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Some New Weak $(H_p - L_p)$-Type Inequality for Weighted Maximal Operators of Partial Sums of Walsh–Fourier Series

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Abstract. In this paper, we introduce some new weighted maximal operators of the partial sums of the Walsh–Fourier series. We prove that for some "optimal" weights these new operators indeed are bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space weak$-L_p(G)$, for $0 < p < 1$. Moreover, we also prove sharpness of this result. As a consequence we obtain some new and well-known results.

Mathematics Subject Classification. 42C10.

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1. Introduction

All symbols used in this introduction can be found in Sect. 2.

It is well-known that the Walsh system does not form a basis in the space $L_1(G)$ (see e.g. [2,30]). Moreover, there exists a function in the dyadic Hardy space $H_1(G)$, such that the partial sums of f are not bounded in the L_1-norm. Uniform and pointwise convergence and some approximation properties of partial sums in $L_1(G)$ norms were investigated by Avdispahić and Memić [1], Gát, Goginava and Tkebuchava [12,13], Nagy [17], Onneweer [18] and Persson, Schipp, Tephnadze and Weisz [21]. Fine [9] obtained sufficient conditions for the uniform convergence which are completely analogous to the Dini–Lipschits conditions. Guličev [15] estimated the rate of uniform convergence of a Walsh–Fourier series2 by using Lebesgue constants and modulus of continuity. These problems for Vilenkin groups were investigated by Blatota, Nagy, Persson and Tephnadze [7] (see also [4–6]), Fridli [10] and Gát [11].

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To study convergence of subsequences of Fejér means and their restricted maximal operators on the martingale Hardy spaces $H_p(G)$ for $0 < p \leq 1/2$, the central role is played by the fact that any natural number $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_j 2^j, \quad n_j \in Z_2 \ (j \in \mathbb{N}), \tag{1}$$

where only a finite numbers of n_j differ from zero and their important characters $[n]$, $|n|$, $\rho(n)$ and $V(n)$ are defined by

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \quad \rho(n) = |n| - [n] \tag{2}$$

and

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|, \quad \text{for all} \ \ n \in \mathbb{N}$$

In particular, (see [8,16,22])

$$\frac{V(n)}{8} \leq \|D_n\|_1 \leq V(n)$$

from which it follows that, for any $F \in L_1(G)$, there exists an absolute constant c such that the following inequality holds:

$$\|S_n F\|_1 \leq cV(n) \|F\|_1 .$$

Moreover, for any $f \in H_1(G)$ (see [26])

$$\|S_n F\|_{H_1} \leq cV(n) \|F\|_{H_1} .$$

For $0 < p < 1$ in Refs. [24,25], the weighted maximal operator $\tilde{S}^{*,p}$, defined by

$$\tilde{S}^{*,p} F := \sup_{n \in \mathbb{N}} \frac{|S_n F|}{(n+1)^{1/p-1}} \tag{3}$$

was investigated and it was proved that the following inequalities hold:

$$\left\| \tilde{S}^{*} F \right\|_p \leq c_p \|F\|_{H_p}$$

Moreover, it was also proved that the rate of the sequence $\{(n+1)^{1/p-1}\}$ given in the denominator of (3) can not be improved.

In Refs. [26,27] (see also [3]), it was proved that if $F \in H_p(G)$, then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n F\|_{H_p} \leq c_p 2^{\rho(n)(1/p-1)} \|F\|_{H_p} ,$$

which implies that

$$\left\| \frac{S_n F}{2^{\rho(n)(1/p-1)}} \right\|_p \leq c_p \|F\|_{H_p} .$$

Moreover, if $0 < p < 1$, $\{n_k : \ k \geq 0\}$ is any increasing sequence of positive integers such that

$$\sup_{k\in\mathbb{N}} \rho\left(n_k\right) = \infty$$

and $\Phi : \mathbb{N}_+ \to [1, \infty)$ is any nondecreasing function, satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{2^{\rho(n_k)(1/p-1)}}{\Phi\left(n_k\right)} = \infty,$$

then there exists a martingale $F \in H_p(G)$, such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{S_{n_k}F}{\Phi\left(n_k\right)} \right\|_{\text{weak}-L_p} = \infty.$$

