Second-order PDEs in 4D with half-flat conformal structure

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Abstract

We study second-order PDEs in 4D for which the conformal structure defined by the characteristic variety of the equation is half-flat (self-dual or anti-self-dual) on every solution. We prove that this requirement implies the Monge-Ampère property. Since half-flatness of the conformal structure is equivalent to the existence of a nontrivial dispersionless Lax pair, our result explains the observation that all known scalar second-order integrable dispersionless PDEs in dimensions four and higher are of Monge-Ampère type. Some partial classification results of Monge-Ampère equations in 4D with half-flat conformal structure are also obtained.

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1 Introduction

Let $u(x)$ be a function of the independent variable $x = (x_1, \dots, x_d) \in M$. This paper is mainly devoted to the case of dimension four, i.e., $d = \dim M = 4$. We consider the general class of second-order PDEs of the form

$$F(x, u, Du, D^2u) = 0 \tag{1}$$

where $Du = \{u_\alpha\}$, $D^2u = \{u_{\alpha\beta}\}$ denote the collection of all first- and second-order partial derivatives of u (we use the notation $u_\alpha = \partial_{x_\alpha} u$, etc). The goal of this paper is to study the implications of integrability on the form of the equation.

1.1 Formulation of the problem and the main results

We will assume that equation (1) is *non-degenerate* in the sense that its characteristic variety defined by the equation

$$\sum_{\alpha\le\beta} \frac{\partial F}{\partial u_{\alpha\beta}}\, p_\alpha p_\beta = 0 \tag{2}$$

is a non-degenerate quadric on every solution, i.e., for every second jet $j^2u = (x, u, Du, D^2u)$ that satisfies (1). Similarly, we will say that equation (1) has rank k if quadratic form (2) has rank k on every solution. In the non-degenerate case quadratic form (2) gives rise to the conformal structure $[g]$ represented by $g = g_{\alpha\beta}dx_\alpha dx_\beta$ where $(g_{\alpha\beta})$ is the inverse to the matrix of the above quadratic form. We will be interested in PDEs (1) in 4D whose conformal structure is half-flat (self-dual or anti-self-dual) on every solution (see Section 1.3). Examples thereof include heavenly type equations such as the second heavenly equation

$$u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0 \tag{3}$$

which governs self-dual Ricci-flat geometry [21]. Note that equation (3) is of Monge-Ampère type, which is also the case for all other known integrable PDEs of type (1) for $d \ge 4$.

Recall that a Monge-Ampère equation is a linear combination of all possible minors of the Hessian matrix $Hess(u)$ with coefficients being arbitrary functions of the first jet $j^1u = (x, u, Du)$. Our main result explains this observation.

Theorem 1 *For a non-degenerate PDE (1) in dimension $d = 4$, half-flatness of the conformal structure $[g]$ on every solution implies the Monge-Ampère property.*

It was demonstrated in [10] that for a 4D dispersionless Hirota type equation $F(D^2u) = 0$, the following conditions are equivalent:

(a) conformal structure $[g]$ is half-flat on every solution;

(b) the equation is integrable by the method of hydrodynamic reductions.

Furthermore, it was shown that either of the above conditions implies the Monge-Ampère property. Thus, Theorem 1 is a generalization of the results of [10]. We emphasize that, unlike the method of hydrodynamic reductions, the half-flatness test is fully contact-invariant and applies to general second-order PDEs (1).

It was demonstrated recently in [4] that in 4D, half-flatness of $[g]$ is equivalent to the existence of a non-trivial dispersionless Lax pair in parameter-dependent vector fields. This leads to the following result.

Theorem 2 *For a non-degenerate PDE (1) in dimension $d = 4$, integrability via a non-trivial dispersionless Lax pair implies the Monge-Ampère property.*

Attempts to generalise this result to higher dimensions meet an immediate obstacle: all known multi-dimensional ($d > 4$) PDEs possessing a dispersionless Lax pair are degenerate. For instance, both the 6-dimensional version of the second heavenly equation [26, 22],

$$u_{15} + u_{26} + u_{13}u_{24} - u_{14}u_{23} = 0,$$

as well as the 8-dimensional generalisation of the general heavenly equation [15],

$$(u_{x_1y_2} - u_{x_2y_1})(u_{x_3y_4} - u_{x_4y_3}) + (u_{x_2y_3} - u_{x_3y_2})(u_{x_1y_4} - u_{x_4y_1}) + (u_{x_3y_1}u_{x_1y_3})(u_{x_2y_4} - u_{x_4y_2}) = 0,$$

have quadratic forms (2) of rank 4. We however have the following generalisation of Theorem 2.

