relative to this guiding centre the orbit of the fluid particle will be constant in time, and the guiding centre may be regarded as the mean position of the fluid particle. The mean motion of the fluid particle will then be identical to the motion of its guiding centre. If this *mean* motion of the fluid is steady, the (fixed) mean stream lines will give the paths of the guiding centres.

In the perturbation model presented in this chapter the position of a fluid particle is described in terms of the deviation from its (constant) equilibrium position, the solution being valid only for small deviations. In the long run the fluid particle may drift far away from this equilibrium position, and the solution will in general no longer be valid. If having (mean) steady state, this limitation is immaterial as far as the stream lines are concerned. Because – since the stream lines in this case will be curves fixed in space (independent of time) – we may choose any moment of time as reference point for our considerations. By choosing as reference point the moment when the guiding centres are in the (initial) equilibrium positions of the fluid particles, the perturbation model solution will be valid and the mean stream lines determined from this solution will give the mean paths of the fluid particles.

Since our concern is the case of steady mean motion, in which the (mean) stream lines will be curves fixed in space, it will be convenient to return to a description of the fluid as a function of space – an Eulerian description. To identify the individual fluid particles we labelled each particle by its initial equilibrium position. As far as the stream lines are concerned, we may – in our case – redefine the functions from referring to a particular fluid particle to giving the function value in the particular corresponding point of space. In other words, we may redefine the labels *(a,c)* as to represent positions in a fixed coordinate system.

Taking the time average of the expression (2.76) for the Eulerian velocity components, we recognize that

$$\left.\begin{aligned} \overline{u^{(2)}} &= \varepsilon^2 \overline{\psi_c^{(2)}} + \varepsilon^2 \frac{\partial(x^{(1)},\psi^{(1)})}{\partial(a,c)} = \varepsilon^2 \overline{x_t^{(2)}} - u_S \\ \overline{\mathrm{w}^{(2)}} &= -\varepsilon^2 \overline{\psi_a^{(2)}} + \varepsilon^2 \frac{\partial(z^{(1)},\psi^{(1)})}{\partial(a,c)} = \varepsilon^2 \overline{z_t^{(2)}} - \mathrm{w}_S\ , \end{aligned}\right\} \tag{2.83}$$

where u_s and w_s are the components of the Stokes drift. With no Eulerian mean velocity of second order ($\overline{u^{(2)}} = \overline{w^{(2)}} = 0$), the particle drift is just the Stokes drift. Hence, the curves in the *(a,c)*-plane, resulting from claiming $\overline{\psi^{(2)}}$ = *constant* (as a function of *a* and *c*), will represent the paths of the fluid's guiding centres or the mean stream lines of the fluid.

According to the above considerations, $\overline{x_t^{(2)}}$ will give the mean motion of the guiding centers in the steady state case, with *(a,c)* now denoting (fixed) positions of space instead of individual particles.

8. Concluding remarks

In this chapter we have established the mathematical tool to be applied to the special cases studied in the next chapters. The equations are given in a more compact and symmetrical form than earlier presentations, which is considered as an improvement. First-order solutions periodic in time and mean, steady state are basic assumptions in obtaining this. The stream function plays an important role in the model.

CHAPTER III

SURFACE WAVES DAMPED BY ELASTIC SURFACE FILM

The theory explaining the damping effect of oil upon water waves is presented in chapter III, after a historical review on the subject. The elastic property of the oil is crucial. A mathematical formulation is given, based upon the model established in chapter II. The (first-order) wave problem is solved, and the mean, tangential stress at the surface is determined.

1. Wave-damping by oil - a historical review

The calming effect of pouring oil into a furious sea has been well-known to sailors and fishermen since ancient times. A proper explanation to the phenomenon has, however, been lacking quite up to modern time, despite numerous - more or less appreciable - attempts. The slow development may partly be ascribed to confusion caused by the fact that there are two different, but strongly related, effects of oil on water. One effect is the calming of breaking storm waves, now known to be connected to the lower rate of generation of wind waves, the other effect is the damping of waves generated on still waters (in absence of wind). In this section we shall give a historical review of how an understanding of the subject did develop. The review is basicly a summary of a review article by John C. Scott [1978]. Other sources have been historical reviews by J. Lucassen [1968] and C.H. Giles [1969].

Already the old Greek philosophers were aware of the calming effect of oil on storm waves. This is evident from the writings of Plutarch and Pliny the Elder. The Antique's Greeks were busy traders by sea - carrying especially *olive oil*, so they probably would have discovered the effect themselves if it had not already been known to them.

Going back to the original classic writings one may find good attempts to explain the phenomenon. Although written in a form very unfamiliar to us today, some of it may be interpreted as to contain a remarkable insight - close up to what we today appreciate as being the correct explanation. Plutarch refers, for instance, in his *Natural Phenomena,* Aristotle's belief that "the role of oil is in causing a smoothness of the surface, so that the wind can make no impression and raise no swell". Here the question "How does oil calm the waves?" is changed into "How does oil smoothen the surface?" - the last formulation being much the kind of question raised by scientists of today. It represents a more materialistic way of approaching the problem than commonly met among the classic nature philosophers, who did usually address the inherent properties of the different materials and their affinity to each other, and hence appreciated a more spiritual approach.

The subject has attracted attention - both popular and among scientists - with varying intensity during history. Also the Church did at times interfere in the debate, as for instance Simon Majolus, Bishop of Volturara - who in his Dies Caniculares tells the tale of a miracle of St. Aidan in which holy oil was given to sailors setting out on a voyage, for use if storm should arise. It is reported that the desired effect was produced, the sea ceasing its fury, and Bishop Majolus is concerned "lest such 'miracles' should appear to detract from the miracle of Jesus Christ in stilling a storm without the need for oil". Such concern made representatives of the established Church deny the existence of the effect as a whole.

A new "golden age" of the subject arose in the last half of the eighteenth century. It was initiated by the scientist and statesman Benjamin Franklin. His interest in the subject did arise on a journey from America to England in 1757. Sailing in convoy with other ships, he noticed that the wakes of some ships were much smoother than of the others, a phenomenon the captain explained by greasy garbage thrown overboard. Later Franklin performed an experiment in a pond in Clapham Common near London, addressing this effect of grease or oil on water waves: On a windy day, with ripples all over the surface of the pond, he poured a teaspoonful of olive oil into the pond. All of a sudden the whole surface was like a mirror, he tells. He was astonished both by the swiftness with which the oil spread, and by the calming effect on the waves. This incident initiated a whole series of experiments performed by both Benjamin Franklin himself and others on the effect of oil on water waves. Franklin reported the results of his experiments to the scientific society, and his great contribution to the field is that he was the first to introduce indisputable facts on the subject to the scientific world, "applying the principles of observation, experiment and theoretical construction in a new area". He was aware of that the effect was not a new discovery, though - he refers both to Pliny and others, and also cites many instances of its common use.

Franklin's theories did, however, not bring the state of the art any further. He tried to explain why the oil spreads so quickly on water by using the concepts of adhesion and cohesion, but did not succeed. Also his tentative solution regarding the reduced ripple has no root in reality; he argued that the oil acted to reduce the friction between the air and the water, just as the function of oil in a mechanical machinery. This has later proven to be wrong.

The subject was heavily discussed during a period of several years. In 1775 it was even offered a prize for the best set of answers to a series of questions concerning the most effective use of the effect, and reported experiments. It is not known if the prize was ever rewarded.

A few years after Franklin performed and reported his experiments, Franz Carl Achard describes the probably first laboratory scale experiments on wave breaking. He did, however, ignore the wind. In these experiments waves were made mechanically, and then oil was added. Achard's explanation to the damping of the waves was that the oil represented a load on the water surface which hindered the lifting up of water necessary to cause waves, and so there would be less wave motion. L'Abbé Mann produced at this time another "fluffy" explanation to the generation of waves by wind, probably inspired by the so-called phlogiston theory which was quite popular at the time. He argued that water has a natural affinity to air - and so *wants* to be rushed along with the wind. The damping effect of oil is then explained by the oil separating the water from the air, and since the oil does not have this natural affinity to air, this separation prevents the waves from being excited in the same way. Robinet (1807) did propose the explanation to the effect of oil removing the wind-produced ripple, that the oil fills the wave troughs and so eliminates the wave form of the surface.

About this time a wave theory was beginning to develop, and monographs were published both by Gerstner (Theorie der Wellen, 1809) and the Weber brothers (Wellenlehre, auf experimente gegründet, 1825). It is therefore rather surprising that the interest in the effect of oil on water waves seems to fade away at the very same time.

Today we know that the *surface tension* is crucial as regards the influence of oil on water waves. This concept was not properly established until around 1850. The Italian scientist Marangoni discovered in 1871 the existence of changes in surface tension in wave motion. Variations in surface tension result in surface *elasticity*, and Marangoni discovered the effect of elasticity on surface motion, including the damping effect of oil on surface waves. With this discovery part of the wave-damping problem, especially damping in absence of wind, reached its satisfactory explanation. The fundamental theory for the decay of short waves by oil (or surface films) is often called "the Marangoni wave damping theory" or "the Marangoni effect" after him. (Other phenomena are also called "the Marangoni effect".)

Already the Weber brothers (1825) had recognized that the effect of oil on water has to be devided into two separate effects - the damping effect of waves in the absence of wind, and the effect of oil as to reduce the stress with which the wind acts upon the water surface. This was, however, not appreciated among most scientists until much later, which has caused considerable confusion.

The damping effect of an elastic oil film on water waves is given its first satisfactory explanation by O. Reynolds in 1880 as "owing to the surface tension (...) varying inversely as the thickness of oil, thus introducing tangential stiffness into the oil-sheet, which prevented the oil (from) taking up the tangential motion of the water beneath". Reynolds, however, still linked the two aspects of the effect. Ten years later Lord Rayleigh describes the mechanism of swell damping correctly, and in 1895 Horace Lamb gives the first mathematical expression: *"... in oscillatory waves any portion of the surface is alternately contracted and extended... . The consequent variations in tension produce an alternating drag on the water, with a consequent increase in the rate of dissipation of energy."*

While Lamb probably assumed the damping rate to increase monotonically between the two extremes of zero and infinite elasticity, R. Dorrestein did show [1951] that there exists an intermediate maximum damping rate. He also demonstrated that the damping rate can be very sensitive to small values of elasticity.

The damping effect of oil on storm waves has been tried used to reduce damage in harbours. Such experiments were not, however, approved by everybody - so did, for instance, Joannes Le Francq van Berkhey publish a brochure where he warned that reducing the waves in one region must make the sea more dangerous elsewhere. The Scottish businessman Shields rediscovered the effect at the end of the nineteenth century. He performed fullscale experiments in Peterhead harbour - during a hurricane - in 1882, with good results. His experiments got a lot of publisity, and led to renewed interest also among scientists.

Van der Mensbrugghe proved in 1882 Franklin's friction-reduction theory to fail - both pure water and pure oil got more ripples on the surface when exposed to wind than water with oily surface. He proposed that the reduced surface tension in the last case caused the reduced growth of ripples, but we now know that this is also wrong. Aitken produced the explanation that the damping of ripples was due to the *changes* in surface tension - that the elastic film exposed resistance against such variations, or against stretching and compression - which again led to less ripple. Aitken also proposed that this effect could work on the surface of large waves (swell), and hence prevent breaking.

In the last decades the interest in the field has increased after a long period of minor activity. The renewed interest is strongly related to the efforts of establishing an efficient earth surveillance by means of satellites. The great oceans are of special importance, and since huge ocean areas may be covered by organic surface films (plancton, oil spill etc.), more knowledge about the effects of such films is needed.

Fig. 3.1. Ripples damped by organic surface film along the shore.
Photo from Kaldfjord, Kvaløya (outside Tromsø).

Today the following is established (valid for open sea and deep water): Air flow over the crests of large amplitude waves produces an aerodynamic drag - depending on the aerodynamic roughness of the crest. A strong drag will cause a considerable horizontal force to apply to the crests. This steepens the wave so that the probability for the wave to break grows, also it transfers energy to the wave so that if it does break, the water thrown off will be in a more violent motion (more kinetic energy). The effect of oil is to reduce the roughness - the ripples - of the surface of the wave, and hence reduce the drag, so that the probability of breaking will be reduced and thrown-off water will be less harmful. The energy from the wind is transferred to the large wave via the ripples by wave-wave interactions. The answer to *how* oil affects the wind-generation of short wavelength disturbances lies in the mechanical properties of an oil slick, such as the dilational elasticity which results from the *changes* in surface tension with compression and extension of the surface. (This is an effect of having an oil surface upon water, pure oil or pure water does not show this effect.) This elasticity does probably stabilize the surface against disturbances, hence the wind is less able to generate ripples.

In the following we shall discuss the plain damping effect of oil - in absence of wind. In this context the process of energy transfer from the wind to the waves is of less interest. Rather we shall look in detail into the properties of an oil slick, and its effect upon capillary surface waves.

2. Surface-active monolayer and capillary waves

Oil poured onto the water will - if not hindered - rapidly spread out and constitute a monomolecular surface layer, commonly referred to as a monolayer. Most oils are surface-active fluids, also called surfactants; they affect the dynamic properties of the water surface. Characteristic for most of these are long molecules with one end being hydrophile and the other hydrophobic. The hydrophile end will seek contact with the water, whereas the hydrophobic end will seek to be as far from the water as possible, hence such molecules tend to line up normal to the surface (see fig.(3.2,a) below). A stretching of the surface will force the molecules to lay more or less down (fig.(3.2,b)), resulting in less surfactant material per unit surface area.

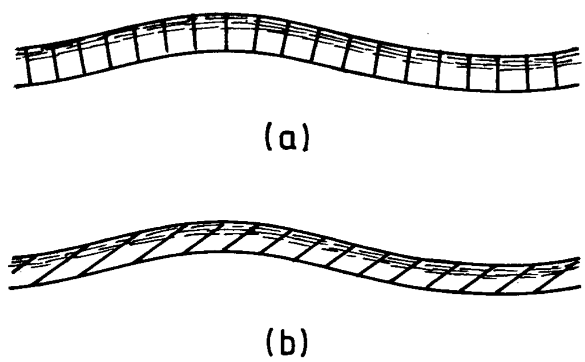

Fig. 3.2. Behavior of surface-active molecules exposed to streching of the surface.

E.H. Lucassen-Reynders and J. Lucassen [1969] have given a thorough theory of the effect of a surface-active monolayer. We shall refer those parts interesting for an *insoluble* oil slick, which will be our concern.

Already Benjamin Franklin [1774] had recognized that short waves are more affected by oil film than are longer waves. Lord Kelvin proposed that the properties of capillary waves (ripples) are mainly determined by the surface tension. By the work of Lucassen-Reynders and Lucassen and others, ripples now represent one of the few cases in which the relation between the dynamic properties of a surface and liquid flow can be predicted completely.

The important observation to make is that one has a surface monolayer of surface-active material with *elastic* properties. The elastic properties cause resistance in the monolayer against the periodic expansions and contractions of the surface accompanying wave motion. This will again alter the pattern of liquid flow immediately beneath the surface, increasing the rate of energy dissipation due to viscous friction. Hence there will be higher damping than for the pure liquid surface.

The surface boundary conditions are crucial in determining the surface motion. Stress acting from within the liquid must be exactly balanced by any stress due to surface and external forces. A surface-active monolayer will not have any effect upon the *normal* component of the stress tensor. However, the periodic contractions and expansions of the surface accompanying wave motion will cause periodic deviations in the surface coverage from its equilibrium value at the resting surface. Since the surface tension depends upon the surface coverage, the wave motion will cause a spatially varying surface tension. This again introduces a varying, non-zero tangential surface stress to appear in the *tangential* component of the stress tensor.

Generally, the state of stress in a surface during deformation should be described by a tensor rather than by an isotropic surface tension. The components of this tensor must depend on the extent and rate of surface deformation, in a relationship involving the resistance of the surface against changes in both area and shape. Either of these types may be expressed in terms of an elasticity and a viscosity. A very common assumption is that the changes in shape are negligible compared to the changes in area, and we shall be concentrating on this case. This implies the very important simplification that the surface stress tensor becomes *isotropic*. The locally varying surface stress may then be described by the gradient of a scalar surface tension.

We shall look a bit more into the physical situation in which we may consider the surface as purely elastic. Such behaviour is observed when the variation in surface tension resulting from local surface contraction or expansion is virtually instantaneous, that is; when any relaxation processes that can cause time dependent variations in the surface tension are either very fast or very slow with respect to the time scale of an experiment, i.e. the inverse ripple frequency. Two conditions must be met for this to be the case: Equilibrium between the surface tension and the locally varying surfactant concentration must be quickly established, and diffusional interchange between bulk and surface during expansion or contraction should be negligible. If diffusional interchange is absent or sufficiently slow for this to be true, the monolayer behaves as insoluble – even though not insoluble by ordinary standards which in general refer to much longer time periods. With these conditions fulfilled the surface will act as purely elastic.

Lucassen-Reynders and Lucassen performed calculations that show – especially for large deviations from an ideal (= inviscid) surface – that the compressibility decreases steeply upon addition of surfactant, which explains the stronger damping.

If one or both of the above mentioned conditions are *not* fulfilled, we have a so-called viscoelastic surface. Rapid changes in surface area may cause the measureable surface tension to have a different value than that of a surface at rest with the same coverage. This effect relies on the reorientation processes of the molecules. Also - if diffusional interchange between surface and bulk during expansion or contraction is not negligible, the surface tension gradients are more or less shortcut, which increases the absolute value of the compressibility.

It may be shown [Lucassen-Reynders and Lucassen, 1969] that the surface dilational elasticity and viscosity can be evaluated as functions of measureable properties if an instantaneous equilibrium between the *local* values of the surface tension, the surfactant concentration and the concentration immediately beneath the surface is assumed. The only effect of diffusion then is to transport matter between the (sub-) surface and deeper layers. In the case of very low frequency or very high concentration there is nothing left of the viscoelastic properties of the surface; any surface tension gradient is completely levelled out by diffusion, and the surface behaves as a pure surface (but with much lower surface tension).

Lucassen-Reynders and Lucassen were able to show that as a function of the surface dilational elasticity, there exists an intermediate maximum in the damping of ripples, and that this maximum is most pronounced for purely elastic surfaces. The maximum occurs when the wave frequency nearly coincides with the frequency of longitudinal, elastic film waves. The surface viscosity lowered the damping only for the most soluble surfactants, but viscosity due to diffusion was able to nearly completely suppress the maximum.

It is important to note that all these reported results are based purely upon hydrodynamic theory, which attributes all energy dissipation to viscous friction in the bulk liquid and not in the surface. The main effect of surfactant monolayers with viscoelastic properties is to *modify the pattern* of liquid flow in the bulk through their effect on the boundary conditions.

Generally, with the surface tension during deformation being expressed by a *tensor*, the stress will be given by its divergence. In this work we shall assume a purely elastic surface, and the surface stress tensor is then reduced to the scalar surface tension. The variation or gradient in the surface tension may in this case be described in terms of the compressibility of the surface and the concentration of surfactant - the compressibility being solely determined by a surface equation of state which gives the equilibrium relationship between the surface tension and the surfactant concentration. Recent studies [Mass and Milgram, 1998] have shown that these conditions are well met when capillary-gravity waves are considered.

3. Mathematical formulation - wave solution

We shall implement the Lagrangian model established in chapter II to the case where the ocean surface is covered by a thin film of surface-active material (surfactant), cf fig.(3.3) on next page. The film is assumed elastic and insoluble, being considered as a monolayer.

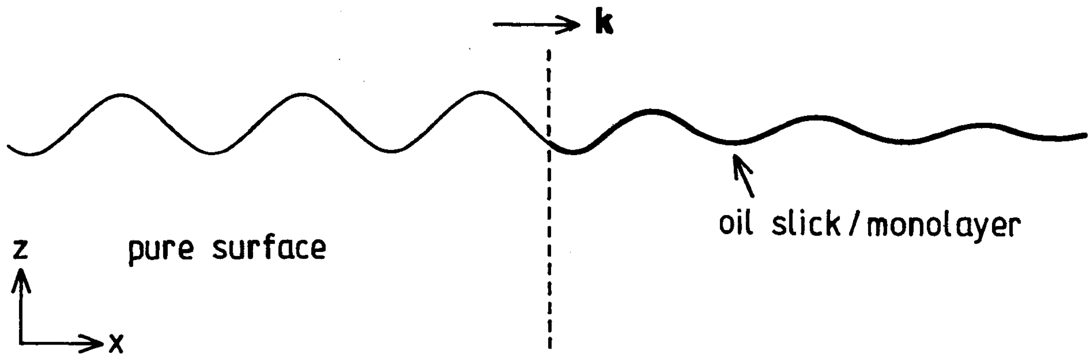

Fig. 3.3. Surface waves moving from pure ocean into an oilslick covered area.

