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Sharpness of some Hardy-type inequalities

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Abstract

The current status concerning Hardy-type inequalities with sharp constants is presented and described in a unified convexity way. In particular, it is then natural to replace the Lebesgue measure *dx* with the Haar measure *dx*/*x*. There are also derived some new two-sided Hardy-type inequalities for monotone functions, where not only the two constants are sharp but also the involved function spaces are (more) optimal. As applications, a number of both well-known and new Hardy-type inequalities are pointed out. And, in turn, these results are used to derive some new sharp information concerning sharpness in the relation between different quasi-norms in Lorentz spaces.

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1 Introduction

The continuous Hardy inequality from 1925 (see [5]) informs us if f is nonnegative p-integrable function on $(0,\infty)$, then f is integrable over the interval $(0,x)$ for each positive x and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \quad p>1. \tag{1.1}$$

The development of the famous Hardy inequality in both discrete and continuous forms during the period 1906 to 1928 has its own history or, as we call it, prehistory. Contributions of mathematicians other than Hardy, such as Landau, Polya, Schur, and Riesz, are important here. The first weighted version of (1.1) was proved by Hardy himself in 1928 (see [6]):

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha\,dx \le \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha\,dx, \tag{1.2}$$

where f is a measurable and nonnegative function on $(0,\infty)$ whenever $\alpha < p-1, p>1$.

In the remarkable further development to which today is called Hardy-type inequalities, in the case of weighted Lebesgue spaces, mostly the Lebesgue measure dx is used (see, for instance, the books [7, 8], and [9] and the references therein). One basic idea in this paper is to use convexity, and then it is more natural to instead use the measure dx/x (=Haar measure when the underlying group is $R+$). Moreover, this way to consider the situation helps us to easier investigate and describe the sharpness in Hardy-type inequalities. In this theory of Hardy-type inequalities (between weighted Lebesgue spaces) we usually have good estimates of the sharp constant (= the operator norm or quasi-norm). However, in very few cases the sharp constant is known.

In this paper we describe and/or derive most of these Hardy-type inequalities in the convexity (dx/x) frame described above. Moreover, we concentrate also on the problem to derive the corresponding reversed inequalities in cones of monotone functions. And still with sharp constants. It turns out that our approach also implies that the sharpness can be further improved in special situations e.g. to not only have sharp constant(s) but also by involving more optimal function spaces, sometimes even with optimal so called target functions involved. In order to illustrate this idea, we present the following introductory example.

Example 1.1 Inequality (1.2) holds also if the interval $(0,\infty)$ is replaced by $(0,\ell)$, $0<\ell\leq\infty$, and still the constant

$$C=\left(\frac{p}{p-1-\alpha}\right)^p$$

is sharp. However, also the following "sharper" inequality is known (see [11] and cf. Theorem 2.3(a) in the book [9]):

$$\int_0^\ell\left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha\,dx\leq\left(\frac{p}{p-1-\alpha}\right)^p\int_0^\ell f^p(x)x^\alpha\left[1-\left(\frac{x}{\ell}\right)^{\frac{p-1-\alpha}{p}}\right]dx,\tag{1.3}$$

where $\alpha<p-1, p>1$, and still the constant $C=(\frac{p}{p-1-\alpha})^p$ is sharp. Moreover, we note that in the cone of nonincreasing functions (1.2) holds in the reversed direction with the constant $C=1$. But indeed (1.3) holds also in the reversed direction with the sharp constant $C=p/(p-1-\alpha)>1$ whenever $\alpha>-1$. See our Theorem 3.2(a). In such a situation when both constants are sharp, we say that the involved weight function

$$g(x):=1-\left(\frac{x}{\ell}\right)^{\frac{p-1-\alpha}{p}}<1,\quad x<\ell,$$

is the "optimal target function".

The paper is organized as follows: In Sect. 2 we present the mentioned convexity approach to derive power weighted Hardy-type inequalities and some of its consequences. Here, and in the sequel, it turns out that this convexity approach makes it natural to present such inequalities by using the Haar measure dx/x instead of the Lebesgue measure dx. In Sect. 3 we derive some new sharp reversed Hardy-type inequalities on cones of monotone functions. Section 4 is used to present and discuss some new applications e.g. concerning two-sided Hardy-type inequalities where both constants are sharp and, moreover, the actual inequalities are further sharpened by pointing out (more) optimal involved function spaces. These results, in their turn, make it possible to derive some new results concerning comparisons of different norms in Lorentz spaces. Finally, in Sect. 5 we give some concluding remarks and present and/or derive some further sharp Hardy-type inequalities.

2 A convexity approach to derive sharp power weighted Hardy-type inequalities

The fact that the concept of convexity can be used to prove several inequalities, both classical and new ones, was of course known by Hardy himself. For example, in the famous book [7] this concept and the more or less equivalent Jensen inequality were frequently used. Hence, it may be surprising that Hardy himself never discovered that also his famous inequality in both original (see (1.1)) and power weighted form (see e.g. (1.2)) follows more or less directly as described below. Concerning convexity and its applications e.g. to prove inequalities, we refer to the recent book [10], the papers [11, 12], and the references therein.

2.1 An elementary convexity proof

First we note that for $p > 1$

$$\begin{aligned} &\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \\ &\quad \Leftrightarrow \\ &\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)\,dy\right)^p \frac{dx}{x} \le 1\cdot\int_0^\infty g^p(x)\frac{dx}{x}, \end{aligned} \tag{2.1}$$

where $f(x) = g(x^{1-1/p})x^{-1/p}$.