In this paper, we prove that the weighted maximal operator of the partial sums of the Walsh–Fourier defined by

$$\sup_{n\in\mathbb{N}} \frac{|S_n F|}{2^{\rho(n)(1/p-1)}}$$

is bounded from the martingale Hardy space $H_p(G)$ to the space weak $-$ $L_p(G)$, for $0 < p < 1$. We also prove the sharpness of this result (see Theorem 2). As a consequence, we obtain both some new and well-known results.

This paper is organized as follows: In order not to disturb our discussions later on some preliminaries are presented in Sect. 2. The main results and some of its consequences can be found in Sect. 3. The detailed proofs of the main results are given in Sect. 4.

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is $1/2$.

Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2's. The elements of G are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots), \quad \text{where} \ \ x_k = 0 \vee 1.$$

It is easy to give a base for the neighborhood of $x \in G :$

$$I_0\left(x\right) := G, \ I_n(x) := \{y \in G : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \ (n \in \mathbb{N}).$$

Denote $I_n := I_n\left(0\right), \quad \overline{I_n} := G \backslash I_n$ and

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G, \quad \text{for} \ \ n \in \mathbb{N}.$$

Then it is easy to prove that

$$\overline{I_M} = \bigcup_{s=0}^{M-1} I_s \backslash I_{s+1}. \tag{4}$$

The norms (or quasi-norms) of the Lebesgue space $L_p(G)$ and the weak Lebesgue space $L_{p,\infty}(G)$, $(0<p<\infty)$ are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p \, \mathrm{d}\mu \quad \text{and} \quad \|f\|_{\text{weak}-L_p}^p := \sup_{\lambda>0} \lambda^p \mu(f>\lambda).$$

The k-th Rademacher function $r_k(x)$ is defined by

$$r_k(x) := (-1)^{x_k} \qquad (x \in G, k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G by

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see e.g. [14, 22]).

If $f \in L_1(G)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Walsh system in the usual manner:

$$\widehat{f}(k) := \int_G f w_k \mathrm{d}\mu \qquad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \qquad (n \in \mathbb{N}_+)$$

$$D_n := \sum_{k=0}^{n-1} w_k \qquad (n \in \mathbb{N}_+).$$

Recall that (see [19, 20, 22])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases} \tag{5}$$

and

$$D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k \left(D_{2^{k+1}} - D_{2^k}\right), \quad \text{for} \quad n = \sum_{i=0}^{\infty} n_i 2^i. \tag{6}$$

Moreover, we have the following lower estimate (see [20]):

Lemma 1. *Let $n \in \mathbb{N}$ and $[n] \neq |n|$. Then,*

$$|D_n(x)| = \left|D_{n-2^{|n|}}(x)\right| \geq \frac{2^{[n]}}{4}, \quad \text{for} \quad x \in I_{[n]+1}\left(e_{[n]}\right).$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by ζ_n $(n \in \mathbb{N})$.

Denote by $F = (F_n, n \in \mathbb{N})$ the martingale with respect to $\digamma_n$ $(n \in \mathbb{N})$ (see e.g. [28]).

The maximal function F^* of a martingale F is defined by

$$F^* := \sup_{n \in \mathbb{N}} |F_n|.$$

In the case $f \in L_1(G)$, the maximal function f^* is given by

$$f^*(x) := \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\,\mathrm{d}\mu(u) \right|.$$

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G)$ consists of all martingales for which

$$\|F\|_{H_p} := \|F^*\|_p < \infty.$$

It is easy to check that for every martingale $F = (F_n, n \in \mathbb{N})$ and every $k \in \mathbb{N}$ the limit

$$\widehat{F}(k) := \lim_{n\to\infty} \int_G F_n(x)\, w_k(x)\,\mathrm{d}\mu(x)$$

exists and it is called the k-th Walsh–Fourier coefficients of F.