For infinite depth and width, with gravity as only external body force, it was seen (eqs. (2.32)-(2.34)) that in a wave solution of the linear problem the velocity components are given by

$$\left.\begin{aligned} x_t^{(1)} &= \left(i\,k\,A\,e^{kc} + m\,C\,e^{mc}\right)e^{i(ka-\omega t)} \\ z_t^{(1)} &= \left(k\,A\,e^{kc} - i\,k\,C\,e^{mc}\right)e^{i(ka-\omega t)} \end{aligned}\right\}, \qquad (3.1)$$

where the parameter m must fulfil (cf eq. (2.37))

$$m^2 = k^2 - i\,\frac{\omega}{\nu}\,. \qquad (3.2)$$

The first-order pressure $p^{(1)}$ is obtained from the Bernoulli equation (2.35),

$$\varphi_t + g\,z^{(1)} + \frac{1}{\rho}\,p^{(1)} = 0\,, \qquad (3.3)$$

with the velocity potential φ in this case given as (cf eq. (2.33))

$$\varphi = A\,e^{kc}\,e^{i(ka-\omega t)}\,. \qquad (3.4)$$

The surface is given by $c=0$ at all times. The coefficients A and C are to be determined by the dynamic boundary conditions at the surface (cf eqs. (2.25)-(2.26)),

$$\left.\begin{aligned} p^{(1)} - 2\mu z_{tc}^{(1)} &= -\sigma z_{aa}^{(1)} \\ \mu\left(x_{ct}^{(1)} + z_{at}^{(1)}\right) &= \left[\frac{\partial \sigma}{\partial s}\right]^{(1)} \end{aligned}\right\}, \quad c = 0. \tag{3.5}$$

Since the surfactant is assumed insoluble, the total amount of surfactant must be conserved, even though the surface is being periodically contracted and extended due to the wave motion. We define a function Γ as to express the concentration of surfactant, i.e. the number of surfactant molecules per unit surface area. Letting Γ_0 denote the surfactant concentration in the undisturbed case, conservation of surfactant may be expressed by the equation

$$\Gamma_0 da = \Gamma ds\,, \tag{3.6}$$

where *da* represents the initial (horizontal) surface element and *ds* the distorted surface element. Assuming the distortion of the surface element *da* to cause only minor change in the length of the element, see fig. 3.4,

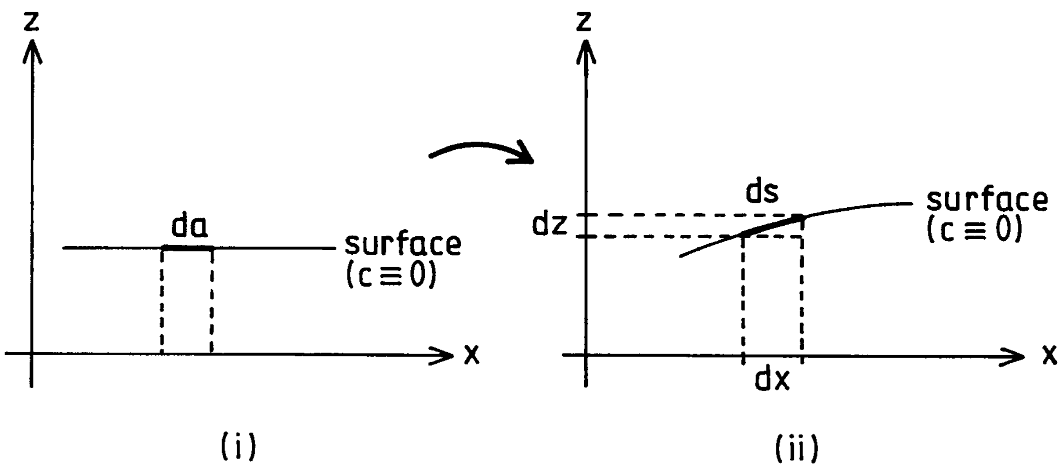

Fig. 3.4. Distortion of a surface element (*da*).

ds may be expressed as

$$ds = \sqrt{(dx)^2 + (dz)^2} = \sqrt{x_a^2 + z_a^2}\,\Big|_{c=0} da\,. \tag{3.7}$$

Combining these expressions, the surfactant concentration Γ becomes

$$\Gamma = \frac{\Gamma_0}{\sqrt{x_a^2 + z_a^2}\,\Big|_{c=0}}\,. \tag{3.8}$$

The surface tension σ depends on *x* through the surfactant concentration Γ ($\sigma = \sigma(\Gamma)$), hence the gradient in the surface tension may be expressed as

$$\frac{\partial\sigma}{\partial x} = \frac{d\sigma}{d\Gamma}\frac{\partial\Gamma}{\partial x} = \frac{d\sigma}{d\Gamma}\frac{\partial(\Gamma,z)}{\partial(a,c)} \approx \left(\frac{d\sigma}{d\Gamma}\right)_0 \frac{\partial(\Gamma,z)}{\partial(a,c)} = \left(\frac{d\sigma}{d\Gamma}\right)_0 \Gamma_a z_c\,, \tag{3.9}$$

where the last equality is valid since Γ is independent of *c* (*c* ≡ *0* at the surface). Again the subscript "0" means the value taken in the undisturbed case. The compressibility C_s of the surface may be defined as the ratio of the relative change in surfactant concentration to the change in film pressure, the film pressure being the difference ($\sigma_w - \sigma$) between the surface tension without and with film present. Mathematically the compressibility may be expressed as

$$C_s \equiv \left[\frac{d\,\ln\Gamma}{d(\sigma_w - \sigma)}\right] \approx \left[\frac{d\,\ln\Gamma}{d(\sigma_w - \sigma)}\right]_0 = -\frac{1}{\Gamma_0}\left[\frac{d\Gamma}{d\,\sigma}\right]_0\,, \tag{3.10}$$

and will be assumed constant. Combining eqs. (3.9) and (3.10) gives the gradient in surface tension as

$$\frac{\partial\sigma}{\partial x} = -\frac{1}{C_s\Gamma_0}\Gamma_a z_c\,. \tag{3.11}$$

Expanding the function x by powers of ε (cf (eq. (2.1)), the first-order approximation of Γ becomes (from eq. (3.8))

$$\Gamma \approx \Gamma_0 \left(1 - \varepsilon\, x_a^{(1)}\right)\Big|_{c=0} . \tag{3.12}$$

Then eq.(3.11) becomes - to the first order in ε -

$$\frac{\partial \sigma}{\partial x} = \frac{1}{C_s} \varepsilon\, x_{aa}^{(1)} \Big|_{c=0} . \tag{3.13}$$

This agrees with the result of Weber and Sætra [1995].

To the first order the surface tension σ in eq. (3.5) may be replaced by the unperturbed value σ_0 (which still may depend on *a*). Hence, in terms of the unperturbed surface tension σ_0 and the compressibility C_s , the first-order surface boundary conditions (3.5) become

$$\left.\begin{aligned} p^{(1)} - 2\mu\, z_{tc}^{(1)} &= -\sigma_0\, z_{aa}^{(1)} \\ \mu\left(x_{ct}^{(1)} + z_{at}^{(1)}\right) &= \frac{1}{C_s} x_{aa}^{(1)} \end{aligned}\right\} c = 0 . \tag{3.14}$$

The velocity components are periodic in time when ω is real. Without lack of generality we may assume this to be the case also for the position components $x^{(1)}$ and $z^{(1)}$, which then become

$$x^{(1)} = \frac{i}{\omega} x_t^{(1)}, \quad z^{(1)} = \frac{i}{\omega} z_t^{(1)} . \tag{3.15}$$

Inserting into the boundary conditions (3.14) expressions obtained from eqs. (3.1), (3.3), (3.4) and (3.15) for the respective functions, the following set of equations results:

$$\left\{-i\omega A+\frac{i\,g\,k}{\omega}\left(A-iC\right)+2\nu\,k^2\left(A-i\frac{m}{k}C\right)\right\}e^{i(ka-\omega t)}$$

$$=-\frac{i\sigma_0 k^3}{\rho\omega}\left(A-iC\right)e^{i(ka-\omega t)}\ , \tag{3.16}$$

$$\left\{i\,k^2 A+m^2 C+i\,k^2\left(A-iC\right)\right\}e^{i(ka-\omega t)}=\frac{k^3}{\mu C_s\omega}\left(A-i\frac{m}{k}C\right)e^{i(ka-\omega t)}. \tag{3.17}$$

Eqs.(3.16) and (3.17) - or rather their real parts - should be valid for all positive values of *a* and all times *t*; hence, the two equations determining *A* and *C* become (after some manipulation)

$$\left.\begin{array}{l}\left\{1-\left(\dfrac{\omega_k}{\omega}\right)^2+i\,(k\delta)^2\right\}A+\left\{i\left(\dfrac{\omega_k}{\omega}\right)^2+\left(\dfrac{m}{k}\right)(k\delta)^2\right\}C=0\\[2ex]\left\{1-i\,(k\delta)\beta\right\}(k\delta)A+\left\{i\beta-(k\delta)^2\beta-i\,(k\delta)\left(\dfrac{m}{k}\right)\right\}C=0\end{array}\right\}. \tag{3.18}$$

In eq. (3.18) new parameters ω_k, β and δ have been introduced, defined by

$$\omega_k^2\equiv gk+\frac{\sigma_0}{\rho}k^3\ , \tag{3.19}$$

$$\beta\equiv 2\mu\,C_s\omega/\left(k^2\delta\right) \tag{3.20}$$

and

$$\delta\equiv\sqrt{2\nu/\omega}\ . \tag{3.21}$$

We recognize ω_k as the familiar angular frequency of a surface wave with wave number k, β is a dimensionless number*) and δ is the characteristic thickness of the viscid surface layer.

Either of the eqs.(3.18) may be chosen to express the unknown coefficient C in terms of the coefficient A. The second equation gives us

$$C = -\frac{1 - i(k\delta)\beta}{\left[i\beta - (k\delta)^2 \beta - i(k\delta)\left(\frac{m}{k}\right) \right]} (k\delta) A \,. \tag{3.22}$$

Assuming our surface wave to be considerably damped within a few wavelengths and the viscid surface layer (in the water) to be thin compared to the wavelength, we shall have

$$|k\delta|^2 << 1 \,. \tag{3.23}$$

Neglecting terms of this order, we find from eq. (3.2) that the parameter m may be approximated by

$$m \approx (1 - i)/\delta \,. \tag{3.24}$$

To this accuracy eq. (3.22) for the coefficient C is reduced to

$$C \approx \frac{1 - i(k\delta)\beta}{\beta - 1 + i}\, i\,(k\delta) A \,, \tag{3.25}$$

and then the (approximate) first-order velocity components become (from eqs. (3.1)),

*) β is in fact the quotient between the two dimensionless numbers $(\mu C_s \omega)/k$ and $k\delta$ possible to construct from the five parameters μ, C_s, ω, k and δ, and will be a number characterizing the behaviour of the surface.

$$\left.\begin{aligned} x_t^{(1)} &\approx \left\{1+(1-i)\frac{(1-i(k\delta)\beta)}{(\beta-1+i)}\,e^{(m-k)c}\right\}A\,i\,k\,e^{kc+i(ka-\omega t)} \\ z_t^{(1)} &\approx \left\{1+(k\delta)\frac{(1-i(k\delta)\beta)}{(\beta-1+i)}\,e^{(m-k)c}\right\}A\,k\,e^{kc+i(ka-\omega t)}\ . \end{aligned}\right\} \tag{3.26}$$

In the two limiting cases – no film (infinite compressibility) and incompressible film (vanishing compressibility) – it is seen by eq. (3.25) that the coefficient C is reduced to, respectively,

$$C \approx (k\delta)^2 A \qquad \text{(no film)} \tag{3.27}$$

and

$$C \approx \frac{(1-i)}{2}(k\delta)A \quad \text{(incompressible film)}. \tag{3.28}$$

A first-order solution to this problem was presented by this author in an early report [Foss, 1990]. The present work represents an improved presentation; exactly the same solution results, though. The solution agrees with the one reported by Weber and Sætra [1995].

4. The dispersion relation

To have a non-trivial solution the determinant of the system (3.18) of equations for the coefficients A and C must equal zero,

$$\begin{aligned} &\left\{1-\left(\frac{\omega_k}{\omega}\right)^2 + i\,(k\delta)^2\right\}\left\{i\beta-(k\delta)^2\beta - i\,(k\delta)\left(\frac{m}{k}\right)\right\} \\ &\qquad -\{1-i\,(k\delta)\beta\}\left\{i\left(\frac{\omega_k}{\omega}\right)^2 + (k\delta)^2\left(\frac{m}{k}\right)\right\}(k\delta) = 0\ . \end{aligned} \tag{3.29}$$

Eq. (3.29) is the dispersion relation of the system. It agrees with the dispersion relation reported by Dysthe and Rabin [1986]. Equivalently eq. (3.29) may be written

$$\left\{\beta-\left(\frac{m}{k}\right)(k\delta)+(k\delta)\right\}\left(\frac{\omega_k}{\omega}\right)^2=\left[1+\alpha(k\delta)^2\right]\beta-\left(\frac{m}{k}\right)(k\delta), \qquad (3.30)$$

the parameter α being defined as

$$\alpha\equiv 2i-\left(\frac{m}{k}\right)(k\delta)^2-(k\delta)^2. \qquad (3.31)$$

This dispersion relation is in perfect agreement with the dispersion relation of small-amplitude sinusoidal surface waves given by Hansen and Ahmad [1971], if film viscosity is neglected.

By approximating the parameter *m* by the expression (3.24), neglecting terms even of order *(kδ)* relatively to unity, we obtain the approximate dispersion relation

$$(\beta-1+i)\left[\left(\frac{\omega_k}{\omega}\right)^2-1\right]=0. \qquad (3.32)$$

Eq.(3.32) may be fulfilled in two different manners – if the parameter β equals *(1-i)* or if the angular frequency ω equals ω_k:

$$\beta=1-i \quad \text{or} \quad \omega^2=\omega_k^2. \qquad (3.33)$$

Hence, two solutions or wave modes may exist, with dispersion relations given by (inserting eqs.(3.19) and (3.20) for ω_k and β, respectively)

$$\omega^{3/2}\equiv\omega_L^{3/2}=\frac{(1-i)k_L^2}{\sqrt{2\nu}\,\rho\,C_s} \qquad (3.34)$$

and

$$\omega^2\equiv\omega_k^2=gk+\frac{\sigma_0}{\rho}k^3. \qquad (3.35)$$

The subscript "L" of ω and k in eq.(3.34) indicates that this wave mode corresponds to the elastic, longitudinal wave which is proved to exist in such a film-covered surface, cf Lucassen-Reynders and Lucassen [1969], Dysthe and Rabin [1986]. There is exact agreement between the dispersion relation eq.(3.34) and their results. The longitudinal wave is heavily damped, which is easily seen from eq.(3.34) by letting either k_L or ω_L be real.

We shall be concentrating on the transversal surface wave. A more accurate dispersion relation than the one given in eq.(3.35) is needed, however, for our purpose. We therefore return to the exact dispersion relation (eq.(3.30)), from which the ratio $(\omega/\omega_k)^2$ is obtained as

$$\left(\frac{\omega}{\omega_k}\right)^2 = 1 + (k\delta)\frac{[1-\alpha(k\delta)\beta]}{\left[1+\alpha(k\delta)^2\right]\beta - \left(\frac{m}{k}\right)(k\delta)}. \qquad (3.36)$$

Approximation to order $(k\delta)$ gives

$$\left(\frac{\omega}{\omega_k}\right)^2 \approx 1 + (k\delta)\frac{1-2i\,(k\delta)\beta}{\beta-1+i}, \qquad (3.37)$$

and by taking the square root of eq. (3.37), we obtain the approximate dispersion relation

$$\omega \approx \omega_k + \tfrac{1}{2}(k\delta)\frac{1-2i(k\delta)\beta}{\beta-1+i}\,\omega_k, \qquad (3.38)$$

(cf Dysthe and Rabin [1986]). The angular frequency ω may be obtained from this equation by iteration - starting by letting $\omega = \omega_k$ in the expressions for β and δ.

We shall study the approximate dispersion relation (eq.(3.38)) further for two special cases: i) damping in time and ii) damping in space.

i) Damping in time

In the case of damping only in time, the wavenumber k is real whereas the angular frequency ω is allowed to have also an imaginary part. Writing ω as

$$\omega = \omega_0 - i\gamma, \qquad \omega_0, \gamma \quad \text{real and positive,} \tag{3.39}$$

γ will be denoting the damping rate of the wave. Real wave number k implies real angular frequency ω_k (eq. (3.19)), and so an approximate expression for the damping rate γ may be obtained from the imaginary part of eq. (3.38),

$$\gamma \approx -\operatorname{Im}\left\{ \tfrac{1}{2}(k\delta)\frac{1-2i(k\delta)\beta}{\beta-1+i}\,\omega_k \right\}. \tag{3.40}$$

This result corresponds exactly to the damping rate to lowest order reported by Weber and Sætra [1995] and first derived by Dorrestein [1951].

In the limit of no film, where the compressibility is great, the damping rate γ becomes

$$\gamma_\infty \approx 2\nu k^2, \quad \beta \to \infty, \tag{3.41}$$

in agreement with the well-established theory of viscid surface waves [Lamb, 1932].

When the compressibility is small, the damping rate γ becomes

$$\gamma \approx \left\{ \frac{\nu k^2 \omega_k}{2} \right\}^{1/2} \operatorname{Im}\frac{(1-\beta+i)}{(1-\beta)^2+1}, \qquad \beta \text{ small}. \tag{3.42}$$

This expression agrees perfectly well with the results reported by Dysthe and Rabin [1986] and others. In the incompressible limit $(\beta = 0)$ eq. (3.42) is further reduced, giving $\gamma_0 \approx \omega_k(k\delta)/4$, as shown by Lamb [1932].

The damping rate γ obtained here has a (local) maximum for an intermediate value of β, as the general theory predicts. From eq. (3.40) (or eq. (3.36)) it may be seen that to leading order in $(k\delta)$ this value of β coincides with a β also allowing the elastic longitudinal wave mode to exist (cf eq. (3.33)). This maximum in damping rate is commonly explained as a result of the two wave modes interacting, giving kind of an inverse resonance effect.

ii) Damping in space

In the case of only spatial damping the angular frequency ω is real, whereas the wave number k is allowed to have an imaginary part,

$$k = k_0 + i\kappa, \quad k_0, \kappa \text{ real, positive}. \tag{3.43}$$

The parameter κ denotes the spatial damping rate of the wave, for which an expression is to be derived.

We return to the approximate dispersion relation of our system, eq. (3.38). Since the angular frequency ω in case of only spatial damping should be real, its imaginary part - as given by this dispersion relation - must vanish,

$$\mathrm{Im}\,\omega = \mathrm{Im}\,\omega_k \left\{ 1 + \tfrac{1}{2}(k\delta)\frac{1 - 2i(k\delta)\beta}{\beta - 1 + i} \right\} = 0. \tag{3.44}$$

The angular frequency ω_k was defined in eq. (3.19), by which it is seen that complex k gives complex ω_k. In terms of a real angular frequency ω_0, defined by

$$\omega_0^2 \equiv g k_0 + \frac{\sigma_0}{\rho} k_0^3, \tag{3.45}$$

and the associated phase and group velocities $\mathrm{v}_{ph,0}$ and $\mathrm{v}_{g,0}$,

$$\left. \begin{aligned} \mathrm{v}_{ph,0} &\equiv \frac{\omega_0}{k_0} \\ \mathrm{v}_{g,0} &\equiv \frac{\partial \omega_0}{\partial k_0} = \frac{1}{2\omega_0}\left(g + 3\frac{\sigma_0}{\rho} k_0^2 \right) \end{aligned} \right\}, \tag{3.46}$$

neglecting terms of relative order $(\kappa/k_0)^2$, the angular frequency ω_k becomes

$$\omega_k \approx \omega_0 \left\{ 1 + i\left(\frac{\kappa}{k_0}\right)\left(\frac{\mathrm{v}_{g,0}}{\mathrm{v}_{ph,0}}\right) \right\}. \tag{3.47}$$

Inserting this expression into eq. (3.44), we obtain the equation

$$\operatorname{Im}\omega_0 \left\{ 1 + i\left(\frac{\kappa}{k_0}\right)\left(\frac{\mathrm{v}_{g,0}}{\mathrm{v}_{ph,0}}\right) \right\} \left\{ 1 + \tfrac{1}{2}(k\delta)\frac{1 - 2i(k\delta)\beta}{\beta - 1 + i} \right\} = 0\,. \tag{3.48}$$

Solving this with respect to the damping rate κ, we obtain to leading order

$$\left.\begin{array}{c} \left(\dfrac{\kappa}{k_0}\right) \approx \dfrac{1}{2}\left(\dfrac{\mathrm{v}_{ph,0}}{\mathrm{v}_{g,0}}\right)\dfrac{(k_0\delta)\left[1 - 2(k_0\delta)\beta_0 + 2(k_0\delta)\beta_0^2\right]}{\left[(\beta_0 - 1)^2 + 1\right]} \\ or \\ \kappa \approx \tfrac{1}{2}\omega_0 (k_0\delta)\,\dfrac{\left[1 - 2(k_0\delta)\beta_0 + 2(k_0\delta)\beta_0^2\right]}{\left[(\beta_0 - 1)^2 + 1\right]\mathrm{v}_{g,0}}\,. \end{array}\right\} \tag{3.49}$$

(β_0 denotes the real part of β.)

In the extreme cases of no film ($\beta \to \infty$) and incompressible film ($\beta \to 0$), eq. (3.49) gives the approximate damping rates

$$\left(\frac{\kappa}{k_0}\right) \approx \left(\frac{\mathrm{v}_{ph,0}}{\mathrm{v}_{g,0}}\right)(k_0\delta)^2 \quad or \quad \kappa \approx \frac{2\nu k_0^2}{\mathrm{v}_{g,0}} \qquad \text{(no film)}, \tag{3.50}$$

and

$$\left.\begin{array}{c} \left(\dfrac{\kappa}{k_0}\right) \approx \dfrac{1}{4}\left(\dfrac{\mathrm{v}_{ph,0}}{\mathrm{v}_{g,0}}\right)(\mathrm{k}_0\delta) \\ or \\ \kappa \approx \dfrac{\left(\nu k_0^2 \omega/2\right)^{1/2}}{2\,\mathrm{v}_{g,0}} \end{array}\right\} \quad \text{(incompressible film)}. \tag{3.51}$$

Eq. (3.51) agrees with the result derived by Weber and Førland [1989]. As in the time damping case we shall have maximum damping for an intermediate value of β.

In both types of damping we observe that the damping rate (γ or κ) has a maximum for $Re\beta=1$, and this maximum value is twice the value of the damping rate for incompressible film ($\beta=0$).

The relation between the temporal and spatial damping rates has been studied by M. Gaster [1962], who found the formulae

$$\gamma/\kappa = \mathrm{v}_g\,, \tag{3.52}$$

v_g being the group velocity of the wave. Inspection shows that the approximate expressions outlined in this section for the damping rates to leading order are in good agreement with Gaster's formulae, if also the exact group velocity v_g is replaced by the approximate value $\mathrm{v}_{g,0}$.

5. The time-averaged tangential surface stress

In the terminology of chapter I the tangential fluid stress at the surface, τ, is defined as $\tau \equiv \underline{\hat{t}} \cdot \underline{\underline{P}}_S \cdot \underline{\hat{n}}$, where $\underline{\hat{t}}$ and $\underline{\hat{n}}$ are unit vectors tangential and normal to the surface, respectively, and $\underline{\underline{P}}_S$ is the stress tensor evaluated at the surface.

From chapter II (eqs. (2.26) and (2.28)) the first and second order approximations of this stress in Lagrangian description, $\tau^{(1)}$ and $\tau^{(2)}$, respectively, are expressed as

$$\tau^{(1)} = -\mu\left(x_{ct}^{(1)} + z_{at}^{(1)}\right),\ \ c = 0\,, \tag{3.53}$$

and

$$\tau^{(2)} = -\mu\left\{x_{ct}^{(2)} + z_{at}^{(2)} + \frac{\partial\left(x^{(1)}, x_t^{(1)}\right)}{\partial(a,c)} + \frac{\partial\left(z_t^{(1)}, z^{(1)}\right)}{\partial(a,c)} - 2\left(x_{at}^{(1)} - z_{ct}^{(1)}\right) z_a^{(1)}\right\},\ \ c = 0\,. \tag{3.54}$$

We want to determine the time-average of these stresses in the case of elastic surface film.

For first-order velocities $x_t^{(1)}$ and $z_t^{(1)}$ being periodic in time (real angular frequency ω), the ime-averaged first-order stress, $\overline{\tau^{(1)}}$, vanishes:

$$\overline{\tau^{(1)}} \equiv -\mu\left(\overline{x_{ct}^{(1)}} + \overline{z_{at}^{(1)}}\right)\Big|_{c=0} = 0 . \tag{3.55}$$

The expression for the time-averaged second-order stress, $\overline{\tau^{(2)}}$, becomes – from eq. (3.54) –

$$\overline{\tau^{(2)}} = -\mu\left\{\overline{x_{ct}^{(2)}} + \overline{z_{at}^{(2)}} + 2\,\overline{x_a^{(1)}\left(x_{ct}^{(1)} + z_{at}^{(1)}\right)}\right\}\Big|_{c=0} . \tag{3.56}$$

(In obtaining eq. (3.56) also the functions $x^{(1)}$ and $z^{(1)}$ were assumed periodic in time (eq. (3.15)), further it was taken advantage of the first-order continuity equation (2.8) and the property (2.44) fulfilled by periodic functions.)

In terms of the stream function ψ introduced in section 5 of chapter II, the mean second-order stress $\overline{\tau^{(2)}}$ in eq.(3.56) may be written

$$\overline{\tau^{(2)}} = -\mu\left\{\overline{\psi_{cc}^{(2)}} - \overline{\psi_{aa}^{(2)}} + 2\,\overline{x_a^{(1)}\left(x_{ct}^{(1)} + z_{at}^{(1)}\right)}\right\},\ c = 0 . \tag{3.57}$$

Near the surface the second-order velocities will vary much stronger vertically than horizontally. Assuming this to be the case also for the mean (second-order) velocities, the mean second-order stream function $\overline{\psi^{(2)}}$ will fulfil

$$\left|\overline{\psi_{aa}^{(2)}}\right| << \left|\overline{\psi_{cc}^{(2)}}\right| . \tag{3.58}$$

Hence we may neglect $\overline{\psi_{aa}^{(2)}}$ in the expression (3.57) for the second-order stress $\overline{\tau^{(2)}}$, obtaining the simpler expression

$$\overline{\tau^{(2)}} \approx -\mu\left\{\overline{\psi_{cc}^{(2)}} + 2\,\overline{x_a^{(1)}\left(x_{ct}^{(1)} + z_{at}^{(1)}\right)}\right\},\ c = 0 . \tag{3.59}$$

To determine the second-order mean tangential stress $\overline{\tau^{(2)}}$ the only task will then be to find $\overline{\psi_{cc}^{(2)}}$ evaluated at the surface, since all the first-order functions present in eq.(3.59) are known.