This means that Hardy's inequality (1.1) is equivalent to (2.1) for $p > 1$ and, thus, that Hardy's inequality can be proved in the following simple way (see form (2.1)): By Jensen's inequality and Fubini's theorem we have that

$$\begin{aligned} \int_0^\infty \left(\frac{1}{x}\int_0^x g(y)\,dy\right)^p \frac{dx}{x} &\le \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)\,dy\right)\frac{dx}{x} \\ &= \int_0^\infty g^p(y)\int_y^\infty \frac{dx}{x^2}\,dy = \int_0^\infty g^p(y)\frac{dy}{y}. \end{aligned} \tag{2.2}$$

Remark 2.1 By instead making the substitution

$$f(t) = g\left(t^{\frac{p-1-\alpha}{p}}\right)t^{-\frac{1+\alpha}{p}}$$

in (1.2), we see that also this inequality is equivalent to (2.1). Indeed, by modifying and analyzing the proof above, we find that:

(a) Hardy's inequalities (1.1) and (1.2) hold also for $p < 0$ (because the function $\varphi(u) = u^p$ is convex also for $p < 0$) and hold in the reversed direction for $0 < p < 1$ (with sharp constants $(\frac{p}{1-p})^p$ and $(\frac{p}{\alpha+1-p})^p, \alpha > p-1$, respectively);

(b) Inequalities (1.1) and (1.2) are equivalent since both are equivalent to (2.1);

(c) Inequality (2.1) holds also with equality for $p = 1$, which gives us a possibility to interpolate and get more information about the mapping properties of the Hardy operator. In particular, we can use interpolation theory to see that in fact the Hardy operator H maps each interpolation space I between $L_1((0,\infty),\frac{dx}{x})$ and $L_\infty((0,\infty),\frac{dx}{x})$ into B i.e. that the following more general Hardy-type inequality holds for some positive constant C:

$$\|Hf\|_I \le C\|f\|_I.$$

2.2 An essential generalization

For the finite interval case, we need the following extension of our basic (convexity) form of Hardy's inequality presented in Sect. 2.2.

Lemma 2.2 *Let g be a nonnegative and measurable function on $(0,\ell), 0<\ell\le\infty$.*

(a) If $p<0$ or $p\ge 1$, then

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)\,dy\right)^p \frac{dx}{x} \le 1\cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}. \tag{2.3}$$

(In the case $p<0$, we assume that $g(x)>0, 0<x\le\ell$).

(b) If $0<p\le 1$, then (2.3) holds in the reversed direction.

(c) The constant $C=1$ is sharp in both (a) and (b).

Proof (a) The proof only consists of an obvious modifications of the proof presented in Sect. 2.1 (see (2.2)).

(b) Since Jensen's inequality holds in the reversed direction for the concave function

$$\phi(u)=u^p, \quad 0<p\le 1,$$

the proof follows in the same way.

(c) Assume (2.3) with the constant 1 replaced by some constant $c, 0<c<1$. By applying (2.3) with the test functions $g(x)=x^a, 0\le x\le\ell, a>0$, a simple calculation shows that

$$(ap+1)(a+1)^{-p}\le c<1,$$

so by choosing a sufficiently small, we get a contradiction, and the proof is complete concerning (a). The proof of the sharpness of (b) is obtained by making an obvious modification of this argument, so the proof is complete. □

The following equivalence theorem holds.

Theorem 2.3 *Let $0<\ell\le\infty$, let $p\ge 1$ or $p<0$, and let f be a nonnegative and measurable function. Then*

(a) The inequality

$$\int_0^\ell \left(\int_0^x f(y)\,dy\right)^p x^{-\alpha}\frac{dx}{x} \le \left(\frac{p}{\alpha}\right)^p \int_0^\ell (xf(x))^p x^{-\alpha}\left[1-\left(\frac{x}{\ell}\right)^{\frac{\alpha}{p}}\right]\frac{dx}{x} \tag{2.4}$$

holds for all measurable functions f, each $\ell, 0<\ell\le\infty$, and all α in the following cases:

$$(a_1) \quad p\ge 1, \alpha>0,$$

(a_2) $p < 0, \alpha < 0$.

(b) *The inequality*

$$\int_\ell^\infty \left(\int_x^\infty f(y)\,dy\right)^p x^\alpha \frac{dx}{x} \le \left(\frac{p}{\alpha}\right)^p \int_\ell^\infty f^p(x)x^\alpha \left[1 - \left(\frac{\ell}{x}\right)^{\frac{\alpha}{p}}\right] \frac{dx}{x} \tag{2.5}$$

holds for all measurable functions f, each $\ell, 0 \le \ell < \infty$, and all α in the following cases:

(c_1) $p \ge 1, \alpha > 0$,

(c_2) $p < 0, \alpha < 0$.

(c) *Inequalities* (2.4) *and* (2.5) *are sharp and the statements in* (a) *and* (b) *are equivalent for all permitted α.*

Proof The proof can be done by just using Lemma 2.2(a) and (c) and doing similar calculations and substitutions as those in the proof of Theorem 2.4 in [11] in the case $l = \infty$ and $\frac{dx}{x}$ replaced by dx (see also Theorem 7.10 in the book [9]), so we omit the details. □

Remark 2.4 For the case $l = \infty$, inequalities (2.4) and (2.5) were formulated, proved, and applied in this convexity form in the new book [13]. This fact has further inspired us to reformulate our results in this convexity (dx/x) way, which not only contributes to a better understanding but is also more suitable for such applications in modern harmonic analysis (see the new book [13]).

For the case $0 < p < 1$, the corresponding equivalence theorem reads as follows.

