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Numerical calculation of Casimir forces

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Abstract

In this thesis a set of regularized boundary integral equation are introduced that can be used to calculate the Casimir force induced by a two dimensional scalar field. The boundary integral method is compared to the functional integral method and mode summation where possible. Comparisons are done for the case of two parallel plates, two concentric circles and two adjacent circles.

The results indicate that the boundary integral method correctly predicts the geometry dependence of the Casimir force on the test problems, but that its value is missing a factor of two compared to the functional integral method and mode summation. After applying various validation procedures on the numerical implementation including a powerful test based on artificial sources, it is concluded that with high probability the missing factor of two is lost somewhere in the theory leading up to the regularized boundary integral equations.

Introduction

The Casimir effect was first predicted by Casimir and Polder in 1948 [1]. The effect is only measurable on small length scales and is often seen as an attractive force between objects with no charge. The first reported experimental measurement of the Casimir effect was in 1958 by M.J. Sparnaay [2]. He tried to measure the force between two parallel plates. Due to systematic errors his results had 100% uncertainty. It was first in 1997 that S.K. Lamoreaux [3] completed the first successful measurement of the Casimir force, between a plate and a sphere, with only 5% uncertainty. In 1998 U. Mohideen and Anushree Roy [4] measured the Casimir force between a plate and a sphere with only 1% uncertainty.

Both the Casimir force and the van der Waals forces are quantum effects that can cause attraction between neutral bodies. The van der Waals force can induce a dipole moment between two nonpolar molecules and at short distances ($<$10nm) cause attractive forces. The primary difference between these two theories is that van der Waals forces are non-relativistic in nature. The van der Waals forces disappear at larger separations (100nm) where relativistic effects must be considered. At these separations the Casimir effect dominates the van der Waals forces and are the primary source of attraction or repulsion.[5; 6]

As nanotechnology finds more and more applications the need to understand physics at the micro- or nano- scales will increase. Microelectromechanical systems (MEMS) are small electrical systems that can function as actuators, sensors or routers. Examples of these are: Accelerometers and gyroscopes in cars or smartphones. At these small scales the Casimir force can cause components to stick together and be a hindrance, or it could provide new functionality such as produceing levitation under certain conditions. [7; 8]

There are several methods that can be used to calculate the Casimir energy, a few of these are: Mode summation with the argument theorem [9; 10], Finite Difference Time Domain (FDTD) methods [11] and functional integral methods [12; 13].

The method of mode summation with the argument principles has been very successful in calculating the Casimir effect. Its primary application is on systems with a symmetry such as for two parallel plates or concentric circles/spheres/cylinders. For applications such as designing a MEMS it is impossible to only be restricted to symmetrical designs. Thus there is a need for methods that calculate the Casimir effect for arbitrary configurations.

The FDTD methods are grid based methods that discretize the problem space into a finite grid. The relevant functions are then evaluated on the grid and time is iterated forward. This method can handle arbitrary configurations as long as the equations allow for such a solution method. These types of methods are very popular in areas such as electrodynamics, fluid mechanics, geology and

weather prediction.

Methods based on functional integrals are able to handle arbitrary configurations of objects. This theory is based on Feynmann's idea to integrate over weighted classical paths, the problem is then to solve these infinite dimensional integrals. These can either be solved using some numerical scheme or by functional determinants.

The object of this thesis is to use the boundary integral method to calculate the Casimir force for an arbitrary configuration of objects. This method is most efficient when used on linear equations and boundaries with piecewise linear material coefficients. This is exactly the situation that will be examined in this thesis. This method avoids a lot of unnecessary calculations because the equations are only solved on the boundaries. For multiple objects with varying distances it is possible to ignore the empty space between objects. Methods based on FDTD will not have this option because they must grid the entire problem space. The boundary integral method will in addition regularize the singularities before the method is implemented, thus providing stability to the calculations.

The boundary integral method outputs the force on each discretized piece of every object. This provides valuable visual information on how the forces are affecting each object. It is also possible to simplify the equations if there exists isometric transformations on some objects.

The relationships between force and energy always require us to find the change in energy with respect to a parameter. Thus a minimum of two evaluations of the energy is required to find an estimate for the force. When the problem size is large, the computational time will be considerable, and it would be advantageous to have a direct method for finding the force. The boundary integral method presented in this thesis calculates the Casimir force directly for any compact geometry. Computationally this method is based on filling and solving a set of linear equations and these type of operations scale well on clusters.

Chapter 1 presents the theory behind using the boundary integral method to find the Casimir force on an object. The boundary integral method uses Green's functions to calculate the Casimir force directly. Chapter 2 introduces the functional integral method that will be used to verify the boundary integral method. This method uses the theory of functional integrals to calculate the energy of the system. Chapter 3 gives an algorithmic overview of the two methods and their complexity. In order to validate the boundary integral method the results will be compared to two other methods. For the symmetric situations the method of mode expansion and the argument principle can be used to find a simple formula for the Casimir energy. This is done with parallel plates and concentric circles in chapter 4. Chapter 5 explains how the Casimir force is calculated from the Casimir energy. The Casimir force can be calculated either directly from the boundary integral method or through the Casimir energy with the functional integral method. Chapter 6 introduces the test cases and the test results, these include: parallel plates, concentric circles and adjacent circles. A conclusion is drawn in chapter 7 on the validity of the boundary integral method and all the test results from chapter 6 are summarized in chapter 7. Appendix A and B contains an implementation of the boundary element and functional integral methods to zero dimensional parallel plates on the line. This test case is only included for comparisons. Appendix C contains calculations where an

attempt is made to find the Casimir energy for two parallel plates using the method of mode summation with a similar regularization as was used in section 4.2

Contents

Chapter 1

Boundary integral method

The first step is to produce an equation for the Green's function. The Green's function will then be related to the stress tensor by point splitting and through this the force on an object will be defined. The next step is to use the equation for the Green's function to produce an integral relation for compact objects. The relation must be regularized because of the singularities in the Green's functions. Through a regularized limiting process the integral equations are derived. These equations are solved using the method of moments and the boundaries are discretized in order to calculate the necessary matrix elements. The boundary integral method is implemented and the output will be the stress on each object.

1.1 Green's function

Consider a massless neutral scalar field $\hat{\varphi}$ with field equation

$$\begin{aligned} \hat{\varphi}_{tt} - c^2\nabla^2\hat{\varphi} &= 0 \\ \hat{\varphi}|_{Q_j} &= 0 \end{aligned} \tag{1.1}$$

The equal time bosonic commutation relations are

$$\begin{aligned} [\hat{\varphi}(\mathbf{x},t), \hat{\varphi}(\mathbf{x}',t)] &= 0 \\ [\hat{\varphi}_t(\mathbf{x},t), \hat{\varphi}(\mathbf{x}',t)] &= i\hbar\delta(\mathbf{x}-\mathbf{x}') \end{aligned} \tag{1.2}$$

The generators for the algebra of observables are $\hat{\varphi}(\mathbf{x})$ and the time evolution of the scalar field is given by

$$\hat{\varphi}(\mathbf{x},t) = e^{i\frac{t}{\hbar}\hat{H}}\hat{\varphi}(\mathbf{x})e^{-i\frac{t}{\hbar}\hat{H}} \tag{1.3}$$

where $\hat{H}$ is the Hamiltonian for the system. Extend the field operators into complex time with the rotation $t = -is$. This will change the partial derivative into

$$\partial_t = \partial_s\frac{ds}{dt} = i\partial_s \tag{1.4}$$

and the field equation above changes into

$$\begin{aligned} \hat{\varphi}_{ss} + c^2\nabla^2\hat{\varphi} &= 0 \\ \hat{\varphi}|_{Q_j} &= 0 \end{aligned} \tag{1.5}$$

with the commutation relations.

$$\begin{aligned}[\hat{\varphi}(\mathbf{x},s),\hat{\varphi}(\mathbf{x}',s)] &= 0 \\ [\hat{\varphi}_s(\mathbf{x},s),\hat{\varphi}(\mathbf{x}',s)] &= \delta(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{1.6}$$

Select units such that $\hbar = c = k = 1$, where k is the *Boltzman constant*. The basic Green's function is described by the time ordered product

$$\mathcal{D}(\mathbf{x},s,\mathbf{x}',s') =< \mathrm{T}[\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',s')] > \tag{1.7}$$

where it is assumed that the quantum field is in a state of thermal equilibrium at temperature T. Letting $\beta = 1/T$ results in

$$\mathcal{D}(\mathbf{x},s,\mathbf{x}',s') = \mathrm{Tr}(\frac{1}{Z}e^{-\beta\hat{H}}T[\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',s')]) \tag{1.8}$$

By defining

$$\begin{aligned}\mathcal{D}^+(\mathbf{x},s,\mathbf{x}',s') &=< \hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',s') > \\ \mathcal{D}^-(\mathbf{x},s,\mathbf{x}',s') &=< \hat{\varphi}(\mathbf{x}',s')\hat{\varphi}(\mathbf{x},s) >\end{aligned} \tag{1.9}$$

the Green's function can be reformulated as

$$\mathcal{D}(\mathbf{x},s,\mathbf{x}',s') = \begin{cases} \mathcal{D}^+(\mathbf{x},s,\mathbf{x}',s') & s > s' \\ \mathcal{D}^-(\mathbf{x},s,\mathbf{x}',s') & s < s' \end{cases} \tag{1.10}$$

First observe that the Green's function is periodic in β:

$$\begin{aligned}\mathcal{D}^+(\mathbf{x},s+\beta,\mathbf{x}',s') &= \mathrm{Tr}\left(\frac{1}{Z}e^{-\beta\hat{H}}e^{(s+\beta)\hat{H}}\hat{\varphi}(\mathbf{x})e^{-(s+\beta)\hat{H}}\hat{\varphi}(\mathbf{x}',s')\right) \\ &= \mathrm{Tr}\left(\frac{1}{Z}e^{s\hat{H}}\hat{\varphi}(\mathbf{x})e^{-s\hat{H}}e^{-\beta\hat{H}}\hat{\varphi}(\mathbf{x}',s')\right) \\ &= \mathrm{Tr}\left(\frac{1}{Z}\hat{\varphi}(\mathbf{x},s)e^{-\beta\hat{H}}\hat{\varphi}(\mathbf{x}',s')\right)\end{aligned} \tag{1.11}$$

A property of the trace is that: $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$. Thus the operators can be moved to the right to show that

$$\mathcal{D}^+(\mathbf{x},s+\beta,\mathbf{x}',s') = \mathrm{Tr}\left(\frac{1}{Z}e^{-\beta\hat{H}}\hat{\varphi}(\mathbf{x}',s')\hat{\varphi}(\mathbf{x},s)\right) = \mathcal{D}^-(\mathbf{x},s,\mathbf{x}',s') \tag{1.12}$$

The same argument works for $\mathcal{D}^-(\mathbf{x},s,\mathbf{x}',s'+\beta)$. Thus

$$\begin{aligned}\mathcal{D}^+(\mathbf{x},s+\beta,\mathbf{x}',s') &= \mathcal{D}^-(\mathbf{x},s,\mathbf{x}',s') \\ \mathcal{D}^-(\mathbf{x},s,\mathbf{x}',s'+\beta) &= \mathcal{D}^+(\mathbf{x},s,\mathbf{x}',s')\end{aligned} \tag{1.13}$$

These are the Kubo-Martin-Schwinger (KMS) boundary conditions. Since $\hat{H}$ is independent of s it is possible to show that

$$\begin{aligned}\mathcal{D}^+(\mathbf{x},s,\mathbf{x}',s') &= \mathrm{Tr}\left(\frac{1}{Z}e^{-\beta\hat{H}}\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',s')\right) \\ &= \mathrm{Tr}\left(\frac{1}{Z}e^{-\beta\hat{H}}e^{s\hat{H}}\hat{\varphi}(\mathbf{x})e^{-s\hat{H}}e^{s'\hat{H}}\hat{\varphi}(\mathbf{x}')e^{-s'\hat{H}}\right) \\ &= \mathrm{Tr}\left(\frac{1}{Z}e^{-\beta\hat{H}}e^{(s-s')\hat{H}}\hat{\varphi}(\mathbf{x})e^{-(s-s')\hat{H}}\hat{\varphi}(\mathbf{x}')\right) \\ &= \mathcal{D}^+(\mathbf{x},s-s',\mathbf{x}',0)\end{aligned} \tag{1.14}$$

similarly for $\mathcal{D}^-(\cdots)$

$$\mathcal{D}^-(\mathbf{x}, s, \mathbf{x}', s') = \cdots = \mathcal{D}^-(\mathbf{x}, s - s', \mathbf{x}', 0) \tag{1.15}$$

Introduce a new Green's function based on the above properties

$$\mathcal{D}(\mathbf{x}, \mathbf{x}', s) = \begin{cases} \mathcal{D}^+(\mathbf{x}, s, \mathbf{x}', 0) & s > 0 \\ \mathcal{D}^-(\mathbf{x}, s, \mathbf{x}', 0) & s < 0 \end{cases} \tag{1.16}$$

for the new Green's function

$$\begin{array}{ll} \mathcal{D}(\mathbf{x}, \mathbf{x}', s - s') = \mathcal{D}^+(\mathbf{x}, s - s', \mathbf{x}', 0) = \mathcal{D}^+(\mathbf{x}, s, \mathbf{x}', s') & s > s' \\ \mathcal{D}(\mathbf{x}, \mathbf{x}', s - s') = \mathcal{D}^-(\mathbf{x}, s - s', \mathbf{x}', 0) = \mathcal{D}^-(\mathbf{x}, s, \mathbf{x}', s') & s < s' \end{array} \tag{1.17}$$

and thus

$$\mathcal{D}(\mathbf{x}, \mathbf{x}', s - s') = \mathcal{D}(\mathbf{x}, s, \mathbf{x}', s') \quad \forall s, s' \tag{1.18}$$

Let us explore some properties for the new Green's function. Let $|n>$ be a complete set of eigenstates for $\hat{H}$.

First for $s > 0$

$$\begin{aligned} \mathcal{D}(\mathbf{x}, \mathbf{x}', s) &= \mathcal{D}^+(\mathbf{x}, s, \mathbf{x}', 0) \\ &= \mathrm{Tr}\left(\frac{1}{Z} e^{-\beta\hat{H}} e^{s\hat{H}} \hat{\varphi}(\mathbf{x}) e^{-s\hat{H}} \hat{\varphi}(\mathbf{x}')\right) \\ &= \sum_n < n| \frac{1}{Z} e^{-\beta\hat{H}} e^{s\hat{H}} \hat{\varphi}(\mathbf{x}) e^{-s\hat{H}} \hat{\varphi}(\mathbf{x}') |n> \\ &= \sum_{nn'} \frac{1}{Z} e^{-(\beta-s)E_n} e^{-sE_{n'}} < n|\hat{\varphi}(\mathbf{x})|n' >< n'|\hat{\varphi}(\mathbf{x}')|n > \end{aligned} \tag{1.19}$$

thus $\mathcal{D}(\mathbf{x}, \mathbf{x}', s)$ only exists for $0 \leq s \leq \beta$.

For $s < 0$

$$\begin{aligned} \mathcal{D}(\mathbf{x}, \mathbf{x}', s) &= \mathcal{D}^-(\mathbf{x}, s, \mathbf{x}', 0) \\ &= \mathrm{Tr}\left(\frac{1}{Z} e^{-\beta\hat{H}} \hat{\varphi}(\mathbf{x}') e^{s\hat{H}} \hat{\varphi}(\mathbf{x}) e^{-s\hat{H}}\right) \\ &= \sum_n < n| \frac{1}{Z} e^{-\beta\hat{H}} \hat{\varphi}(\mathbf{x}') e^{s\hat{H}} \hat{\varphi}(\mathbf{x}) e^{-s\hat{H}} |n> \\ &= \sum_{nn'} \frac{1}{Z} e^{-(\beta+s)E_n} e^{sE_{n'}} < n|\hat{\varphi}(\mathbf{x}')|n' >< n'|\hat{\varphi}(\mathbf{x})|n > \end{aligned} \tag{1.20}$$

thus $\mathcal{D}(\mathbf{x}, \mathbf{x}', s)$ only exists for $-\beta \leq s \leq 0$.

It is clear that $\mathcal{D}(\mathbf{x}, \mathbf{x}', s)$ only exists for $s \in [-\beta, \beta]$. With the KMS boundary conditions

$$\mathcal{D}(\mathbf{x}, \mathbf{x}', s + \beta) = \mathcal{D}^+(\mathbf{x}, s + \beta, \mathbf{x}', 0) = \mathcal{D}^-(\mathbf{x}, s, \mathbf{x}', 0) = \mathcal{D}(\mathbf{x}, \mathbf{x}', s) \tag{1.21}$$

Thus $\mathcal{D}(\mathbf{x}, \mathbf{x}', s)$ is determined by its values in the interval $-\beta \leq s \leq 0$. Observe that $\mathcal{D}(\mathbf{x}, \mathbf{x}', s)$ is a Green's function for the operator defining equation (1.5).

First note that

$$\mathcal{D}(\mathbf{x}, \mathbf{x}', s) = \theta(s) < \hat{\varphi}(\mathbf{x}, s)\hat{\varphi}(\mathbf{x}', 0) > + \theta(-s) < \hat{\varphi}(\mathbf{x}', 0)\hat{\varphi}(\mathbf{x}, s) > \tag{1.22}$$

where $\theta(s)$ is the Heavyside step function. Differentiate once with respect to s to get

$$\begin{aligned}\partial_s\mathcal{D}(\mathbf{x},\mathbf{x}',s) &= \delta(s) < \hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',0) > + \theta(s) < \partial_s\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',0) > \\ &\quad - \delta(s) < \hat{\varphi}(\mathbf{x}',0)\hat{\varphi}(\mathbf{x},s) > + \theta(-s) < \hat{\varphi}(\mathbf{x}',0)\partial_s\hat{\varphi}(\mathbf{x},s) > \\ &= \delta(s)[\hat{\varphi}(\mathbf{x},s),\hat{\varphi}(\mathbf{x}',0)] + \theta(s) < \partial_s\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',0) > \\ &\quad + \theta(-s) < \hat{\varphi}(\mathbf{x}',0)\partial_s\hat{\varphi}(\mathbf{x},s) > \\ &= \theta(s) < \partial_s\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',0) > + \theta(-s) < \hat{\varphi}(\mathbf{x}',0)\partial_s\hat{\varphi}(\mathbf{x},s) >\end{aligned} \tag{1.23}$$

Where the commutation relations for bosons in (1.6) were applied. With a second partial derivative this becomes

$$\begin{aligned}\partial_{ss}\mathcal{D}(\mathbf{x},\mathbf{x}',s) &= \delta(s) < \partial_s\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',0) > \\ &\quad + \theta(s) < \partial_{ss}\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',0) > - \delta(s) < \hat{\varphi}(\mathbf{x}',0)\partial_s\hat{\varphi}(\mathbf{x},s) > \\ &\quad + \theta(-s) < \hat{\varphi}(\mathbf{x}',0)\partial_{ss}\hat{\varphi}(\mathbf{x},s) > \\ &= \delta(s)[\partial_s\hat{\varphi}(\mathbf{x},s),\hat{\varphi}(\mathbf{x}',0)] + \theta(s) < (-\nabla^2_{\mathbf{x}})\hat{\varphi}(\mathbf{x},s)\hat{\varphi}(\mathbf{x}',0) > \\ &\quad + \theta(-s) < \hat{\varphi}(\mathbf{x}',0)(-\nabla^2_{\mathbf{x}})\hat{\varphi}(\mathbf{x},s) > \\ &= \delta(s)\delta(\mathbf{x}-\mathbf{x}') - \nabla^2_{\mathbf{x}}\mathcal{D}(\mathbf{x},\mathbf{x}',s)\end{aligned} \tag{1.24}$$

Where the defining equation in (1.5) and the commutation relations from (1.6) were used. The Green's function satisfies the equation

$$\partial_{ss}\mathcal{D}(\mathbf{x},\mathbf{x}',s) + \nabla^2_{\mathbf{x}}\mathcal{D}(\mathbf{x},\mathbf{x}',s) = \delta(s)\delta(\mathbf{x}-\mathbf{x}') \tag{1.25}$$

This equation is valid for any temperature T and, the Green's function $\mathcal{D}(\mathbf{x},\mathbf{x}')$, is periodic in s with period $\beta = 1/T$. Thus $\mathcal{D}(\mathbf{x},\mathbf{x}',s)$ can be written as a Fourier series where the frequencies are called *Matsubara* frequencies. However the problems of this thesis will only consider the case where $T \to 0$. Thus the period of the Fourier series will be infinite and a Fourier transform in s will produce

$$\begin{aligned}\nabla^2_{\mathbf{x}}\mathcal{D}(\mathbf{x},\mathbf{x}',\omega) - \omega^2\mathcal{D}(\mathbf{x},\mathbf{x}',\omega) &= \delta(\mathbf{x}-\mathbf{x}') \\ \mathcal{D}(\mathbf{x},\mathbf{x}',\omega)|_{Q_j} &= 0\end{aligned} \tag{1.26}$$

1.2 Force

The *Lagrangian* for the classical wave equation $\varphi_{tt} - \nabla^2\varphi = 0$ is given by

$$\mathcal{L} = \frac{1}{2}\varphi_t^2 - \frac{1}{2}(\varphi_x^2 + \varphi_y^2) \tag{1.27}$$

The stress-energy tensor is calculated from

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L} \tag{1.28}$$

using a signature $\eta^{\mu\nu} = \{1,-1,-1\}$ will give

$$\begin{aligned} T^{00} &= \frac{1}{2}\varphi_t^2 + \frac{1}{2}(\varphi_x^2+\varphi_y^2) & T^{02} &= \varphi_t\varphi_y \\ T^{11} &= -\frac{1}{2}\varphi_t^2 + \frac{1}{2}(-\varphi_x^2+\varphi_y^2) & T^{10} &= -\varphi_x\varphi_t \\ T^{22} &= -\frac{1}{2}\varphi_t^2 + \frac{1}{2}(\varphi_x^2-\varphi_y^2) & T^{12} &= -\varphi_x\varphi_y \\ T^{01} &= \varphi_t\varphi_x & T^{20} &= -\varphi_y\varphi_t \\ & & T^{21} &= -\varphi_y\varphi_x \end{aligned} \tag{1.29}$$