If $F := (S_{2^n} f : n \in \mathbb{N})$ is a regular martingale, generated by $f \in L_1(G)$, then (see e.g. [20,23,28])

$$\widehat{F}(k) = \widehat{f}(k), \quad k \in \mathbb{N}.$$

A bounded measurable function a is called p-atom, if there exists a dyadic interval I, such that

$$\int_I a\mathrm{d}\mu = 0, \ \ \|a\|_\infty \le \mu(I)^{-1/p}, \quad \mathrm{supp}\ (a) \subset I.$$

The dyadic Hardy martingale spaces $H_p(G)$ for $0 < p \le 1$ have an atomic characterization. Namely, the following holds (see [20,28,29]):

Lemma 2. *A martingale $F = (F_n, n \in \mathbb{N})$ belongs to $H_p(G)$ $(0 < p \le 1)$ if and only if there exists a sequence $(a_k,\ k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that for every $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty, \tag{7}$$

Moreover, $\|F\|_{H_p} \backsim \inf\left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decomposition of F of the form (7).

3. The Main Results with Applications

Our first main result reads:

Theorem 1. *Let $0 < p < 1$, $f \in H_p(G)$, n be defined by (1) and $\rho(n)$ be defined by (2). Then, the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by*

$$\widetilde{S}^{*,\nabla} F = \sup_{n\in\mathbb{N}} \frac{|S_n F|}{2^{\rho(n)(1/p-1)}}, \tag{8}$$

is bounded from the martingale Hardy space $H_p(G)$ to the space weak$-L_p(G)$.

Our second main result shows that Theorem 1 can not be improved in general, because it is sharp in some special senses:

Theorem 2. (a) *Let* $0<p<1$, n *be defined by* (1), $\rho(n)$ *be defined by* (2) *and* $\widetilde{S}^{*,\nabla}$ *is defined by* (8). *Then, there exists a sequence* $\{f_n, n\in\mathbb{N}\}$ *of* p*-atoms, such that*

$$\sup_{n\in\mathbb{N}}\frac{\left\|\widetilde{S}^{*,\nabla}f_n\right\|_p}{\|f_n\|_{H_p(G)}}=\infty.$$

(b) *Let* $0<p<1$, n *be defined by* (1) *and* $\rho(n)$ *be defined by* (2). *If* $\varphi:\mathbb{N}\to[1,\infty)$ *is a nondecreasing function, satisfying the condition*

$$\overline{\lim_{n\to\infty}}\frac{2^{\rho(n)(1/p-1)}}{\varphi(n)}=\infty, \tag{9}$$

then there exists a sequence $\{f_n, n\in\mathbb{N}\}$ *of* p*-atoms, such that*

$$\sup_{n\in\mathbb{N}}\frac{\left\|\sup_{k\in\mathbb{N}}\frac{|S_kf_n|}{\varphi(k)}\right\|_{weak-L_p(G)}}{\|f_n\|_{H_p(G)}}=\infty.$$

Theorem 1 implies the following result of Weisz [29] (see also [28]):

Corollary 1. *Let* $0<p<1$ *and* $f\in H_p(G)$. *Then the maximal operator* $S^{*,\triangle}$ *defined by*

$$S^{*,\triangle}F:=\sup_{n\in\mathbb{N}}|S_{2^n}F|$$

is bounded from the Hardy space $H_p(G)$ *to the Lebesgue space weak*$-L_p(G)$ *(and, thus, to the Lebesgue space* $L_p(G)$*).*

Moreover, Theorems 1 and 2 imply the following results (see [20]):

Corollary 2. *Let* $0<p<1$ *and* $f\in H_p(G)$. *Then, the maximal operator* $\widetilde{S}^{*,\nabla}$, *defined by*

$$S^{*,\nabla}F:=\sup_{k\in\mathbb{N}}|S_{n_k}F|$$

is bounded from the Hardy space $H_p(G)$ *to the Lebesgue space weak*$-L_p(G)$ *if and only if condition*

$$\sup_{k\in\mathbb{N}}\rho(n_k)<c<\infty$$

is fulfilled.