Theorem 2.5 *Let $0 < \ell \le \infty$, let $0 < p < 1, \alpha > 0$, and let f be a nonnegative and measurable function. Then*

(a) *inequality* (2.5) *holds in the reversed direction for all $\ell, 0 \le \ell < \infty$;*

(b) *inequality* (2.5) *holds in the reversed direction for all $\ell, 0 \le \ell < \infty$;*

(c) *all inequalities in* (a) *and* (b) *are sharp and equivalent for all $\alpha > 0$.*

Proof By instead using Lemma 2.2(b) and (c), the proof is step by step similar to that of Theorem 2.3, so we omit the details. □

3 Reversed sharp Hardy inequalities for monotone functions

For the proof of our main results in this section, we need the following lemma.

Lemma 3.1 *Let $p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and let f be a nonnegative and measurable function on (a,b), $\infty \le a < b \le \infty$.*

(a) *Let f be nonincreasing on (a,b), $\infty < a < b \le \infty$. If $p \ge 1$, then*

$$\left(\int_a^b f(y)\,dy\right)^p \ge p\int_a^b (y-a)^{p-1}\big(f(y)\big)^p\,dy. \tag{3.1}$$

If $0 < p \le 1$, then (3.1) *holds in the reversed direction.*

(*b*) *Let* f *be nondecreasing on* (a,b), $-\infty \le a < b < \infty$. *If* $p \ge 1$, *then*

$$\left(\int_a^b f(y)\,dy\right)^p \ge p\int_a^b (b-y)^{p-1}\big(f(y)\big)^p\,dy. \tag{3.2}$$

If $0 < p \le 1$, *then* (3.2) *holds in the reversed direction.*

(*c*) *The constant* p *is sharp in all these four inequalities. In fact, we have even equality in* (3.1) *for the function* $f(y) = A\chi_{(a,c)}(y)$ *for some* $c \in (a,b)$ *and* $A > 0$. *Moreover, equality in* (3.2) *holds if* $f(y) = A\chi_{(c,b)}(y)$ *for some* $c \in (a,b)$ *and* $A > 0$.

Proofs of various variants of this lemma can be found in many places (see e.g. [3]), but for the readers' convenience, we include a simple proof of just this variant.

Proof First assume that $-\infty < a < b < \infty$. Next we observe that the proof of (b) can be reduced to that of (a) by putting $g(y) = f(a+b-y)$. Hence, it is sufficient to prove (a). Moreover, by making suitable coordinate transformations, we conclude that it is sufficient to consider the case $(a,b) = (0,1)$. Therefore, we consider a nonnegative, measurable, and nonincreasing function f on $(0,1)$.

Let

$$F(x) = \int_0^x f(y)\,dy.$$

Then $F(0) = 0$ and, for almost all $x \in (0,1)$, if $p \ge 1$ then

$$\frac{d}{dx}\big(F(x)\big)^p = pf(x)\big(F(x)\big)^{p-1} \ge px^{p-1}\big(f(x)\big)^p.$$

By integrating from 0 to 1, we find that

$$\left(\int_0^1 f(y)\,dy\right)^p = \big(F(1)\big)^p \ge p\int_0^1 y^{p-1}f(y)\,dy.$$

The same argument shows that this inequality holds in the reversed direction if $0 < p \le 1$. We conclude that (a) and (b) are proved. It is obvious that we have equality in inequalities (3.1) and (3.2) and their reversed versions for $0 < p \le 1$ for the claimed test functions

$$f(y) = A\chi_{(a,c)}(y) \quad \text{and} \quad f(y) = A\chi_{(c,b)}(y),$$

respectively.

The proof of the cases $a = -\infty$ or $b = \infty$ follows by just doing a limit procedure, so the proof is complete. □

First we consider the case when f is nonincreasing and note that then such a reversed inequality has meaning only if $0 < \alpha < p$ (since if not the involved integrals diverge for all nontrivial functions f).

Our first main result reads as follows.

Theorem 3.2 *Let* $p > 0$, $0 < \alpha < p$, *and let* f *be a measurable, nonnegative, and nonincreasing function on* $(0,\ell)$, $0 < l \le \infty$.

(*a*) *If* $p \geq 1$, *then*

$$\int_0^\ell \left(\int_0^x f(y)\,dy\right)^p x^{-\alpha}\frac{dx}{x} \geq \frac{p}{\alpha}\int_0^\ell \bigl(xf(x)\bigr)^p x^{-\alpha}\left(1-\left(\frac{x}{\ell}\right)^\alpha\right)\frac{dx}{x}. \tag{3.3}$$

(*b*) *If* $0 < p \leq 1$, *then* (3.3) *holds in the reversed direction.*

(*c*) *The constant* $C = p/\alpha$ *is sharp in both* (*a*) *and* (*b*), *and equality appears for each function* $f(x) = A\chi_{(0,c)}(x)$ *for some* $c \in (0,l)$ *and* $A > 0$.

Proof (a) By using Lemma 3.1 and Fubini's theorem, we find that

$$\begin{aligned}
&\int_0^\ell \left(\int_0^x f(y)\,dy\right)^p x^{-\alpha}\frac{dx}{x} \\
&\quad\geq p\int_0^\ell \left(\int_0^x y^{p-1}\bigl(f(y)\bigr)^p\,dy\right)x^{-\alpha}\frac{dx}{x} \\
&\quad= p\int_0^\ell \bigl(yf(y)\bigr)^p\left(\int_y^\ell x^{-\alpha-1}\,dx\right)\frac{dy}{y} \\
&\quad= \frac{p}{\alpha}\int_0^\ell \bigl(yf(y)\bigr)^p\bigl(y^{-\alpha}-\ell^{-\alpha}\bigr)\frac{dy}{y} \\
&\quad= \frac{p}{\alpha}\int_0^\ell \bigl(yf(y)\bigr)^p y^{-\alpha}\left(1-\left(\frac{y}{\ell}\right)^\alpha\right)\frac{dy}{y}.
\end{aligned}$$

(b) Only one inequality is used in the proof of (a) and, according to Lemma 3.1, this inequality holds in the reversed direction in this case, so also (b) is proved.