In chapter II a partial differential equation for the mean, second-order stream function $\overline{\psi^{(2)}}$ was established, eq. (2.71). By the assumption (3.58) we may neglect the partial derivatives of $\overline{\psi^{(2)}}$ with respect to the horizontal variable in this equation. We are left with the approximate equation

$$\overline{\psi_{cccc}^{(2)}} = \frac{1}{\nu}\left(\overline{Q_{2a}} - \overline{Q_{1c}}\right), \tag{3.60}$$

to be valid reasonably well near the surface *c = 0.*

Because of the chosen form of the first-order functions, eq. (3.60) is easily integrated to obtain the particular solution, $\overline{\psi_{part}^{(2)}}$. At the surface *(c=0)* $\overline{\psi_{part,cc}^{(2)}}$ is found to be - to the leading order in *(kδ)* -

$$\left.\overline{\psi_{part,cc}^{(2)}}\right|_{c=0} \approx -\frac{k_0^3|A|^2 e^{-2\kappa a}\left\{1 - 2\left[2\left(\frac{\kappa}{k_0}\right) + (k_0\delta)\right]\beta_0 + 2(k_0\delta)\beta_0^2\right\}}{\omega\delta|\beta - 1 + i|^2}. \tag{3.61}$$

(β_0 still denotes the real part of β (cf section 4).)

In the extreme cases of no film $(\beta_0 \to \infty)$ and incompressible film $(\beta_0 = 0)$ eq. (3.61) is reduced to

$$\left.\overline{\psi_{part,cc}^{(2)}}\right|_{c=0} \approx -\frac{2k_0^3|A|^2 e^{-2\kappa a}(k_0\delta)}{\omega\delta}, \quad \beta_0 \to \infty, \tag{3.62}$$

and

$$\left.\overline{\psi_{part,cc}^{(2)}}\right|_{c=0} \approx -\frac{k_0^3|A|^2 e^{-2\kappa a}}{2\omega\delta}, \quad \beta_0 = 0, \tag{3.63}$$

respectively.

Longuet-Higgins introduced the term "virtual wave stress" [1969] to denote the contribution to the mass transport gradient resulting from the wave-induced vorticity associated with viscid wave motion. In our terminology this vorticity is given by $\nabla_L^2 \overline{\psi} \approx \varepsilon^2 \overline{\psi_{cc}^{(2)}}$, in which case the virtual wave stress, τ_{virt}, will be

$$\tau_{virt} \approx -\mu\varepsilon^2 \overline{\psi_{cc}^{(2)}}\Big|_{c=0} . \tag{3.64}$$

We find that inserting the expressions (3.62)-(3.63) into eq. (3.64) for $\overline{\psi_{cc}^{(2)}}\Big|_{c=0}$ gives excellent agreement with the virtual wave stresses obtained by Longuet-Higgins [1969] in the no film-case and by Weber and Førland [1989] in the incompressible film-case, respectively. To the leading order there is also agreement between the result obtained from expression (3.61) and the virtual stress obtained by Weber and Sætra [1995] for the case of an elastic film.

The second term in expression (3.59) becomes, also to leading order,

$$2\,\overline{x_a^{(1)}\left(x_{ct}^{(1)} + z_{at}^{(1)}\right)}\Big|_{c=0} \approx -\frac{2k_0^3|A|^2 e^{-2\kappa a}\left\{o\left((k_0\delta)^2\right) + \left(\dfrac{\kappa}{k_0}\right)\beta_0 + (k_0\delta)\beta_0^2\right\}}{\omega\delta|\beta - 1 + i|^2}, \tag{3.65}$$

and so we have

$$2\,\overline{x_a^{(1)}\left(x_{ct}^{(1)} + z_{at}^{(1)}\right)}\Big|_{c=0} \approx -\frac{2k_0^3|A|^2 e^{-2\kappa a}(k_0\delta)}{\omega\delta}, \quad \beta_0 \to \infty \tag{3.66}$$

and

$$2\,\overline{x_a^{(1)}\left(x_{ct}^{(1)} + z_{at}^{(1)}\right)}\Big|_{c=0} \approx -\frac{2k_0^3|A|^2 e^{-2\kappa a}}{2\omega\delta} O(k_0\delta)^2, \quad \beta_0 = 0. \tag{3.67}$$

Expression (3.66) agrees with the contribution from the Stokes drift in the case of no film [Longuet-Higgins, 1969], except for the exponential damping factor $e^{-2\kappa a}$. It seems natural to interpret the expression (3.65) as giving the Stokes drift contribution in the general case.

The sum of the virtual wave stress and the Stokes drift contribution,

$$\overline{\tau_{part}^{(2)}} \approx \frac{\mu k_0^3 |A|^2 e^{-2\kappa a}}{\omega\delta |\beta - 1 + i|^2} \left\{ 1 - 2\left[\left(\frac{\kappa}{k_0} \right) + (k_0\delta) \right] \beta_0 + 4\,(k_0\delta)\beta_0^2 \right\}, \tag{3.68}$$

is interpreted as the mean, tangential, wave-induced stress of the fluid at the surface. The no film-stress obtained from this,

$$\overline{\tau_{part,\infty}^{(2)}} \approx 4\,(k_0\delta) \frac{\mu k_0^3 |A|^2}{\omega\delta} e^{-2\kappa a}, \quad |(k\delta)\beta| >> 1, \tag{3.69}$$

agrees perfectly with the result of Longuet-Higgins. In the inelastic case the virtual wave stress totally dominates the wave-induced stress; it exceeds the Stokes drift contribution by a factor $(k_0\delta)^{-2}$, so that

$$\overline{\tau_{part,0}^{(2)}} \approx -\mu \overline{\psi_{cc}^{(2)}}\Big|_{c=0} \approx \frac{1}{\omega\delta} \frac{\mu k_0^3 |A|^2}{(\beta_0 - 1)^2 + 1} e^{-2\kappa a}, \quad |(k\delta)\beta| << 1. \tag{3.70}$$

The value of the mean, tangential stress $\overline{\tau_{part}}$ for *Reβ=1* is twice its value for *β=0*, a result parallel to the one obtained for the damping rates γ and κ in section 4.

6. Concluding remarks

In this chapter we have applied the mathematical tool established in chapter II to study the case of capillary-gravity waves propagating through elastic surface film. The (first-order) wave solution has been determined. For the case of spatially damped waves and mean, steady state, the mean, tangential fluid stress at the surface is expressed in terms of the mean stream function and the first-order solution. The particular solution of the mean stream function at the surface is determined. It is interpreted as giving the virtual wave stress [Longuet-Higgins, 1969]. The expression resulting from the first-order terms is interpreted as the Stokes drift contribution. Adding these together, the *wave-induced* mean, tangential fluid stress at the surface is obtained. The validity of this will be further discussed in chapter V and VI.

CHAPTER IV

SURFACE WAVES DAMPED BY VISCID SURFACE LAYER

In cold ocean waters surface layers of grease ice often form. Grease ice exhibits viscid fluid-like properties, having a damping effect on the surface waves. In chapter IV this is given a mathematical formulation, based upon the model established in chapter II. The (first-order) wave problem is solved, and the mean horizontal motion within the surface layer is determined.

1. Grease ice

In cold areas a large part of the ocean surface is covered by ice. At the ice edge the seasonal variations cause new ice to be formed regularly. The ice forming process is described in detail by Weeks and Ackly [1986]. The photo below (fig. 4.1) shows a typical situation with sea ice.

Fig. 4.1. Outside the western coast of Greenland, on Feb. the 27th 1993. The ocean is partly covered by new ice and grease ice. Photo by H. Valeur, Danish Meteorological Institute.

Normal, salt seawater has highest density at a lower temperature than its freezing point. Cooling of the surface therefore causes an unstable vertical density distribution. Convective mixing will occur, and the lower (and warmer) water will be transported to the surface where the heat will be dissipated. When the surface temperature becomes low enough for ice crystals to form, a thick (possibly several meters) upper layer of the sea will already have its temperature lowered to the freezing point.

Under calm, non-turbulent conditions the ice forming process starts with small spheres of pure ice. These spheres grow into circular discs, then forming star-like crystals. Eventually, the crystals will overlap and freeze together, forming a continuous thin ice skim – "sheet ice". Under more realistic conditions wind- and waveinduced turbulence will cause the initial crystals to be "stirred" throughout a depth of several meters [Martin and Kauffman, 1981]. Discoidal growth of ice crystals is then favoured – star-like crystals will not occur to any extent. The result is a surface layer where lots of thin, disc-formed ice crystals of about 1 mm diameter are quite homogeneously distributed throughout the depth of the layer. This is known as "frazil ice".

By the action of wind and waves the frazil ice may be pushed together, forming "grease ice". Grease ice is a soupy mass of frazil ice, which exhibits viscid fluid-like mechanical properties. Martin and Kauffman [1981] proposed that at least 40% of the water volume must consist of frazil ice before more "solid" behavior takes place. Surface layers of grease ice may reach a thickness of the order of one meter. Grease ice has low reflectivity, which makes it possible to optically distinct from other kinds of ice.

Frazil ice was believed to account for only 5% of the total ice cover [Martin, 1979]. Later works indicate, however, that frazil ice is much more widespread. Hence, frazil ice - and consequently also grease ice - are important factors in the dynamics of the upper oceans.

The present study was inspired by an experiment reported by Seeley Martin and Peter Kauffman [1981]. In laboratory they grew ice in a tank of salt water. A paddle at one end of the tank produced surface waves. Disc-like crystals formed and rose to the surface, where they were "caught" by the wave motion and advected toward the far end of the tank. As the ice concentration increased, a surface layer of grease ice formed. Grease ice was seen to be a very efficient wave absorber, and the waves were heavily damped as they propagated into this surface layer. A schematic drawing of the experiment is shown in fig. 4.2:

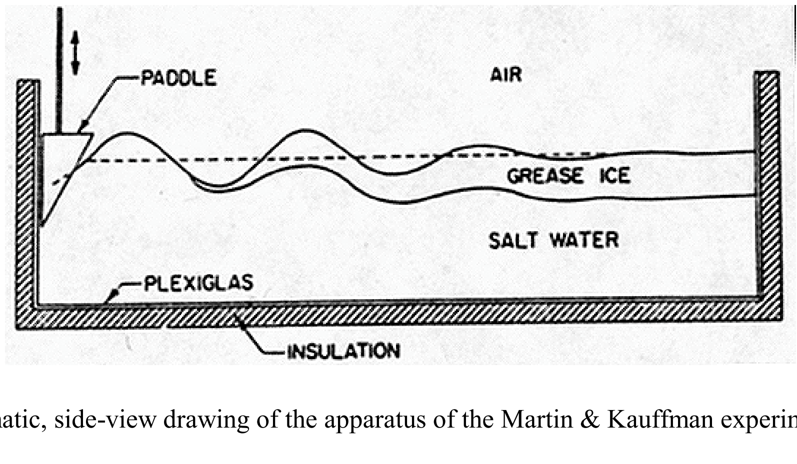

Fig. 4.2. A schematic, side-view drawing of the apparatus of the Martin & Kauffman experiment [1981, fig. 7].

The ice area was observed to have a fluid (grease ice) region with strong relative motions and a region where the ice acted more like a solid. A mean mass circulation was observed within the grease ice: A streaming velocity on the surface of decreasing magnitude toward the solid ice, downwelling below the surface, and a return flow near the bottom of the grease ice layer. Below the ice no motion was observed. The observed circulation is schematically sketched in fig. 4.3:

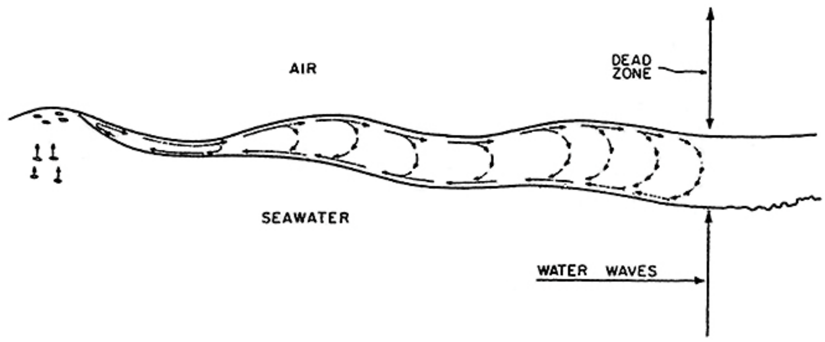

Fig. 4.3. A drawing of the mean circulation in the grease ice, the Martin & Kauffman experiment [1981, fig. 9].

A qualitative explanation for the observed mean circulation was presented by Martin and Kauffman: The strong decay generates a vorticity. In the highly viscid fluid this will induce a horizontal flow immediately below the surface boundary layer. Since this flow decreases as the amplitude decays, there must be some downwelling for mass to be conserved. No mean motion was observed below the grease ice layer, hence – again by the mass conservation arguement – a horizontal return flow at the lower surface of the layer is also required. Upwelling will occur only at the leading edge, as observed.

To see if the mean mass circulation observed by Martin and Kauffman may be reproduced theoretically, we shall in the following be modelling wave damping in a grease ice layer, determining the resulting mean, horizontal flow.

2. A two-layer model

J.E. Weber has studied mean wave-induced drift in terms of the Lagrangian approach in several papers. In 1987 he published a work [Weber, 1987] on gravity waves in a slightly viscid ocean with a thin surface layer of concentrated brash-like ice. In that work the drift *below* the ice layer was determined, and a jet-like flow was found just below the ice cover.

In a paper from 1990 [Weber and Førland, 1990] the effect of air on the drift velocity of water waves was studied, applying a *two-layer* Lagrangian model. The effect was found to be analogous to that of a surface film, only weaker. Again the drift in the water (lower layer) was the object of interest.

In our case the upper ocean is assumed to consist of grease ice. A two-layer model is applied to study the mean flow *within* the grease ice (upper layer). The effect of air is neglected. We shall consider the grease ice as a highly viscid fluid, assuming it to constitute a surface layer covering a much less viscid fluid (normal ocean water). The surface layer is exposed to an incoming train of permanent surface waves as illustrated in fig. (4.4):

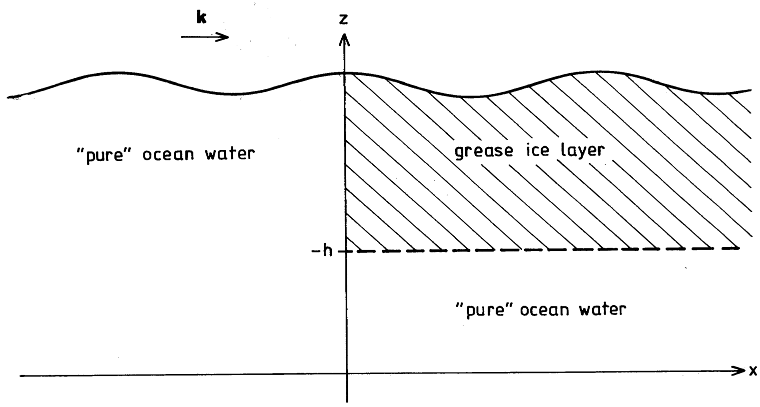

Fig. 4.4. Surface waves moving from pure ocean into a grease ice covered area.

We shall consider short gravity waves. Compared to the work of Weber [1987], our waves are at the opposite end of the scale. In the case of Weber the thickness of the surface layer is much less than the wavelength, whereas the thickness in our case is of the order of the wavelength. The other important difference is that Weber concentrates on the mean motion *below* the surface layer, we concentrate on the mean motion *within* this layer. Weber and Førland [1990] studied the drift velocity of water waves, taking into account the effect of the viscid air above. The air is less viscid than the water. Hence also this case is in a way "the opposite" of ours, in our case the upper fluid is most viscid and we study the drift velocity within this.

The viscosity of the upper layer is assumed so much greater than the viscosity of the ocean water that the viscosity of the latter may be neglected. Both fluids are assumed to be incompressible, Newtonian fluids.

Using the formalism of chapter II on each fluid, we have the following set of equations:

Upper layer (viscid):

$$\left.\begin{aligned} x_{tt} &= -\frac{1}{\rho}\frac{\partial(p,z)}{\partial(a,c)} + \nu\left(\nabla^2 u\right)_L \\ z_{tt} + g &= -\frac{1}{\rho}\frac{\partial(x,p)}{\partial(a,c)} + \nu\left(\nabla^2 \mathrm{w}\right)_L \\ \frac{\partial(x,z)}{\partial(a,c)} &= 1 \end{aligned}\right\} \qquad (4.1)$$

Ocean water (inviscid):

$$\left.\begin{aligned} X_{tt} &= -\frac{1}{\rho_0}\frac{\partial(P,Z)}{\partial(a,c)} \\ Z_{tt}+g &= -\frac{1}{\rho_0}\frac{\partial(X,P)}{\partial(a,c)} \\ \frac{\partial(X,Z)}{\partial(a,c)} &= 1 \end{aligned}\right\} \tag{4.2}$$

(Capital letters are used to denote the functions associated with the lower ocean water to avoid confusion.)

The densities of the upper fluid and the ocean water are denoted by ρ and ρ_0, respectively. In most of the following analysis, however, the difference will be neglected and so ρ will be put equal to ρ_0. We shall also make the essential assumption that there will be no mixing between the two fluids - that is, a fluid particle being at one moment an upper fluid particle will always remain an upper fluid particle, and a fluid particle being at one moment an ocean water particle will remain an ocean water particle.

3. The first-order problem

Following chapter II, the first-order problem has a standard wave solution

$$\left.\begin{aligned} x_t^{(1)} &= \varphi_a + \tilde{\psi}_c \\ z_t^{(1)} &= \varphi_c - \tilde{\psi}_a \\ X_t^{(1)} &= \Theta_a \\ Z_t^{(1)} &= \Theta_c \ , \end{aligned}\right\} \tag{4.3}$$

with the functions $\varphi, \tilde{\psi}$ and Θ given as

$$\left.\begin{aligned} \varphi &= \left(A\,e^{kc} + B\,e^{-kc}\right)e^{i(ka-\omega t)} \\ \tilde{\psi} &= \left(C\,e^{mc} + D\,e^{-mc}\right)e^{i(ka-\omega t)} \\ \Theta &= E\,e^{kc}\ e^{i(ka-\omega t)}\ . \end{aligned}\right\} \tag{4.4}$$

(In the expression for Θ in eqs.(4.4) there is only one term, since Θ must vanish as $c \to -\infty$ (deep-water waves)).

This solution implies exponentially decaying wave amplitude. The experimental results of

Martin and Kauffman [1981] apparently showed *linear* decay as the waves propagated into the grease ice layer. However, by inspection of those results it may be seen that in the actual interval our exponentially decaying amplitude may be well approximated by the linearly decaying amplitude of Martin and Kauffman. Hence, the exponential decay model is considered appropriate.

The pressures are determined by the equations

$$\left.\begin{aligned} \varphi_t + g\,z^{(1)} + \frac{p^{(1)}}{\rho} &= 0 \\ \Theta_t + g\,Z^{(1)} + \frac{P^{(1)}}{\rho_0} &= 0\,. \end{aligned}\right\} \tag{4.5}$$

We consider (short) gravity waves, therefore both surface tension and tension in the interface between the viscid upper layer and the ocean water below are neglected. The dynamic boundary conditions at the surface *c=0* and at an interface *c=-h* then become (cf chapter II)

$$p^{(1)} - 2\mu\, z^{(1)}_{tc} = \begin{cases} 0\,, & c = 0 \\ P^{(1)}, & c = -h \end{cases} \tag{4.6}$$

$$\mu\left(x^{(1)}_{ct} + z^{(1)}_{at}\right) = 0, \qquad c = 0, -h\,. \tag{4.7}$$

In addition, the velocity component normal to the boundary must be continuous at the boundary, hence

$$z^{(1)}_t = Z^{(1)}_t, \qquad c = -h\,. \tag{4.8}$$

By the five equations in (4.6)-(4.8) the unknown coefficients *A*, *B, C, D* and *E* in the functions $\varphi, \tilde{\psi}$ and Θ may be determined. Eliminating the pressure terms by eqs.(4.5), also assuming $z^{(1)}$ and $Z^{(1)}$ to be periodic in time so that

$$\left.\begin{aligned} z^{(1)} &= \frac{i}{\omega} z^{(1)}_t \\ Z^{(1)} &= \frac{i}{\omega} Z^{(1)}_t \end{aligned}\right\}, \tag{4.9}$$

the equations (4.6)-(4.8) may be rewritten in terms of the functions $\varphi, \tilde{\psi}$ and Θ by means of eqs. (4.3), giving

$$\varphi_t + \frac{ig}{\omega}\left(\varphi_c - \tilde{\psi}_a\right) + 2\nu\left(\varphi_{cc} - \tilde{\psi}_{ac}\right) = \begin{cases} 0, & c = 0 \\ \left(\frac{\rho_0}{\rho}\right)\left[\Theta_t + \frac{ig}{\omega}\Theta_c\right], & c = -h, \end{cases} \tag{4.10}$$

$$2\varphi_{ac} + \tilde{\psi}_{cc} - \tilde{\psi}_{aa} = 0, \quad c = 0, -h, \tag{4.11}$$

$$\varphi_c - \tilde{\psi}_a = \Theta_c, \quad c = -h. \tag{4.12}$$

The function Θ in eq. (4.10) may be eliminated by means of eqs.(4.4) and (4.12). Hence, we are left with the four equations

$$\left.\begin{aligned} &\varphi_t + \frac{ig}{\omega}\left(\varphi_c - \tilde{\psi}_a\right) + 2\nu\left(\varphi_{cc} - \tilde{\psi}_{ac}\right) = \begin{cases} 0, & c = 0 \\ -i\left(\frac{\rho_0}{\rho}\right)\left(\frac{\omega}{k} - \frac{g}{\omega}\right)\left(\varphi_c - \tilde{\psi}_a\right), & c = -h \end{cases} \\ &2\varphi_{ac} + \tilde{\psi}_{cc} - \tilde{\psi}_{aa} = 0, \qquad c = 0, \ -h \end{aligned}\right\} \tag{4.13}$$

to determine the coefficients *A, B, C* and *D*.

Taking advantage of the special forms (4.4) chosen for the functions φ and $\tilde{\psi}$, also substituting expressions for φ_c obtained from the last equation in (4.13) into the first, the set (4.13) of equations becomes

$$\left.\begin{aligned} (1+i\xi)\,\varphi - \frac{1}{\xi}\frac{gk}{\omega^2}\tilde{\psi} + \frac{\xi}{k}\tilde{\psi}_c &= 0, \quad c = 0 \\ (1+i\xi)\,\varphi - \frac{s}{\xi}\tilde{\psi} + \frac{\xi}{k}\tilde{\psi}_c &= 0, \quad c = -h \\ \frac{\xi}{k}\varphi_c - (1+i\xi)\,\tilde{\psi} &= 0, \quad c = 0, -h \end{aligned}\right\}, \tag{4.14}$$

where we have introduced the parameters ξ and *s* defined by

$$\xi \equiv 2\nu k^2 / \omega \tag{4.15}$$

and

$$s \equiv \frac{gk}{\omega^2} - \frac{\rho_0}{\rho}\left(\frac{gk}{\omega^2} - 1\right), \tag{4.16}$$

respectively. (Observe that the first two equations in (4.14) have identical form if $\rho_0=0$, or if $\rho_0=\rho$ and $\omega^2=gk$ as for ordinary gravity waves. Also note that the parameter ξ introduced here equals $(k\delta)^2$ in the terminology of chapter III .)

