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Some new refinements of the Young, Hölder, and Minkowski inequalities

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Abstract

We prove and discuss some new refined Hölder inequalities for any $p > 1$ and also a reversed version for $0 < p < 1$. The key is to use the concepts of superquadraticity, strong convexity, and to first prove the corresponding refinements of the Young and reversed Young inequalities. Refinements of the Minkowski and reversed Minkowski inequalities are also given.

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1 Introduction

The classical Young inequality reads

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \tag{1}$$

where a and b are nonnegative numbers, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, [5].

The reversed version reads

$$ab \ge \frac{a^p}{p} + \frac{b^q}{q}, \qquad a, b > 0, \qquad 0 < p < 1, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{2}$$

The first observation is that both (1) and (2) are simple consequences of the convexity of the function $\varphi(x) = e^x$. Indeed,

$$ab = e^{\log a + \log b} = e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q} \le \frac{1}{p}e^{\log a^p} + \frac{1}{q}e^{\log b^q} = \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Moreover, (2) follows from (1) by just juggling with the parameters and numbers. First, use (1) with $p_1 = \frac{1}{p}$ and $q_1 = -\frac{q}{p}$ (so, $p_1, q_1 > 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$) and after that replace a by $(ab)^p$ and b by b^{-p}.

One recent idea to derive refinements of inequalities is to use the concept of superquadraticity introduced in [1].

Definition 1.1 A function $\varphi : [0,\infty) \to \mathbf{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbf{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi\bigl(|y-x|\bigr) \geq C_x(y-x)$$

for all $y \geq 0$.

We say that f is subquadratic if $-f$ is superquadratic.

Some guiding ideas for introducing this concept (in connection to refining the Hölder inequality) can be found in the earlier paper [9], where in particular the following refinement of the Hölder inequality was proved:

Proposition 1.2 ([9, Theorem 1.1]) *Let $p \geq 2$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any nonnegative μ-measurable functions f and g,*

$$\int fg\,d\mu \leq \left(\int f^p\,d\mu - \int \left|f - g^{q-1}\frac{\int fg\,d\mu}{\int g^q\,d\mu}\right|^p d\mu\right)^{\frac{1}{p}} \cdot \left(\int g^q\,d\mu\right)^{\frac{1}{q}}. \tag{3}$$

In this paper we prove some other refinements of the Hölder inequality, where we do not have the restriction $p \geq 2$ and where the refinements are not only made to the first factor as in (3) (see Theorems 3.1 and 4.1). Based on the ideas above, we will use "natural" quasiconvex function $\varphi(x) = e^x - 1 - x$ or, more generally,

$$\varphi(x) = e^x - \sum_{k=0}^{n} \frac{x^k}{k!}, \quad n \geq 1.$$

In fact, we have the following useful characterization in a special case.

Lemma 1.3 ([2, Lemma 2.2]) *Let $\varphi : [0,\infty) \to \mathbf{R}$ be a continuously differentiable function with $\varphi(0) = \varphi'(0) = 0$ and φ' convex. Then φ is superquadratic.*

We also need the following Jensen-type inequality.

Theorem 1.4 ([1, Theorem 2.3]) *Let (Ω,μ) be a probability measure space. Then the inequality*

$$\varphi\left(\int_\Omega f(s)\,d\mu(s)\right) \leq \int_\Omega \left(\varphi\bigl(f(s)\bigr) - \varphi\left(\left|f(s) - \int_\Omega f(t)\,d\mu(t)\right|\right)\right) d\mu(s) \tag{4}$$

holds for all nonnegative μ-integrable functions f if and only if φ is superquadratic. Moreover, (4) holds in the reversed direction if and only if φ is subquadratic.

If φ is a nonnegative superquadratic function, then φ is convex (see [1, Lemma 2.2]) and, since the term $\varphi(|f(s) - \int_\Omega f(t)\,d\mu(t)|)$ is nonnegative, inequality (4) can be continued by $\leq \int_\Omega \varphi(f(s))\,d\mu(s)$, and we get a refinement of the Jensen inequality.

The paper is organized as follows. In Sect. 2 we present our refinements of both Young and reversed Young inequalities (see Theorems 2.2 and 2.4). Our corresponding refinements of the Hölder and the reversed Hölder inequalities are given in Sect. 3 (see Theorems 3.1 and 3.2) while results related to the Minkowski inequality are given in Sect. 4. Finally, Sect. 5 gives some concluding remarks and results. In particular, we derive some similar refinements by using the concept of strong convexity (see Lemma 5.4 and Theorem 5.5). The results obtained in these two ways are also compared.

2 Refined Young inequality

Let us first state the following auxiliary statements about superquadratic functions.

Lemma 2.1 *Let p and q be numbers such that $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.*

a) If φ is a superquadratic function on $[0,\infty)$, then for any $x,y\in[0,\infty)$ the following inequality holds:

$$\varphi\left(\frac{x}{p}+\frac{y}{q}\right)\leq\frac{1}{p}\varphi(x)+\frac{1}{q}\varphi(y)-\frac{1}{p}\varphi\left(\frac{|x-y|}{q}\right)-\frac{1}{q}\varphi\left(\frac{|x-y|}{p}\right). \tag{5}$$

b) For any $k\geq 2$, the following inequality holds:

$$V_k(x,y;p):=\left(\frac{x}{p}+\frac{y}{q}\right)^k-\frac{x^k}{p}-\frac{y^k}{q}+|x-y|^k\left(\frac{1}{pq^k}+\frac{1}{p^kq}\right)\leq 0 \tag{6}$$

for any $x,y\in[0,\infty)$.