Remark 1. The statement in Corollary 2 holds also if the space weak$-L_p(G)$ is replaced by $L_p(G)$.

Corollary 3. (a) *Let* $0<p<1$ *and* $f\in H_p(G)$. *Then, the weighted maximal operator defined by*

$$\sup_{n\in\mathbb{N}}\frac{\left|S_{2^n+2^{n/2}}F\right|}{2^{\frac{n}{2}(1/p-1)}}$$

is bounded from the Hardy space $H_p(G)$ *to the Lebesgue space weak*$-L_p(G)$.

(b) (*Sharpness*) *Let* $\varphi : \mathbb{N} \to [1, \infty)$ *be a nondecreasing function, satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{2^{\frac{n}{2}(1/p-1)}}{\varphi(n)} = \infty.$$

Then, there exists sequence $\{f_n, n \in \mathbb{N}\}$ *of* p*-atoms, such that*

$$\sup_{n\in\mathbb{N}} \frac{\left\| \frac{S_{2^n+2^{n/2}} f_n}{\varphi\left(2^n+2^{n/2}\right)} \right\|_{weak-L_p(G)}}{\|f_n\|_{H_p(G)}} = \infty.$$

Corollary 4. (a) *Let* $0 < p < 1$ *and* $f \in H_p(G)$. *Then, the weighted maximal operator defined by*

$$\sup_{n\in\mathbb{N}} \frac{|S_{2^n+1}F|}{2^{n(1/p-2)}}$$

is bounded from the Hardy space $H_p(G)$ *to the Lebesgue space* $weak-L_p(G)$.

(b) (*Sharpness*) *Let* $\varphi : \mathbb{N} \to [1, \infty)$ *is a nondecreasing function, satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{2^{n(1/p-1)}}{\varphi(n)} = \infty.$$

Then, there exists sequence $\{f_n, n \in \mathbb{N}\}$ *of* p*-atoms, such that*

$$\sup_{n\in\mathbb{N}} \frac{\left\| \frac{S_{2^n+1} f_n}{\varphi(2^n+1)} \right\|_{weak-L_p(G)}}{\|f_n\|_{H_p(G)}} = \infty.$$

Finally, we note that Theorem 1 implies the following result of Tephnadze [27]:

Corollary 5. (a) *Let* $0 < p < 1$ *and* $f \in H_p(G)$. *Then, the weighted maximal operator* $\overset{\sim *,p}{S}$, *defined by* (3) *is bounded from the martingale Hardy space* $H_p(G)$ *to the Lebesgue space* $weak-L_p(G)$.

(b) *Let* $\{\varphi_n\}$ *be any nondecreasing sequence satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-1}}{\varphi_n} = \infty.$$

Then, there exists a martingale $f \in H_p(G)$, *such that*

$$\sup_{n\in\mathbb{N}} \left\| \frac{S_n f}{\varphi_n} \right\|_p = \infty.$$

4. Proofs of the Theorems

Proof of Theorem 1. Since σ_n is bounded from L_∞ to L_∞, by Lemma 2, the proof of Theorem 1 will be complete, if we prove that

$$t^p \mu \left\{ x \in \overline{I_M} : \widetilde{S}^{*,\nabla} a(x) \geq t \right\} \leq c_p < \infty, \quad t \geq 0 \tag{10}$$

for every p-atom a. In this paper, c_p (or C_p) denotes a positive constant depending only on p but which can be different in different places.