4. The dispersion relation

To have a unique solution, the determinant of the system (4.14) must equal zero, that is

$$\det\begin{Bmatrix} 1+i\xi & 1+i\xi & -\frac{1}{\xi}\left(\frac{gk}{\omega^2} - \xi^2\frac{m}{k}\right) & -\frac{1}{\xi}\left(\frac{gk}{\omega^2} + \xi^2\frac{m}{k}\right) \\ \xi & -\xi & -(1+i\xi) & -(1+i\xi) \\ (1+i\xi)e^{-kh} & (1+i\xi)e^{kh} & -\frac{1}{\xi}\left(s - \xi^2\frac{m}{k}\right)e^{-mh} & -\frac{1}{\xi}\left(s + \xi^2\frac{m}{k}\right)e^{mh} \\ \xi\, e^{-kh} & -\xi\, e^{kh} & -(1+i\xi)e^{-mh} & -(1+i\xi)e^{mh} \end{Bmatrix} = 0\,. \tag{4.17}$$

Eq.(4.17) gives us the dispersion relation of the system.

For the special case of $h \to \infty$ and $B=D\equiv 0$ (one single viscid fluid), eq. (4.17) is reduced to

$$\det\begin{Bmatrix} (1+i\xi) & -\frac{1}{\xi}\left(\frac{gk}{\omega^2} - \xi^2\frac{m}{k}\right) \\ \xi & -(1+i\xi) \end{Bmatrix} = 0. \tag{4.17 a}$$

This equation agrees with the dispersion relation presented by Chandrasekhar [1961] for viscous surface gravity-capillary waves, if surface tension is neglected. (See also Jenkins and Jacobs [1997], eq. 27.)

In the present analysis we shall assume the parameter ξ to be small (much less than 1),

$$|\xi| << 1\,. \tag{4.18}$$

The quotient *(m/k)* is then of order $1/\xi^{1/2}$. Neglecting terms of relative order $\xi^{3/2}$, also assuming the quotient ρ_0/ρ to not differ much from unity, the determinant (4.17) is reduced to

$$\det\begin{Bmatrix} 1+i\xi & 1+i\xi & -\frac{1}{\xi}\frac{gk}{\omega^2} & -\frac{1}{\xi}\frac{gk}{\omega^2} \\ \xi & -\xi & -(1+i\xi) & -(1+i\xi) \\ (1+i\xi)e^{-kh} & (1+i\xi)e^{kh} & -\frac{s}{\xi}e^{-mh} & -\frac{s}{\xi}e^{mh} \\ \xi e^{-kh} & -\xi e^{kh} & -(1+i\xi)e^{-mh} & -(1+i\xi)e^{mh} \end{Bmatrix} = 0. \tag{4.19}$$

An approximate dispersion relation is given by

$$\begin{aligned}\left(e^{mh}-e^{-mh}\right)\Bigg\{&\left(1-\frac{\rho_0}{\rho}\right)\left(e^{kh}-e^{-kh}\right)\left(\frac{gk}{\omega^2}\right)^2+2\frac{\rho_0}{\rho}\left[e^{kh}+i\xi\left(e^{kh}+e^{-kh}\right)\right]\left(\frac{gk}{\omega^2}\right)\\ &-\frac{\rho_0}{\rho}(1+2i\xi)\left(e^{kh}+e^{-kh}\right)-(1+4i\xi)\left(e^{kh}-e^{-kh}\right)\Bigg\}=0.\end{aligned} \tag{4.20}$$

Regarding eq.(4.20) as an equation in *(gk/ω^2)* with constant coefficients (this is an approximation, the coefficients are *not* constants), we get the two approximate solutions (still to order $\xi^{3/2}$)

$$\frac{gk}{\omega^2}=1+2i\xi\frac{\left(1-e^{-2kh}\right)}{\left[1-\left(1-\frac{\rho_0}{\rho}\right)e^{-2kh}\right]} \tag{4.21}$$

and

$$\begin{aligned}\frac{gk}{\omega^2}=-\frac{1}{\left(1-\frac{\rho_0}{\rho}\right)\left(1-e^{-2kh}\right)}\Bigg\{&\left(1+\frac{\rho_0}{\rho}\right)-\left(1-\frac{\rho_0}{\rho}\right)e^{-2kh}\\ &+2i\xi\frac{\left[\left(1-\left(1-\frac{\rho_0}{\rho}\right)e^{-2kh}\right)^2+\left(\frac{\rho_0}{\rho}\right)^2e^{-2kh}\right]}{\left[1-\left(1-\frac{\rho_0}{\rho}\right)e^{-2kh}\right]}\Bigg\}.\end{aligned} \tag{4.22}$$

The solution (4.21) corresponds to the surface wave which we shall consider, whereas the solution (4.22) corresponds to an internal wave mode which may exist in a fluid with layers of different density.

In the special case $\rho_0/\rho = 1$ we have only one solution,

$$\frac{gk}{\omega^2} \approx 1 + 2i\xi\left(1 - e^{-2kh}\right). \tag{4.23}$$

Writing eq. (4.23) in the form

$$k = \frac{\omega^2}{g}\left[1 + 2i\xi\left(1 - e^{-2kh}\right)\right], \tag{4.24}$$

we observe that an approximate solution for k may be achieved by iteration, initially substituting k by $k_0=\omega^2/g$ at the right-hand side of eq.(4.24),

$$k = k_0\left[1 + 2i\xi_0\left(1 - e^{-2k_0h}\right)\right] \equiv k_0 + i\kappa, \tag{4.25}$$

with the parameter ξ_0 being an approximation to ξ, given as

$$\xi_0 = \frac{2\nu\omega^3}{g^2} = \frac{2\nu k_0^2}{\omega} \approx \xi. \tag{4.26}$$

From eq. (4.24) we see that for real angular frequency ω, the wave number k will have an imaginary part of order ξ relatively to the real part.

5. The first-order solution

The first-order solution is determined by solving the set (4.14) of equations. Neglecting terms of order $\xi^{3/2}$, the terms involving the function $\tilde{\psi}_c$ vanish and the unknown coefficients C an D may easily be eliminated. We are left with two equations in the unknown coefficients A and B,

$$\left.\begin{aligned}&\left(1+i\xi\right)\varphi-\frac{gk}{\omega^2}\frac{1}{k\left(1+i\xi\right)}\varphi_c=0, \qquad c=0\\&\left(1+i\xi\right)\varphi-\frac{s}{k\left(1+i\xi\right)}\varphi_c=0, \qquad c=-h\end{aligned}\right\}, \tag{4.27}$$

or – inserting the functions φ and φ_c from eq. (4.4) and neglecting terms of order ξ^2 –

$$\left.\begin{aligned}&\left(1+2i\xi-\frac{gk}{\omega^2}\right)A+\left(1+2i\xi+\frac{gk}{\omega^2}\right)B=0\\&\left(1+2i\xi-s\right)e^{-kh}A+\left(1+2i\xi+s\right)e^{kh}B=0\end{aligned}\right\}. \tag{4.28}$$

The determinant of this system of equations equals eq. (4.20) obtained in the last section, except for the factor *(e^{mh}-e^{-mh})*. (This was to be expected, as the same approximations *($gk \approx \omega^2$)* were made.)

With gk/ω^2 as in eq. (4.21), the coefficient *B* becomes (in terms of *A*)

$$B\approx-\frac{1}{\left[1-\left(1-\frac{\rho_0}{\rho}\right)e^{-2kh}\right]}i\xi\left(\frac{\rho_0}{\rho}\right)e^{-2kh}A\,. \tag{4.29}$$

The coefficients *C* and *D* become to the leading order – from the last equation in (4.14) –

$$C\approx\frac{\left(1-e^{-(m+k)h}\right)}{\left(1-e^{-2mh}\right)}\xi A \tag{4.30}$$

and

$$D\approx\frac{\left(1-e^{-(m-k)h}\right)}{\left(1-e^{-2mh}\right)}e^{-(m+k)h}\xi A\,. \tag{4.31}$$

Eqs. (4.30)-(4.31) show that the coefficients *C* and *D*, and hence the function $\tilde{\psi}$, are independent of ρ_0/ρ to this order of approximation.

We are primarily interested in the streaming motion depending on the function $\tilde{\psi}$, not the potential flow ($\sim\varphi$). Since $\tilde{\psi}$ is independent of ρ_0/ρ, a reasonable simplification in the following will be to assume $\rho = \rho_0$. In that case the coefficient *B* becomes

$$B \approx -i\xi e^{-2kh} A, \quad \rho = \rho_0 . \tag{4.32}$$

The unknown coefficient *E* in the pure water function Θ is determined from eq. (4.12) and becomes – to leading order -

$$E = \left\{ 1 - i\xi \frac{\left(1 - \dfrac{\rho_0}{\rho}\right)\left(1 - e^{-2kh}\right)}{\left[1 - \left(1 - \dfrac{\rho_0}{\rho}\right) e^{-2kh}\right]} \right\} A . \tag{4.33}$$

Hence, in the case of $\rho = \rho_0$, the functions $\varphi, \tilde{\psi}$ and Θ are given by - to the relative order ξ -

$$\left.\begin{aligned} \varphi &= \left[1 - i\xi\, e^{-2k(h+c)}\right] A e^{kc} e^{i(ka-\omega t)} \\ \tilde{\psi} &= \left[\left(1 - e^{-(m+k)h}\right) e^{mc} + \left(1 - e^{-(m-k)h}\right) e^{-(m+k)h} e^{-mc}\right] \frac{\xi\, A e^{i(ka-\omega t)}}{\left(1 - e^{-2mh}\right)} \\ \Theta &= A e^{kc} e^{i(ka-\omega t)} \end{aligned}\right\}, \tag{4.34}$$

and so the first-order velocities become

$$\begin{aligned} x_t^{(1)} = ik\, A e^{kc} e^{i(ka-\omega t)} \Big\{ 1 - i\xi \Big[e^{-2k(c+h)} \\ + \frac{(m/k)}{\left(1 - e^{-2mh}\right)} \left(\left(1 - e^{-(m+k)h}\right) e^{(m-k)c} - \left(1 - e^{-(m-k)h}\right) e^{-(m+k)(c+h)}\right) \Big] \Big\}, \end{aligned} \tag{4.35}$$

$$z_t^{(1)} = k\,Ae^{kc}e^{i(ka-\omega t)}\Big\{1 + i\xi\Big[e^{-2k(c+h)} - \frac{1}{\left(1-e^{-2mh}\right)}\Big(\left(1-e^{-(m+k)h}\right)e^{(m-k)c} + \left(1-e^{-(m-k)h}\right)e^{-(m+k)(c+h)}\Big)\Big]\Big\}, \tag{4.36}$$

$$\left.\begin{aligned} X_t^{(1)} &= ik\,Ae^{kc}e^{i(ka-\omega t)} \\ Z_t^{(1)} &= \;k\,Ae^{kc}e^{i(ka-\omega t)} \end{aligned}\right\}. \tag{4.37}$$

To the most leading order the expressions for the functions in the upper layer (eqs. (4.34)-(4.36)) are reduced to

$$\left.\begin{aligned} \varphi &\approx Ae^{kc}e^{i(ka-\omega t)} \\ \tilde{\psi} &\approx \left[e^{mc} + e^{-m(h+c)}e^{-kh}\right]\xi\,Ae^{i(ka-\omega t)} \end{aligned}\right\}, \tag{4.38}$$

$$\left.\begin{aligned} x_t^{(1)} &\approx ik\,Ae^{kc}e^{i(ka-\omega t)} \\ z_t^{(1)} &\approx k\,Ae^{kc}e^{i(ka-\omega t)} \end{aligned}\right\}. \tag{4.39}$$

6. The second-order problem

The equations for the second-order motion in the upper layer are given in eqs.(2.13) and (2.17). As we are assuming gravity waves, surface tension is neglected. The second-order dynamic boundary conditions at the surface are then given by eqs.(2.27)-(2.28) with $\sigma=0$. In that case both the normal and the tangential stress should vanish at the surface. Using the first-order boundary condition (4.7) and the continuity equation (2.8), these conditions may be expressed as

$$p^{(2)} - 2\mu\left\{z_{ct}^{(2)} + \frac{\partial\left(x^{(1)}, z_t^{(1)}\right)}{\partial(a,c)}\right\} = 0, \quad c = 0, \tag{4.40}$$

$$x_{ct}^{(2)} + z_{at}^{(2)} - \left(3x_{ta}^{(1)}z_a^{(1)} - 2x_{tc}^{(1)}x_a^{(1)} + x_c^{(1)}x_{ta}^{(1)}\right) = 0, \quad c = 0, \tag{4.41}$$

in exact agreement with the boundary conditions stated by Ünlüata and Mei [1970] for an equivalent problem.

We shall be considering the second-order *time-averaged* case only. Taking the time-average of eq.(4.40), the Jacobian vanishes because of eqs.(2.8), (4.7) and (2.44). Similarly, the time-averaged terms in the parenthesis of eq.(4.41) cancel each other. We shall assume mean, steady state. Then the surface may not have any mean, second-order, vertical velocity relatively to itself; that is, the condition

$$\overline{z_t^{(2)}} = 0, \quad c = 0 \tag{4.42}$$

must also be fulfilled. Hence, the time-averaged, second-order surface boundary conditions are

$$\left.\begin{aligned} &\overline{z_t^{(2)}} = 0 \\ &\overline{p^{(2)}} - 2\mu\,\overline{z_{ct}^{(2)}} = 0 \\ &\overline{x_{tc}^{(2)}} + \overline{z_{at}^{(2)}} = 0 \end{aligned}\right\} \quad c = 0\,. \tag{4.43}$$

Having mean, steady state the mean thickness of the viscid layer must be constant in time. Then the lower boundary of the upper layer, identified by $c = -h$, may not have any mean, second-order, vertical velocity relatively to itself, that is

$$\overline{z_t^{(2)}} = 0, \quad c = -h\,. \tag{4.44}$$

If a slow, mean variation in the thickness of the viscid layer was allowed, eq. (4.44) would not be valid. The overall dynamics would in that case be much more complex.

In the interface between a fluid of high viscosity and inviscid water a reasonable approximation is to claim no mean, tangential, second-order stress, which gives – again by eqs.(2.8) and (4.7) – the condition

$$\overline{x_{tc}^{(2)}} + \overline{z_{at}^{(2)}} = 0, \quad c = -h\,. \tag{4.45}$$

This condition implies that we assume the interaction among the upper and lower fluid to be negligible, as far as the second-order, mean motion is considered.
Finally – the mean, normal second-order stress in the upper layer must be balanced by the dynamic pressure in the (inviscid) water at the interface c=-h, that is

$$\overline{p^{(2)}} - 2\mu\,\overline{z_{ct}^{(2)}} = \overline{P^{(2)}}, \quad c = -h\,. \tag{4.46}$$

In determining the mean, second-order motion we shall again assume the mean, second-order acceleration to be negligible (cf eq. (2.48)). The task, then, will be to solve eq. (2.49), subject to the boundary conditions (4.43)-(4.46).

Following chapter II, we introduce a helping function H (cf eq. (2.50)). The function H should fulfil eq. (2.52), which right-hand side is explicitly given in eq. (2.54). Substituting the first-order solution from the previous section, we obtain the equation to be fulfilled by the function H in the present case. We find that this equation may be written in the form

$$\begin{aligned}\nabla_L{}^2 H &= \frac{1}{\nu}\left(\overline{Q_{2a}} - \overline{Q_{1c}}\right)\\ &= \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\left\{\alpha_1\tilde{\psi}\tilde{\psi}_c^* - \alpha_2\varphi\tilde{\psi}^* - \xi\alpha_3\varphi_c\tilde{\psi}_c^* + 2k^{*2}\xi^2\varphi\varphi_c^*\right\},\end{aligned} \tag{4.47}$$

in which the coefficients α_1, α_2 and α_3 denote

$$\left.\begin{aligned}\alpha_1 &\equiv 3k^2 - 3ik^{*2}\xi + ik^2\xi + 2k^{*2}\xi^2\\ \alpha_2 &\equiv 2|k|^2k^*\xi + 2ik^3 + 3|k|^2k\xi - 2ik^{*3}\xi^2\\ \alpha_3 &\equiv 2k^* + k - 2i\,k^{*2}\xi/k\,,\end{aligned}\right\} \tag{4.48}$$

and the stars mean complex conjugation.

The solution for H is expressed in terms of a particular solution H_{part} and a homogeneous solution H_{hom},

$$H = H_{part} + H_{\mathrm{hom}} + K_1 c + K_2\,, \tag{4.49}$$

the real K_1 and K_2 being constants of integration. The functions H_{part} and H_{hom} become

$$H_{part} = \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\left\{\frac{\left(4\kappa^2 + m^2 + m^{*2}\right)\tilde{\psi}\tilde{\psi}_c^* - 2m^{*2}\alpha\tilde{\psi}_c\tilde{\psi}^*}{\left[4\kappa^2 + \left(m+m^*\right)^2\right]\left[4\kappa^2 + \left(m-m^*\right)^2\right]}\alpha_1\right.$$

$$-\frac{\left[\left(4\kappa^2 + k^2 + m^{*2}\right)\alpha_2 - 2\left(k\,m^*\right)^2\xi\alpha_3\right]\varphi\tilde{\psi}^* - \left[2\alpha_2 - \left(4\kappa^2 + k^2 + m^{*2}\right)\xi\alpha_3\right]\varphi_c\tilde{\psi}_c^*}{\left[4\kappa^2 + \left(k+m^*\right)^2\right]\left[4\kappa^2 + \left(k-m^*\right)^2\right]} \quad (4.50)$$

$$\left. + \frac{2k^{*3}\xi^2\left[\left|A\right|^2 e^{(k+k^*)c} - \left|B\right|^2 e^{-(k+k^*)c}\right]e^{-2\kappa a}}{\left[4\kappa^2 + \left(k+k^*\right)^2\right]}\right\}$$

and

$$H_{\mathrm{hom}} = e^{-2\kappa a}\,\mathrm{Re}\left(C_1 e^{2i\kappa c}\right), \quad (4.51)$$

respectively, where C_1 is a (complex) constant of integration.

Introducing the mean, second-order stream function $\overline{\psi^{(2)}}$, we obtain from eq. (2.72) (with the constant included in H)

$$\nabla_L^2\overline{\psi^{(2)}} = H, \quad c \le 0\,. \quad (4.52)$$

As for H, the solution for $\overline{\psi^{(2)}}$ is expressed in terms of a particular solution $\overline{\psi_{part}^{(2)}}$ and a homogeneous solution $\overline{\psi_{\mathrm{hom}}^{(2)}}$,

$$\overline{\psi^{(2)}} = \overline{\psi_{part}^{(2)}} + \overline{\psi_{\mathrm{hom}}^{(2)}} + K_3 c\,. \quad (4.53)$$

The functions $\overline{\psi_{part}^{(2)}}$ and $\overline{\psi_{\mathrm{hom}}^{(2)}}$ become

$$\overline{\psi_{part}^{(2)}} = \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\left\{\frac{\left[\left(4\kappa^2+m^2+m^{*2}\right)^2+4m^2m^{*2}\right]\tilde{\psi}\tilde{\psi}_c^* - 4m^{*2}\left(4\kappa^2+m^2+m^{*2}\right)\tilde{\psi}_c\tilde{\psi}^*}{\left[4\kappa^2+\left(m+m^*\right)^2\right]^2\left[4\kappa^2+\left(m-m^*\right)^2\right]^2}\alpha_1\right.$$

$$- \frac{1}{\left[4\kappa^2+\left(k+m^*\right)^2\right]^2\left[4\kappa^2+\left(k-m^*\right)^2\right]^2} \times$$

$$\times \left[\left[\left(4\kappa^2+k^2+m^{*2}\right)^2\alpha_2 + 4\left(km^*\right)^2\alpha_2 - 4\left(km^*\right)^2\left(4\kappa^2+k^2+m^{*2}\right)\xi\alpha_3\right]\varphi\tilde{\psi}^*\right.$$

$$\left. - \left[4\left(4\kappa^2+k^2+m^{*2}\right)\alpha_2 - 4\left(km^*\right)^2\xi\alpha_3 - \left(4\kappa^2+k^2+m^{*2}\right)^2\xi\alpha_3\right]\varphi_c\tilde{\psi}_c^*\right]$$

$$\left. + \frac{2k^{*3}\xi^2\left[\left|A\right|^2e^{(k+k^*)c} - \left|B\right|^2e^{-(k+k^*)c}\right]e^{-2\kappa a}}{\left[4\kappa^2+\left(k+k^*\right)^2\right]^2}\right\}$$

$$- \; e^{-2\kappa a}\,\mathrm{Re}\left\{C_1\frac{ic}{4\kappa}e^{2i\kappa c}\right\} + \left(\frac{1}{6}K_1c^3 + \frac{1}{2}K_2c^2\right) \tag{4.54}$$

and

$$\overline{\psi_{\mathrm{hom}}^{(2)}} = e^{-2\kappa a}\,\mathrm{Re}\left\{C_2e^{2i\kappa c}\right\}, \tag{4.55}$$

respectively, again the constants C_2 (complex) and K_3 (real) are integration constants.

By the boundary conditions (4.43)-(4.45) $\overline{\psi^{(2)}}$ must fulfil the conditions

$$\left.\begin{aligned}\overline{\psi_a^{(2)}} &= 0\\ \overline{\psi_{aa}^{(2)}} - \overline{\psi_{cc}^{(2)}} &= 0\end{aligned}\right\} \qquad c = 0, -h\,. \tag{4.56}$$

Eqs. (4.56) determine the constants K_1, K_2, K_3, C_1 and C_2. Since these equations should be valid for all $a>0$, both K_1 and K_2 must equal zero. We assume the mean, second-order horizontal motion $\overline{x_t^{(2)}} = \overline{\psi_c^{(2)}}$ to vanish in the limit $a \to \infty$, that is

$$\overline{x_t^{(2)}} = \overline{\psi_c^{(2)}} \to 0, \qquad a \to \infty\,, \tag{4.57}$$

from which it follows that also K_3 must equal zero. The constants C_1 and C_2 are determined in appendix A.

The mean, second-order stream function $\overline{\psi^{(2)}}$ becomes – to the leading order –

$$\overline{\psi^{(2)}} \approx \frac{\xi_0 |A|^2}{4\nu} e^{-2\kappa a} \Big\{ e^{2k_0 c} + \frac{c}{h}\left(e^{-2k_0 h} - 1\right) - 1 + \frac{ck_0^2}{h}(h+c)\left[(h-c)\,e^{-2k_0 h} + (h+c)\right]$$
$$+ \frac{ck_0^2}{2h}(h+c)\left[(h-c)\,e^{-2k_0 h} - (h+c)\right]\xi_0^{1/2} \Big\}. \tag{4.58}$$

(For details – again see appendix A.)

The contour of this stream function was plotted for selected values of the parameters. All the plots showed the same characteristics. A typical plot is shown in fig. 4.5.