Proof a) Using Theorem 1.4 with point measures $\frac{1}{p}$ and $\frac{1}{q}$ at the points x and y, respectively, we get (5).

b) By Lemma 1.3, the function $\varphi(x)=x^k$ is superquadratic for $k\geq 2$. Hence, inequality (6) is a simple consequence of inequality (5) for this particular power function. □

Our refined Young inequality reads:

Theorem 2.2 *Let $a,b\geq 1$, $p,q>1$ where $\frac{1}{p}+\frac{1}{q}=1$ and $n\in\mathbf{N}$, $n\geq 2$. Then*

$$\begin{aligned}
ab&\leq\frac{a^p}{p}+\frac{b^q}{q}-\left(\frac{1}{p}e^{\frac{1}{q}|\log\frac{a^p}{b^q}|}+\frac{1}{q}e^{\frac{1}{p}|\log\frac{a^p}{b^q}|}-\frac{2|\log\frac{a^p}{b^q}|}{pq}-1\right)\\
&\quad+\sum_{k=2}^{n}\frac{1}{k!}V_k(p\log a,q\log b;p)\\
&\leq\frac{a^p}{p}+\frac{b^q}{q}-\left(\frac{1}{p}e^{\frac{1}{q}|\log\frac{a^p}{b^q}|}+\frac{1}{q}e^{\frac{1}{p}|\log\frac{a^p}{b^q}|}-\frac{2|\log\frac{a^p}{b^q}|}{pq}-1\right)\\
&\quad+\sum_{k=2}^{n-1}\frac{1}{k!}V_k(p\log a,q\log b;p)\\
&\leq\cdots\\
&\leq\frac{a^p}{p}+\frac{b^q}{q}-\left(\frac{1}{p}e^{\frac{1}{q}|\log\frac{a^p}{b^q}|}+\frac{1}{q}e^{\frac{1}{p}|\log\frac{a^p}{b^q}|}-\frac{2|\log\frac{a^p}{b^q}|}{pq}-1\right)\\
&\leq\frac{a^p}{p}+\frac{b^q}{q}-\left(e^{\frac{2|\log\frac{a^p}{b^q}|}{pq}}-\frac{2|\log\frac{a^p}{b^q}|}{pq}-1\right)\\
&\leq\frac{a^p}{p}+\frac{b^q}{q},
\end{aligned} \tag{7}$$

where V_k is defined in (6) and with the convention that the sum $\sum_{k=2}^{1}$ is equal to 0.

Proof By Lemma 1.3, the function $\varphi(x) = e^x - \sum_{k=0}^{n} \frac{x^k}{k!}$, $n \geq 1$, is superquadratic. By applying (5) with this function, we obtain (after some elementary calculations) that

$$e^{(\frac{x}{p}+\frac{y}{q})} \leq \frac{e^x}{p} + \frac{e^y}{q} - \left(\frac{1}{p}e^{\frac{1}{q}|x-y|} + \frac{1}{q}e^{\frac{1}{p}|x-y|} - \frac{2|x-y|}{pq} - 1\right) + \sum_{k=2}^{n} \frac{1}{k!} V_k(x,y;p). \tag{8}$$

Since, by (6), $V_n(x,y;p) \leq 0$ and

$$\sum_{k=2}^{n} \frac{1}{k!} V_k(x,y;p) = \sum_{k=2}^{n-1} \frac{1}{k!} V_k(x,y;p) + \frac{1}{n!} V_n(x,y;p),$$

then, for any $n \geq 2$, the following chain of inequalities holds:

$$\begin{aligned} e^{(\frac{x}{p}+\frac{y}{q})} &\leq \frac{e^x}{p} + \frac{e^y}{q} - \left(\frac{1}{p}e^{\frac{1}{q}|x-y|} + \frac{1}{q}e^{\frac{1}{p}|x-y|} - \frac{2|x-y|}{pq} - 1\right) + \sum_{k=2}^{n} \frac{1}{k!} V_k(x,y;p) \\ &\leq \frac{e^x}{p} + \frac{e^y}{q} - \left(\frac{1}{p}e^{\frac{1}{q}|x-y|} + \frac{1}{q}e^{\frac{1}{p}|x-y|} - \frac{2|x-y|}{pq} - 1\right) + \sum_{k=2}^{n-1} \frac{1}{k!} V_k(x,y;p) \\ &\leq \cdots \leq \frac{e^x}{p} + \frac{e^y}{q} - \left(\frac{1}{p}e^{\frac{1}{q}|x-y|} + \frac{1}{q}e^{\frac{1}{p}|x-y|} - \frac{2|x-y|}{pq} - 1\right), \end{aligned} \tag{9}$$

with the convention that $\sum_{k=2}^{1}$ is equal to 0.

From the classical Young inequality (1) applied with $a = b = e^{\frac{|x-y|}{pq}}$, we get

$$e^{\frac{2|x-y|}{pq}} \leq \frac{1}{p}e^{\frac{1}{q}|x-y|} + \frac{1}{q}e^{\frac{1}{p}|x-y|},$$

and the last line in (9) can be followed by

$$\leq \frac{e^x}{p} + \frac{e^y}{q} - \left(e^{\frac{2|x-y|}{pq}} - \frac{2|x-y|}{pq} - 1\right).$$

Since $e^t - t - 1 \geq 0$ for all $t \geq 0$ and using this estimate for $t = \frac{2|x-y|}{pq}$, we obtain the following continuation of the chain of inequalities:

$$\frac{e^x}{p} + \frac{e^y}{q} - \left(e^{\frac{2|x-y|}{pq}} - \frac{2|x-y|}{pq} - 1\right) \leq \frac{e^x}{p} + \frac{e^y}{q}. \tag{10}$$

Putting in (8), (9), and (10) $x = p\log a$ and $y = q\log b$, we obtain (7). The proof is complete. □

Remark 2.3 Let us interchange the numbers a and b in (7). For the sake of simplicity, we write only the last three inequalities from the whole chain (7). Then we get the following inequalities:

$$\begin{aligned} ab &\le \frac{b^p}{p} + \frac{a^q}{q} - \left(\frac{1}{p} e^{\frac{1}{q}|\log \frac{b^p}{a^q}|} + \frac{1}{q} e^{\frac{1}{p}|\log \frac{b^p}{a^q}|} - \frac{2|\log \frac{b^p}{a^q}|}{pq} - 1\right) \\ &\le \frac{b^p}{p} + \frac{a^q}{q} - \left(e^{\frac{2|\log \frac{b^p}{a^q}|}{pq}} - \frac{2|\log \frac{b^p}{a^q}|}{pq} - 1\right) \\ &\le \frac{b^p}{p} + \frac{a^q}{q}. \end{aligned} \tag{11}$$

