Application of Dual Quaternions to the problem of trajectory tracking with quadrotor-gimbal platform

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***Abstract*—We address the problem of state feedback trajectory tracking of the composite quadrotor-gimbal platform using the dual quaternion framework by extending the previuous result in [1] to the composite case. More precisely; we model the composite system using dual quaternion coordinates and derive the error dynamics which by inserting a PD+ based control law has equilibrium points that is shown to be uniformly practical asymptoticly stable (UPAS).**

***Index Terms*—Dual-Quaternion, UAV, Nonlinear Control, Composite System**

I. Introduction

Pose control of a rigid body in 3-D space is an important challenge with broad impact to a number of mechanical systems including, but not limited to; satellites, autonomous underwater vehicles, unmanned aerial vehicles and robot manipulators [2]. The commonly used Newton-Euler framework completely describe the motion of a rigid body in six-degrees-of-freedom (6-DOF), however the rotational and translational movement is often considered separately, thus control algorithms are designed separately. Concurrent position and attitude control is especially relevant in applications such as formation flying, aerial towing and near-earth environment inspection such as power-line inspection. In these scenarios it is imperative that the 6-DOF coupled motion of a rigid body is taken into account [3]. The advantages realizing simultaneous 6-DOF control, as suppose to 3+3 DOF, are greatest in systems where translational and rotation motion is highly coupled [4]; e.g. underactuated systems as fixed-wing aerial vehicles and quadrotors. Further, several authors state that dual quaternions is the most compact and efficient way to express motion i 3-D space [5]–[8], e.g. in [9] the author notes that dual quaternion algebra, which are isomorphic to the even subalgebra of Euclidean projective geometric algebra of order three, *is the smallest known algebra that can model Euclidean transformations in a structure preserving manner*. Moreover, pose control laws based on dual quaternions includes the coupling between rotation and translation [10], [11], and as it is noted in [12], dual quaternions allow pose control laws to be written compactly as a single control law. There are however some disadvantages; due to topological constraints it is impossible to design a continuous feedback that globally stabilizes the pose of a rigid body [13] and the unit dual quaternion group is endowed with a double representation of every pose in the configuration manifold which may lead to unwinding.

There excists some application of dual quaternions to composite systems; in [14] dual quaternions is used for modeling and control of an unmanned aerial manipulator consisting of a quadrotor serially coupled with a three-link manipulator, however, dual quaternions is only used for kinematic control and quadrotor dynamics in terms of dual quaternions is not considered. In [15] and [16] the authors present a hierarchical control law for quadrotor stabilization and aerial manipulator tracking. The dual quaternion logarithm is used to generate the desired force vector and subsequently the desired attitude trajectory, however, they do not consider Coriolis acceleration or centrifugal acceleration in their dynamics model. Quadrotor-gimbal composite systems has been studied previously in litarature; in [17] the authors present a nonlinear velocity controller that achieves global uniform ultimate boundedness of the velocity errors in the gimbal camera frame. In recent work we presented a novel control strategy for the quadrotor platform using the dual quaternion framework for the problem of trajectory tracking, uniform practical asymptotic stability if the equlibrium points was shown using a PD+ based control law [1]. In the presented work, we address the problem of state feedback trajectory tracking of the composite quadrotor-gimbal platform using the dual quaternion framework by extending the previuous result in [1] to the composite case. More precisely; we model the composite system using dual quaternion coordinates, augment it using a additional reference frame, derive the error dynamics which by insterting a PD+ based control law has closed loop equilibrium points that uniformly asymptoticly stable, which implies uniform practical asymptotic stability of the real system.

II. Preliminaries

A. Notation and reference frames

Throughout this paper scalar values are denoted in normal face, vectors in lowercase boldface while matrices are written in capital boldface letters. The time derivative is denoted as $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$, the Euclidean norm is denoted by $\|\cdot\|$ while the supremum norm is denoted as $|\cdot|_\infty$. Note that $\mathbf{I}_{n\times n}$ denotes an $n\times n$ identity matrix while $\mathbf{0}_{n\times m}$ denotes an $n\times m$ matrix of zeros. Different reference frames is used throughout this paper, denoted by superscripts; $\mathcal{F}^b$ is the quadrotor body frame defined with its x-axis pointing forward between two of the quadrotors motors[1], the z-axis is pointing downward and its y-axis completing the right-hand system. $\mathcal{F}^g$ denotes the gimbal frame defined in a similar manner with its x-axis pointing forward and z-axis down, $\mathcal{F}^d$ denotes the desired frame and $\mathcal{F}^n$ is the standard North-East-Down (NED) frame which is assumed to be inertial. The rotation matrix from $\mathcal{F}^b$ to $\mathcal{F}^n$ is denoted as $\mathbf{R}_b^n \in SO(3)$, where

$$SO(3) := \{\mathbf{R} \in \mathbb{R}^{3\times 3} : \mathbf{R}^\top \mathbf{R} = \mathbf{I}_{3\times 3}, \det(\mathbf{R}) = 1\} \tag{1}$$

is the special orthogonal group of order three. In this work we use quaternions to parametrize $SO(3)$, and the equivalent attitude quaternion representing rotations from $\mathcal{F}^b$ to $\mathcal{F}^n$ is denoted as $\mathbf{q}_{n,b}$. The homogeneous transformation matrix from $\mathcal{F}^b$ to $\mathcal{F}^n$ is denoted as $\mathbf{T}_b^n \in SE(3)$, where

$$SE(3) := \left\{\mathbf{T} \in \mathbb{R}^{4\times 4} : \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}, \mathbf{R} \in SO(3), \mathbf{p} \in \mathbb{R}^3 \right\} \tag{2}$$

is the group of proper Euclidean motion in three dimensional space. In this work we will use dual quaternions to parametrize $SE(3)$, and the equivalent pose dual quaternion is denoted $\hat{\mathbf{q}}_{n,b}$. Angular velocity is denoted $\boldsymbol{\omega}_{b,c}^a \in \mathbb{R}^3$, ie. the angular velocity of $\mathcal{F}^c$ relative $\mathcal{F}^b$ referenced in $\mathcal{F}^a$. For any arbitrary vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$, we denote the cross-product operator as $\mathbf{S}(\mathbf{v}_1)\mathbf{v}_2 = \mathbf{v}_1 \times \mathbf{v}_2$. A function $\alpha : \mathbb{R}_{0\geq} \to \mathbb{R}_{0\geq}$ is of class $\mathcal{K}$ if α is strictly increasing, continuous and $\alpha(0) = 0$. Moreover, α is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded.

