A TWO-COMPONENT NONLINEAR VARIATIONAL WAVE SYSTEM

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Abstract. We derive a novel two-component generalization of the nonlinear variational wave equation as a model for the director field of a nematic liquid crystal with a variable order parameter. The equation admits classical solutions locally in time. We prove that a special semilinear case is globally well-posed. We show that a particular long time asymptotic expansion around a constant state in a moving frame satisfy the two-component Hunter–Saxton system.

1. Introduction

The nonlinear variational wave equation $\psi_{tt} - c(\psi)(c(\psi)\psi_x)_x = 0$ was derived by Saxton [30] as a model of the director field of a nematic liquid crystal from Ericksen–Leslie theory with Oseen–Frank potential. The nonlinear variational wave equation has recieved wide attention [3, 6, 15, 17] due to mathematical challenges in the form of wavebreaking in finite time, and distinct ways to extend the solution to a global weak solution. In this context wavebreaking means that either the time or the space derivative becomes unbounded at certain points, while the solution remains Hölder continuous.

In the Ericksen–Leslie theory of nematic liquid crystals the configuration is described by a director field $\mathbf{n}$ which gives the local orientation of the rods, and an order parameter field s which gives the local degree of orientation [13, 25]. The scalar order parameter s can be derived from the Q-tensor of both the macroscopic Landau–de Gennes theory and the mean-field Maier–Saupe theory in the uniaxial case [1, 27]. When the nonlinear variational wave equation was first derived the degree of orientation was assumed constant [30]. Here we will proceed as in the derivation of the nonlinear wave equation, but account for variable degree of orientation. For some results concerning the well-posedness of models including the order parameter s in the Ericksen–Leslie systems see [7, 26]. The mathematical analysis of the relationship between equilibrium states of the full Q-tensor Landau–de Gennes model and Oseen–Frank potentials is an active field of research [12, 14].

Furthermore, from the nonlinear wave equation Hunter and Saxton [18] derived the equation $u_{tx} + (uu_x)_x = \frac{1}{2}u_x^2$ as an asymptotic equation for small perturbations in the long time regime in a moving frame. The Hunter–Saxton equation share many of the features of the nonlinear variational wave equation such as wavebreaking, conservative and dissipative weak solutions, and Hölder continuity [2, 4, 5, 8, 11, 20, 21, 33, 34]. In addition it exhibited novel features such as complete integrability and interpretation as a geodesic flow [19, 24]. It also proved easier to work with due to there being only one family of characteristics and existence of explicit solutions.

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A two-component generalization for the Hunter–Saxton equation was derived from two-component Camassa–Holm [10], and independently form the Gurevich–Zybin system [29]. The two-component generalization is similar to the Hunter–Saxton equation since wavebreaking is possible, there are both conservative and dissipative weak solutions, and u is Hölder continuous [16, 28, 32]. However when the second variable is nonzero almost everywhere initially, there will be no wave breaking [28], and in that sense the introduction of a second variable regularizes the equation. The two-component Hunter–Saxton system has, however, not been shown to be related to the theory of nematic liquid crystals, and one of the aims of this paper is to establish that the two-component Hunter–Saxton system can indeed be derived from the theory of nematic liquid crystals, and that the second variable is related to the order parameter s.

In Section 2 we will perform an original derivation of a novel two-component system of nonlinear wave equations. Moreover, we show that a similar asymptotic expansion to the one by Hunter and Saxton [18] yields the two-component Hunter–Saxton equation. In Section 3 existence of global solutions is shown in the case of constant wave speed, and local solutions is shown to exist in general. The proofs rely on fixed point iterations and standard theory for evolution equations [22].

2. Derivation of the equations

In Ericksen–Leslie theory a nematic liquid crystal in a domain Ω is desribed by a director field $\mathbf{n} : [0,T] \times \Omega \to \mathbb{RP}^2$ and an order parameter $s : [0,T] \times \Omega \to (-\frac{1}{2}, 1)$ [13, 25]. The Lagrangian of the system is then [13] given by

$$\mathcal{L} = -\frac{1}{2}(s\mathbf{n})_t^2 + W_2(s, \nabla s, \mathbf{n}, \nabla \mathbf{n}) + W_0(s),$$

with the potential energy term W_2

$$\begin{aligned} W_2(s, \nabla s, \mathbf{n}, \nabla \mathbf{n}) &= (K_1 + L_1 s)\, s^2 \left(\nabla \cdot \mathbf{n}\right)^2 + (K_2 + L_2 s)\, s^2 \left(\mathbf{n} \cdot \nabla \times \mathbf{n}\right)^2 \\ &\quad + (K_3 + L_3 s)\, s^2 \left|\mathbf{n} \times \nabla \times \mathbf{n}\right|^2 \\ &\quad + ((K_2 + K_4) + (L_2 + L_4)s)\, s^2 \left[\operatorname{tr} \nabla \mathbf{n}^2 - (\nabla \cdot \mathbf{n})^2\right] \\ &\quad + (\kappa_1 + \lambda_1 s)\, |\nabla s|^2 + (\kappa_2 + \lambda_2 s)\, (\nabla s \cdot \mathbf{n})^2 \\ &\quad + (\kappa_3 + \lambda_3 s)\, (\nabla \cdot \mathbf{n})\, (\nabla s \cdot \mathbf{n}) + (\kappa_4 + \lambda_4 s)\, \nabla s \cdot ((\mathbf{n} \cdot \nabla)\mathbf{n})\,, \end{aligned}$$

and some potential function W_0. We are interested in the case $\mathbf{n} = (\cos\psi, \sin\psi, 0)$, and s and ψ depends on t and x only. Then the expression for W_2 can be written

$$\begin{aligned} W_2 &= s^2 \psi_x^2 \left((K_1 + L_1 s)\sin^2\psi + (K_3 + L_3 s)\cos^2\psi\right) \\ &\quad + s_x^2 \left((\kappa_1 + \lambda_1 s)\sin^2\psi + (\tilde{\kappa}_2 + \tilde{\lambda}_2 s)\cos^2\psi\right) \\ &\quad - s_x \psi_x \sin 2\psi \left(\frac{\kappa_3 + \kappa_4}{2} + \frac{\lambda_3 + \lambda_4}{2} s\right), \end{aligned}$$

where $\tilde{\kappa}_2 = \kappa_1 + \kappa_2$ and $\tilde{\lambda}_2 = \lambda_1 + \lambda_2$. We simplify the problem by considering the case where

$$\begin{gathered} \kappa_3 = \kappa_4 = \lambda_3 = \lambda_4 = 0, \\ L_1 = L_3 = \lambda_1 = \tilde{\lambda}_2 = 0, \\ K_1 = \kappa_1, \qquad K_3 = \tilde{\kappa}_2. \end{gathered}$$

Then with the definition $c(\psi)^2 = K_1 \sin^2\psi + K_3 \cos^2\psi$, we get the Lagrangian density

$$\begin{aligned}\mathcal{L}^{2NVW} &= -\frac{1}{2}s_t^2 - \frac{1}{2}s^2\psi_t^2 + \frac{1}{2}s^2 c(\psi)^2\psi_x^2 + \frac{1}{2}c(\psi)^2 s_x^2 + W_0(s)\\ &= -\frac{1}{2}(s\mathbf{n})_t^2 + \frac{1}{2}c(\mathbf{n})^2(s\mathbf{n})_x^2 + W_0(s). \end{aligned}\tag{2.2}$$

We define the two-component nonlinear variational wave system to be the Euler–Lagrange equations for (2.2), namely,

$$s^2\left(\psi_{tt} - c(\psi)\left(c(\psi)\psi_x\right)_x\right) + 2s\left(\psi_t s_t - c(\psi)^2\psi_x s_x\right) + c(\psi)c'(\psi)s_x^2 = 0, \tag{2.3a}$$

$$s_{tt} - c(\psi)\left(c(\psi)s_x\right)_x - c(\psi)c'(\psi)\psi_x s_x - s\left(\psi_t^2 - c(\psi)^2\psi_x^2\right) + W_0'(s) = 0, \tag{2.3b}$$

with $c(\psi)^2 = K_1\sin^2\psi + K_3\cos^2\psi$. We define the energy density

$$\mathcal{E} = \frac{1}{2}\left(s^2(\psi_t^2 + c(\psi)^2\psi_x^2) + (s_t^2 + c(\psi)^2 s_x^2)\right) + W_0(s), \tag{2.4}$$

and the energy density flux

$$\mathcal{F} = s^2\psi_t\psi_x + s_t s_x. \tag{2.5}$$

For classical solutions of (2.3) the energy density and energy density flux satisfy the equations

$$\mathcal{E}_t - \left(c(\psi)^2\mathcal{F}\right)_x = 0, \tag{2.6a}$$

$$\mathcal{F}_t - \left(\mathcal{E} - 2W_0(s)\right)_x = 0. \tag{2.6b}$$

The energy $E(t) = \int_{\mathbb{R}} \mathcal{E}(t,x)\mathrm{d}x$ is conserved.

We will now derive the two-component Hunter–Saxton system from (2.2). To follow the work of Hunter and Saxton [18] we introduce $\psi(t,x) = \psi_0 + \varepsilon u(\varepsilon t, x - c_0 t)$ and $s = s_0 + \varepsilon r(\varepsilon t, x - c_0 t)$ with $c_0^2 = K_1\sin^2\psi_0 + K_3\cos^2\psi_0$. Then expansion of (2.2) in powers of ε gives

$$\begin{aligned}\mathcal{L}^\varepsilon &= W_0(s_0) + \varepsilon W_0'(s_0)r + \varepsilon^2\frac{1}{2}W_0''(s_0)r^2\\ &\quad + \varepsilon^3\left(c_0 r_t r_x + s_0^2 c_0 u_t u_x + s_0^2\,(cc')_0\, u u_x^2 + (cc')_0\, u r_x^2 + \frac{1}{6}W_0'''(s_0)r^3\right) + O(\varepsilon^4).\end{aligned}$$

If we select s_0 such that $W_0'(s_0) = W_0''(s_0) = W'''(s_0) = 0$, the third order terms in the above Lagrangian gives (possibly by rescaling)

$$\mathcal{L}^{2HS} = u_t u_x + u u_x^2 + r_t r_x + u r_x^2. \tag{2.7}$$

Proposition 2.1. *The two-component Hunter–Saxton system*

$$\begin{aligned}(u_t + uu_x)_x &= \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2,\\ \rho_t + (u\rho)_x &= 0,\end{aligned}$$

is the Euler–Lagrange equations for the Lagrangian density (2.7) *with* $\rho = r_x$.

3. The two-component nonlinear variational wave system

3.1. The semilinear case. We will first consider the case $K_1 = K_3$, that is the wave speed c is independent of ψ. We will assume $s \geq 0$ such that we can introduce the complex variable $\zeta = se^{i\psi}$, and (2.3) reduces to

$$\zeta_{tt} - c^2\zeta_{xx} + \frac{W_0'(|\zeta|)}{|\zeta|}\zeta = 0, \tag{3.1}$$

which is a semilinear wave equation. Recall that the physical interpretation of ψ is an angle, and thus any solutions ψ and $\psi + n \cdot 2\pi$ should be considered equal from an application point of view. We have to assume $s \geq 0$ since $(-s, \psi + \pi)$ and (s, ψ) are distinct. Energy density defined by (2.4) takes the form $\mathcal{E} = \frac{1}{2}|\zeta_t|^2 + \frac{1}{2}c^2|\zeta_x|^2 + W_0(|\zeta|)$ and energy density flux given by (2.5) takes the form $\mathcal{F} = \frac{1}{2}\left(\bar{\zeta}_t\zeta_x + \zeta_t\bar{\zeta}_x\right)$, with bar indicating complex conjugation. The conservation laws (2.6) then reduce to

$$\frac{\partial}{\partial t}\mathcal{E} - c^2\frac{\partial}{\partial x}\mathcal{F} = 0, \tag{3.2a}$$

$$\frac{\partial}{\partial t}\mathcal{F} - \frac{\partial}{\partial x}\left(\mathcal{E} - 2W_0(|\zeta|)\right) = 0. \tag{3.2b}$$

Hence both $\int_{\mathbb{R}} \mathcal{E}\mathrm{d}x$ and $\int_{\mathbb{R}} \mathcal{F}\mathrm{d}x$ are conserved for any classical solution with bounded energy. We define the function spaces for the solutions in the next definition.