From this the conservation equations are given by

$$\partial_t T^{0\nu} + \partial_x T^{1\nu} + \partial_y T^{2\nu} = 0 \tag{1.30}$$

where $\mu = 0$ gives energy conservation and $\mu = 1, 2$ gives momentum conservation. The equation for energy conservation is

$$\partial_t(\frac{1}{2}\varphi_t^2 + \frac{1}{2}(\varphi_x^2 + \varphi_y^2)) + \partial_x(-\varphi_x\varphi_t) + \partial_y(-\varphi_y\varphi_t) = 0 \tag{1.31}$$

or

$$\partial_t \rho_e + \nabla \cdot S_e = 0 \tag{1.32}$$

where

$$S_e = -\varphi_t \nabla\varphi \tag{1.33}$$

is the energy flux tensor and

$$\rho_e = \frac{1}{2}\varphi_t^2 \mathrm{I} + \frac{1}{2}\mathrm{Tr}(\nabla\varphi\nabla\varphi)\mathrm{I} \tag{1.34}$$

is the energy density. The equations of momentum conservation are

$$\begin{aligned} \partial_t(\varphi_t\varphi_x) + \partial_x(-\frac{1}{2}\varphi_t^2 + \frac{1}{2}(-\varphi_x^2 + \varphi_y^2)) + \partial_y(-\varphi_y\varphi_x) = 0 \\ \partial_t(\varphi_t\varphi_y) + \partial_x(-\varphi_x\varphi_y) + \partial_y(-\frac{1}{2}\varphi_t^2 + \frac{1}{2}(\varphi_x^2 - \varphi_y^2)) = 0 \end{aligned} \tag{1.35}$$

or

$$\partial_t \rho + \nabla \cdot S = 0 \tag{1.36}$$

where ρ is the momentum density given by

$$\rho = \varphi_t \nabla\varphi \tag{1.37}$$

and S is the momentum flux. The momentum flux can be written as follows

$$S(\mathbf{x}, t) = -\nabla\varphi\nabla\varphi + \frac{1}{2}\mathrm{Tr}(\nabla\varphi\nabla\varphi)\mathrm{I} - \frac{1}{2}\varphi_t^2\mathrm{I} \tag{1.38}$$

where the dyadic product of vectors has been used to simplify the notation. The quantum stress tensor is defined by point splitting

$$\hat{S}(\mathbf{x}, t) = \lim_{\substack{\mathbf{x}' \to \mathbf{x} \\ t' \to t}} \left(-\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'} + \frac{1}{2}\mathrm{Tr}(\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'})\mathrm{I} - \frac{1}{2}\partial_t\partial_{t'}\mathrm{I} \right) \hat{\varphi}(\mathbf{x}, t)\hat{\varphi}(\mathbf{x}', t') \tag{1.39}$$

and find the expectation value of the ordered product to get

$$S_q(\mathbf{x}, t) = \lim_{\substack{\mathbf{x}' \to \mathbf{x} \\ t' \to t}} \left(-\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'} + \frac{1}{2}\mathrm{Tr}(\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'})\mathrm{I} - \frac{1}{2}\partial_t\partial_{t'}\mathrm{I} \right) \mathcal{D}(\mathbf{x}, t, \mathbf{x}', t') \tag{1.40}$$

Where $\mathcal{D}(\mathbf{x}, t, \mathbf{x}', t')$ is the Green's function defined earlier. Using the definitions in the previous section: $t = -iu$, $t' = -iu'$ and $s = u - u'$ this changes S_q into

$$S_q(\mathbf{x}) = \lim_{\substack{\mathbf{x}' \to \mathbf{x} \\ s \to 0}} \left(-\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'} + \frac{1}{2}\mathrm{Tr}(\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'})\mathrm{I} - \frac{1}{2}\partial_{ss}\mathrm{I} \right) \mathcal{D}(\mathbf{x}, \mathbf{x}', s) \tag{1.41}$$

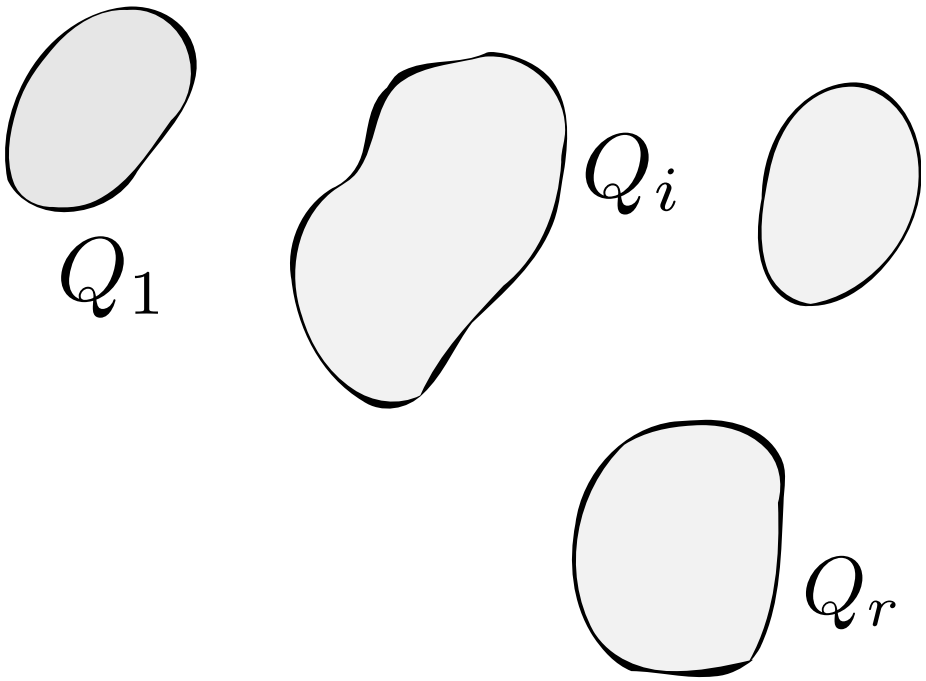

Figure 1.1: Illustration of the objects with shaded interiors and marked boundaries Q_i.

The Fourier transform in time results in

$$S_q(\mathbf{x},\omega) = \lim_{\mathbf{x}'\to\mathbf{x}} \left(-\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'} + \frac{1}{2}\mathrm{Tr}(\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'})\mathrm{I} + \frac{1}{2}\omega\mathrm{I} \right) \mathcal{D}(\mathbf{x},\mathbf{x}',\omega) \tag{1.42}$$

and the quantum stress tensor is given by the Fourier transform evaluated at zero

$$S_q(\mathbf{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_q(\mathbf{x},\omega) \tag{1.43}$$

Figure 1.1 illustrates the situation and for object i with volume V_i and boundary Q_i the net force from this object is

$$\begin{aligned} \mathbf{F}_i = \frac{\partial P}{\partial t} = \partial_t \int_{V_i} dV\, \rho(\mathbf{x},t) = -\int_{V_i} dV\, \nabla\cdot S_q(\mathbf{x},t) \\ = -\oint_{Q_i} dl\, S_q\cdot\mathbf{n} \end{aligned} \tag{1.44}$$

where $\mathbf{n}$ is a normal vector pointing out of Q_i and into V_0. Because the total system is stationary the sum of all forces is zero: $\sum_j \mathbf{F}_j = 0$

The unit normal and tangent vectors, $\mathbf{n}$ and $\mathbf{t}$, span $\mathbb{R}^2$ at any point along the curves Q_i. With respect to this basis the unit vector are

$$\begin{aligned} \mathbf{e}_x &= (\mathbf{e}_x\cdot\mathbf{t})\mathbf{t} + (\mathbf{e}_x\cdot\mathbf{n})\mathbf{n} \\ \mathbf{e}_y &= (\mathbf{e}_y\cdot\mathbf{t})\mathbf{t} + (\mathbf{e}_y\cdot\mathbf{n})\mathbf{n} \end{aligned} \tag{1.45}$$

The gradient changes into

$$\nabla_{\mathbf{x}} \to (\mathbf{t}\cdot\nabla_{\mathbf{x}})\mathbf{t} + (\mathbf{n}\cdot\nabla_{\mathbf{x}})\mathbf{n} = \mathbf{t}\partial_{\mathbf{t}} + \mathbf{n}\partial_{\mathbf{n}} \tag{1.46}$$

and the double gradient is given by

$$\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'} = \mathbf{t}\mathbf{t}'\partial_{\mathbf{t}\mathbf{t}'} + \mathbf{t}\mathbf{n}'\partial_{\mathbf{t}\mathbf{n}'} + \mathbf{t}'\mathbf{n}\partial_{\mathbf{t}'\mathbf{n}} + \mathbf{n}\mathbf{n}'\partial_{\mathbf{n}\mathbf{n}'} \tag{1.47}$$

Because $\mathcal{D}(\mathbf{x},\mathbf{x}',\omega) = 0$ when $\mathbf{x},\mathbf{x}' \in Q_j$ the tangential derivatives are

$$\partial_{\mathbf{t}}\mathcal{D}|_{Q_j} = \partial_{\mathbf{t}'}\mathcal{D}|_{Q_j} = 0 \tag{1.48}$$

Thus for $\mathbf{x},\mathbf{x}' \in Q_j$

$$\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'}\mathcal{D}(\mathbf{x},\mathbf{x}',\omega) \to \mathbf{n}\mathbf{n}'\partial_{\mathbf{n}\mathbf{n}'}\mathcal{D}(\mathbf{x},\mathbf{x}',\omega) \tag{1.49}$$

The stress tensor defined in equation (1.42) will for points on Q_j be

$$S_q(\mathbf{x},\omega) = \lim_{\mathbf{x}'\to\mathbf{x}} \left(\frac{1}{2}\mathrm{Tr}(\mathbf{n}\mathbf{n}')\mathrm{I} - \mathbf{n}\mathbf{n}'\right)\partial_{\mathbf{n}\mathbf{n}'}\mathcal{D}(\mathbf{x},\mathbf{x}',\omega) \tag{1.50}$$

and the force will be given by

$$\begin{aligned}
\mathbf{F}_i &= -\oint_{Q_i} dl\, S_q\cdot\mathbf{n} = -\frac{1}{2\pi}\oint_{Q_i} dl_{\mathbf{x}} \int_{-\infty}^{\infty} d\omega\, S_q(\mathbf{x},\omega)\cdot\mathbf{n} \\
&= -\frac{1}{2\pi}\oint_{Q_i} dl_{\mathbf{x}} \int_{-\infty}^{\infty} d\omega \lim_{\mathbf{x}'\to\mathbf{x}} \left(\frac{1}{2}\mathrm{Tr}(\mathbf{n}\mathbf{n}')\mathrm{I} - \mathbf{n}\mathbf{n}'\right)\partial_{\mathbf{n}\mathbf{n}'}\mathcal{D}(\mathbf{x},\mathbf{x}',\omega)\cdot\mathbf{n} \\
&= -\frac{1}{2\pi}\oint_{Q_i} dl_{\mathbf{x}} \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2}\mathrm{Tr}(\mathbf{n}\mathbf{n})\mathrm{I} - \mathrm{I}\right)\mathbf{n}\cdot\partial_{\mathbf{n}\mathbf{n}}\mathcal{D}(\mathbf{x},\mathbf{x},\omega) \\
&= \frac{1}{4\pi}\oint_{Q_i} dl_{\mathbf{x}} \int_{-\infty}^{\infty} d\omega\, \mathbf{n}\cdot\partial_{\mathbf{n}\mathbf{n}}\mathcal{D}(\mathbf{x},\mathbf{x},\omega)
\end{aligned} \tag{1.51}$$

Where $\mathrm{Tr}(\mathbf{n}\mathbf{n}) = 1$ for unit normals. The force on each object from the vacuum is thus given by

$$\begin{aligned}
\mathbf{F}_i &= \oint_{Q_i} dl_{\mathbf{x}}\, \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega\, \mathbf{n}(\mathbf{x})\cdot\partial_{\mathbf{n}\mathbf{n}'}\mathcal{D}(\mathbf{x},\mathbf{x},\omega) \\
&= \oint_{Q_i} dl_{\mathbf{x}}\, \mathbf{n}(\mathbf{x})\cdot\mathbf{p}(\mathbf{x})
\end{aligned} \tag{1.52}$$

Where the normals $\mathbf{n}$ are directed out of each compact object.

1.3 Regularized boundary integral equations

Finding the force $\mathbf{F}_i$ has now been reduced to finding the double normal derivative of the Green's function $\mathcal{D}(\mathbf{x},\mathbf{x},\omega)$. Take the gradient of equation (1.26) with respect to the primed variable in order to find an equation for this quantity.

$$\nabla^2_{\mathbf{x}}\mathcal{E}(\mathbf{x},\mathbf{x}',\omega) - \omega^2\mathcal{E}(\mathbf{x},\mathbf{x}',\omega) = \nabla_{\mathbf{x}'}\delta(\mathbf{x}-\mathbf{x}') \tag{1.53}$$

where $\mathcal{E}(\mathbf{x},\mathbf{x}',\omega) = \nabla_{\mathbf{x}'}\mathcal{D}(\mathbf{x},\mathbf{x}',\omega)$. This equation has the same boundary conditions as equation (1.26), that is

$$\mathcal{E}(\mathbf{x},\mathbf{x}',\omega) = 0 \quad \text{when} \quad \mathbf{x} \in Q_i \tag{1.54}$$

Consider the operator $\mathcal{L}$ given by

$$\mathcal{L} = \nabla^2 - \omega^2 \tag{1.55}$$

The equation for our free Green's function is

$$\mathcal{L}D_0(\mathbf{x},\mathbf{x}'',\omega) = \delta(\mathbf{x}-\mathbf{x}'') \tag{1.56}$$

A Green's function that satisfies this is equation is given by

$$D_0(\mathbf{x},\mathbf{x}'',\omega) = -\frac{1}{2\pi}K_0(\omega||\mathbf{x}-\mathbf{x}''||) \tag{1.57}$$

Where K_0 is a modified Bessel function.

The *Divergence theorem* and *Green's second identity* are required to produce the integral formulation of the boundary value problem 1.53 and 1.54

$$\begin{aligned}\int_{V_0} dV\,(D_0\mathcal{L}\mathcal{E} - \mathcal{E}\mathcal{L}D_0) &= \int_{V_0} dV\,(D_0\nabla^2\mathcal{E} - \mathcal{E}\nabla^2 D_0)\\ &\vdots\\ &= -\sum_\alpha \oint_{Q_\alpha} ds\,(D_0\nabla\mathcal{E} - \mathcal{E}\nabla D_0)\cdot\mathbf{n}\end{aligned} \tag{1.58}$$

The normal vector $\mathbf{n}$ should point out of each compact object Q_α and into V_0. Inserting equations (1.53) and (1.56) into the above relation gives

$$\int_{V_0} dV\,(D_0\nabla_{\mathbf{x}'}\delta(\mathbf{x}-\mathbf{x}') - \mathcal{E}\delta(\mathbf{x}-\mathbf{x}'')) = -\sum_\alpha \oint_{Q_\alpha} ds\,(D_0\nabla\mathcal{E} - \mathcal{E}\nabla D_0)\cdot\mathbf{n} \tag{1.59}$$

Notice that $\nabla_{\mathbf{x}'}$ is independent of the integration variable $\mathbf{x}$, this can then be extracted from the integral and basic delta function identities gives

$$\nabla_{\mathbf{x}'}D_0(\mathbf{x}',\mathbf{x}'') - \mathcal{E}(\mathbf{x}'',\mathbf{x}') = -\sum_\alpha \oint_{Q_\alpha} ds\,(D_0\nabla\mathcal{E} - \mathcal{E}\nabla D_0)\cdot\mathbf{n} \tag{1.60}$$

Because of the boundary condition $\mathcal{E}(\mathbf{x},\mathbf{x}') = 0$ when $\mathbf{x} \in Q_\alpha$ the integral will be simplified into

$$\nabla_{\mathbf{x}'}D_0(\mathbf{x}',\mathbf{x}'') - \mathcal{E}(\mathbf{x}'',\mathbf{x}') = -\sum_\alpha \oint_{Q_\alpha} ds\,D_0(\mathbf{x},\mathbf{x}'')\nabla_{\mathbf{x}}\mathcal{E}(\mathbf{x},\mathbf{x}')\cdot\mathbf{n} \tag{1.61}$$

This integral relation is valid for any $\mathbf{x}', \mathbf{x}'' \in V_0$.

Take the limit of the above relation when $\mathbf{x}'$ and $\mathbf{x}''$ approach the boundaries Q_j. How this limit is performed is the first part of our regularization. Start by letting $\mathbf{x}'' \to Q_i$. Because of the boundary conditions equation (1.61) turns into

$$\nabla_{\mathbf{x}'}D_0(\mathbf{x}',\mathbf{x}'') = -\sum_\alpha \oint_{Q_\alpha} ds\,D_0(\mathbf{x},\mathbf{x}'')\nabla_{\mathbf{x}}\mathcal{E}(\mathbf{x},\mathbf{x}')\cdot\mathbf{n} \tag{1.62}$$

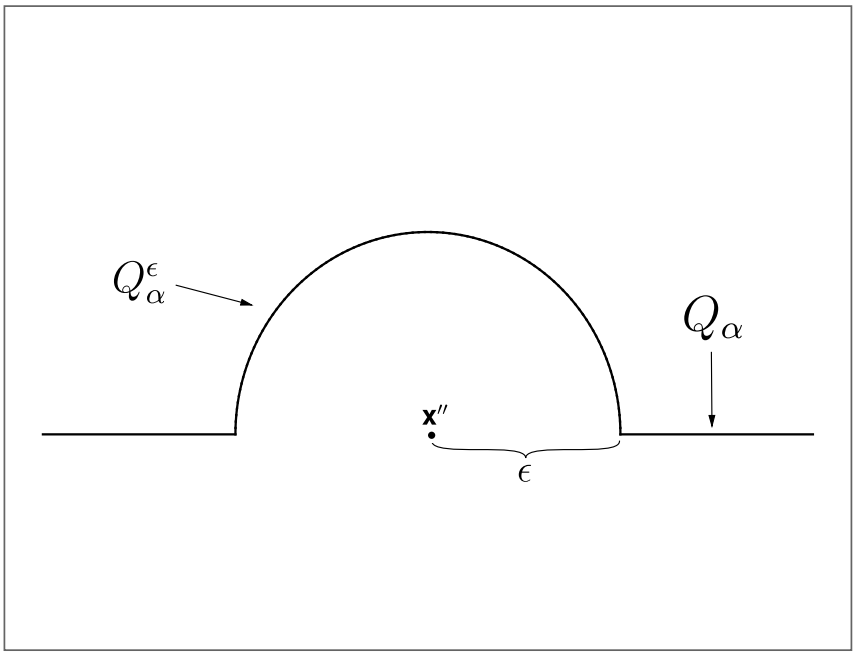

Figure 1.2: Illustration of the contour around each singularity of $D_0(\mathbf{x}, \mathbf{x}'')$

The integral on the right hand side pass right through a singularity of the Green's function $D_0(\mathbf{x}, \mathbf{x}'')$. This problem is handled by extending the contour to include these points as shown in figure 1.2. These extensions will be parametrized as circles with radius ϵ. To get back to the original contour it is simply a matter of letting the radius go to zero.

$$Q_\alpha^\epsilon : \gamma(\theta) = \mathbf{x}'' + \epsilon(\cos(\theta), \sin(\theta)) \quad \theta \in [0, \pi] \tag{1.63}$$

With this contour the integral will give a finite contribution for all these singularities The line integral is split into the contribution from integrating around Q_α and the extensions Q_α^ϵ

$$\begin{aligned} \nabla_{\mathbf{x}'} D_0(\mathbf{x}', \mathbf{x}'') = -\sum_\alpha \Bigg(& \mathrm{PV}_{\mathbf{x}''} \int_{Q_\alpha} ds\, D_0(\mathbf{x}, \mathbf{x}'') \nabla_{\mathbf{x}} \mathcal{E}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{n} \\ & + \lim_{\epsilon \to 0} \int_{Q_\alpha^\epsilon} ds\, D_0(\mathbf{x}, \mathbf{x}'') \nabla_{\mathbf{x}} \mathcal{E}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{n} \Bigg) \end{aligned} \tag{1.64}$$

The contribution from integrating along the Q_α^ϵ is given by

$$\int_{Q_\alpha^\epsilon} ds\, D_0(\mathbf{x}, \mathbf{x}'') \nabla \mathcal{E}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{n} = \int_0^\pi (\epsilon d\theta) D_0(\gamma(\theta), \mathbf{x}'') \nabla \mathcal{E}(\gamma(\theta), \mathbf{x}') \cdot \mathbf{n} \tag{1.65}$$

Where $\gamma(\theta)$ was defined in equation (1.63). Thus the limit

$$\begin{aligned}\lim_{\epsilon\to 0}\epsilon D_0(\gamma(\theta),\mathbf{x}'') &= \lim_{\epsilon\to 0} -\frac{1}{2\pi}\epsilon K_0(\omega||\gamma(\theta)-\mathbf{x}''||)\\ &= \lim_{\epsilon\to 0} -\frac{1}{2\pi}\epsilon K_0(\omega\epsilon)\end{aligned} \tag{1.66}$$

Since $\lim_{\epsilon\to 0}\epsilon K_0(\omega\epsilon) = 0$ the contribution from $\epsilon D_0(\gamma(\theta),\mathbf{x}'') \to 0$. Thus all contributions from the extended contour vanish and the following equation remains

$$-\nabla_{\mathbf{x}'} D_0(\mathbf{x}',\mathbf{x}'') = \sum_\alpha \mathrm{PV}_{\mathbf{x}''} \int\limits_{Q_\alpha} ds\, D_0(\mathbf{x},\mathbf{x}'')\nabla_{\mathbf{x}}\mathcal{E}(\mathbf{x},\mathbf{x}')\cdot\mathbf{n} \tag{1.67}$$

Where the Cauchy principal value integrals are to be taken around each of the singularities in $D_0(\mathbf{x}',\mathbf{x}'')$

The problem is now how to take the limit $\mathbf{x}' \to Q_i$. If $\mathbf{x}' \to \mathbf{y}$ where $\mathbf{y} \in Q_i$ but $\mathbf{y} \neq \mathbf{x}''$ there is no problem. Since the right side is defined by a Green's function it is obvious that there is a singularity when $\mathbf{x}' \to \mathbf{x}''$. Let us start by investigating what happens if $\mathbf{x}'$ approaches $\mathbf{x}''$ along some arbitrary path $\mathbf{x}'(t)$ close to $\mathbf{x}''$ but still inside V_0. Define

$$\mathbf{x}'(t) = \mathbf{x}'' + t\dot{\mathbf{x}}'(0) + \frac{1}{2}t^2\ddot{\mathbf{x}}'(0) + \ldots \tag{1.68}$$

With this the left side of equation (1.67) will be

$$\begin{aligned}\nabla_{\mathbf{x}'} D_0(\mathbf{x}',\mathbf{x}'') &= \frac{\omega}{2\pi}\frac{\mathbf{x}'-\mathbf{x}''}{||\mathbf{x}'-\mathbf{x}''||}K_1(\omega||\mathbf{x}'-\mathbf{x}''||)\\ &\overset{\approx}{\mathbf{x}'\to\mathbf{x}''} \frac{\omega}{2\pi}\frac{\dot{\mathbf{x}}'(0)}{||\dot{\mathbf{x}}'(0)||}K_1(\omega t||\dot{\mathbf{x}}'(0)||)\end{aligned} \tag{1.69}$$

From a small argument the Bessel function approximates to $K_1(x) \approx 1/x$ thus

$$\nabla_{\mathbf{x}'} D_0(\mathbf{x}',\mathbf{x}'') \approx \frac{1}{2\pi}\frac{\dot{\mathbf{x}}'(0)}{t||\dot{\mathbf{x}}'(0)||^2} \tag{1.70}$$

Thus when $t \to 0$ this will diverge. To regularize this limit: first let $\mathbf{x}' \to Q_i$ and then let $\mathbf{x}' \to \mathbf{x}''$ along this curve. The equations are regularized by stipulating that the limit is to be taken along only a small subset of possible curves through $\mathbf{x}''$.