B. Quaternions and dual quaternions

The set of quaternions can be defined as

$$\mathbb{H} := \{\mathbf{q} = \eta + \varepsilon_1 i + \varepsilon_2 j + \varepsilon_3 k : \quad \eta, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}\} \tag{3}$$

with i, j, k being the well known quaternions basis elements. Quaternions constitute a real vector space which is isomorphic to $\mathbb{R}^4$ through the isomorphism $\zeta : \mathbb{H} \to \mathbb{R}^4$ defined as

$$\zeta(\eta + \varepsilon_1 i + \varepsilon_2 j + \varepsilon_3 k) = \begin{bmatrix} \eta \\ \boldsymbol{\varepsilon} \end{bmatrix} \tag{4}$$

with $\boldsymbol{\varepsilon} = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3]^\top$. The product of two quaternions, $\mathbf{q} = [\eta_q \ \boldsymbol{\varepsilon}_q^\top]^\top$ and $\mathbf{p} = [\eta_p \ \boldsymbol{\varepsilon}_p^\top]^\top$, is calculated as

$$\mathbf{p} \otimes \mathbf{q} = \begin{bmatrix} \eta_p \eta_q - \boldsymbol{\varepsilon}_p^\top \boldsymbol{\varepsilon}_q \\ \eta_p \boldsymbol{\varepsilon}_q + \eta_q \boldsymbol{\varepsilon}_p + \mathbf{S}(\boldsymbol{\varepsilon}_p)\boldsymbol{\varepsilon}_q \end{bmatrix} \tag{5}$$

while the quaternion conjugate is given as $\mathbf{q}^* := [\eta \ -\boldsymbol{\varepsilon}^\top]^\top$. The subset of quaternions that possess norm equal to one are known as unit quaternions, $\mathbb{H}_u := \{\mathbf{q} \in \mathbb{H} : \|\mathbf{q}\| = 1\}$ and topologically they form the $3-sphere$ S^3 in $\mathbb{R}^4$ [18],

$$S^3 := \{\mathbf{q} \in \mathbb{R}^4 : \|\mathbf{q}\| = 1\}, \tag{6}$$

which under quaternion multiplication forms an associative and distributive, but non-abelian Lie Group $Spin(3)$ [19]. This group has its inverse defined by the quaternion conjugate, it is a double cover of $SO(3)$, and thus the map $\gamma : S^3 \to SO(3)$ is a 2-to-1 homomorphism, defined as $\gamma(\mathbf{q}) = \{\pm\mathbf{q}\}$. Unit quaternions can be used to represent rigid body attitude and the attitude kinematics is modeled by the differential equation

$$\dot{\mathbf{q}}_{n,b} = \mathbf{T}(\mathbf{q}_{n,b})\boldsymbol{\omega}_{n,b}^b \tag{7}$$

where $\dot{\mathbf{q}}_{n,b} \in \mathbb{R}^4$, $\boldsymbol{\omega}_{n,b}^b \in \mathbb{R}^3$ and $\mathbf{T}(\mathbf{q}_{n,b}) \in \mathbb{R}^{4\times 3}$ is defined as

$$\mathbf{T}(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} -\boldsymbol{\varepsilon}^\top \\ \eta \mathbf{I}_{3\times 3} + \mathbf{S}(\boldsymbol{\varepsilon}) \end{bmatrix}. \tag{8}$$

Vectors in $\mathbb{R}^3$ can be represented using pure quaternions, $\mathbf{q} \in \mathbb{H}_p = \{\mathbf{q} \in \mathbb{H} : \eta = 0\}$, by a trivial isomorphism. For two pure quaternions, $\mathbf{v}, \mathbf{u} \in \mathbb{R}^4$, the cross product is just the cross product between the vectors in $\mathbb{R}^3$, such that

$$\mathbf{q} \times \mathbf{p} = \begin{bmatrix} 0 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{S}(\boldsymbol{\varepsilon}_p) \end{bmatrix} \mathbf{q} := \mathbf{S}_q(\mathbf{p})\mathbf{q}. \tag{9}$$

The set of dual numbers is defined as

$$\mathbb{D} := \{\hat{z} = z_p + \epsilon z_s : \quad z_p, z_d \in \mathbb{R}\} \tag{10}$$

with $\epsilon := \{\epsilon \neq 0, \epsilon^2 = 0\}$ being the dual operator, not to be confused with the quaternion vector element $\boldsymbol{\varepsilon}$. Dual quaternions are a combination of quaternions and dual numbers, they can be seen as a quaternion where each element is a dual number or conversely a dual number where each element is a quaternion. We employ the latter and define the set of dual quaternions as

$$\mathbb{DH} := \{\hat{\mathbf{q}} \in \mathbb{H} \times \mathbb{D} : \quad \hat{\mathbf{q}} = \mathbf{q}_p + \epsilon \mathbf{q}_d, \quad \mathbf{q}_p, \mathbf{q}_d \in \mathbb{H}\} \tag{11}$$

where $\mathbf{q}_p$ is denoted as the primary part and $\mathbf{q}_d$ is denoted the dual part. Dual quaternions can be seen to constitute a real vector space which is isomorphic to $\mathbb{R}^8$ through the isomorphism $\hat{\zeta} : \mathbb{DH} \to \mathbb{R}^8$ defined as

$$\hat{\zeta}(\mathbf{q}_p + \epsilon \mathbf{q}_d) = \begin{bmatrix} \mathbf{q}_p \\ \mathbf{q}_d \end{bmatrix}. \tag{12}$$

The product of two dual quaternions is calculated as

$$\hat{\mathbf{q}} \otimes \hat{\mathbf{p}} = \mathbf{q}_p \otimes \mathbf{p}_p + \epsilon(\mathbf{q}_p \otimes \mathbf{p}_d + \mathbf{q}_d \otimes \mathbf{p}_p) \tag{13}$$

which can be seen to be a semi-direct product between the primary and dual part. The dual quaternion conjugate is given as

$$\hat{\mathbf{q}}^* = \mathbf{q}_p^* + \epsilon \mathbf{q}_d^* \tag{14}$$

where $(\cdot)^*$ is the quaternion conjugate. The subset of dual quaternions that satisfy the norm constraint $\|\hat{\mathbf{q}}\| = 1$ is denoted the set of unit dual quaternions

$$\mathbb{DH}_u := \{\hat{\mathbf{q}} \in \mathbb{DH} : \mathbf{q}_p \in \mathbb{H}_u, \mathbf{q}_p \otimes \mathbf{q}_d^* + \mathbf{q}_d \otimes \mathbf{q}_p^* = 0\}. \tag{15}$$