Theorem 3 *For a rank 4 PDE (1) in any dimension $d \geq 4$, integrability via a non-trivial dispersionless Lax pair implies the Monge-Ampère property.*

It should be noted that in 3D the existence of a nontrivial dispersionless Lax pair does not imply the Monge-Ampère property. Generic integrable second-order equations in 3D have transcendental dependence on 2-jets, see [11].

Remark 1. The non-degeneracy condition has a simple geometric interpretation in any dimension d. Without any loss of generality assume that equation (1) is of dispersionless Hirota type,

$$F(D^2u) = 0. \tag{4}$$

The corresponding linearised equation is

$$\sum_{\alpha\leq\beta} \frac{\partial F}{\partial u_{\alpha\beta}}\, v_{\alpha\beta} = 0, \tag{5}$$

here $\alpha, \beta \in \{1, \dots, n\}$. Equation (4) can be viewed as the equation of a hypersurface $\mathcal{E}$ in the Lagrangian Grassmannian Λ (locally identified with $n \times n$ symmetric matrices $u_{\alpha\beta}$). For any point $L \in \mathcal{E}$ the tangent space $T_L\Lambda \simeq S^2L^*$ (which can also be identified with the space of $n \times n$ symmetric matrices $v_{\alpha\beta}$) contains two ingredients:

(a) the Veronese cone $V = \{p \odot p : p \in L^*\}$ of rank 1 matrices;

(b) the tangent hyperplane $T_L\mathcal{E}$ defined by equation (5);

here $\odot$ denotes symmetric product. In this language, the non-degeneracy of (4) means that $T_L\mathcal{E}$ is not tangential to V.

Moreover, consider the bilinear form on $L^* \simeq T^*M$ associated to (2) at the points of $T_L\mathcal{E} \cap V$:

$$c_F(p, q) = \sum \frac{1}{1 + \delta_{\alpha\beta}} \frac{\partial F}{\partial u_{\alpha\beta}}\, p_\alpha q_\beta, \quad p, q \in L^*.$$

Since $T_pV = \{p \odot q : p, q \in L^*\}$ for $p \neq 0$, the kernel of this form is

$$\operatorname{Ker} c_F = \{p \in L^* : T_pV \subset T_L\mathcal{E}\}.$$

This is a linear subspace in L^* of dimension equal to $\operatorname{corank} c_F = d - k$.

Remark 2. Although the converse of Theorem 1 is not true, our proof implies a somewhat stronger result which says that if equation (1) has half-flat conformal structure on every solution then the dependence of F on the second-order partial derivatives of u is essentially the same as for 4D integrable symplectic Monge-Ampère equations classified in [6]. That is, freezing the first jet of u (by giving the variables x, u, Du arbitrary constant values) we obtain an *integrable* symplectic Monge-Ampère equation. Still, a complete classification of PDEs (1) with half-flat conformal structure is out of reach at present.

In Section 2 we discuss examples and partial classification results of PDEs with half-flat conformal structure. These are obtained as deformations of some known integrable equations. For example, consider the so-called general heavenly equation [24],

$$\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{23}u_{14} = 0, \quad \alpha + \beta + \gamma = 0,$$

which, according to [6], is the most generic integrable symplectic Monge-Ampère equation in 4D. Note that without the condition $\alpha + \beta + \gamma = 0$ the above equation is not integrable, which can be explained on the basis of Theorem 1: this condition makes it a PDE of Monge-Ampère type. We will show that the general heavenly equation possesses the following deformation with half-flat conformal structure:

$$(x_1 - x_2)(x_3 - x_4)u_{12}u_{34} + (x_3 - x_1)(x_2 - x_4)u_{13}u_{24} + (x_2 - x_3)(x_1 - x_4)u_{23}u_{14} = 0.$$

It will also be shown that these two equations are contact non-equivalent.

Proofs of Theorems 1, 2 and 3 will be given in Section 3. In Section 4 we discuss translation non-invariance of generic integrable second-order PDEs.