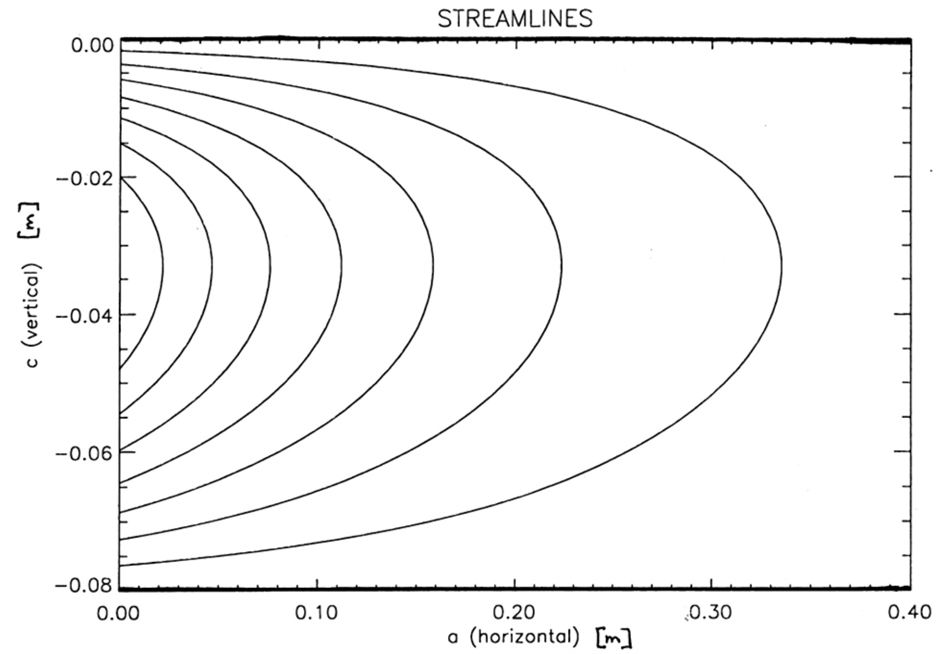

Fig. 4.5. Contour of the stream function $\overline{\psi^{(2)}}$.

The mean, second-order horizontal drift $\overline{x_t^{(2)}}$ is given as $\overline{\psi_c^{(2)}}$. To the leading order this drift becomes (cf eq. (A.19) in appendix A)

$$\overline{x_t^{(2)}} = \overline{\psi_c^{(2)}} \approx \frac{\xi_0 k_0 |A|^2}{2\nu} e^{-2\kappa a} \quad \times$$

$$\times \quad \Big\{ e^{2k_0 c} + \frac{1}{2(k_0 h)}\left(e^{-2k_0 h} - 1\right) + \frac{k_0}{2h}\left[\left(2c^2 + 4hc + h^2\right) - \left(2c^2 - h^2\right) e^{-2k_0 h}\right]$$

$$- 2\xi_0^{1/2}\Big[e^{(m_r + k_0)c}(\cos m_i c - \sin m_i c)$$

$$- e^{2k_0 c - (m_r + k_0)(h+c)}(\cos m_i (h+c) + \sin m_i (h+c))\Big] \qquad (4.59)$$

$$- \xi_0^{1/2} \frac{k_0}{4h}\left[\left(2c^2 + 4hc + h^2\right) + \left(2c^2 - h^2\right) e^{-2k_0 h}\right]\Big\}.$$

(Although being neglected in the approximate expression (4.58) for the stream function, terms of order $\xi \exp(mc)$ must be included when the differentiated stream function is to be obtained, since the contribution from those terms will be of order $\xi^{1/2} \exp(mc)$ in the differentiated function.)

At the boundaries *c=0* and *c=-h* this mean horizontal drift becomes

$$\overline{x_t^{(2)}}(c=0) \approx \frac{\xi_0 k_0 |A|^2}{2\nu} e^{-2\kappa a} \Big\{ 1 + \frac{1}{2}(k_0 h)\left(1 + e^{-2k_0 h}\right) - \frac{1}{2(k_0 h)}\left(1 - e^{-2k_0 h}\right)$$

$$- \left[2 + \frac{1}{4}(k_0 h)\left(1 - e^{-2k_0 h}\right)\right]\xi_0^{1/2}\Big\}$$

$$= \frac{\xi_0 |A|^2}{4\nu h} e^{-2\kappa a} \Big\{ (k_0 h)^2 + 2(k_0 h) - 1 + \left[(k_0 h)^2 + 1\right] e^{-2k_0 h}$$

$$- \frac{1}{2}(k_0 h)\left[(k_0 h) + 8 - (k_0 h) e^{-2k_0 h}\right]\xi_0^{1/2}\Big\}$$

$$\approx \frac{\xi_0 |A|^2}{4\nu h} e^{-2\kappa a} \left\{ (k_0 h)^2 + 2(k_0 h) - 1 + \left[(k_0 h)^2 + 1\right] e^{-2k_0 h}\right\} \qquad (4.60)$$

and

$$\overline{x_t^{(2)}}(c=-h) \approx \frac{\xi_0 k_0 |A|^2}{2\nu} e^{-2\kappa a}\Big\{ e^{-2k_0h} - \frac{1}{2(k_0h)}\left(1-e^{-2k_0h}\right) - \frac{1}{2}(k_0h)\left(1+e^{-2k_0h}\right)$$

$$+\left[2e^{-2k_0h} + \frac{1}{4}(k_0h)\left(1-e^{-2k_0h}\right)\right]\xi_0^{1/2}\Big\}$$

$$= -\frac{\xi_0|A|^2}{4\nu h} e^{-2\kappa a}\Big\{ (k_0h)^2 + 1 + \left[(k_0h)^2 - 2(k_0h) - 1\right]e^{-2k_0h}$$

$$-\frac{1}{2}(k_0h)\left[(k_0h) + \left(8-(k_0h)\right)e^{-2k_0h}\right]\xi_0^{1/2}\Big\}$$

$$\approx -\frac{\xi_0|A|^2}{4\nu h} e^{-2\kappa a}\Big\{ (k_0h)^2 + 1 + \left[(k_0h)^2 - 2(k_0h) - 1\right]e^{-2k_0h}\Big\}, \tag{4.61}$$

respectively. A typical velocity profile is shown in fig. 4.6.

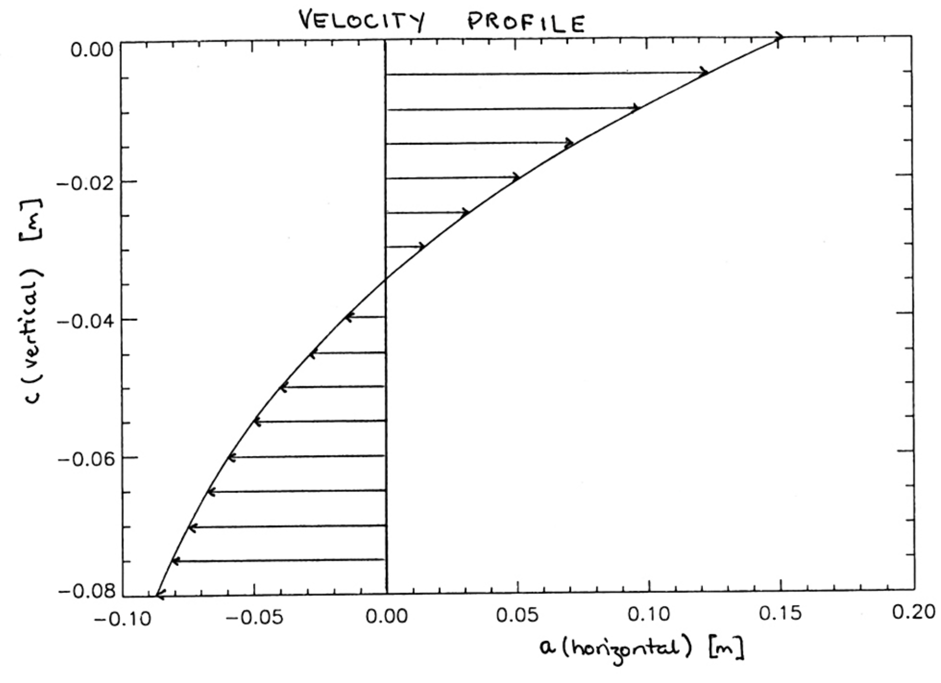

Fig. 4.6. Velocity profile of the mean, horizontal velocity $\overline{x_t^{(2)}}$.

The profile shows a forward mean motion at the top and a return flow at the bottom of the viscid layer. Hence, this theoretical model is able to qualitatively reproduce the experimental results reported by Martin and Kauffman [1981].

It is interesting to compare this mean drift with the Stokes drift of inviscid water. The surface Stokes drift is given as $\mathrm{v}_{S0} = a^2\omega k$ (cf eq. (1.22)), with a denoting the wave amplitude. In our terminology the wave amplitude is given by

$$\left|\varepsilon z^{(1)}\right| \approx \frac{\varepsilon k_0 |A|}{\omega}. \tag{4.62}$$

The total mean drift velocity, $\varepsilon^2 \overline{x_t^{(2)}}$, may then be written – to most leading order –

$$\begin{aligned}
\varepsilon^2 \overline{x_t^{(2)}} &\approx \frac{\varepsilon^2 \xi_0 k_0 |A|^2}{2\nu} e^{-2\kappa a} \Bigg\{ e^{2k_0 c} + \\
&\quad + \frac{1}{2(k_0 h)} \left[(k_0 h)^2 + 2(k_0 c)^2 + 4(k_0 c)(k_0 h) - 1 + \left((k_0 h)^2 - 2(k_0 c)^2 + 1 \right) e^{-2k_0 h} \right] \Bigg\} \\
&= \mathrm{v}_{S0} e^{-2\kappa a} \Bigg\{ e^{2k_0 c} + \\
&\quad + \frac{1}{2(k_0 h)} \left[(k_0 h)^2 + 2(k_0 c)^2 + 4(k_0 c)(k_0 h) - 1 + \left((k_0 h)^2 - 2(k_0 c)^2 + 1 \right) e^{-2k_0 h} \right] \Bigg\}.
\end{aligned} \tag{4.63}$$

Eq. (4.63) shows that there may be a considerable modification of the drift velocity, compared to the Stokes drift. The modification is strongly depending upon the size of (k_0h). For (k_0h) =1, which is equivalent to a thickness $h = \lambda / 2\pi$ (λ denoting the wavelength), the surface drift is almost identical to the Stokes drift, but for (k_0h)=2π, i.e. a thickness which equals the wavelength, the surface drift becomes approximately 4 times the inviscid Stokes drift. This modification is caused by the modified boundary conditions. To fulfil those, a mean Eulerian drift has been induced. By eq.(4.59) we see that the "viscid drift", the drift velocity represented by the terms involving the viscosity, is weaker by a factor $\xi_0^{1/2}$ and is of less importance. However, the viscid terms give important contributions to the velocity gradients.

7. Concluding remarks

In this chapter the model established in chapter II has been applied to waves propagating into a surface layer of viscid grease ice. The first-order wave solution and the time-averaged, second-order stream function in case of mean, steady state have been determined. From the stream function the mean, horizontal drift was obtained, and it is seen to be qualitatively in agreement with reported experimental results.

CHAPTER V

THE RADIATION STRESS CONCEPT

In this chapter the concept of radiation stress is presented, and the dynamics associated with wave damping by oil film and by grease ice as considered in chapter III and IV is discussed within this concept.

1. Radiation stress

In the early 1960's M.S. Longuet-Higgins and R.W. Stewart [1960-1962] introduced the concept of radiation stress in the context of surface waves. The radiation stress is defined as the (excess) momentum flux due to the presence of the waves, and is similar to the radiation pressure in electromagnetic theory. This type of pressure or stress is present in all kind of wave motion. Longuet-Higgins and Stewart demonstrated [1964] how many wave phenomena – change in mean sea level due to storm waves, interaction of waves with steady currents, steepening of short gravity waves on the crests of longer waves, among others – may be studied in a very straightforward and intuitive manner exploiting this concept. They suggested that the radiation stress might prove to be a most convenient feature in the study of non-linear dynamic effects. However, despite the promising results the concept seems to have achieved only moderate attention.

The reason for the efficiency of the radiation stress concept becomes evident through the treatment of the dynamical conservation equations in O.M. Phillips' monograph "The dynamics of the upper ocean" [Phillips, 1977, p. 62-78], in which the radiation stress is an important feature. We shall present the theory for momentum conservation following those lines. The analysis is performed within the Eulerian picture.

An equation for the total mean momentum flux is achieved by integrating the Navier-Stokes equation (1.1) over the depth of the fluid, from a solid bottom at a depth $z=-h$ to the surface at $z=\eta$. For an incompressible fluid ($\nabla \cdot \underline{v}=0$), the integrated equation may be written – by repeatedly use of the Leibniz' theorem for interchanging integration and differentiation –

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\rho \underline{v}\, dz+\nabla_H\cdot\int_{-h}^{\eta}\left(\rho \underline{v}\,\underline{v}+\underline{\underline{P}}\right)dz-\rho \underline{v}_S\frac{\partial\eta}{\partial t}-\nabla_H\eta\cdot\left(\rho \underline{v}\,\underline{v}+\underline{\underline{P}}\right)_S$$

$$-\nabla_H h\cdot\left(\rho \underline{v}\,\underline{v}+\underline{\underline{P}}\right)_B+\underline{e}_z\cdot\left(\rho \underline{v}\,\underline{v}+\underline{\underline{P}}\right)_S-\underline{e}_z\cdot\left(\rho \underline{v}\,\underline{v}+\underline{\underline{P}}\right)_B=-\underline{e}_z\rho\, g\left(\eta+h\right). \tag{5.1}$$

The symbol ∇_H denotes the horizontal part of the gradient operator and the subscripts S and B indicate that the functions are to be evaluated at the surface and at the bottom, respectively. We recognize ($\underline{e}_z-\nabla_H\eta$) as a vector normal to the surface and ($\underline{e}_z+\nabla_H h$) as a vector normal to the bottom, and hence – by the kinematic boundary conditions (eqs. (1.3)-(1.4)) – eq. (5.1) is reduced to

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\rho \underline{v}\, dz+\nabla_H\cdot\int_{-h}^{\eta}\left(\rho \underline{v}\,\underline{v}+\underline{\underline{P}}\right)dz+\left(\underline{e}_z-\nabla_H\eta\right)\cdot\underline{\underline{P}}_S$$
$$-\left(\underline{e}_z+\nabla_H h\right)\cdot\underline{\underline{P}}_B=-\underline{e}_z\rho\, g\left(\eta+h\right). \tag{5.2}$$

Further analysis will be restricted to the *horizontal* component of this momentum equation,

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\rho \underline{v}_H\, dz+\nabla_H\cdot\int_{-h}^{\eta}\left(\rho \underline{v}_H\,\underline{v}_H+\underline{\underline{P}}_H\right)dz+\left[\ \left(\underline{e}_z-\nabla_H\eta\right)\cdot\underline{\underline{P}}_S\ \right]_H$$
$$-\left[\ \left(\underline{e}_z+\nabla_H h\right)\cdot\underline{\underline{P}}_B\ \right]_H=0, \tag{5.3}$$

where $\underline{v}_H$ represents the horizontal velocity. Some simplifications will be introduced in the following. This will not, however, reduce the power of the analysis.

For an incompressible and inviscid fluid the stress tensor $\underline{\underline{P}}$ is just

$$\underline{\underline{P}}=p\underline{\underline{I}}, \tag{5.4}$$

in which case eq. (5.3) is reduced to

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\rho \underline{v}_H\,dz + \nabla_H \cdot \int_{-h}^{\eta}\left(\rho \underline{v}_H\,\underline{v}_H + p\underline{\underline{I}}_H\right)dz - p_S \nabla_H \eta - p_B \nabla_H h = 0\,. \tag{5.5}$$

Neglecting surface tension, also the surface pressure p_S vanishes (cf eq. (1.3)). Hence, for an incompressible fluid, neglecting viscosity and surface tension, the horizontal component of the averaged and depth integrated Navier-Stokes equation is

$$\frac{\partial}{\partial t}\overline{\int_{-h}^{\eta}\rho \underline{v}_H\,dz} + \nabla_H \cdot \overline{\int_{-h}^{\eta}\left(\rho \underline{v}_H\,\underline{v}_H + p\underline{\underline{I}}_H\right)dz} - \overline{p_B}\nabla_H h = 0\,. \tag{5.6}$$

The mean value of the first integral is recognized as the mean, total horizontal momentum $\underline{M}$ of the fluid per unit surface area (cf eq. (1.23)), or – equivalently – the mean mass flux associated with the horizontal motion. Assuming the motion to consist of a fluctuating motion superimposed upon a mean motion, the total velocity $\underline{v}$ may be expressed as the sum of the mean velocity $\underline{V}$ and the fluctuating velocity $\underline{v}'$,

$$\underline{v} = \underline{V} + \underline{v}'\,. \tag{5.7}$$

For algebraic simplicity, the mean velocity $\underline{V}$ is assumed to be independent of depth. The mean value of the second integral in eq. (5.6) may then be written

$$\begin{aligned}\overline{\int_{-h}^{\eta}\left(\rho \underline{v}_H\,\underline{v}_H + p\underline{\underline{I}}_H\right)dz} &= \rho \underline{V}_H\,\underline{V}_H\left(h+\overline{\eta}\right) + \underline{V}_H\,\underline{M}' + \underline{M}'\,\underline{V}_H \\ &+ \overline{\int_{-h}^{\eta}\left(\rho \underline{v}'_H\,\underline{v}'_H + p\underline{\underline{I}}_H\right)dz}\,,\end{aligned} \tag{5.8}$$

with $\overline{\eta}$ denoting the mean surface elevation and $\underline{M}'$ the horizontal *wave* momentum,

$$\underline{M}' = \overline{\int_{-h}^{\eta}\rho \underline{v}'_H\,dz}\,. \tag{5.9}$$

The mean, total, horizontal momentum $\underline{M}$ is the sum of $\underline{M}'$ and the momentum $\hat{\underline{M}}$ of the mean, horizontal motion,

$$\underline{\hat{M}} = \overline{\int_{-h}^{\eta} \rho \underline{V}_H dz} = \underline{V}_H \rho\left(h + \overline{\eta}\right) , \tag{5.10}$$

so that

$$\underline{M} = \overline{\int_{-h}^{\eta} \rho \underline{v}_H dz} = \underline{M}' + \underline{\hat{M}} . \tag{5.11}$$

Introducing a (fictive) *steady*, horizontal flow, with mean momentum $\underline{\widetilde{M}}$ equal to the mean momentum $\underline{M}$ of the actual horizontal motion, and a mean transport velocity $\underline{\widetilde{U}}$ defined by

$$\underline{\widetilde{U}} = \underline{\widetilde{M}} / \left[\rho\left(h + \overline{\eta}\right)\right], \tag{5.12}$$

eq. (5.8) may alternatively be written

$$\overline{\int_{-h}^{\eta} \left(\rho \underline{v}_H \underline{v}_H + p \underline{\underline{I}}_H\right) dz} = \underline{\widetilde{U}}\,\underline{\widetilde{M}} - \frac{\underline{M}'\underline{M}'}{\rho(h + \overline{\eta})} + \overline{\int_{-h}^{\eta} \left(\rho \underline{v}'_H \underline{v}'_H + p \underline{\underline{I}}_H\right) dz} , \tag{5.13}$$

with $\underline{\widetilde{U}}\,\underline{\widetilde{M}}$ representing the momentum flux of the *steady* motion.

In case of progressive waves and zero mean, vertical motion the mean bottom pressure $\overline{p_B}$ may be approximated by

$$\overline{p_B} = \rho g \left(h + \overline{\eta}\right), \tag{5.14}$$

valid to the second order. The bottom pressure term in eq. (5.6) may then be expressed by

$$\overline{p_B}\nabla_H h = \rho g\left(h+\bar{\eta}\right)\nabla_H h$$
$$= \nabla_H\left\{\frac{1}{2}\rho g\left(h+\bar{\eta}\right)^2\right\} - \rho g\left(h+\bar{\eta}\right)\nabla_H \bar{\eta}\,. \tag{5.15}$$

Defining a new quantity $\underline{\underline{S}}_H$ by

$$\underline{\underline{S}}_H = \overline{\int_{-h}^{\eta}(\rho\underline{v}'_H\underline{v}'_H + p\underline{\underline{I}}_H)\,dz} - \frac{1}{2}\rho g\,(h+\bar{\eta})^2\,\underline{\underline{I}}_H - \frac{\overline{\underline{M}'\underline{M}'}}{\rho\,(h+\bar{\eta})}, \tag{5.16}$$

this quantity may be interpreted as representing the *excess momentum flux* or *radiation stress* resulting from the fluctuating motion. The equation for the mean, horizontal momentum, eq. (5.6), may now be written in the form

$$\frac{\partial\underline{\tilde{M}}}{\partial t} + \nabla_H\cdot\left\{\underline{\tilde{U}}\,\underline{\tilde{M}} + \underline{\underline{S}}_H\right\} = \underline{F}_H\,, \tag{5.17}$$

with

$$\underline{F}_H = -\rho\,g\,\left(h+\bar{\eta}\right)\nabla_H\bar{\eta} \tag{5.18}$$

representing the net horizontal force per unit area arising from the (mean) slope of the free surface. Eq. (5.17) expresses the balance of the total, mean, horizontal momentum per unit area.

Longuet-Higgins and Stewart assert that if $\partial S_{xx}/\partial x \neq 0$, the radiation stress is expected to generate or modify effects on a greater scale than the waves themselves. The momentum flux of the steady motion may often easily be calculated, hence eq. (5.17) illustrates how the radiation stress may have the potential of being useful in analysing wave phenomena.

We shall calculate the radiation stress S_{xx} in case of a standard, inviscid two-dimensional wave with surface elevation η given by eq. (1.17), applying the analysis outlined above. If the surface tension σ may not be neglected, the surface pressure term in eq. (5.5) will not vanish,

$$p_S \nabla_H \eta \to p_S \frac{\partial \eta}{\partial x} = \kappa_S \sigma \frac{\partial \eta}{\partial x} \approx -\sigma \frac{\partial^2 \eta}{\partial x^2} \frac{\partial \eta}{\partial x} = -\sigma \frac{\partial}{\partial x} \left\{ \frac{1}{2} \left(\frac{\partial \eta}{\partial x} \right)^2 \right\}. \qquad (5.19)$$

(The surface pressure p_S is obtained from the surface boundary condition (1.3), with constant surface tension.)

The radiation stress S_{xx} is achieved from eq. (5.16), with the mean surface elevation $\overline{\eta}$ assumed to be zero:

$$S_{xx} = \overline{\int_{-h}^{\eta} (\rho u^2 + p) dz} - \frac{1}{2} \rho g h^2 - \frac{M'^2}{\rho h} + \frac{\sigma}{2} \overline{\left(\frac{\partial \eta}{\partial x} \right)^2}. \qquad (5.20)$$

The horizontal velocity component u and the vertical velocity component w are given as (for moderate depth h)

$$\left. \begin{aligned} u &= a\omega \frac{\cosh k(z+h)}{\sinh kh} \cos(kx - \omega t) \\ w &= a\omega \frac{\sinh k(z+h)}{\sinh kh} \sin(kx - \omega t) \end{aligned} \right\}. \qquad (5.21)$$

An expression for the dynamic pressure p is achieved by integrating the vertical component of the Navier-Stokes equation in the form (1.7) (with the kinematic viscosity ν=0). This results in, assuming $p = \kappa_S \sigma$ at the surface $z=\eta$,

$$p = \rho g(\eta - z) + \kappa_S \sigma + \frac{\partial}{\partial t} \int_z^{\eta} \rho w dz + \frac{\partial}{\partial x} \int_z^{0} \rho u w dz - \rho w^2. \qquad (5.22)$$

Hence, to the second order in the wave steepness *(ak)*, the radiation stress S_{xx} becomes – since the M'^2-term in this case will be of the fourth order –

$$S_{xx} = \int_{-h}^{0} \rho \overline{u^2} dz + \frac{1}{2}\rho g \overline{\eta^2} - \sigma \overline{\eta \frac{\partial^2 \eta}{\partial x^2}} - \int_{-h}^{0} \rho \overline{w^2} dz + \frac{\sigma}{2} \overline{\left(\frac{\partial \eta}{\partial x}\right)^2} . \tag{5.23}$$

Inserting the expressions (5.21) for the velocity components u and w, and (1.17) for the surface elevation η, also applying the dispersion relation which for moderate depth h is

$$\omega^2 = gk\left(1 + \frac{\sigma k^2}{\rho g}\right)\tanh kh , \tag{5.24}$$

the radiation stress S_{xx} becomes

$$S_{xx} = \frac{1}{2}\rho g a^2 \left\{ \frac{2kh}{\sinh 2kh}\left(1 + \frac{\sigma k^2}{\rho g}\right) + \frac{1}{2}\left(1 + 3\frac{\sigma k^2}{\rho g}\right)\right\}, \tag{5.25}$$

cf Longuet-Higgins and Stewart [1964].