Consider what happens to equation (1.53) when $\mathbf{x}' \to \mathbf{x}''$ along the curve Q_i. First observe that given a complete set of eigenfunctions $\varphi_n(\mathbf{x})$ for the Helmholz equation in (1.26) any reasonable function can be expressed as

$$f(\mathbf{x}) = \sum_n c_n\varphi_n(\mathbf{x}) = \sum_n \int dV_{\mathbf{x}'}\, f(\mathbf{x}')\varphi_n(\mathbf{x}')\varphi_n(\mathbf{x}) \tag{1.71}$$

Thus formally

$$\delta(\mathbf{x}-\mathbf{x}') = \sum_n \varphi_n(\mathbf{x}')\varphi_n(\mathbf{x}) \tag{1.72}$$

The eigenfunctions satisfy the boundary conditions such that $\varphi_n(\mathbf{x}) = 0$ when $\mathbf{x} \to Q_i$. Let us expand our gradient in terms of normal and tangent derivatives

$\nabla_{\mathbf{x}'} \to \mathbf{t}'\partial_{\mathbf{t}'} + \mathbf{n}'\partial_{\mathbf{n}'}$. The boundary conditions imply that $\partial_{\mathbf{t}'}\varphi_n(\mathbf{x}') = 0$ and thus the gradient on the right hand side of equation (1.53) will be

$$\begin{aligned}\nabla_{\mathbf{x}'}\delta(\mathbf{x}-\mathbf{x}') &= \sum_n \nabla_{\mathbf{x}'}\varphi_n(\mathbf{x}')\varphi_n(\mathbf{x}) \to \sum_n (\mathbf{t}'\partial_{\mathbf{t}'} + \mathbf{n}'\partial_{\mathbf{n}'})\varphi_n(\mathbf{x}')\varphi_n(\mathbf{x}) \\ &= \sum_n \mathbf{n}'\partial_{\mathbf{n}'}\varphi_n(\mathbf{x}')\varphi_n(\mathbf{x}) = \mathbf{n}'\partial_{\mathbf{n}'}\delta(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{1.73}$$

Thus as $\mathbf{x}'$ approach the curve Q_i the gradient on the right hand side of the equation changes into a normal derivative and equation (1.67) changes into

$$-\mathbf{n}'\partial_{\mathbf{n}'}D_0(\mathbf{x}',\mathbf{x}'') = \sum_\alpha \mathrm{PV}_{\mathbf{x}''}\int_{Q_\alpha} ds\, D_0(\mathbf{x},\mathbf{x}'')\nabla_{\mathbf{x}}\mathcal{E}(\mathbf{x},\mathbf{x}')\cdot\mathbf{n} \tag{1.74}$$

As $\mathbf{x}' \to \mathbf{x}''$ along Q_i the left side will now be given by

$$\begin{aligned}\mathbf{n}'\cdot\nabla_{\mathbf{x}'}D_0(\mathbf{x}',\mathbf{x}'') &= \frac{\omega}{2\pi}\frac{\mathbf{n}'\cdot(\mathbf{x}'-\mathbf{x}'')}{||\mathbf{x}'-\mathbf{x}''||}K_1(\omega||\mathbf{x}'-\mathbf{x}''||) \\ &\approx \frac{1}{2\pi}\frac{\mathbf{n}'\cdot(\mathbf{x}'-\mathbf{x}'')}{||\mathbf{x}'-\mathbf{x}''||^2}\end{aligned} \tag{1.75}$$

Let the function $\Theta(t)$ parametrize the curve. Assuming that t' and t'' are defined by

$$\Theta(t') = \mathbf{x}' \quad \Theta(t'') = \mathbf{x}'' \tag{1.76}$$

The tangent at $\mathbf{x}'$ is given by $\Theta'(t')$ and naturally the tangent satisfies the relation $\mathbf{n}'\cdot\Theta'(t') = \mathbf{n}(\Theta(t'))\cdot\Theta'(t') = 0$.

Expand around $\mathbf{x}''$ as $\mathbf{x}'' \approx \mathbf{x}'$ and define $\Delta t = t''-t'$. From the parametrization it is clear that

$$\begin{aligned}\mathbf{x}'-\mathbf{x}'' &= \Theta(t''+\Delta t) - \Theta(t'') \\ &\approx \Theta(t'') + \Theta'(t'')\Delta t + \frac{1}{2}\Theta''(t'')\Delta t^2 - \Theta(t'') \\ &= \Theta'(t'')\Delta t + \frac{1}{2}\Theta''(t'')\Delta t^2\end{aligned} \tag{1.77}$$

so

$$\mathbf{n}'\cdot(\mathbf{x}'-\mathbf{x}'') \approx \frac{1}{2}\mathbf{n}(\Theta(t''))\cdot\Theta''(t'')\Delta t^2 \tag{1.78}$$

And for the norm

$$\begin{aligned}||\mathbf{x}'-\mathbf{x}''||^2 &\approx \left(\Theta'(t'')\Delta t + \frac{1}{2}\Theta''(t'')\Delta t^2\right)\cdot\left(\Theta'(t'')\Delta t + \frac{1}{2}\Theta''(t'')\Delta t^2\right) \\ &\approx \Theta'(t'')\cdot\Theta'(t'')\Delta t^2\end{aligned} \tag{1.79}$$

Thus when $\mathbf{x}' \to \mathbf{x}''$ along the curve Q_i

$$\begin{aligned}\mathbf{n}'\cdot\nabla_{\mathbf{x}'}D_0(\mathbf{x}',\mathbf{x}'') &= \frac{\omega}{2\pi}\frac{\mathbf{n}'\cdot(\mathbf{x}'-\mathbf{x}'')}{||\mathbf{x}'-\mathbf{x}''||}K_1(\omega||\mathbf{x}'-\mathbf{x}''||) \\ &\approx \frac{1}{2\pi}\frac{\mathbf{n}'\cdot(\mathbf{x}'-\mathbf{x}'')}{||\mathbf{x}'-\mathbf{x}''||^2} \approx \frac{1}{4\pi}\frac{\mathbf{n}(\Theta(t''))\cdot\Theta''(t'')}{\Theta'(t'')\cdot\Theta'(t'')}\end{aligned} \tag{1.80}$$

Thus the proposed regularization has canceled the singularity on the left hand side of equation (1.67). The factor that appears above is proportional to the *curvature* of the curve at the point $\mathbf{x}''$.

The unknown in equation (1.67) is $\nabla_{\mathbf{x}}\mathcal{E}(\mathbf{x},\mathbf{x}')$, but it is $\partial_{\mathbf{nn}}\mathcal{D}(\mathbf{x},\mathbf{x})$ that is required to compute the force in equation (1.52). Changing the basis to the normal and tangent vectors at each point gives $\nabla_{\mathbf{x}} \to \mathbf{t}\partial_{\mathbf{t}} + \mathbf{n}\partial_{\mathbf{n}}$ and $\nabla_{\mathbf{x}'} \to \mathbf{t}'\partial_{\mathbf{t}'} + \mathbf{n}'\partial_{\mathbf{n}'}$ for the primed variables. Thus

$$\begin{aligned}\mathbf{n}\cdot\nabla_{\mathbf{x}}\mathcal{E}(\mathbf{x},\mathbf{x}') &= \mathbf{n}\cdot\nabla_{\mathbf{x}}\nabla_{\mathbf{x}'}\mathcal{D}(\mathbf{x},\mathbf{x}') \\ &\to \mathbf{n}\cdot(\mathbf{t}\partial_{\mathbf{t}}+\mathbf{n}\partial_{\mathbf{n}})(\mathbf{t}'\partial_{\mathbf{t}'}+\mathbf{n}'\partial_{\mathbf{n}'})\mathcal{D}(\mathbf{x},\mathbf{x}') \\ &= \mathbf{n}'\partial_{\mathbf{nn}'}\mathcal{D}(\mathbf{x},\mathbf{x}')\end{aligned} \tag{1.81}$$

And equation (1.74) changes into

$$-\mathbf{n}'\partial_{\mathbf{n}'}D_0(\mathbf{x}',\mathbf{x}'') = \sum_\alpha \mathrm{PV}_{\mathbf{x}''}\int\limits_{Q_\alpha} ds\, D_0(\mathbf{x},\mathbf{x}'')\mathbf{n}'\partial_{\mathbf{nn}'}\mathcal{D}(\mathbf{x},\mathbf{x}') \tag{1.82}$$

Note that $\mathbf{n}'$ is common on both sides and can be canceled

$$-\partial_{\mathbf{n}'}D_0(\mathbf{x}',\mathbf{x}'') = \textstyle\sum_\alpha \mathrm{PV}_{\mathbf{x}''}\int_{Q_\alpha} ds\, D_0(\mathbf{x},\mathbf{x}'')\partial_{\mathbf{nn}'}\mathcal{D}(\mathbf{x},\mathbf{x}') \quad \begin{smallmatrix}\mathbf{x}''\in Q_i\\ \mathbf{x}'\in Q_j\end{smallmatrix} \tag{1.83}$$

The problem has now been reduced to finding a scalar function. This will then later be multiplied by the the normals to produce the force on each line segment. Observe that when $\omega \to \infty$ the free Green's function $D_0(\mathbf{x},\mathbf{x}'') \to 0$. This will make the equations decouple and the resulting system is

$$-\partial_{\mathbf{n}'}D_0(\mathbf{x}',\mathbf{x}'') = \mathrm{PV}_{\mathbf{x}''}\textstyle\int_{Q_i} ds\, D_0(\mathbf{x},\mathbf{x}'')\partial_{\mathbf{nn}'}D_i(\mathbf{x},\mathbf{x}') \quad \begin{smallmatrix}\mathbf{x}'',\mathbf{x}'\in Q_i\\ i\in 1\ldots r\end{smallmatrix} \tag{1.84}$$

These equations will be called the *self stress equations*. To regularize the force in equation (1.52) the solution to the self stress equation will be subtracted. This will remove the high frequency contribution from the force and the resulting force will be redefined as the correct force for this problem.

Define the regularized density as $\Delta_i(\mathbf{x},\mathbf{x}')$

$$\Delta_i(\mathbf{x},\mathbf{x}') = \partial_{\mathbf{nn}'}\mathcal{D}(\mathbf{x},\mathbf{x}') - \partial_{\mathbf{nn}'}D_i(\mathbf{x},\mathbf{x}') \quad \mathbf{x},\mathbf{x}'\in Q_i \tag{1.85}$$

When the regularized density is inserted back into equation (1.83) it is convenient to separate the equations based on the points $\mathbf{x}'$ and $\mathbf{x}''$

$$\begin{aligned}&\mathrm{PV}_{\mathbf{x}''}\int\limits_{Q_i} ds\, D_0(\mathbf{x},\mathbf{x}'')\Delta_i(\mathbf{x},\mathbf{x}') \\ &\quad + \sum_{\alpha,\alpha\neq i}\int\limits_{Q_\alpha} ds\, D_0(\mathbf{x},\mathbf{x}'')\partial_{\mathbf{nn}'}\mathcal{D}(\mathbf{x},\mathbf{x}') = 0\end{aligned} \quad \mathbf{x}'',\mathbf{x}'\in Q_i \tag{1.86}$$

$$\begin{aligned}&\mathrm{PV}_{\mathbf{x}''}\int\limits_{Q_i} ds\, D_0(\mathbf{x},\mathbf{x}'')\Delta_i(\mathbf{x},\mathbf{x}') \\ &\quad + \sum_{\alpha,\alpha\neq i}\int\limits_{Q_\alpha} ds\, D_0(\mathbf{x},\mathbf{x}'')\partial_{\mathbf{nn}'}\mathcal{D}(\mathbf{x},\mathbf{x}') \\ &= -\partial_{\mathbf{n}'}D_0(\mathbf{x}',\mathbf{x}'') - \int\limits_{Q_i} ds\, D_0(\mathbf{x},\mathbf{x}'')\partial_{\mathbf{nn}'}D_i(\mathbf{x},\mathbf{x}')\end{aligned} \quad \begin{smallmatrix}\mathbf{x}''\in Q_j\\ \mathbf{x}'\in Q_i\\ j\neq i\end{smallmatrix} \tag{1.87}$$

1.4 Discretization

To solve equations (1.86) and (1.87) the integrals will discretized. The resulting sums can then be organized into matrix form and the matrix can then be inverted to find the solution.

The integral equation will be solved using the method of moments. Each smooth curve Q_i will be approximated by a piecewise linear curve. There are other options that could be used, but this is the simplest. Let I_k^i be the k^{th} linear piece of the piecewise linear curve that approximate Q_i. The integrals are then changed into

$$\begin{array}{l} \int_{Q_i} ds\, D_0(\mathbf{x}, \mathbf{s}_{k''}) \partial_{\mathbf{nn}'} \mathcal{D}(\mathbf{x}, \mathbf{s}_{k'}) \approx \sum_k \int_{I_k^i} ds\, D_0(\mathbf{x}, \mathbf{s}_{k''}) \partial_{\mathbf{nn}'} \mathcal{D}(\mathbf{x}, \mathbf{s}_{k'}) \\ \approx \sum_k \partial_{\mathbf{nn}'} \mathcal{D}(\mathbf{s}_k, \mathbf{s}_{k'}) \int_{I_k^i} ds\, D_0(\mathbf{x}, \mathbf{s}_{k''}) = \sum_k x_{kk'}^{ij'} a_{kk''}^{ij''} \end{array} \tag{1.88}$$

where $\mathbf{s}_k$ is the midpoint of the line element I_k^i

For this approximation to be good it is required that I_k^i is small enough such that $\partial_{\mathbf{nn}'} \mathcal{D}(\mathbf{x}, \mathbf{s}_{k'})$ is approximately constant in each subinterval I_k^i. Approximate functions defined on the curve Q_i by their values at the midpoints of the line elements I_k^i.

To simplify the matrix formulas the following definitions will be helpful

$$x_{kk'}^{ij} = \begin{cases} \partial_{\mathbf{nn}'} \mathcal{D}(\mathbf{s}_k, \mathbf{s}_{k'}) & \mathbf{s}_k \in Q_i, \mathbf{s}_{k'} \in Q_j, j \neq i \\ \Delta_i(\mathbf{s}_k, \mathbf{s}_{k'}) & \mathbf{s}_k, \mathbf{s}_{k'} \in Q_i \end{cases} \tag{1.89}$$

$$a_{kk''}^{ij} = \begin{cases} \int_{I_k^i} dl_{\mathbf{x}}\, D_0(\mathbf{x}, \mathbf{s}_{k''}) & \mathbf{s}_{k''} \in Q_j, j \neq i \\ \text{PV}_{\mathbf{s}_{k''}} \int_{I_k^i} dl_{\mathbf{x}}\, D_0(\mathbf{x}, \mathbf{s}_{k''}) & \mathbf{s}_{k''} \in Q_i \end{cases} \tag{1.90}$$

and for the right side

$$y_{k'k''}^{ij} = -\partial_{\mathbf{n}'} D_0(\mathbf{s}_{k'}, \mathbf{s}_{k''}) \quad \mathbf{s}_{k'} \in Q_i, \mathbf{s}_{k''} \in Q_j \tag{1.91}$$

$$b_{kk'}^{ij} = \partial_{\mathbf{nn}'} D_i(\mathbf{s}_k, \mathbf{s}_{k'}) \quad \mathbf{s}_k, \mathbf{s}_{k'} \in Q_i \tag{1.92}$$

To form a system of equations there are two options: Either let $\mathbf{x}' \to Q_i$ and then let $\mathbf{x}'' \to Q_j$ for $\forall j \neq i$ or let $\mathbf{x}'' \to Q_i$ and then let $\mathbf{x}' \to Q_j$ for $\forall j \neq i$. The above calculations are unaffected by this limit and the only place where the limit might cause a problem is in the Green's function $\partial_{\mathbf{nn}'} \mathcal{D}(\mathbf{x}, \mathbf{x}')$. Observe that from equation (1.7)

$$\mathcal{D}(\mathbf{x}, t, \mathbf{x}', t') = < \mathrm{T}[\hat{\varphi}(\mathbf{x}, t) \hat{\varphi}(\mathbf{x}', t')] > = \begin{cases} < \hat{\varphi}(\mathbf{x}, t) \hat{\varphi}(\mathbf{x}', t') > & t > t' \\ < \hat{\varphi}(\mathbf{x}', t') \hat{\varphi}(\mathbf{x}, t) > & t < t' \end{cases} \tag{1.93}$$

Thus there there is no problem in interchanging $(\mathbf{x}, t) \leftrightarrow (\mathbf{x}', t')$.

Consider $\mathbf{x}'' \to Q_j$ and then $\mathbf{x}' \to Q_i$ for $\forall j \neq i$.

When there are r objects equations (1.86) and (1.87) will give the system

$$\begin{aligned}
\sum_k a^{11}_{kk''}x^{1i}_{kk'} + \ldots + a^{i1}_{kk''}x^{ii}_{kk'} + \ldots + a^{r1}_{kk''}x^{ri}_{kk'} &= y^{i1}_{k'k''} - \sum_k a^{i1}_{kk''}b^{ii}_{kk'} \\
&\vdots \\
\sum_k a^{1,i-1}_{kk''}x^{1i}_{kk'} + \ldots + a^{i,i-1}_{kk''}x^{ii}_{kk'} + \ldots + a^{r,i-1}_{kk''}x^{ri}_{kk'} &= y^{i,i-1}_{k'k''} - \sum_k a^{i,i-1}_{kk''}b^{i,i}_{kk'} \\
\sum_k a^{1i}_{kk''}x^{1i}_{kk'} + \ldots + a^{ii}_{kk''}x^{ii}_{kk'} + \ldots + a^{ri}_{kk''}x^{ri}_{kk'} &= 0 \\
\sum_k a^{1,i+1}_{kk''}x^{1i}_{kk'} + \ldots + a^{i,i+1}_{kk''}x^{ii}_{kk'} + \ldots + a^{r,i+1}_{kk''}x^{ri}_{kk'} &= y^{i,i+1}_{k'k''} - \sum_k a^{i,i+1}_{kk''}b^{ii}_{kk'} \\
&\vdots \\
\sum_k a^{1r}_{kk''}x^{1i}_{kk'} + \ldots + a^{ir}_{kk''}x^{ii}_{kk'} + \ldots + a^{rr}_{kk''}x^{ri}_{kk'} &= y^{ir}_{k'k''} - \sum_k a^{ir}_{kk''}b^{ii}_{kk'}
\end{aligned} \tag{1.94}$$

and the self stress equation in (1.84) for object i is

$$\sum_k a^{ii}_{kk''}b^{ii}_{kk'} = y^{ii}_{k'k''} \tag{1.95}$$

Let us express this as the product of block matrices, to do this the variables are transposed: $A^{ij} = (a^{ij}_{kk''})^T$, $X^{ij} = x^{ij}_{kk'}$, $Y^{ij} = (y^{ij}_{k'k''})^T$ and $B^{ii} = b^{ii}_{kk'}$. Thus the sums become the regular matrix products and the equations are represented as

$$\begin{bmatrix} A^{11} & \cdots & A^{i1} & \cdots & A^{r1} \\ \vdots & \ddots & \vdots & & \vdots \\ A^{1i} & \cdots & A^{ii} & \cdots & A^{ri} \\ \vdots & & \vdots & \ddots & \vdots \\ A^{1r} & \cdots & A^{ir} & \cdots & A^{rr} \end{bmatrix} \begin{bmatrix} X^{1i} \\ \vdots \\ X^{ii} \\ \vdots \\ X^{ri} \end{bmatrix} = \begin{bmatrix} Y^{i1} - A^{i1}B^{ii} \\ \vdots \\ Y^{i,i-1} - A^{i,i-1}B^{ii} \\ 0 \\ Y^{i,i+1} - A^{i,i+1}B^{ii} \\ \vdots \\ Y^{ir} - A^{ir}B^{ii} \end{bmatrix} \tag{1.96}$$

These are r block matrix equations for r block matrix unknowns. For the case of two object equation (1.96) simplifies into

$$\begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix} \begin{bmatrix} X^{11} \\ X^{21} \end{bmatrix} = \begin{bmatrix} 0 \\ Y^{12} - A^{12}B^{11} \end{bmatrix} \tag{1.97}$$

and

$$\begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix} \begin{bmatrix} X^{12} \\ X^{22} \end{bmatrix} = \begin{bmatrix} Y^{21} - A^{21}B^{22} \\ 0 \end{bmatrix} \tag{1.98}$$

Consider $\mathbf{x}'' \to Q_i$ and then $\mathbf{x}' \to Q_j$ for $\forall j \neq i$.

Write down equations (1.86) and (1.87) for a total of r objects where the goal is the force on object i

$$\begin{aligned}
&\sum_k a^{1i}_{kk''}x^{11}_{kk'} + \ldots + a^{ii}_{kk''}x^{i1}_{kk'} + \ldots + a^{ri}_{kk''}x^{r1}_{kk'} = y^{1i}_{k'k''} - \sum_k a^{1i}_{kk''}b^{11}_{kk'} \\
&\qquad\qquad\qquad\qquad \vdots \\
&\sum_k a^{1i}_{kk''}x^{1,i-1}_{kk'} + \ldots + a^{i-1,i}_{kk''}x^{i-1,i-1}_{kk'} + \ldots + a^{ri}_{kk''}x^{r,i-1}_{kk'} = y^{i-1,i}_{k'k''} - \sum_k a^{i-1,i}_{kk''}b^{i-1,i-1}_{kk'} \\
&\qquad \sum_k a^{1i}_{kk''}x^{1i}_{kk'} + \ldots + a^{ii}_{kk''}x^{ii}_{kk'} + \ldots + a^{ri}_{kk''}x^{ri}_{kk'} = 0 \\
&\sum_k a^{1i}_{kk''}x^{1,i+1}_{kk'} + \ldots + a^{i+1,i}_{kk''}x^{i+1,i+1}_{kk'} + \ldots + a^{ri}_{kk''}x^{r,i+1}_{kk'} = y^{i+1,i}_{k'k''} - \sum_k a^{i+1,i}_{kk''}b^{i+1,i+1}_{kk'} \\
&\qquad\qquad\qquad\qquad \vdots \\
&\sum_k a^{1i}_{kk''}x^{1r}_{kk'} + \ldots + a^{ii}_{kk''}x^{ir}_{kk'} + \ldots + a^{ri}_{kk''}x^{rr}_{kk'} = y^{ri}_{k'k''} - \sum_k a^{ri}_{kk''}b^{rr}_{kk'}
\end{aligned} \tag{1.99}$$

and the self stress equation in (1.84) for object i is

$$\sum_k a^{ii}_{kk''}b^{ii}_{kk'} = y^{ii}_{k'k''} \tag{1.100}$$

These are r linear block matrix equations for r^2 unknowns $x^{ij}_{kk'}$. If the equations for the force on the other $r-1$ objects are included there will be a total of r^2 equations for r^2 unknowns.