1.2 The Monge-Ampère property

Let us represent equation (1) in evolutionary form,

$$u_{11} = f(x, u, Du, u_{1i}, u_{ij}) \tag{6}$$

where the Latin indices take values $i, j, k \in \{2, 3, 4\}$. The proof of Theorem 1 will require explicit differential constraints for the right-hand side f that are equivalent to the Monge-Ampère property. These have only been known in low dimensions [3, 23, 5, 13]. In full generality they were obtained recently in [10]. In 4D the Monge-Ampère conditions consist of several groups of equations for f. First of all, for every $i \in \{2, 3, 4\}$ one has the relations

$$f_{u_{ii}} f_{u_{1i}u_{1i}} + f_{u_{ii}u_{ii}} = 0, \quad f_{u_{1i}} f_{u_{1i}u_{1i}} + 2 f_{u_{1i}u_{ii}} = 0.$$

Secondly, for every pair of distinct indices $i \neq j \in \{2,3,4\}$ one has the relations

$$
\begin{gathered}
f_{u_{1j}} f_{u_{1i}u_{1i}} + 2 f_{u_{1i}} f_{u_{1i}u_{1j}} + 2 f_{u_{1i}u_{ij}} + 2 f_{u_{1j}u_{ii}} = 0, \\
f_{u_{ij}} f_{u_{1i}u_{1i}} + 2 f_{u_{ii}} f_{u_{1i}u_{1j}} + 2 f_{u_{ii}u_{ij}} = 0, \\
f_{u_{jj}} f_{u_{1i}u_{1i}} + f_{u_{ii}} f_{u_{1j}u_{1j}} + 2 f_{u_{ij}} f_{u_{1i}u_{1j}} + 2 f_{u_{ii}u_{jj}} + f_{u_{ij}u_{ij}} = 0.
\end{gathered}
$$

Furthermore, for every triple of distinct indices $i \neq j \neq k \in \{2,3,4\}$ one has the relations

$$
\begin{gathered}
f_{u_{1k}} f_{u_{1i}u_{1j}} + f_{u_{1j}} f_{u_{1i}u_{1k}} + f_{u_{1i}} f_{u_{1j}u_{1k}} + f_{u_{1i}u_{jk}} + f_{u_{1j}u_{ik}} + f_{u_{1k}u_{ij}} = 0, \\
f_{u_{jk}} f_{u_{1i}u_{1i}} + 2 f_{u_{ik}} f_{u_{1i}u_{1j}} + 2 f_{u_{ij}} f_{u_{1i}u_{1k}} + 2 f_{u_{ii}} f_{u_{1j}u_{1k}} + 2 f_{u_{ii}u_{jk}} + 2 f_{u_{ij}u_{ik}} = 0.
\end{gathered}
$$

Due to the contact invariance of the Monge-Ampère class, this system of 25 relations is invariant under arbitrary contact transformations.

1.3 Half-flatness and Lax pairs

In 4D, the key invariant of a conformal structure $[g]$ is its Weyl tensor W. Let us introduce its self-dual and anti-self-dual parts, $W_\pm = \frac{1}{2}(W \pm *W)$, where $*$ is the Hodge star operator defined as

$$
*W^i_{jkl} = \frac{1}{2}\sqrt{\det g}\; g^{ia} g^{bc} \epsilon_{ajbd} W^d_{ckl}.
$$

A conformal structure is said to be half-flat (self-dual or anti-self-dual) if W_- or W_+ vanishes. Note that the conditions of self-duality and anti-self-duality are switched under the change of orientation. Integrability of the conditions of half-flatness by the twistor construction is due to Penrose [20] who observed that half-flatness of $[g]$ is equivalent to the existence of a 3-parameter family of totally null surfaces; see also [7] for a more direct treatment of half-flat structures in specially adapted coordinates. Thus, for the second heavenly equation (3) we have

$$
g = dx_1 dx_3 + dx_2 dx_4 - u_{22} dx_3^2 + 2u_{12} dx_3 dx_4 - u_{11} dx_4^2,
$$

and direct calculation shows that the conformal structure $[g]$ is half-flat on every solution of (3).