It can be shown that under multiplication unit dual quaternions form the group $S^3 \ltimes \mathbb{R}^3$ which double covers $SE(3)$ [20], such that

$$S^3 \ltimes \mathbb{R}^3 := \{\hat{\mathbf{q}} \in \mathbb{R}^8 : \mathbf{q}_p \in S^3, \mathbf{q}_s \in \mathbb{R}^4\}. \tag{16}$$

[1]This corresponds to the x-configuration of the quadrotor as opposed to the pluss-configuration, the choice of which is arbitrary.

Vectors in $\mathbb{R}^6$ can be represented using pure dual quaternions, $\hat{\mathbf{q}} \in \mathbb{DH}_p = \{\hat{\mathbf{q}} \in \mathbb{DH} : \mathbf{q}_p, \mathbf{q}_s \in \mathbb{H}_p\}$. For two pure dual quaternions, $\hat{\mathbf{v}}, \hat{\mathbf{u}} \in \mathbb{R}^8$, the cross product is defined as

$$\begin{aligned}\hat{\mathbf{v}} \times \hat{\mathbf{u}} &= \mathbf{v}_p \times \mathbf{u}_p + \epsilon(\mathbf{v}_p \times \mathbf{u}_d + \mathbf{v}_d \times \mathbf{u}_p) \\ &= \begin{bmatrix} \mathbf{S}_q(\mathbf{v}_p) & 0 \\ \mathbf{S}_q(\mathbf{v}_d) & \mathbf{S}_q(\mathbf{v}_p) \end{bmatrix} \hat{\mathbf{u}} = \hat{\mathbf{S}}(\hat{\mathbf{v}})\hat{\mathbf{u}}.\end{aligned} \tag{17}$$

III. Modeling and Problem statement

In the remaining we omitt explicit statements on the use of isomorphisms between $\mathbb{R}^6$ and $\mathbb{R}^8$ for the sake of brevity.

A. Quadrotor-Gimbal system model

The pose of the quadrotor is represented using a dual quaternion, $\hat{\mathbf{q}}_{n,b} \in \mathbb{R}^8$, defined as

$$\hat{\mathbf{q}}_{n,b} = \mathbf{q}_{n,b} + \epsilon\frac{1}{2}\mathbf{p}^n \otimes \mathbf{q}_{n,b} = \mathbf{q}_{n,b} + \epsilon\frac{1}{2}\mathbf{q}_{n,b} \otimes \mathbf{p}^b \tag{18}$$

where $\mathbf{p}^n, \mathbf{p}^b \in \mathbb{R}^4$ and $\mathbf{q}_{n,b} \in S^3$. By a similar relation as in (7) the dual quaternion kinematics is modeled as

$$\dot{\hat{\mathbf{q}}}_{n,b} = \hat{\mathbf{T}}(\hat{\mathbf{q}}_{n,b})\hat{\boldsymbol{\omega}}^b_{n,b} \tag{19}$$

where $\hat{\boldsymbol{\omega}}^b_{n,b} \in \mathbb{R}^6$ is the velocity screw of the quadrotor[2] defined as $\hat{\boldsymbol{\omega}}^b_{n,b} = \boldsymbol{\omega}^b_{n,b} + \epsilon\mathbf{v}^b$ with $\mathbf{v}^b = \dot{\mathbf{p}}^b + \mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{p}^b$. The matrix $\hat{\mathbf{T}}(\hat{\mathbf{q}}_{n,b}) \in \mathbb{R}^{8\times 6}$ is defined as

$$\hat{\mathbf{T}}(\hat{\mathbf{q}}_{n,b}) = \begin{bmatrix} \mathbf{T}(\mathbf{q}_p) & \mathbf{0}_{4\times 3} \\ \mathbf{T}(\mathbf{q}_d) & \mathbf{T}(\mathbf{q}_p) \end{bmatrix} \tag{20}$$

with $\mathbf{T}(\cdot)$ given in (8). In this work we model dynamics by employ the approach presented in [21]. Expanding the dual quaternions into $\mathbb{R}^8$ using the isomporhisim $\hat{\zeta}$ allows for screws to be mapped into co-screws by multiply them with the inertia matrix, named the dual inertia matrix, defined as

$$\hat{\mathbf{M}}^b = \begin{bmatrix} 0 & \mathbf{0}_{1\times 3} & 1 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 1} & m\mathbf{I}_3 \\ 1 & \mathbf{0}_{1\times 3} & 0 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{J}^b & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} \end{bmatrix} \tag{21}$$

where $\mathbf{J} = \mathrm{diag}\{\mathrm{J_{xx}}, \mathrm{J_{yy}}, \mathrm{J_{zz}}\}$ is the inertia matrix and m is the mass of the quadrotor. The dynamic model for the quadrotor in the dual quaternion framework then becomes

$$\hat{\mathbf{M}}^b\dot{\hat{\boldsymbol{\omega}}}^b_{n,b} = \hat{\mathbf{f}}^b_u - \hat{\mathbf{f}}^b_g - \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^b_{n,b})\hat{\mathbf{M}}^b\hat{\boldsymbol{\omega}}^b_{n,b} \tag{22}$$

where $\hat{\mathbf{f}}^b_u = \mathbf{f}^b_T + \epsilon\boldsymbol{\tau}^b$ is the dual control force[3] given as

$$\mathbf{f}^b_T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ T \end{bmatrix} \qquad \boldsymbol{\tau}^b = \begin{bmatrix} 0 \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} \tag{23}$$

and $\hat{\mathbf{f}}^b_g = \mathbf{q}^*_{n,b} \otimes \mathbf{f}^n_g \otimes \mathbf{q}_{n,b} + \epsilon\mathbf{0}$ is the gravitational force. Similarly; assuming that the gimbal is rigidily attached to the quadrotor, the pose, kinematics, and dynamics of the gimbal frame relative the body frame is represented as