By combining (11) with Theorem 2.2, we get the following inequalities:

$$\begin{aligned} ab \le \min\biggl\{ &\frac{a^p}{p} + \frac{b^q}{q} - \left(\frac{1}{p} e^{\frac{1}{q}|\log \frac{a^p}{b^q}|} + \frac{1}{q} e^{\frac{1}{p}|\log \frac{a^p}{b^q}|} - \frac{2|\log \frac{a^p}{b^q}|}{pq} - 1\right), \\ &\frac{b^p}{p} + \frac{a^q}{q} - \left(\frac{1}{p} e^{\frac{1}{q}|\log \frac{b^p}{a^q}|} + \frac{1}{q} e^{\frac{1}{p}|\log \frac{b^p}{a^q}|} - \frac{2|\log \frac{b^p}{a^q}|}{pq} - 1\right)\biggr\}, \end{aligned} \tag{12}$$

$$\begin{aligned} ab \le \min\biggl\{ &\frac{a^p}{p} + \frac{b^q}{q} - \left(e^{\frac{2|\log \frac{a^p}{b^q}|}{pq}} - \frac{2|\log \frac{a^p}{b^q}|}{pq} - 1\right), \\ &\frac{b^p}{p} + \frac{a^q}{q} - \left(e^{\frac{2|\log \frac{b^p}{a^q}|}{pq}} - \frac{2|\log \frac{b^p}{a^q}|}{pq} - 1\right)\biggr\}, \end{aligned} \tag{13}$$

$$ab \le \min\left\{\frac{a^p}{p} + \frac{b^q}{q}, \frac{b^p}{p} + \frac{a^q}{q}\right\}. \tag{14}$$

It will be interesting if we can say something about the inequalities (12)–(14) compared with the corresponding inequalities in (11). The comparison related to (14) is recently discussed also in [4, p. 57].

Let a and b be a real numbers such that $1 \le a \le b$. Let us consider a function $h(x) := \frac{x^q}{q} - \frac{x^p}{p}$, $p \ge q \ge 1$. Then $h'(x) = x^{q-1}(1 - x^{p-q}) \le 0$ for $x \ge 1$, i.e., h is nonincreasing on $[1, \infty)$, and for $1 \le a \le b$ we have $h(a) \ge h(b)$, i.e., $\frac{a^p}{p} + \frac{b^q}{q} \le \frac{b^p}{p} + \frac{a^q}{q}$. Hence, if $1 \le a \le b$, then inequality (14) gives the following refined Young inequality:

$$ab \le \min\left\{\frac{a^p}{p} + \frac{b^q}{q}, \frac{b^p}{p} + \frac{a^q}{q}\right\} = \frac{a^p}{p} + \frac{b^q}{q}.$$

Similar comparisons related to inequalities (12) and (13) are still open problems.

Guided by the arguments in our introduction, we can also derive the following refined version of the reversed Young inequality (2).

Theorem 2.4 *Let $a, b \ge 1$, $0 < p < 1$, $q < 0$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $n \in \mathbf{N}$, $n \ge 2$. Then*

$$\begin{aligned} ab &\ge \frac{a^p}{p} + \frac{b^q}{q} + S - \sum_{k=2}^{n} \frac{1}{pk!} V_k\left(\log(ab), q \log b; \frac{1}{p}\right) \\ &\ge \frac{a^p}{p} + \frac{b^q}{q} + S - \sum_{k=2}^{n-1} \frac{1}{pk!} V_k\left(\log(ab), q \log b; \frac{1}{p}\right) \ge \cdots \ge \end{aligned}$$

$$\begin{aligned}&\geq \frac{a^p}{p}+\frac{b^q}{q}+S \\ &\geq \frac{a^p}{p}+\frac{b^q}{q}+\frac{1}{p}e^{-\frac{2p^2}{q}|\log ab^{1-q}|}+\frac{2p}{q}\left|\log ab^{1-q}\right|-\frac{1}{p} \\ &\geq \frac{a^p}{p}+\frac{b^q}{q},\end{aligned} \tag{15}$$

where

$$S := e^{-\frac{p}{q}|\log ab^{1-q}|}-\frac{1}{q}e^{p|\log ab^{1-q}|}+\frac{2p}{q}\left|\log ab^{1-q}\right|-\frac{1}{p},$$

V_k *is defined in* (6), *and with the convention that the sum* $\sum_{k=2}^{1}$ *is equal to* 0.

Proof Consider the chain of inequalities in (7). First, we replace p by $\frac{1}{p}>1$ and q by $-\frac{q}{p}$. After that we replace a by $(ab)^p$ and b with b^{-p}. Then, by (7) we have that

$$\begin{aligned}a^p = (ab)^p b^{-p} &\leq pab-\frac{p}{q}b^q-pS+\sum_{k=2}^{n}\frac{1}{k!}V_k\left(\log(ab),q\log b;\frac{1}{p}\right) \\ &\leq pab-\frac{p}{q}b^q-pS+\sum_{k=2}^{n-1}\frac{1}{k!}V_k\left(\log(ab),q\log b;\frac{1}{p}\right)\leq\cdots\leq \\ &\leq pab-\frac{p}{q}b^q-pS\leq pab-\frac{p}{q}b^q-pT \\ &\leq pab-\frac{p}{q}b^q,\end{aligned} \tag{16}$$

where $T := e^{-\frac{2p^2}{q}|\log ab^{1-q}|}+\frac{2p}{q}|\log ab^{1-q}|-\frac{1}{p}$. Dividing in (16) by p and adding $\frac{1}{q}b^q$, we get the following chain of inequalities:

$$\begin{aligned}\frac{1}{p}a^p+\frac{1}{q}b^q &\leq ab-S+\sum_{k=2}^{n}\frac{1}{pk!}V_k\left(\log(ab),q\log b;\frac{1}{p}\right) \\ &\leq ab-S+\sum_{k=2}^{n-1}\frac{1}{pk!}V_k\left(\log(ab),q\log b;\frac{1}{p}\right)\leq\cdots\leq \\ &\leq ab-S\leq ab-T \\ &\leq ab.\end{aligned} \tag{17}$$