Equations with half-flat conformal structure are known to possess dispersionless Lax pairs, that is, vector fields X, Y depending on $j^2 u$ and an auxiliary parameter λ such that the Frobenius integrability condition $[X, Y] \in \mathrm{span}(X, Y)$ holds identically modulo the equation (and its differential consequences). For the second heavenly equation we have

$$
X = \partial_4 + u_{22}\partial_2 - u_{12}\partial_1 + \lambda\partial_1, \quad Y = \partial_3 - u_{12}\partial_2 + u_{22}\partial_1 - \lambda\partial_2,
$$

here $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$. Integral surfaces of the involutive distribution $\langle X, Y\rangle$ give totally null surfaces of the corresponding conformal structure $[g]$, thus providing an alternative proof of its half-flatness. This is a particular case of the general result of [4] relating half-flatness to the existence of a dispersionless Lax pair. The main technical observation of [4] is that any nontrivial Lax pair must be characteristic, i.e. null with respect to the conformal structure defined by the characteristic variety.

Lax pairs in vector fields play a key role in the twistor-theoretic technique [1, 20, 9, 8, 25], dispersionless d-bar method [2] and the novel inverse scattering approach [18, 19] to PDEs with half-flat conformal structure.

2 Examples and classification results

In this section we provide examples and partial classification results of integrable second-order PDEs in 4D. Integrability is manifested via the half-flatness condition. All our examples are obtained as translationally non-invariant deformations of the following heavenly type Monge-Ampère equations appearing in self-dual Ricci-flat geometry:

1. $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$ (second heavenly equation [21]);
2. $u_{13}u_{24} - u_{14}u_{23} = 1$ (first heavenly equation [21]);
3. $\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0$ (general heavenly equation [24]), $\alpha + \beta + \gamma = 0$.

In all cases we present dispersionless Lax pairs in λ-dependent vector fields. We refer to [16] for further examples of this kind.

2.1 Equations of the second heavenly type

Here we consider equations of the form

$$u_{13} + u_{24} + f(x, u, Du)(u_{11}u_{22} - u_{12}^2) = 0$$

where f is to be determined from the requirement of half-flatness. The corresponding conformal structure can be represented as

$$g = dx_1dx_3 + dx_2dx_4 - fu_{22}dx_3^2 + 2fu_{12}dx_3dx_4 - fu_{11}dx_4^2,$$

note that $\det g = 1$. With the choice $\sqrt{\det g} = 1$ one can show that the requirement of vanishing of the anti-self-dual part W_- leads to a system of differential constraints for the function f with the general solution

$$f = c_0(x_1x_3 + x_2x_4) + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5$$

where the constants c_i satisfy a single quadratic constraint $c_1c_3+c_2c_4-c_0c_5 = 0$ (on the contrary, the vanishing of the self-dual part W_+ leads to the degenerate case $f = 0$).

Modulo linear transformations of the independent variables one can bring f to one of the four normal forms summarised in Table 1 below.

Table 1: Equations of the second heavenly type

Function f	Lax pair X, Y
$f = x_1x_3 + x_2x_4$	$X = \partial_4 + (x_1x_3 + x_2x_4)(u_{11}\partial_2 - u_{12}\partial_1) + \frac{x_1-\lambda x_2}{x_4+\lambda x_3}\partial_1,$ $Y = \partial_3 + (x_1x_3 + x_2x_4)(u_{22}\partial_1 - u_{12}\partial_2) - \frac{x_1-\lambda x_2}{x_4+\lambda x_3}\partial_2$
$f = x_1$	$X = \partial_4 + x_1(u_{11}\partial_2 - u_{12}\partial_1) + \frac{x_1}{x_4-\lambda}\partial_1,$ $Y = \partial_3 + x_1(u_{22}\partial_1 - u_{12}\partial_2) - \frac{x_1}{x_4-\lambda}\partial_2$
$f = x_3$	$X = \partial_4 + x_3(u_{11}\partial_2 - u_{12}\partial_1) + \frac{\lambda-x_2}{x_3}\partial_1,$ $Y = \partial_3 + x_3(u_{22}\partial_1 - u_{12}\partial_2) - \frac{\lambda-x_2}{x_3}\partial_2$
$f = 1$	$X = \partial_4 + u_{11}\partial_2 - u_{12}\partial_1 + \lambda\partial_1,$ $Y = \partial_3 + u_{22}\partial_1 - u_{12}\partial_2 - \lambda\partial_2$