For $r = 2$ objects the force on object $i = 1$ is given by

$$\begin{aligned}
&\sum_k a^{11}_{kk''}x^{11}_{kk'} + a^{21}_{kk''}x^{21}_{kk'} = 0 \\
&\sum_k a^{11}_{kk''}x^{12}_{kk'} + a^{21}_{kk''}x^{22}_{kk'} = y^{21}_{k'k''} - \sum_k a^{21}_{kk''}b^{22}_{kk'}
\end{aligned} \tag{1.101}$$

and $i = 2$

$$\begin{aligned}
&\sum_k a^{12}_{kk''}x^{11}_{kk'} + a^{22}_{kk''}x^{21}_{kk'} = y^{12}_{k'k''} - \sum_k a^{12}_{kk''}b^{11}_{kk'} \\
&\sum_k a^{12}_{kk''}x^{12}_{kk'} + a^{22}_{kk''}x^{22}_{kk'} = 0
\end{aligned} \tag{1.102}$$

Let us express this as the product of block matrices, to do this the variables are transposed: $A^{ij} = (a^{ij}_{kk''})^T$, $X^{ij} = x^{ij}_{kk'}$, $Y^{ij} = (y^{ij}_{k'k''})^T$ and $B^{ii} = b^{ii}_{kk'}$. Thus the sums become the regular matrix products and the equations are represented as

$$\begin{bmatrix} A^{11} & 0 & A^{21} & 0 \\ 0 & A^{11} & 0 & A^{21} \\ A^{12} & 0 & A^{22} & 0 \\ 0 & A^{12} & 0 & A^{22} \end{bmatrix} \begin{bmatrix} X^{11} \\ X^{12} \\ X^{21} \\ X^{22} \end{bmatrix} = \begin{bmatrix} 0 \\ Y^{21} - A^{21}B^{22} \\ Y^{12} - A^{12}B^{11} \\ 0 \end{bmatrix} \tag{1.103}$$

It is clear that rows 1 and 3 form the same equations as for object 1 in the previous limit and rows 2 and 4 are the same equations as for object 2 in the previous limit.

1.5 Applications

The goal is to find the force on object i from equation (1.52) with the renormalized force.

$$\mathbf{F}_i = \oint_{Q_i} dl_{\mathbf{x}}\, \mathbf{n}(\mathbf{x}) \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega\, \Delta_i(\mathbf{x}, \mathbf{x}, \omega) \tag{1.104}$$

Where the normals should point out of each compact object Q_i and $\Delta_i(\mathbf{x}, \mathbf{x}, \omega)$ is the renormalized density that can be found from equation (1.96). The equation that define the problem in the case of two objects is

$$\begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix} \begin{bmatrix} X^{11} \\ X^{21} \end{bmatrix} = \begin{bmatrix} 0 \\ Y^{12} - A^{12}B^{11} \end{bmatrix} \tag{1.105}$$

and

$$\begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix} \begin{bmatrix} X^{12} \\ X^{22} \end{bmatrix} = \begin{bmatrix} Y^{21} - A^{21}B^{22} \\ 0 \end{bmatrix} \tag{1.106}$$

Where the self stress B^{11} and B^{22} is given by

$$\begin{aligned} A^{11}B^{11} &= Y^{11} \\ A^{22}B^{22} &= Y^{22} \end{aligned} \tag{1.107}$$

From the above equations it is clear that, for each ω, the self stress is found by solving for B^{11} and B^{22}. After this the matrices X^{11} and X^{22} are solved with the given right hand side.

1.6 Matrices

The problem is to investigate compact objects that will be defined through a piecewise linear parametrization. Given an object with points $\mathbf{P}_k^i \in \mathbb{R}^2$, $k \in 1 \cdots N$ along the border of the object. Define the piecewise linear parametrization of object i as

$$\gamma^i(s) = \mathbf{P}_{k-1}^i + (\mathbf{P}_k^i - \mathbf{P}_{k-1}^i)\tfrac{s-\alpha_{k-1}}{\Delta} \qquad s \in J_k = [\alpha_{k-1}, \alpha_k] \tag{1.108}$$

Where the total parametrization interval for $\gamma^i(s)$ is $[-L/2, L/2]$, divided into N sections J_k. The edges of each parametrization interval and their centers are defined as

$$\begin{aligned} \alpha_k &= -L/2 + k \cdot \Delta & k &= 0 \ldots N \\ s_k &= -L/2 + (k - 1/2) \cdot \Delta & k &= 1 \ldots N \end{aligned} \tag{1.109}$$

where $\Delta = L/N = \alpha_k - \alpha_{k-1}$. The number of discretization intervals is N and the discretization parameter $\alpha \in [-L/2, L/2]$ for some k. The length, e_k^i, of each piece of the piecewise linear curve approximating Q_i is

$$e_k^i \equiv \int_{I_k} dl_{\mathbf{x}} = \int_{\alpha_{k-1}}^{\alpha_k} ds\, \frac{1}{\Delta} ||\mathbf{P}_k^i - \mathbf{P}_{k-1}^i|| = ||\mathbf{P}_k^i - \mathbf{P}_{k-1}^i|| \tag{1.110}$$

The next step is to calculate the matrix elements, these are given by formula (1.90) and (1.91)

$$a^{ij}_{kk''} = \begin{cases} \int_{I^i_k} dl_{\mathbf{x}}\, D_0(\mathbf{x}, \mathbf{s}_{k''}) & \mathbf{s}_{k''} \in Q_j, j \neq i \\ \mathrm{PV}_{\mathbf{s}_{k''}} \int_{I^i_k} dl_{\mathbf{x}}\, D_0(\mathbf{x}, \mathbf{s}_{k''}) & \mathbf{s}_{k''} \in Q_i \end{cases} \tag{1.111}$$

$$y^{ij}_{k'k''} = -\partial_{\mathbf{n}'} D_0(\mathbf{s}_{k'}, \mathbf{s}_{k''}) \qquad \mathbf{s}_{k'} \in Q_i, \mathbf{s}_{k''} \in Q_j \tag{1.112}$$

Consider these in turn.

1.6.1 Matrix elements: $y^{ij}_{k'k''}$

The matrix elements $y^{ij}_{k'k''}$ elements from equation (1.91) will in general be

$$\begin{aligned} y^{ij}_{k'k''} &= -\partial_{\mathbf{n}'} D_0(\mathbf{s}_{k'}, \mathbf{s}_{k''}) = -\mathbf{n}' \cdot \nabla_{\mathbf{s}_{k'}} D_0(\mathbf{s}_{k'}, \mathbf{s}_{k''}) \\ &= \frac{1}{2\pi} \mathbf{n}' \cdot \nabla_{\mathbf{s}_{k'}} K_0(\omega ||\mathbf{s}_{k'} - \mathbf{s}_{k''}||) \\ &= -\frac{\omega}{2\pi} \frac{\mathbf{n}' \cdot (\mathbf{s}_{k'} - \mathbf{s}_{k''})}{||\mathbf{s}_{k'} - \mathbf{s}_{k''}||} K_1(\omega ||\mathbf{s}_{k'} - \mathbf{s}_{k''}||) \end{aligned} \tag{1.113}$$

The only point where this could pose a problem is when $i = j$ and $k' = k''$. For small arguments the Bessel functions are approximated by $K_1(x) \approx 1/x$ and thus

$$y^{ii}_{k'k''} \approx -\frac{1}{2\pi} \frac{\mathbf{n}' \cdot (\mathbf{s}_{k'} - \mathbf{s}_{k''})}{||\mathbf{s}_{k'} - \mathbf{s}_{k''}||^2} \tag{1.114}$$

But this problem has already been solved. Thus with equation (1.80) it is easy to see that the the diagonal elements are equal to

$$y^{ii}_{kk} \approx -\frac{1}{4\pi} \frac{\mathbf{n}(\Theta(t'')) \cdot \Theta''(t'')}{\Theta'(t'') \cdot \Theta'(t'')} \tag{1.115}$$

1.6.2 Matrix elements: $a^{ij}_{kk''}$

In general the matrix elements are

$$a^{ij}_{kk''} = \mathrm{PV}_{\mathbf{s}_{k''}} \int\limits_{I^i_k} dl_{\mathbf{x}}\, D_0(\mathbf{x}, \mathbf{s}_{k''}) = -\frac{1}{2\pi} \mathrm{PV}_{\mathbf{s}_{k''}} \int\limits_{I^i_k} dl_{\mathbf{x}}\, K_0(\omega ||\mathbf{x} - \mathbf{s}_{k''}||) \tag{1.116}$$

where $\mathbf{s}_{k''} \in C_j$.

This integral can be solved by using a center approximation or some quadrature in all cases but when $i = j$ and $k = k''$. Observe first that for this case

$$\begin{aligned} ||\mathbf{x}(s) - \mathbf{s}_{k''}|| &= ||\mathbf{P}^i_{k-1} + (\mathbf{P}^i_k - \mathbf{P}^i_{k-1}) \frac{s - \alpha_{k-1}}{\Delta} \\ &\quad - \mathbf{P}^i_{k-1} - (\mathbf{P}^i_k - \mathbf{P}^i_{k-1}) \frac{s_{k''} - \alpha_{k-1}}{\Delta} || \\ &= ||(\mathbf{P}^i_k - \mathbf{P}^i_{k-1}) \frac{s - s_{k''}}{\Delta} || = e^i_k \frac{|s - s_{k''}|}{\Delta} \end{aligned} \tag{1.117}$$

where e_k^i is the length of the parametrized line element. Thus the above integral is

$$\begin{aligned} a_{kk}^{ii} &= -\frac{1}{2\pi}\mathrm{PV}_{\mathbf{s}_{k''}} \int\limits_{I_k^i} dl_{\mathbf{x}}\, K_0(\omega||\mathbf{x}-\mathbf{s}_{k''}||) \\ &= -\frac{1}{2\pi}\mathrm{PV}_{s_{k''}} \int\limits_{\alpha_{k-1}}^{\alpha_k} \left(\frac{e_k^i}{\Delta} ds\right) K_0(|s-s_{k''}|\omega e_k^i/\Delta) \\ &= -\frac{e_k^i}{2\pi\Delta}\mathrm{PV}_{s_{k''}} \int\limits_{\alpha_{k-1}}^{\alpha_k} ds\, K_0(|s-s_{k''}|B) \end{aligned} \tag{1.118}$$

Where the constants are gathered in $B = \omega e_k^i/\Delta$. Make a change to coordinates such that our integration interval is small

$$\begin{aligned} \theta(s) &= -\Delta/2 + s - \alpha_{k-1} \\ d\theta &= ds \end{aligned} \tag{1.119}$$

and thus it is then possible to use the small argument approximation: $K_0(x) \approx C - log(x)$. Where $C = \log(2) - \gamma_{\mathrm{em}}$ and $\gamma_{\mathrm{em}} = 0.577215..$ is the Euler-Mascheroni constant.

$$\begin{aligned} a_{kk}^{ii} &\approx -\frac{e_k^i}{2\pi\Delta}\mathrm{PV}_0 \int\limits_{-\Delta/2}^{\Delta/2} d\theta\, (C - \log(|\theta|B)) \\ &= -\frac{e_k^i}{2\pi\Delta}\left(C\Delta - \mathrm{PV}_0 \int\limits_{-\Delta/2}^{\Delta/2} d\theta\, \log(|\theta|B)\right) \\ &= -\frac{e_k^i}{2\pi\Delta}\lim_{\epsilon\to 0}\left(C\Delta - \int\limits_{-\Delta/2}^{-\epsilon} d\theta\, \log(-\theta B) - \int\limits_{\epsilon}^{\Delta/2} d\theta\, \log(\theta B)\right) \\ &= -\frac{e_k^i}{2\pi\Delta}\lim_{\epsilon\to 0}\left(C\Delta - \int\limits_{\epsilon}^{\Delta/2} d\theta\, \log(\theta B) - \int\limits_{\epsilon}^{\Delta/2} d\theta\, \log(\theta B)\right) \\ &= -\frac{e_k^i}{2\pi\Delta}\lim_{\epsilon\to 0}\left(C\Delta - 2\int\limits_{\epsilon}^{\Delta/2} d\theta\, \log(\theta B)\right) \\ &= -\frac{e_k^i}{2\pi\Delta}\lim_{\epsilon\to 0}\left(C\Delta - \frac{2}{B}\int\limits_{B\epsilon}^{\Delta B/2} dy\, \log(y)\right) \\ &= -\frac{e_k^i}{2\pi\Delta}\left(C\Delta - \Delta\log\left(\frac{1}{2}B\Delta\right) + \Delta\right) \\ &= -\frac{e_k^i}{2\pi}\left(C + 1 - \log\left(\frac{1}{2}\omega e_k^i\right)\right) \end{aligned} \tag{1.120}$$

These are the diagonal elements of the matrix a_{kk}^{ii}.

1.7 Symmetry simplifications

Given that $\mathcal{D}(\mathbf{x},\mathbf{x}')$ is the unique solution to equation (1.83)

$$-\partial_{\mathbf{n}(\mathbf{x}')}D_0(\mathbf{x}',\mathbf{x}'') = \mathrm{PV}_{\mathbf{x}''}\int\limits_Q ds\, D_0(\mathbf{x},\mathbf{x}'')\partial_{\mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x}')}\mathcal{D}(\mathbf{x},\mathbf{x}') \tag{1.121}$$

where $Q = \cup_\alpha Q_\alpha$ and $\mathbf{x}',\mathbf{x}'' \in Q$.

Introducing a new varable

$$P(\mathbf{x},\mathbf{x}') = \partial_{\mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x}')}\mathcal{D}(\mathbf{x},\mathbf{x}') \tag{1.122}$$

where $P(\mathbf{x},\mathbf{x}')$ is a solution to

$$-\partial_{\mathbf{n}(\mathbf{x}')}D_0(\mathbf{x}',\mathbf{x}'') = \mathrm{PV}_{\mathbf{x}''}\int\limits_Q ds\, D_0(\mathbf{x},\mathbf{x}'')P(\mathbf{x},\mathbf{x}') \tag{1.123}$$

If g is an isometry that preserves Q then $B(\mathbf{x},\mathbf{x}') = P(g^{-1}\mathbf{x}, g^{-1}\mathbf{x}')$ is also a solution to equation (1.123).

From the right hand side

$$\begin{aligned}
&\mathrm{PV}_{\mathbf{x}''}\int\limits_Q ds\, D_0(\mathbf{x},\mathbf{x}'')\partial_{\mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x}')}B(\mathbf{x},\mathbf{x}') \\
&= \mathrm{PV}_{g^{-1}\mathbf{x}''}\int\limits_Q ds\, D_0(g^{-1}\mathbf{x}, g^{-1}\mathbf{x}'')P(g^{-1}\mathbf{x}, g^{-1}\mathbf{x}') \\
&= \mathrm{PV}_{g^{-1}\mathbf{x}''}\int\limits_Q ds\, D_0(\mathbf{y}, g^{-1}\mathbf{x}'')P(\mathbf{y}, g^{-1}\mathbf{x}') \\
&= -\partial_{\mathbf{n}(g^{-1}\mathbf{x}')}D_0(g^{-1}\mathbf{x}', g^{-1}\mathbf{x}'') \\
&= -\nabla_{(g^{-1}\mathbf{x}')}D_0(g^{-1}\mathbf{x}', g^{-1}\mathbf{x}'')\cdot\mathbf{n}(g^{-1}\mathbf{x}')
\end{aligned} \tag{1.124}$$

because the isometric transformation preserves the norm such that $D_0(g\mathbf{x}, g\mathbf{x}'') = D_0(\mathbf{x},\mathbf{x}'')$

From the chain rule

$$\nabla_{\mathbf{x}'}D_0(g^{-1}\mathbf{x}', g^{-1}\mathbf{x}'') = (\nabla_{\mathbf{y}}D_0(\mathbf{y}, g^{-1}\mathbf{x}''))|_{\mathbf{y}=g^{-1}\mathbf{x}'}D(g^{-1})(\mathbf{x}') \tag{1.125}$$

and for

$$\nabla_{g^{-1}\mathbf{x}'}D_0(g^{-1}\mathbf{x}', g^{-1}\mathbf{x}'') = \nabla_{\mathbf{x}'}D_0(g^{-1}\mathbf{x}', g^{-1}\mathbf{x}'')Dg(g^{-1}\mathbf{x}') \tag{1.126}$$

Observe that

$$\begin{aligned}
\mathbf{x}' &= g(g^{-1}\mathbf{x}') \\
\mathrm{I} &= Dg(g^{-1}\mathbf{x}')D(g^{-1})(\mathbf{x}')
\end{aligned} \tag{1.127}$$

Thus the above equation simplifies into

$$\begin{aligned}
&\mathrm{PV}_{\mathbf{x}''}\int\limits_Q ds\, D_0(\mathbf{x},\mathbf{x}'')\partial_{\mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x}')}B(\mathbf{x},\mathbf{x}') \\
&= -\nabla_{(g^{-1}\mathbf{x}')}D_0(g^{-1}\mathbf{x}', g^{-1}\mathbf{x}'')\cdot\mathbf{n}(g^{-1}\mathbf{x}') \\
&= -\nabla_{\mathbf{x}'}D_0(g^{-1}\mathbf{x}', g^{-1}\mathbf{x}'')\cdot Dg(g^{-1}\mathbf{x}')\mathbf{n}(g^{-1}\mathbf{x}') \\
&= -\nabla_{\mathbf{x}'}D_0(g^{-1}\mathbf{x}', g^{-1}\mathbf{x}'')\cdot\mathbf{n}(\mathbf{x}') \\
&= -\nabla_{\mathbf{x}'}D_0(\mathbf{x}',\mathbf{x}'')\cdot\mathbf{n}(\mathbf{x}')
\end{aligned} \tag{1.128}$$

Because the solution is unique

$$P(\mathbf{x}, \mathbf{x}') = B(\mathbf{x}, \mathbf{x}') \tag{1.129}$$

and thus

$$P(\mathbf{x}, \mathbf{x}') = P(g^{-1}\mathbf{x}, g^{-1}\mathbf{x}') \tag{1.130}$$

or

$$P(\mathbf{x}, g\mathbf{x}') = P(g^{-1}\mathbf{x}, \mathbf{x}') \tag{1.131}$$

With the solution of equation (1.123) at one point $\mathbf{x}'$ it is possible to find the solution at another point $g\mathbf{x}$, connected by the isometry g. If such an isometry is found it would greatly decrease the time required to find the force.

1.8 Source test

In order to test for possible human or numerical errors in the boundary element solver the problem can be modified slightly. This modification should be small and make the transition back into the original equations natural. Start with a modification of equation (1.26):

$$\nabla^2_{\mathbf{x}} \mathcal{D}(\mathbf{x}, \mathbf{x}') - \omega^2 \mathcal{D}(\mathbf{x}, \mathbf{x}') = \rho(\mathbf{x}, \mathbf{x}') \tag{1.132}$$

where $\mathcal{D}(\mathbf{x}, \mathbf{x}') = 0$ when $\mathbf{x}, \mathbf{x}' \in Q$ and $\rho(\mathbf{x}, \mathbf{x}')$ is arbitrary. Using the same procedure as above will result in the following equation instead of equation (1.83).

$$-\int_{V_0} dV\, D_0(\mathbf{x}, \mathbf{x}'')\partial_{\mathbf{n}'}\rho(\mathbf{x}, \mathbf{x}') = \sum_{\alpha} \mathrm{PV}_{\mathbf{x}''} \int_{Q_\alpha} ds\, D_0(\mathbf{x}, \mathbf{x}'')\partial_{\mathbf{n}\mathbf{n}'} D(\mathbf{x}, \mathbf{x}') \tag{1.133}$$

For $\mathbf{x}'' \in Q_i, \mathbf{x}' \in Q_j$ and where the volume integral is over the whole domain V_0 modulo the interior volumes V_i.

Because the source, $\rho(\mathbf{x}, \mathbf{x}')$, is arbitrary the regularization in equation (1.73) will impose a condition. To get the above equation the source must satisfy $\partial_{\mathbf{t}'}\rho(\mathbf{x}, \mathbf{x}') = 0$. This will be resolved if $\rho(\mathbf{x}, \mathbf{x}') = 0$ when $\mathbf{x}' \in Q_j$

After discretization the equation for object 1 will be

$$\begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix} \begin{bmatrix} X^{11} \\ X^{21} \end{bmatrix} = \begin{bmatrix} H^{11} \\ H^{12} \end{bmatrix} \tag{1.134}$$

and for object 2

$$\begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix} \begin{bmatrix} X^{12} \\ X^{22} \end{bmatrix} = \begin{bmatrix} H^{21} \\ H^{22} \end{bmatrix} \tag{1.135}$$

where the right hand side is given by

$$H^{ij} = -\int_{V_0} dV\, D_0(\mathbf{x}, \mathbf{x}'')\partial_{\mathbf{n}'}\rho(\mathbf{x}, \mathbf{x}') \quad \mathbf{x}'' \in i, \mathbf{x}' \in j \tag{1.136}$$

The above equations are very similar to the ones that are solved in the original problem. The main difference is that it does not take into account the self stress, but this is only a minor difference since the goal here is to test the equation solver. Alternatively the self stress regularization could be included in this problem, but this would remove any exact solution that could be found

here. Thus the code from the original solver can be used on this problem with minimal modifications.

The parallel plate geometry from section (6.1) will be used to test the code with the following function

$$\mathcal{D}(\mathbf{x},\mathbf{x}') = \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x'}{a}\right)e^{-\left(y^2+y'^2\right)} \tag{1.137}$$

where each plate is located at $x = \pm a/2$ and has length L.

This satisfies the boundary conditions $\mathcal{D}(\mathbf{x},\mathbf{x}') = 0$ when $\mathbf{x},\mathbf{x}'$ are on the plates. From this function the source is found to be

$$\begin{aligned}\rho(\mathbf{x},\mathbf{x}') &= \nabla^2\mathcal{D}(\mathbf{x},\mathbf{x}') - \omega^2\mathcal{D}(\mathbf{x},\mathbf{x}') \\ &\vdots \\ &= -\left(\omega^2 + \left(\frac{\pi}{a}\right)^2 + 2\left(1-2y^2\right)\right)\mathcal{D}(\mathbf{x},\mathbf{x}')\end{aligned} \tag{1.138}$$

and

$$\begin{aligned}&\partial_{\mathbf{n}'}\rho(\mathbf{x},\mathbf{x}') \\ &= -n'_x\frac{\pi}{a}\left(\omega^2 + \left(\frac{\pi}{a}\right)^2 + 2\left(1-2y^2\right)\right)\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x'}{a}\right)e^{-\left(y^2+y'^2\right)}\end{aligned} \tag{1.139}$$

With this on the right side the solution will be

$$\begin{aligned}\partial_{\mathbf{n}\mathbf{n}'}\mathcal{D}(\mathbf{x},\mathbf{x}') &= n_x^2\partial_{xx'}\mathcal{D}(\mathbf{x},\mathbf{x}') \\ &= n_x^2\left(\frac{\pi}{a}\right)^2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x'}{a}\right)e^{-\left(y^2+y'^2\right)}\end{aligned} \tag{1.140}$$

When this expression is evaluated on either plate it is reduced to

$$\partial_{\mathbf{n}\mathbf{n}'}\mathcal{D}(\mathbf{x},\mathbf{x}') = \left(\frac{\pi}{a}\right)^2 e^{-2y^2} \tag{1.141}$$

and if it is evaluated at $y = 0$

$$a^2\partial_{\mathbf{n}\mathbf{n}'}\mathcal{D}(\mathbf{x},\mathbf{x}') = \pi^2 \tag{1.142}$$

Thus as a test for the code it is possible to solve the above system using the given source for a few values of a and then compare this to the exact solution above. Figure (1.3) shows the result of running the source conditions through the solver and figure (1.4) shows more details on the error.