$$\begin{aligned}\hat{\mathbf{q}}_{b,g} &= \mathbf{q}_{b,g} + \epsilon\frac{1}{2}\mathbf{p}^b_g \otimes \mathbf{q}_{b,g} = \mathbf{q}_{b,g} + \epsilon\frac{1}{2}\mathbf{q}_{b,g} \otimes \mathbf{p}^g_g \\ \dot{\hat{\mathbf{q}}}_{b,g} &= \hat{\mathbf{T}}(\hat{\mathbf{q}}_{b,g})\hat{\boldsymbol{\omega}}^g_{b,g} \\ \hat{\mathbf{M}}^g\dot{\hat{\boldsymbol{\omega}}}^g_{b,g} &= \hat{\mathbf{f}}^g_u - \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^g_{b,g})\hat{\mathbf{M}}^g\hat{\boldsymbol{\omega}}^g_{b,g}\end{aligned} \tag{24}$$

where $\mathbf{p}^b_g \in \mathbb{R}^4$ is a constant vector describing the gimbal position in $\mathcal{F}^b$ and $\hat{\mathbf{f}}^g_u = \mathbf{0} + \epsilon\boldsymbol{\tau}^g$.

Combining (18) and (24) the quadrotor-gimbal composed system pose is given as

$$\hat{\mathbf{q}}_{n,g} = \hat{\mathbf{q}}_{n,b} \otimes \hat{\mathbf{q}}_{b,g}. \tag{25}$$

Further, deriving the velocity screw of the composed system to be $\hat{\boldsymbol{\omega}}^g_{n,g} = \hat{\mathbf{q}}^*_{b,g} \otimes \hat{\boldsymbol{\omega}}^b_{n,b} \otimes \hat{\mathbf{q}}_{b,g} + \hat{\boldsymbol{\omega}}^g_{b,g}$ the equations of motion of the composed system is stated as

$$\begin{aligned}\dot{\hat{\mathbf{q}}}_{n,g} &= \hat{\mathbf{T}}(\hat{\mathbf{q}}_{n,g})\hat{\boldsymbol{\omega}}^g_{n,g} \\ \dot{\hat{\boldsymbol{\omega}}}^g_{n,g} &= \hat{\mathbf{q}}^*_{b,g} \otimes \dot{\hat{\boldsymbol{\omega}}}^b_{n,b} \otimes \hat{\mathbf{q}}_{b,g} + \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^g_{n,b})\hat{\boldsymbol{\omega}}^g_{b,g} + \dot{\hat{\boldsymbol{\omega}}}^g_{b,g}.\end{aligned} \tag{26}$$

B. Problem formulation

The tracking control problem can be stated as; let $\hat{\mathbf{q}}_{n,d} : \mathbb{R}_{\geq 0} \to \mathbb{R}^8$ be a given, two-times continuously differentiable bounded time-varying desired trajectory, i.e.

$$\max\{|\hat{\mathbf{q}}_{\mathrm{n,d}}|_\infty,\ |\hat{\boldsymbol{\omega}}^{\mathrm{d}}_{\mathrm{n,d}}|_\infty,\ |\dot{\hat{\boldsymbol{\omega}}}^{\mathrm{d}}_{\mathrm{n,d}}|_\infty\} \leq \beta_{\mathrm{d}} \tag{27}$$

Define the tracking error in dual quaternion coordinates as

$$\begin{aligned}\hat{\mathbf{q}}_e &:= \hat{\mathbf{q}}^*_{n,d} \otimes \hat{\mathbf{q}}_{n,g} = \mathbf{q}_e + \epsilon\frac{1}{2}\mathbf{q}_e \otimes \mathbf{p}^g_e \\ &:= \mathbf{q}_{e,p} + \epsilon\mathbf{q}_{e,d} = \begin{bmatrix} \eta_e \\ \boldsymbol{\varepsilon}_e \end{bmatrix} + \epsilon \begin{bmatrix} \eta_{ed} \\ \boldsymbol{\varepsilon}_{ed} \end{bmatrix}\end{aligned} \tag{28}$$

and, due to the double cover $S^3 \ltimes \mathbb{R}^3$ of $SE(3)$, define

$$\hat{\mathbf{q}}_{e\pm} := \begin{bmatrix} (1 \mp \eta_e) \\ \boldsymbol{\varepsilon}_e \end{bmatrix} + \epsilon\frac{1}{2}\mathbf{q}_e \otimes \mathbf{p}^g_e \tag{29}$$

with error kinematics and dynamics

$$\begin{aligned}\dot{\hat{\mathbf{q}}}_{e\pm} =& \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})\hat{\boldsymbol{\omega}}^g_e \\ \dot{\hat{\boldsymbol{\omega}}}^b_e =& \hat{\mathbf{q}}^*_{b,g} \otimes (\hat{\mathbf{M}}^b)^{-1}(\hat{\mathbf{f}}^b_u + \hat{\mathbf{f}}^b_g - \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^b_{n,b})\hat{\mathbf{M}}\hat{\boldsymbol{\omega}}^b_{n,b}) \otimes \hat{\mathbf{q}}_{b,g} \\ &+ \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^g_{n,b})\hat{\boldsymbol{\omega}}^g_{b,g} + (\hat{\mathbf{M}}^g)^{-1}(\hat{\mathbf{f}}^g_u - \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^g_{b,g})\hat{\mathbf{M}}^g\hat{\boldsymbol{\omega}}^g_{b,g}) \\ &- \dot{\hat{\boldsymbol{\omega}}}^g_{n,d}\end{aligned} \tag{30}$$

where $\hat{\boldsymbol{\omega}}^g_e = \hat{\boldsymbol{\omega}}^g_{n,g} - \hat{\boldsymbol{\omega}}^g_{n,d}$ and

$$\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm}) = \begin{bmatrix} \mathbf{T}_{e\pm}(\mathbf{q}_r) & \mathbf{0}_{3\times 4} \\ \mathbf{T}(\mathbf{q}_d) & \mathbf{T}(\mathbf{q}_r) \end{bmatrix} \tag{31}$$

with

$$\mathbf{T}_{e\pm}(\mathbf{q}_r) = \frac{1}{2}\begin{bmatrix} \pm\boldsymbol{\varepsilon}^\top \\ \eta\mathbf{I}_{3\times 3} + \mathbf{S}(\boldsymbol{\varepsilon}) \end{bmatrix}. \tag{32}$$

Then, design a feedback control law, $\hat{\mathbf{u}}^g(\hat{\mathbf{f}}^b_u, \hat{\mathbf{f}}^g_u)$, that stabilizes the origin for the system (30).