It can be demonstrated that contact symmetry algebras of PDEs corresponding to cases 1 and 2 of Table 1 depend on three arbitrary functions of two variables. A more detailed analysis revealed that these cases are in fact point-equivalent. The explicit link is provided by the formulae

$$X_1 = x_1 + \frac{x_2 x_4}{x_3}, \qquad X_2 = \frac{x_2}{x_3}, \qquad X_3 = -\frac{1}{x_3}, \qquad X_4 = \frac{x_4}{x_3}, \qquad U = x_3 u;$$

here $u(x_i)$ and $U(X_i)$ satisfy PDEs corresponding to cases 1 and 2, respectively. Similarly, contact symmetry algebras of PDEs correspoding to cases 3 and 4 depend on four arbitrary functions of two variables. These PDEs are related by the point transformation

$$X_1 = x_1 x_3, \qquad X_2 = \frac{x_2}{x_3^2}, \qquad X_3 = -\frac{1}{2x_3^2}, \qquad X_4 = x_4, \qquad U = x_3 u + \frac{x_1 x_2^2}{2x_3};$$

here $u(x_i)$ and $U(X_i)$ satisfy PDEs corresponding to cases 3 and 4, respectively. Thus, Table 1 contains only two contact non-equivalent cases.

2.2 Equations of the first heavenly type

Here we consider equations of the form

$$u_{13}u_{24} - u_{14}u_{23} = f(x, u, Du).$$

The corresponding conformal structure can be represented as

$$g = u_{13}dx_1dx_3 + u_{14}dx_1dx_4 + u_{23}dx_2dx_3 + u_{24}dx_2dx_4.$$

Note that by virtue of the equation we have $\det g = \frac{1}{16}f^2$ so that $\sqrt{\det g} = \frac{1}{4}f$. With this choice of the square root the requirement of vanishing of the anti-self-dual part W_- leads to a system of differential constraints for f which we do not present here explicitly due to its complexity (on the contrary, the vanishing of the self-dual part W_+ leads to the degenerate case $f = 0$). The classification of functions f satisfying the condition $W_- = 0$ is performed modulo the following point transformations which leave the class under study form-invariant (this considerably reduces the number of cases):

(a) changes of variables $x_1 \to a(x_1, x_2),\ x_2 \to b(x_1, x_2),\ x_3 \to p(x_3, x_4),\ x_4 \to q(x_3, x_4)$;

(b) simultaneous permutations $\{x_1 \leftrightarrow x_3\},\ \{x_2 \leftrightarrow x_4\}$;

(c) translations $u \to u + f(x_1, x_2) + g(x_3, x_4)$;

(d) rescalings $u \to cu$.

The classification results are summarised in Table 2 below:

Table 2: Equations of the first heavenly type

Function f	Lax pair X, Y
$f = \frac{u_1u_3}{(x_2-x_4)^2}$	$X = u_{13}\partial_4 - u_{14}\partial_3 + \frac{u_3(\lambda-x_2)}{(\lambda-x_4)(x_2-x_4)}\partial_1$, $Y = u_{23}\partial_4 - u_{24}\partial_3 + \frac{u_3(\lambda-x_2)}{(\lambda-x_4)(x_2-x_4)}\partial_2$
$f = u_1$	$X = u_{13}\partial_4 - u_{14}\partial_3 + (\lambda - x_2)\partial_1$, $Y = u_{23}\partial_4 - u_{24}\partial_3 + (\lambda - x_2)\partial_2$
$f = 1$	$X = u_{13}\partial_4 - u_{14}\partial_3 + \lambda\partial_1$, $Y = u_{23}\partial_4 - u_{24}\partial_3 + \lambda\partial_2$

Contact non-equivalence of the corresponding PDEs can be seen from the structure of their symmetries: one can show that contact symmetry algebras of PDEs corresponding to cases 1-3 of Table 2 depend on two, three and four arbitrary functions of two variables, respectively.

