VANISHING OF THE ENTROPY PSEUDONORM
FOR CERTAIN INTEGRABLE SYSTEMS

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ABSTRACT. We introduce the notion of entropy pseudonorm for an action of
$\mathbb{R}^n$ and prove that it vanishes for the group actions associated with a large
class of integrable Hamiltonian systems.

1. ENTROPY PSEUDONORM

Let W be a smooth manifold and $\Phi : (\mathbb{R}^n, +) \to \mathrm{Diff}(W)$ a smooth action on
it. Assume there exists a compact Φ-invariant exhaustion of W, i.e., $W = \bigcup_{i=1}^{\infty} K_i$
with compact Φ-invariant sets K_i. Define the following function on $\mathbb{R}^n$ (where h_{top}
is the topological entropy):

$$\rho_\Phi(v) = h_{\mathrm{top}}(\Phi(v)), \quad v \in \mathbb{R}^n.$$

This function is a pseudonorm on $\mathbb{R}^n$: $\rho_\Phi(v)$ is well defined because, under our
hypothesis, the entropy h_d of [8] does not depend on the distance function d on W,
homogeneity is standard, and the triangle inequality follows from the Hu formula
[16]. We call ρ_Φ the **entropy pseudonorm**.

We will investigate it in the case of the Poisson action corresponding to an
integrable Hamiltonian system on a symplectic manifold (W^{2n}, ω). Namely, let
(W^{2n}, ω) possess pairwise Poisson-commuting functions $I_1, I_2, \ldots, I_n$, which are
functionally independent almost everywhere.

Denote by φ_i^τ the time τ shift along the Hamiltonian vector field of the function
I_i. The maps φ_i^τ commute and therefore generate the Poisson action of the group
$(\mathbb{R}^n, +)$,

$$\Phi(\tau_1, \ldots, \tau_n) \stackrel{\mathrm{def}}{=} \varphi_1^{\tau_1} \circ \cdots \circ \varphi_n^{\tau_n} : W^{2n} \to W^{2n},$$

with the corresponding momentum map $\Psi = (I_1, \ldots, I_n) : W^{2n} \to \mathbb{R}^n$; see [1].

The entropy pseudonorm ρ_Φ vanishes in the following important cases (we discuss
these conditions at the end of the paper, in Section 3):

- Williamson-Vey-Eliasson-Ito nondegenerate singularities [12, 18];
- Taimanov nondegeneracy condition [33].

In the first case vanishing of topological entropy of the Hamiltonian flow was
proved in [29]; in the second case, in [33]. Since there is nothing special about

the Hamiltonian in these situations, it can be changed to any of the integrals and $\rho_\Phi \equiv 0$ follows. Also in [28, 9] vanishing of h_{top} was proven for the cases of

- Systems integrable with periodic integrals;
- Collectively integrable systems (the definition is in [14]).

It is not difficult to see that in both cases the entropy pseudonorm ρ_Φ vanishes as well.

Note that Liouville integrability does not imply vanishing of topological entropy; see [7] (more examples in [10]). For these examples the entropy pseudonorm is nonzero, but degenerate. However, it is possible to construct integrable examples [20] such that ρ_Φ is a norm.

In the present paper we prove vanishing of the entropy pseudonorm for another class of integrable systems. These systems have recently been actively studied in mathematical physics in the framework of the theory of separation of variables. In different contexts they are called Benenti systems [17], L-systems [3, 4], co-factor systems [23], or quasi-bi-Hamiltonian systems [11]. The Benenti systems are certain integrable Hamiltonian systems on T^*M with Hamiltonian of the form $H = K_g + V$, where $K_g : T^*M \to \mathbb{R}$, $K_g(x,p) = \langle p, p \rangle_g$, is twice the kinetic energy corresponding to a Riemannian metric g and $V : M \to \mathbb{R}$ is a potential. Important feature of these systems is that every integral is a sum of a function quadratic in momenta and a function on M. Moreover, the quadratic forms corresponding to the quadratic-in-momenta terms are simultaneously diagonalizable. We will provide precise definitions, the conditions on the metric g and potential V as well as formulas for integrals in Section 2.1, where we also explain how these systems are related to the theory of geodesic equivalence.