It is easily seen that in terms of the energy density E,

$$E = \frac{\rho \omega^2 a^2}{2k}\coth kh = \frac{1}{2}\rho g a^2 \left(1 + \frac{\sigma k^2}{\rho g}\right), \tag{5.26}$$

the group velocity $v_g = \partial\omega/\partial k$ and the phase velocity $v_{ph} = \omega/k$, the radiation stress in this case may be expressed

$$S_{xx} = E\left\{\frac{v_g}{v_{ph}} + \frac{kh}{\sinh 2kh}\right\}. \tag{5.27}$$

The radiation stress obtained here may also be a good approximation in case of a slightly viscid fluid, where the vorticity may be assumed to contribute only in thin boundary layers. The contribution from the vorticity to the integral over the total depth of the fluid will then be negligible, except for extreme cases.

For deep-water waves ($kh >> 1$) expression (5.27) is reduced to

$$S_{xx} = E\, \mathrm{v}_g / \mathrm{v}_{ph}\,, \tag{5.28}$$

and the interpretation of the radiation stress is obvious – as representing the wave momentum E/v_{ph} transported by the group velocity v_g.

The radiation stress concept may be used in determining the possible "set-up" of the mean surface when deep-water gravity waves encounter a sloping beach, see fig. 5.1 below.

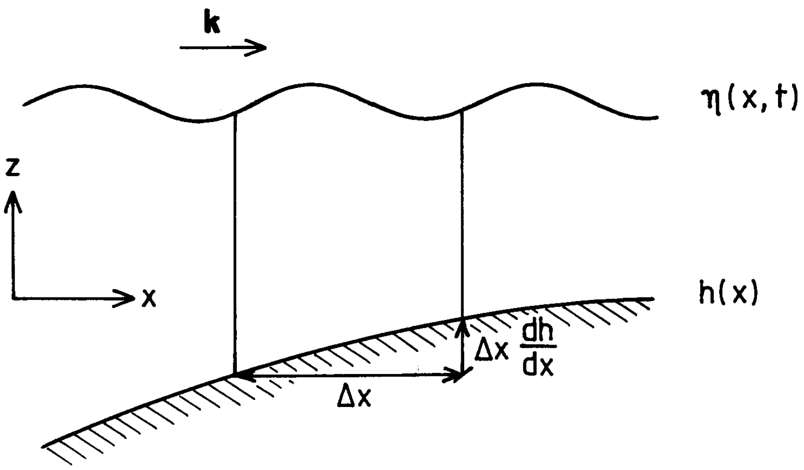

Fig. 5.1. Deep-water gravity waves encountering a sloping beach.

For a steady state ($\partial/\partial t \equiv 0$), the total mean momentum (or mass flux) $\underline{M} = \underline{\widetilde{M}}$ is constant. Since no mass flux is possible through the beach, $\underline{M}$ must be zero. But then the velocity $\underline{\widetilde{U}}$ must also be zero (cf eq. (5.12)), and eq. (5.17) is reduced to

$$\nabla_H \cdot \underline{\underline{S}}_H = -\rho\, g \left(h + \overline{\eta}\right) \nabla_H \overline{\eta}\,. \tag{5.29}$$

Assuming no variation in the y-direction and the mean surface elevation $\overline{\eta}$ to be small compared to the depth h, eq. (5.29) becomes

$$\frac{\partial S_{xx}}{\partial x} = -\rho\, g \left(h + \overline{\eta}\right) \frac{\partial \overline{\eta}}{\partial x} \approx -\rho\, g\, h \frac{\partial \overline{\eta}}{\partial x}\,, \tag{5.30}$$

or – equivalently –

$$\frac{\partial \bar{\eta}}{\partial x} = -\frac{1}{\rho g h} \frac{\partial S_{xx}}{\partial x} . \tag{5.31}$$

Longuet-Higgins and Stewart [1962,1964] showed that this equation may be exactly integrated, giving the mean level $\bar{\eta}$ of the surface as

$$\bar{\eta} = -\frac{a^2 k}{2 \sinh 2kh} . \tag{5.32}$$

These examples illustrate how the radiation stress concept may be exploited in analysing the dynamics of a wave field in a convenient manner, especially in the case of a mean, steady state, providing a nice tool for studying the overall dynamics of surface waves. Hence, the concept of radiation stress may greatly simplify the analysis of these complex physical situations. In the following it will be demonstrated how the radiation stress concept may be applied to obtain insight into the overall dynamics of the cases studied in chapter III and IV.

2. The elastic monolayer case

In the elastic monolayer case of chapter III a steady train of plane surface waves were travelling into an area covered by thin (monomolecular), elastic film. The film hinders the free motion of the water, causing strong damping of the waves. The main interaction is assumed to take place between the elastic surface and the water just beneath.

When implementing the radiation stress analysis of section 1 to this case, some adjustments are needed. In the two-dimensional case (no motion in the y-direction), eq. (5.5) is reduced to

$$\frac{\partial}{\partial t} \int_{-h}^{\eta} \rho u dz + \frac{\partial}{\partial x} \int_{-h}^{\eta} (\rho u^2 + p) dz - p_S \frac{\partial \eta}{\partial x} - p_B \frac{\partial h}{\partial x} = 0 . \tag{5.33}$$

From the dynamic surface condition (1.3) we obtain – allowing for a varying surface tension σ –

$$-\frac{p_S \eta_x}{\sqrt{1+\eta_x^2}} = -\kappa_s \sigma \frac{\eta_x}{\sqrt{1+\eta_x^2}} - \frac{\partial \sigma}{\partial x} + \frac{\eta_x}{\sqrt{1+\eta_x^2}} \frac{(\sigma_x \eta_x - \sigma_z)}{\sqrt{1+\eta_x^2}} . \tag{5.34}$$

Also assuming horizontal bottom ($\partial h / \partial x = 0$), the averaged eq. (5.33) becomes

$$\frac{\partial}{\partial t}\overline{\int_{-h}^{\eta}\rho u dz}+\frac{\partial}{\partial x}\overline{\int_{-h}^{\eta}(\rho u^2+p)dz}-\overline{\sigma\kappa_s\eta_x}-\frac{\partial\overline{\sigma}}{\partial x}\overline{\sqrt{1+\eta_x^2}}+\overline{\frac{\eta_x(\sigma_x\eta_x-\sigma_z)}{\sqrt{1+\eta_x^2}}}=0\,. \qquad (5.35)$$

If the surface tension σ is not explicitly depending on the vertical coordinate z, the last three terms of eq. (5.35) may be written – to the second order in the wave steepness –

$$\sigma\kappa_s\eta_x+\sigma_x\sqrt{1+\eta_x^2}-\frac{\sigma_x\eta_x^2}{\sqrt{1+\eta_x^2}}\approx-\sigma\eta_x\eta_{xx}+\sigma_x\left(1+\frac{1}{2}\eta_x^2\right)-\sigma_x\eta_x^2$$

$$=-\sigma\eta_x\eta_{xx}+\sigma_x\left(1-\frac{1}{2}\eta_x^2\right)=\frac{\partial}{\partial x}\left\{\sigma\left(1-\frac{1}{2}\eta_x^2\right)\right\}, \qquad (5.36)$$

reducing eq. (5.35) to

$$\frac{\partial}{\partial t}\overline{\int_{-h}^{\eta}\rho u dz}+\frac{\partial}{\partial x}\overline{\int_{-h}^{\eta}(\rho u^2+p)dz}-\frac{\partial}{\partial x}\overline{\left\{\sigma\left(1-\frac{1}{2}\eta_x^2\right)\right\}}=0\,. \qquad (5.37)$$

The first integral in eq. (5.37) is again recognized as the *total,* horizontal, mean momentum M (cf eq. (1.23)). Following section 1, the radiation stress S_{xx} is in this case given as (assuming the mean surface elevation, $\overline{\eta}$, to be zero)

$$S_{xx}=\overline{\int_{-h}^{\eta}(\rho u^2+p)dz}-\frac{M'^2}{\rho h}+\frac{1}{2}\overline{\sigma\eta_x^2}\,, \qquad (5.38)$$

where M' is the mean, horizontal momentum associated with the fluctuating motion. Eq. (5.37) may then be written – cf eq. (5.17) –

$$\frac{\partial\widetilde{M}}{\partial t}+\frac{\partial}{\partial x}\left\{\widetilde{U}\widetilde{M}+S_{xx}\right\}=\frac{\partial\overline{\sigma}}{\partial x}\,, \qquad (5.39)$$

where a steady flow $\widetilde{U}$ with momentum $\widetilde{M}=M$ is introduced, and $\partial\overline{\sigma}/\partial x$ is the (induced) stress in the film. $\widetilde{U}\widetilde{M}$ is the momentum flux tensor of the equivalent steady flow, which for pure wave motion will be of fourth order in the wave steepness. The radiation stress S_{xx} will be of second order. Having mean steady state ($\partial / \partial t \equiv 0$) the momentum balance equation (5.39) may therefore be approximated by the equation

$$\frac{\partial S_{xx}}{\partial x} - \frac{\partial \overline{\sigma}}{\partial x} = 0 \ . \tag{5.40}$$

This shows that for steady state the gradient of the radiation stress should balance the (induced) mean, tangential stress within the elastic film. Or – steady state is possible if the radiation stress gradient balances the (induced) film stress.

We shall examine the result of chapter III in the light of this analysis. For deep-water waves the radiation stress S_{xx} was given by the formulae (5.28). Since the group velocity v_g equals the ratio between the damping rate γ in time and κ in space (cf eq. (3.52)), and the energy density E for surface waves generally is given as

$$E = \rho\omega^2 \overline{\eta^2} / k \ , \tag{5.41}$$

the radiation stress S_{xx} becomes

$$S_{xx} = \rho\gamma\omega\overline{\eta^2} / \kappa \, . \tag{5.42}$$

By the terminology of chapter III the mean, square surface elevation $\overline{\eta^2}$ may be expressed as

$$\overline{\eta^2} = \frac{1}{2}\left|\varepsilon z^{(1)}\right|^2_{c=0} \ , \tag{5.43}$$

to the second order in the expansion parameter ε. By eqs. (3.15) and (3.26) eq. (5.43) becomes

$$\begin{aligned} \overline{\eta^2} &\approx \frac{|\varepsilon k|^2}{2\omega^2}|A - iC|^2 e^{-2\kappa a} \\ &\approx \frac{|\varepsilon kA|^2 e^{-2\kappa a}}{2\omega^2|\beta - 1 + i|^2}\left\{2[1-(k_0\delta)] - 2\beta_0\left[1 + 2\left(\frac{\kappa}{k_0}\right) - (k_0\delta)\right] + \beta_0^2\right\} . \end{aligned} \tag{5.44}$$

The time damping rate γ may be determined by the dispersion relation (3.36), as the imaginary part of ω with opposite sign. To leading order the expression for γ becomes (for details – see appendix B)

$$\frac{\gamma}{\omega_0} \approx \frac{(k\delta)}{2|\beta - 1 + i|^2}\left\{1 + (k\delta)(\beta - 1)\left[2\beta - \frac{1}{|\beta - 1 + i|^2}\right]\right\} . \tag{5.45}$$

In this expression both the wave number k and the parameters δ and β are assumed real, and ω_0 is the real part of ω. Inserting eqs. (5.44)-(5.45) into eq. (5.42), we obtain the radiation stress S_{xx} as

$$S_{xx} \approx \frac{\mu\varepsilon^2 k_0^3 |A|^2 e^{-2\kappa a}}{2\omega\kappa\delta|\beta - 1 + i|^2}\left\{1 + (k_0\delta)(\beta_0 - 1)\left[2\beta_0 + \frac{1}{|\beta_0 - 1 + i|^2}\right] - \frac{4\left(\frac{\kappa}{k_0}\right)\beta_0}{|\beta_0 - 1 + i|^2}\right\}, \tag{5.46}$$

where $\beta \approx \beta_0(1 - 2i\frac{\kappa}{k_0})$.

In the case of steady state we may interpret the variable a as the horizontal coordinate, cf the discussion of chapter II, section 7. To leading order differentiation with respect to x may be replaced by differentiation with respect to a in the averaged expressions, and the gradient $\partial S_{xx} / \partial x$ of the radiation stress becomes

$$\frac{\partial S_{xx}}{\partial x} \rightarrow \frac{\partial S_{xx}}{\partial a} = -2\,\kappa\, S_{xx} . \tag{5.47}$$

Combining eqs. (5.46)-(5.47) finally gives us the gradient of the radiation stress $\partial S_{xx} / \partial x$ as

$$\frac{\partial S_{xx}}{\partial x} \approx -\frac{\mu\varepsilon^2 k_0^3 |A|^2 e^{-2\kappa a}}{\omega\delta|\beta - 1 + i|^2}\left\{1 + \frac{\left[(k_0\delta) - 4(\frac{\kappa}{k_0})\right]\beta_0 - (k_0\delta)}{|\beta_0 - 1 + i|^2} - 2(k_0\delta)\beta_0 + 2(k_0\delta)\beta_0^2\right\} . \tag{5.48}$$

Inspection shows that this radiation stress gradient is very similar to (minus) the virtual wave stress obtained in chapter III (eqs. (3.61) – (3.64)).

The analysis in this chapter showed that the radiation stress gradient should balance the mean stress in the film. By eqs. (2.28) and (3.54) this is also the case for (minus) the mean, tangential fluid stress obtained in chapter III. But whereas the radiation stress gradient in eq. (5.48) agrees well with (minus) the virtual wave stress of chapter III, the total mean fluid stress obtained there has a contribution from the Stokes drift in addition to the virtual wave stress. However, in chapter III the mean, tangential fluid stress is determined within Lagrangian fluid description, whereas the radiation stress analysis is carried out within Eulerian fluid description. The discrepancy is probably due to the different approaches, parallel to the mean, Lagrangian drift being a sum of the mean, Eulerian drift and the Stokes drift (eq. (2.82)).

The outline of this section supports several things: First, the assumption of the *surface* stress being the dominating factor in this dynamic situation is supported, since the radiation stress gradient agrees with the virtual wave stress. Secondly, it supports the validity of the simple formulae (5.28) for the radiation stress in the deep-water case. And thirdly, the analysis and results of chapter III are supported by the agreement with the result of this simplified analysis.

We may conclude that for surface waves in deep water, covered by an elastic, monomolecular surface film, the average wave-induced fluid stress acting at the surface may be calculated in a very straight-forward manner by means of the radiation stress concept.

3. The viscid layer case

For a viscid fluid with no variation in the y-direction, the integrated momentum equation (5.3) becomes

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\rho u dz + \frac{\partial}{\partial x}\int_{-h}^{\eta}\left(\rho u^2 + p - 2\mu\frac{\partial u}{\partial x}\right)dz$$
$$+\left(\underline{e}_z - \frac{\partial \eta}{\partial x}\underline{e}_x\right)\cdot \underline{\underline{P}}_S \cdot \underline{e}_x - \left(\underline{e}_z + \frac{\partial h}{\partial x}\underline{e}_x\right)\cdot \underline{\underline{P}}_B \cdot \underline{e}_x = 0, \qquad (5.49)$$

in which u denotes the horizontal velocity. For gravity waves (surface tension neglected) the surface pressure term in eq. (5.49) vanishes because of the surface boundary condition (1.3) (with $\sigma \equiv 0$). Applied to the case of chapter IV, in which a steady train of plane surface waves were travelling into an area of viscid fluid covering an inviscid ocean, the integration is performed over the thickness h of the viscid layer. The thickness h is supposed to be of the order of a wavelength. The bottom stress tensor $\underline{\underline{P}}_B$ must equal the interface stress tensor $p'_B\underline{\underline{I}}$ of the inviscid fluid (if no tension is assumed in the interface between the two fluids). Hence eq. (5.49) becomes

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\rho u dz+\frac{\partial}{\partial x}\int_{-h}^{\eta}\left(\rho u^2+p-2\mu\frac{\partial u}{\partial x}\right)dz-\frac{\partial h}{\partial x}p_B{}'=0\ . \tag{5.50}$$

The experimental results of Martin and Kauffman [1981] for wave damping in grease ice and the calculations of chapter IV suggested a picture as sketched in fig. 5.2, with a mean flow induced within the viscid fluid.

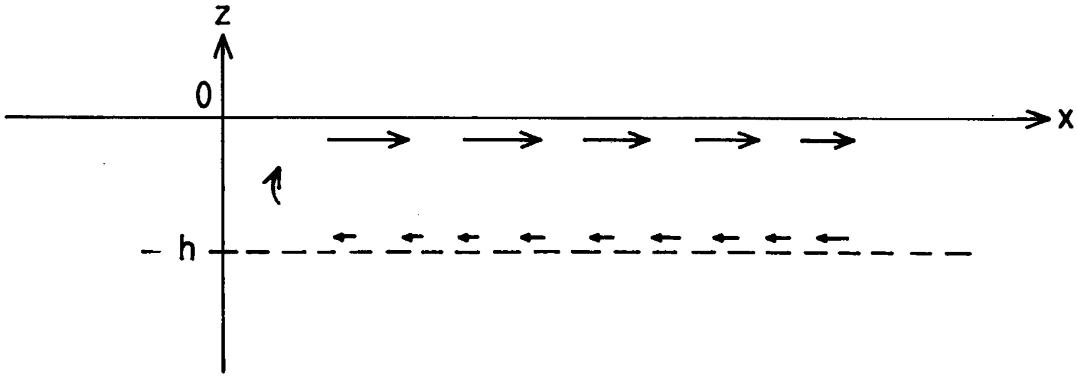

Fig. 5.2. Induced mean flow within the viscid fluid.

A qualitative analysis may illustrate how the radiation stress of the viscid fluid may be interpreted as the driving mechanism of this mean flow. To have "steady state" the total mass flux must be kept constant. This may be obtained by an (induced) mean flow, appropriately distributed. As in section 1 we assume the velocity u to have a mean part U and a fluctuating part u',

$$u=U+u'\ . \tag{5.51}$$

The fluctuating velocity u' is assumed to be of first order, whereas the mean flow U is of second order. To the second order the averaged equation (5.50) becomes

$$\frac{\partial M}{\partial t}+\frac{\partial}{\partial x}\overline{\int_{-h}^{\eta}\left(\rho u'^2+p-2\mu\frac{\partial u'}{\partial x}-2\mu\frac{\partial U}{\partial x}\right)dz}-\overline{\frac{\partial h}{\partial x}p'_B}=0\,, \tag{5.52}$$

or

$$\frac{\partial M}{\partial t}+\frac{\partial}{\partial x}\int_{-h_0}^{0}\rho\overline{u'^2}dz+\frac{\partial}{\partial x}\overline{\int_{-h}^{\eta}\left(p-2\mu\frac{\partial u'}{\partial x}\right)dz}-2\mu\frac{\partial}{\partial x}\int_{-h_0}^{0}\frac{\partial U}{\partial x}dz-\overline{\frac{\partial h}{\partial x}p'_B}=0\ , \tag{5.53}$$

with M denoting the mean, horizontal momentum of the viscid fluid per unit surface area,

$$M = \overline{\int_{-h}^{\eta} \rho u dz}\,, \tag{5.54}$$

and h_0 is the equilibrium bottom level of the viscid fluid. In terms of the radiation stress the momentum equation may be expressed as

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left\{ -2\mu \int_{-h_0}^{0} \frac{\partial U}{\partial x} dz + S_{xx} \right\} = \overline{\frac{\partial h}{\partial x} p'_B}\,, \tag{5.55}$$

with the radiation stress S_{xx} given by

$$S_{xx} = \int_{-h_0}^{0} \rho \overline{u'^2} dz + \overline{\int_{-h}^{\eta} \left(p - 2\mu \frac{\partial u'}{\partial x} \right) dz}\,. \tag{5.56}$$

If the pressure term on the right-hand side of eq. (5.55) may be neglected, the momentum balance for steady state $(\partial / \partial t \equiv 0)$ becomes

$$\frac{\partial}{\partial x}\left\{ -2\mu \int_{-h_0}^{0} \frac{\partial U}{\partial x} dz + S_{xx} \right\} = 0\,, \tag{5.57}$$

which shows the possibility of the radiation stress being balanced by the viscous stress associated with the induced mean second-order flow.

This qualitative analysis is performed within an Eulerian picture, with U representing the mean *Eulerian* flow. As pointed out in the beginning of this thesis (cf "Introduction" and chapter I, section 3), the total mass flow may be considered as having two qualitatively different contributions, the (generalized) Stokes drift, which is a Lagrangian drift, and a mean Eulerian flow. The calculations of chapter IV for the mean, horizontal flow were performed within a Lagrangian picture, and the *total* mass flow was obtained. The results of chapter IV may therefore not be directly put into eq. (5.57). Anyway, this analysis suggests that applying the radiation stress concept may be fruitful also in that context.

4. Radiation stress versus wind stress

The importance of the radiation stress in case of damped waves may be indicated by comparing the radiation stress gradient to the wind stress. Many ocean surface phenomena, with or without surface material involved, have commonly been related to wind stress acting at the surface. Phenomena as piling-up of grease ice and drift of an oilslick are examples of this, and the wind has been considered as the main factor in the process of ice-packing. However, it is quite possible that the wave motion itself, through the radiation stress, may be just as important. We shall demonstrate qualitatively how these effects compare to each other, by estimating the ratio between the tangential surface wind stress and the radiation stress gradient.

The wind stress expresses the flux by which momentum is transferred from the wind velocity field into the ocean. The mean, horizontal wind stress τ_a due to a steady, horizontal wind is given by (e.g. Phillips [1977], p. 128)

$$\tau_a = \rho_a u_*^2, \tag{5.58}$$

where ρ_a is the density of the air and u_* is known as the friction velocity of the wind field. For gravity waves and a fully developed sea Phillips gives the relation [1977, p.161]

$$\frac{g^2\overline{\eta^2}}{u_*^4} \sim C\frac{Fg}{u_*^2} \tag{5.59}$$

between the mean square surface elevation $\overline{\eta^2}$ and the wind fetch F (=the distance over which the wind has acted upon the ocean surface). Eq. (5.59) is a semi-empiric formulae, in which the coefficient C has to be determined experimentally. It has been estimated to be of the order 10^{-4} [Phillips 1977, p. 162].

A rough estimate of the radiation stress is obtained from formulae (5.28), assuming deep-water gravity waves:

$$S_{xx} = \mathrm{v}_g E/\mathrm{v}_{ph} = \rho g\overline{\eta^2}/2\,. \tag{5.60}$$

For waves exponentially damped in space the gradient becomes

$$\frac{\partial S_{xx}}{\partial x} = -2\kappa S_{xx}, \tag{5.61}$$

κ being the spatial damping coefficient. Defining

$$\tau_w = \left|\frac{\partial S_{xx}}{\partial x}\right|, \tag{5.62}$$

an estimate for the ratio of the stresses τ_w and τ_a is obtained as

$$\frac{\tau_w}{\tau_a} \sim C \frac{\rho}{\rho_a} \kappa F \,. \tag{5.63}$$

Applying the experimental value 10^{-4} for C and 10^3 for (ρ/ρ_a) further gives

$$\frac{\tau_w}{\tau_a} \sim \frac{\kappa F}{10} \,. \tag{5.64}$$

Hence, the estimated ratio between the two stresses depends solely on the two parameters wind fetch F and damping coefficient κ. With a characteristic damping length $L = 1/\kappa$, we have

$$\frac{\tau_w}{\tau_a} \sim \frac{F}{10\,L} \,. \tag{5.65}$$

At this rough estimate the radiation stress gradient may exceed the wind stress if the fetch F exceeds the characteristic damping length L of the wave with more than a factor 10. Assuming $L \sim 10\,\lambda$, this means that for a 10-meter wave the fetch must be greater than 1 km, whereas for a 1-meter or a 100-meter wave the corresponding distances will be 100 m and 10 km, respectively. This indicates that in the open ocean where the fetch may take on great values and a fully developed sea commonly exists, the radiation stress gradient indeed has the potential of exceeding the wind stress.