2.3 Equations of the general heavenly type

Here we classify equations of the form

$$u_{13}u_{24} - u_{14}u_{23} = f(x, u, Du)(u_{13}u_{24} - u_{12}u_{34}).$$

In this case the explicit form of $[g]$, as well as the system of differential constraints for f coming from the requirement of half-flatness are rather lengthy so we skip the details. One can show that the generic such f is given by the formula

$$f = \frac{(\phi_1 - \phi_2)(\phi_3 - \phi_4)}{(\phi_1 - \phi_4)(\phi_3 - \phi_2)}$$

where $\phi_i = \phi_i(x_i, u_i)$ are arbitrary functions of the indicated variables. If $\phi_i \neq const$ one can set $\phi_i = x_i$ via a suitable contact transformation, thus, normal forms depend on how many functions among ϕ_i are non-constant (five different forms altogether).

PDEs corresponding to such normal forms are contact non-equivalent. Indeed, when s functions among ϕ_i are constant, the contact symmetry algebra depends on s arbitrary functions of two variables, $0 \leq s \leq 4$. The two limiting cases are summarised in Table 3 below.

Table 3: Equations of the general heavenly type

Function f	Lax pair X, Y
$f = \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)}$	$X = (x_4 - x_3)(x_1 + \lambda)u_{34}\partial_1 + (x_3 - x_1)(x_4 + \lambda)u_{13}\partial_4 + (x_1 - x_4)(x_3 + \lambda)u_{14}\partial_3,$ $Y = (x_4 - x_3)(x_2 + \lambda)u_{34}\partial_2 + (x_3 - x_2)(x_4 + \lambda)u_{23}\partial_4 + (x_2 - x_4)(x_3 + \lambda)u_{24}\partial_3$
$f = c,\ c \neq 0, 1$	$X = u_{34}\partial_1 - u_{13}\partial_4 + \lambda(u_{14}\partial_3 - u_{34}\partial_1),$ $Y = u_{23}\partial_4 - u_{34}\partial_2 + (1 - c)\lambda(u_{34}\partial_2 - u_{24}\partial_3)$

In the last case (which is equivalent to the general heavenly equation) the contact symmetry algebra depends on four arbitrary functions of two variables, while in the first case the contact symmetry algebra involves a certain number of functions of one variable only. This confirms contact non-equivalence of the corresponding equations.

3 Proofs of the main results

Proof of Theorem 1 is based on direct computation of the Weyl tensor of the conformal structure $[g]$,

$$W_{ijkl} = R_{ijkl} - w_{ik}g_{jl} - w_{jl}g_{ik} + w_{jk}g_{il} + w_{il}g_{jk} = 0, \tag{7}$$

where $R_{ijkl} = g_{is}R^s_{jkl}$ is the Riemann curvature tensor, $w_{ij} = \frac{1}{2}R_{ij} - \frac{R}{12}g_{ij}$ is the 4-dimensional Schouten tensor, R_{ij} is the Ricci tensor and R is the scalar curvature. The key calculation (performed in Mathematica) can be split into several steps:

- Calculate the Weyl tensor W of the conformal structure $[g]$ defined by the characteristic variety of equation (6). Note that since $[g]$ depends on no more than second-order partial derivatives of u, components of W will depend on partial derivatives of u up to the order at most four. Furthermore, fourth-order partial derivatives of u will enter linearly.

- Restrict W to a solution of (6), that is, use (6) and its differential consequences to eliminate from W all partial derivatives of u of the form $u_{11}, Du_{11}, D^2u_{11}$.

- Calculate the anti-self-duality condition $W_- = 0$ (the self-duality case $W_+ = 0$ will be essentially the same).

- Take coefficients of W_- at the remaining fourth-order partial derivatives of u, that is, at u_{ijkl}, and equate them to zero. This will result in a system of second-order relations for the right-hand side f of equation (6), 30 linearly independent equations altogether. These relations will involve partial derivatives of f with respect to the variables u_{ij} only.

- Show that 30 second-order relations for f obtained at the previous step contain all of the 25 relations from Section 1.2 that characterise Monge-Ampère equations in 4D. The remaining 5 relations are the additional necessary conditions for integrability; they are equivalent to the requirement that the equation obtained from (6) by freezing the first jet of u (by giving the variables x, u, Du arbitrary constant values) results in an integrable Hirota type equation (we will not use these extra relations in what follows).