Hence

$$S-\sum_{k=2}^{n}\frac{1}{pk!}V_k\left(\log(ab),q\log b;\frac{1}{p}\right)\geq S-\sum_{k=2}^{n-1}\frac{1}{pk!}V_k\left(\log(ab),q\log b;\frac{1}{p}\right)\geq S\geq T\geq 0,$$

and after some calculations we get the chain of inequalities in (15). The proof is complete. □

3 Refined Hölder inequality

Here and in the following sections, we denote a positive measure space on $(0,\infty)$ by (E,μ). If $S\subseteq E$, then as usual we denote

$$\|f\|_{p,S}=\left(\int_S \left|f(x)\right|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

for any real $p\neq 0$ and measurable function f. If $S=E$, we simply write $\|f\|_p$. Our refined version of Hölder inequality reads:

Theorem 3.1 *Let $p,q>1$ be real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Let f, g be functions, which are positive and finite a.e. on E. Let a subset E_1 be defined as*

$$E_1:=\left\{x\in E:\frac{f(x)}{\|f\|_p}\geq 1,\frac{g(x)}{\|g\|_q}\geq 1\right\}.$$

Then, provided that the involved integrals are finite, we have that

$$\int_E f(x)g(x)\,d\mu(x)\leq \|f\|_p\|g\|_q(1-A)\leq \|f\|_p\|g\|_q(1-B)\leq \|f\|_p\|g\|_q, \tag{18}$$

where

$$A:=\int_{E_1}\left(\frac{1}{p}e^{\frac{k(x)}{q}}+\frac{1}{q}e^{\frac{k(x)}{p}}-\frac{2}{pq}k(x)-1\right)d\mu(x),$$
$$B:=\int_{E_1}\left(e^{\frac{2k(x)}{pq}}-\frac{2}{pq}k(x)-1\right)d\mu(x)$$

with

$$k(x):=\left|\log\frac{f^p(x)\|g\|_q^q}{\|f\|_p^p g^q(x)}\right|.$$

Proof First, using the third, fourth, and fifth inequalities in (7) with a and b replaced by $\frac{f(x)}{\|f\|_p}$ and $\frac{g(x)}{\|g\|_q}$, respectively, we find that for $x\in E_1$,

$$\begin{aligned}\frac{f(x)}{\|f\|_p}\cdot\frac{g(x)}{\|g\|_q}&\leq \frac{1}{p}\frac{f^p(x)}{\|f\|_p^p}+\frac{1}{q}\frac{g^q(x)}{\|g\|_q^q}-\left(\frac{1}{p}e^{\frac{k(x)}{q}}+\frac{1}{q}e^{\frac{k(x)}{p}}-\frac{2}{pq}k(x)-1\right)\\&\leq \frac{1}{p}\frac{f^p(x)}{\|f\|_p^p}+\frac{1}{q}\frac{g^q(x)}{\|g\|_q^q}-\left(e^{\frac{2k(x)}{pq}}-\frac{2}{pq}k(x)-1\right)\\&\leq \frac{1}{p}\frac{f^p(x)}{\|f\|_p^p}+\frac{1}{q}\frac{g^q(x)}{\|g\|_q^q}.\end{aligned}$$

By integrating over E_1, we get that

$$\begin{aligned}&\frac{\int_{E_1}f(x)g(x)\,d\mu(x)}{\|f\|_p\|g\|_q}\\&\quad\leq \frac{\int_{E_1}f^p(x)\,d\mu(x)}{p\|f\|_p^p}+\frac{\int_{E_1}g^q(x)\,d\mu(x)}{q\|g\|_q^q}-\int_{E_1}\left(\frac{1}{p}e^{\frac{k(x)}{q}}+\frac{1}{p}e^{\frac{k(x)}{q}}-\frac{2}{qp}k(x)-1\right)d\mu(x)\end{aligned}$$

$$\leq \frac{\int_{E_1} f^p(x)\,d\mu(x)}{p\|f\|_p^p} + \frac{\int_{E_1} g^q(x)\,d\mu(x)}{q\|g\|_q^q} - \int_{E_1}\left(e^{\frac{2k(x)}{pq}} - \frac{2}{pq}k(x) - 1\right)d\mu(x)$$
$$\leq \frac{\int_{E_1} f^p(x)\,d\mu(x)}{p\|f\|_p^p} + \frac{\int_{E_1} g^q(x)\,d\mu(x)}{q\|g\|_q^q}.$$

Moreover, on $E\backslash E_1$ we use just (1) in a similar way and obtain that

$$\frac{\int_{E\backslash E_1} f(x)g(x)\,d\mu(x)}{\|f\|_p\|g\|_q} \leq \frac{\int_{E\backslash E_1} f^p(x)\,d\mu(x)}{p\|f\|_p^p} + \frac{\int_{E\backslash E_1} g^q(x)\,d\mu(x)}{q\|g\|_q^q}.$$

By just adding the two previous inequalities, using the additivity of the integral, $\int_E = \int_{E_1} + \int_{E\backslash E_1}$, the equality $\frac{1}{p} + \frac{1}{q} = 1$, and multiplying with $\|f\|_p\|g\|_q$, we get (18). The proof is complete. □

Our corresponding refinement of the reversed Hölder inequality reads:

Theorem 3.2 *Let $p \in (0,1)$ and $q < 0$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be functions, which are positive and finite a.e. on E. Let*

$$E_1 := \left\{x \in E : \frac{f(x)}{\|f\|_p} \geq 1, \frac{g(x)}{\|g\|_q} \geq 1\right\}.$$