We may assume that a is an arbitrary p-atom, with support I, $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $S_n a(x) = 0$, when $n < 2^M$. Therefore, we can suppose that $n \geq 2^M$. Since $\|a\|_\infty \leq 2^{M/p}$, we obtain that

$$\begin{aligned}\frac{|S_n a(x)|}{2^{\rho(n)(1/p-1)}} &\leq \frac{1}{2^{\rho(n)(1/p-1)}} \|a\|_\infty \int_{I_M} |D_n(x+t)| \mu(t) \\ &\leq \frac{1}{2^{\rho(n)(1/p-1)}} 2^{M/p} \int_{I_M} |D_n(x+t)| \mu(t).\end{aligned}$$

Let $x \in I_s \backslash I_{s+1}$, $0 \leq s < [n] \leq M$ or $0 \leq s \leq M < [n]$. Then, it is easy to see that $x + t \in I_s \backslash I_{s+1}$ for $t \in I_M$ and if we combine (5) and (6) we get that $D_n(x+t) = 0$, for $t \in I_M$ so that

$$\frac{|S_n a(x)|}{2^{\rho(n)(1/p-1)}} = 0. \tag{11}$$

Let $I_s \backslash I_{s+1}$, $[n] \leq s \leq M$ or $[n] \leq s \leq M$. Then, it is easy to see that $x + t \in I_s \backslash I_{s+1}$ for $t \in I_M$ and if we again combine (5) and (6), we find that $D_n(x+t) \leq c2^s$, for $t \in I_M$ and

$$\begin{aligned}\frac{|S_n a(x)|}{2^{\rho(n)(1/p-1)}} &\leq c_p 2^{M/p} \frac{2^{s-M}}{2^{\rho(n)(1/p-1)}} \\ &\leq c_p \frac{2^{[n](1/p-1)+s+M(1/p-1)}}{2^{|n|(1/p-1)}} \leq c_p 2^{[n](1/p-1)+s} \leq c_p 2^s.\end{aligned} \tag{12}$$

By applying (11) and (12) for any $x \in I_s \backslash I_{s+1}$, $0 \leq s < M$, we find that

$$\widetilde{S}^{*,\nabla} a(x) = \sup_{n \in \mathbb{N}} \left(\frac{|S_n a(x)|}{2^{\rho(n)(1/p-1)}} \right) \leq C_p 2^{s/p}. \tag{13}$$

It immediately follows that for $s \leq M$, we have the following estimate

$$\widetilde{S}^{*,\nabla} a(x) \leq C_p 2^{M/p} \text{ for any } x \in I_s \backslash I_{s+1}, \quad s = 0, 1, \ldots, M$$

and also that

$$\mu\left\{x \in I_s \backslash I_{s+1} : \widetilde{S}^{*,\nabla} a(x) > C_p 2^{k/p}\right\} = 0, \quad k = M, M+1, \ldots \tag{14}$$

By combining (4) and (13), we get that

$$\left\{x \in \overline{I_N} : \widetilde{S}^{*,\nabla} a(x) \geq C_p 2^{k/p}\right\} \subset \bigcup_{s=k}^{M-1} \left\{x \in I_s \backslash I_{s+1} : \widetilde{S}^{*,\nabla} a(x) \geq C_p 2^{k/p}\right\}$$

and

$$\mu\left\{x \in \overline{I_M} : \widetilde{S}^{*,\nabla} a(x) \geq C_p 2^{k/p}\right\} \leq \sum_{s=k}^{M-1} \frac{1}{2^s} \leq \frac{2}{2^k}. \tag{15}$$