(c) In view of the proofs above, this sharpness statement follows by using Lemma 3.1, but we also verify this directly: Let $f(x) = A\chi_{(0,c)}(x)$, $c \in (0,l)$. Then

$$\begin{aligned}
\frac{p}{\alpha}\int_0^\ell \bigl(xf(x)\bigr)^p x^{-\alpha}\left(1-\left(\frac{x}{\ell}\right)^\alpha\right)\frac{dx}{x} &= \frac{p}{\alpha}A^p\int_0^c x^{p-\alpha-1}\left(1-\left(\frac{x}{\ell}\right)^\alpha\right)dx \\
&= A^p\frac{p}{\alpha}\left(\frac{c^{p-\alpha}}{p-\alpha}-\frac{1}{\ell^\alpha}\frac{c^p}{p}\right) := I.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_0^\ell \left(\int_0^x f(y)\,dy\right)^p x^{-\alpha}\frac{dx}{x} &= A^p\int_0^c x^{p-\alpha-1}\,dx + A^p c^p\int_c^\ell x^{-\alpha-1}\,dx \\
&= A^p\frac{c^{p-\alpha}}{p-\alpha}+\frac{A^pc^p}{\alpha}\bigl(c^{-\alpha}-\ell^{-\alpha}\bigr) \\
&= A^p\frac{p}{\alpha}c^{p-\alpha}-A^p\frac{c^p}{\alpha} = I.
\end{aligned}$$

We conclude that the constant p/α is sharp in both (a) and (b) with equality for

$$f(x) = A\chi_{(0,c)}(x), \quad c \in (0,l),$$

so also (c) is proved. □

As already mentioned, inequality (3.3) has no meaning in the cone of nonincreasing functions if $\alpha \geq p$. But it is not so if we instead restrict to the cone of nondecreasing functions. But in this case the "target function"

$$f(x) = \left(1 - \left(\frac{x}{\ell}\right)^{\alpha}\right)$$

is different and connected to the truncated β_α function defined as follows:

$$\beta_\alpha = \beta_\alpha(u,v) = \int_\alpha^1 t^{u-1}(1-t)^{v-1}, \quad 0 \leq \alpha < 1.$$

In particular, β_0 coincides with the usual β function $\beta(u,v)$.

Our next main result reads as follows.

Theorem 3.3 *Let $\alpha \geq p > 0$ and let f be a measurable, nonnegative, and nondecreasing function on $(0,\ell)$, $0 < \ell \leq \infty$.*

(a) If $p \geq 1$, then

$$\int_0^\ell \left(\int_0^x f(y)\,dy\right)^p x^{-\alpha}\frac{dx}{x} \geq \frac{p}{\alpha}\int_0^\ell \bigl(xf(x)\bigr)^p x^{-\alpha} T(x)\frac{dx}{x}, \tag{3.4}$$

where

$$T(x) := \alpha\beta_{\frac{x}{\ell}}(p, \alpha - p + 1), \quad x \leq l.$$

(b) If $0 < p \leq 1$, then (3.4) holds in the reversed direction.

(c) The constant $\frac{p}{\alpha}$ is sharp in both (a) and (b), and equality appears if

$$f(x) = A\chi_{(c,l)}(x) \quad \textit{for some } c \in (0,l) \textit{ and } A > 0.$$

Proof (a) By using again Lemma 3.1 and Fubini's theorem, we obtain that

$$\begin{aligned} I &:= \int_0^\ell \left(\int_0^x f(y)\,dy\right)^p x^{-\alpha}\frac{dx}{x} \geq p\int_0^\ell \int_0^x (x-y)^{p-1}\bigl(f(y)\bigr)^p\,dy x^{-\alpha}\frac{dx}{x} \\ &= p\int_0^\ell \bigl(f(y)\bigr)^p \int_y^\ell (x-y)^{p-1}x^{-\alpha}\frac{dx}{x}\,dy. \end{aligned}$$

We make the transformation $t = \frac{y}{x}$ in the inner integral and get that

$$\begin{aligned} I &\geq p\int_0^\ell \bigl(f(y)\bigr)^p \int_{y/l}^1 (1-t)^{p-1}\left(\frac{y}{t}\right)^{p-\alpha-1}\frac{dt}{t}\,dy \\ &= p\int_0^\ell \bigl(yf(y)\bigr)^p y^{-\alpha}\int_{y/l}^1 (1-t)^{p-1}t^{\alpha-p}\,dt\frac{dy}{y} \\ &= \frac{p}{\alpha}\int_0^\ell \bigl(yf(y)\bigr)^p y^{-\alpha}T(y)\frac{dy}{y}. \end{aligned}$$

(b) Since the only inequality used above holds in the reversed direction in this case (see Lemma 3.1), the proof of (b) follows in the same way.