Theorem 1. *Let M be a compact connected manifold. Then the entropy pseudo-norm of the action Φ associated with any Benenti integrable Hamiltonian system on T^*M vanishes: $\rho_\Phi \equiv 0$.*

The class of mechanical systems covered by Theorem 1 contains Lagrange spinning tops, the von Neumann system, the Braden system, Bogoyavlensky systems, some Manakov systems, and many other quadratically integrable (Stäckel) systems. For most of them our vanishing result is new.

For geodesic flows ($V \equiv 0$) vanishing of $h_{\text{top}}(H)$ was proven in our earlier paper [21]. Theorem 1 generalizes the result of [21] in the following two directions. First, it includes the potential energy in the picture. Second, it shows that the topological entropy of the Hamiltonian flow of every integral (not only the Hamiltonian) vanishes.

Our result has topological implications (for example, due to [31]). We discussed some of them in [21] (see also [25, 26, 27]). In particular, if a manifold M possesses a Benenti-integrable Hamiltonian system, then it must be covered by a product of a rationally elliptic manifold and a torus: $M^n = Q^k \tilde{\times} T^{n-k}$. Indeed, by the description from Section 2.1 we know that the kinetic energy K_g corresponding to a Benenti system is also an integrable Hamiltonian and, moreover, for the corresponding Riemannian metric g, there exists another, $\bar{g}$, which is geodesically equivalent to g and is strictly nonproportional to it. Thus the claim follows from Theorem 7 of [21].

There are also many dynamical implications. Actually, for large values of the Hamiltonian H and the integrals I_t (with $t \leq \min \text{Sp}(L)$ to achieve positivity of the

quadratic forms; see Section 2.1), these functions behave like Riemannian metrics on M because of the Maupertuis principle [1, 5]. Their geodesic flows commute and have zero topological entropy. This implies, for all of them, a subexponential growth of the volume of balls in the universal cover (Manning's inequality), of the number of geodesic arcs joining two generic points (Mañé's formula), etc.; see [30] (Sections 3.4, 3.5, and 4.2).

An outline of the proof for Theorem 1 will be presented in Section 2.2. We describe singular orbits of the Poisson action and show that the restriction of our integrable system to every singular orbit is a subsystem of a Benenti system on a manifold of smaller dimension. Then we apply induction on the dimension. Again as in the case of geodesic flows, the set of singular points can be very complicated: If $n \stackrel{\text{def}}{=} \dim(M) > 2$, then there exists a singular point over every point of M and the set of singular points in $\Psi^{-1}(c)$ can project to a fractal in M^n of Hausdorff dimension $> n - 1$.

We also discuss vanishing of other entropies. The well-known entropy for the group action $h_{\text{top}}(\Phi)$ [13] vanishes for Benenti systems by elementary reasons: $h_{\text{top}}(\Phi) = 0$. Actually, if this entropy is positive, then (directly from the definition) all entropies of subgroup actions are infinite; in particular, $h_{\text{top}}(H) = +\infty$, which is incorrect.

However, there is another definition of the entropy for a group action $h_U(\Phi)$, which behaves naturally with respect to restrictions to subgroup actions [15]. Here U is the cube $[-1, 1]^n \subset \mathbb{R}^n$ that defines the strongly regular system $(U, 2U, 3U, \dots)$ exhausting our group $(\mathbb{R}^n, +)$. With respect to this definition and the action Φ of $\mathbb{R}^n$ associated to a Benenti integrable system we have (see [15] for the definition and properties)

$$h_U(\Phi) = 0.$$

This follows from the inequalities

$$\max_{1 \leq k \leq n} h_{\text{top}}(I_k) \leq \sup_{v \in U} \rho_\Phi(v) \leq h_U(\Phi) \leq n \cdot \max_{1 \leq k \leq n} h_{\text{top}}(I_k).$$

Here the first inequality is obvious, the second is Proposition 2.6 from [15], and the third easily follows from the definition of h_U. Thus vanishing of the entropy pseudonorm ρ_Φ is equivalent to vanishing of the entropy $h_U(\Phi)$.