Then, provided that the involved integrals are finite, we have that

$$\int_E f(x)g(x)\,d\mu(x) \geq \|f\|_p\|g\|_q(1+C) \geq \|f\|_p\|g\|_q(1+D) \geq \|f\|_p\|g\|_q, \tag{19}$$

where

$$C := \int_{E_1}\left(e^{-\frac{p}{q}r(x)} - \frac{1}{q}e^{pr(x)} - \frac{2p}{q}r(x) - \frac{1}{p}\right)d\mu(x),$$
$$D := \int_{E_1}\left(\frac{1}{p}e^{\frac{2p^2}{q}r(x)} - \frac{2p}{q}r(x) - \frac{1}{p}\right)d\mu(x),$$

with

$$r(x) := \left|\log\frac{f(x)g^{1-q}(x)}{\|f\|_p\|g\|_q^{1-q}}\right|.$$

Proof By using (15) instead of (7), the proof is step by step similar to that of Theorem 3.1. Hence, we omit the details. □

Remark 3.3 If we denote

$$E_c := \left\{x \in E : \frac{f(x)}{\|f\|_p} \geq c^{1/p}, \frac{g(x)}{\|g\|_q} \geq c^{1/q}\right\}, \quad c > 0,$$

then in the same way we can state alternative formulations of Theorems 3.1 and 3.2, where E_1 is replaced by E_c and, in the inequalities (18) and (19), A, B, C, and D are replaced by cA, cB, cC, and cD, respectively.

The following theorem gives another refinement of the Hölder inequality, also based on the usage of our refinement of the Young inequality. We consider a subset $F \subseteq E$ consisting of positive functions f and g which are bounded by some positive constants and construct refinements involving these bounds.

Theorem 3.4 *Let $p,q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be functions, which are positive and finite a.e. on E and bounded on $F \subseteq E$ by positive constants, $0 < \mu(F) < \infty$. Denote $c := \frac{m}{M\mu(F)}$, where $m := \min\{\inf_{x\in F} f^p(x), \inf_{x\in F} g^q(x)\}$ and $M := \max\{\sup_{x\in F} f^p(x), \sup_{x\in F} g^q(x)\}$.*

Then, provided that the involved integrals are finite, we have that

$$\begin{aligned} \int_E f(x)g(x)\,d\mu(x) &\leq \|f\|_{p,F}\|g\|_{q,F} + \|f\|_{p,E\setminus F}\|g\|_{q,E\setminus F} - cA_1\|f\|_{p,F}\|g\|_{q,F} \\ &\leq \|f\|_p\|g\|_q - cA_1\|f\|_{p,F}\|g\|_{q,F} \\ &\leq \|f\|_p\|g\|_q - cB_1\|f\|_{p,F}\|g\|_{q,F} \leq \|f\|_p\|g\|_q, \end{aligned} \tag{20}$$

where

$$\begin{aligned} A_1 &:= \int_F \left(\frac{1}{p}e^{\frac{k_1(x)}{q}} + \frac{1}{q}e^{\frac{k_1(x)}{p}} - \frac{2}{pq}k_1(x) - 1\right) d\mu(x), \\ B_1 &:= \int_F \left(e^{\frac{2k_1(x)}{pq}} - \frac{2}{pq}k_1(x) - 1\right) d\mu(x), \end{aligned}$$

with

$$k_1(x) := \left|\log \frac{f^p(x)\|g\|_{q,F}^q}{\|f\|_{p,F}^p g^q(x)}\right|.$$

Proof Denote $\tilde{f}(x) := \frac{f(x)}{c^{1/p}\|f\|_{p,F}}$, $\tilde{g}(x) := \frac{g(x)}{c^{1/q}\|g\|_{q,F}}$. From the definition of the constants m and M, we get that

$$\begin{aligned} &m \leq f^p(x) \leq M, \qquad m\mu(F) \leq \|f\|_{p,F}^p \leq M\mu(F) = \frac{m}{c}, \\ &m \leq g^q(x) \leq M, \qquad m\mu(F) \leq \|g\|_{q,F}^q \leq M\mu(F) = \frac{m}{c}. \end{aligned}$$

Hence, $\frac{f(x)}{c^{1/p}\|f\|_{p,F}} \geq 1$, i.e., $\tilde{f}(x) \geq 1$ on F, and similarly, $\frac{g(x)}{c^{1/q}\|g\|_{q,F}} \geq 1$, i.e., $\tilde{g}(x) \geq 1$ on F.

Putting in (7) $a = \tilde{f}(x)$, $b = \tilde{g}(x)$, and integrating over F, we find that

$$\begin{aligned} \int_F \tilde{f}(x)\tilde{g}(x)\,d\mu(x) \leq{}& \frac{\|\tilde{f}\|_{p,F}^p}{p} + \frac{\|\tilde{g}\|_{q,F}^q}{q} \\ &- \int_F \left(\frac{1}{p}e^{\frac{1}{q}k_1(x)} + \frac{1}{q}e^{\frac{1}{p}k_1(x)} - \frac{2}{pq}k_1(x) - 1\right) d\mu(x). \end{aligned}$$

Since $\frac{\|\tilde{f}\|_{p,F}^p}{p} + \frac{\|\tilde{g}\|_{q,F}^q}{q} = \frac{1}{c}$, we conclude that

$$\int_F \tilde{f}(x)\tilde{g}(x)\,d\mu(x) \leq \frac{1}{c} - \int_F \left(\frac{1}{p}e^{\frac{1}{q}k_1(x)} + \frac{1}{q}e^{\frac{1}{p}k_1(x)} - \frac{2}{pq}k_1(x) - 1\right) d\mu(x).$$