Definition 3.1. *Let $\zeta^* \in \mathbb{C}$ with $|\zeta^*| < 1$, and*

$$X_{\zeta^*} = \{(\zeta, \sigma) \mid \zeta - \zeta^* \in H^1(\mathbb{R}), \sigma \in L^2(\mathbb{R})\},$$

and define

$$\begin{aligned}
\mathcal{E}(\zeta, \sigma) &= \frac{1}{2}|\sigma|^2 + \frac{1}{2}|\zeta_x|^2 + W_0(|\zeta|),\\
\mathcal{F}(\zeta, \sigma) &= \frac{1}{2}\sigma\bar{\zeta}_x + \frac{1}{2}\bar{\sigma}\zeta_x,
\end{aligned}$$

and note that both $\mathcal{E}$ and $\mathcal{F}$ are real valued whenever defined. Define the metric space $\left(X_{E,\zeta^*}, d_{X_{E,\zeta^*}}\right)$ by

$$\begin{aligned}
X_{E,\zeta^*} &= \{(\zeta, \sigma) \in X_{\zeta^*} \mid \|\mathcal{E}\|_{L^1} \leq E\},\\
d_{X_{E,\zeta^*}}\left((\zeta_1, \sigma_1), (\zeta_2, \sigma_2)\right) &= \|\zeta_1 - \zeta_2\|_{H^1(\mathbb{R})} + \|\sigma_1 - \sigma_2\|_{L^2(\mathbb{R})}\\
&\quad + \|W_0(|\zeta_1|) - W_0(|\zeta_2|)\|_{L^1(\mathbb{R})},
\end{aligned}$$

and denote

$$\begin{aligned}
\mathcal{D}_{T,E} = \Big\{&\zeta : [0,T] \times \mathbb{R} \to \mathbb{C} \mid \zeta - \zeta^* \in C\left([0,T], H^1(\mathbb{R}, \mathbb{C})\right) \cap C^1\left([0,T], L^2(\mathbb{R}, \mathbb{C})\right)\\
&\text{and } (\zeta(t), \zeta_t(t)) \in X_{E,\zeta^*} \text{ for all } t \in [0,T]\Big\},\\
\mathcal{D}_{\infty,E} = \{&\zeta : [0,\infty) \times \mathbb{R} \to \mathbb{C} \mid \zeta|_{[0,T]\times\mathbb{R}} \in \mathcal{D}_{T,E} \text{ for all } T \geq 0\}.
\end{aligned}$$

One should keep in mind that even though ζ^* is omitted from the notation in $\mathcal{D}_{T,E}$ it is still a part of the definition. In the case $T < \infty$ we equip $\mathcal{D}_{T,E}$ with the metric $d_{\mathcal{D}_{T,E}}$ induced from

$$\|\zeta\| = \sup_{t\in[0,T]}\left(\|\zeta(t) - \zeta^*\|_{H^1(\mathbb{R})} + \|\zeta_t(t)\|_{L^2(\mathbb{R})} + \|W_0(|\zeta(t)|)\|_{L^1(\mathbb{R})}\right).$$

In the case of $\mathcal{D}_{\infty,E}$ *we use the topology generated by the open sets* $B_{T,\tilde{\zeta},\delta} = \left\{\zeta \in \mathcal{D}_{\infty,E} \mid d_{\mathcal{D}_{T,E}}\left(\zeta,\tilde{\zeta}\right) < \delta\right\}$.

Remark 3.2. *Both metric spaces* $\left(X_{E,\zeta^*}, d_{X_{E,\zeta^*}}\right)$ *and* $\left(\mathcal{D}_{T,E}, d_{\mathcal{D}_{T,E}}\right)$ *are complete metric spaces. The metric space* $\left(X_{E,\zeta^*}, d_{X_{E,\zeta^*}}\right)$ *is not a vector space since we will require* $W_0 : [0,1) \to [0,\infty)$, *and finite energy thus implicitly impose* $\|\zeta\|_\infty \leq 1$. *Furthermore,* ζ^* *must satisfy* $W_0(|\zeta^*|) = 0$ *for the space* X_{E,ζ^*} *to be nonempty.*

To analyze solutions we will need conditions on the function W_0. From [13] and [31] we get that W_0 tends to infinity as s tends to 1 or $-\frac{1}{2}$, and that $W_0'(0) = 0$. We will assume that $W_0(s) \to \infty$ rapidly enough as $s \to 1$ to be able to bound $\|W_0(|\zeta|)\|_\infty$ in terms of the total energy E. In addition we will assume that W_0 is non-negative and that W_0 is well behaved close to $s = 0$, and also close to any zeros of W_0. We will require $s = 0$ to be a stationary point.

Definition 3.3. *We define a non-negative function* $W_0 \in C^4([0,1))$ *to be admissible if the following holds.*

- $\mathbf{A}_1$ *There exists a nonzero finite number of* s^* *such that* $W_0(s^*) = 0$. *Furthermore, for all zeros* s^* *of* W_0,
$$\lim_{s\to s^*} \frac{W_0(s)}{(s-s^*)^2} = \frac{1}{2}W_0''(s^*) \in [0,\infty).$$
- $\mathbf{A}_2$ *The function is second order close to zero in the sense that*
$$\lim_{s\to 0} \frac{W_0(s) - W_0(0)}{s^2} = \frac{1}{2}W_0''(0) \in \mathbb{R}.$$
- $\mathbf{A}_3$ *The function tends to infinity quickly enough as* s *tends to* 1,
$$\lim_{s\uparrow 1} W_0(s) = \infty,$$
$$\int_s^1 W_0(u)(1-u)\,du = \infty.$$
- $\mathbf{A}_4$ *The function tends monotonically to infinity sufficiently close to* 1. *Particularly, there exists* $\tilde{s} \in [0,1)$ *such that* $W_0(s) > 0$ *and* $W_0'(s) > 0$ *for all* $s \in (\tilde{s}, 1)$.

Proposition 3.4. *Let* W_0 *be admissible in the sense of Definition 3.3. Then there exist positive constants* c_E, C_E, *depending on* W_0 *and* E *only, such that for any* $(\zeta,\sigma) \in X_{E,\zeta^*}$ *we have*
$$\|W_0(|\zeta|)\|_{L^\infty(\mathbb{R})} \leq C_E,$$
$$\|\zeta\|_{L^\infty(\mathbb{R})} \leq c_E < 1.$$
Moreover for $s \in [0, c_E]$ *there is a positive constant* k_E *such that*
$$W_0'(s)^2 \leq k_E W_0(s). \tag{3.4}$$
Furthermore, we can define
$$L_E = \sup_{s\in[0,c_E]} |W_0'(s)|, \tag{3.5a}$$
$$L_E' = \sup_{s\in[0,c_E]} \left|\frac{W_0'(s)}{s}\right|, \tag{3.5b}$$

$$L''_E = \sup_{s\in[0,c_E]} |W''_0(s)|. \tag{3.5c}$$

Finally, $\zeta \mapsto \frac{W'_0(|\zeta|)}{|\zeta|}\zeta$ *is* C^1 *in the open unit disc in two real dimensions.*

Proof. Since $\frac{1}{2}\|\sigma\|_2^2 + \frac{1}{2}c^2\|\zeta_x\|_2^2 + \|W_0(|\zeta|)\|_1 \leq E$ we have that $|\zeta(x_1) - \zeta(x_2)| \leq \frac{\sqrt{2E_1}}{c}\sqrt{|x_1 - x_2|}$ where $E_1 = E - \frac{1}{2}\|\sigma\|_2^2 - \|W_0(|\zeta|)\|_1$. Since ζ is continuous we have that there exists $\hat{x}$ such that $|\zeta(\hat{x})| = \|\zeta\|_{L^\infty(\mathbb{R})} \leq 1$. Then we have that

$$|\zeta(x)| \geq \begin{cases} \|\zeta\|_\infty - \frac{\sqrt{2E_1}}{c}\sqrt{|x-\hat{x}|}, & \hat{x} - \frac{\|\zeta\|_\infty^2 c^2}{2E_1} < x < \hat{x} + \frac{\|\zeta\|_\infty^2 c^2}{2E_1}, \\ 0, & \text{else.} \end{cases}$$

In particular, by $\mathbf{A}_4$, we have for $\tilde{s}$ such that $W'_0(s) > 0$ for all $s > \tilde{s}$ that

$$\begin{aligned} \int_{\mathbb{R}} W_0(|\zeta(x)|)\,\mathrm{d}x &\geq \int_{\hat{x}-\frac{c^2}{2E_1}(\|\zeta\|_\infty - \tilde{s})^2}^{\hat{x}+\frac{c^2}{2E_1}(\|\zeta\|_\infty - \tilde{s})^2} W_0(|\zeta(x)|)\,\mathrm{d}x \\ &\geq \int_{\hat{x}-\frac{c^2}{2E_1}(\|\zeta\|_\infty - \tilde{s})^2}^{\hat{x}+\frac{c^2}{2E_1}(\|\zeta\|_\infty - \tilde{s})^2} W_0\left(\|\zeta\|_\infty - \frac{\sqrt{2E_1}}{c}\sqrt{|x-\hat{x}|}\right)\mathrm{d}x \\ &= \int_{\tilde{s}}^{\|\zeta\|_\infty} W_0(u)\left(\|\zeta\|_\infty - u\right)\,\mathrm{d}u. \end{aligned}$$

By the monotone convergence theorem and $\mathbf{A}_3$ we have that

$$\lim_{S\to 1^-}\int_s^S W_0(u)\,(S-u)\,\mathrm{d}u = \int_s^1 W_0(u)\,(1-u)\,\mathrm{d}u = \infty.$$

Hence we have the inequality

$$\int_{\tilde{s}}^{\|\zeta\|_\infty} W_0(u)\left(\|\zeta\|_\infty - u\right)\,\mathrm{d}u \leq E - E_1,$$

which proves the existence of c_E, C_E such that $\|\zeta\|_\infty = c_E < 1$ and $\|W_0(|\zeta|)\|_\infty = \sup_{0\leq s\leq c_E} W_0(s) = C_E < \infty$.

To prove that $W_0'^2 \leq k_E W_0$, note that by $\mathbf{A}_1$ whenever W_0 tends to zero that

$$\lim_{s\to s^*}\frac{W'_0(s)^2}{W_0(s)} = 2W''_0(s^*) \in [0,\infty).$$

In between zeros of W_0 the fraction $\frac{W_0'^2}{W_0}$ is continuous. Since there is only a finite number of points where W_0 is zero and $[0, c_E]$ is bounded, $\frac{W_0'^2}{W_0}$ has to be bounded as well by a constant dependent on c_E. Since W_0 is non-negative we get the desired inequality (3.4).