2. Definitions and sketch of the proof

2.1. Benenti systems and geodesically equivalent metrics.
Let $g, \bar{g}$ be two Riemannian metrics on a connected manifold M of dimension n, and $\flat_g : TM \to T^*M$, $\sharp^{\bar{g}} : T^*M \to TM$ the corresponding bundle morphisms. We regard the bundle morphism $\sharp^{\bar{g}} \circ \flat_g : TM \to TM$ as a $(1,1)$-tensor.

The metrics $g, \bar{g}$ are called **geodesically equivalent** if every geodesic of $\bar{g}$, regarded as an unparameterized curve, is a geodesic of g. They are said to be **strictly nonproportional** at $P \in M^n$ if the spectrum $\text{Sp}(\sharp^{\bar{g}} \circ \flat_g) \subset \mathbb{R}_+$ is simple at P.

Consider the $(1,1)$-tensor $L \stackrel{\text{def}}{=} (\sharp^{\bar{g}} \circ \flat_g) / \sqrt[n+1]{\det(\sharp^{\bar{g}} \circ \flat_g)} : TM \to TM$. For every $t \in \mathbb{R}$, consider the $(1,1)$-tensor $S_t \stackrel{\text{def}}{=} \det(L - t\,\text{Id})\,(L - t\,\text{Id})^{-1}$. The family S_t is polynomial in t of degree $n - 1$.

We will always identify the tangent and the cotangent bundle of M with the help of $\flat_g$. This identification gives us a symplectic form and a Poisson structure on TM.

Theorem 2 ([24, 25]). *If g, $\bar{g}$ are geodesically equivalent, then for all $t_1, t_2 \in \mathbb{R}$ the functions*

$$I_{t_i} : TM \to \mathbb{R}, \quad I_{t_i}(v) \stackrel{\text{def}}{=} g(S_{t_i}(v), v)$$

are commuting integrals for the geodesic flow of g.

If, in addition, the metrics are strictly nonproportional at one point, then this is so for almost every point. Consequently, for all $t_1 < \cdots < t_n$ the integrals I_{t_i} are functionally independent almost everywhere so that the geodesic flow of g is Liouville-integrable.

It is possible to add potential energy to the picture. In local coordinates, it was done in [2] (see also [3, 4, 6]); other approaches are given in [17] and [11].

Let g and $\bar{g}$ be geodesically equivalent Riemannian metrics on M^n. A smooth function $V : M^n \to \mathbb{R}$ will be called **compatible** with respect to g and $\bar{g}$ if the 1-form

$$dV \circ (L - \text{trace}(L)\text{Id})$$

is exact. For every pair of geodesically equivalent metrics that are not affine equivalent, we can prove the existence of a nonconstant compatible V (actually of a continuum-dimensional family).

It is possible to show that if there exists a compatible function V, then there exists a family V_t, $t \in \mathbb{R}$, of smooth functions on M^n such that the following two conditions are fulfilled:

$$(1) \qquad \begin{cases} V_t & \text{is a polynomial in } t \text{ of degree } \leq n-1, \\ dV \circ S_t = dV_t & \text{for every } t \in \mathbb{R}. \end{cases}$$

The potential V determines the family V_t up to (addition of) a constant polynomial $P(t)$ of degree $\leq n-1$. Note that the family V_t also defines the function V up to a constant. In fact, the function V is the coefficient of t^{n-1}.

Locally, the existence of such V_t was explained in [4]. From the normal form for the functions V_t, given in Theorem 4 below, it is clear that, near generic points, we have a great deal of freedom in choosing the functions V and V_t: They depend on arbitrary n functions of one variable. Globally on M the existence of such V_t is nontrivial, for instance because the functions X_i from Theorem 4 can have singularities near the bifurcation points of the spectrum $\text{Sp}(L)$ of L.