In view of (14) and (15), we can conclude that

$$2^k \mu\left\{x \in \overline{I_N} : \widetilde{S}^{*,\nabla} a(x) \geq C_p 2^{k/p}\right\} < c_p < \infty,$$

which shows that (10) holds and the proof of is complete. $\square$

Proof of Theorem 2. a) Set

$$f_{n_k}(x) = D_{2^{n_k+1}}(x) - D_{2^{n_k}}(x), \quad n_k \geq 3.$$

It is evident that

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, & \text{if } i = 2^{n_k}, \ldots, 2^{n_k+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have that

$$S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{2^{n_k}}(x), & \text{if } \ i = 2^{n_k}, \ldots, 2^{n_k+1}-1, \\ f_{n_k}(x), & \text{if } \ i \geq 2^{n_k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Since

$$D_{j+2^{n_k}}(x) - D_{2^{n_k}}(x) = w_{2^{n_k}} D_j(x), \quad j = 1, 2, \ldots, 2^{n_k}, \tag{17}$$

from (5), it follows that

$$\begin{aligned} \|f_{n_k}\|_{H_p} &= \left\| \sup_{n \in \mathbb{N}} S_{2^n} f_{n_k} \right\|_p = \|D_{2^{n_k+1}} - D_{2^{n_k}}\|_p \\ &= \|D_{2^{n_k}}\|_p \leq 2^{n_k(1-1/p)}. \end{aligned} \tag{18}$$

Let $q_{n_k}^s \in \mathbb{N}$ be such that

$$2^{n_k} < q_{n_k}^s < 2^{n_k+1}, \quad \text{where } \ [q_{n_k}^s] = s, \quad s = 0, \ldots, n_k - 1.$$

By combining (16) and (17), we can conclude that

$$\left| S_{q_{n_k}^s} f_{n_k}(x) \right| = \left| D_{q_{n_k}^s}(x) - D_{2^{n_k}}(x) \right| = \left| D_{q_{n_k}^s - 2^{n_k} - 1}(x) \right|$$

Let $x \in I_{s+1}(e_s)$. By using Lemma 1, we find that

$$\left| S_{q_{n_k}^s} f_{n_k}(x) \right| \geq c 2^s \tag{19}$$

so that

$$\frac{\left| S_{q_{n_k}^s} f_{n_k}(x) \right|}{2^{(1/p-1)\rho\left(q_{n_k}^s\right)}} \geq \frac{c 2^{s/p}}{2^{n_k(1/p-1)}}.$$

Hence,

$$\begin{aligned} &\int_G \left(\sup_{n \in \mathbb{N}} \frac{|S_n f_{n_k}(x)|}{2^{(1/p-1)\rho(n)}} \right)^p \mathrm{d}\mu(x) \\ &\geq \sum_{s=0}^{n_k-1} \int_{I_{s+1}(e_s)} \left(\frac{\left| S_{q_{n_k^s}} f_{n_k}(x) \right|}{2^{(1/p-1)\rho\left(q_{n_k}^s\right)}} \right)^p \mathrm{d}\mu(x) \\ &\geq c_p \sum_{s=0}^{n_k-1} \frac{1}{2^s} \frac{2^s}{2^{n_k(1-p)}} \geq \frac{C_p}{2^{n_k(1-p)}} \sum_{s=1}^{n_k} 1 \geq \frac{C_p n_k}{2^{n_k(1-p)}}. \end{aligned} \tag{20}$$

Finally, by combining (18) and (20), we obtain that

$$\frac{\left(\int_G\left(\sup_{n\in\mathbb{N}}\frac{\left|S_nf_{n_k}(x)\right|}{2^{(1/p-1)\rho(n)}}\right)^p \mathrm{d}\mu\left(x\right)\right)^{1/p}}{\|f_{n_k}\|_{H_p}}\geq\frac{\left(\frac{C_pn_k}{2^{n_k(1-p)}}\right)^{1/p}}{2^{n_k(1-1/p)}}\geq c_pn_k^{1/p}\to\infty,\quad \text{as}\quad k\to\infty,$$

so the proof of part a) is complete.