The wind stress is known as being capable of "pulling together" grease ice and also of packing ice. The above estimate indicates the importance of the radiation stress in such a process. Since the radiation stress even may exceed the wind stress, it is reason to believe that the wave motion itself may have considerable influence upon the ice concentration and should not be neglected in this context.

J.E. Weber made a similar conclusion, in his study [1987] of wave-induced drift induced by gravity waves in a viscid ocean propagating through a thin layer of highly viscid brash-like ice. He determined the mean, viscous stress exerted on the ice, also based upon a qualitative analysis, showing that this stress might be comparable to the stress induced by moderate winds.

However, both observations and more rigorous analysis are needed to obtain more reliable results.

CHAPTER VI

SUMMARY, DISCUSSION AND CONCLUSIONS

In this final chapter we summarize and discuss our results. We also try to draw some general conclusions.

1. Aim of study

In this study focus has been on ocean surface waves propagating into a wave damping area. As pointed out in the introductory chapter, obtaining better understanding of the dynamic processes involved is of great importance. Through the years, several authors have studied the topic, approaching it from different points of view. The complexity of the field makes a complete theory difficult to obtain, however, and progress still has to be made by studying special cases.

The aim of our study was to contribute to the understanding of the dynamic effects *induced* by the decaying surface waves. This was to be achieved through a theoretical investigation of two selected cases: waves propagating into an oilfilm-covered area and waves propagating into viscid grease ice. In both cases we shall have a decaying wave field, implying loss of wave momentum. Since momentum cannot vanish, the lost wave momentum must have been transferred to the surroundings. How this transfer will manifest itself, depends upon the actual dynamic case. Assuming the manifestations in our cases to be a mean wave induced stress in the film and a mean wave induced flow in the grease ice, we aimed at theoretically determining those.

2. Summary of the results

A mathematical model, based upon Lagrangian description of *fluid particle* motion and perturbation theory, had been presented as convenient for this kind of problem by several authors [Pierson 1962, Chang 1969, Weber 1983 a & b]. We developed the model further and obtained, by extended use of Jacobian determinants, a very compact and symmetrical set of formal equations. In our form the equations became easier to handle, and consequently errors were easier to avoid. This was a considerable advantage in the subsequent work. Before reaching this compact form, the expressions to be operated were rather lengthy. The advantage of keeping the Jacoby determinant notation in the expressions has not been reported earlier, to this author's knowledge. Under the only constraint of first-order functions being periodic in time, several formal calculations were possible to perform. This was a great advantage. In the time-averaged second-order equations this made further simplifications possible. The resulting equations became quite easy to interpret.

In Eulerian fluid description the concept of stream function and stream lines is well established as a convenient analytical tool. Our investigations demonstrated that introduction of a stream function was just as fruitful in our Lagrangian fluid description, regarding the consequences for the calculations as well as interpretation of the solutions.

Based upon the formal model the difference between Lagrangian and Eulerian drift was analysed. It was demonstrated that the Lagrangian drift may be interpreted as the sum of a generalized Stokes drift and the Eulerian drift, in agreement with, for instance, Phillips' discussion [1977, p. 43].

A thourough discussion was performed on the validity of our solution in the long run. This discussion was important and necessary, since the perturbation solution – by assumption – will be valid only for small deviations from the initial state whereas a fluid particle in the long run may drift far out of this range. Our analysis showed that in case of steady state, the results will still be valid.

The first-order Lagrangian equations were solved by assuming a plane-wave-type of solution, as is also the common procedure having Eulerian equations. Inspection of the obtained formal solution showed that the *first-order* Lagrangian solution may be interpreted as the corresponding Eulerian solution, despite the different approach. This implies that the parameters amplitude, wave number and orbital frequency in the Lagrangian solution may be associated with the corresponding observable features of the Eulerian solution, and may be interpreted as representing those.

The developed general model was applied to the two selected special cases. In both cases mean, steady state and spatially damped waves being periodic in time were assumed, and no external forces except gravity were allowed.

The oilfilm case was modelled as an elastic, surface-active, insoluble and monomolecular surface film covering a deep, viscid ocean. Our model reproduced the first-order results obtained by other authors: the damping rate determined by Dorrestein [1951], the dispersion relation of Dysthe and Rabin [1986], the wave solution of Weber and Sætra [1995] in case of damping in time. This was to be expected, as the first-order model did not introduce any new aspects.

The mean, second-order, tangential wave stress was determined analytically for spatial damping. The stress had two parts. The first part was interpreted as the "virtual wave stress" introduced by Longuet-Higgins [1969]. It agreed with Longuet-Higgins' result in the no film-case, with Weber and Førland [1989] for inelastic film and with Weber and Sætra [1995] for elastic film. The other part was interpreted as the fluid stress associated with the (generalized) Stokes drift. In the inelastic case the virtual wave stress was the far dominating part. It was less dominating for intermediate elasticity, and in the no film-case the two parts were equal, again reproducing the result of Longuet-Higgins [1969]. The stress showed similar dependence of the surfactant compressibility as the damping rate did, with maximum for intermediate compressibility.

In the grease ice case a layer, the order of a wavelength thick, of highly viscid fluid was assumed to cover a deep, inviscid ocean. No mixing was allowed between the two fluids. Gravity waves were assumed, neglecting both surface tension and tension in the interface between upper and lower fluid. The dispersion relation for the system was determined. In the limit of one single viscid fluid it was found to agreee with the dispersion relation given by Chandrasekhar [1961] (with surface tension neglected), and we also regained the "usual" damping rate for temporal damping. Jenkins and Jacobs have more recently [1997] discussed wave damping by a linear two-layer model for two viscid fluids. Their work is parallel to the linear part of our work if viscosity of lower fluid approaches zero, and in that limit the results agree well.

Under the assumption of no mean, tangential, second-order stress at the interface and creeping motion, we analytically determined the mean, second-order, horizontal flow of the upper layer. For typical parameter values it was seen to give a forward drift at the surface and a return flow at the bottom of the viscid layer. Qualitatively, this is in agreement with the experimental results obtained by Martin and Kauffman [1981].

The expression for the mean flow at the surface was reduced to the well-known Stokes drift if we let the damping rate approach zero and assumed the viscid layer to be several wavelengths thick.

The two special cases were finally studied within the "radiation stress concept" of Longuet-Higgins and Stewart [1960-64]. To a good approximation the radiation stress gradient agreed with the stream function contribution or virtual wave stress determined through the thorough analysis of chapter III. The mean flow determined in chapter IV could not be reproduced in a similar way, but the analysis demonstrated for both cases the importance of the radiation stress in the overall dynamic picture. It was also demonstrated that the radiation stress under certain conditions may be a dynamic factor just as important as the more commonly considered wind stress.

3. Discussion

The model developed in chapter II naturally has its limitations. An important limitation is that it presupposes only small deviations from the equilibrium positions of the fluid particles. Further, the second-order equations are developed under the basic assumption of the first-order functions being periodic in time. If this assumption fails to be fulfilled, many of the manipulations performed upon the equations will not be valid. When applying the model and our results, these limitations must be kept in mind.

In the cases we have studied in this work, mean, steady state has been assumed. It may, however, be questioned if this is possible to obtain in a real case. The rate of mean wave momentum loss must then necessarily be compensated. Compensation is a priori possible through a mean stress applied at the boundaries and through induced mean (bulk) flow. We believe that in the film case a mean, elastic stress will be induced within the film by the wave motion, exactly compensating the rate of mean wave momentum loss so that mean, steady state may be achieved. And in the grease ice case we believe that the wave momentum loss may be compensated by an induced, second-order, mean flow. The total mean momentum of the fluid will then be constant, and mean, steady state may be established.

Our analysis is carried out to the second order in the - assumed small - parameter ε, being valid only for relatively small displacements from the reference position of the fluid particle. As far as *steady,* mean drift is studied, no interpretation problems should arise (cf ch. II, sect. 7). It will be possible to consider the second-order, mean, Lagrangian flow from an Eulerian point of view, and by this be able to get a picture of the mean, steady flow.

The rather simple interpretation possible in the mean, steady case is based upon the following: Since we have mean, steady state, the mean flow pattern will be constant in time. Therefore, when a fluid particle has drifted (in mean) as far as the perturbation analysis allows to still be valid, we simply may find its further path by considering another particle for which this area is within the "allowed" distance from the reference position of *that* particle. Since mean, steady state is assumed, the paths must be identical.

For drift attenuated in time, however, this is not possible. After some time the fluid particle will have drifted too far from its reference position for the second-order approximation to be valid. The mean second-order function cannot any longer be trusted as a good approximation. Since we do not have steady state, we cannot simply "switch" to another particle to obtain an appropriate approximation. This aspect has not been given much attention by other authors applying this approach.

Weber and Førland have studied *attenuated,* mean drift [Weber 1983 b, Weber and Førland 1989, Førland 1989]. The hodographs they present show how the mean, second-order, horizontal drift developes in time. In the work of Weber and Sætra [1995] a similar analysis is performed. However, in the light of the comment above, one should be careful when interpreting those results. If the fluid particle has drifted far from its reference position it is not obvious that the obtained drift still actually is the mean flow of the fluid. I consider this problem as not trivial and believe that it should be studied further.

A basic assumption for the second-order equations is "creeping motion". By "creeping motion" is commonly meant friction-dominated flow, in which the mean acceleration of a fluid particle should be negligible compared to the viscous forces exposed to it, and the motion essentially results from a balance between pressure (stress) and friction. This may not be fulfilled in general. But with the additional constraint of steady state there cannot be any mean acceleration, and the assumption of creeping motion should be appropriate. And in the very viscid grease ice it should be a reasonable assumption that the viscid forces will dominate. Hence, this limitation should be acceptable and eq. (2.48) be valid.

In chapter III (the film case) we made some assumptions which validity ought to be commented on. The film was assumed insoluble. This will normally be appropriate for oil film. However, for fluids of which solubility has to be taken into account, our results must be taken only as preliminary.

The assumption of a monomolecular surface layer is also a restriction. Calculations by Jenkins and Jacobs [1997] indicate that the effect of surface elasticity generally dominates the wave damping, but that a finite surface layer causes significant changes in the damping when the thickness of the layer exceeds 100 μm. By combining their work with ours, more general results should be obtainable.

We assumed the film elasticity to be dominating compared to film viscosity. In a recent work [Mass and Milgram, 1998] this is studied in detail. The results show that the dynamic behavior of the film is determined by the *static* elasticity, and that the surface viscosity safely may be neglected. This supports that our simplification is a good approximation in most cases.

The mean, tangential fluid stress at the surface was expressed in terms of the mean, second-order stream function and products of the first-order functions. Considering the partial differential equation for the stream function as a driven equation, the particular solution may be interpreted as a direct result of the driving force; that is, a direct result of the wave motion. It is our belief that when applying this particular solution for the stream function when determining the mean, tangential fluid stress, neglecting an additional, homogeneous solution possibly needed to fulfil the actual boundary conditions, we obtained the *wave induced* fluid stress at the surface. Although obtained through a somewhat different approach, our results proved to be in good agreement with results of other authors, something we consider as a strong support to the validity of our results and considerations.

The virtual wave stress was obtained directly from the particular solution for the mean, second-order stream function. This shows the advantage of applying the stream function concept. It should be noted that whereas the virtual wave stress was reduced by an order of magnitude when going from inelastic film to no film, we had the opposite situation for the Stokes drift contribution, the two contributions being of equal importance in the no film-case.

By assuming mean, steady state we have implicitly assumed that the film somehow was kept in place. Any additional boundary conditions associated with this is not believed to affect our solution, since only the *particular* solution for the mean stream function was considered. It should be noticed that the integrand was independent of time, therefore also the particular solution would be so. Then there was no need of including any acceleration term in the equation, and the a priori assumption of "creeping motion" was redundant here.

Weber and Sætra [1995] have studied the elastic film case under somewhat different assumptions. They assume temporally damped waves, in which case steady state is not possible. Further, they allow the film to be freely moving and determine the (non-zero) horizontal surface (or film) drift. Weber and Sætra found the vertical gradient of the mean, horizontal drift to always equal zero, whatever compressibility. This result was considered to be reasonable, as a freely moving, elastic surface should not be able to – in an average sense – impose a net horizontal stress on the underlying fluid when there was no mean stress above it. In our case spatially damped waves and mean, steady state were assumed. The film was assumed to be kept in place (no mean horizontal film drift). In such a case the wave motion will be associated with a mean, non-zero, horizontal stress in the fluid. It is our belief that the total fluid stress is supported at the surface by an induced mean, elastic stress within the film, and that – as long as the fluid stress is of moderate strength – these stresses balance each other, making mean, steady state possible. If the fluid stress is greater than the maximum stress possible in the film, the film will start drifting.

The result of chapter III may be applied to determine the maximum film stress experimentally. By performing an experiment in which the wave parameters are varied, observing when the film no longer is able to "resist" the fluid stress, the film stress may be computed through the formulae of chapter III. It should be possible to exploit this to explore the film properties.

In the viscid fluid layer case (ch. IV) we seek a complete solution, assuming the viscid stress to be totally compensated by induced flow. It is important to note that we neglect any tension in the surface as well as in the interface between the two fluids. The approximation implies that the mean, tangential viscid stress (to second order) vanishes both in the surface and in the interface between the grease ice layer and the clean ocean below. This approximation will normally be appropriate for gravity waves. For wavelengths approaching the capillary-gravity area, however, the approximation will be less valid. If the viscid layer thickness exceeds a quarter of a wavelength, most of the wave energy will be within this layer. For thicknesses of the order of a wavelength as we assume, we believe that the most important dynamics is likely to take place within the grease ice layer, and that neglecting the interface tension and stress makes little difference. For thin layers, however, this may be of importance. Jenkins and Dysthe [1997] have looked closer into that case. It might be possible to estimate the size of the error introduced by our approximation by means of that work.

The dispersion relation from the first-order solution was seen to have two possible solutions as long as we allowed the density of the viscid layer to differ from the clean ocean density. One of these solutions represented a wave mode of very low frequency compared to the ordinary gravity wave and was identified as an internal wave. The existence of an internal wave was to be expected: Internal waves are commonly found in the ocean when variations in temperature and salinity produce density gradients, and layers of different density could therefore a priori be supposed to cause internal waves. When we in the subsequent analysis assumed the two densities to be equal, the effect was probably just to loose track of the internal wave. In our context that would be of minor importance.

Førland and Weber [1989] studied the effect of air on the drift velocity. This may have some resemblance to our case of grease ice/water, although we let the upper fluid be the more viscid and restrict it to only a wavelength's thickness. In contrast to our approach, Førland and Weber consider their case as a variant of the film case.

In 1995 Piedra-Cueva published a paper [Piedra-Cueva, 1995] in which the drift velocity of spatially decaying waves in a two-layer viscid system is studied. Piedra-Cueva was specially interested in water-mud systems and studied in particular the influence of viscosity of lower layer on the drift velocity. In that matter his aspect is almost the opposite of ours, as we completely neglected viscosity in lower layer. The general model is, however, equal to ours, and Piedra-Cueva is also assuming steady state. Piedra-Cueva looks at two different physical situations: waves in closed channels, in which it is appropriate to impose a zero net mass flux condition; and waves in an unbounded domain, where the steady-state mass flux and the set-up of both the free surface and upper-lower layer interface may be non-zero. A rigid bottom is assumed. The second-order equations are in exact agreement with ours.

We concentrated upon the mean flow in the upper, viscid grease ice layer. Grease ice and ocean water have similar densities, and our final analysis did assume them to be equal. The basic results will probably also apply for the more accurate case of *nearly* equal densities. The results may contribute to explain the experimental results of Martin and Kauffman [1981]. The mean, second-order, horizontal motion in the viscid layer was found to have opposite direction at the top and bottom of the layer. This result indicates that in the case of limited horizontal dimensions we would have obtained a mean circulation of fluid particles within the viscid layer. This would be in agreement of what Martin and Kauffman [1981] have reported from their laboratory experiment.

In chapter V we analysed the two special cases within the "radiation stress concept". The general analysis for the film case showed that the radiation stress gradient should balance the mean (induced) film stress. The mean, tangential fluid stress in chapter III was also supposed to balance this film stress, accordingly it should agree with the radiation stress gradient. However, the analysis in chapter V is performed within an Eulerian approach, whereas the analysis in chapter III is Lagrangian. This complicates a direct comparison of the results. The stress determined in chapter III was seen to have two parts, the virtual wave stress and the Stokes drift contribution. The radiation stress gradient determined in chapter V agreed with the virtual wave stress. It is tempting to explain this as a result of the different approaches, parallel to the mean, Lagrangian drift being the sum of the mean, Eulerian drift and the Stokes drift. If this is correct, the results of chapter III and V agree with each other.

In our opinion the analysis in the grease ice case, despite less rigid, demonstrated quite convincingly that the radiation stress is a dynamic factor that should not be neglected, especially not in non-linear problems.

Finally, the restriction throughout this thesis to one single wave component should be commented on. We believe that extension to a real wave field, which in general will contain a large spectrum of wave components, in principle will be quite straightforward, but statistical treatment would be needed.

4. Conclusions

In this thesis we have studied selected parts of the upper ocean dynamics. Most of the analysis was performed within Lagrangian fluid description. Through perturbation analysis, assuming the first-order wave solution to be periodic in time, we obtained a very compact and symmetrical form of the basic formal equations for the time-averaged second-order problem, not reported earlier.

The model was applied to two – qualitatively different – cases of surface waves travelling into an area of heavy damping. The damping was caused by an elastic surface film and by a surface layer of very viscid fluid. Under the assumption of mean, steady state we determined the mean, tangential surface stress of the fluid and the mean, horizontal flow within the viscid layer, respectively.

The model proved to be very well suited for these problems and allowed for analytical solutions. Also it managed to reproduce results obtained by other authors, which supports its reliability. We believe that the result of the surface film case may be exploited in exploring the dynamics of surface films. In the viscid layer case we have theoretically reproduced the experimental results of Martin and Kauffman [1981]. The two cases were also briefly analysed within the radiation stress concept. It was seen – in a qualitative manner – that the radiation stress is a factor which should not be neglected. The analysis supported the claim of this concept being a very convenient tool in overall non-linear dynamic studies.

By this thesis some additional insight into the field of decaying ocean surface waves hopefully has been achieved. We feel that the aim of our study is reasonably well obtained, and hope our work may contribute a little bit to the further progress of the huge and complex field of upper ocean dynamics.

APPENDIX A

Appendix to chapter IV
Calculation of the stream function $\overline{\psi^{(2)}}$ (eq. 4.58)

For convenience we shall express eq. (4.54) in the form

$$\overline{\psi^{(2)}_{part}} \equiv e^{-2\kappa a} l(c) - e^{-2\kappa a}\,\mathrm{Re}\left\{ C_1 \frac{ic}{4\kappa} e^{2i\kappa c} \right\} + \frac{1}{6} K_1 c^3 + \frac{1}{2} K_2 c^2 , \tag{A.1}$$

with the function *l(c)* denoting the part of $\overline{\psi^{(2)}_{part}}$ originating from H_{part}, except for the factor *exp(-2κa)*, that is

$$l(c)e^{-2\kappa a} \equiv \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\left\{ l_1 \tilde{\psi}\tilde{\psi}_c^* + l_2 \tilde{\psi}_c \tilde{\psi}^* + l_3 \varphi \tilde{\psi}^* + l_4 \varphi_c \tilde{\psi}_c^* + l_5 \left[|A|^2 e^{2k_0 c} - |B|^2 e^{-2k_0 c} \right] e^{-2\kappa a} \right\}, \tag{A.2}$$

where

$$l_1 \equiv \frac{\left[\left(4\kappa^2 + m^2 + m^{*2}\right)^2 + 4m^2 m^{*2} \right]\alpha_1}{\left[4\kappa^2 + \left(m + m^*\right)^2\right]^2 \left[4\kappa^2 + \left(m - m^*\right)^2\right]^2} \tag{A.3}$$

$$l_2 \equiv -\frac{4m^{*2}\left(4\kappa^2 + m^2 + m^{*2}\right)\alpha_1}{\left[4\kappa^2 + \left(m + m^*\right)^2\right]^2 \left[4\kappa^2 + \left(m - m^*\right)^2\right]^2} \tag{A.4}$$

$$l_3 \equiv -\frac{\left[\left(4\kappa^2 + k^2 + m^{*2}\right)^2 \alpha_2 + 4\left(km^*\right)^2 \alpha_2 - 4\left(km^*\right)^2 \left(4\kappa^2 + k^2 + m^{*2}\right)\xi\alpha_3 \right]}{\left[4\kappa^2 + \left(k + m^*\right)^2\right]^2 \left[4\kappa^2 + \left(k - m^*\right)^2\right]^2} \tag{A.5}$$

$$l_4 \equiv \frac{\left[4\left(4\kappa^2 + k^2 + m^{*2}\right)\alpha_2 - 4\left(km^*\right)^2 \xi\alpha_3 - \left(4\kappa^2 + k^2 + m^{*2}\right)^2 \xi\alpha_3 \right]}{\left[4\kappa^2 + \left(k + m^*\right)^2\right]^2 \left[4\kappa^2 + \left(k - m^*\right)^2\right]^2} \tag{A.6}$$

$$l_5 \equiv \frac{2k^{*3}\xi^2}{\left[4\kappa^2 + \left(k + k^*\right)^2\right]^2} \quad . \tag{A.7}$$

The first and second derivatives of *l(c)* then become

$$l'(c)e^{-2\kappa a} = \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\Big\{l_1\left(\tilde{\psi}_c\tilde{\psi}_c^* + m^{*2}\tilde{\psi}\tilde{\psi}^*\right) + l_2\left(m^2\tilde{\psi}\tilde{\psi}^* + \tilde{\psi}_c\tilde{\psi}_c^{\ *}\right) + l_3\left(\varphi_c\tilde{\psi}^* + \varphi\tilde{\psi}_c^{\ *}\right)$$

$$+ l_4\left(k^2\varphi\tilde{\psi}_c^* + m^{*2}\varphi_c\tilde{\psi}^*\right) + 2k_0 l_5\left[|A|^2 e^{2k_0c} + |B|^2 e^{-2k_0c}\right]e^{-2\kappa a}\Big\} \tag{A.8}$$

and

$$l''(c)e^{-2\kappa a} = \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\Big\{l_1\left[(m^2 + m^{*2})\tilde{\psi}\tilde{\psi}_c^{\ *} + 2m^{*2}\tilde{\psi}_c\tilde{\psi}^*\right] + l_2\left[\left(m^2 + m^{*2}\right)\tilde{\psi}_c\tilde{\psi}^* + 2m^2\tilde{\psi}\tilde{\psi}_c^{\ *}\right]$$

$$+ l_3\left[\left(k^2 + m^{*2}\right)\varphi\tilde{\psi}^* + 2\varphi_c\tilde{\psi}_c^{\ *}\right] + l_4\left[\left(k^2 + m^{*2}\right)\varphi_c\tilde{\psi}_c^* + 2k^2 m^{*2}\varphi\tilde{\psi}^*\right] \tag{A.9}$$

$$+ 4k_0^{\ 2} l_5\left[|A|^2 e^{2k_0c} - |B|^2 e^{-2k_0c}\right]e^{-2\kappa a}\Big\}.$$