Theorem 3 ([3, 4, 11, 6]). *Let g, $\bar{g}$ on a connected M^n be geodesically equivalent. Suppose V is compatible with respect to $g, \bar{g}$. Consider a family V_t of functions satisfying conditions (1). Then for all $t_1, t_2 \in \mathbb{R}$ the functions $\hat{I}_{t_i} \stackrel{\text{def}}{=} I_{t_i} + V_{t_i}$ are commuting integrals for the Hamiltonian system with Hamiltonian $K_g + V$, where K_g is twice the kinetic energy corresponding to g. If, in addition, the metrics are strictly nonproportional at least at one point, then for all $t_1 < \cdots < t_n$ the integrals $\hat{I}_{t_i}$ are functionally independent almost everywhere.*

We will call a **Benenti system** the integrable system on TM^n generated by the integrals $\hat{I}_{t_1}, \ldots, \hat{I}_{t_n}$ from Theorem 3 provided the metrics g, $\bar{g}$ are strictly nonproportional at least at one (and hence at almost every) point.

2.2. **Idea of the proof of Theorem 1.** We use induction on the dimension. If the dimension of the manifold is $n < 2$, Theorem 1 is trivial. Assume that Theorem 1 is true for every dimension less than n and consider $\dim M = n$.

Suppose the topological entropy of the Hamiltonian flow corresponding to an integral $\hat{I}_t$ is not zero. Then, by the variational principle, there exists an ergodic $\hat{I}_t$-invariant Borel probability measure μ such that $h_\mu(g) \neq 0$ [19]. By ergodicity the support $\mathrm{Supp}(\mu)$ is contained in a connected component of some fiber $\Psi^{-1}(c)$. If $\mathrm{Supp}(\mu)$ contains a point P with $\mathrm{rank}(d\hat{I}_{t_1}, \ldots, d\hat{I}_{t_n}) = n$, then the orbit of P is diffeomorphic to a cylinder over torus, $\Phi(\mathbb{R}^n, P) \simeq T^k \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$ [1]. By the implicit function theorem, a small neighborhood of a point P in $\mathrm{Supp}(\mu)$ lies in the orbit $\Phi(\mathbb{R}^n, P)$. Since $\mathrm{Supp}(\mu)$ is a closed invariant subset and its point P cannot be wandering, the support is diffeomorphic to a subtorus T^l, $l \leq k$, for which the flow of $\hat{I}_t$ is conjugate to a standard linear flow. This implies that the entropy h_μ vanishes.

Now suppose that every point of $\mathrm{Supp}(\mu)$ is **singular**, so that

$$\mathrm{rank}(d\hat{I}_{t_1}, \ldots, d\hat{I}_{t_n}) \leq k < n$$

and the rank equals k almost everywhere on the support of μ. In this case, we can reduce the dimension. Namely, there exists a closed proper submanifold $N^k \subset M^n$ with induced Benenti system and a subgroup $\mathbb{R}^k \subset \mathbb{R}^n$ with $\tilde{\Phi} = \Phi_{|\mathbb{R}^k}$ such that $\mathrm{Supp}(\mu)$ is $\tilde{\Phi}$-invariant and $\tilde{\Phi}_{|\mathrm{Supp}(\mu)}$ is a subsystem of the Poisson action corresponding to the Benenti system on N^k. Then $h_\mu = 0$ by the induction assumption.

This is actually the main point of the proof. Precisely the same logic was used in [21]. To a certain extent, not only the statement, but also most of the proofs from [21] can be generalized to our more general setting. In Sections 2.3, 2.4 we will explain how to construct these closed submanifolds under the additional assumption that all eigenvalues of L are nonconstant. This additional assumption makes the proof much shorter (for instance, because in this case we can take $k = n - 1$; in the paper [21] the longest part was dedicated to dealing with constant eigenvalues of L), so that we can hope to make the main ideas of the proof clear to everyone.

2.3. **Benenti systems and singular points in Levi-Civita coordinates.**

Theorem 4 (follows from [22, 6, 4]). *Let g and $\bar{g}$ be geodesically equivalent Riemannian metrics on M^n. Suppose they are strictly nonproportional at $P \in M^n$. Let the function V be compatible with respect to g and $\bar{g}$, and suppose the functions V_t satisfy conditions* (1).