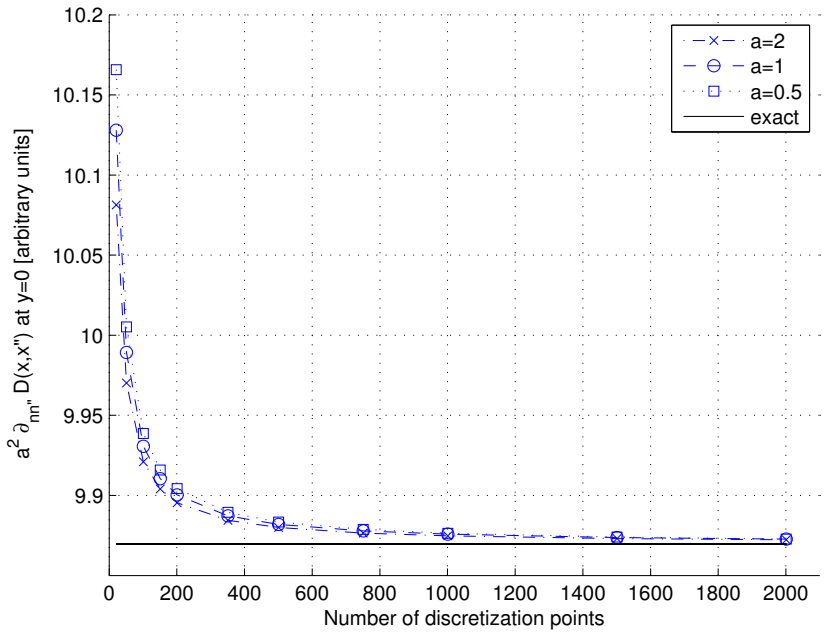

Figure 1.3: The output of the solver using the source initial conditions for a few values of a. The exact solution is π^2 for all a.

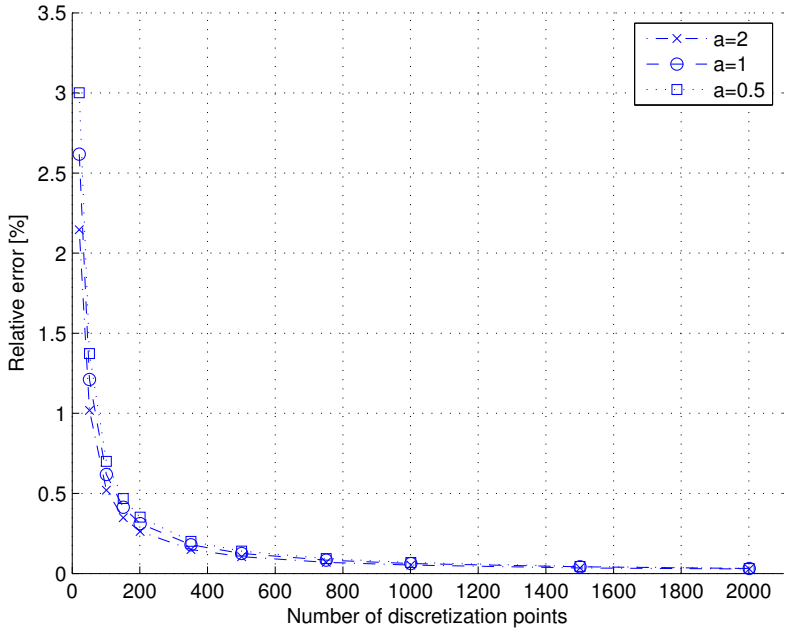

Figure 1.4: The relative error in the output of the solver using the source initial conditions in figure (1.3) for a few values of a.

Chapter 2

Functional Integral method

A method used to calculate the Casimir energy using functional integrals is developed by T. Emig, N. Graham, R.L. Jaffe and M. Kardar [12]. Their article was used as a basis for the implementation of this method. The primary difference between the method described in the article and the following implementation is the use of a spherical basis in the former.

In the following sections the theory needed to calculate the functional integrals will be developed. The object is to show that the energy can be expressed as a functional integral of an exponential. Finally a well known formula will be used to solve the integral.

The final formula for the Casimir energy will involve calculating a determinant of a matrix that contains all the information relating to the geometry of the problem. This geometry will be defined by discretizing the boundary of each object.

2.1 Background

Consider the Lagrangian density

$$\mathcal{L}(\varphi) = \frac{1}{2}(\varphi_t^2 - \nabla\varphi^2) \tag{2.1}$$

and two field configurations $\varphi(\mathbf{x}, t') = a(\mathbf{x})$ and $\varphi(\mathbf{x}, t) = b(\mathbf{x})$. The action associated with these field configurations are defined in terms of the Lagrangian density by

$$S[\varphi] = \int_{t'}^{t} ds \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathcal{L}(\varphi(\mathbf{x}, s)) \tag{2.2}$$

The probability of ending up in configuration $b(\mathbf{x})$ when starting in $a(\mathbf{x})$ is found by summing over all possible connecting paths. The propagator is this transition probability amplitude and is given by

$$G(b, t, a, t') = \int D\varphi\, e^{\frac{i}{\hbar}S[\varphi]} \tag{2.3}$$

Take note that for $S[\varphi]$ of order 1 the primary contribution to the propagator will come from the stationary path. This is the classical path found by taking

the variation of the action and setting this equal to zero. If $S[\varphi]$ is of order $\hbar$, the propagator will be considerably influenced by paths that vary substantially from the classical path. Note that a scaling of the Lagrangian will influence this directly, but if $\hbar$ is scaled this will alter the results from all possible measurable quantities, since they are calculated as expectation values. This is why it is important to make sure to use the proper scaling. The models in thesis only have to be consistent as no result will be compared to any physical measurements.

The interesting configurations are $a(\mathbf{x}, t') = b(\mathbf{x}, t) = 0$. This restriction will be implemented on the set of paths by considering all paths that start at the zero configuration and connects back to the zero configuration during a time T and then in the end let $T \to \infty$. The possible fields will further be restricted by boundary conditions on some space time boundary C.

After these constraints it is clear that the integration should be over T-periodic fields that satisfy the boundary conditions on C. Denote this transition probability amplitude by

$$Z[C, T] = \int D\varphi_{C,T}\, e^{\frac{i}{\hbar}S[\varphi]} \tag{2.4}$$

Consider a scalar quantum field $\hat{\varphi}$ that is a quantization of the scalar field φ. The field equation for $\hat{\varphi}$ is given by

$$\hat{\varphi}_{tt} - \nabla^2\hat{\varphi} = 0 \tag{2.5}$$

The field $\hat{\varphi}$ satisfies the equal time commutation conditions for bosons

$$\begin{aligned} [\hat{\varphi}(\mathbf{x}, t), \hat{\varphi}(\mathbf{x}', t)] &= 0 \\ [\hat{\varphi}_t(\mathbf{x}, t), \hat{\varphi}(\mathbf{x}', t)] &= i\hbar\delta(\mathbf{x} - \mathbf{x}') \end{aligned} \tag{2.6}$$

Let $\hat{H}$ be the energy operator and $\{\Psi_\alpha\}$ be a complete set of energy eigenstates for the scalar field.

$$\hat{H}\Psi_\alpha = E_\alpha\Psi_\alpha \tag{2.7}$$

Assuming completeness the configuration basis $\{\Psi_\varphi\}$ can be formally expanded in an energy basis Ψ_α.

$$\Psi_\varphi = \sum_\alpha \Psi_\varphi[\alpha]\Psi_\alpha \tag{2.8}$$

Where $\Psi_\varphi[\alpha]$ is a functional on the space of classical configurations. Let Ψ_0 be the ground state of the field. The vacuum to vacuum transition amplitude is then given by

$$\begin{aligned} Z[C, T] &= (\Psi_0, e^{-\frac{i}{\hbar}\hat{H}(t-t')}\Psi_0) \\ &= \sum_{\alpha,\beta} \Psi_0^*[\alpha]\Psi_0[\beta](\Psi_\alpha, \Psi_\beta)e^{-\frac{i}{\hbar}E_\beta T} \\ &= \sum_\alpha \Psi_0^*[\alpha]\Psi_0[\alpha]e^{-\frac{i}{\hbar}E_\alpha T} \end{aligned} \tag{2.9}$$

Observe that if this functions is rotated into the complex plane with $T = -is$

$$Z[C, -is] = \sum_\alpha \Psi_0^*[\alpha]\Psi_0[\alpha]e^{-\frac{s}{\hbar}E_\alpha} \tag{2.10}$$

Assume that there exists a lowest energy state E_0 and that all other E_α are higher (and thus contribute less to the sum for large s). In other words it is assumed that the spectrum of the energy operator is bounded from below.

$$Z[C, -is] \approx \Psi_0^*[0]\Psi_0[0]e^{-\frac{s}{\hbar}E_0} \tag{2.11}$$

Take the logarithm on both sides to get

$$\ln\left(Z[C, -is]\right) \approx \ln|\Psi_0[0]|^2 - \tfrac{s}{\hbar}E_0 \qquad s >> \hbar/E_0 \tag{2.12}$$

When s is large the constant factor will disappear and the only remaining part is the propagator

$$E_0 = -\lim_{s\to\infty}\frac{\hbar}{s}\ln Z[C, -is] \tag{2.13}$$

E_0 is the lowest energy level possible for the quantum system. This level is called vacuum energy, ground state energy or *Casimir* energy. The plan is to find an expression for $\ln Z[C, T]$ for large T and evaluate this at $T = -is$.

2.2 Spatial boundary conditions

It is necessary to define how the boundary conditions in the functional integral will be implemented. Consider functions on the plane $\varphi(\mathbf{x})$ and a functional integral

$$\int D\varphi K[\varphi] \tag{2.14}$$

Where $K[\varphi]$ is some functional. Let C be some curve in the plane and assume that the integration should be restricted to functions such that $\varphi|_C = 0$.

Define a parametrization of the curve C by

$$\gamma : [a, b] \to C \tag{2.15}$$

Discretize the curve C with N+1 points chosen as

$$\alpha_j = a + j\Delta t \qquad j = 0, 1, \ldots N \tag{2.16}$$

where $\Delta t = \frac{b-a}{N}$.

Let $t_j \in I_j = [\alpha_{j-1}, \alpha_j]$, $j = 1, \cdots N$ be the midpoints of each interval I_j. Then

$$t_j = a + (j - \frac{1}{2})\Delta t \tag{2.17}$$

Let $\varphi_j = \varphi(\underline{\gamma}(t_j))$. Inserting a product of delta functions into the integral in equation (2.14) will give

$$\int D\varphi \prod_{j=1}^{N}\delta(\varphi_j)K[\varphi] \tag{2.18}$$

Thus the only contribution to the integral is when $\varphi_j = \varphi(\underline{\gamma}(t_j)) = 0$. When $N \to \infty$ the parametrization will be dense on C and the integral will be restricted to functions such that $\varphi|_C = 0$. Define the functional delta function $\delta(\varphi|_C)$ by

$$\lim_{N\to\infty}\int D\varphi \prod_{j=1}^{N}\delta(\varphi_j)K[\varphi] \equiv \int D\varphi\,\delta(\varphi|_C)K[\varphi] \tag{2.19}$$

For each j the standard delta representation is

$$\delta(\varphi_j) = \frac{1}{2\pi}\int da_j e^{ia_j\varphi_j} \tag{2.20}$$

so

$$\prod_{j=1}^{N}\delta(\varphi_j) = \prod_{j=1}^{N}\frac{1}{2\pi}\int da_j\, e^{ia_j\varphi_j} = \int\left(\prod_{j=1}^{N}\frac{da_j}{2\pi}\right)e^{i\sum_j a_j\varphi_j} \tag{2.21}$$

Take a step back and look at a function α, defined on C by

$$\alpha(\underline{\gamma}(t_j)) = \frac{a_j}{||\underline{\gamma}'(t_j)||\Delta t} \tag{2.22}$$

On the curve C parametrized by $\underline{\gamma}$ the derivative can be approximated by the backwards difference.

The norm of the derivative is

$$||\underline{\gamma}'(t_j)|| \approx \frac{1}{\Delta t}||\underline{\gamma}(\alpha_j) - \underline{\gamma}(\alpha_{j-1})|| \tag{2.23}$$

inserting this into the product formula above gives

$$\prod_{j=1}^{N}\delta(\varphi_j) = \int\prod_{j=1}^{N}\left(\frac{||\underline{\gamma}(t_j)||\Delta t}{2\pi}\right)d\alpha(t_j)e^{i\sum_j \alpha(\underline{\gamma}(t_j))\varphi(\underline{\gamma}(t_j))||\underline{\gamma}'(t_j)||\Delta t} \tag{2.24}$$

formally

$$\prod_{j=1}^{N}\left(\frac{||\underline{\gamma}(t_j)||\Delta t}{2\pi}\right)d\alpha(t_j) \to D\alpha \qquad N\to\infty \tag{2.25}$$

Thus the final formal expression for the delta functional is

$$\delta(\varphi|_C) = \int D\alpha e^{i\int_C ds\,\alpha\varphi} \tag{2.26}$$

Inserting this expression into equation (2.19) to get a formal expression for the functional integral with boundary conditions

$$\int D\varphi D\alpha e^{i\int_C ds\,\alpha\varphi}K[\varphi] \tag{2.27}$$

The same approach can be used to define delta functions $\delta(\varphi|_{C'})$ where C' is a space time boundary.

The boundary conditions in equation (2.4) are now changed to

$$Z[C,T] = \int D\varphi_T DJ_T e^{\frac{i}{\hbar}\left(S[\varphi]+\int_C dsJ\varphi\right)} \tag{2.28}$$

Where the exponent of the delta functional is measured in units of $\hbar$ for convenience.

2.3 Periodic boundary conditions

The differentials $D\varphi_T$ and DJ_T indicates that the integration is over T-periodic functions. Since φ and J are T-periodic they can be expanded as Fourier series.

$$\begin{aligned} \varphi(\mathbf{x},t) &= \sum_{n=-\infty}^{\infty} \varphi_n(\mathbf{x}) e^{2\pi i n t/T} \\ J(\mathbf{x},t) &= \sum_{n=-\infty}^{\infty} J_n(\mathbf{x}) e^{2\pi i n t/T} \end{aligned} \tag{2.29}$$

Note that if the fields are real the coefficients satisfy $\varphi_{-n} = \varphi_n^*$ and $J_{-n} = J_n^*$. This will be used later when considering a real scalar field. The result of this simplification will be that the resulting Casimir energy is half of the one found from using complex fields.

Their differentials are

$$\begin{aligned} D\varphi_T &= \prod_{n=-\infty}^{\infty} D\varphi_n \\ DJ_T &= \prod_{n=-\infty}^{\infty} DJ_n \end{aligned} \tag{2.30}$$

There should be a Jacobian here, but this will be canceled later when the Casimir energy is renormalized.

All boundaries will be fixed in time, so the domain is given by $C = Q \times [0,T]$. Start with the integral inside the exponential in equation (2.28).

$$\begin{aligned} \frac{i}{\hbar}\int_C ds\, \varphi(\mathbf{x},t) J(\mathbf{x},t) &= \frac{i}{\hbar}\int_C ds \sum_{m,n} \varphi_n(\mathbf{x}) J_m(\mathbf{x}) e^{2\pi i (n+m)t/T} \\ &= \frac{i}{\hbar}\sum_{m,n}\int_Q dA\, \varphi_n(\mathbf{x}) J_m(\mathbf{x}) \int_0^T dt\, e^{2\pi i (n+m)t/T} \\ &= \frac{iT}{\hbar}\sum_n \int_Q dA\, \varphi_n(\mathbf{x}) J_{-n}(\mathbf{x}) \end{aligned} \tag{2.31}$$

This extra term modifies the action $S[\varphi]$ to account for boundary conditions. Consider $S[\varphi]$ defined by equation (2.2) with the *real* massless scalar field given by equation (2.1)

$$\begin{aligned} S[\varphi] &= \int_{t'}^{t} ds \int_{\mathbb{R}^2} d^2\mathbf{x}\, \mathcal{L}(\varphi(\mathbf{x},s)) = \int_0^T ds \int_{\mathbb{R}^2} d^2\mathbf{x}\, \frac{1}{2}(\varphi_t^2 - \nabla\varphi\nabla\varphi) \\ &= \int_0^T ds \int_{\mathbb{R}^2} d^2\mathbf{x}\, \frac{1}{2}\sum_{n,m}\Bigg(\frac{2\pi i n}{T}\frac{2\pi i m}{T}\varphi_n(\mathbf{x})\varphi_m(\mathbf{x}) e^{2\pi i (n+m)t/T} \\ &\qquad - \nabla\varphi_n(\mathbf{x})\nabla\varphi_m(\mathbf{x}) e^{2\pi i (n+m)t/T}\Bigg) \\ &= \sum_n \int_{\mathbb{R}^2} d^2\mathbf{x}\, \frac{T}{2}\left(\left(\frac{2\pi n}{T}\right)^2 \varphi_n(\mathbf{x})\varphi_{-n}(\mathbf{x}) - \nabla\varphi_n(\mathbf{x})\nabla\varphi_{-n}(\mathbf{x})\right) \end{aligned} \tag{2.32}$$

Including the above calculations in the propagator given by equation (2.28).

$$\begin{aligned} Z[Q,T] &= \int D\varphi_T DJ_T e^{\frac{i}{\hbar}\left(S[\varphi]+\int_C ds J\varphi\right)} \\ &= \int \prod_{m=-\infty}^{\infty} D\varphi_m DJ_m e^{\frac{iT}{\hbar}\sum_n \left(\int dV\, \frac{1}{2}\left(\left(\frac{2\pi n}{T}\right)^2 \varphi_n(\mathbf{x})\varphi_{-n}(\mathbf{x}) \right.\right.} \\ &\qquad \left.\left. - \nabla\varphi_n(\mathbf{x})\nabla\varphi_{-n}(\mathbf{x})\right)+\int_Q dA\, \varphi_n J_{-n}\right) \\ &= \prod_{n=-\infty}^{\infty} \int D\varphi_n DJ_n e^{\frac{iT}{\hbar}\left(\int dV\, \frac{1}{2}\left(\left(\frac{2\pi n}{T}\right)^2 \varphi_n(\mathbf{x})\varphi_{-n}(\mathbf{x}) \right.\right.} \\ &\qquad \left.\left. - \nabla\varphi_n(\mathbf{x})\nabla\varphi_{-n}(\mathbf{x})\right)+\int_Q dA\, \varphi_n J_{-n}\right) \end{aligned} \tag{2.33}$$

so

$$\begin{aligned} \ln Z[Q,T] = \sum_{n=-\infty}^{\infty} \ln \int D\varphi_n DJ_n e^{\frac{iT}{\hbar}\left(\int dV\, \frac{1}{2}\left(\left(\frac{2\pi n}{T}\right)^2 \varphi_n(\mathbf{x})\varphi_{-n}(\mathbf{x}) \right.\right.} \\ \left.\left. - \nabla\varphi_n(\mathbf{x})\nabla\varphi_{-n}(\mathbf{x})\right)+\int_Q dA\, \varphi_n J_{-n}\right) \end{aligned} \tag{2.34}$$

Consider this sum for large T and introduce a continuous variable k discretized by $k_n = 2\pi n/T$ and the spacing $\Delta k_n = 2\pi/T << 1$. Then the sum becomes an integral

$$\sum_{n=-\infty}^{\infty} f_n = \frac{1}{\Delta k_n} \sum_{n=-\infty}^{\infty} \Delta k_n\, f_n \approx \frac{T}{2\pi} \int_{-\infty}^{\infty} dk\, f_k \tag{2.35}$$

Then for large T

$$\begin{aligned} \ln Z[Q,T] &= \frac{T}{2\pi} \int_{-\infty}^{\infty} dk\, \ln \int D\varphi_k DJ_k e^{\frac{iT}{\hbar}\left(\int dV\, \frac{1}{2}\left(k^2\varphi_k(\mathbf{x})\varphi_{-k}(\mathbf{x}) \right.\right.} \\ &\qquad \left.\left. - \nabla\varphi_k(\mathbf{x})\nabla\varphi_{-k}(\mathbf{x})\right)+\int_Q dA\, \varphi_k J_{-k}\right) \\ &= \frac{T}{2\pi} \int_{-\infty}^{0} dk\, \ln \int D\varphi_k DJ_k e^{\frac{iT}{\hbar}\left(\int dV\, \frac{1}{2}\left(k^2\varphi_k(\mathbf{x})\varphi_{-k}(\mathbf{x}) \right.\right.} \\ &\qquad \left.\left. - \nabla\varphi_k(\mathbf{x})\nabla\varphi_{-k}(\mathbf{x})\right)+\int_Q dA\, \varphi_k J_{-k}\right) \\ &\quad + \frac{T}{2\pi} \int_{0}^{\infty} dk\, \ln \int D\varphi_k DJ_k e^{\frac{iT}{\hbar}\left(\int dV\, \frac{1}{2}\left(k^2\varphi_k(\mathbf{x})\varphi_{-k}(\mathbf{x}) \right.\right.} \\ &\qquad \left.\left. - \nabla\varphi_k(\mathbf{x})\nabla\varphi_{-k}(\mathbf{x})\right)+\int_Q dA\, \varphi_k J_{-k}\right) \\ &= \frac{T}{2\pi} \int_{0}^{\infty} dk\, \ln \int D\varphi_k DJ_k D\varphi_k^* DJ_k^* e^{\frac{iT}{\hbar}\left(\int dV \left(k^2\varphi_k(\mathbf{x})\varphi_k^*(\mathbf{x}) \right.\right.} \\ &\qquad \left.\left. - \nabla\varphi_k(\mathbf{x})\nabla\varphi_k^*(\mathbf{x})\right)+\int_Q dA\, (\varphi_k(\mathbf{x})J_k^*(\mathbf{x})+\varphi_k^*(\mathbf{x})J_k(\mathbf{x}))+c.c.\right) \end{aligned} \tag{2.36}$$

This can be summarized into a few simple formulas:

$$\ln Z[Q,T] = \frac{T}{2\pi} \int_{0}^{\infty} dk\, \ln B_Q(k) \tag{2.37}$$

where

$$B_Q(k) = \int D\varphi_k DJ_k D\varphi_k^* DJ_k^* e^{\frac{iT}{\hbar}\hat{S}[\varphi_k,\varphi_k^*,J_k,J_k^*]} \tag{2.38}$$

and the modified action $\hat{S}[\ldots]$ is

$$\begin{aligned}\hat{S}[\varphi_k,\varphi_k^*,J_k,J_k^*] = &\int dV\ \left(k^2\varphi_k(\mathbf{x})\varphi_k^*(\mathbf{x}) - \nabla\varphi_k(\mathbf{x})\nabla\varphi_k^*(\mathbf{x})\right) \\ &+ \int\limits_Q dA\ (\varphi_k(\mathbf{x})J_k^*(\mathbf{x}) + \varphi_k^*(\mathbf{x})J_k(\mathbf{x})) + c.c.\end{aligned} \tag{2.39}$$

It will be necessary to evaluate $Z[C,T]$ on the imaginary time axis $T = -is$.

$$\ln Z[C,-is] = -\frac{is}{2\pi}\int\limits_0^\infty dk\ \ln B_Q(k)|_{T=-is} \tag{2.40}$$

Assuming that $B_Q(k)$ has no poles in the complex plane it is possible to rotate the integration curve to the imaginary axis with $k = i\kappa$

$$\ln Z[C,-is] = \frac{s}{2\pi}\int\limits_0^\infty d\kappa\ \ln B_Q(i\kappa)|_{T=-is} \tag{2.41}$$

This is the current expression for the propagator. The next step is to examine how the boundary conditions modify the action.