We show (3.5b). Note that by $\mathbf{A}_2$ we have

$$W_0(s) = W_0(0) + \frac{1}{2}W_0^{(2)}(0)s^2 + \frac{1}{6}W_0^{(3)}(0)s^3 + \int_0^s \frac{1}{6}W_0^{(4)}(u)(s-u)^3\,\mathrm{d}u,$$

and thus

$$\left|\frac{W'_0(s)}{s}\right| \leq |W_0^{(2)}(0)| + \|W_0^{(3)}\|_{L^\infty([0,c_E])}c_E.$$

To prove that $\zeta \mapsto \frac{W'_0(|\zeta|)}{|\zeta|}\zeta$ is continuously differentiable observe that

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{W'_0(s)}{s} = \frac{1}{2}W_0^{(3)}(0) - \frac{1}{s^2}\int_0^s \frac{1}{2}W_0^{(4)}(u)(s-u)^2\,\mathrm{d}u + \frac{1}{s}\int_0^s W_0^{(4)}(u)(s-u)\,\mathrm{d}u.$$

Then we can show that the derivative of $\frac{W_0'(|\zeta|)}{|\zeta|}\zeta$ is bounded and continuous

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\frac{W_0'(s)}{s}(s)\right| \leq \frac{1}{2}\left|W_0^{(3)}(0)\right| + \frac{3}{2}\left\|W_0^{(4)}\right\|_{L^\infty([0,s])} s,$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=s_1}\frac{W_0'(s)}{s} - \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=s_2}\frac{W_0'(s)}{s}\right| \leq \frac{5}{2}\left\|W_0^{(4)}\right\|_{L^\infty([0,\max\{s_1,s_2\}])}|s_1-s_2|\,.$$

Thus $\zeta \mapsto \frac{W_0'(|\zeta|)}{|\zeta|}\zeta$ is C^1 in the open unit disc in two dimensions. □

Remark 3.5. *Note that for ζ_1 and ζ_2 in X_{E,ζ^*} we have that*

$$\begin{aligned}\int_{\mathbb{R}} W_0(|\zeta_1(x)|)\,\mathrm{d}x &\leq \int_{\mathbb{R}}\frac{1}{2}L_E''|\zeta_1-\zeta^*|^2\,\mathrm{d}x\\ &\leq \frac{1}{2}L_E''\|\zeta_1-\zeta^*\|_2^2,\end{aligned}$$

and similarly, since $W_0(|\zeta|) = \int_{|\zeta^*|}^{|\zeta|} W_0''(u)(|\zeta|-u)\,\mathrm{d}u$,

$$\begin{aligned}\int_{\mathbb{R}}|W_0(|\zeta_1|)-W_0(|\zeta_2|)|\,\mathrm{d}x &= \int_{\mathbb{R}}\left|\int_{|\zeta^*|}^{|\zeta_1|}W_0''(u)\left(|\zeta_1|-|\zeta_2|\right)\,\mathrm{d}u\right.\\ &\qquad\left. + \int_{|\zeta_2|}^{|\zeta_1|}W_0''(u)\left(|\zeta_2|-u\right)\,\mathrm{d}u\right|\,\mathrm{d}x\\ &\leq L_E''\|\zeta_1-\zeta_2\|_2^2 + L_E''\|\zeta_1-\zeta^*\|_2\|\zeta_1-\zeta_2\|_2.\end{aligned}$$

We can now prove local well posedness of local strong solutions. First we define what strong solutions are.

Definition 3.6. *We define a local strong solution with initial data $(\zeta_0,\zeta_{t0}) \in X_{E,\zeta^*}$ to be $\zeta \in \mathcal{D}_{T,E}$ such that*

$$\begin{aligned}\zeta(t,x) &= \frac{1}{2}\left(\zeta_0(x+ct)+\zeta_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\zeta_{t0}(y)\,\mathrm{d}y\\ &\quad - \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}\frac{W_0'(|\zeta(s,y)|)}{|\zeta(s,y)|}\zeta(s,y)\,\mathrm{d}y\mathrm{d}s,\\ \zeta_t(t,x) &= \frac{c}{2}\left(\zeta_{x0}(x+ct)-\zeta_{x0}(x-ct)\right) + \frac{1}{2}\left(\zeta_{t0}(x+ct)+\zeta_{t0}(x-ct)\right)\\ &\quad - \frac{1}{2}\int_0^t\left[\frac{W_0'(|\zeta(s,x-c(t-s))|)}{|\zeta(s,x-c(t-s))|}\zeta(s,x-c(t-s))\right.\\ &\qquad\left. + \frac{W_0'(|\zeta(s,x+c(t-s))|)}{|\zeta(s,x+c(t-s))|}\zeta(s,x+c(t-s))\right]\mathrm{d}s,\end{aligned} \tag{3.6}$$

for all $t \leq T$. If ζ is a strong solution for any $T \geq 0$ we define ζ to be a global solution.

From the definition of strong solutions we immediately get continuous dependence on initial data, uniqueness, and a semigroup property.

Lemma 3.7. *For any strong solutions $\zeta_1,\zeta_2 \in \mathcal{D}_{T,E'}$ in the sense of Definition 3.6 with initial data $(\zeta_{10},\zeta_{t10}),(\zeta_{20},\zeta_{t20}) \in X_E$ we have that there for any time $t \leq T$ exists constants $C_{E'}(t)$ such that*

$$d_{X_{E',\zeta^*}}(\zeta_1(t),\zeta_2(t)) \leq C_{E'}(t)d_{X_{E,\zeta^*}}\left((\zeta_{10},\zeta_{t10}),(\zeta_{20},\zeta_{t20})\right),$$

and hence

$$d_{\mathcal{D}_{T,E'}}(\zeta_1,\zeta_2) \le C_{E'}(T) d_{X_{E,\zeta^*}}\left((\zeta_{10},\zeta_{t10}),(\zeta_{20},\zeta_{t20})\right),$$

which implies uniqueness of solution and continuous dependence on initial data. Moreover, if $\zeta \in \mathcal{D}_{T,E'}$ is a solution with initial data $(\zeta_0,\zeta_{t0}) \in X_{E,\zeta^*}$, then $\bar\zeta \in \mathcal{D}_{T-s,E'}$ given by $\bar\zeta(t) = \zeta(t+s)$ is a strong solution with initial data $(\zeta(s),\zeta_t(s)) \in X_{E',\zeta^*}$.

Proof. We need to estimate the difference of the integral in (3.6) with two $\zeta_1,\zeta_2 \in \mathcal{D}_{T,E'}$. First for $z_1,z_2 \in \mathbb{C}$ with $0<|z_1|<|z_2|<1$ we have

$$\begin{aligned}\left|\frac{W'(|z_1|)}{|z_1|}z_1 - \frac{W'(|z_2|)}{|z_2|}z_2\right| &\le \left|\frac{W'(|z_1|)}{|z_1|}\left(z_1 - \frac{|z_1|}{|z_2|}z_2\right) + \frac{z_2}{|z_2|}\left(W'(|z_1|)-W'(|z_2|)\right)\right| \\ &\le \left|\frac{W'(|z_1|)}{|z_1|}\right| |z_1-z_2| + \sup_{|z_1|\le s\le|z_2|}|W''(s)|\,|z_1-z_2|\end{aligned}$$

Thus, for $\xi_1,\xi_2 \in \mathcal{D}_{T,Q}$, we get

$$\left|\frac{W'(|\xi_1|)}{|\xi_1|}\xi_1 - \frac{W'(|\xi_2|)}{|\xi_2|}\xi_2\right| \le (L'_Q + L''_Q)|\xi_1-\xi_2|. \tag{3.7}$$

Thus

$$\left|\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} \frac{W'_0(|\xi_1(s,y)|)}{|\xi_1(s,y)|}\xi_1(s,y) - \frac{W'_0(|\xi_2(s,y)|)}{|\xi_2(s,y)|}\xi_2(s,y)\,\mathrm{d}y\right|$$

$$\le (t-s)(L'_Q+L''_Q)\|\xi_1(s)-\xi_2(s)\|_{L^\infty(\mathbb{R})}, \tag{3.8}$$

$$\left|\frac{\partial}{\partial x}\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} \frac{W'(|\xi_1(s,y)|)}{|\xi_1(s,y)|}\xi_1(s,y) - \frac{W'(|\xi_2(s,y)|)}{|\xi_2(s,y)|}\xi_2(s,y)\,\mathrm{d}y\right|$$

$$\le 2(L'_Q+L''_Q)\|\xi_1(s)-\xi_2(s)\|_{L^\infty(\mathbb{R})}. \tag{3.9}$$

The integrals in (3.6) can be interpreted as convolutions

$$\int_{x-ct}^{x+ct} f(y)\,\mathrm{d}y = \left(\mathbf{1}_{[-ct,ct]} * f\right)(x),$$

and then Young's inequality implies that for any $p \in [1,\infty]$,

$$\left\|\int_{x-ct}^{x+ct} f(y)\,\mathrm{d}y\right\|_p \le 2ct\|f\|_p. \tag{3.10}$$

Application of (3.7) and (3.10) to the expression on the left hand side of (3.8) gives

$$\left\|\int_{x-c(t-s)}^{x+c(t-s)} \frac{W'_0(|\xi_1(s,y)|)}{|\xi_1(s,y)|}\xi_1(s,y) - \frac{W'_0(|\xi_2(s,y)|)}{|\xi_2(s,y)|}\xi_2(s,y)\,\mathrm{d}y\right\|_2$$

$$\le 2c(t-s)(L'_Q+L''_Q)\|\xi_1(s)-\xi_2(s)\|_2. \tag{3.11}$$

From the strong formulation of the equation (3.6) and estimates (3.8), (3.9), (3.11) we get

$$\begin{aligned}\|\zeta_1(t)-\zeta_2(t)\|_\infty &\le \|\zeta_{10}-\zeta_{20}\|_\infty + \sqrt{\frac{t}{2c}}\|\zeta_{1t0} - +\zeta_{2t0}\|_2 \\ &\quad + \int_0^t (t-s)(L'_{E'}+L''_{E'})\|\zeta_1(s)-\zeta_2(s)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s, \\ \|\zeta_1(t)-\zeta_2(t)\|_2 &\le \|\zeta_{10}-\zeta_{20}\|_2 + t\|\zeta_{1t0} - +\zeta_{2t0}\|_2\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^t (t-s)(L'_{E'} + L''_{E'})\|\zeta_1(s) - \zeta_2(s)\|_2 \,\mathrm{d}s, \\
\|\zeta_{1x}(t) - \zeta_{2x}(t)\|_2 &\le \|\zeta_{10x} - \zeta_{20x}\|_2 + \frac{1}{c}\|\zeta_{1t0} - \zeta_{2t0}\|_2 \\
&\quad + \frac{1}{c}\int_0^t (L'_{E'} + L''_{E'})\|\zeta_1(s) - \zeta_2(s)\|_2 \,\mathrm{d}s, \\
\|\zeta_{1t}(t) - \zeta_{2t}(t)\|_2 &\le c\|\zeta_{10x} - \zeta_{20x}\|_2 + \|\zeta_{1t0} - \zeta_{2t0}\|_2 \\
&\quad + \int_0^t (L'_{E'} + L''_{E'})\|\zeta_1(s) - \zeta_2(s)\|_2 \,\mathrm{d}s.
\end{aligned}
$$