(b) Under condition (9), we can choose $q_{n_k}^{s_k}\in\mathbb{N}$ for some $0\leq s_k<n_k$ such that

$$2^{n_k}<q_{n_k}^{s_k}<2^{n_k+1},\quad \text{where}\ \ [q_{n_k}]=s_k$$

and

$$\lim_{k\to\infty}\frac{2^{\rho(q_{n_k}^{s_k})(1/p-1)}}{\varphi\left(q_{n_k}^{s_k}\right)}=\infty.$$

Let $x\in I_{s_k+1}\left(e_{s_k}\right)$. By using (19) we get that

$$\left|S_{q_{n_k}^{s_k}}f_{n_k}\left(x\right)\right|\geq c2^{s_k}$$

so that

$$\frac{\left|S_{q_{n_k}^{s_k}}f_{n_k}\left(x\right)\right|}{\varphi\left(q_{n_k}^{s_k}\right)}\geq\frac{c2^{s_k}}{\varphi\left(q_{n_k}^{s_k}\right)}.$$

Hence, we find that

$$\mu\left\{x\in G:\frac{\left|S_{q_{n_k}^{s_k}}f_{n_k}\left(x\right)\right|}{\varphi\left(q_{n_k}^{s_k}\right)}\geq\frac{c2^{s_k}}{\varphi\left(q_{n_k}^{s_k}\right)}\right\}\geq\mu\left(I_{s_k+1}(e_{s_k})\right)>c/2^{s_k}. \tag{21}$$

By combining (18) and (21), we get that

$$\begin{aligned}&\frac{\frac{c2^{s_k}}{\varphi\left(q_{n_k}^{s_k}\right)}\left(\mu\left\{x\in G:\frac{\left|S_{q_{n_k}^{s_k}}f_{n_k}(x)\right|}{\varphi\left(q_{n_k}^{s_k}\right)}\geq\frac{c2^{s_k}}{\varphi\left(q_{n_k}^{s_k}\right)}\right\}\right)^{1/p}}{\|f_{n_k}\|_{H_p(G)}}\\
&\quad\geq\frac{c_p2^{s_k}}{\varphi\left(q_{n_k}^{s_k}\right)2^{n_k(1-1/p)}}\frac{1}{2^{s_k/p}}\\
&\quad=\frac{c_p2^{n_k(1/p-1)}}{2^{s_k(1/p-1)}\varphi\left(q_{n_k}^{s_k}\right)}\\
&\quad=\frac{c_p2^{\rho\left(q_{n_k}^{s_k}\right)(1/p-1)}}{\varphi\left(q_{n_k}^{s_k}\right)}\to\infty\qquad\text{as}\qquad k\to\infty,\end{aligned}$$

so also part (b) is proved and the proof is complete. □

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Declarations

Conflict of Interest The authors declare that they have no competing interests.

References

[1] Avdispahić, M., Memić, N.: On the Lebesgue test for convergence of Fourier series on unbounded Vilenkin groups. Acta Math. Hung. **129**(4), 381–392 (2010)

[2] Bary, N.K.: Trigonometric Series. Gos. Izd. Fiz. Mat. Lit, Moscow (1961). (**Russian**)

[3] Baramidze, D., Persson, L.-E., Tephnadze, G.: Some new $(H_p - L_p)$ type inequalities for weighted maximal operators of partial sums of Walsh–Fourier series. Positivity **27**(3), 38 (2023)

[4] Blahota, I.: Approximation by Vilenkin–Fourier sums in $L_p(G_m)$. Acta Acad. Paed. Nyireg. **13**, 35–39 (1992)

[5] Blahota, I., Nagy, K.: Approximation by θ-means of Walsh–Fourier series. Anal. Math. **44**, 57–71 (2018)

[6] Blahota, I., Nagy, K., Tephnadze, G.: Approximation by Marcinkiewicz θ-means of double Walsh–Fourier series. Math. Inequal. Appl. **22**(3), 837–853 (2019)

[7] Blahota, I., Nagy, K., Persson, L.E., Tephnadze, G.: A sharp boundedness result concerning maximal operators of Vilenkin–Fourier series on martingale Hardy spaces. Georgian Math. J. **26**(3), 351–360 (2019)