Multiplying the above inequality by $c\|f\|_{p,F}\|g\|_{q,F}$, we obtain that

$$\int_F f(x)g(x)\,d\mu(x) \le (1-cA_1)\|f\|_{p,F}\|g\|_{q,F}.$$

Next we use the classical Hölder inequality with the set of integration $E\backslash F$ and have that

$$\begin{aligned}\int_E f(x)g(x)\,d\mu(x) &= \int_F f(x)g(x)\,d\mu(x) + \int_{E\backslash F} f(x)g(x)\,d\mu(x)\\ &\le (1-cA_1)\|f\|_{p,F}\|g\|_{q,F} + \|f\|_{p,E\backslash F}\|g\|_{q,E\backslash F},\\ \int_E f(x)g(x)\,d\mu(x) &\le \|f\|_{p,F}\|g\|_{q,F} + \|f\|_{p,E\backslash F}\|g\|_{q,E\backslash F} - cA_1\|f\|_{p,F}\|g\|_{q,F},\end{aligned}$$

and the first inequality in (20) is proved. Next we will prove that

$$\|f\|_{p,F}\|g\|_{q,F} + \|f\|_{p,E\backslash F}\|g\|_{q,E\backslash F} \le \|f\|_p\|g\|_q. \tag{21}$$

Consider a function $h(y,z) = y^\alpha z^{1-\alpha} + (1-y)^\alpha(1-z)^{1-\alpha}$, $0 \le y,z \le 1$, $\alpha \in (0,1)$. Since

$$h'_y(y,z) = \alpha\left[\left(\frac{y}{z}\right)^{\alpha-1} - \left(\frac{1-y}{1-z}\right)^{\alpha-1}\right],$$

we get that $h'_y(y,z) = 0$ when $y = z$. It is easy to see that this is the maximum of h and since $h(z,z) = 1$, we conclude that $h(y,z) \le 1$.

Taking $\alpha = \frac{1}{p}$, $y = \frac{\|f\|_{p,F}^p}{\|f\|_p^p}$, and $z = \frac{\|g\|_{q,F}^q}{\|g\|_q^q}$, we have that $1-\alpha = \frac{1}{q}$, $1-y = \frac{\|f\|_{p,E\backslash F}^p}{\|f\|_p^p}$, and $1-z = \frac{\|g\|_{q,E\backslash F}^q}{\|g\|_q^q}$.

Hence,

$$\frac{\|f\|_{p,F}\|g\|_{q,F}}{\|f\|_p\|g\|_q} + \frac{\|f\|_{p,E\backslash F}\|g\|_{q,E\backslash F}}{\|f\|_p\|g\|_q} \le 1,$$

so (21) and thus the second inequality in (20) is proved. The proof of the third inequality in (20) is a standard application of the Young inequality as in the proof of Theorem 2.2, and the fourth inequality is trivial, so the proof is complete. □

Corollary 3.5 *Let $p,q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f,g be functions, which are positive and bounded on E, $0 < \mu(E) < \infty$. Let m and M be positive constants such that*

$$m \le f^p(x) \le M, \qquad m \le g^q(x) \le M,$$

for all $x \in E$. Denote $c := \frac{m}{M\mu(E)}$.

Then, provided that involved integrals are finite, we have that

$$\int_E f(x)g(x)\,d\mu(x) \le \|f\|_p\|g\|_q(1-cA_2) \le \|f\|_p\|g\|_q(1-cB_2) \le \|f\|_p\|g\|_q,$$

where

$$A_2 := \int_E \left(\frac{1}{p}e^{\frac{k(x)}{q}} + \frac{1}{q}e^{\frac{k(x)}{p}} - \frac{2}{pq}k(x) - 1\right)d\mu(x),$$

$$B_2 := \int_E \left(e^{\frac{2k(x)}{pq}} - \frac{2}{pq} k(x) - 1 \right) d\mu(x),$$

with $k(x)$ defined in Theorem 3.1.

Proof Putting $F = E$ in Theorem 3.4, we get the statement of this corollary. □

4 Refined Minkowski inequality

Note that in the previous section we defined numbers A, B, C, D, and functions k and r, which depend on subsets E_1 or E and functions f and g, i.e., $A = A_{E_1,f,g}$, $k = k_{E,f,g}$, etc. We use the corresponding notation in the following results concerning refinements of the Minkowski inequality. For instance, a number $B_{G,g,h}$ which appeared in Theorem 4.1 is equal to $B_{G,g,h} := \int_G (e^{\frac{2k(x)}{pq}} - \frac{2}{pq} k(x) - 1)\, d\mu(x)$ with $k(x) = k_{E,g,h}(x) := |\log \frac{g^p(x)\|h\|_q^q}{\|g\|_p^p h^q(x)}|$.

Theorem 4.1 *Let $p > 1$ be a real number, f, g be functions, which are positive and finite a.e. on E, and*

$$G = \left\{ x \in E : \frac{f(x)}{\|f\|_p} \geq 1, \frac{g(x)}{\|g\|_p} \geq 1, \frac{(f(x)+g(x))^{p-1}}{\|f+g\|_p^{p-1}} \geq 1 \right\}. \tag{22}$$

Then, provided that the involved integrals are finite, we have that

$$\begin{aligned} \|f+g\|_p &\leq \|f\|_p(1 - A_{G,f,h}) + \|g\|_p(1 - A_{G,g,h}) \\ &\leq \|f\|_p(1 - B_{G,f,h}) + \|g\|_p(1 - B_{G,g,h}) \leq \|f\|_p + \|g\|_p, \end{aligned} \tag{23}$$

where $h = (f+g)^{p-1}$, $A_{G,f,h}$, $A_{G,g,h}$, $B_{G,f,h}$, and $B_{G,g,h}$ are defined in Theorem 3.1, but with G defined by (22). In $A_{G,f,h}$ and $B_{G,f,h}$, a function $k := k_{E,f,h}$ appears, while a function $k := k_{E,g,h}$ occurs in $A_{G,g,h}$ and $B_{G,g,h}$.

Proof We have that

$$\begin{aligned} \|f+g\|_p^p &= \int_E f(f+g)^{p-1}\, d\mu + \int_E g(f+g)^{p-1}\, d\mu \\ &\leq \left(\int_E f^p\, d\mu \right)^{\frac{1}{p}} \left(\int_E (f+g)^{(p-1)q}\, d\mu \right)^{\frac{1}{q}} (1 - A_{G,f,h}) \\ &\quad + \left(\int_E g^p\, d\mu \right)^{\frac{1}{p}} \left(\int_E (f+g)^{(p-1)q}\, d\mu \right)^{\frac{1}{q}} (1 - A_{G,g,h}), \end{aligned}$$

where in the last inequality, we have used the statement of Theorem 3.1 applied with the functions f and $h := (f+g)^{p-1}$ with conjugate exponents $p > 1$ and $q = \frac{p}{p-1} > 1$ for the first integral and the statement of the same theorem applied with the functions g and $(f+g)^{p-1}$ for the second integral.