Grönwall's inequality then gives

$$
\|\zeta_1(t) - \zeta_2(t)\|_\infty \le \left(\|\zeta_{10} - \zeta_{20}\|_\infty + \sqrt{\frac{t}{2c}}\|\zeta_{1t0} - \zeta_{2t0}\|_2\right) e^{\frac{1}{2}M_{E'}t^2}, \tag{3.13a}
$$

$$
\|\zeta_1(t) - \zeta_2(t)\|_2 \le \left(\|\zeta_{10} - \zeta_{20}\|_2 + t\|\zeta_{1t0} - \zeta_{2t0}\|_2\right) e^{\frac{1}{2}M_{E'}t^2}, \tag{3.13b}
$$

$$
\begin{aligned}
\|\zeta_{1t}(t) - \zeta_{2t}(t)\|_2 &\le \|\zeta_{1t0} - \zeta_{2t0}\|_2 + c\|\zeta_{10x} - \zeta_{20x}\|_2 \\
&\quad + M_{E'}t\left(\|\zeta_{10} - \zeta_{20}\|_2 + t\|\zeta_{1t0} - \zeta_{2t0}\|_2\right) e^{\frac{1}{2}M_{E'}t^2},
\end{aligned} \tag{3.13c}
$$

$$
\begin{aligned}
\|\zeta_{1x}(t) - \zeta_{2x}(t)\|_2 &\le \frac{1}{c}\|\zeta_{1t0} - \zeta_{2t0}\|_2 + \|\zeta_{10x} - \zeta_{20x}\|_2 \\
&\quad + \frac{M_{E'}}{c}t\left(\|\zeta_{10} - \zeta_{20}\|_2 + t\|\zeta_{1t0} - \zeta_{2t0}\|_2\right) e^{\frac{1}{2}(M_{E'})t^2},
\end{aligned} \tag{3.13d}
$$

with $M_{E'} = L'_{E'} + L''_{E'}$. It remains to estimate $\int_{\mathbb{R}} |W_0(|\zeta_1(t)|) - W_0(|\zeta_2(t)|)|\mathrm{d}x$. We have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(W_0(|\zeta_1|) - W_0(|\zeta_2|)\right) &= W_0'(|\zeta_1|)\frac{\bar{\zeta}_1\zeta_{1t} + \zeta_1\bar{\zeta}_{1t}}{2|\zeta_1|} \\
&\quad - W_0'(|\zeta_2|)\frac{\bar{\zeta}_2\zeta_{2t} + \zeta_2\bar{\zeta}_{2t}}{2|\zeta_2|} \\
&= \left(W_0'(|\zeta_1|) - W_0'(|\zeta_2|)\right)\frac{\bar{\zeta}_1\zeta_{1t} + \zeta_1\bar{\zeta}_{1t}}{2|\zeta_1|} \\
&\quad + W_0'(|\zeta_2|)\left(\frac{\bar{\zeta}_1\zeta_{1t} + \zeta_1\bar{\zeta}_{1t}}{2|\zeta_1|} - \frac{\bar{\zeta}_2\zeta_{2t} + \zeta_2\bar{\zeta}_{2t}}{2|\zeta_2|}\right).
\end{aligned}
$$

Further note that

$$
\begin{aligned}
\left(\frac{\bar{\zeta}_1\zeta_{1t} + \zeta_1\bar{\zeta}_{1t}}{2|\zeta_1|} - \frac{\bar{\zeta}_2\zeta_{2t} + \zeta_2\bar{\zeta}_{2t}}{2|\zeta_2|}\right) &= \frac{1}{|\zeta_2|}\left[\frac{1}{2}\left(|\zeta_2| - |\zeta_1|\right)\left(\frac{\bar{\zeta}_1}{|\zeta_1|}\zeta_{1t} + \frac{\bar{\zeta}_1}{|\zeta_1|}\bar{\zeta}_{1t}\right)\right. \\
&\quad + \frac{1}{2}\Big((\bar{\zeta}_1 - \bar{\zeta}_2)\zeta_{1t} + \bar{\zeta}_2(\zeta_{1t} - \zeta_{2t}) \\
&\quad \left.+ (\zeta_1 - \zeta_2)\bar{\zeta}_{1t} + \zeta_2(\bar{\zeta}_{1t} - \bar{\zeta}_{2t})\Big)\right],
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|W_0(|\zeta_1|) - W_0(|\zeta_2|)\|_1 &\le \int_{\mathbb{R}} |W_0'(|\zeta_1|) - W_0'(|\zeta_2|)|\left|\frac{\bar{\zeta}_1\zeta_{1t} + \zeta_1\bar{\zeta}_{1t}}{2|\zeta_1|}\right| \,\mathrm{d}x \\
&\quad + 2\int_{\mathbb{R}}\left|\frac{W_0'(|\zeta_2|)}{|\zeta_2|}\right| |\zeta_1 - \zeta_2|\,|\zeta_{1t}| \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}} \left| \frac{W_0'(|\zeta_2|)}{|\zeta_2|} \right| |\zeta_{1t} - \zeta_{2t}| \, |\zeta_2| \, \mathrm{d}x \\
&\leq (L'_{E'} + 2L''_{E'}) \sqrt{E'} \|\zeta_1 - \zeta_2\|_2 \\
&+ \sqrt{k_{E'}E'} \|\zeta_{1t} - \zeta_{2t}\|_2,
\end{aligned} \tag{3.14}
$$

where (t) has been omitted to enhance readability. Now, (3.14) together with (3.13) implies that there are constants $C_{E'}(t)$, increasing in both E' and t, such that

$$
d_{\mathcal{D}_{T,E}} \left(\zeta_1(t), \zeta_2(t)\right) \leq C_{E'}(t) d_{X_{E,\zeta^*}} \left((\zeta_{10}, \zeta_{1t0}), (\zeta_{20}, \zeta_{2t0})\right).
$$

The semigroup property, if $\zeta \in \mathcal{D}_{T,E'}$ is a solution with initial data $(\zeta_0, \zeta_{t0}) \in X_{E,\zeta^*}$, then $\bar{\zeta} \in \mathcal{D}_{T-s,E'}$ given by $\bar{\zeta}(t) = \zeta(t+s)$ is the strong solution with initial data $(\zeta(s), \zeta_t(s)) \in X_{E',\zeta^*}$, can be verified by direct computation from the strong form (3.6) of the equation. □

We are now ready to prove that the Cauchy problem of (3.1) is locally well posed for strong solutions as defined in Definition 3.6.

Proposition 3.8. *Assume that W_0 satisfies the conditions in Definition 3.3. Then given initial data $(\zeta_0, \zeta_{t0}) \in X_{E,\zeta^*}$ and $E' > E$ there exists a unique strong solution in the sense of Definition 3.6 in $\mathcal{D}_{T,E'}$ depending Lipschitz continuously on the initial data. The existence time T can be made to depend on E and E' only.*

Proof. Uniqueness and continuity with respect to initial data is proven in Lemma 3.7. We will establish local existence of a strong solution by a fixed point argument on $\mathcal{D}_{T,E'}$ for a to be specified T. We will now assume that $0 < E < E'$ and that $(\zeta_0, \zeta_{t0}) \in X_E$, and let $\hat{\zeta} \in \mathcal{D}_{T,E'}$ with $(\hat{\zeta}(0), \hat{\zeta}_t(0)) = (\zeta_0, \zeta_{t0})$. Then let (ζ, ζ_t), be given by Duhamel's principle

$$
\begin{aligned}
\zeta(t,x) = {}& \frac{1}{2}\left(\zeta_0(x-ct) + \zeta_0(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \zeta_{t0}(y) \, \mathrm{d}y \\
&- \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{W_0'(|\hat{\zeta}(s,y)|)}{|\hat{\zeta}(s,y)|} \hat{\zeta}(s,y) \, \mathrm{d}y \, \mathrm{d}s,
\end{aligned} \tag{3.15a}
$$

$$
\begin{aligned}
\zeta_t(t,x) = {}& \frac{c}{2}\left(\zeta_{x0}(x+ct) - \zeta_{x0}(x-ct)\right) + \frac{1}{2}\left(\zeta_{t0}(x+ct) + \zeta_{t0}(x-ct)\right) \\
&- \frac{1}{2}\int_0^t \left[\frac{W_0'(|\hat{\zeta}(s,x-c(t-s))|)}{|\hat{\zeta}(s,x-c(t-s))|} \hat{\zeta}(s,x-c(t-s)) \right. \\
&\left. + \frac{W_0'(|\hat{\zeta}(s,x+c(t-s))|)}{|\hat{\zeta}(s,x+c(t-s))|} \hat{\zeta}(s,x+c(t-s)) \right] \mathrm{d}s.
\end{aligned} \tag{3.15b}
$$

To prove that the strong solution (3.15) is a member of $\mathcal{D}_{T,E'}$ we will need norm estimates on the solutions.

Let $\zeta \in \mathcal{D}_{T,Q}$ be given. Then we have directly from Proposition 3.4 that

$$
\left| \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} \frac{W_0'(|\zeta(s,y)|)}{|\zeta(s,y)|} \zeta(s,y) \, \mathrm{d}y \right| \leq L_Q(t-s),
$$

$$
\left| \frac{\partial}{\partial x} \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} \frac{W_0'(|\zeta(s,y)|)}{|\zeta(s,y)|} \zeta(s,y) \, \mathrm{d}y \right| \leq \frac{1}{c} L_Q,
$$

Thus, by the estimates on W_0 from Proposition 3.4 and (3.10), the solution satisfies the estimates

$$\begin{aligned}\|\zeta(t)-\zeta^*\|_\infty &\le \|\zeta_0-\zeta^*\|_\infty + \sqrt{\frac{t}{2c}}\|\zeta_{t0}\|_2 + \frac{1}{2}t^2 L_{E'},\\ \|\zeta(t)-\zeta^*\|_2 &\le \|\zeta_0-\zeta^*\|_2 + t\|\zeta_{t0}\|_2 + \frac{1}{2}t^2\sqrt{k_{E'}E'},\\ \|\zeta_x(t)\|_2 &\le \|\zeta_{x0}\|_2 + \frac{1}{c}\|\zeta_{t0}\|_2 + \frac{2}{c}t\sqrt{k_{E'}E'},\\ \|\zeta_t(t)\|_2 &\le c\|\zeta_{x0}\|_2 + \|\zeta_{t0}\|_2 + t\sqrt{k_{E'}E'}.\end{aligned}$$

We need to show that the energy is bounded. Note that (3.15) and Proposition 3.4 implies that

$$\|\zeta(t)-\zeta(s)\|_\infty \le \sqrt{|t-s|}\left(\|\zeta_{t0}\|_2 + c\|\zeta_{0x}\|_2\right) + |t-s|L'_{E'} + s|t-s|L'_{E'},$$

and since $(\|\zeta_{t0}\|_2 + c\|\zeta_{0x}\|_2) \le \sqrt{2E}$, we have that for $t \le \frac{\sqrt{2E+4L'_{E'}(c_{E'}-c_E)}-\sqrt{2E}}{2L'_{E'}}$ we can bound $\|\zeta(t)\|_\infty \le c_{E'}$. Similarly by Young's inequality we get

$$\begin{aligned}\|\zeta(t)-\zeta_0\|_2 &\le t\left(\|\zeta_{t0}\|_2 + c\|\zeta_{0x}\|_2\right) + \frac{1}{2}t^2 L_{E'}\\ &\le t\sqrt{2E} + \frac{1}{2}t^2 L_{E'}.\end{aligned}$$

Note that since $\zeta_{\rm lin}(t,x) = \frac{1}{2}\left(\zeta_0(x+ct) - \zeta_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\zeta_{t0}(y)\mathrm{d}y$ is a solution to the linear wave equation we have that

$$\|\zeta_{{\rm lin},t}(t)\|_2^2 + c^2\|\zeta_{{\rm lin},x}(t)\|_2^2 = \|\zeta_{t0}\|_2^2 + c^2\|\zeta_{0x}\|_2^2.$$

Then by we get

$$\begin{aligned}E(t) &= \int_{\mathbb{R}} \mathcal{E}(t,x)\,\mathrm{d}x\\ &\le \left\|\frac{1}{2}|\zeta_t(t)|^2 + \frac{1}{2}c^2|\zeta_x(t)|^2\right\|_1 + \|W_0(|\zeta_0|)\|_1 + \|W_0(|\zeta(t)|) - W_0(|\zeta_0|)\|_1\\ &\le \frac{1}{2}\|\zeta_{t0}\|_2^2 + \frac{1}{2}c^2\|\zeta_{0x}\|_2^2 + \|W_0(|\zeta_0|)\|_1\\ &\quad + 2\sqrt{2Ek_{E'}E'}t + 2k_{E'}E't^2 + \frac{1}{2}L_{E'}\left(\sqrt{2E}t + \frac{1}{2}L'_{E'}t^2\right)^2\\ &\le E + 2\sqrt{2Ek_{E'}E'}t + (2k_{E'}E' + L_{E'}E)t^2 + \frac{1}{2}\sqrt{2E}L_{E'}L'_{E'}t^3 + \frac{1}{8}L_{E'}{L'_{E'}}^2t^4.\end{aligned}$$

Thus for T small enough we have that the solution map $\Phi : \hat{\zeta} \mapsto \zeta$ defined by (3.15) maps elements in $\mathcal{D}_{T,E'}$ to elements in $\mathcal{D}_{T,E'}$. The choice of T can be made to depend on E and E' only.