[8] Blahota, I., Persson, L.E., Tephnadze, G.: Two-sided estimates of the Lebesgue constants with respect to Vilenkin systems and applications. Glasg. Math. J. **60**(1), 17–34 (2018)

[9] Fine, N.I.: On Walsh function. Trans. Am. Math. Soc. **65**, 372–414 (1949)

[10] Fridli, S.: Approximation by Vilenkin–Fourier series. Acta Math. Hung. **47**(1–2), 33–44 (1986)

[11] Gát, G.: Best approximation by Vilenkin-like systems. Acta Acad. Paed. Nyireg. **17**, 161–169 (2001)

[12] Gát, G., Goginava, U., Tkebuchava, G.: Convergence in measure of logarithmic means of quadratical partial sums of double Walsh–Fourier series. J. Math. Appl. **323**(1), 535–549 (2006)

[13] Goginava, U., Tkebuchava, G.: Convergence of subsequence of partial sums and logarithmic means of Walsh–Fourier series. Acta Sci. Math. **72**, 159–177 (2006)

[14] Golubov, B.I., Efimov, A.V., Skvortsov, V.A.: Walsh Series and Transforms. (Russian) Nauka, Moscow (1987), English transl, Mathematics and its Applications (Soviet Series), 64. Kluwer Academic Publishers Group, Dordrecht (1991)

[15] Guličev, N.V.: Approximation to continuous functions by Walsh–Fourier series. Anal. Math. **6**, 269–280 (1980)

[16] Lukomskii, S.F.: Lebesgue constants for characters of the compact zero-dimensional Abelian groups. East J. Approx. **15**(2), 219–231 (2009)

[17] Nagy, K.: Approximation by Cesáro means of negative order of Walsh–Kaczmarz-Fourier series. East J. Approx. **16**(3), 297–311 (2010)

[18] Onneweer, C.W.: On L-convergence of Walsh–Fourier series. Int. J. Math. Sci. **1**, 47–56 (1978)

[19] Paley, A.: A remarkable series of orthogonal functions. Proc. Lond. Math. Soc. **34**, 241–279 (1932)

[20] Persson, L.E., Tephnadze, G., Weisz, F.: Martingale Hardy Spaces and Summability of One-Dimenional Vilenkin–Fourier Series. Birkhäuser/Springer, Boston (2022)

[21] Persson, L.-E., Schipp, F., Tephnadze, G., Weisz, F.: An analogy of the Carleson–Hunt theorem with respect to Vilenkin systems. J. Fourier Anal. Appl. **28**(48), 1–29 (2022)

[22] Schipp, F., Wade, W., Simon, P., Pál, J.: Walsh Series, An Introduction to Dyadic Harmonic Analysis. Akademiai Kiadó, Budapest (1990)

[23] Simon, P.: A note on the of the Sunouchi operator with respect to Vilenkin systems. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **431**, 101–116 (2001)

[24] Tephnadze, G.: On the Vilenkin–Fourier coefficients. Georgian Math. J. **20**(1), 169–177 (2013)

[25] Tephnadze, G.: On the partial sums of Vilenkin–Fourier series. J. Contemp. Math. Anal. **49**(1), 23–32 (2014)

[26] Tephnadze, G.: On the partial sums of Walsh-Fourier series. Colloq. Math. **141**(2), 227–242 (2015)

[27] Tephnadze, G.: On the convergence of partial sums with respect to Vilenkin system on the martingale Hardy spaces. J. Contemp. Math. Anal. **53**(5), 294–306 (2018)

[28] Weisz, F.: Martingale Hardy Spaces and Their Applications in Fourier Analysis. Springer, Berlin (1994)

[29] Weisz, F.: Hardy spaces and Cesáro means of two-dimensional Fourier series. Bolyai Soc. Math. Stud. **5**, 353–367 (1996)

[30] Zygmund, A.: Trigonometric Series, vol. 1. Cambridge University Press, Cambridge (1959)

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