(c) Choose the test function

$$f(x) = A\chi_{(c,l)}(x), \quad c \in (0,l).$$

Then, in view of the proofs of (a) and (b), for any $p > 0$, the right-hand side of (3.4) is equal to

$$\begin{aligned} I &:= \int_0^\ell \int_0^x (x-y)^{p-1} A^p \big(\chi_{(c,l)}(y)\big)^p \, dy x^{-\alpha} \frac{dx}{x} \\ &= pA^p \int_c^\ell \int_c^x (x-y)^{p-1} \, dy x^{-\alpha} \frac{dx}{x} \\ &= A^p \int_c^\ell (x-c)^p x^{-\alpha} \frac{dx}{x}. \end{aligned}$$

Moreover, the left-hand side of (3.4) is equal to

$$A^p \int_0^\ell \int_0^x \big(\chi_{(c,l)}(y) \, dy\big)^p x^{-\alpha} \frac{dx}{x} = A^p \int_c^\ell (x-c)^p x^{-\alpha} \frac{dx}{x} = I,$$

so we have equality in (3.4) and the reversed inequality for $0 < p \leq 1$ for all $p > 0$.

The proof is complete. □

Example 3.4 For the case $l = \infty$, we obtain the sharp inequality

$$\int_0^\infty \left(\int_0^x f(y) \, dy\right)^p x^{-\alpha} \frac{dx}{x} \geq p\beta(p, \alpha - p + 1) \int_0^\infty \big(xf(x)\big)^p x^{-\alpha} \frac{dx}{x}$$

for all nondecreasing functions f. This inequality holds in the reversed direction when $0 < p \leq 1$ and the constant is sharp also then. Hence, by just changing notations, we see that our result generalizes also a result in [3].

Hence, we have investigated all cases concerning the usual (arithmetic mean) Hardy operator, so we turn to the dual situation (cf. Theorem 2.3(c)), and here the only nontrivial situation is to study the nonincreasing case.

Our main result for this case reads as follows.

Theorem 3.5 *Let $p > 0, \alpha > 0$, and f be a measurable, nonnegative, and nonincreasing function on (ℓ, ∞), $0 \leq \ell < \infty$.*

(a) If $p \geq 1$, then

$$\int_\ell^\infty \left(\int_x^\infty f(y) \, dy\right)^p x^\alpha \frac{dx}{x} \geq \frac{p}{\alpha} \int_\ell^\infty \big(xf(x)\big)^p x^\alpha T_0(x) \frac{dx}{x}, \tag{3.5}$$

where

$$T_0(x) := \alpha \beta_{\frac{\ell}{x}}(p, \alpha), \quad x \geq l.$$

(*b*) *If* $0 < p \le 1$, *then* (3.5) *holds in the reversed direction.*

(*c*) *The constant* p/α *is sharp in both* (*a*) *and* (*b*), *and equality appears in both* (*a*) *and* (*b*) *if*

$$f(x) = A\chi_{(\ell,c)}(x) \quad \textit{for some } c \in (\ell,\infty) \textit{ and } A > 0.$$

Proof (a) By again applying Lemma 3.1 and Fubini's theorem, we get that

$$\begin{aligned}
I &:= \int_\ell^\infty \left(\int_x^\infty f(y)\,dy\right)^p x^\alpha \frac{dx}{x} \ge p\int_\ell^\infty (y-x)^{p-1}\big(f(y)\big)^\beta\,dy x^\alpha \frac{dx}{x} \\
&= p\int_\ell^\infty \big(f(y)\big)^p \int_\ell^y (y-x)^{p-1}x^{\alpha-1}\,dx\,dy \\
&= p\int_\ell^\infty \big(f(y)\big)^p y^{p-1}\int_\ell^y \left(1-\frac{x}{y}\right)^{p-1} x^{\alpha-1}\,dx\,dy.
\end{aligned}$$

Thus, by making the transformation $t = x/y$ in the inner integral, we can conclude that

$$\begin{aligned}
I &\ge p\int_\ell^\infty \big(f(y)y\big)^p y^\alpha \int_{l/y}^1 (1-t)^{p-1}t^{\alpha-1}\frac{dy}{y} \\
&= \frac{p}{\alpha}\int_\ell^\infty \big(f(y)y\big)^p y^\alpha T_0(x)\frac{dy}{y}.
\end{aligned}$$

(b) The proof follows in the same way since the only inequality used in (a) now holds in the reversed direction.

(c) Similar as in the proof of Theorem 3.3(c), we can easily verify that we indeed have equality in inequality (3.5) (and the reversed inequality when $0 < p \le 1$) for every function

$$f(x) = A\chi_{(\ell,c)}(x), \quad c \in (\ell,\infty) \text{ and } A > 0.$$

Hence, also the sharpness is proved. □

Example 3.6 Let f, p, and α be defined as in Theorem 3.5. If $p \ge 1$, then

$$\int_0^\infty \left(\int_x^\infty f(y)\,dy\right)^p x^\alpha \frac{dx}{x} \ge p\beta(p,\alpha)\int_0^\infty \big(xf(x)\big)^p x^\alpha \frac{dx}{x},$$

where $f(x)$ is a nonnegative and nonincreasing function. The inequality holds in the reversed direction when $0 < p \le 1$ and the constant $p\beta(p,\alpha)$ is sharp in both cases. Hence, Theorem 3.5 may be regarded also as a generalization of another result in [3].

4 Applications

By combining Theorem 2.3(a), (b), and (c) with Theorem 3.2, we obtain the following sharp two-sided estimates.

Theorem 4.1 . *Let* $p > 0$, $0 < \alpha < p$, $0 < \ell \le \infty$ *and let* f *be a measurable, nonnegative, and nonincreasing function on* $(0,\ell)$.