Then in a small neighborhood $U \subset M^n$ of P there exist coordinates (called Levi-Civita coordinates) such that the metrics g, $\bar{g}$ and the functions V_t are given by the formulas

$$(2) \qquad ds_g^2 \;=\; \sum_{i=1}^{n} (-1)^{i-1} \prod_{j \neq i} (\lambda_j - \lambda_i)\, dx_i^2,$$

$$(3) \qquad ds_{\bar{g}}^2 \;=\; \sum_{i=1}^{n} \frac{(-1)^{i-1}}{\lambda_i \prod_\alpha \lambda_\alpha} \prod_{j \neq i} (\lambda_j - \lambda_i)\, dx_i^2,$$

$$(4) \qquad V_t \;=\; \sum_{i=1}^{n} (-1)^{i-1} X_i \prod_{j \neq i} \frac{\lambda_j - t}{\lambda_j - \lambda_i},$$

where, for every i, λ_i and X_i are functions of one variable x_i. If in a neighborhood of almost every point the metrics g, $\bar{g}$ and the functions V_t are given by (2)–(4), then the metrics are geodesically equivalent and the functions satisfy conditions (1) with respect to some compatible function V.

In the Levi-Civita coordinate system, L is diagonal $\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$. We will always assume that, at every point, the eigenvalues λ_i of L are indexed according to their value, so that $\lambda_i(P) \leq \lambda_{i+1}(P)$ for every $P \in M^n$ and every $1 \leq i \leq n-1$.

We see that the metric g and the tensor L define the Levi-Civita coordinate system up to a shift of the origin and a change of the direction of coordinate axes. Indeed, the vector $v_i \overset{\text{def}}{=} \frac{\partial}{\partial x_i}$ is determined up to sign by the conditions

$$\begin{cases} Lv_i &=& \lambda_i v_i, \\ g(v_i, v_i) &=& (-1)^{i-1} \prod_{j \neq i} (\lambda_j - \lambda_i). \end{cases}$$

In this coordinate system the integrals $\hat{I}_t = I_t + V_t$ (as functions of the cotangent bundle) are given by

$$(5) \qquad \hat{I}_t = \sum_{i=1}^{n} (-1)^{i-1} (p_i^2 + X_i) \prod_{j \neq i} \frac{\lambda_j - t}{\lambda_j - \lambda_i}.$$

Note that the functions X_i can be invariantly obtained from V_t and λ_i, namely

$$X_i(P) = (-1)^{i-1} V_{\lambda_i(P)}(P).$$

Let $\hat{I}'_t = \frac{d}{dt}\hat{I}_t$. For every t the function $\hat{I}'_t$ is a linear combination of the integrals $I_{t_1}, \ldots, I_{t_n}$.

Corollary 1. *Let the metrics g and $\bar{g}$ be geodesically equivalent on a closed connected manifold M and strictly nonproportional at $P \in M$. If a point $(P, \xi) \in TM$ is singular with respect to the Poisson action Φ corresponding to the integrals $\hat{I}_{t_i} = I_{t_i} + V_{t_i}$, then $d_{(P,\xi)}\hat{I}_{\tilde{\lambda}_i} = 0$ for some i, where $\tilde{\lambda}_i \overset{\text{def}}{=} \lambda_i(P)$. In addition, in Levi-Civita coordinates the i-th component of ξ vanishes: $\xi_i = 0$.*

Proof. Consider the function $\hat{I}_{\lambda_i(x)}$. Expressed in Levi-Civita coordinates on the cotangent bundle T^*M, it equals $(-1)^{i-1} \left(p_i^2 + X_i(x_i) \right)$, so that its differential is $(-1)^{i-1} \left(2p_i\, dp_i + X'_i(x_i)dx_i \right)$. On the other hand,

$$d\hat{I}_{\tilde{\lambda}_i} = d\hat{I}_{\lambda_i(x_i)} - \hat{I}'_{\tilde{\lambda}_i} d\lambda_i(x_i) = (-1)^{i-1} 2p_i dp_i + \left((-1)^{i-1} X'(x_i) - \lambda'_i(x_i) \hat{I}'_{\tilde{\lambda}_i} \right) dx_i.$$