To prove that Φ is a contraction let $\hat{\zeta}_1, \hat{\zeta}_2 \in \mathcal{D}_{T,E'}$ with coinciding initial data (ζ_0, ζ_{t0}), and denote $\zeta_1 = \Phi(\hat{\zeta}_1), \zeta_2 = \Phi(\hat{\zeta}_2)$. Then by (3.8), (3.9), (3.11), and (3.15), we have for $p = 2, \infty$,

$$\begin{aligned}\sup_{t\in[0,T]}\|\zeta_1(t)-\zeta_2(t)\|_p &\le (L'_{E'} + L''_{E'})\int_0^t (t-s)\|\hat{\zeta}_1(s)-\hat{\zeta}_2(s)\|_p\,\mathrm{d}s,\\ \sup_{t\in[0,T]}\|\zeta_{1,t}(t)-\zeta_{2,t}(t)\|_2 &\le (L'_{E'} + L''_{E'})\int_0^t \|\hat{\zeta}_1(s)-\hat{\zeta}_2(s)\|_2\,\mathrm{d}s,\end{aligned}$$

$$\sup_{t\in[0,T]} \|\zeta_{1,x}(t)-\zeta_{2,x}(t)\|_2 \le \frac{1}{c}\left(L'_{E'}+L''_{E'}\right)\int_0^t \|\hat{\zeta}_1(s)-\hat{\zeta}_2(s)\|_2\,\mathrm{d}s.$$

Hence for T small enough Φ is a contraction on $\mathcal{D}_{T,E'}$. Thus there exists a unique fixed point in $\mathcal{D}_{T,E'}$ which is a strong solution in the sense of (3.6). □

If the initial data is smooth, the local strong solution is in fact a classical solution. The energy is then a conserved quantity.

Theorem 3.9. *Assume that W_0 satisfies the conditions in Definition 3.3. Then given smooth initial data $(\zeta_0,\zeta_{t0}) \in X_{E,\zeta^*}\cap(C^2(\mathbb{R})\times C^1(\mathbb{R}))$ there is a unique global classical smooth solution in $\mathcal{D}_{\infty,E}$. Moreover, the energy is conserved $E(t)=\int_R \mathcal{E}(t,x)\,\mathrm{d}x = E(0)$, and the classical solution can be extended to a global classical solution in $\mathcal{D}_{\infty,E}$.*

Proof. Given the initial data $(\zeta_0,\zeta_{t0}) \in X_{E,\zeta^*}\cap(C^2(\mathbb{R})\times C^1(\mathbb{R}))$ choose any $E'>E$ and let $\zeta\in\mathcal{D}_{T,E'}$ be the unique strong solution given implicitly by (3.6). To upgrade the regularity of the solution we employ a bootstrapping argument. To that end define

$$I(t,x)=\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} \frac{W_0'(|\zeta(\tau,y)|)}{|\zeta(\tau,y)|}\zeta(\tau,y)\,\mathrm{d}y\mathrm{d}\tau.$$

The first derivatives of I are

$$\begin{aligned}
\frac{\partial}{\partial t}I(t,x) &= c\int_0^t \frac{W_0'(|\zeta(\tau,x+c(t-\tau))|)}{|\zeta(\tau,x+c(t-\tau))|}\zeta(\tau,x+c(t-\tau)) \\
&\qquad + \frac{W_0'(|\zeta(\tau,x-c(t-\tau))|)}{|\zeta(\tau,x-c(t-\tau))|}\zeta(\tau,x-c(t-\tau))\,\mathrm{d}\tau,\\
\frac{\partial}{\partial x}I(t,x) &= \int_0^t \frac{W_0'(|\zeta(\tau,x+c(t-\tau))|)}{|\zeta(\tau,x+c(t-\tau))|}\zeta(\tau,x+c(t-\tau)) \\
&\qquad - \frac{W_0'(|\zeta(\tau,x-c(t-\tau))|)}{|\zeta(\tau,x-c(t-\tau))|}\zeta(\tau,x-c(t-\tau))\,\mathrm{d}\tau,
\end{aligned}$$

which are continuous since ζ is continuous by assumption. Hence $\zeta\in C^1([0,\infty)\times\mathbb{R})$. Similarly the second derivatives are

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}I(t,x) &= 2c\frac{W_0'(|\zeta(t,x)|)}{|\zeta(t,x)|}\zeta(t,x) \\
&\quad + c^2\int_0^t \frac{\partial}{\partial x}\frac{W_0'(|\zeta(\tau,x+c(t-\tau))|)}{|\zeta(\tau,x+c(t-\tau))|}\zeta(\tau,x+c(t-\tau)) \\
&\qquad - \frac{\partial}{\partial x}\frac{W_0'(|\zeta(\tau,x-c(t-\tau))|)}{|\zeta(\tau,x-c(t-\tau))|}\zeta(\tau,x-c(t-\tau))\,\mathrm{d}\tau,\\
\frac{\partial^2}{\partial t\partial x}I(t,x) &= c\int_0^t \frac{\partial}{\partial x}\frac{W_0'(|\zeta(\tau,x+c(t-\tau))|)}{|\zeta(\tau,x+c(t-\tau))|}\zeta(\tau,x+c(t-\tau)) \\
&\qquad + \frac{\partial}{\partial x}\frac{W_0'(|\zeta(\tau,x-c(t-\tau))|)}{|\zeta(\tau,x-c(t-\tau))|}\zeta(\tau,x-c(t-\tau))\,\mathrm{d}\tau,\\
\frac{\partial^2}{\partial x^2}I(t,x) &= \int_0^t \frac{\partial}{\partial x}\frac{W_0'(|\zeta(\tau,x+c(t-\tau))|)}{|\zeta(\tau,x+c(t-\tau))|}\zeta(\tau,x+c(t-\tau)) \\
&\qquad - \frac{\partial}{\partial x}\frac{W_0'(|\zeta(\tau,x-c(t-\tau))|)}{|\zeta(\tau,x-c(t-\tau))|}\zeta(\tau,x-c(t-\tau))\,\mathrm{d}\tau,
\end{aligned}$$

which are continuous by Proposition 3.4 since ζ is continuously differentiable. Thus ζ given by

$$\zeta(t,x) = \frac{1}{2}\left(\zeta_0(t,x+ct) + \zeta_0(t,x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \zeta_{t0}(y)\,\mathrm{d}y - \frac{1}{2c}I(t,x),$$

is a sum of $C^2([0,\infty)\times\mathbb{R})$ functions and is thus $C^2([0,\infty)\times\mathbb{R})$. We get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\zeta(t,x) - c^2\frac{\partial^2}{\partial t^2}\zeta(t,x) &= -\frac{\partial^2}{\partial t^2}\frac{1}{2c}I(t,x) + c^2\frac{\partial^2}{\partial x^2}\frac{1}{2c}I(t,x)\\ &= -\frac{W_0'(|\zeta(t,x)|)}{|\zeta(t,x)|}\zeta(t,x)\end{aligned}$$

pointwise.

Since ζ is a classical solution direct computation yields (3.2) pointwise, and hence we get a conservation law for energy. The iteration scheme in Proposition 3.8 can now be repeated with initial data $(\zeta(nT),\zeta_t(nT)) \in X_{E,\zeta^*} \cap (C^2(\mathbb{R})\times C^1(\mathbb{R}))$ to get a unique classical solution for $t\in[0,(n+1)T]$ for any $n\in\mathbb{N}$. Uniqueness and continuity follow from Lemma 3.7. □

Theorem 3.10. *Let $(\zeta_0,\zeta_{t0}) \in X_{E,\zeta^*}$, and assume that W_0 satisfies the conditions in Definition 3.3. Then there exists a unique global strong energy conserving solution $(\zeta,\zeta_t) \subseteq \mathcal{D}_{\infty,E}$ of (3.1) depending continuously on the initial data $(\zeta_0,\zeta_{t0}) \in X_{E,\zeta^*}$.*

Proof. Let $(\zeta_0,\zeta_{t0}) \in X_{E,\zeta^*}$ be given, and construct $(\zeta_0^\varepsilon,\zeta_{t0}^\varepsilon) \in C^2(\mathbb{R})\times C^1(\mathbb{R})$ by convolution with a Friedrich's mollifier. We want to show that for ε small enough there exists $\bar{E} \geq E$ such that $(\zeta_0^\varepsilon,\zeta_{t0}^\varepsilon) \in X_{\bar{E},\zeta^*}$ and $(\zeta_0^\varepsilon,\zeta_{t0}^\varepsilon) \to (\zeta_0,\zeta_{t0})$ in $X_{\bar{E},\zeta^*}$. It remains to show that $\|W_0(|\zeta_0^\varepsilon|)\|_1 < +\infty$ and $\|W_0(|\zeta_0^\varepsilon|) - W_0(|\zeta_0|)\|_1 \to 0$. We have that $\|\zeta_0^\varepsilon - \zeta_0\|_\infty \leq \|\zeta_{0x}\|_2\sqrt{\varepsilon}$, and $\|\zeta_0^\varepsilon\|_\infty \leq \|\zeta_0\|_\infty \leq c_E$. Thus from Remark 3.5 the estimates

$$\|W_0(|\zeta_0^\varepsilon|)\|_1 \leq \frac{1}{2}L_E''\|\zeta_0^\varepsilon - \zeta^*\|_2^2 < +\infty,$$

and

$$\|W_0(|\zeta_0^\varepsilon|) - W_0(|\zeta_0|)\|_1 \leq \frac{1}{2}L_E'\|\zeta_0^\varepsilon - \zeta_0\|_2^2 \to 0,$$

hold. Since $\|\zeta_{0x}^\varepsilon\|_2 \leq \|\zeta_{0x}\|_2$ and $\|\zeta_{t0}^\varepsilon\|_2 \leq \|\zeta_{t0}\|_2$ it follows that the energy

$$E^\varepsilon = \frac{1}{2}\|\zeta_{t0}^\varepsilon\|_2^2 + \frac{1}{2}c^2\|\zeta_{0x}^\varepsilon\|_2^2 + \|W_0(|\zeta_0^\varepsilon|)\|_1 \leq E + \frac{1}{2}L_E'\|\zeta_0^\varepsilon - \zeta_0\|_2^2.$$

Thus given $0<\delta$ we get for all $\varepsilon<\delta$ the energy is bounded by $E + \frac{1}{2}L_E'\|\zeta_0^\delta - \zeta_0\|_2^2$, and hence $(\zeta_0^\varepsilon,\zeta_{t0}^\varepsilon) \in X_{\bar{E},\zeta^*}$ and $(\zeta_0^\varepsilon,\zeta_{t0}^\varepsilon) \to (\zeta_0,\zeta_{t0})$ in $X_{\bar{E},\zeta^*}$ for $\bar{E} = E + \frac{1}{2}L_E'\|\zeta_0^\delta - \zeta_0\|_2^2$. Let $\zeta^\varepsilon \in \mathcal{D}_{\infty,\bar{E}}$ be the classical solution given by Theorem 3.9. Then by Lemma 3.7 ζ^ε converges in $\mathcal{D}_{T,\bar{E}}$ for all $T>0$, and hence also in $\mathcal{D}_{\infty,\bar{E}}$. Let $\zeta \in \mathcal{D}_{\infty,\bar{E}}$ be the limit. Since for each ε the energy is conserved, and the energies converge, the energy in the limit is also conserved.