If $p > 1$, *then*

$$\left(\frac{p}{\alpha}\right)^{1/p} I_1 \le I_2 \le \frac{p}{\alpha} I_1, \tag{4.1}$$

where

$$I_1 = \left(\int_0^\ell \big(xf(x)\big)^p x^{-\alpha} \left(1 - \left(\frac{x}{\ell}\right)^\alpha \right) \frac{dx}{x} \right)^{1/p}$$

and

$$I_2 = \int_0^\ell \left(\int_0^x f(y)\,dy \right)^p x^{-\alpha} \frac{dx}{x}.$$

If $0 < p \le 1$, *then* (4.1) *holds in the reversed direction. Moreover, both constants* $(p/\alpha)^{1/p}$ *and* p/α *are sharp for all* $p > 0$.

Remark 4.2 (a) This means that the equivalence $I_2 \approx I_1$ holds and the corresponding "optimal target function "is $g(x) = 1 - (\frac{x}{\ell})^\alpha$.

(b) In the lower inequality we can even have equality, while in the above inequality the sharpness follows by choosing a sequence of nonincreasing functions (a well-known fact from the theory of Hardy-type inequalities).

Remark 4.3 Many crucial objects in different mathematical areas are nondecreasing (e.g. in Lorentz spaces, interpolation theory, approximation theory, and harmonic analysis). Hence, in particular, Theorem 4.1 can be useful to obtain some more precise versions of known results in each of these areas. We illustrate this fact only in the theory of Lorentz spaces but aim to later also use our result to improve some results in the modern harmonic analysis as presented in the new book [13].

Let f^* denote the nonincreasing rearrangement of a function f on a measure space (Ω, μ). The Lorentz spaces $L^{p,q}$, $0 < p, q < \infty$ are defined by using the quasi-norm (norm when $p > 1$, $q \ge 1$)

$$\|f\|_{p,q}^* := \left(\int_0^\infty \big(f^*(t) t^{1/p}\big)^q \frac{dt}{t} \right)^{1/q}. \tag{4.2}$$

It is well known that for the case $p > 1$ this quasi-norm is equivalent to the following one equipped with the usual Hardy operator:

$$\|f\|_{p,q}^{**} := \left(\int_0^\infty \left(\int_0^t f^*(u)\,du \right)^q t^{-q/p'} \frac{dt}{t} \right)^{1/q}.$$

Moreover, we have the following more precise estimates:

$$\big(p'\big)^{1/q} \|f\|_{p,q}^* \le \|f\|_{p,q}^{**} \le p' \|f\|_{p,q}^* \tag{4.3}$$

if $q > 1$, and the reversed inequalities hold if $0 < q \le 1$. However, by using Theorem 4.1, we not only get the sharp estimates in (4.3) but also the following more precise statement.

Corollary 4.4 *With the notations and assumptions above, $p > 1$ and $0 < \ell \le \infty$, we have that*

$$\left(p'\right)^{1/q} I_\ell^* \le I_\ell^{**} \le p' I_\ell^*, \tag{4.4}$$

where $q > 1$,

$$I_\ell^* := \left(\int_0^\ell \left(f^*(t) t^{1/p}\right)^q \left(1 - \left(\frac{t}{\ell}\right)^{q/p'}\right) \frac{dt}{t} \right)^{1/q}$$

and

$$I_\ell^{**} := \left(\int_0^\ell \left(\int_0^t f^*(u)\, du \right)^q t^{-q/p'} \frac{dt}{t} \right)^{1/q}.$$

If $0 < q \le 1$, then the inequalities in (4.4) hold in the reversed directions. Both constants $(p')^{1/q}$ and p' are sharp for all $q > 0$.

Proof Just apply Theorem 4.1 with p replaced by q and α replaced by q/p'. □

Remark 4.5 Note that (4.3) is obtained by just using (4.4) with $l = \infty$, so in particular, both constants in (4.3) (and the reversed inequalities for $0 \le q \le 1$) are sharp.

Remark 4.6 For the case $0 < p \le 1$, it is known that the quasi-norm $\|f\|_{p,q}^*$ is equivalent to the following quasi-norm $\|f\|_{p,q}^{**}$ equipped with the dual Hardy operator:

$$\|f\|_{p,q}^{**} := \left(\int_0^\infty \left(\int_t^\infty f^*(u)\, du \right)^q t^{-q/p'} \frac{dt}{t} \right)^{1/q}.$$

By instead using Theorem 2.3(c), (d), and (e) with $l = 0$ combined with Example 3.6, we obtain that if $0 < p \le 1$, then

$$\left(q\beta\left(q, -q/p'\right)\right)^{1/q} \|f\|_{p,q}^* \le \|f\|_{p,q}^{**} \le -p' \|f\|_{p,q}^* \tag{4.5}$$

if $q \ge 1$ and the reversed inequalities hold if $0 < q \le 1$. Both constants $(q\beta(q, -q/p'))^{1/q}$ and $-p'$ are sharp for all $q > 0$.

Remark 4.7 A more general statement like that in Corollary 4.4 involving sharp constants in both inequalities can be formulated, where the integrals $\int_0^\infty$ are replaced by the integrals $\int_\ell^\infty$, $0 \le \ell < \infty$. In particular, this gives a similar generalization of (4.5). However, in this case the result looks less nice since the two target functions $1 - (\frac{x}{\ell})^\alpha$ and $\alpha \beta_{\frac{\ell}{x}}(p, \alpha)$ do not coincide.