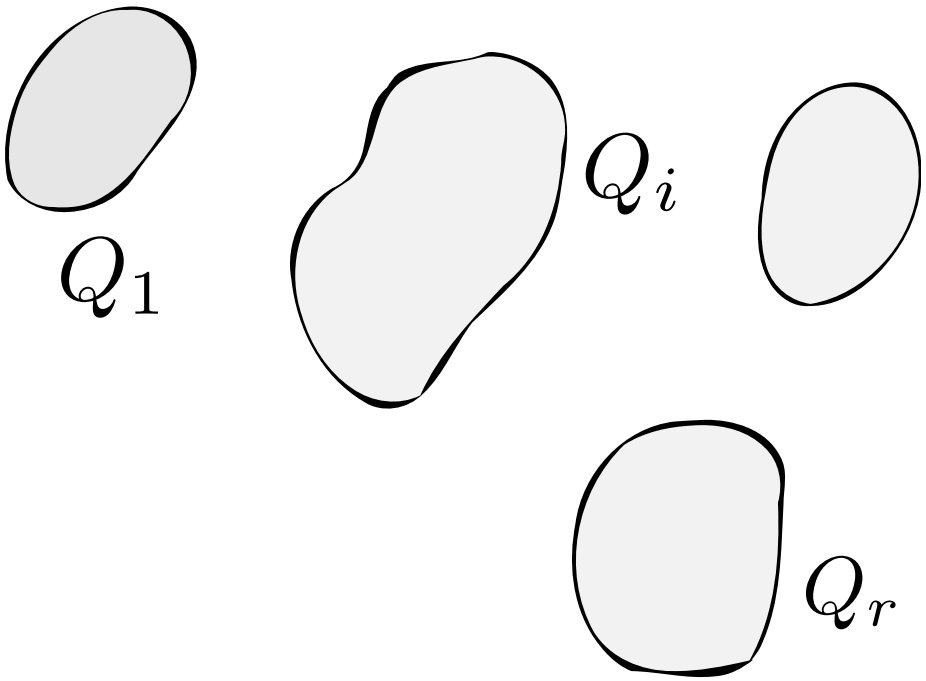

Figure 2.1: Illustration of the objects with marked interiors for the problem.

2.4 Modified action

Consider a type of problem where there are $\alpha = 1, \ldots, ..r$ disjoint compact pieces each with volume V_α and surface Q_α. V_0 is the complement of all objects, $V_0 = (\cup_i V_i)^c$, this will be the exterior for this problem. Figure 2.1 illustrates the situations. Denote the source on the surface Q_α as J_n^α. The calculations that lead up to equations (2.38) and (2.39) are modified slightly to include multiple disjoint boundaries Q_α.

This results in

$$B_Q(k) = \int D\varphi_k D\varphi_k^* \prod_{\alpha=1}^{r} DJ_k^\alpha DJ_k^{\alpha *} e^{\frac{iT}{\hbar}\hat{S}[\varphi_k, \varphi_k^*, \{J_k^\alpha\}_\alpha, \{J_k^{\alpha *}\}_\alpha]} \tag{2.42}$$

and

$$\begin{aligned} &\hat{S}[\varphi_k, \varphi_k^*, \{J_k^\alpha\}_\alpha, \{J_k^{\alpha *}\}_\alpha] \\ &= \int dV \, \left(k^2 \varphi_k(\mathbf{x})\varphi_k^*(\mathbf{x}) - \nabla\varphi_k(\mathbf{x})\nabla\varphi_k^*(\mathbf{x})\right) \\ &\qquad + \sum_{\alpha=1}^{r} \int_{Q_\alpha} dA \, \left(\varphi_k(\mathbf{x}) J_k^{\alpha *}(\mathbf{x}) + \varphi_k^*(\mathbf{x}) J_k^\alpha(\mathbf{x})\right) + c.c. \end{aligned} \tag{2.43}$$

In order to find the classical field solution $\varphi_{cl,k}$ for the action one should take the variational derivative of $\hat{S}[\ldots]$ from equation (2.43) with respect to φ_k^*.

Consider a variation from the classical solution

$$
\begin{aligned}
&\hat{S}[\varphi_{cl,k}, \varphi^*_{cl,k} + \delta\varphi^*_k, \{J^\alpha_k\}, \{J^{\alpha *}_k\}] \\
&= \int\limits_{\mathbb{R}^n} dV \, \left(k^2 \varphi_{cl,k}(\mathbf{x}) \left(\varphi^*_{cl,k} + \delta\varphi^*_k\right)(\mathbf{x}) - \nabla\varphi_{cl,k}(\mathbf{x}) \nabla \left(\varphi^*_{cl,k} + \delta\varphi^*_k\right)(\mathbf{x})\right) \\
&\qquad + \sum_{\alpha=1}^{r} \int\limits_{Q_\alpha} dA \, \left(\varphi_{cl,k}(\mathbf{x}) J^{\alpha *}_k(\mathbf{x}) + \left(\varphi^*_{cl,k}(\mathbf{x}) + \delta\varphi^*_k(\mathbf{x})\right) J^\alpha_k\right) + c.c. \\
&= \hat{S}[\varphi_{cl,k}, \varphi^*_{cl,k}, \{J^\alpha_k\}, \{J^{\alpha *}_k\}] \\
&\qquad + \int\limits_{\mathbb{R}^n} dV \, \left(k^2 \varphi_{cl,k}(\mathbf{x}) \delta\varphi^*_k(\mathbf{x}) - \nabla\varphi_{cl,k}(\mathbf{x}) \nabla\delta\varphi^*_k(\mathbf{x})\right) \\
&\qquad + \sum_{\alpha=1}^{r} \int\limits_{Q_\alpha} dA \, \delta\varphi^*_k(\mathbf{x}) J^\alpha_k(\mathbf{x}) + c.c. \qquad (2.44) \\
&= \hat{S}[\varphi_{cl,k}, \varphi^*_{cl,k}, \{J^\alpha_k\}, \{J^{\alpha *}_k\}] \\
&\qquad + \int\limits_{V_0} dV \, \left(k^2 \varphi_{cl,k}(\mathbf{x}) \delta\varphi^*_k(\mathbf{x}) - \nabla\varphi_{cl,k}(\mathbf{x}) \nabla\delta\varphi^*_k(\mathbf{x})\right) \\
&\qquad + \sum_{\alpha=1}^{r} \int\limits_{V_\alpha} dV \, \left(k^2 \varphi_{cl,k}(\mathbf{x}) \delta\varphi^*_k(\mathbf{x}) - \nabla\varphi_{cl,k}(\mathbf{x}) \nabla\delta\varphi^*_k(\mathbf{x})\right) \\
&\qquad + \sum_{\alpha=1}^{r} \int\limits_{Q_\alpha} dA \, \delta\varphi^*_k(\mathbf{x}) J^\alpha_k(\mathbf{x}) + c.c.
\end{aligned}
$$

Use Green's identity to change the volume integrals into boundary integrals.

$$
\int\limits_V dV \, \nabla f \nabla g = \int\limits_{\partial V} dA \, \partial_{\mathbf{n}} f \cdot g - \int\limits_V dV \, \nabla^2 f \cdot g \qquad (2.45)
$$

Where the normal $\mathbf{n}$ should be oriented to point out of the volume V. In order to define the normal derivative on the boundaries Q_α it is helpful to separate the solutions in V_0 and V_α with

$$
\begin{aligned}
\varphi_-(\mathbf{t}) &= \lim_{\substack{\mathbf{x}\to\mathbf{t} \\ \mathbf{x}\in V_\alpha \\ \mathbf{t}\in Q_\alpha}} \varphi_{cl,k}(\mathbf{x}) \\
\varphi_+(\mathbf{t}) &= \lim_{\substack{\mathbf{x}\to\mathbf{t} \\ \mathbf{x}\in V_0 \\ \mathbf{t}\in Q_\alpha}} \varphi_{cl,k}(\mathbf{x})
\end{aligned} \qquad (2.46)
$$

Thus

$$\begin{aligned}
&\hat{S}[\ldots,\varphi^*_{cl,k}+\delta\varphi^*_k,\ldots]\\
&=\hat{S}[\ldots]+\int\limits_{\mathbb{R}^n} dV\ \left(k^2\varphi_{cl,k}(\mathbf{x})+\nabla^2\varphi_{cl,k}(\mathbf{x})\right)\delta\varphi^*_k(\mathbf{x})\\
&\qquad+\sum_{\alpha=1}^{r}\int\limits_{Q_\alpha} dA\,\partial_{\mathbf{n}}\varphi_+(\mathbf{x})\delta\varphi^*_k(\mathbf{x})-\sum_{\alpha=1}^{r}\int\limits_{Q_\alpha} dA\,\partial_{\mathbf{n}}\varphi_-(\mathbf{x})\delta\varphi^*_k(\mathbf{x})\\
&\qquad+\sum_{\alpha=1}^{r}\int\limits_{Q_\alpha} dA\,\delta\varphi^*_k(\mathbf{x})J^\alpha_k(\mathbf{x})+c.c.\\
&=\hat{S}[\ldots]+\int\limits_{\mathbb{R}^n} dV\ \left(k^2\varphi_{cl,k}(\mathbf{x})+\nabla^2\varphi_{cl,k}(\mathbf{x})\right)\delta\varphi^*_k(\mathbf{x})\\
&\qquad+\sum_{\alpha=1}^{r}\int\limits_{Q_\alpha} dA\ \left(J^\alpha_k(\mathbf{x})-\Delta\partial_{\mathbf{n}}\varphi_{cl,k}(\mathbf{x})\right)\delta\varphi^*_k(\mathbf{x})+c.c.
\end{aligned}\tag{2.47}$$

Where $\Delta\partial_{\mathbf{n}}\varphi_{cl,k}(\mathbf{x})=\partial_{\mathbf{n}}\varphi_-(\mathbf{x})-\partial_{\mathbf{n}}\varphi_+(\mathbf{x})$. Since $\delta\varphi^*$ is arbitrary

$$\begin{aligned}
\nabla^2\varphi_{cl,k}+k^2\varphi_{cl,k}&=0 \qquad &\mathbf{x}\notin Q_\alpha\\
\Delta\varphi_{cl,k}&=0 &\mathbf{x}\in Q_\alpha\\
\Delta\partial_{\mathbf{n}}\varphi_{cl,k}&=J^\alpha_k &\mathbf{x}\in Q_\alpha
\end{aligned}\tag{2.48}$$

Thus the stationary field $\varphi_{cl,k}$ satisfy a scattering problem with fixed sources J^α_k on the curves Q_α. Use the solution $\varphi_{cl,k}$ to change variables in the functional integral.

$$\begin{aligned}
\varphi_k&=\varphi_{cl,k}+\theta_k\\
\varphi^*_k&=\varphi^*_{cl,k}+\theta^*_k
\end{aligned}\tag{2.49}$$

The action from equation (2.43) is changed into

$$\begin{aligned}
&\hat{S}[\varphi_{cl,k}+\theta_k,\varphi^*_{cl,k}+\theta^*_k,\{J^\alpha_k\},\{J^{\alpha*}_k\}]\\
&=\int\limits_{\mathbb{R}^n} dV\ \left(k^2(\varphi_{cl,k}+\theta_k)(\varphi^*_{cl,k}+\theta^*_k)-\nabla(\varphi_{cl,k}+\theta_k)\nabla(\varphi^*_{cl,k}+\theta^*_k)\right)\\
&\qquad+\sum_\alpha\int\limits_{Q_\alpha} dA\ \left(J^{\alpha*}_k(\varphi_{cl,k}+\theta_k)+J^\alpha_k(\varphi^*_{cl,k}+\theta^*_k)\right)+c.c.
\end{aligned}\tag{2.50}$$

Since $\varphi_{cl,k}$ is a solution to (2.48) the action simplifies into

$$\begin{aligned}
&\hat{S}[\varphi_{cl,k}+\theta_k,\varphi^*_{cl,k}+\theta^*_k,\{J^\alpha_k\},\{J^{\alpha*}_k\}]\\
&=\hat{S}[\ldots]+\int\limits_{\mathbb{R}^n} dV\ \left(k^2\theta_k\theta^*_k-\nabla\theta_k\nabla\theta^*_k\right)+\sum_\alpha\int\limits_{Q_\alpha} dA\ \left(J^{\alpha*}_k\theta_k+J^\alpha_k\theta^*_k\right)+c.c.
\end{aligned}\tag{2.51}$$

Notice that the second part is geometry independent and will later cancel when the regularized Casimir energy is calculated. The same applies to the first part of the action itself in equation (2.43) and it is advantageous to redefine the

action in order to only account for the geometry dependent integrals.

$$\hat{S}[\varphi_{cl,k}, \varphi^*_{cl,k}, \{J^\alpha_k\}, \{J^{\alpha *}_k\}] = \sum_\alpha \int\limits_{Q_\alpha} dA \, \left(J^{\alpha *}_k \varphi_{cl,k} + J^\alpha_k \varphi^*_{cl,k} \right) + c.c. \tag{2.52}$$

The functional integrals in equation (2.42) will only involve integrals over the sources J^α_k and $J^{\alpha *}_k$. Everything else will cancel when the energy is renormalized. Redefine $B_Q(k)$ here for reference.

$$B_Q(k) = \int \prod_{\alpha=1}^{r} DJ^\alpha_k DJ^{\alpha *}_k e^{\frac{iT}{\hbar} \hat{S}[\varphi_k, \varphi^*_k, \{J^\alpha_k\}_\alpha, \{J^{\alpha *}_k\}_\alpha]} \tag{2.53}$$

2.5 Scattering solutions

The equations in (2.48) are linear and thus the solution can be written as a linear superposition.

$$\varphi_{cl,k} = \sum_\beta \varphi_\beta \tag{2.54}$$

Where φ_β is the solution when $J^\alpha_k = 0$ for $\alpha \neq \beta$ and in effect there is a single scattering object Q_β. The action $\hat{S}[\ldots]$ from equation (2.52) can be written as

$$\hat{S}[\varphi_{cl,k}, \varphi^*_{cl,k}, \{J^\alpha_k\}, \{J^{\alpha *}_k\}] = \sum_{\alpha\beta} S_{\alpha\beta} \tag{2.55}$$

where

$$S_{\alpha\beta} = \int\limits_{Q_\alpha} dA \, \left(J^{\alpha *}_k \varphi_{\beta,k} + J^\alpha_k \varphi^*_{\beta,k} \right) \tag{2.56}$$

For each object V_α choose a coordinate system $\mathcal{O}_\alpha$ with coordinates denoted $\mathbf{x}_\alpha$. The origin of the coordinate system is inside V_α. With respect to $\mathcal{O}_\alpha$ choose a complete set of functions $\{a^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha)\}$ defined on the curve Q_α. In general $\mathbf{i}_\alpha$ will be a multi index. For a complete set of functions, $a^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha)$, any reasonable function f on Q_α can be written as

$$f(\mathbf{x}_\alpha) = \textstyle\sum_{\mathbf{i}_\alpha} f_{\mathbf{i}_\alpha} a^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) \qquad \mathbf{x}_\alpha \in Q_\alpha \tag{2.57}$$

Let $G^\alpha(\mathbf{x}_\alpha, \mathbf{x}'_\alpha)$ be the Green's function for the operator $\mathcal{L} = \nabla^2 + k^2$ (outgoing at infinity) in the coordinate system $\mathcal{O}_\alpha$.

Thus

$$\mathcal{L}_\alpha G^\alpha(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) = -\delta_\alpha(\mathbf{x}_\alpha - \mathbf{x}'_\alpha) \tag{2.58}$$

where $\mathcal{L}_\alpha$ and δ_α are the operator $\mathcal{L}$ and the delta distribution in the the coordinates $\mathcal{O}_\alpha$.

The plan is to expand the action $S_{\alpha\beta}$ in the basis $a^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha)$. Consider the two cases $S_{\alpha\alpha}$ and $S_{\alpha\beta}$, for $\alpha \neq \beta$, separately.

2.5.1 Expanding φ_α over $\mathcal{O}_\alpha$

Expand all functions connected to object V_α in the basis $\mathcal{O}_\alpha$.

By completeness this implies

$$J^\alpha(\mathbf{x}_\alpha) = \sum_{\mathbf{j}_\alpha} z^\alpha_{\mathbf{j}_\alpha} a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) \tag{2.59}$$

and

$$G^\alpha(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) = \sum_{\mathbf{i}_\alpha} G^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}'_\alpha) a^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) \tag{2.60}$$

The properties of the Green's functions let us us write the each φ_α in (2.54) as

$$\varphi_\alpha(\mathbf{x}_\alpha) = \int\limits_{Q_\alpha} dA_{\mathbf{x}'_\alpha}\, G^\alpha(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) J^\alpha(\mathbf{x}'_\alpha) \tag{2.61}$$

Combine this to get

$$\begin{aligned}
\varphi_\alpha(\mathbf{x}_\alpha) &= \int\limits_{Q_\alpha} dA_{\mathbf{x}'_\alpha} \sum_{\mathbf{i}_\alpha} G^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}'_\alpha) a^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) \sum_{\mathbf{j}_\alpha} z^\alpha_{\mathbf{j}_\alpha} a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha) \\
&= \sum_{\mathbf{i}_\alpha, \mathbf{j}_\alpha} z^\alpha_{\mathbf{j}_\alpha} a^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) \int\limits_{Q_\alpha} dA_{\mathbf{x}'_\alpha}\, G^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}'_\alpha) a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha) \\
&= \sum_{\mathbf{i}_\alpha, \mathbf{j}_\alpha} G^\alpha_{\mathbf{i}_\alpha \mathbf{j}_\alpha} z^\alpha_{\mathbf{j}_\alpha} a^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha)
\end{aligned} \tag{2.62}$$

where

$$G^\alpha_{\mathbf{i}_\alpha \mathbf{j}_\alpha} = \int\limits_{Q_\alpha} dA_{\mathbf{x}'_\alpha}\, G^\alpha_{\mathbf{i}_\alpha}(\mathbf{x}'_\alpha) a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha) \tag{2.63}$$

With this the integrals in our action can be expanded

$$\begin{aligned}
&\int\limits_{Q_\alpha} dA_{\mathbf{x}_\alpha}\, J^{\alpha *}_k(\mathbf{x}_\alpha) \varphi_\alpha(\mathbf{x}_\alpha) \\
&= \int\limits_{Q_\alpha} dA_{\mathbf{x}_\alpha} \sum_{\mathbf{i}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} a^{\alpha *}_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) \sum_{\mathbf{j}_\alpha, \mathbf{k}_\alpha} G^\alpha_{\mathbf{j}_\alpha, \mathbf{k}_\alpha} z^\alpha_{\mathbf{k}_\alpha} a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) \\
&= \sum_{\mathbf{j}_\alpha, \mathbf{i}_\alpha, \mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} G^\alpha_{\mathbf{j}_\alpha, \mathbf{k}_\alpha} z^\alpha_{\mathbf{k}_\alpha} \int\limits_{Q_\alpha} dA_{\mathbf{x}_\alpha}\, a^{\alpha *}_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) \\
&= \sum_{\mathbf{j}_\alpha, \mathbf{i}_\alpha, \mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} D^\alpha_{\mathbf{i}_\alpha, \mathbf{j}_\alpha} G^\alpha_{\mathbf{j}_\alpha, \mathbf{k}_\alpha} z^\alpha_{\mathbf{k}_\alpha} \\
&= \sum_{\mathbf{i}_\alpha, \mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} H^\alpha_{\mathbf{i}_\alpha \mathbf{k}_\alpha} z^\alpha_{\mathbf{k}_\alpha}
\end{aligned} \tag{2.64}$$

where

$$D^\alpha_{\mathbf{i}_\alpha \mathbf{j}_\alpha} = \int\limits_{Q_\alpha} dA_{\mathbf{x}_\alpha}\, a^{\alpha *}_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) \tag{2.65}$$

and

$$H^{\alpha}_{\mathbf{i}_\alpha \mathbf{k}_\alpha} = \sum_{\mathbf{j}_\alpha} D^{\alpha}_{\mathbf{i}_\alpha \mathbf{j}_\alpha} G^{\alpha}_{\mathbf{j}_\alpha \mathbf{k}_\alpha} \tag{2.66}$$

Note that if the basis is orthonormal this will result in

$$D^{\alpha}_{\mathbf{i}_\alpha, \mathbf{j}_\alpha} = \delta_{\mathbf{i}_\alpha, \mathbf{j}_\alpha} \tag{2.67}$$

2.5.2 Expanding φ_α over $\mathcal{O}_\beta$

Evaluate the field φ_α in coordinates system $\mathcal{O}_\beta$ by evaluating the basis functions

$$\varphi_\alpha(\mathbf{x}_\beta) \equiv \varphi_\alpha(\mathbf{x}_\alpha(\mathbf{x}_\beta)) \tag{2.68}$$

And complete the same evaluation as in the previous section.