[2]From screw theory this is known as a twist, i.e. the angular velocity around an axis and the linear velocity along it.

[3]From screw theory this is known as a wrench, i.e. the combination of force and torque acting on a rigid body

Remark 1. *The quadrotor is underactuated with four actuators for six degrees of freedom, while the composite system is overactuated with seven actuators.*

Remark 2. *Following [22], we define two sets* $\hat{\boldsymbol{q}}_{e+} \in S^3_{e+} \ltimes \mathbb{R}^3 := \{[1-\eta_e,\ \boldsymbol{\varepsilon}_e^\top,\ \boldsymbol{q}_{e,d}^\top]^\top : \eta_e \geq 0, \hat{\boldsymbol{q}}_e \in S^3 \ltimes \mathbb{R}^3\}$ *and* $\hat{\boldsymbol{q}}_{e-} \in S^3_{e-} \ltimes \mathbb{R}^3 := \{[1+\eta_e,\ \boldsymbol{\varepsilon}_e^\top,\ \boldsymbol{q}_{e,d}^\top]^\top : \eta_e \leq 0, \hat{\boldsymbol{q}}_e \in S^3 \ltimes \mathbb{R}^3\}$. *Thus,* $\hat{\boldsymbol{q}}_{e\pm} \in S^3_{e+} \ltimes \mathbb{R}^3 \cup S^3_{e-} \ltimes \mathbb{R}^3 = S^3_e \ltimes \mathbb{R}^3 := \{[1-|\eta_e|,\ \boldsymbol{\varepsilon}_e^\top,\ \boldsymbol{q}_{e,d}^\top]^\top : \hat{\boldsymbol{q}}_e \in S^3 \ltimes \mathbb{R}^3\}$.

IV. Main Results

A. Virtual frames

As the quadrotor is underactuated we introduce a virtual frame that maps two of the rotational actuators onto the translational error, facilitating control design. The pose of frame $\mathcal{F}^c$ is defined as

$$\hat{\mathbf{q}}_{g,c} = \hat{\mathbf{q}}^*_{b,g} \otimes \hat{\mathbf{q}}_{b,c} \tag{33}$$

with

$$\hat{\mathbf{q}}_{b,c} = \mathbf{q}_{b,g} + \epsilon\frac{1}{2}\boldsymbol{\Delta}^b \otimes \mathbf{q}_{b,g} \tag{34}$$

where the constant vector $\boldsymbol{\Delta}^b = [0\ 0\ 0\ \Delta]^\top$, $\Delta \in \mathbb{R}$. Note that the kinematics and dynamics of this frame can be found to be

$$\begin{aligned}
\dot{\hat{\mathbf{q}}}_{g,c} &= \hat{\mathbf{T}}(\hat{\mathbf{q}}_{g,c})\hat{\boldsymbol{\omega}}^c_{g,c}\\
\hat{\boldsymbol{\omega}}^c_{g,c} &= \hat{\boldsymbol{\omega}}^c_{b,c} - \hat{\mathbf{q}}^*_{g,c} \otimes \hat{\boldsymbol{\omega}}^g_{b,g} \otimes \hat{\mathbf{q}}_{g,c}\\
\dot{\hat{\boldsymbol{\omega}}}^c_{g,c} &= \dot{\hat{\boldsymbol{\omega}}}^c_{b,c} - \hat{\mathbf{q}}^*_{g,c} \otimes \dot{\hat{\boldsymbol{\omega}}}^g_{b,g} \otimes \hat{\mathbf{q}}_{g,c} - \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^c_{b,g})\hat{\boldsymbol{\omega}}^c_{g,c}
\end{aligned} \tag{35}$$

where $\dot{\hat{\boldsymbol{\omega}}}^c_{b,c} = \dot{\hat{\boldsymbol{\omega}}}^g_{b,g}$. Using this frame we compose a augmented system as

$$\hat{\mathbf{q}}_{n,c} = \hat{\mathbf{q}}_{n,b} \otimes \hat{\mathbf{q}}_{b,g} \otimes \hat{\mathbf{q}}_{g,c} \tag{36}$$

and derive its kinematics

$$\dot{\hat{\mathbf{q}}}_{n,c} = \hat{\mathbf{T}}(\hat{\mathbf{q}}_{n,c})\hat{\boldsymbol{\omega}}^c_{n,c} \tag{37}$$

with $\hat{\boldsymbol{\omega}}^c_{n,c} = \hat{\boldsymbol{\omega}}^c_{n,b} + \hat{\boldsymbol{\omega}}^c_{b,g} + \hat{\boldsymbol{\omega}}^c_{g,c}$. Further, we derive the dynamics of the composed system and after inserting (35) one finds

$$\begin{aligned}
\dot{\hat{\boldsymbol{\omega}}}^c_{n,c} =& \hat{\mathbf{q}}^*_{b,c} \otimes (\hat{\mathbf{M}}^b)^{-1}(\hat{\mathbf{f}}^b_u + \hat{\mathbf{f}}^b_g - \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^b_{n,b})\hat{\mathbf{M}}\hat{\boldsymbol{\omega}}^b_{n,b}) \otimes \hat{\mathbf{q}}_{b,c}\\
&+ \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^c_{n,b})\hat{\boldsymbol{\omega}}^c_{b,c} + (\hat{\mathbf{M}}^g)^{-1}(\hat{\mathbf{f}}^g_u - \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^g_{b,g})\hat{\mathbf{M}}^g\hat{\boldsymbol{\omega}}^g_{b,g}).
\end{aligned} \tag{38}$$