We only give the following final example related to Remark 4.7 and the well-known inequality: If $0 < p < 1$, then

$$\int_0^\infty \left(\frac{1}{x} \int_x^\infty f(y)\, dy \right)^p dy \le \frac{\pi p}{\sin \pi p} \int_0^\infty f^p(x)\, dx \tag{4.6}$$

for all functions as defined in Theorem 3.5.

Example 4.8 Let $0 < p < 1$ and let f be a measurable, nonnegative, and nonincreasing function on $(\ell, \infty), 0 \leq l < \infty$. If $0 < p \leq 1$, then

$$\int_\ell^\infty \left(\frac{1}{x}\int_0^\infty f(y)\,dy\right)^p dx \leq p \int_\ell^\infty f^p(x)\beta_{\frac{\ell}{x}}(p, 1-p)\,dx,$$

and the constant p is sharp. This is just Theorem 3.5(b) with $\alpha = 1 - p$.

In particular, for $l = \infty$ this inequality coincides with (4.6) since

$$\beta(p, 1-p) = \pi / \sin \pi p,$$

so the constant $\frac{\pi p}{\sin \pi p}$ in (4.6) is sharp.

5 Some further results and final remarks

First we remark that e.g. Hardy's inequality (2.4) has no meaning in the limit case $\alpha = 0$. However, by restricting to the interval $(0, 1)$ and involving some suitable logarithms, Bennett in 1973 succeeded to prove such an inequality when he developed his well-known theory for real interpolation between the (fairly close) spaces L and $LLogL$ on $(0, 1)$, see [2] and cf. also [3]. This result has been generalized by other authors, but the so far most precise results were derived in [1]. Here we state a little more general form of this result in our dx/x terminology and with the interval $(0, 1)$ replaced by $(0, \ell), 0 < \ell < \infty$.

Theorem 5.1 *Let $\alpha > 0, p \geq 1$, and f be a nonnegative and measurable function on $(0, \ell), 0 < \ell < \infty$. Then*

$$\begin{aligned} &\alpha^{p-1}\left(\int_0^\ell f(x)\,dx\right)^p + \alpha^p \int_0^\ell \left[\log\left(\frac{\ell}{x}\right)\right]^{\alpha p - 1}\left(\int_0^x f(y)\,dy\right)^p \frac{dx}{x} \\ &\quad \leq \int_0^\ell x^p \left[\log\left(\frac{\ell}{x}\right)\right]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}. \end{aligned} \tag{5.1}$$

Both constants α^{p-1} and α^p in (5.1) *are sharp. For the case $p = 1$, we have even equality in* (5.1).

Proof The proof can be done by just modifying step by step the convexity arguments in the proof in [1] for the case $l = 1$. Hence, we omit the details. □

Remark 5.2 (5.1) is one of the few inequalities we know containing two constants, both of which are sharp. In the original paper [2] only the case $p > 1$ was considered and with one constant involved (the first term in (5.1) was missed), and the sharpness was not discussed at all.

The corresponding result for the case $0 < p \leq 1$ is the following.

Theorem 5.3 *Let $0 < p \leq 1$, $\alpha > 0$, and f be a nonnegative and measurable function on $(0, \ell), 0 < \ell < \infty$. Then* (5.1) *holds in the reversed direction.*

Proof The proof is exactly the same as that of Theorem 5.1, the only difference is that here we use the corresponding concavity arguments, so we leave out the details. □

By using Theorems 5.1 and 5.3 with $f(x) = g(1/x)x^{-2}$ and making obvious variable transformations and changes in the notations, we also get the following "dual" version.

Theorem 5.4 *Let* $\alpha, p > 0$ *and* f *be a nonnegative and measurable function on* $(0,\ell), 0 < \ell < \infty$.

(*a*) *If* $p > 1$, *then*

$$\alpha^{p-1}\left(\int_\ell^\infty f(x)\,dx\right)^p + \alpha^p \int_\ell^\infty \left[\log(xe/\ell)\right]^{\alpha p-1}\left(\int_x^\infty f(y)\,dy\right)^p \frac{dx}{x} \le \int_\ell^\infty x^p\left[\log(xe/\ell)\right]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}. \tag{5.2}$$

Both constants α^{p-1} *and* α^p *in* (5.2) *are sharp.*

(*b*) *If* $0 < p \le 1$, *then* (5.2) *holds in the reversed direction, and also here both constants* α^{p-1} *and* α^p *are sharp.*

Next we pronounce that all sharp inequalities we presented so far are for the case $q = p$. Very little concerning sharp constants is known for other cases. Let us illustrate this problem by mentioning the fact that by applying the general theory in Hardy-type inequalities (see e.g. the book [9]) in a power weighted case, we get in our dx/x frame the following.

Example 5.5 The inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)\,dt\right)^q x^{-\alpha}\frac{dx}{x}\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (xf(x))^p x^{-\beta}\frac{dx}{x}\right)^{\frac{1}{p}} \tag{5.3}$$

holds for some finite constant $C > 0$ for $1 < p \le q < \infty$ if and only if

$$\beta > 0 \quad \text{and} \quad \frac{\alpha}{q} = \frac{\beta}{p}. \tag{5.4}$$

Remark 5.6 For the case $p = q$, we have already pointed out the sharp constant, but for the case $1 < p < q < \infty$ this has been a fairy long lasted open question since Bliss in 1930 solved it for $\beta = p - 1$ (see [4]). It was finally solved in 2015 in the paper [12] and in our dx/x frame their result reads as follows.