Thus if a linear combination $\sum \mu_i d\hat{I}_{\tilde{\lambda}_i}$ vanishes, then for every $\mu_i \neq 0$ the corresponding $d\hat{I}_{\tilde{\lambda}_i}$ vanishes. Then its dp_i and dx_i components vanish, yielding $p_i = 0$ (which implies $\xi_i = 0$). $\qquad\square$

2.4. Submanifolds $\mathcal{M}_i$ and Sing_i. In this section we assume that g and $\bar{g}$ are geodesically equivalent metrics on a closed connected M^n, that every eigenvalue λ_i of L is nonconstant, and that the functions V_t satisfy (1) with respect to a compatible V. For $i = 1, \ldots, n-1$, put

$$\mathrm{Reg}_i = \{x \in M : \lambda_i(x) \neq \lambda_{i+1}(x)\},$$

and for $i = 0$, let $\mathrm{Reg}_0 = M^n$. At every point of $\mathrm{Reg}_i \cap \mathrm{Reg}_{i-1}$ the eigenvalue λ_i is simple. In particular, at every point of $\mathrm{Reg} \overset{\text{def}}{=} \bigcap_i \mathrm{Reg}_i$, the eigenvalues $\lambda_1, \ldots, \lambda_n$ are mutually distinct.

For every $x \in \mathrm{Reg}_i \cap \mathrm{Reg}_{i-1}$ denote by $\mathcal{D}_i(x) \subset T_x M^n$ the subspace spanned by the eigenspaces corresponding to λ_j, $j \neq i$. The distribution $\mathcal{D}_i$ is smooth. By Theorem 4, it is integrable in $\mathrm{Reg}_i \cap \mathrm{Reg}_{i-1}$. Denote by $\mathcal{M}_i(P)$ its integral manifold containing $P \in M^n$ (beware of our using the notation $D_{\mathsf{C}(i)}$ and $M_{\mathsf{C}(i)}$ in [21] instead of the present $\mathcal{D}_i$ and $\mathcal{M}_i$). By Theorem 4 the functions λ_i and X_i are constant along $\mathcal{M}_i(P)$.

For $i = 1, \dots, n-1$, let $\bar{\lambda}_i = \frac{1}{2} \left(\max_{x \in M} \lambda_i + \min_{x \in M} \lambda_{i+1} \right)$. By Corollary 1 from [25], for every point $P \in M^n$ we have $\lambda_i(P) \leq \bar{\lambda}_i \leq \lambda_{i+1}(P)$. Consider

$$\mathrm{Sing}_i \stackrel{\text{def}}{=} \{ P \in M^n : (\lambda_i(P) - \bar{\lambda}_i)(\lambda_{i+1}(P) - \bar{\lambda}_i) = 0 \}.$$

In [25] (see Theorem 5 there) it was proven that if Sing_i is nonempty, then, under the assumption that all λ_i are nonconstant, it is a connected submanifold of codimension 1. Moreover, almost all points of Sing_i belong to Reg and the intersection $\mathrm{Sing}_i \cap \mathrm{Reg}_i$ ($\mathrm{Sing}_i \cap \mathrm{Reg}_{i+1}$, respectively) is a finite union of leaves $\mathcal{M}_i$ ($\mathcal{M}_{i+1}$, respectively).

In [21] (see Lemma 2 there) we proved that if the function λ_i is not constant, then every $\mathcal{M}_i(P)$ is a closed submanifold or is a part of Sing_i or Sing_{i+1}. Combining this observation and Corollary 1, we obtain that the projection of every singular orbit belongs to some compact $\mathcal{M}_i$ or to one of Sing_i. Since our measure μ from §2.2 is ergodic, $\mathrm{Supp}(\mu)$ belongs to the closure of an orbit. Then the projection of $\mathrm{Supp}(\mu)$ belongs to a compact submanifold of smaller dimension.