By the contruction of $\hat{\mathbf{q}}_{b,c}$ we now have that

$$\hat{\mathbf{q}}^*_{b,c} \otimes (\hat{\mathbf{M}}^b)^{-1}\hat{\mathbf{f}}^b_u \otimes \hat{\mathbf{q}}_{b,c} = \begin{bmatrix} \mathbf{q}^*_{b,c} \otimes (\mathbf{J}^b)^{-1}\boldsymbol{\tau}^b \otimes \mathbf{q}_{b,c} \\ \mathbf{q}^*_{b,c} \otimes \begin{bmatrix} \Delta\tau_y/J_{yy} \\ -\Delta\tau_x/J_{xx} \\ T/m \end{bmatrix} \otimes \mathbf{q}_{b,c} \end{bmatrix}$$

which shows that the quadrotor rotational actuators has been mapped onto the translational error. Defining a control screw for the composed system as

$$\hat{\mathbf{u}}^c(\hat{\mathbf{f}}^b_u, \hat{\mathbf{f}}^g_u) = \begin{bmatrix} (\mathbf{J}^g)^{-1}\boldsymbol{\tau}^g \\ \mathbf{q}^*_{b,c} \otimes \begin{bmatrix} \Delta\tau_y/J_{yy} \\ -\Delta\tau_x/J_{xx} \\ T/m \end{bmatrix} \otimes \mathbf{q}_{b,c} \end{bmatrix} \tag{39}$$

we restate (38) suitable for control design

$$\begin{aligned}
\dot{\hat{\boldsymbol{\omega}}}^c_{n,c} =& \hat{\mathbf{u}}^c - \hat{\mathbf{q}}^*_{b,c} \otimes (\hat{\mathbf{M}}^b)^{-1}(\hat{\mathbf{f}}^b_g + \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^b_{n,b})\hat{\mathbf{M}}\hat{\boldsymbol{\omega}}^b_{n,b}) \otimes \hat{\mathbf{q}}_{b,c}\\
&+ \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^c_{n,b})\hat{\boldsymbol{\omega}}^c_{b,c} - (\hat{\mathbf{M}}^g)^{-1}(\hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^g_{b,g})\hat{\mathbf{M}}^g\hat{\boldsymbol{\omega}}^g_{b,g}) + \boldsymbol{\delta}(\hat{\mathbf{u}}^c)
\end{aligned} \tag{40}$$

where

$$\hat{\delta}(\hat{\mathbf{u}}^c) = \begin{bmatrix} \mathbf{q}^*_{b,c} \otimes \begin{bmatrix} \tau_x/J_{xx} \\ \tau_y/J_{yy} \\ 0 \\ \mathbf{0} \end{bmatrix} \otimes \mathbf{q}_{b,c} \end{bmatrix}. \tag{41}$$

B. Error kinematics

With the new augmented system we redefine the tracking error, error kinematics and dynamics as

$$\hat{\mathbf{q}}_e = \hat{\mathbf{q}}^*_{n,d} \otimes \hat{\mathbf{q}}_{n,c} := \mathbf{q}_e + \epsilon\frac{1}{2}\mathbf{q}_e \otimes \mathbf{p}^c_e \tag{42}$$

with error kinematics and dynamics

$$\begin{aligned}
\dot{\hat{\mathbf{q}}}_{e\pm} =& \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})\hat{\boldsymbol{\omega}}^c_e\\
\dot{\hat{\boldsymbol{\omega}}}^c_e =& \hat{\mathbf{u}}^c - \hat{\mathbf{q}}^*_{b,c} \otimes (\hat{\mathbf{M}}^b)^{-1}(\hat{\mathbf{f}}^b_g + \hat{\boldsymbol{\omega}}^b_{n,b} \times \hat{\mathbf{M}}^b\hat{\boldsymbol{\omega}}^b_{n,b}) \otimes \hat{\mathbf{q}}_{b,c}\\
&+ \hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^c_{n,b})\hat{\boldsymbol{\omega}}^c_{b,c} - (\hat{\mathbf{M}}^g)^{-1}(\hat{\mathbf{S}}(\hat{\boldsymbol{\omega}}^g_{b,g})\hat{\mathbf{M}}^g\hat{\boldsymbol{\omega}}^g_{b,g})\\
&+ \hat{\delta}(\hat{\mathbf{u}}^c) - \dot{\hat{\boldsymbol{\omega}}}^c_{n,d}
\end{aligned} \tag{43}$$

It has been shown in [1] that achieving asymptotic tracking for the augmented system is equivalent to achieving practical asymptotic tracking for the real system.

C. PD+ controller

The following proposition establishes uniform asymptotic stability of the closed-loop augmented system under a modified PD+ controller.

Proposition 1. *Let* $\hat{\boldsymbol{q}}_{eq} \in S^3_e \ltimes \mathbb{R}^3$ *and* $\mathrm{sgn}(\eta_{e,\mathrm{p}}(\mathrm{t}_0)) = \mathrm{sgn}(\eta_{e,\mathrm{p}}(\mathrm{t}))$ *for all* $t \geq t_0$, *let the desired trajectory,* $\hat{\boldsymbol{q}}_{n,d}$, *satisfy (27), then the equilibrium points* $(\hat{\boldsymbol{q}}_{e\pm}, \hat{\boldsymbol{\omega}}^c_e) = (\boldsymbol{0}, \boldsymbol{0})$ *of the system (43), in closed-loop with the control law*

$$\begin{aligned}
\hat{\boldsymbol{u}}^c =& \hat{\boldsymbol{q}}^*_{b,c} \otimes (\hat{\boldsymbol{M}}^b)^{-1}(\hat{\boldsymbol{f}}^b_g + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}^b_{n,b})\hat{\boldsymbol{M}}^b\hat{\boldsymbol{\omega}}^b_{n,b}) \otimes \hat{\boldsymbol{q}}_{b,c}\\
&- \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}^c_{n,b})\hat{\boldsymbol{\omega}}^c_{b,c} - \hat{\delta}(\hat{\boldsymbol{u}}^c) + \hat{\boldsymbol{q}}^*_e \otimes \dot{\hat{\boldsymbol{\omega}}}^z_{n,z} \otimes \hat{\boldsymbol{q}}_e + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}^c_{n,z})\hat{\boldsymbol{\omega}}^c_e\\
&+ (\hat{\boldsymbol{M}}^g)^{-1}(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}^g_{b,g})\hat{\boldsymbol{M}}^g\hat{\boldsymbol{\omega}}^g_{b,g}) - \boldsymbol{K}_p\tilde{\boldsymbol{\varepsilon}} - \boldsymbol{K}_d\hat{\boldsymbol{\omega}}_e
\end{aligned} \tag{44}$$

where $\tilde{\boldsymbol{\varepsilon}}^\top = 2\hat{\boldsymbol{q}}_{eq}^\top\hat{\boldsymbol{T}}_e(\hat{\boldsymbol{q}}_{eq})$, $\boldsymbol{K}_p$, $\boldsymbol{K}_d$ *are positive feedback control gain matrices, are uniformly asymptotically stable (UAS).*