First from φ_α in equation (2.61)

$$\varphi_\alpha(\mathbf{x}_\beta) = \varphi_\alpha(\mathbf{x}_\alpha(\mathbf{x}_\beta)) = \int_{Q_\alpha} dA_{\mathbf{x}'_\alpha}\, G^{\alpha}(\mathbf{x}_\alpha(\mathbf{x}_\beta), \mathbf{x}'_\alpha) J^{\alpha}(\mathbf{x}'_\alpha) \tag{2.69}$$

Define

$$G^{\beta\alpha}(\mathbf{x}_\beta, \mathbf{x}'_\alpha) = G^{\alpha}(\mathbf{x}_\alpha(\mathbf{x}_\beta), \mathbf{x}'_\alpha) \tag{2.70}$$

where the series expansion is given by

$$G^{\beta\alpha}(\mathbf{x}_\beta, \mathbf{x}'_\alpha) = \sum_{\mathbf{i}_\beta} G^{\beta\alpha}_{\mathbf{i}_\beta}(\mathbf{x}'_\alpha) a^{\alpha}_{\mathbf{i}_\beta}(\mathbf{x}_\beta) \tag{2.71}$$

Use this in the integral (2.69) to get

$$\begin{aligned} \varphi_\alpha(\mathbf{x}_\beta) &= \int_{Q_\alpha} dA_{\mathbf{x}'_\alpha} \sum_{\mathbf{i}_\beta} G^{\beta\alpha}_{\mathbf{i}_\beta}(\mathbf{x}'_\alpha) a^{\beta}_{\mathbf{i}_\beta}(\mathbf{x}_\beta) \sum_{\mathbf{j}_\alpha} z^{\alpha}_{\mathbf{j}_\alpha} a^{\alpha}_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha) \\ &= \sum_{\mathbf{i}_\beta, \mathbf{j}_\alpha} G^{\beta\alpha}_{\mathbf{i}_\beta \mathbf{j}_\alpha} z^{\alpha}_{\mathbf{j}_\alpha} a^{\beta}_{\mathbf{i}_\beta}(\mathbf{x}_\beta) \end{aligned} \tag{2.72}$$

where

$$G^{\beta\alpha}_{\mathbf{i}_\beta \mathbf{j}_\alpha} = \int_{Q_\alpha} dA_{\mathbf{x}'_\alpha}\, G^{\beta\alpha}_{\mathbf{i}_\beta}(\mathbf{x}'_\alpha) a^{\alpha}_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha) \tag{2.73}$$

Expand the integrals in our action

$$\begin{aligned} &\int_{Q_\alpha} dA_{\mathbf{x}_\alpha}\, J^{\alpha *}_k(\mathbf{x}_\alpha) \varphi_\beta(\mathbf{x}_\alpha) \\ &= \int_{Q_\alpha} dA_{\mathbf{x}_\alpha} \sum_{\mathbf{i}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} a^{\alpha *}_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) \sum_{\mathbf{j}_\alpha, \mathbf{k}_\beta} G^{\alpha\beta}_{\mathbf{j}_\alpha \mathbf{k}_\beta} z^{\beta}_{\mathbf{k}_\beta} a^{\alpha}_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) \\ &= \sum_{\mathbf{i}_\alpha, \mathbf{j}_\alpha, \mathbf{k}_\beta} z^{\alpha *}_{\mathbf{i}_\alpha} G^{\alpha\beta}_{\mathbf{j}_\alpha \mathbf{k}_\beta} z^{\beta}_{\mathbf{k}_\beta} \int_{Q_\alpha} dA_{\mathbf{x}_\alpha}\, a^{\alpha *}_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) a^{\alpha}_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) \\ &= \sum_{\mathbf{i}_\alpha, \mathbf{j}_\alpha, \mathbf{k}_\beta} z^{\alpha *}_{\mathbf{i}_\alpha} D^{\alpha}_{\mathbf{i}_\alpha \mathbf{j}_\alpha} G^{\alpha\beta}_{\mathbf{j}_\alpha \mathbf{k}_\beta} z^{\beta}_{\mathbf{k}_\beta} \\ &= \sum_{\mathbf{i}_\alpha, \mathbf{k}_\beta} z^{\alpha *}_{\mathbf{i}_\alpha} K^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta} z^{\beta}_{\mathbf{k}_\beta} \end{aligned} \tag{2.74}$$

where $D^{\alpha}_{\mathbf{i}_\alpha \mathbf{j}_\alpha}$ was defined in equation (2.65).

And

$$K^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta} = \sum_{\mathbf{j}_\alpha} D^{\alpha}_{\mathbf{i}_\alpha \mathbf{j}_\alpha} G^{\alpha\beta}_{\mathbf{j}_\alpha \mathbf{k}_\beta} \tag{2.75}$$

Thus

$$\begin{aligned} S &= \sum_{\alpha} S_{\alpha\alpha} + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} S_{\alpha\beta} \\ &= \sum_{\alpha} \sum_{\mathbf{i}_\alpha, \mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} H^{\alpha}_{\mathbf{i}_\alpha, \mathbf{k}_\alpha} z^{\alpha}_{\mathbf{k}_\alpha} + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \sum_{\mathbf{i}_\alpha, \mathbf{k}_\beta} z^{\alpha *}_{\mathbf{i}_\alpha} K^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta} z^{\beta}_{\mathbf{k}_\beta} + c.c. \end{aligned} \tag{2.76}$$

The action is comprised of two parts: the interaction, $K^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta}$, and the self energy $H^{\alpha}_{\mathbf{i}_\alpha, \mathbf{k}_\alpha}$. These matrices contain the integrals of the Green's function with the basis functions given by (2.63) and (2.73). When the frequency, k, is large the interaction part of the action will disappear and the action will only depend on the self energy.

Thus

$$S \to S_\infty \equiv \sum_{\alpha} \sum_{\mathbf{i}_\alpha, \mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} H^{\alpha}_{\mathbf{i}_\alpha, \mathbf{k}_\alpha} z^{\alpha}_{\mathbf{k}_\alpha} + c.c. \tag{2.77}$$

It is important to note that the integrals defining $B_Q(i\kappa)$ in equation (2.53) are Gaussian. This is the only functional integral that can be solved exactly.

2.6 Gaussian Integrals

This elementary 1D-integral is found in any textbook [14] (p. 271)

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}} \tag{2.78}$$

a product of integrals with different coefficients a_j and variables x_j will be

$$\int_{\mathbb{R}^n} d\mathbf{x}\, e^{-\frac{1}{2}\sum_j a_j x_j^2} = (2\pi)^{n/2} \left(\sqrt{\prod_j a_j} \right)^{-1/2} \tag{2.79}$$

Let A be a $n \times n$ diagonal matrix with the coefficients a_j on the diagonal. Then

$$< \mathbf{x}, A\mathbf{x} >= \sum_j a_j x_j^2 \tag{2.80}$$

and $\det(A) = \prod_j a_j$. The above integral is

$$\int_{\mathbb{R}^n} d\mathbf{x}\, e^{-\frac{1}{2}<\mathbf{x}, A\mathbf{x}>} = (2\pi)^{n/2} \det(A)^{-1/2} \tag{2.81}$$

This will hold for any real symmetric matrix. Because any such matrix can be reduced to diagonal form with an orthogonal change of variables.

How to define this relation for functional integrals? First define a inner product of two functions $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ by.

$$< \varphi, \psi >= \int d\mathbf{x}\, \varphi(\mathbf{x})\psi(\mathbf{x}) \tag{2.82}$$

Let A be a self-adjoint operator and consider

$$\int D\varphi\, e^{-\frac{1}{2}<\varphi, A\varphi>} \tag{2.83}$$

Since A is self-adjoint there is a complete set of orthonormal eigenfunctions $\{\varphi_i\}$ with corresponding eigenvalues $\{\lambda_i\}$. From a change of variables

$$\varphi = \sum_i a_i \varphi_i \tag{2.84}$$

Then

$$< \varphi, A\varphi >= \sum_i \lambda_i a_i^2 \tag{2.85}$$

assuming that A is a positive operator such that that all $\lambda_i > 0$ for all i. Denote the Jacobian as J and thus under a *linear* change of variables then

$$\begin{aligned} &\int D\varphi\, e^{-\frac{1}{2}<\varphi, A\varphi>} \\ &= J \int \prod_i da_i\, e^{-\frac{1}{2}\lambda_i a_i^2} = J \prod_i \int da_i e^{-\frac{1}{2}\lambda_i a_i^2} \\ &= J \prod_i (2\pi)^{1/2} \left(\prod_i \lambda_i \right)^{-1/2} \propto \det(A)^{-1/2} \end{aligned} \tag{2.86}$$

for complex variables there is a similar result.

Multiplying equation (2.78) with itself using two different integration variables. The resulting formula is

$$\int \int dxdy\, e^{-\frac{a}{2}(x^2+y^2)} = \frac{2\pi}{a} \tag{2.87}$$

now change the variable to z and z^* by

$$\begin{aligned} z &= \tfrac{1}{\sqrt{2}}(x+iy) \qquad & z^* &= \tfrac{1}{\sqrt{2}}(x-iy) \\ dx &= \tfrac{1}{\sqrt{2}}(dz+dz^*) \qquad & dy &= \tfrac{1}{\sqrt{2}i}(dz-dz^*) \end{aligned} \tag{2.88}$$

The volume element changes into

$$dx \wedge dy = \frac{1}{2i}(dz+dz^*) \wedge (dz-dz^*) = -i\, dz \wedge dz^* \tag{2.89}$$

and the integral changes into

$$\int \frac{dz^*}{\sqrt{2\pi}i} \frac{dz}{\sqrt{2\pi}i}\, e^{-az^*z} = \frac{1}{a} \tag{2.90}$$

By introducing complex fields φ, ψ with a standard complex inner product. Then with a similar reasoning as above it is possible to show that for a self adjoint positive operator

$$\int D\varphi D\varphi^*\, e^{-<\varphi^*, A\varphi>} = \det(A)^{-1} \tag{2.91}$$

2.7 Casimir energy

Everything is now ready to produce a formula for the Casimir energy. It is convenient to first reorganize the action in equation (2.76) into

$$\begin{aligned}
S &= \sum_{\alpha}\left(\sum_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} H^{\alpha}_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\alpha}_{\mathbf{k}_\alpha} + c.c.\right) + \sum_{\substack{\alpha\beta\\ \alpha\neq\beta}}\left(\sum_{\mathbf{i}_\alpha\mathbf{k}_\beta} z^{\alpha *}_{\mathbf{i}_\alpha} K^{\alpha\beta}_{\mathbf{i}_\alpha\mathbf{k}_\beta} z^{\beta}_{\mathbf{k}_\beta} + c.c.\right)\\
&= \sum_{\alpha}\left(\sum_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} H^{\alpha}_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\alpha}_{\mathbf{k}_\alpha} + z^{\alpha}_{\mathbf{i}_\alpha} H^{\alpha *}_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{k}_\alpha}\right)\\
&\quad + \sum_{\substack{\alpha\beta\\ \alpha\neq\beta}}\left(\sum_{\mathbf{i}_\alpha\mathbf{k}_\beta} z^{\alpha *}_{\mathbf{i}_\alpha} K^{\alpha\beta}_{\mathbf{i}_\alpha\mathbf{k}_\beta} z^{\beta}_{\mathbf{k}_\beta} + z^{\alpha}_{\mathbf{i}_\alpha} K^{\alpha\beta *}_{\mathbf{i}_\alpha\mathbf{k}_\beta} z^{\beta *}_{\mathbf{k}_\beta}\right)\\
&= \sum_{\alpha}\sum_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha}\left(H^{\alpha}_{\mathbf{i}_\alpha\mathbf{k}_\alpha} + H^{\alpha *}_{\mathbf{k}_\alpha\mathbf{i}_\alpha}\right) z^{\alpha}_{\mathbf{k}_\alpha}\\
&\quad + \sum_{\substack{\alpha\beta\\ \alpha\neq\beta}}\sum_{\mathbf{i}_\alpha\mathbf{k}_\beta} z^{\alpha *}_{\mathbf{i}_\alpha}\left(K^{\alpha\beta}_{\mathbf{i}_\alpha\mathbf{k}_\beta} + K^{\beta\alpha *}_{\mathbf{k}_\beta\mathbf{i}_\alpha}\right) z^{\beta}_{\mathbf{k}_\beta}\\
&= \sum_{\alpha}\sum_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\alpha *}_{\mathbf{i}_\alpha} T^{\alpha}_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\alpha}_{\mathbf{k}_\alpha} + \sum_{\substack{\alpha\beta\\ \alpha\neq\beta}}\sum_{\mathbf{i}_\alpha\mathbf{k}_\beta} z^{\alpha *}_{\mathbf{i}_\alpha} U^{\alpha}_{\mathbf{i}_\alpha\mathbf{k}_\alpha} z^{\beta}_{\mathbf{k}_\beta}
\end{aligned} \tag{2.92}$$

where

$$\begin{aligned}
T^{\alpha}_{\mathbf{i}_\alpha\mathbf{k}_\alpha} &= H^{\alpha}_{\mathbf{i}_\alpha\mathbf{k}_\alpha} + H^{\alpha *}_{\mathbf{k}_\alpha\mathbf{i}_\alpha}\\
U^{\alpha\beta}_{\mathbf{i}_\alpha\mathbf{k}_\beta} &= K^{\alpha\beta}_{\mathbf{i}_\alpha\mathbf{k}_\beta} + K^{\beta\alpha *}_{\mathbf{k}_\beta\mathbf{i}_\alpha}
\end{aligned} \tag{2.93}$$

Note that the $T^{\alpha}_{\mathbf{i}_\alpha\mathbf{k}_\alpha}$ and $U^{\alpha\beta}_{\mathbf{i}_\alpha\mathbf{k}_\beta}$ are self adjoint.

Evaluate the functional integral from equation (2.53) with the above action

$$B_Q(i\kappa) = \int \prod_{\alpha=1}^{r} DJ^{\alpha} DJ^{\alpha *}\; e^{\frac{iT}{\hbar} S}\Big|_{T=-is} \tag{2.94}$$

and make a *linear* change of variables of the form

$$J^{\alpha}(\mathbf{x}_\alpha) = \sum_{\mathbf{i}_\alpha} z^{\alpha}_{\mathbf{i}_\alpha} a^{\alpha}_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) \tag{2.95}$$

and thus equation (2.91) will result in

$$B_Q(i\kappa) = J\int \prod_{\alpha=1}^{r}\prod_{\mathbf{i}_\alpha} Dz^{\alpha}_{\mathbf{i}_\alpha} Dz^{\alpha *}_{\mathbf{i}_\alpha} e^{-<z^{\alpha *}, A z^{\alpha}>} \propto \det(A_Q(i\kappa))^{-1} \tag{2.96}$$

Where the Jacobian, J, is constant and the matrix A is

$$A_Q(i\kappa) = \frac{s}{\hbar}\begin{bmatrix} -T^1 & -U^{12} & \cdots & -U^{1r}\\ -U^{21} & \ddots & & \vdots\\ \vdots & & & \\ & & & -U^{r-1,r}\\ -U^{r1} & \cdots & -U^{r,r-1} & -T^r \end{bmatrix} \tag{2.97}$$

Subtract the high frequency part to regularize the Casimir energy. At high frequencies, the only contribution to the energy will be from the self energy of each object. Thus it is convenient to subtract this contribution because the interesting part is the interaction energy. Let E_∞ denote the self energy calculated using only the high frequency contribution S_∞ from equation (2.77).

The functional integral that will have to be solved for the high frequency part is

$$B_\infty(i\kappa) = \int \prod_{\alpha=1}^{r} DJ^\alpha DJ^{\alpha *}\, e^{\frac{iT}{\hbar} S_\infty}\Big|_{T=-is} \tag{2.98}$$

and after a linear change of variables the integrals can be calculated as

$$B_\infty(i\kappa) = J \int \prod_{\alpha=1}^{r} \prod_{\mathbf{i}_\alpha} Dz_{\mathbf{i}_\alpha}^{\alpha} Dz_{\mathbf{i}_\alpha}^{\alpha *} e^{-<z^{\alpha *}, A z^\alpha>} \propto \det(A_\infty(i\kappa))^{-1} \tag{2.99}$$

Where the Jacobian J is constant and the matrix A is

$$A_\infty(i\kappa) = \frac{s}{\hbar} \begin{bmatrix} -T^1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & -T^r \end{bmatrix} \tag{2.100}$$

The regularized Casimir energy can now be calculated from the ground state energy in equation (2.41)

$$\begin{aligned} \mathcal{E} &= E_0 - E_\infty \\ &= -\lim_{s\to\infty} \left(\frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \ln B_Q(i\kappa) - \frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \ln B_\infty(i\kappa) \right)\Big|_{T=-is} \\ &= -\lim_{s\to\infty} \frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \ln \frac{B_Q(i\kappa)}{B_\infty(i\kappa)}\Big|_{T=-is} \\ &= -\lim_{s\to\infty} \frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \ln \frac{\det(A_Q(i\kappa))^{-1}}{\det(A_\infty(i\kappa))^{-1}} \\ &= \lim_{s\to\infty} \frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \ln \frac{\det(A_Q(i\kappa))}{\det(A_\infty(i\kappa))} \end{aligned} \tag{2.101}$$

Using the relation

$$\tfrac{\det(A)}{\det(B)} = \det(A)\det(B^{-1}) = \det(B^{-1}A) \qquad \det(B) \neq 0 \tag{2.102}$$

and since $A_\infty(i\kappa)$ is diagonal the energy is simplified into

$$\mathcal{E} = \frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \ln \det(M_Q(i\kappa)) \tag{2.103}$$

where

$$M_Q(i\kappa) = \begin{bmatrix} \mathbf{1} & \cdots & (T^1)^{-1}U^{1r} \\ \vdots & \ddots & \vdots \\ (T^r)^{-1}U^{r1} & \cdots & \mathbf{1} \end{bmatrix} \tag{2.104}$$

For the special case of two objects this will be

$$\mathcal{E} = \frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \ln\det(\mathbf{1} - (T^1)^{-1} U^{12} (T^2)^{-1} U^{21}) \tag{2.105}$$

2.8 Applications

To calculate the Casimir energy given in equation (2.103)

$$\mathcal{E} = \frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \ln\det(M_Q(i\kappa)) \tag{2.106}$$

From the formula of $M_Q(i\kappa)$ is dependent on the matrices T^i and U^{ij} given by equation (2.93). For real fields these are

$$\begin{aligned} T^\alpha_{\mathbf{i}_\alpha \mathbf{k}_\alpha} &= H^\alpha_{\mathbf{i}_\alpha \mathbf{k}_\alpha} + H^{\alpha *}_{\mathbf{k}_\alpha \mathbf{i}_\alpha} = H^\alpha_{\mathbf{i}_\alpha \mathbf{k}_\alpha} + H^\alpha_{\mathbf{k}_\alpha \mathbf{i}_\alpha} \\ U^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta} &= K^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta} + K^{\beta\alpha *}_{\mathbf{k}_\beta \mathbf{i}_\alpha} = K^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta} + K^{\beta\alpha}_{\mathbf{k}_\beta \mathbf{i}_\alpha} \end{aligned} \tag{2.107}$$

Look to equations (2.66) and (2.75) to calculate $H^\alpha_{\mathbf{i}_\alpha, \mathbf{k}_\alpha}$ and $K^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta}$

$$\begin{aligned} H^\alpha_{\mathbf{i}_\alpha \mathbf{k}_\alpha} &= \sum_{\mathbf{j}_\alpha} D^\alpha_{\mathbf{i}_\alpha \mathbf{j}_\alpha} G^\alpha_{\mathbf{j}_\alpha \mathbf{k}_\alpha} \\ K^{\alpha\beta}_{\mathbf{i}_\alpha \mathbf{k}_\beta} &= \sum_{\mathbf{i}_\alpha} D^\alpha_{\mathbf{i}_\alpha \mathbf{j}_\alpha} G^{\alpha\beta}_{\mathbf{j}_\alpha \mathbf{k}_\beta} \end{aligned} \tag{2.108}$$

Where the $D^\alpha_{\mathbf{k}_\alpha, \mathbf{j}_\alpha}$ is the inner product matrix given by equation (2.65) as

$$D^\alpha_{\mathbf{i}_\alpha \mathbf{j}_\alpha} = \int_{Q_\alpha} dA_{\mathbf{x}_\alpha} \, a^{\alpha *}_{\mathbf{i}_\alpha}(\mathbf{x}_\alpha) a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) \tag{2.109}$$

$G^\alpha_{\mathbf{j}_\alpha, \mathbf{k}_\alpha}$ and $G^{\alpha\beta}_{\mathbf{j}_\alpha \mathbf{k}_\beta}$ are given by their definitions in equation (2.63) and (2.73)

$$\begin{aligned} G^\alpha_{\mathbf{j}_\alpha \mathbf{k}_\alpha} &= \int_{Q_\alpha} dA_{\mathbf{x}'_\alpha} \, G^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha) a^\alpha_{\mathbf{k}_\alpha}(\mathbf{x}'_\alpha) \\ G^{\beta\alpha}_{\mathbf{j}_\beta \mathbf{k}_\alpha} &= \int_{Q_\alpha} dA_{\mathbf{x}'_\alpha} \, G^{\beta\alpha}_{\mathbf{j}_\beta}(\mathbf{x}'_\alpha) a^\alpha_{\mathbf{k}_\alpha}(\mathbf{x}'_\alpha) \end{aligned} \tag{2.110}$$

Insert the definitions of $G^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha)$ and $G^{\beta\alpha}_{\mathbf{j}_\beta}(\mathbf{x}'_\alpha)$ to get a clear formula. These are defined as the coefficients of the series for the Green's function in the basis $a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha)$. Choose the basis such that $D^\alpha_{\mathbf{i}_\alpha \mathbf{j}_\alpha} = \delta_{\mathbf{i}_\alpha \mathbf{j}_\alpha}$ then the coefficients are found with the usual method.

$$\begin{aligned} G^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}'_\alpha) &= \int dA_{\mathbf{x}_\alpha} \, G^\alpha(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) a^\alpha_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) \\ G^{\beta\alpha}_{\mathbf{j}_\beta}(\mathbf{x}'_\alpha) &= \int dA_{\mathbf{x}_\beta} \, G^\alpha(\mathbf{x}_\beta, \mathbf{x}'_\alpha) a^\beta_{\mathbf{j}_\beta}(\mathbf{x}_\beta) \end{aligned} \tag{2.111}$$

Thus the double integrals

$$\begin{aligned} G^{\alpha}_{\mathbf{j}_\alpha \mathbf{k}_\alpha} &= \int_{Q_\alpha}\int_{Q_\alpha} dA_{\mathbf{x}'_\alpha} dA_{\mathbf{x}_\alpha}\, a^{\alpha}_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) G^{\alpha}(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) a^{\alpha}_{\mathbf{k}_\alpha}(\mathbf{x}'_\alpha) \\ G^{\beta\alpha}_{\mathbf{j}_\beta \mathbf{k}_\alpha} &= \int_{Q_\alpha}\int_{Q_\beta} dA_{\mathbf{x}'_\alpha} dA_{\mathbf{x}_\beta}\, a^{\beta}_{\mathbf{j}_\beta}(\mathbf{x}_\beta) G^{\alpha}(\mathbf{x}_\beta, \mathbf{x}'_\alpha) a^{\alpha}_{\mathbf{k}_\alpha}(\mathbf{x}'_\alpha) \end{aligned} \tag{2.112}$$

The last thing that is needed is the Green's functions G^α and the basis functions $a^\alpha_{\mathbf{i}_\alpha}$. The Green's function for this problem is given by equation (2.58) and the two dimensional solution to this equation is a modified Bessel function.

$$G^{\alpha}(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) = -\frac{1}{2\pi} K_0(\kappa ||\mathbf{x}_\alpha - \mathbf{x}'_\alpha||) \tag{2.113}$$

2.9 Discretization

Consider compact objects whose boundary curves Q_i will be approximated by a piecewise linear curve. The piecewise linear curve is determined by N points $\mathbf{P}^i_k$ placed on the curve Q_i. A parameterization for this piecewise linear curve is

$$\gamma^i(s) = \mathbf{P}^i_{k-1} + (\mathbf{P}^i_k - \mathbf{P}^i_{k-1}) \tfrac{s-\alpha_{k-1}}{\Delta} \qquad s \in I_k = [\alpha_{k-1}, \alpha_k] \tag{2.114}$$

The edges of each parametrization interval and their centers are defined as

$$\begin{aligned} \alpha_k &= -L/2 + k \cdot \Delta \\ s_k &= -L/2 + (k - 1/2) \cdot \Delta \end{aligned} \tag{2.115}$$

where $\Delta = L/N = \alpha_k - \alpha_{k-1}$. The number of discretization intervals is N and the discretization parameter $\alpha \in [-L/2, L/2]$ for some k. Integrating over a single parametrization interval gives

$$\int_{C_k} dl \;=\; \int_{\alpha_{k-1}}^{\alpha_k} ds\, \frac{1}{\Delta} ||\mathbf{P}^i_k - \mathbf{P}^i_{k-1}|| = ||\mathbf{P}^i_k - \mathbf{P}^i_{k-1}|| \equiv e^i_k \tag{2.116}$$

Where the length of each parametrization interval is defined as e^i_k.