We need to show that the limit is a strong solution. Note that for any T the smooth solutions ζ^ε converges to ζ uniformly in $[0,T]\times\mathbb{R}$, which implies

$$\begin{aligned}\zeta(t,x) &= \lim_{\varepsilon\to 0}\zeta^\varepsilon(t,x)\\ &= \lim_{\varepsilon\to 0}\Bigg(\frac{1}{2}\left(\zeta_0^\varepsilon(x+ct) + \zeta_0^\varepsilon(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\zeta_{t0}^\varepsilon(y)\,\mathrm{d}y\end{aligned}$$

$$- \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{W_0'(|\zeta^\varepsilon(s,y)|)}{|\zeta^\varepsilon(s,y)|}\zeta^\varepsilon(s,y)\,\mathrm{d}y\mathrm{d}s\Big),$$
$$= \frac{1}{2}\left(\zeta_0(x+ct)+\zeta_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\zeta_{t0}(y)\,\mathrm{d}y$$
$$- \frac{1}{2c}\lim_{\varepsilon\to 0}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{W_0'(|\zeta^\varepsilon(s,y)|)}{|\zeta^\varepsilon(s,y)|}\zeta^\varepsilon(s,y)\,\mathrm{d}y\mathrm{d}s.$$

Since

$$\left|\frac{W_0'(|\zeta^\varepsilon(s,y)|)}{|\zeta^\varepsilon(s,y)|}\zeta^\varepsilon(s,y)\right| \leq L_{\bar{E}},$$

the dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{W_0'(|\zeta^\varepsilon(s,y)|)}{|\zeta^\varepsilon(s,y)|}\zeta^\varepsilon(s,y)\,\mathrm{d}y\mathrm{d}s$$
$$= \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{W_0'(|\zeta(s,y)|)}{|\zeta(s,y)|}\zeta(s,y)\,\mathrm{d}y\mathrm{d}s.$$

Similarily, we have that $\zeta_t^\varepsilon(t) \to \zeta_t(t)$ uniformly in $L^2(\mathbb{R})$ for all $t \leq T$. Then, keeping in mind that for each t the limit and equality is in the sense of $L^2(\mathbb{R})$,

$$\begin{aligned}
\zeta_t(t,x) &= \lim_{\varepsilon\to 0}\zeta_t^\varepsilon(t,x)\\
&= \lim_{\varepsilon\to 0}\Bigg(\frac{c}{2}\left(\zeta_{x0}^\varepsilon(x+ct)-\zeta_{x0}^\varepsilon(x-ct)\right) + \frac{1}{2}\left(\zeta_{t0}^\varepsilon(x+ct)+\zeta_{t0}^\varepsilon(x-ct)\right)\\
&\quad - \frac{1}{2}\int_0^t \Bigg[\frac{W_0'(|\zeta^\varepsilon(s,x-c(t-s))|)}{|\zeta^\varepsilon(s,x-c(t-s))|}\zeta^\varepsilon(s,x-c(t-s))\\
&\qquad + \frac{W_0'(|\zeta^\varepsilon(s,x+c(t-s))|)}{|\zeta^\varepsilon(s,x+c(t-s))|}\zeta^\varepsilon(s,x+c(t-s))\Bigg]\,\mathrm{d}s\Bigg),\\
&= \frac{c}{2}\left(\zeta_{x0}(x+ct)-\zeta_{x0}(x-ct)\right) + \frac{1}{2}\left(\zeta_{t0}(x+ct)+\zeta_{t0}(x-ct)\right)\\
&\quad - \frac{1}{2}\lim_{\varepsilon\to 0}\int_0^t \Bigg[\frac{W_0'(|\zeta^\varepsilon(s,x-c(t-s))|)}{|\zeta^\varepsilon(s,x-c(t-s))|}\zeta^\varepsilon(s,x-c(t-s))\\
&\qquad + \frac{W_0'(|\zeta^\varepsilon(s,x+c(t-s))|)}{|\zeta^\varepsilon(s,x+c(t-s))|}\zeta^\varepsilon(s,x+c(t-s))\Bigg]\,\mathrm{d}s.
\end{aligned}$$

From (3.7) we get that

$$\left\|\int_0^t \frac{W_0'(|\zeta^\varepsilon(s)|)}{|\zeta^\varepsilon(s)|}\zeta^\varepsilon(s) - \frac{W_0'(|\zeta(s)|)}{|\zeta(s)|}\zeta(s)\,\mathrm{d}s\right\|_2 \leq (L_{\bar{E}}' + L_{\bar{E}}'')\int_0^t \|\zeta^\varepsilon(s)-\zeta(s)\|_2\,\mathrm{d}s$$
$$\to 0,$$

and hence ζ is a strong solution. Uniqueness and continuous dependence on initial data is proven in Lemma 3.7. □

Remark 3.11. *Note that if ζ is a solution of* (3.1) *in $\mathcal{D}_{\infty,E}$, then so is its complex conjugate $\bar{\zeta}$ and ζ modulated by a constant phase $e^{i\theta}\zeta$, $\theta \in \mathbb{R}$. Uniqueness of solutions then implies that if ψ initially is constant, $\psi_0(x) = \psi^*$ with $\psi_{t0} = 0$, then the solution is of the form $\zeta(t,x) = s(t,x)e^{i\psi^*}$.*

3.2. **The quasilinear case.** We will now consider the case $K_1 \neq K_3$, that is c is a smooth function of ψ instead of a constant. The main difficulty for (2.3) compared to (3.1) is that (2.3) is quasilinear since the terms $c(\psi)^2\psi_{xx}$ and $c(\psi)^2 s_{xx}$ are nonlinear. The method used in the proof of Theorem 3.10 relied heavily on (3.1) being semilinear.

Remark 3.12. *Let $\psi^* \in \mathbb{R}$ be such that $c'(\psi^*) = 0$, and let $(s_0 e^{i\psi^*}, s_{t0} e^{i\psi^*}) \in X_{E,\zeta^*}$ for some E, and $s_0 \in C^2(\mathbb{R}), s_{t0} \in C^1(\mathbb{R})$. Then by Theorem 3.9 and Remark 3.11 there is a unique global classical solution to $\zeta_{tt} - c(\psi^*)^2\zeta_{xx} + \frac{W_0'(|\zeta|)}{|\zeta|}\zeta = 0$ on the form $\zeta(t,x) = s(t,x)e^{i\psi^*}$. The pair (ψ^*, s) is then also a global classical solution to* (2.3)*.*

Remark 3.13. *Note that a similar generic construction as in Remark 3.12 with constant $s = s^*$ is not possible due to the presence of the term $-s(\psi_t^2 - c(\psi)^2\psi_x^2)$ in the second equation of* (2.3)*. A reduction of* (2.3) *to to the scalar nonlinear variational wave equation $\psi_{tt} - c(\psi)\left(c(\psi)\psi_x\right)_x = 0$ thus has to involve changing the potential W_0. One could for example try to choose initial data such that $s_0(x) = s^*, s_{t0} = 0$ where $s^* \neq 0$ is a zero of W_0 and look for a limit as ε tends to 0 of solutions $(\psi^\varepsilon, s^\varepsilon)$ to* (2.3) *with the scaled potential $W_0^\varepsilon = \frac{1}{\varepsilon}W_0$.*

We will only give a local existence theorem for (2.3). The two-component nonlinear wave equation (2.3) is degenerate in the sense that the first equation vanishes whenever $s = 0$. We will thus assume that s is bounded away from zero. Continuity then implies that s is of definite sign, and to avoid having to impose new conditions on W_0 we will restrict s to be positive. It is possible to accomodate negative s as well by imposing conditions on W_0 for $s < 0$ similar to the conditions in 3.3.

Theorem 3.14. *Assume that $W_0|_{[0,1)}$ is admissible in the sense of 3.3. Then given initial data $(\psi_0, \psi_{t0}, s_0, s_{t0})$ satisfying*

$$
\begin{aligned}
(\psi_0, \psi_{t0}, s_0, s_{t0}) \in W^{3,\infty}(\mathbb{R}) \times W^{2,\infty}(\mathbb{R}) &\times W^{3,\infty}(\mathbb{R}) \times W^{2,\infty}(\mathbb{R}),\\
\frac{1}{s_0} &\in L^\infty(\mathbb{R}),\\
s_0 &> 0,\\
\int_{\mathbb{R}} \frac{1}{2}s_0^2(\psi_{t0}^2 + c(\psi_0)^2\psi_{0x}^2) + \frac{1}{2}(s_{t0}^2 + c(\psi_0)^2 s_{0x}^2) &+ W_0(s_0)\,\mathrm{d}x < +\infty,
\end{aligned}
$$

there exists a unique short time classical solution of (2.3)*.*

Proof. Local existence follows from the standard approach taken to semigroups of nonlinear evolution equations. Here we will diverge from the standard semigroup approach by considering linear operators on Banach spaces, and instead solve the linear equations by characteristics.