The last step is to explain that the dynamics on $\mathrm{Supp}(\mu)$ is a subsystem of a certain Benenti system on this submanifold. Almost every point of Sing_i belongs to $\mathcal{M}_i \cup \mathcal{M}_{i+1}$, so it is sufficient to consider only $\mathcal{M}_i$. Since $\mathcal{M}_i \cap \mathrm{Reg}$ is dense in $\mathcal{M}_i$, we can use Levi-Civita coordinates. Since $T^* \mathcal{M}_i$ is a symplectic submanifold of $T^* M^n$, the claim follows from the fact that the restrictions of the integrals $\hat{I}_t$ to $T^* \mathcal{M}_i$ are linear combinations of the integrals of the induced Benenti system on $\mathcal{M}_i$. This assertion can be checked in Levi-Civita coordinates using formula (5).

In fact, the family V_t is determined up to a constant polynomial of degree $n-1$. Since the function X_i is constant on $\mathcal{M}_i(P)$, we can assume without loss of generality that $X_i = 0$ on $\mathcal{M}_i(P)$. Since the coordinate p_i vanishes along $T^* \mathcal{M}_i(P)$ and the function λ_i is constant on $\mathcal{M}_i(P)$, the restriction of the integral $\hat{I}_t = I_t + V_t$ to $T^* \mathcal{M}_i(P)$ is equal to

$$
\begin{aligned}
\hat{I}_t &= \sum_{k \neq i} (-1)^{k-1} (p_k^2 + X_k) \prod_{j \neq k} \frac{\lambda_j - t}{\lambda_j - \lambda_k} \\
&= (\lambda_i - t) \cdot \left[\sum_{k \neq i} \frac{(-1)^{k-1-\theta(k-i)} (p_k^2 + X_k)}{|\lambda_i - \lambda_k|} \prod_{j \neq k, i} \frac{\lambda_j - t}{\lambda_j - \lambda_k} \right],
\end{aligned}
$$

where $\theta(x)$ is the Heaviside function, so that $\psi(k) = k - \theta(k-i)$ enumerates $\{1, \dots, n\} \setminus \{i\}$.

By the direct calculation we check that, for every t, the above expression in square brackets is a linear combinations of the integrals $\hat{I}^{\text{new}}_{\tau_1}, \dots, \hat{I}^{\text{new}}_{\tau_{n-1}}$ of the Benenti system corresponding to geodesically equivalent metrics $g^{\text{new}} \stackrel{\text{def}}{=} g_{|\mathcal{M}_i(P)}$, $\bar{g}^{\text{new}} \stackrel{\text{def}}{=} \lambda_i \bar{g}_{|\mathcal{M}_i(P)}$ on $\mathcal{M}_i(P)$ and the family $V^{\text{new}}_t : \mathcal{M}_i(P) \to \mathbb{R}$, $V^{\text{new}}_t \stackrel{\text{def}}{=} \frac{1}{\lambda_i - t} V_{t|\mathcal{M}_i(P)}$. The facts that the metrics g^{new}, $\bar{g}^{\text{new}}$ are geodesically equivalent and strictly non-proportional at least at one point and that the family V_t satisfies condition (1)

follow from Theorem 4, because on $\mathcal{M}_i(P)$ the coefficient $|\lambda_k - \lambda_i|$ depends on x_k only and therefore can be "hidden" in the corresponding dx_k and X_k. Thus our system is a subsystem of a Benenti system and the induction hypothesis finishes the proof of Theorem 1.

3. Discussion

It is clear that all the difficulties with vanishing of entropies for integrable systems are due to a complicated singularity set (we explained essentially in §2.2 that the set of regular points bears no entropy), as positive-entropy examples of [7, 20, 9, 10] demonstrate. In all good cases, where vanishing of the entropy has been proven, some stratification of singularities was achieved; see, e.g., [28, 29, 33, 9, 21]. Here we formulate a scheme for most vanishing results.