Since

$$\left(\int_E (f+g)^{(p-1)q}\, d\mu \right)^{\frac{1}{q}} = \left(\int_E (f+g)^p\, d\mu \right)^{\frac{1}{p}(p-1)} = \|f+g\|_p^{p-1},$$

dividing the above inequality by $\|f+g\|_p^{p-1}$, we get

$$\|f+g\|_p \le \|f\|_p(1-A_{G,f,h}) + \|g\|_p(1-A_{G,g,h}).$$

Thus the first inequality in (23) is proved. The proof of the second inequality is completely similar so we omit the details. The third inequality is trivial, so the proof is complete. □

By instead using the inequalities in Theorem 3.2 and the method given in the proof of Theorem 4.1, the corresponding refined version of the reversed Minkowski inequality can be proved.

Theorem 4.2 *Let $p \in (0,1)$ be a real number, f and g be functions, which are positive and finite a.e. on E, and let G is defined by (22). Then, provided that the involved integrals are finite, we have that*

$$\begin{aligned}\|f+g\|_p &\ge \|f\|_p(1+C_{G,f,h}) + \|g\|_p(1+C_{G,g,h}) \\ &\ge \|f\|_p(1+D_{G,f,h}) + \|g\|_p(1+D_{G,g,h}) \ge \|f\|_p + \|g\|_p,\end{aligned}$$

where $h=(f+g)^{p-1}$, $C_{G,f,h}$, $C_{G,g,h}$, $D_{G,f,h}$, and $D_{G,g,h}$ are defined in Theorem 3.1. In $C_{G,f,h}$ and $D_{G,f,h}$, a function $r := r_{E,f,h}$ appears, while a function $r := r_{E,g,h}$ occurs in $C_{G,g,h}$ and $D_{G,g,h}$.

5 Concluding remarks and results

It is well known that also by using the concept of strong convexity we can derive refined versions of classical inequalities, see, e.g., [7] and the references therein.

Definition 5.1 ([6, 8]) Let I be an interval of the real line. A function $\varphi : I \to \mathbf{R}$ is called a strongly convex function with modulus $c > 0$ if

$$\varphi\bigl(\lambda x + (1-\lambda)y\bigr) \le \lambda\varphi(x) + (1-\lambda)\varphi(y) - c\lambda(1-\lambda)(x-y)^2 \tag{24}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

For applications the following lemma is useful.

Lemma 5.2 ([6]) *The function φ is strongly convex with modulus c if and only if $f(x) = \varphi(x) - cx^2$ is convex.*

The function $\varphi(x) = e^x$ is not only convex but also strongly convex with modulus c on the interval $[\log 2c, \infty)$. As a consequence of that fact, we have the following refinements of the Young inequality.

Lemma 5.3 *Let $a, b > 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} - \frac{\min\{a^p, b^q\}}{2pq}\log^2\frac{a^p}{b^q} \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{25}$$

Furthermore, if $a, b \ge 1$, then we have the following further refinement:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} - \frac{\min\{a^p, b^q\}}{2pq}\log^2\frac{a^p}{b^q} \le \frac{a^p}{p} + \frac{b^q}{q} - \frac{1}{2pq}\log^2\frac{a^p}{b^q} \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{26}$$

Proof For a given p, let us fix both $a, b > 0$. Denote $c := \frac{\min\{a^p, b^q\}}{2}$. Let $x = \log a^p$, $y = \log b^q$. Then $x \geq \log 2c$, $y \geq \log 2c$. Using the strong convexity of $\varphi(x) = e^x$ on $[\log 2c, \infty)$ with modulus c and putting in (24) $\varphi(x) = e^x$ and $\lambda = \frac{1}{p}$, we get the wanted inequality (25).

If $a, b \geq 1$, then $\min\{a^p, b^q\} \geq 1$ and (26) holds. The proof is complete. □

Inequality (25) is already known in the literature, see, for example, [3, Theorem 3]. The refinement of the reversed Young inequality is given in the following lemma. It is proved by the same method as that described in the proof of Theorem 2.4.

Lemma 5.4 *Let $a, b > 0$, $0 < p < 1$, $q < 0$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \geq \frac{a^p}{p} + \frac{b^q}{q} - \frac{p}{2q} \min\{ab, b^q\} \log^2 (ab^{1-q}) \geq \frac{a^p}{p} + \frac{b^q}{q}.$$

Moreover, if $a, b \geq 1$, then

$$ab \geq \frac{a^p}{p} + \frac{b^q}{q} - \frac{p}{2q} \log^2 (ab^{1-q}) \geq \frac{a^p}{p} + \frac{b^q}{q} - \frac{p}{2q} \min\{ab, b^q\} \log^2 (ab^{1-q}) \geq \frac{a^p}{p} + \frac{b^q}{q}.$$

Applying the above-mentioned refinements of the Young inequality, we can state the following refinement of the Hölder inequality.