Proof. In the following we only consider, without loss of generality, the positive equilibrium point, i.e. $\hat{\mathbf{q}}_{eq} = \hat{\mathbf{q}}_{e+}$ and $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e+})$. The closed-loop kinematics and dynamics, resulting from inserting (44) into (43), is

$$\begin{aligned}
\dot{\hat{\mathbf{q}}}_{eq} &= \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\hat{\boldsymbol{\omega}}^c_e\\
\dot{\hat{\boldsymbol{\omega}}}^c_e &= -\mathbf{K}_p\tilde{\boldsymbol{\varepsilon}} - \mathbf{K}_d\hat{\boldsymbol{\omega}}^c_e
\end{aligned} \tag{45}$$

Consider the radially unbounded Lyapunov function candidate

$$V(\hat{\mathbf{q}}_{eq}, \hat{\boldsymbol{\omega}}^c_e) = \hat{\mathbf{q}}_{eq}^\top\mathbf{K}_p\hat{\mathbf{q}}_{eq} + \frac{1}{2}(\hat{\boldsymbol{\omega}}^c_e)^\top\hat{\boldsymbol{\omega}}^c_e. \tag{46}$$

We show that there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(\mathbf{x}) \leq V(\mathbf{x}) \leq \overline{\alpha}(\mathbf{x})$. Defining $\boldsymbol{\chi} = [\hat{\mathbf{q}}_{eq}^\top \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) \ (\hat{\boldsymbol{\omega}}_e^c)^\top]^\top$ and utilizing Lemma (3.2) in [1], we obtain

$$p_m \|\boldsymbol{\chi}\|^2 \leq V(\hat{\mathbf{q}}_{eq}, \hat{\boldsymbol{\omega}}_e^c) \leq p_M \|\boldsymbol{\chi}\|^2 \tag{47}$$

for some $p_M > p_m > 0$. Thus choosing $\underline{\alpha}(\hat{\mathbf{q}}_{eq}, \hat{\boldsymbol{\omega}}_e^c) = p_m\|\boldsymbol{\chi}\|^2$ and $\overline{\alpha}(\hat{\mathbf{q}}_{eq}, \hat{\boldsymbol{\omega}}_e^c) = p_M\|\boldsymbol{\chi}\|^2$ ensures the existence of such functions. Evaluating the time derivative of V along the closed-loop trajectories generated by (45) yields

$$\begin{aligned} \dot{V}(\hat{\mathbf{q}}_{eq}, \hat{\boldsymbol{\omega}}_e^c) =& 2\hat{\mathbf{q}}_{eq}^\top \hat{\mathbf{K}}_p \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\hat{\boldsymbol{\omega}}_e^c \\ &+ (\hat{\boldsymbol{\omega}}_e^c)^\top(-\hat{\mathbf{K}}_p\tilde{\varepsilon} - \hat{\mathbf{K}}_d\hat{\boldsymbol{\omega}}_e^c) \\ =& (\hat{\mathbf{K}}_p\tilde{\varepsilon})^\top \hat{\boldsymbol{\omega}}_e^c - (\hat{\boldsymbol{\omega}}_e^c)^\top \hat{\mathbf{K}}_p\tilde{\varepsilon} - (\hat{\boldsymbol{\omega}}_e^c)^\top \hat{\mathbf{K}}_d \hat{\boldsymbol{\omega}}_e^c \\ =& -(\hat{\boldsymbol{\omega}}_e^c)^\top \hat{\mathbf{K}}_d \hat{\boldsymbol{\omega}}_e^c \leq 0. \end{aligned} \tag{48}$$

We conclude, by Theorem 4.8 in [23], that the equilibrium point $(\hat{\mathbf{q}}_{eq}, \hat{\boldsymbol{\omega}}_e^c) = (\mathbf{0}, \mathbf{0})$ is uniformly stable and the solutions are uniformly bounded.

To show uniform asymptotic stability we invoke Matrosov's theorem, as stated in [24], by introducing the auxiliary function

$$W(\hat{\mathbf{q}}_{eq}, \hat{\boldsymbol{\omega}}_e^c) = \tilde{\varepsilon}^\top \hat{\boldsymbol{\omega}}_e^c = 2\hat{\mathbf{q}}_{eq}^\top \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\hat{\boldsymbol{\omega}}_e^c \tag{49}$$

which is continuous in both arguments and depends on time through the bounded reference function $\hat{\mathbf{q}}_d$. Differentiation of the auxiliary function yields

$$\dot{W} = 2\dot{\hat{\mathbf{q}}}_{eq}^\top \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\hat{\boldsymbol{\omega}}_e^c + 2\hat{\mathbf{q}}_{eq}^\top \dot{\hat{\mathbf{T}}}_e(\hat{\mathbf{q}}_{eq})\hat{\boldsymbol{\omega}}_e^c + \tilde{\varepsilon}^\top \dot{\hat{\boldsymbol{\omega}}}_e^c \tag{50}$$

and after inserting (45) one can varify that on the set $E = \{\dot{V} = 0\} = \{\hat{\boldsymbol{\omega}}_e^c = \mathbf{0}\}$,

$$\dot{W} = -\tilde{\varepsilon}^\top \mathbf{K}_p \tilde{\varepsilon} \tag{51}$$

That is, $\dot{W}$ is non-zero definite on E. Thus all conditions of Matrosov's theorem are satisfied, and $(\hat{\mathbf{q}}_{e\pm}, \hat{\boldsymbol{\omega}}_e^c) \to (\mathbf{0}, \mathbf{0})$ asymptotically. The proof for the negative equilibrium point, $\hat{\mathbf{q}}_{eq} = \hat{\mathbf{q}}_{e-}$ and $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e-})$ is performed in the same way. It follows that the dual equilibrium points $\hat{\mathbf{q}}_{eq} \in S_e^3 \ltimes \mathbb{R}^3$ are UAS. $\square$

D. Practical stability and damping

In the above section we establish practical stability of the equilibrium points of the closed loop augmented system, there is however no damping for the two rotational degrees of freedom that is used to solve the translational tracking problem. To counter this we add a damping term to the controller in Propostion 1 defined as

$$\hat{\omega}_t = \mathbf{0} + \epsilon \mathbf{S}(\omega_{n,b}^c)\boldsymbol{\Delta}^c \tag{52}$$

such that the closed loop dynamics now become

$$\dot{\hat{\omega}}_e^c = -\mathbf{K}_p\tilde{\varepsilon} - \mathbf{K}_d\hat{\omega}_e^c - \hat{\omega}_t. \tag{53}$$