Inserting the expressions for m, ξ, α_1, α_2 and α_3 from eqs. (2.37), (4.15) and (4.48) into eqs. (A.3)-(A.7), we obtain – to leading order in the parameters $\xi_0 \equiv \frac{2\nu k_0^2}{\omega}$ and $\frac{\kappa}{k_0}$ –

$$l_1 \approx \frac{\xi_0^2\alpha_1}{16k_0^4} \approx \frac{3\xi_0^2}{16k_0^2}\left[1 + 2i\,(\tfrac{\kappa}{k_0}) - \tfrac{2}{3}i\xi_0\right] \tag{A.10}$$

$$l_2 \approx -\frac{\xi_0^3(\xi_0 + 2i)\alpha_1}{32k_0^4} \approx -\frac{3\xi_0^3}{32k_0^2}\left[\tfrac{7}{3}\xi_0 - 4\tfrac{\kappa}{k_0} + 2i\right] \tag{A.11}$$

$$l_3 \approx \frac{\xi_0^2}{4k_0^4}\left[(1 - 4i\xi_0)\alpha_2 - 2k_0^2\xi_0\left(2 - i(3\xi_0 - 8\tfrac{\kappa}{k_0})\right)\alpha_3\right] \approx \frac{\xi_0^2}{4k_0}\left[\xi_0 - 6(\tfrac{\kappa}{k_0}) + 2i\right] \tag{A.12}$$

$$l_4 \approx \frac{\xi_0^3}{4k_0^6}\left[2(\xi_0 + i)\alpha_2 + k_0^2\left(1 + 2i(\tfrac{\kappa}{k_0}) - 4i\xi_0\right)\alpha_3\right] \approx -\frac{\xi_0^3}{4k_0^3}\left[1 + 7i(\tfrac{\kappa}{k_0})\right] \tag{A.13}$$

$$l_5 \approx \frac{\xi_0^2}{8k_0}\left[1 - i(\tfrac{\kappa}{k_0})\right]. \tag{A.14}$$

l_1, l_3 and l_5 are all seen to be of order ξ_0^2, whereas l_2 and l_4 are of order ξ_0^3. Since *m/k* is of order $\xi_0^{-1/2}$, φ of order 1, $\tilde{\psi}$ of order ξ_0, φ_c of order $k\varphi$ and $\tilde{\psi}_c$ of order $m\tilde{\psi}$, we have – still to the leading order in the parameters ξ_0 and $\dfrac{\kappa}{k_0}$ –

$$l(c) \approx \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\left\{l_5|A|^2 e^{2k_0c}\right\}, \tag{A.15}$$

$$l'(c)e^{-2\kappa a} \approx \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\left\{l_3\varphi\tilde{\psi}_c^* + 2k_0 l_5|A|^2 e^{2k_0c-2\kappa a}\right\} \tag{A.16}$$

and

$$l''(c)e^{-2\kappa a} \approx \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\left\{2l_1 m^{*2}\tilde{\psi}_c\tilde{\psi}^* + l_3\left(m^{*2}\varphi\tilde{\psi}^* + 2\varphi_c\tilde{\psi}_c^*\right)\right.$$
$$\left. + l_4 m^{*2}\varphi_c\tilde{\psi}_c^* + 4k_0^2 l_5|A|^2 e^{2k_0c-2\kappa a}\right\}. \tag{A.17}$$

By inserting the approximate expressions for the parameters l_1-l_5 and the functions φ and ψ we obtain

$$l(c) \approx \frac{\xi_0}{4\nu}|A|^2 e^{2k_0c}, \tag{A.18}$$

$$l'(c) \approx \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\left\{l_3 m^*\left[AC^* e^{(k+m^*)c} - AD^* e^{(k-m^*)c}\right] + 2k_0 l_5|A|^2 e^{2k_0c}\right\}$$

$$\approx \frac{k_0\xi_0}{2\nu}|A|^2 e^{2k_0c} \times \tag{A.19}$$

$$\times\left\{1 - 2\xi_0^{1/2}\left[e^{(m_r-k_0)c}\left(\cos m_i c - \sin m_i c\right) - e^{-(m_r+k_0)(h+c)}\left(\cos m_i(h+c) + \sin m_i(h+c)\right)\right]\right\}$$

and

$$l''(c) \approx \frac{2}{\nu}\mathrm{Re}\frac{k}{\xi}\Big\{2l_1 m^{*2} m\Big[|C|^2 e^{2m_r c} - |D|^2 e^{-2m_r c} - \Big(CD^* e^{-2im_i c} - C^* D e^{2im_i c}\Big)\Big]$$

$$+ l_3\Big[m^{*2}\Big(AC^* e^{(k+m^*)c} + AD^* e^{(k-m^*)c}\Big) + 2km^*\Big(AC^* e^{(k+m^*)c} - AD^* e^{(k-m^*)c}\Big)\Big]$$

$$+ l_4 m^{*2} km^*\Big[AC^* e^{(k+m^*)c} - AD^* e^{(k-m^*)c}\Big] + 4k_0^2 l_5 |A|^2 e^{2k_0 c}\Big\} \qquad \text{(A.20)}$$

$$\approx \frac{k_0^2 \xi_0}{\nu}|A|^2 e^{2k_0 c} \times$$

$$\times\Big\{1 - 2\Big[e^{(m_r-k_0)c}\cos m_i c + e^{-(m_r+k_0)(h+c)}\cos m_i(h+c)\Big]$$
$$+ \xi_0^{1/2}\Big[\tfrac{3}{2}e^{2(m_r-k_0)c} - \tfrac{3}{2}e^{-2(m_r+k_0)(h+c)} - 3e^{-m_r h - k_0(h+2c)}\sin m_i(h+2c)$$
$$+ e^{(m_r-k_0)c}\big(\sin m_i c - \cos m_i c\big) + e^{-(m_r+k_0)(h+c)}\big(\cos m_i(h+c) - 3\sin m_i(h+c)\big)\Big]\Big\}.$$

It is understood that the parameters m_r and m_i are the real and imaginary part of the parameter m, respectively. (In these expressions also terms of relative order $e^{-m_r h}$ are neglected, and we have assumed κh<<1.)

In the limits *c=0* and *c=-h* the expressions (A.18)-(A-20) become

$$l(0) \approx \frac{\xi_0 |A|^2}{4\nu}, \qquad \text{(A.21)}$$

$$l(-h) \approx \frac{\xi_0 |A|^2}{4\nu} e^{-2k_0 h}, \qquad \text{(A.22)}$$

$$l'(0) \approx \frac{k_0 \xi_0}{2\nu}|A|^2\Big\{1 - 2\xi_0^{1/2}\Big[1 - e^{-(m_r+k_0)h}\big(\cos m_i h + \sin m_i h\big)\Big]\Big\}$$
$$\approx \frac{k_0 \xi_0}{2\nu}|A|^2\big(1 - 2\xi_0^{1/2}\big), \qquad \text{(A.23)}$$

$$l'(-h) \approx \frac{k_0 \xi_0}{2\nu}|A|^2 e^{-2k_0 h}\Big\{1 + 2\xi_0^{1/2}\Big[1 - e^{-(m_r-k_0)h}\big(\cos m_i h + \sin m_i h\big)\Big]\Big\}$$
$$\approx \frac{k_0 \xi_0}{2\nu}|A|^2 e^{-2k_0 h}\big(1 + 2\xi_0^{1/2}\big), \qquad \text{(A.24)}$$

$$l''(0) \approx -\frac{\xi_0 k_0^2 |A|^2}{\nu}\left\{1 + 2e^{-(m_r+k_0)h}\cos m_i h - \xi^{1/2}\left[\tfrac{1}{2} - e^{-(m_r+k_0)h}\left(6\sin m_i h - \cos m_i h\right)\right]\right\}$$
$$\approx -\frac{\xi_0 k_0^2 |A|^2}{\nu}\left[1 - \tfrac{1}{2}\xi_0^{1/2}\right] \qquad \text{(A.25)}$$

and

$$l''(-h) \approx -\frac{\xi_0 k_0^2 |A|^2}{\nu} e^{-2k_0 h}\left\{1 + 2e^{-(m_r-k_0)h}\cos m_i h + \xi_0^{1/2}\left[\tfrac{1}{2} - e^{-(m_r-k_0)h}\left(2\sin m_i h - \cos m_i h\right)\right]\right\}$$
$$\approx -\frac{\xi_0 k_0^2 |A|^2}{\nu} e^{-2k_0 h}\left(1 + \tfrac{1}{2}\xi_0^{1/2}\right). \qquad \text{(A.26)}$$

It should be noted that in *l"(c)* the terms involving the rotational function $\tilde{\psi}$ contributes to equal order as do the terms involving only the "potential flow" function φ, causing a negative value (eqs. (A.25)-(A.26)).

In terms of *l(c)* the mean, second-order stream function $\overline{\psi^{(2)}}$ may be written (from eq. (4.53), also inserting eqs. (4.55) and (A.1))

$$\overline{\psi^{(2)}} = e^{-2\kappa a}\left\{l(c) - \mathrm{Re}\left[C_1 \frac{ic}{4\kappa} e^{2i\kappa c} - C_2 e^{2i\kappa c}\right]\right\} + \frac{1}{6}K_1 c^3 + \frac{1}{2}K_2 c^2 + K_3 c$$

$$= e^{-2\kappa a}\left\{l(c) + \frac{c}{4\kappa}\mathrm{Re}\,C_1 \sin 2\kappa c + \mathrm{Re}\,C_2 \cos 2\kappa c\right. \qquad \text{(A.27)}$$

$$\left. + \frac{c}{4\kappa}\mathrm{Im}\,C_1 \cos 2\kappa c - \mathrm{Im}\,C_2 \sin 2\kappa c\right\} + \frac{1}{6}K_1 c^3 + \frac{1}{2}K_2 c^2 + K_3 c\,,$$

and so the boundary conditions in eqs. (4.56) become equivalent to the conditions

$$\left.\begin{aligned}
&-2\kappa e^{-2\kappa a}\left\{\, l(0)+\operatorname{Re}C_2\right\}=0\\
&-2\kappa e^{-2\kappa a}\left\{l(-h)+\operatorname{Re}\left[\left(C_2+\frac{ih}{4\kappa}C_1\right)e^{-2i\kappa h}\right]\right\}=0\\
&e^{-2\kappa a}\left\{\,4\kappa^2 l(0)+8\kappa^2\operatorname{Re}C_2-l''(0)-\operatorname{Re}C_1\right\}-K_2=0\\
&e^{-2\kappa a}\left\{\,4\kappa^2 l(-h)-l''(-h)-\operatorname{Re}\left[\,C_1(1-2i\kappa h)\,e^{-2i\kappa h}\right]+8\kappa^2\operatorname{Re}\left[\,C_2 e^{-2i\kappa h}\right]\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+K_1 h-K_2=0\,.
\end{aligned}\right\}\quad\text{(A.28)}$$

Since these equations should be valid for all $a>0$, both K_1 and K_2 must equal zero. Hence – the real and imaginary parts of the complex constants C_1 and C_2 become

$$\left.\begin{aligned}
&\operatorname{Re}C_1=-\left[\,4\kappa^2 l(0)+l''(0)\right]\\
&\operatorname{Re}C_2=-\,l(0)\\
&\operatorname{Im}C_1=\frac{1}{\sin 2\kappa h}\left\{\left[4\kappa^2 l(0)+l''(0)\right]\cos 2\kappa h-\left[4\kappa^2 l(-h)+l''(-h)\right]\right\}\\
&\operatorname{Im}C_2=\frac{h}{4\kappa\sin^2 2\kappa h}\left\{\left[4\kappa^2 l(0)+l''(0)\right]-\left[4\kappa^2 l(-h)+l''(-h)\right]\cos 2\kappa h\right.\\
&\qquad\qquad\qquad\left.-\frac{4\kappa}{h}l(-h)\sin 2\kappa h+\frac{2\kappa}{h}l(0)\sin 4\kappa h\right\}.
\end{aligned}\right\}\quad\text{(A.29)}$$

By means of the equations (A.29) and the approximations

$$\left.\begin{aligned}
&\sin 2\kappa c\approx 2\kappa c\\
&\cos 2\kappa c\approx 1-2(\kappa c)^2
\end{aligned}\right\}\quad\text{(A.30)}$$

we may – to the leading order – express the mean, second-order stream function $\overline{\psi^{(2)}}$ in terms of the helping function $l(c)$ as

$$
\begin{aligned}
\overline{\psi^{(2)}} &\approx e^{-2\kappa a}\Big\{ l(c) - l(0)\Big[1 + \tfrac{c}{h} + \tfrac{\kappa^2 c}{h}\big(h^2 + c^2\big)\Big] - \tfrac{c}{4h}(h+c)^2\, l''(0) \\
&\quad + \tfrac{c}{h}\Big[1 + \kappa^2\big(c^2 - h^2\big)\Big] l(-h) + \tfrac{c}{4h}\big(c^2 - h^2\big)\, l''(-h)\Big\} \\
&\approx e^{-2\kappa a}\Big\{ l(c) - \big(1 + \tfrac{c}{h}\big) l(0) - \tfrac{c}{4h}(h+c)^2\, l''(0) \\
&\quad + \tfrac{c}{h} l(-h) + \tfrac{c}{4h}\big(c^2 - h^2\big)\, l''(-h)\Big\}.
\end{aligned}
\tag{A.31}
$$

To leading order the constants in (A.29) become – from eqs. (A.21)-(A.26), also applying the assumption $\kappa h<<1$ –

$$
\left.
\begin{aligned}
&\mathrm{Re}\, C_1 \approx -l''(0) \approx \frac{\xi_0 k_0^2 |A|^2}{\nu}\Big[1 - \tfrac{1}{2}\xi_0^{1/2}\Big] \\
&\mathrm{Re}\, C_2 = -l(0) \approx -\frac{\xi_0 |A|^2}{4\nu} \\
&\mathrm{Im}\, C_1 \approx \frac{1}{2\kappa h}\big[l''(0) - l''(-h)\big] \approx -\frac{\xi_0 k_0^2 |A|^2}{2\nu\kappa h}\left[1 - e^{-2k_0 h} - \frac{1}{2}\big(1 + e^{-2k_0 h}\big)\xi_0^{1/2}\right] \\
&\mathrm{Im}\, C_2 \approx \frac{1}{16\kappa^3 h}\big[l''(0) - l''(-h)\big] \\
&\qquad \approx -\frac{\xi_0 k_0^2 |A|^2}{16\nu\kappa^3 h}\left[1 - e^{-2k_0 h} - \frac{1}{2}\big(1 + e^{-2k_0 h}\big)\xi_0^{1/2}\right].
\end{aligned}
\right\}
\tag{A.32}
$$

Written out – by inserting the expressions (A.21)-(A.22) and (A.25)-(A.26) for *l* and *l''* at *c=0* and *c=-h* – we obtain eq. (4.58). (To obtain this expression it has also been assumed that $exp(-m_r h)$ is much smaller than one.) The function (4.58) fulfils the claim $\overline{\psi_a^{(2)}} = 0$ at the boundaries *c=0* and *c=-h*.

The first- and second-order partial differentials of $\overline{\psi^{(2)}}$ with respect to the vertical variable *c* become

$$\overline{\psi_c^{(2)}} = e^{-2\kappa a}\left\{ l'(c) + \frac{1}{4\kappa}\operatorname{Re}C_1 \sin 2\kappa c + \frac{c}{2}\operatorname{Re}C_1 \cos 2\kappa c - 2\kappa \operatorname{Re}C_2 \sin 2\kappa c \right.$$

$$\left. + \frac{1}{4\kappa}\operatorname{Im}C_1 \cos 2\kappa c - \frac{c}{2}\operatorname{Im}C_1 \sin 2\kappa c - 2\kappa \operatorname{Im}C_2 \cos 2\kappa c \right\} \qquad \text{(A.33)}$$

$$\approx e^{-2\kappa a}\left\{ l'(c) - \frac{1}{h}l(0) + \frac{1}{h}l(-h) - \frac{1}{4h}(h^2 + 2c^2 + 4hc)l''(0) - \frac{1}{4h}(h^2 - 2c^2)l''(-h) \right\}$$

and

$$\overline{\psi_{cc}^{(2)}} = e^{-2\kappa a}\{\, l''(c) + \operatorname{Re}C_1 \cos 2\kappa c - \kappa c \operatorname{Re}C_1 \sin 2\kappa c - 4\kappa^2 \operatorname{Re}C_2 \cos 2\kappa c$$

$$- \operatorname{Im}C_1 \sin 2\kappa c - \kappa c \operatorname{Im}C_1 \cos 2\kappa c + 4\kappa^2 \operatorname{Im}C_2 \sin 2\kappa c \} \qquad \text{(A.34)}$$

$$\approx e^{-2\kappa a}\left\{ l''(c) + \frac{c}{h}l''(-h) - \frac{(h+c)}{h}l''(0) \right\}.$$

(The terms containing *l(0)* and *l(-h)* contribute only to the order $(\kappa/k_0)^4$ in $\overline{\psi_{cc}^{(2)}}$ and are neglected. Be aware of that the differentiations to obtain $\overline{\psi_c^{(2)}}$ and $\overline{\psi_{cc}^{(2)}}$ must be performed upon $\overline{\psi^{(2)}}$ in the form (A.27), not in the form (A.31).)

To see that $\overline{\psi^{(2)}}$ also fulfils the claim $\overline{\psi_{aa}^{(2)}}$ - $\overline{\psi_{cc}^{(2)}}$ = 0 at the boundaries *c=0* and *c=-h*, more terms than those included in eq. (4.58) have to be included.

Inserting eqs. (A.19), (A.21)-(A.22) and (A.25)-(A.26) into eq. (A.33) then gives $\overline{\psi_c^{(2)}}$ (eq. (4.59)), see next page.

$$\overline{\psi_c^{(2)}} \approx \frac{\xi_0 k_0 |A|^2}{2\nu} e^{-2\kappa a} \quad \times$$

$$\times \quad \Big\{ e^{2k_0 c} + \frac{1}{2(k_0 h)}\left(e^{-2k_0 h} - 1\right) + \frac{k_0}{2h}\left[\left(2c^2 + 4hc + h^2\right) - \left(2c^2 - h^2\right) e^{-2k_0 h}\right]$$

$$- 2\xi_0^{1/2}\Big[\, e^{(m_r + k_0)c} (\cos m_i c - \sin m_i c)$$

$$- e^{2k_0 c - (m_r + k_0)(h+c)} (\cos m_i (h+c) + \sin m_i (h+c))\Big]$$

$$- \xi_0^{1/2} \frac{k_0}{4h}\Big[\left(2c^2 + 4hc + h^2\right) + \left(2c^2 - h^2\right) e^{-2k_0 h}\Big] \Big\}. \qquad \text{(A.35)}$$

APPENDIX B

Appendix to chapter V
Calculation of the temporal damping rate γ (eq. 5.45)

The temporal damping rate γ may be determined from the dispersion relation (3.36), which reads

$$\left(\frac{\omega}{\omega_k}\right)^2 = 1+(k\delta)\frac{[1-\alpha(k\delta)\beta]}{\left[1+\alpha(k\delta)^2\right]\beta-\left(\frac{m}{k}\right)(k\delta)}. \tag{B.1}$$

From eq. (3.2) we have

$$\left(\frac{m}{k}\right)(k\delta) = (1-i)\left[1+\frac{i}{4}(k\delta)^2+o(k\delta)^4\right], \tag{B.2}$$

so that

$$\frac{\omega^2}{\omega_k^2} = 1+(k\delta)\frac{[1-\alpha(k\delta)\beta]}{(\beta-1+i)\left[1+\frac{(4\alpha\beta-1-i)}{4(\beta-1+i)}(k\delta)^2+o(k\delta)^4\right]}$$

$$= 1+(k\delta)\frac{[1-\alpha(k\delta)\beta]}{(\beta-1+i)}\left[1-\frac{(4\alpha\beta-1-i)}{4(\beta-1+i)}(k\delta)^2+o(k\delta)^4\right] \tag{B.3}$$

$$= 1+\frac{(k\delta)}{(\beta-1+i)}\left\{1-\alpha(k\delta)\beta-\frac{(4\alpha\beta-1-i)}{4(\beta-1+i)}(k\delta)^2+o(k\delta)^3\right\} \quad .$$

The parameter α was defined in eq. (3.31) as

$$\alpha \equiv 2i-\left(\frac{m}{k}\right)(k\delta)^2-(k\delta)^2 = 2i-(1-i)(k\delta)+o(k\delta)^2. \tag{B.4}$$

Inserting this into the expression (B.3), we obtain

$$\left(\frac{\omega}{\omega_k}\right)^2 = 1 + \frac{(k\delta)}{(\beta - 1 + i)}\left\{1 - 2i(k\delta)\beta + o(k\delta)^2\right\}. \tag{B.5}$$

Taking the square root of this equation, we get

$$\frac{\omega}{\omega_k} = 1 + \frac{(k\delta)}{2(\beta - 1 + i)}\left[1 - 2i(k\delta)\beta\right] - \frac{(k\delta)^2}{8(\beta - 1 + i)^2} + o(k\delta)^3 . \tag{B.6}$$

In the case of temporal damping ω is complex and k (and hence ω_k) is real. The temporal damping rate γ is defined as minus the imaginary part of ω. Denoting the real part of ω by ω_0, we write

$$\omega \equiv \omega_0 - i\gamma. \tag{B.7}$$

The parameters β and δ in eq. (B.6) both depend on ω. As long as γ is assumed small compared to ω_0, only a minor error will be introduced by replacing ω by ω_0 in those parameters in the relatively small terms in eq. (B.6). The parameters β and δ then become real, and we obtain

$$\begin{aligned}\frac{\omega}{\omega_k} \approx &\left\{1 + \frac{(k\delta)(\beta - 1)}{2|\beta - 1 + i|^2} - \frac{(k\delta)^2\beta}{|\beta - 1 + i|^2}\left[1 + \frac{(\beta - 2)}{8|\beta - 1 + i|^2}\right]\right\} \\ &- i\left\{\frac{(k\delta)}{2|\beta - 1 + i|^2} + (k\delta)^2\left[\frac{\beta(\beta - 1)}{|\beta - 1 + i|^2} - \frac{(\beta - 1)}{4|\beta - 1 + i|^4}\right]\right\} \quad ,\end{aligned} \tag{B.8}$$

where the terms in curly brackets all are real. Hence,

$$\frac{\omega_0}{\omega_k} \approx 1 + \frac{(k\delta)(\beta - 1)}{2|\beta - 1 + i|^2} - \frac{(k\delta)^2\beta}{|\beta - 1 + i|^2}\left[1 + \frac{(\beta - 2)}{8|\beta - 1 + i|^2}\right] \tag{B.9}$$

and

$$\frac{\gamma}{\omega_0} = \frac{\gamma}{\omega_k}\cdot\frac{\omega_k}{\omega_0}$$

$$\approx \frac{(k\delta)}{2|\beta-1+i|^2}\left\{1+(k\delta)(\beta-1)\left[2\beta-\frac{1}{2|\beta-1+i|^2}\right]\right\}\cdot\left\{1-\frac{(k\delta)(\beta-1)}{2|\beta-1+i|^2}\right\} \qquad \text{(B.10)}$$

$$\approx \frac{(k\delta)}{2|\beta-1+i|^2}\left\{1+(k\delta)(\beta-1)\left[2\beta-\frac{1}{|\beta-1+i|^2}\right]\right\} \quad .$$

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