Theorem 5.5 *Let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be positive and finite functions a.e. on E.*

Then, provided that involved integrals are finite, we have that

$$\int_E f(x)g(x)\, d\mu(x) \leq \|f\|_p \|g\|_q \left(1 - \frac{1}{2pq} \int_E l(x) k^2(x)\, d\mu(x) \right) \leq \|f\|_p \|g\|_q, \tag{27}$$

where

$$k(x) := \left| \log \frac{f^p(x) \|g\|_q^q}{\|f\|_p^p g^q(x)} \right| \quad \textit{and} \quad l(x) = \min \left\{ \frac{f^p(x)}{\|f\|_p^p}, \frac{g^q(x)}{\|g\|_q^q} \right\}.$$

Proof For fixed $x \in E$, putting $a = \tilde{f}(x)$, $b = \tilde{g}(x)$ in (25) and integrating it over E, we get that

$$\int_E \tilde{f}(x)\tilde{g}(x)\, d\mu(x) \leq \frac{\|\tilde{f}\|_p^p}{p} + \frac{\|\tilde{g}\|_q^q}{q} - \frac{1}{2pq} \int_E \min\{\tilde{f}^p(x), \tilde{g}^q(x)\} \log^2 \frac{\tilde{f}^p(x)}{\tilde{g}^q(x)}\, d\mu(x).$$

Assuming that $\|\tilde{f}\|_p = 1$, $\|\tilde{g}\|_q = 1$, we find that

$$\int_E \tilde{f}(x)\tilde{g}(x)\, d\mu(x) \leq 1 - \frac{1}{2pq} \int_E \min\{\tilde{f}^p(x), \tilde{g}^q(x)\} \log^2 \frac{\tilde{f}^p(x)}{\tilde{g}^q(x)}\, d\mu(x).$$

Replacing $\tilde{f}(x)$ with $\frac{f(x)}{\|f\|_p}$ and $\tilde{g}(x)$ with $\frac{g(x)}{\|g\|_p}$, we obtain that

$$\int_E \frac{f(x)g(x)}{\|f\|_p \|g\|_q}\, d\mu(x) \leq 1 - \frac{1}{2pq} \int_E \min \left\{ \frac{f^p(x)}{\|f\|_p^p}, \frac{g^q(x)}{\|g\|_q^q} \right\} \log^2 \frac{f^p(x) \|g\|_q^q}{\|f\|_p^p g^q(x)}\, d\mu(x).$$

Multiplying the above inequality with $\|f\|_p\|g\|_q$, we get inequality (27). The proof is complete. □

Finally, we will do some comparisons between the results obtained in Sect. 2 and in Lemma 5.3. Since Theorem 2.2 holds for $a, b \geq 1$, we work only under that condition. The whole term in the third inequality in (7),

$$\frac{a^p}{p} + \frac{b^q}{q} - \left(\frac{1}{p}e^{\frac{1}{q}|\log\frac{a^p}{b^q}|} + \frac{1}{q}e^{\frac{1}{p}|\log\frac{a^p}{b^q}|} - \frac{2|\log\frac{a^p}{b^q}|}{pq} - 1\right),$$

and the middle term in (25),

$$\frac{a^p}{p} + \frac{b^q}{q} - \frac{\min\{a^p, b^q\}}{2pq}\log^2\frac{a^p}{b^q},$$

cannot be compared. Namely, the difference $d_1(a,b)$,

$$d_1(a,b) := \frac{1}{p}e^{\frac{1}{q}|\log\frac{a^p}{b^q}|} + \frac{1}{q}e^{\frac{1}{p}|\log\frac{a^p}{b^q}|} - \frac{2|\log\frac{a^p}{b^q}|}{pq} - 1 - \frac{\min\{a^p, b^q\}}{2pq}\log^2\frac{a^p}{b^q},$$

changes its sign for different values of p, q, a, and b. For example, if $p = 4$, $q = 4/3$, then $d_1(10,3) \approx 60.3$ and $d_1(2,2) \approx -0.3$. Similarly, the difference d_2, which arises from the fourth inequality in (7) and (25), also changes sign. For example, if $p = 20$, $q = \frac{20}{19}$, then $d_2(39,2) \approx 716.7$ and then $d_2(2,8) \approx -28$, where

$$d_2(a,b) := e^{\frac{2|\log\frac{a^p}{b^q}|}{pq}} - \frac{2|\log\frac{a^p}{b^q}|}{pq} - 1 - \frac{\min\{a^p, b^q\}}{2pq}\log^2\frac{a^p}{b^q}.$$

Also, the difference

$$d_3(a,b) := e^{\frac{2|\log\frac{a^p}{b^q}|}{pq}} - \frac{2|\log\frac{a^p}{b^q}|}{pq} - 1 - \frac{1}{2pq}\log^2\frac{a^p}{b^q}$$

changes its sign, for example, if $p = 20$, $q = \frac{20}{19}$, then $d_3(39,2) \approx 851$ and $d_3(2,5) \approx -2.5$.

But the third inequality in (7) can be compared with (26). In fact, let us consider the difference $d_4(a,b)$ defined by

$$d_4(a,b) := \frac{1}{p}e^{\frac{1}{q}|\log\frac{a^p}{b^q}|} + \frac{1}{q}e^{\frac{1}{p}|\log\frac{a^p}{b^q}|} - \frac{2|\log\frac{a^p}{b^q}|}{pq} - 1 - \frac{1}{2pq}\log^2\frac{a^p}{b^q}.$$

This difference contains the term $s := |\log\frac{a^p}{b^q}|$, $s \geq 0$, so we can consider the function

$$f(s) := \frac{1}{p}e^{s/q} + \frac{1}{q}e^{s/p} - \frac{2s}{pq} - 1 - \frac{s^2}{2pq}.$$

Then $f'(s) = \frac{1}{pq}(e^{s/q} + e^{s/p} - 2 - s)$, $f''(s) = \frac{1}{pq}(\frac{1}{q}e^{s/q} + \frac{1}{p}e^{s/p} - 1) \geq 0$, and $f'(s) \geq f'(0) = 0$, which implies that $f(s) \geq f(0) = 0$. So, $d_4(a,b) \geq 0$ for any $a, b \geq 1$, and we have the following chain of refinements:

$$\begin{aligned} ab &\le \frac{a^p}{p} + \frac{b^q}{q} - \left(\frac{1}{p} e^{\frac{1}{q}|\log \frac{a^p}{b^q}|} + \frac{1}{q} e^{\frac{1}{p}|\log \frac{a^p}{b^q}|} - \frac{2|\log \frac{a^p}{b^q}|}{pq} - 1 \right) \\ &\le \frac{a^p}{p} + \frac{b^q}{q} - \frac{1}{2pq} \log^2 \frac{a^p}{b^q} \le \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

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