- To simplify the calculations we utilise the fact that the induced action of the contact group on the space of 1-jets of f has a unique open orbit (its complement consists of 1-jets of degenerate systems). This property plays a key role in the proof by allowing one to assume that all sporadic factors depending on first-order derivatives of f that arise in the process of Gaussian elimination at the previous step, are nonzero. This considerably simplifies the arguments by eliminating non-essential branching.

  More precisely, calculations are simplified considerably if, after having computed the Weyl tensor W (so that all remaining calculations are entirely algebraic), we use the 1-jet of f defined as $f_{u_{14}} = f_{u_{23}} = 1$, while all other first-order partial derivatives of f are zero (the same 1-jet of f should be substituted into all of the 25 Monge-Ampère relations).

This finishes the proof. □

Before passing to the next proof recall that non-triviality of dispersionless Lax pairs in 4D was discussed in detail in [4] (in 3D the non-triviality condition of [4] is somewhat more involved). In higher dimensions $d > 4$ we say that a disperionless Lax pair is non-trivial if the commutativity of the generators of any modification of the Lax pair which is identical on the equation, holds essentially modulo the equation.

Proof of Theorems 2, 3. As mentioned in the Introduction, Theorem 2 follows from the general result of [4]. As for Theorem 3, note that a generic 4D traveling wave reduction of a multi-dimensional integrable PDE of rank 4 will automatically be non-degenerate and integrable, because a generic travelling wave reduction of a non-trivial disperionless Lax pair is itself non-trivial. Hence the reduced PDE must be of Monge-Ampère type by Theorem 1 demonstrated above. Indeed, by [4] the existence of a non-trivial Lax pair in 4D yields half-flatness of the conformal structure on any solution, so Theorem 1 applies. Henceforth, since all traveling wave reductions of a multi-dimensional PDE are of Monge-Ampère type, by the argument similar to that in [10] we conclude that the initial equation in d dimensions should be of the Monge-Ampère type as well. □

4 Translational invariance for integrable equations

Although some PDEs obtained in Section 2 contain explicit dependence on the independent variables $x = (x_1, \ldots, x_d)$, each of them can be put into translationally invariant form by a suitable contact transformation. Indeed, the necessary and sufficient condition for this, in general dimension d, is the existence of a d-dimensional commutative subalgebra in the algebra of contact symmetries of the equation, which acts simply transitively in J^1M. This subalgebra is responsible for translations of the independent variables and can be contactly mapped to $\langle \partial_{x_1}, \ldots, \partial_{x_d} \rangle$.

Proposition 4 *There exist nondegenerate integrable PDEs in 3D and 4D that are not contact equivalent to any translationally invariant equation.*

In other words, there exist examples of integrable PDEs with essential dependence on the independent variables. Note that PDEs containing explicit dependence on the independent variables have appeared previously, see e.g. [12, 7, 16, 17]. However, to our knowledge, the observation that this dependence can be essential is apparently new.

Proof. We use the above observation on the existence of d-dimensional Abelian subalgebra of contact symmetries as a necessary condition. We consider the cases $d = 3, 4$ in turn.

Case d = 3. The following 3D equation was obtained in [17] as an integrable deformation of the Veronese web equation:

$$(x_1 - x_2)u_3u_{12} + (x_2 - x_3)u_1u_{23} + (x_3 - x_1)u_2u_{13} = 0. \tag{8}$$

It has a Lax pair $[X, Y] = 0$ where

$$X = \partial_1 - \frac{x_3 - \lambda}{x_1 - \lambda}\frac{u_1}{u_3}\partial_3, \qquad Y = \partial_2 - \frac{x_3 - \lambda}{x_2 - \lambda}\frac{u_2}{u_3}\partial_3.$$

The contact symmetry algebra of equation (8) is generated by vector fields

$$X_f = f(u)\partial_u, \qquad Y_0 = \partial_{x_1} + \partial_{x_2} + \partial_{x_3}, \qquad Y_1 = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}.$$

One can easily see that this algebra does not contain any three-dimensional commutative subalgebra. Indeed, Y_0 and Y_1 do not commute, though both commute with X_f, and in the algebra $\langle X_f \rangle$ of vector fields on the line the maximal Abelian subalgebra has dimension 1. Thus, PDE (8) is not contact-equivalent to any translationally invariant equation.