We consider systems on noncompact W^{2n}, but such that the variational principle holds. This is, for instance, the case when W^{2n} admits an exhaustion by compact invariant sets (other cases are discussed in [32]). Define the following Φ-invariant subsets of W^{2n}:

$$\Sigma_k = \{x \in W^{2n} \,|\, \text{rank}(d_x\Phi) = k\}.$$

Theorem 5. *Suppose for every $k < n$ we can decompose $\Sigma_k = \Sigma_k^+ \cup \Sigma_k^-$, where Σ_k^+ is a closed invariant subset of Σ_k and Σ_k^- consists of nonrecurrent points of $\Phi(v)$ for almost every $v \in \mathbb{R}^n$. Suppose the momentum map $\Psi : \Sigma_k^+ \to \mathbb{R}^n$ can be factorized to the composition of continuous maps $\pi_k : \Sigma_k^+ \to A_k$ to a Hausdorff space A_k and $\sigma_k : A_k \to \mathbb{R}^n$ such that each fiber $\Sigma_k^\alpha = \pi_k^{-1}(\alpha)$ is a Φ-invariant k-dimensional submanifold of W^{2n}. Then the entropy pseudonorm vanishes: $\rho_\Phi \equiv 0$.*

Proof. By the variational principle it suffices to prove $h_\mu(\Phi(v)) = 0$ for almost every v and every $\Phi(v)$-invariant ergodic measure μ. By ergodicity Ψ is constant on the support of μ. Consequently, it suffices to prove that for every k the system has zero entropy on $\Psi^{-1}(c) \cap \Sigma_k$. Since $\mathcal{R}(v) \cap \Sigma_k \subset \Sigma_k^+$, where $\mathcal{R}(v)$ denotes the set of $\Phi(v)$-recurrent points, it is enough to show vanishing of entropy on the set $\Sigma_k^+ \cap \Psi^{-1}(c)$.

This set is foliated by the strata Σ_k^α, and hence μ must be supported on a single connected component of it only. This component possesses a transitive Poisson $\mathbb{R}^k$-action and so is isomorphic to a torus T^k (it cannot be a cylinder $T^{k-l} \times \mathbb{R}^l$ because it consists of recurrent points) with quasi-periodic dynamics and hence $h_\mu(\Phi(v)|\Sigma_k^+ \cap \Psi^{-1}(c)) = 0$ implying the claim. □

The assumptions of Theorem 5 are satisfied for integrable systems with Williamson-Vey-Eliasson-Ito nondegenerate singularities (see [12, 18]) or with Taimanov nondegeneracy condition [33].

Indeed, Taimanov's condition is as follows: for every $(n - k) \times n$ matrix v_{ij} of rank $n - k$ the set

(6) $$\left\{ \xi \in \Sigma_k : \; \xi \neq 0 \;\; \text{and} \;\; \sum_{j=1}^{n} v_{ij} \, d_\xi(I_j) = 0 \;\; \forall i = 1, \ldots, n - k \right\}$$

is a submanifolds of TM of dimension k. Clearly, if the Taimanov condition is fulfilled, the space of the connected components of the sets

(7) $$L_k(c) \stackrel{\text{def}}{=} \Sigma_k \cap \Psi^{-1}(c) = \{\xi \in \Sigma_k : I_1 = c_1, \ldots, I_n = c_n\}$$

with induced topology is a Hausdorff space, so that the assumptions of Theorem 5 are automatically fulfilled.

The definition of Williamson-Vey-Eliasson-Ito nondegenerate singularities is complicated and can be found, for example, in Section 1.8 of the book [5]. It is a generalization of the notion "Morse singularity" for integrable systems. Locally, the structure of the momentum mapping near Williamson-Vey-Eliasson-Ito nondegenerate points is given by Eliasson-Ito theorem, which is a generalization of the Morse lemma to the context of integrable Hamiltonian systems (Section 1.8.3 of [5]). From this description, it is clear that the space of connected components of the sets $L_k(c)$ given by (7) with induced topology is again a Hausdorff space, so that the assumptions of Theorem 5 are fulfilled.

Our induction approach implies that the singularities of Benenti systems are also stratified in the manner of Theorem 5. It is feasible that a kind of good stratification is necessary for vanishing of the entropies.

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