In particular, inspired by [23], the system (2.3) can be rewritten as a quasilinear symmetric hyperbolic system and the results in [22] applied. Indeed, introduce the variables

$$
\begin{aligned}
R &= \psi_t + c(\psi)\psi_x, \quad & L &= \psi_t - c(\psi)\psi_x\\
U &= s_t + c(\psi)s_x, \quad & V &= s_t - c(\psi)s_x,
\end{aligned}
$$

the system (2.3) can be written as a quasilinear symmetric hyperbolic system

$$
\psi_t = \frac{1}{2}\left(R + L\right), \tag{3.18a}
$$

$$s_t = \frac{1}{2}(U+V), \tag{3.18b}$$

$$L_t + c(\psi)L_x = \frac{c'(\psi)}{4c(\psi)}\left(L^2 - R^2 - \left(\frac{U-V}{s}\right)^2\right) - \frac{1}{s}(VR+UL), \tag{3.18c}$$

$$V_t + c(\psi)V_x = \frac{c'(\psi)}{2c(\psi)}L(V-U) + sRL - W_0'(s), \tag{3.18d}$$

$$R_t - c(\psi)R_x = \frac{c'(\psi)}{4c(\psi)}\left(R^2 - L^2 - \left(\frac{U-V}{s}\right)^2\right) - \frac{1}{s}(VR+UL), \tag{3.18e}$$

$$U_t - c(\psi)U_x = \frac{c'(\psi)}{2c(\psi)}R(U-V) + sRL - W_0'(s). \tag{3.18f}$$

We introduce the space for the solutions

$$\begin{aligned} X_{E,Q} = \Big\{ & Z = (\psi, s, L, V, R, U) \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^6) \mid \\ & \|Z\|_{W^{1,\infty}(\mathbb{R})} \leq Q, \quad \left\|\frac{1}{s}\right\|_{L^\infty(\mathbb{R})} \leq Q, \\ & \int_{\mathbb{R}} \frac{1}{4}s^2\left(R^2 + L^2\right) + \frac{1}{4}\left(U^2 + V^2\right) + W_0(s)\,\mathrm{d}x \leq E \Big\}. \end{aligned}$$

Note that for elements of $X_{E,Q}$ the variable s is of definite sign since s is continuous and $\frac{1}{s}$ is bounded. Similarly to Proposition 3.4, we are able to bound $\|s\|_\infty$ in terms of E and Q. We have for $Z \in X_{E,Q}$ that $\|s\|_\infty \leq \min\{1, Q\}$, and since s is continuous there exists x^* such that $s(x^*) = \|s\|_\infty$. From the definition we also get $\|s_x\|_\infty \leq Q$. By $\mathbf{A}_1$ and $\mathbf{A}_4$ of Definition 3.3 we then have that $W_0(s(x)) \geq W_0(\|s\|_\infty - Q|x - x^*|)$ for $|x - x^*| \leq \frac{\|s\|_\infty - \tilde{s}}{Q}$ where $\tilde{s}$ is the largest $\tilde{s}$ with $W_0(\tilde{s}) = 0$. We then get that

$$E \geq \|W_0(s)\|_1 \geq 2Q \int_{\tilde{s}}^{\|s\|_\infty} W_0(u)\,\mathrm{d}u,$$

and $\mathbf{A}_3$ of Definition 3.3 then implies that there must be a positive constant $c_{E,Q} < 1$ such that $\|s\|_\infty \leq c_{E,Q}$. Note that this in turn implies that W_0, W_0', W_0'', and W_0''' are bounded in terms of E and Q.

We define our solution space $\mathcal{D}_{T,E,Q}$ as follows

$$\mathcal{D}_{T,E,Q} = C([0,T], X_{E,Q}) \cap C^1([0,T], L^\infty(\mathbb{R}, \mathbb{R}^6)).$$

We equip $\mathcal{D}_{T,E,Q}$ with the metric

$$\begin{aligned} d_{\mathcal{D}_{T,E,Q}}(Z_1, Z_2) = & \sup_{0 \leq t \leq T} \|Z_1(t) - Z_2(t)\|_{W^{1,\infty}(\mathbb{R},\mathbb{R}^6)} \\ & + \sup_{0 \leq t \leq T} \|Z_{1,t}(t) - Z_{2,t}(t)\|_{L^\infty(\mathbb{R},\mathbb{R}^6)}, \end{aligned}$$

which renders $\mathcal{D}_{T,E,Q}$ a complete metric space. Formally, we can formulate (3.18) as

$$Z_t - c(\psi)AZ_x = F(Z),$$

where A is a constant symmetric hyperbolic matrix with eigenvalues $-1, 0, 1$ and $F : [-Q, Q] \times [\frac{1}{Q}, c_{E,Q}] \times [-Q, Q]^4 \to \mathbb{R}^6$ is smooth. We want to show existence of short time solutions from a fix point argument. To be able to make a contraction we will further restrict our space for approximate solutions to

$$\mathcal{D}^{\mathrm{lin}}_{T,E,Q} = \left\{ Z \in \mathcal{D}_{T,E,Q} \mid Z(t) \in W^{2,\infty}(\mathbb{R},\mathbb{R}^6), \|Z_{xx}(t)\|_{L^\infty(\mathbb{R},\mathbb{R}^6)} \leq Q \right\}.$$

Let $0 < E < E', 0 < Q < Q'$ and $Z_0 \in X_{E,Q}$ with $\|Z_{0xx}\|_\infty \leq Q$ be the initial data, then for any $\hat{Z} \in \mathcal{D}^{\mathrm{lin}}_{T,E',Q'}$ with $\hat{Z}(0) = Z_0$, the linear system of transport equations

$$Z_t - c(\hat{\psi})AZ_x = F(\hat{Z}) \tag{3.19}$$

can be solved by characteristics. Define the backward characteristics $x_\pm$ at time τ from point (t, x) by

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x_\pm(\tau; t, x) = \mp c\Big(\hat{\psi}(\tau, x_\pm(\tau; t, x))\Big),$$

and note that $x_\pm(\tau; t, x) = x_\pm(\tau; s, x_\pm(s; t, x))$ and $x_\pm(t; t, x)) = x$. Then, by taking the derivative with respect to s, we get the following

$$\frac{\partial}{\partial t} x_\pm(\tau; t, x) \mp c(\hat{\psi}(\tau, x_\pm(\tau; t, x))) \frac{\partial}{\partial x} x_\pm(\tau; t, x) = 0.$$

Thus

$$\begin{aligned}
|x_\pm(\tau_1; t, x) - x_\pm(\tau_2; t, x)| &\leq \|c\|_\infty |\tau_2 - \tau_1|, \\
|x_\pm(\tau; t, x_1) - x_\pm(\tau; t, x_2)| &\leq \exp\left\{ \left\| \frac{1}{c} \right\|_\infty Q' |t - \tau| \right\} |x_1 - x_2|, \\
|x_+(\tau; t, x) - x_-(\tau; t, x)| &\leq 2\|c\|_\infty |t - \tau|.
\end{aligned}$$

With T_τ the solution operator of $Z_t(t) - c(\psi(t+\tau))AZ_x(t) = 0$ we can use Duhamel's principle to write the solution of (3.19) as $T_0(t)Z_0 + \int_0^t T_\tau(t-\tau)F(\hat{Z}(\tau))\,\mathrm{d}\tau$. We need to show that $\|T_0(t)Z_0\|_{W^{2,\infty}(\mathbb{R})}$ is bounded and that the Duhamel operator $DF(\hat{Z})(t) = \int_0^t T_\tau(t-\tau)F(\hat{Z}(\tau))\,\mathrm{d}\tau$ satisfies $DF(\hat{Z}) \in C([0,\infty), W^{2,\infty}(\mathbb{R},\mathbb{R}^6))$, and $\|DF(\hat{Z}_1)(t) - DF(\hat{Z}_2)(t)\|_{W^{1,\infty}(\mathbb{R})} \leq C(t, E', Q')\|\hat{Z}_1 - \hat{Z}_2\|_{W^{1,\infty}(\mathbb{R})}$ for $\hat{Z}_1, \hat{Z}_2 \in \mathcal{D}^{\mathrm{lin}}_{T,E',Q'}$. We get directly from characteristics that $\|T_0(t)Z_0\|_\infty = \|Z_0\|_\infty \leq Q$. Taking successive derivatives of $Z_t - c(\hat{\psi})AZ_x = 0$ with respect to x and employing Gronwall's inequality yields

$$\begin{aligned}
\|T_0(t)Z_0\|_\infty &\leq Q, \\
\|(T_0(t)Z_0)_x\|_\infty &\leq Q\exp\{\|c'\|_\infty Q't\}, \\
\|(T_0(t)Z_0)_{xx}\|_\infty &\leq \big(Q + t(\|c''\|_\infty Q'^2 + \|c'\|_\infty Q')\big)\exp\{\|c'\|_\infty Q't\}.
\end{aligned}$$

Since F is a smooth function, we have that for $\hat{Z} \in \mathcal{D}^{\mathrm{lin}}_{T,E',Q'}$ there is

$$\|F(\hat{Z}(t))\|_{W^{2,\infty}(\mathbb{R})} \leq C_{E',Q'},$$

for some constant $C_{E',Q'}$ depending on E' and Q' only. The Duhamel operator then satisfies $\|DF(\hat{Z})(t)\|_{W^{2,\infty}(\mathbb{R})} \leq C_{E',Q'}t$, and thus $Z(t) = T_0(t)Z_0 + DF(\hat{Z})(t)$ belongs to $\mathcal{D}^{\mathrm{lin}}_{T,E,Q}$ as long as

$$\big(Q + t(\|c''\|_\infty Q'^2 + \|c'\|_\infty Q')\big)\exp\{\|c'\|_\infty Q't\} + C_{E',Q'}t \leq Q'. \tag{3.21}$$

We need that the energy of the solution is bounded by E'. The energy density and energy density flux defined by(2.4) and (2.5), take here the form $\mathcal{E} = \frac{1}{4}s^2(R^2 +$

$L^2) + \frac{1}{4}(U^2+V^2) + W_0(s)$ and $c(\psi)\mathcal{F} = \frac{1}{4}s^2\left(R^2-L^2\right) + \frac{1}{4}\left(U^2-V^2\right)$. Instead of (2.6) we get the balance law

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{E} - \frac{\partial}{\partial x}c(\hat{\psi})c(\psi)\mathcal{F} = -s^2\left(\frac{\hat{U}-\hat{V}}{\hat{s}}\right)^2 - \frac{s^2}{\hat{s}}\left(\hat{V}\hat{R}+\hat{U}\hat{L}\right) + \frac{\hat{U}+\hat{V}}{2}\hat{s}\hat{R}\hat{L} \\ + \frac{c'(\hat{\psi})}{4c(\psi)}\left(U\hat{U}\hat{R} + V\hat{V}\hat{L} - U\hat{R}\hat{V} - V\hat{L}\hat{U}\right) \\ + \frac{\hat{U}+\hat{V}}{4}s\left(R^2+L^2\right) + \frac{\hat{U}+\hat{V}}{2}W_0'(s) \\ - \frac{U+V}{2}W_0'(\hat{s}) - c(\hat{\psi})\frac{ss_x}{2}\left(R^2-L^2\right) \\ - c'(\hat{\psi})\frac{\hat{\psi}_x}{4}\left(s^2(R^2-L^2) + (U^2-V^2)\right).\end{aligned}$$

By Definition 3.3 we have that

$$\begin{aligned}\left\|W_0'(s)\frac{\hat{U}+\hat{V}}{2}\right\|_1 &= \|W_0'(s)\|_2\left\|\frac{\hat{U}+\hat{V}}{2}\right\|_2 \\ &\le \sqrt{k_{E(t)}\|W_0(s)\|_1}\sqrt{E'} \\ &\le \sqrt{k_{E'}E(t)}\sqrt{E'},\end{aligned}$$

and hence as long as $E(t) \le E'$, integration $E(t) = \int_{\mathbb{R}} \mathcal{E}(t,x)\,\mathrm{d}x$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le 4Q'E' + Q'E' + Q'^6E' + Q'^4E' + 4KQ'^2E' + \frac{1}{2}Q'^2E(t) \\ + \|c\|_\infty Q'E(t) + \frac{1}{4}\|c'\|_\infty Q'E(t) + 2\sqrt{k_{E'}}E'.\end{aligned}$$

Hence $E(t) \le E'$ for t such that

$$(E + tK_{Q',E'})\exp\left\{t\left(\frac{1}{2}Q'^2 + \|c\|_\infty Q' + \frac{1}{4}\|c'\|_\infty\right)\right\} \le E', \tag{3.22}$$

with $K_{Q',E'} = 5Q'E' + Q'^4E' + Q'^6E' + 4KQ'^2E' + 2\sqrt{k_{E'}}E'$.