Theorem 5.7 *Let* $1 < p < q < \infty$ *and the parameters* α *and* β *satisfy* (5.4). *Then the sharp constant in* (5.3) *is* $C = C^*_{pq}$, *where*

$$C^*_{pq} = \left(\frac{p-1}{\beta}\right)^{\frac{1}{p'}+\frac{1}{q}}\left(\frac{p'}{q}\right)^{\frac{1}{p}}\left(\frac{\frac{q-p}{p}\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{p}{q-p})\Gamma(\frac{p(q-1)}{q-p})}\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{5.5}$$

Remark 5.8 Some straightforward calculations show that

$$C^*_{pq} \to \frac{p}{\beta} \quad \text{as } q \to p,$$

so indeed we have the expected continuity in the sharp constants as $q \to p$.

In the dual situation we have the following.

Example 5.9 The inequality

$$\left(\int_0^\infty \left(\int_x^\infty f(t)\,dt\right)^q x^\alpha \frac{dx}{x}\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty (xf(x))^p x^\beta \frac{dx}{x}\right)^{\frac{1}{p}} \tag{5.6}$$

holds for $1 < p \leq q < \infty$ and some finite constant $C > 0$ if and only if

$$\beta > 0 \quad \text{and} \quad \frac{\alpha}{q} = \frac{\beta}{p},$$

and the sharp constant is known also in this case (see [12]).

Remark 5.10 All cases when we have equality in (5.3) with $C = C^\star_{pq}$ defined by (5.5) and when we have equality in (5.6) are also known (see again [12]). Hence, it seems to be an interesting open question to derive the corresponding sharp results when the integrals $\int_0^\infty$ are replaced by $\int_0^\ell, 0 < \ell \leq \infty$ or $\int_\ell^\infty, 0 \leq \ell < \infty$, respectively. We aim to investigate this in a forthcoming paper. We use this opportunity to note a misprint in [12]. The condition $\frac{n+\alpha}{p} = \frac{n+\beta}{q}$ in Theorems 4.1 and 4.2 in [12] should be replaced by $\frac{n+\alpha}{q} = \frac{n+\beta}{p}$.

By using the same transformations as those pointed out just before Theorem 5.4, we can transform inequalities involving integrals $\int_0^\ell$ to inequalities involving the integrals $\int_\ell^\infty$. Let us just as one example of this fact restate Theorem 3.2 in this way.

Theorem 5.11 *Let $p > 0, 0 < \alpha < p$ and let $f(x)x^2$ be a measurable, nonnegative, and nondecreasing function on $(0,\ell), 0 \leq \ell < \infty$.*

(*a*) *If $p \geq 1$, then*

$$\left(\int_\ell^\infty \left(\int_x^\infty f(y)\,dy\right)^p x^\alpha \frac{dx}{x}\right)^{\frac{1}{p}} \geq \frac{p}{\alpha}\left(\int_\ell^\infty (xf(x))^p x^\alpha \left(1 - \left(\frac{\ell}{x}\right)^\alpha\right) \frac{dx}{x}\right)^{\frac{1}{p}}. \tag{5.7}$$

(*b*) *If $0 < p \leq 1$, then* (5.7) *holds in the reversed direction.*

(*c*) *The constant p/α is sharp in both* (*a*) *and* (*b*), *and equality appears for any $f(x) = Ax^{-2}\chi_{(c,\infty)}(x)$ for some $c \in (\ell,\infty), A > 0$.*

Remark 5.12 The function $f(x)$ in Theorem 5.11 is an example of a so called quasi-monotone function, which means that $f(x)x^\alpha$ is nonincreasing or nondecreasing for some $\alpha \in R$. It is another interesting open question to investigate all our results concerning monotone functions for such more general quasi-monotone functions. Even in the case with infinite intervals some interesting phenomena appear. See [3] and the references therein for a special case.

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References

1. Barza, S., Persson, L.-E., Samko, N.: Some new sharp limit Hardy-type inequalities via convexity. J. Inequal. Appl. **6**, 11 (2014)
2. Bennett, C.: Intermediate spaces and the class L $\log^+$ L. Ark. Mat. **11**, 215–228 (1973)
3. Bergh, J., Burenkov, V., Persson, L.-E.: Best constants in reversed Hardy's inequalities for quasimonotone functions. Acta Sci. Math. **59**, 221–239 (1994)
4. Bliss, C.A.: An integral inequality. J. Lond. Math. Soc. **5**, 40–46 (1930)
5. Hardy, G.H.: Notes on some points in the integral calculus, LX. Messenger Math. **54**, 150–156 (1925)
6. Hardy, G.H.: Notes on some points in the integral calculus, LXIV. Further inequalities between integrals. Messenger Math. **57**, 12–16 (1928)
7. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1952)
8. Kokilashvili, V., Meskhi, A., Persson, L.-E.: Weighted Norm Inequalities for Integral Transforms with Product Weights. Nova Scientific Publishers, Inc., New York (2010)
9. Kufner, A., Persson, L.-E., Samko, N.: Weighted Inequalities of Hardy Type, 2nd edn. Word Scientific, New Jersey (2017)
10. Niculescu, C.P., Persson, L.-E.: Convex Functions and Their Applications. A Contemporary Approach, 2nd edn. CMS Books in Mathematics. Springer, Berlin (2018)
11. Persson, L.-E., Samko, N.: What should have happened if Hardy had discovered this? J. Inequal. Appl. **29**, 11 (2012)
12. Persson, L.-E., Samko, S.: A note on the best constants in some Hardy inequalities. J. Math. Inequal. **9**(2), 437–447 (2015)
13. Persson, L.-E., Tephnadze, G., Weisz, F.: Martingale Hardy Spaces and Summability of One-Dimensional Vilenkin–Fourier Series. Book Manuscript. Springer, Berlin (2022)

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