Case d = 4. Analogously, in 4D, consider a linear combination of four copies of equation (8):

$$\begin{aligned} & a_4[(x_1-x_2)u_3u_{12}+(x_2-x_3)u_1u_{23}+(x_3-x_1)u_2u_{13}] \\ & +a_3[(x_4-x_2)u_1u_{24}+(x_2-x_1)u_4u_{12}+(x_1-x_4)u_2u_{14}] \\ & +a_2[(x_3-x_4)u_1u_{34}+(x_1-x_3)u_4u_{13}+(x_4-x_1)u_3u_{14}] \\ & +a_1[(x_3-x_2)u_4u_{23}+(x_2-x_4)u_3u_{24}+(x_4-x_3)u_2u_{34}]=0, \end{aligned} \tag{9}$$

where $a_1, \ldots, a_4$ are arbitrary constants (each of them can be normalised to 0 or 1 independently). We emphasize that a linear combination of 3D commuting flows is by no means a 4D integrable PDE in general. However, this is the case for *linearly degenerate* commuting flows of the Veronese web hierarchy. Indeed, equation (9) has a Lax pair $[X, Y] = 0$ with $X = X_1 - b_1X_3$, $Y = X_2 - b_2X_3$, where

$$X_1 = \partial_1 - \tfrac{x_4-\lambda}{x_1-\lambda}\tfrac{u_1}{u_4}\,\partial_4, \qquad X_2 = \partial_2 - \tfrac{x_4-\lambda}{x_2-\lambda}\tfrac{u_2}{u_4}\,\partial_4, \qquad X_3 = \partial_3 - \tfrac{x_4-\lambda}{x_3-\lambda}\tfrac{u_3}{u_4}\,\partial_4,$$

$$b_1 = \tfrac{x_3-\lambda}{x_1-\lambda}\tfrac{a_1u_4-a_4u_1}{a_3u_4-a_4u_3}, \qquad b_2 = \tfrac{x_3-\lambda}{x_2-\lambda}\tfrac{a_2u_4-a_4u_2}{a_3u_4-a_4u_3}.$$

The contact symmetry algebra of equation (9), when all $a_i \neq 0$, is generated by the vector fields

$$X_f = f(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4, u)\partial_u, \qquad Y_0 = \sum_{i=1}^{4} \partial_{x_i}, \qquad Y_1 = \sum_{i=1}^{4} x_i\partial_{x_i}.$$

Again, Y_0 and Y_1 do not commute, and both commute with X_f iff $f = f(u)$. Then, similarly to the 3D case, the maximal Abelian subalgebra has dimension 2. If f contains essential dependence on the first argument $\sum_{i=1}^{4} a_ix_i$, then X_f does not commute with either of Y_0, Y_1. Thus if such f is admissible, the maximal Abelian subalgebra is a part of $\langle X_f\rangle$. This latter Lie algebra contains an infinite-dimensional commutative subalgebra corresponding to all f depending only on the first argument $\sum_{i=1}^{4} a_ix_i$. However, the prolongation of vector fields X_f to J^1M with all admissible f has an orbit $\langle \partial_u, \sum_{i=1}^{4} a_i\partial_{u_i}, \sum_{i=1}^{4} u_i\partial_{u_i}\rangle$ of dimension 3 only. Thus, PDE (9) is not contact-equivalent to any translationally invariant equation. □

Let us finally remark that a trick used in the second half of the proof of Proposition 4 does not work in dimensions $d > 4$. For instance, in 5D one can form the following linear combination of equations from the Veronese web hierarchy,

$$\sum_{(ijklm)=(12345)} \pi_{ijklm}a_{ij}[(x_k - x_l)u_mu_{kl} + (x_l - x_m)u_ku_{lm} + (x_m - x_k)u_lu_{mk}] = 0,$$

where π_σ is the signature of the permutation σ. For generic values of a_{ij} this is a non-degenerate determined PDE, however it is not integrable: a non-trivial dispersionless Lax pair does not exist. The last claim can be verified directly, but it also follows from the general result of [4, Theorem 1]: a dispersionless Lax pair of a determined second-order PDE is necessarily characteristic, which in our context means that its congruence of 2-planes $\langle X, Y\rangle$ is co-isotropic with respect to the canonical conformal structure, and such 2-planes cannot exist in dimensions $d > 4$.

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