We need to show that the solution Z of (3.19) depends Lipschitz continuously on the function $\hat{Z} \in \mathcal{D}_{T,E',Q'}$. Let $\hat{Z}_1, \hat{Z}_2 \in \mathcal{D}_{T,E',Q'}$ and note that

$$\begin{aligned}&\frac{\partial}{\partial \tau}\left(x_\pm^1(\tau;t,x_1) - x_\pm^2(\tau;t,x_2)\right) \\ &= \mp\left(c(\hat{\psi}_1(\tau, x_\pm^1(\tau;t,x_1))) - c(\hat{\psi}_2(\tau, x_\pm^2(\tau;t,x_2)))\right),\end{aligned} \tag{3.23}$$

and thus

$$\begin{aligned}|x_\pm^1(\tau;t,x_1) - x_\pm^2(\tau;t,x_2)| \le |x_1-x_2|e^{\|c'\|_\infty Q'(t-\tau)} \\ + \frac{1}{Q'}\left(e^{\|c'\|_\infty Q'(t-\tau)} - 1\right)\|\hat{\psi}_1 - \hat{\psi}_2\|_{L^\infty([0,t],L^\infty(\mathbb{R}))}.\end{aligned}$$

Differentiation of (3.23), for $x_1 = x_2 = x$, with respect to x and then an application of Gronwall's inequality gives

$$\left|\frac{\partial}{\partial x}x_\pm^1(\tau;t,x) - \frac{\partial}{\partial x}x_\pm^2(\tau;t,x)\right| \le \frac{\|c''\|_\infty Q' + \|c'\|_\infty}{\|c'\|_\infty Q'}\left(\exp\{\|c'\|_\infty Q'(t-\tau)\} - 1\right)$$

$$\times \exp\{\|c'\|_\infty Q'(t-\tau)\}$$
$$\times \sup_{s\in[0,T]} \|\hat{\psi}_1(s) - \hat{\psi}_2(s)\|_{W^{1,\infty}(\mathbb{R})},$$

where we have used that $\|\hat{\psi}_{1,xx}\|_\infty \leq Q'$. We get for the $T_0(t)Z_0$ part the estimate

$$\begin{aligned}
\|T_0^1(t)Z_0 - T_0^2(t)Z_0\|_{W^{1,\infty}(\mathbb{R})} &\leq Q\frac{\|c''\|_\infty Q' + \|c'\|_\infty}{\|c'\|_\infty Q'}\left(\exp\{\|c'\|_\infty Q't\} - 1\right) \\
&\quad \times \exp\{\|c'\|_\infty Q't\} \sup_{s\in[0,T]} \|\hat{\psi}_1(s) - \hat{\psi}_2(s)\|_{W^{1,\infty}(\mathbb{R})} \\
&\quad + Q\left(\exp\{\|c'\|_\infty Q't\} - 1\right) \\
&\quad \times \exp\left\{\left\|\frac{1}{c}\right\|_\infty Q't\right\} \sup_{s\in[0,T]} \|\hat{\psi}_1(s) - \hat{\psi}_2(s)\|_\infty.
\end{aligned}$$

Since F is a smooth function and $\|\hat{Z}_1\|_{W^{2,\infty}(\mathbb{R})}, \|\hat{Z}_2\|_{W^{2,\infty}(\mathbb{R})} \leq Q'$,

$$\begin{aligned}
&\left|\int_0^t F_j(\hat{Z}_1)(\tau, x_\pm^1(\tau;t,x)) - F_j(\hat{Z}_2)(\tau, x_\pm^2(\tau;t,x))\,\mathrm{d}\tau\right| \\
&\leq \left|\int_0^t F_j(\hat{Z}_1)(\tau, x_\pm^1(\tau;t,x)) - F_j(\hat{Z}_2)(\tau, x_\pm^1(\tau;t,x))\,\mathrm{d}\tau\right| \\
&\quad + \left|\int_0^t F_j(\hat{Z}_2)(\tau, x_\pm^1(\tau;t,x)) - F_j(\hat{Z}_2)(\tau, x_\pm^2(\tau;t,x))\,\mathrm{d}\tau\right| \\
&\leq t\|F\|_{\mathrm{Lip}X_{E',Q'}} \exp\{\|c'\|_\infty Q't\} \sup_{s\in[0,T]} \|\hat{Z}_1(s) - \hat{Z}_2(s)\|_\infty.
\end{aligned}$$

For $\frac{\partial}{\partial x}\Big(DF(\hat{Z}_1)(t) - DF(\hat{Z}_2)(t)\Big)$ we estimate

$$\begin{aligned}
&\left|\frac{\partial}{\partial x}\int_0^t F_j(\hat{Z}_1)(\tau, x_\pm^1(\tau;t,x)) - F_j(\hat{Z}_2)(\tau, x_\pm^2(\tau;t,x))\,\mathrm{d}\tau\right| \\
&\leq 6\int_0^t \bigg(\|F\|_{\mathrm{Lip}X_{E',Q'}} \|\hat{Z}_{1,x}(\tau)\|_\infty |\partial_x x_\pm^1(\tau;t,x) - \partial_x x_\pm^2(\tau;t,x)| \\
&\quad + \|F\|_{\mathrm{Lip}X_{E',Q'}} |\partial_x x_\pm^2(\tau;t,x)| \|\hat{Z}_1(\tau) - \hat{Z}_2(\tau)\|_\infty \\
&\quad + \|F\|_{\mathrm{Lip}X_{E',Q'}} \|\hat{Z}_{2,xx}(\tau)\|_\infty |\partial_x x_\pm^2(\tau;t,x)| |x_\pm^1(\tau;t,x) - x_\pm^2(\tau;t,x)| \\
&\quad + \|\nabla_Z F\|_{\mathrm{Lip}X_{E',Q'}} \|\hat{Z}_{1,x}(\tau)\|_\infty \|\hat{Z}_{2,x}(\tau)\|_\infty \\
&\qquad \times |\partial_x x_\pm^2(\tau;t,x)| |x_\pm^1(\tau;t,x) - x_\pm^2(\tau;t,x)| \\
&\quad + \|\nabla_Z F\|_{\mathrm{Lip}X_{E',Q'}} \|\hat{Z}_{2,x}(\tau)\|_\infty |\partial_x x_\pm^2(\tau;t,x)| |\hat{Z}_1(\tau) - \hat{Z}_2(\tau)|\bigg)\,\mathrm{d}\tau \\
&\leq 6\left(\|F\|_{\mathrm{Lip}X_{E',Q'}} \frac{\|c''\|_\infty Q' + \|c'\|_\infty}{\|c'\|_\infty} + \|\nabla F\|_{\mathrm{Lip}X_{E',Q'}} Q'\right) \\
&\quad \times t\exp\{kQ't\}\, d_{\mathcal{D}_{T,E',Q'}}(\hat{Z}_1, \hat{Z}_2),
\end{aligned}$$

where $k = \max\left\{\|c'\|_\infty, \left\|\frac{1}{c}\right\|\right\}$. That is, we have that

$$\|Z_1(t) - Z_2(t)\|_{W^{1,\infty}(\mathbb{R})} \leq C_{E',Q'}\, t\exp\{kQ't\}\, d_{\mathcal{D}_{T,E',Q'}}(\hat{Z}_1, \hat{Z}_2). \tag{3.24}$$

Since Z is continuously differentiable in t, we can do similar estimates for $\|Z_{1,t}(t) - Z_{2,t}(t)\|_{L^\infty(\mathbb{R})}$ directly from the linear system and get

$$\begin{aligned}\|Z_{1,t}(t) - Z_{2,t}(t)\|_\infty &\le \|c'\|_\infty Q \exp\{\|c'\|_\infty Q' t\} \, \|\hat{\psi}_1(t) - \hat{\psi}_2(t)\|_\infty \\ &\quad + \|c\|_\infty \|Z_{1,x}(t) - Z_{2,x}(t)\|_\infty + \left\| F(\hat{Z}_1(t)) - F(\hat{Z}_2(t)) \right\|_\infty \\ &\le \|c'\|_\infty Q \exp\{\|c'\|_\infty Q' t\} \, \|\hat{\psi}_1(t) - \hat{\psi}_2(t)\|_\infty \\ &\quad + C_{E',Q'} t \exp\{kQ't\} \, \|\hat{Z}_1(t) - \hat{Z}_2(t)\|_{W^{1,\infty}(\mathbb{R})} \\ &\quad + \|F\|_{\mathrm{Lip} X_{E',Q'}} \, \|\hat{Z}_1(t) - \hat{Z}_2(t)\|_\infty .\end{aligned}$$

Direct computation of the local Lipschitz constant $\|F\|_{\mathrm{Lip} X_{E',Q'}}$ gives a bound depending on Q' and E' only. Moreover, $\|\hat{Z}_1(t) - \hat{Z}_2(t)\|_\infty \le t \sup_{s\in[0,T]} \|\hat{Z}_{1,t}(s) - \hat{Z}_{2,t}(s)\|_\infty$, and hence

$$\|Z_{1,t}(t) - Z_{2,t}(t)\|_\infty \le K_{t,E',Q'} d_{T,E',Q'} \left(\hat{Z}_1, \hat{Z}_2 \right), \tag{3.25}$$

where

$$\begin{aligned}K_{t,E',Q'} &= \|c\|_\infty C_{E',Q'} t \exp\{kQ't\} + t \, \|F\|_{\mathrm{Lip} X_{E',Q'}} \\ &\quad + t \|c'\|_\infty Q \exp\{\|c'\|_\infty Q' t\} .\end{aligned}$$

A combination of (3.24) and (3.25) yields

$$d_{\mathcal{D}_{T,E',Q'}}(Z_1, Z_2) \le (C_{E',Q'} T \exp\{kQ'T\} + K_{T,E',Q'}) \, d_{\mathcal{D}_{T,E',Q'}}(\hat{Z}_1, \hat{Z}_2),$$

and hence, as long as T satisfies

$$C_{E',Q'} T \exp\{kQ'T\} + K_{T,E',Q'} < 1, \tag{3.26}$$

the map $\hat{Z} \mapsto Z$ is a contraction.

Given initial data $Z_0 \in X_{E,Q}$ with $\|Z_{0,xx}\|_\infty \le Q$, $Q' > Q$, and $E' > E$ one can find $T > 0$ that satisfies satisfies (3.21), (3.22), and (3.26). We can then construct a Cauchy sequence in $\mathcal{D}_{T,E',Q'}$ as follows by letting Z^{n+1} be the solution of (3.19) with right hand side Z^n and wave speed $c(\psi^n)$. Moreover, the limit will satisfy $\mathcal{E}_t - (c(\psi)^2 \mathcal{F})_x = 0$, and thus conserves energy.

We show that the solution is unique in $\mathcal{D}_{T,E,Q}$. Let $Z_1, Z_2 \in \mathcal{D}_{T,E,Q}$ be solutions of (3.18), then

$$(Z_1 - Z_2)_t - c(\psi_2) A \, (Z_1 - Z_2)_x = (c(\psi_1) - c(\psi_2)) \, A Z_{1,x} + F(Z_1) - F(Z_2),$$

and Duhamel's principle implies that

$$\begin{aligned}\|Z_1(t) - Z_2(t)\|_\infty &\le \|Z_1(0) - Z_2(0)\|_\infty \\ &\quad + \left(Q\|c'\|_\infty + \|F\|_{\mathrm{Lip}(X_{E,Q})} \right) \int_0^t \|Z_1(s) - Z_2(s)\|_\infty \, \mathrm{d}s,\end{aligned}$$

and then by Gronwall's inequality

$$\|Z_1(t) - Z_2(t)\|_\infty \le \|Z_1(0) - Z_2(0)\|_\infty \exp\left\{ (Q\|c'\|_\infty + \|F\|_{\mathrm{Lip}(X_{E,Q})}) t \right\},$$

and hence we get uniqueness. □

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