Re-evaluating 46 we have that

$$\begin{aligned} \dot{V}(\hat{\mathbf{q}}_{eq}, \hat{\boldsymbol{\omega}}_e^c) &= -(\hat{\boldsymbol{\omega}}_e^c)^\top \hat{\mathbf{K}}_d \hat{\boldsymbol{\omega}}_e^c - (\hat{\boldsymbol{\omega}}_e^c)^\top \hat{\boldsymbol{\omega}}_t \\ &= -(\hat{\boldsymbol{\omega}}_e^c)^\top \hat{\mathbf{K}}_d \hat{\boldsymbol{\omega}}_e^c - (\hat{\boldsymbol{\omega}}_{e0}^c)^\top \hat{\boldsymbol{\omega}}_t - (\hat{\boldsymbol{\omega}}_t)^\top \hat{\boldsymbol{\omega}}_t \end{aligned} \tag{54}$$

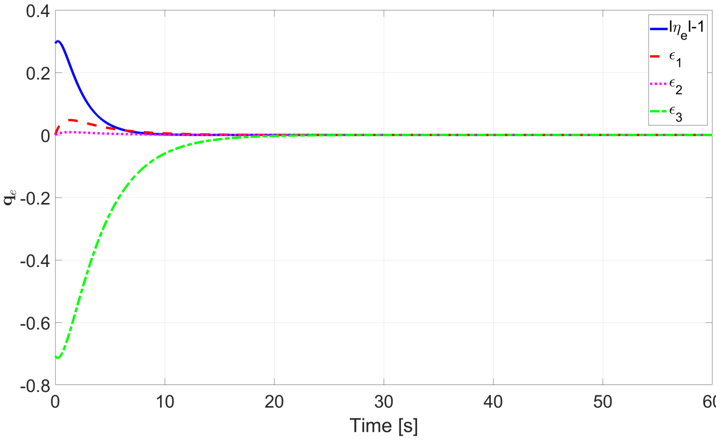

Fig. 1. Quaternion attitude error of the system

where we use the fact that $\hat{\boldsymbol{\omega}}_e^c = \hat{\boldsymbol{\omega}}_{e0}^c + \hat{\boldsymbol{\omega}}_t$ and $\hat{\boldsymbol{\omega}}_{e0}^c$ is the dual velocity error of the original system. Strictly speaking this makes the above stated proof invalid however in practice it may be argued that since $\hat{\boldsymbol{\omega}}_{e0}^c$ converges to $-\hat{\boldsymbol{\omega}}_t$ and is oscillating around zero the only valid equilibrium point is that of $\hat{\boldsymbol{\omega}}_{e0}^c = \mathbf{0}$. This is not shown in the strict mathematical sense and is left for future work.

V. Simulations

In this section, simulation results for a quadrotor-gimbal platform tracking a trajectory are presented to demonstrate the performance of the presented control law in Proposition 1. The quadrotor model is based upon the UiTRotor quadrotor that have a mass of 1.3kg, and moments of inertia for the quadrotor given as $\mathbf{J}^b = \mathrm{diag}\{0.04\ 0.04\ 0.5\}\mathrm{kgm}^2$ and for the gimbal $\mathbf{J}^g = \mathrm{diag}\{0.00023\ 0.00023\ 0.00045\}\mathrm{kgm}^2$. The control gains is given as $\mathbf{K}_p = \mathrm{diag}\{1\ 1\ 1\ 1\ 1\ 0.1\ 0.1\ 0.1\}$, $\mathbf{K}_d = \mathrm{diag}\{1\ 2\ 2\ 2\ 1\ 0.4\ 0.4\ 2\}$. The initial condition for the quadrotor system is

$$\begin{aligned} \hat{\mathbf{q}}_{n,b}(t_0) &= \hat{\mathbf{q}}_I + \varepsilon\frac{1}{2}\hat{\mathbf{q}}_I \otimes [0\ 0\ 5\ \ -5]^\top \\ \hat{\boldsymbol{\omega}}_{n,b}^b(t_0) &= [0\ 0.1\ 0.2\ 0\ 0\ 0\ 0\ 0]^\top \end{aligned}$$

and $\Delta = 1$. We employ a straight-line trajectory with a constant angular velocity reference, similar to that found in [25], defined as

$$\begin{aligned} \mathbf{p}_d^n(t) &= [0\ ((75/4) - (3/4)t)\ 1\ \ -10]^\top \\ \boldsymbol{\omega}_{nd}^d(t) &= [0\ 0.01\ 0.03\ 0.2]^\top \end{aligned}$$

with initial condition $\mathbf{q}_{n,d} = \mathbf{q}_I$. Figure 1 shows the quaternion attitude error for the gimbal; as the gimbal is assumed fully actuated this error goes to zero fairly quick. Figure 2 show the position error for the augmented system; as expected the position error does not completely converge to zero as implied by the practical stability property.

VI. Conclusion and Future work

We proposed a new method to solve the trajectory tracking problem for the underactuated quadrotor platform, including a PD+ based state feedback control law for solving the

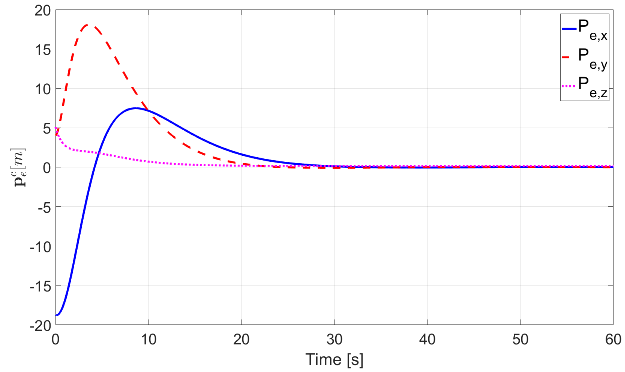

Fig. 2. Position error of the system

tracking problem under this method. It was shown that the equilibria of the closed-loop augmented system are uniformly asymptotically stable, which implied that the equilibria of the closed-loop real system are practically asymptotically stable. Simulations demonstrate the theoretical results, however, they revealed that further work is necessary in order for the method to be implemented on a real quadrotor.

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