Select some basis on our object and to keep everything simple use an orthonormal basis on each parametrization interval.

$$\theta_{ik}(\mathbf{x}) = \begin{cases} 1/\sqrt{e^i_k} & \mathbf{x} \in \gamma_i(I^i_k) \\ 0 & \mathbf{x} \in \gamma_i(I^{i'}_k), i' \neq k \end{cases} \tag{2.117}$$

At this point it is possible to use some spherical basis, as might be convenient in some situations, but the above basis will simplify several integrals and our matrix. With this basis the matrix $D^{\alpha}_{\mathbf{k}_\alpha, \mathbf{j}_\alpha}$ from equation (2.65) will be

$$D^i_{kk'} = \int_{Q_i} dA_{\mathbf{x}_i}\, \theta_{ik'}(\mathbf{x}) \theta_{ik}(\mathbf{x}) = \delta^k_{k'} \tag{2.118}$$

From equation (2.112) it is simple to calculate what the Green's function matrix will contain.

$$\begin{aligned} G^{\alpha}_{\mathbf{j}_\alpha \mathbf{k}_\alpha} &= \int\limits_{Q_\alpha}\int\limits_{Q_\alpha} dA_{\mathbf{x}'_\alpha} dA_{\mathbf{x}_\alpha}\, a^{\alpha}_{\mathbf{j}_\alpha}(\mathbf{x}_\alpha) G^{\alpha}(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) a^{\alpha}_{\mathbf{k}_\alpha}(\mathbf{x}'_\alpha) \\ G^{\beta\alpha}_{\mathbf{j}_\beta \mathbf{k}_\alpha} &= \int\limits_{Q_\alpha}\int\limits_{Q_\beta} dA_{\mathbf{x}'_\alpha} dA_{\mathbf{x}_\beta}\, a^{\beta}_{\mathbf{j}_\beta}(\mathbf{x}_\beta) G^{\alpha}(\mathbf{x}_\beta, \mathbf{x}'_\alpha) a^{\alpha}_{\mathbf{k}_\alpha}(\mathbf{x}'_\alpha) \end{aligned} \tag{2.119}$$

Do each of these in turn

2.9.1 Matrix elements $G^{\alpha}_{\mathbf{j}_\alpha \mathbf{k}_\alpha}$ and $G^{\beta\alpha}_{\mathbf{j}_\beta \mathbf{k}_\alpha}$

The discretization is organized with a singe variable $k = 1, \ldots, N$ and for $\alpha \neq \beta$

$$\begin{aligned} G^{\beta\alpha}_{kk'} &= \int\limits_{Q_\alpha}\int\limits_{Q_\beta} dA_{\mathbf{x}'} dA_{\mathbf{x}}\, a^{\beta}_{k}(\mathbf{x}) G^{\alpha}(\mathbf{x}, \mathbf{x}') a^{\alpha}_{k'}(\mathbf{x}') \\ &= -\frac{1}{2\pi\sqrt{e^{\alpha}_k e^{\beta}_{k'}}} \int\limits_{I^{\alpha}_k}\int\limits_{I^{\beta}_{k'}} dA_{\mathbf{x}'} dA_{\mathbf{x}}\, K_0(\kappa||\mathbf{x} - \mathbf{x}'||) \end{aligned} \tag{2.120}$$

This double integral can be approximated using a midpoint rule or a 2D Gaussian quadrature. Whatever methods is used, this integral is non singular and will contribute to the off diagonal block matrices in $M_Q(i\kappa)$

The next integral will involve integrating over the singularities in the Green's function so it will be necessary to use the *Cauchy* principal integral.

$$\begin{aligned} G^{\alpha}_{kk'} &= \int\limits_{Q_\alpha}\int\limits_{Q_\alpha} dA_{\mathbf{x}'} dA_{\mathbf{x}}\, a^{\alpha}_{k}(\mathbf{x}) G^{\alpha}(\mathbf{x}, \mathbf{x}') a^{\alpha}_{k'}(\mathbf{x}') \\ &= -\frac{1}{2\pi\sqrt{e^{\alpha}_k e^{\alpha}_{k'}}} \int\limits_{I^{\alpha}_k}\int\limits_{I^{\alpha}_{k'}} dA_{\mathbf{x}'} dA_{\mathbf{x}}\, K_0(\kappa||\mathbf{x} - \mathbf{x}'||) \end{aligned} \tag{2.121}$$

There are now two cases, if $k \neq k'$ then $K_0(\ldots)$ will be a smooth function without any singularities. Any numerical integration routine can be used to calculate the double integral.

But if $k = k'$

$$G^{\alpha}_{kk} = -\frac{1}{2\pi e^{\alpha}_k} \mathrm{PV}_{\mathbf{x}} \int\limits_{I^{\alpha}_k}\int\limits_{I^{\alpha}_k} dA_{\mathbf{x}'} dA_{\mathbf{x}}\, K_0(\kappa||\mathbf{x} - \mathbf{x}'||) \tag{2.122}$$

Note that

$$\begin{aligned} ||\mathbf{x} - \mathbf{x}'|| &= ||\mathbf{P}^{\alpha}_{k-1} + (\mathbf{P}^{\alpha}_{k} - \mathbf{P}^{\alpha}_{k-1})\frac{s - \alpha_{k-1}}{\Delta} \\ &\quad - \mathbf{P}^{\alpha}_{k-1} - (\mathbf{P}^{\alpha}_{k} - \mathbf{P}^{\alpha}_{k-1})\frac{s' - \alpha_{k-1}}{\Delta}|| \\ &= ||(\mathbf{P}^{\alpha}_{k} - \mathbf{P}^{\alpha}_{k-1})(\frac{s - s'}{\Delta})|| = \frac{|s - s'|}{\Delta} e^{\alpha}_k \end{aligned} \tag{2.123}$$

Thus when the parametrization is inserted into the integrals

$$\begin{aligned} G^\alpha_{kk} &= -\frac{1}{2\pi e^\alpha_k}\mathrm{PV}_s \int\limits_{I^\alpha_k}\int\limits_{I^\alpha_k} \frac{e^\alpha_k ds'}{\Delta}\frac{e^\alpha_k ds}{\Delta}\, K_0(\kappa\frac{|s-s'|}{\Delta}e^\alpha_k) \\ &= -\frac{e^\alpha_k}{2\pi\Delta^2}\mathrm{PV}_s \int\limits_{I^\alpha_k}\int\limits_{I^\alpha_k} ds' ds\, K_0(\kappa\frac{|s-s'|}{\Delta}e^\alpha_k) \end{aligned} \tag{2.124}$$

Change the variable of integration

$$\begin{aligned} \theta(\alpha_{k-1}) &= -\tfrac{1}{2}\Delta & \theta'(\alpha_{k-1}) &= -\tfrac{1}{2}\Delta \\ \theta(\alpha_k) &= \tfrac{1}{2}\Delta & \theta'(\alpha_k) &= \tfrac{1}{2}\Delta \\ \theta(s) &= -\Delta/2 + s - \alpha_{k-1} & \theta'(s') &= -\Delta/2 + s' - \alpha_{k-1} \\ d\theta &= ds & d\theta' &= ds' \end{aligned} \tag{2.125}$$

The integral is now

$$G^\alpha_{kk} = -\frac{e^\alpha_k}{2\pi\Delta^2}\mathrm{PV}_\theta \int\limits_{-\Delta/2}^{\Delta/2}\int\limits_{-\Delta/2}^{\Delta/2} d\theta' d\theta\, K_0(\kappa\frac{|\theta-\theta'|}{\Delta}e^\alpha_k) \tag{2.126}$$

As $\Delta \to 0$ the integration will be over a vanishing square and the parametrization points will be dense on the surface Q_α. The Taylor expansion of $K_0(x)$ for small arguments is $K_0(x) = C - \log(x)$ where $C = \log(2) - \gamma_{\mathrm{em}}$ and $\gamma_{\mathrm{em}} = 0.577215..$ is the Euler-Mascheroni constant.

With this

$$\begin{aligned} G^\alpha_{kk} &= -\frac{e^\alpha_k}{2\pi\Delta^2}\mathrm{PV}_\theta \int\limits_{-\Delta/2}^{\Delta/2}\int\limits_{-\Delta/2}^{\Delta/2} d\theta' d\theta\, (C - \ln(\kappa e^\alpha_k|\theta-\theta'|/\Delta) \\ &= -\frac{e^\alpha_k}{2\pi\Delta^2}\left(\Delta^2 C - \mathrm{PV}_\theta \int\limits_{-\Delta/2}^{\Delta/2}\int\limits_{-\Delta/2}^{\Delta/2} d\theta' d\theta\, \ln(\kappa e^\alpha_k|\theta-\theta'|/\Delta)\right) \\ &= -\frac{Ce^\alpha_k}{2\pi} + \frac{e^\alpha_k}{2\pi\Delta^2}\mathrm{PV}_\theta \int\limits_{-\Delta/2}^{\Delta/2}\int\limits_{-\Delta/2}^{\Delta/2} d\theta' d\theta\, \ln(\kappa e^\alpha_k|\theta-\theta'|/\Delta) \\ &= -\frac{Ce^\alpha_k}{2\pi} + \frac{e^\alpha_k}{2\pi\Delta^2}\frac{1}{B}\int\limits_{-\Delta/2}^{\Delta/2} d\theta \lim_{\epsilon\to 0}\left(\int\limits_{B\epsilon}^{B(\theta+\Delta/2)} dy\, \ln(y)\right. \\ &\quad \left. + \int\limits_{B\epsilon}^{B(\Delta/2-\theta)} dz\, \ln(z)\right) \end{aligned} \tag{2.127}$$

Where $B = \kappa e_k^\alpha/\Delta$, $y = B(\theta - \theta')$ and $z = B(\theta' - \theta)$

$$\begin{aligned} G_{kk}^\alpha &= -\frac{Ce_k^\alpha}{2\pi} + \frac{e_k^\alpha}{2\pi\Delta^2}\frac{1}{B}\int_{-\Delta/2}^{\Delta/2} d\theta\ (B(\theta+\Delta/2)\ln(B(\theta+\Delta/2)) \\ &\quad -B(\theta+\Delta/2) + B(\Delta/2-\theta)\ln(B(\Delta/2-\theta)) - B(\Delta/2-\theta)) \\ &= -\frac{(C+1)e_k^\alpha}{2\pi} + \frac{e_k^\alpha}{2\pi\Delta^2}\left(\int_{-\Delta/2}^{\Delta/2} d\theta\,(\theta+\Delta/2)\ln(B(\theta+\Delta/2))\right. \\ &\quad \left. -\int_{-\Delta/2}^{\Delta/2} d\theta\,(\Delta/2-\theta)\ln(B(\Delta/2-\theta))\right) \\ &= -\frac{(C+1)e_k^\alpha}{2\pi} + \frac{e_k^\alpha}{2\pi\Delta^2}\frac{2}{B^2}\int_0^{B\Delta} dy\, y\ln(y) \end{aligned} \tag{2.128}$$

where $y = B(\theta + \Delta/2)$ and $z = B(\Delta/2 - \theta)$ and then the two integral were joined by setting $z = y$.

Thus

$$G_{kk}^\alpha = -\frac{e^\alpha}{2\pi}(C + 1 - \ln(B\Delta) + 1/2) = -\frac{e^\alpha}{2\pi}(C + 3/2 - \ln(\kappa e_k^\alpha)) \tag{2.129}$$

Every matrix component of $M_Q(i\kappa)$ has now been calculated.

Chapter 3

Implementation

All the coding was done in *C* using *phread* for parallel support. The implementation of both the boundary element method and the functional integral method have many similarities. This is because both methods use Greens functions: the free Greens function in the boundary integral method and the scattering Greens function in the functional integral method. The primary difference between the two methods is that one solves a linear system and the other calculates a determinant.

Boundary integral method

The boundary integral method is about computing surface integrals by solving a linear system of equations and there are three parts to this algorithm: Filling matrices, multiplying matrices and solving the linear systems. The first two are simple to parallelize and if it is possible to avoid memory races the scaling will be excellent using multiple processors. Solving a linear system of equations in parallel could be done using an iterative method such as generalized minimal residue method (GMRES).

If any geometric symmetries are exploitable by the boundary integral method this would significantly reduce the computational time.

The primary algorithm used in the program is the LU-decomposition (Crout's algorithm) with pivoting. This is a good solver since the methods calls for us to solve the same linear system for several different input vectors. Thus the LU-factorization can be calculated once and the input can be varied without the need to recalculate the matrix. For small matrices this algorithm is fast, but for larger matrices it might be better to use an iterative algorithm such as GMRES. But due to the limited matrix size of our problems, the LU-decomposition will be more than sufficient.

If there are r objects all discretized using N points (for simplicity, in practical problems this might vary for different objects), then the algorithm requires the following steps to evaluate the density function once.

- Fill $r(r+1)$ block matrices of size $N \times N$
- Solve r linear systems of size $N \times N$ with N different r.h.s. each
- Compute $r-1$ block matrix multiplications of size $N \times N$

- Solve a single $rN \times rN$ linear system for N different r.h.s.

Functional integral method

To calculate the density function this method has to find a determinant of a matrix. This matrix is comprised of smaller block matrices that must first be inverted and multiplied. Since all the matrices will be relatively small it is possible to use LU-decomposition for all relevant matrix operations.

If there are r objects all discretized using N points, then the algorithm requires the following steps to evaluate the density function once.

- Fill r^2 block matrices of size $N \times N$
- Solve $r(r-1)$ linear systems of size $N \times N$ for N different r.h.s each but where there are only r different matrices on the left.
- Find the determinant of a $rN \times rN$ block matrix

Comparisons

Since both methods are based on Greens functions, the main matrices will be filled by similar elements. Start by comparing the differences in the above procedures:

- Fill the r.h.s. of the boundary integral method. This requires that r additional block matrices be filled.
- The functional integral method has to solve $r(r-2)$ more linear systems of size $N \times N$ for N different r.h.s. each
- The boundary integral method has to multiply $r-1$ block matrices.
- Either solve a linear system or find a determinant.

Using LU-factorization it is possible to efficiently solve the equations for multiple r.h.s. by performing the LU-factorization once for each matrix and using back substitution for each r.h.s. This method can also be used to calculate the determinant of a matrix. For the LU algorithm the complexity for large N is in the order of $(2/3)N^3$. The back substitution is performed with an asymptotic complexity in the order of $2N^2$.

A comparison of the asymptotic complexity in each method (if the LU-factorization is used) yields

- Boundary integral method
  - Fill block matrices: $cr(r+1)N^2$ where c is some constant
  - Solve linear systems: $((2/3)r^3 + 2r^2)N^3 + (8/3)rN^3$
  - Matrix multiplication: $(r-1)N^3$ (standard formula)
  - Total: $(2r^3 + 6r^2 + 11r - 3)N^3/3$
- Functional integral method
  - Fill block matrices: cr^2N^2 where c is some constant

- Solve linear systems: $r(2/3)N^3 + 2r^2(r-1)N^3$
- Determinant: $(2/3)(rN)^3$
- Total: $(8r^3 + 6r^2 - 4r)N^3/3$

Thus it is clear that both of these methods have very similar asymptotic complexity at least when it is reasonable to use the LU-factorization in both methods. For very large matrices on clusters or similar the LU-decomposition will be unpractical. In this regime iterative methods such as GMRES will probably be the most efficient method for solving linear system.

Chapter 4

Mode summation

A common method to find the Casimir energy is mode summation. In situations with a high degree of symmetry it is possible for finding the Casimir energy using this method. The exact solutions can then be compared to the other methods.

4.1 Parallel plates

Consider two parallel non-dispersive plates separated by a distance a, the defining equation for a scalar field is the wave equation. In each separate region let $c(\mathbf{x})$ be constant

$$\varphi_{tt}(\mathbf{x},t) - c(x)^2\nabla^2\varphi(\mathbf{x},t) = 0 \tag{4.1}$$

where

$$c(x) = \begin{cases} c_1 & x \in (0,a) \\ c_2 & \text{else} \end{cases} \tag{4.2}$$

and

$$\varphi(\mathbf{x}) = \begin{cases} \varphi_2(\mathbf{x}) & \mathbf{x} \in (-\infty,0) \\ \varphi_1(\mathbf{x}) & \mathbf{x} \in (0,a) \\ \varphi_2(\mathbf{x}) & \mathbf{x} \in (a,\infty) \end{cases} \tag{4.3}$$

At each boundary the solution φ should be continuous.

$$\begin{aligned} \lim_{\mathbf{x}\to 0^-} \varphi_2(\mathbf{x}) &= \lim_{\mathbf{x}\to 0^+} \varphi_1(\mathbf{x}) \\ \lim_{\mathbf{x}\to a^-} \varphi_1(\mathbf{x}) &= \lim_{\mathbf{x}\to a^+} \varphi_2(\mathbf{x}) \end{aligned} \tag{4.4}$$

and that no energy is deposited into the boundaries. This condition comes from the energy flux found in equation (1.33)

$$S_e = -c(\mathbf{x})^2\varphi_t\nabla\varphi \tag{4.5}$$

Assuming that the normal component of S_e is continuos across each boundary then

$$\begin{aligned} \lim_{\mathbf{x}\to 0^-} -c_2^2\varphi_{2t}\partial_{\mathbf{n}}\varphi_2(\mathbf{x}) &= \lim_{\mathbf{x}\to 0^+} -c_1^2\varphi_{1t}\partial_{\mathbf{n}}\varphi_1(\mathbf{x}) \\ \lim_{\mathbf{x}\to a^-} -c_1^2\varphi_{1t}\partial_{\mathbf{n}}\varphi_1(\mathbf{x}) &= \lim_{\mathbf{x}\to a^+} -c_2^2\varphi_{2t}\partial_{\mathbf{n}}\varphi_2(\mathbf{x}) \end{aligned} \tag{4.6}$$

Assuming that φ_t is continuos across the boundary

$$\begin{aligned} \lim_{\mathbf{x}\to 0^-} -c_2^2\partial_{\mathbf{n}}\varphi_2(\mathbf{x}) &= \lim_{\mathbf{x}\to 0^+} -c_1^2\partial_{\mathbf{n}}\varphi_1(\mathbf{x}) \\ \lim_{\mathbf{x}\to a^-} -c_1^2\partial_{\mathbf{n}}\varphi_1(\mathbf{x}) &= \lim_{\mathbf{x}\to a^+} -c_2^2\partial_{\mathbf{n}}\varphi_2(\mathbf{x}) \end{aligned} \tag{4.7}$$

And assuming that the same boundary conditions apply to the quantum fields.

Starting with a Fourier transform in the time domain.

$$\omega^2\varphi(\mathbf{x}) + c(x)^2\nabla^2\varphi(\mathbf{x}) = 0 \tag{4.8}$$

and then another Fourier transform with wavenumber k will give

$$-\varphi''(x) + \left(k^2 - \left(\frac{\omega}{c(x)}\right)^2\right)\varphi(x) = 0 \tag{4.9}$$

Divide the space into three regions

I $\to x < 0$

II $\to 0 < x < a$

III $\to x > a$

The following equations are found for each region

I $-\varphi''(x) + q_2^2\varphi = 0$

II $-\varphi''(x) + q_1^2\varphi = 0$

III $-\varphi''(x) + q_2^2\varphi = 0$

where $q_j^2 = \left(k^2 - (\omega/c_j)^2\right)$. The solution in each region is

I $\varphi(x) = Ae^{q_2 x} + Be^{-q_2 x}$

II $\varphi(x) = Ce^{q_1 x} + De^{-q_1 x}$

III $\varphi(x) = Ee^{q_2 x} + Fe^{-q_2 x}$

In order to get a mode of frequency ω it is necessary to have $E = B = 0$. The boundary conditions give us for $x = 0$

$$\begin{aligned} A &= C + D \\ c_2^2 q_2 A &= c_1^2 q_1 (C - D) \end{aligned} \tag{4.10}$$

and for $x = a$

$$\begin{aligned} Ce^{q_1 a} + De^{-q_1 a} &= Fe^{-q_2 a} \\ c_1^2 q_1 \left(Ce^{q_1 a} - De^{-q_1 a}\right) &= -c_2^2 q_2 Fe^{-q_2 a} \end{aligned} \tag{4.11}$$

Eliminate A and F will result in

$$\begin{aligned} C + D &= \frac{c_1^2 q_1}{c_2^2 q_2}(C - D) \\ Ce^{q_1 a} + De^{-q_1 a} &= -\frac{c_1^2 q_1}{c_2^2 q_2}\left(Ce^{q_1 a} - De^{-q_1 a}\right) \end{aligned} \tag{4.12}$$

define $\alpha = \frac{c_1^2 q_1}{c_2^2 q_2}$ and the following system emerges

$$\begin{bmatrix} (1-\alpha) & (1+\alpha) \\ (1+\alpha)e^{q_1 a} & (1-\alpha)e^{-q_1 a} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.13}$$

This will have a non trivial solution only if the determinant is zero.

$$f(\omega, k) = (1-\alpha)^2 e^{-q_1 a} - (1+\alpha)^2 e^{q_1 a} = 0 \tag{4.14}$$

or equivalently

$$g(\omega, k) = e^{-q_2 a}\left((c_1^2 q_1 + c_2^2 q_2)^2 e^{q_1 a} - (c_1^2 q_1 - c_2^2 q_2)^2 e^{-q_1 a}\right) = 0 \tag{4.15}$$

This relation determines the possible frequencies ω. But in order to derive the Lifshitz formula, the system must be dispersive. Since any amount of dispersion is sufficient, the high frequency contribution to the force will be modified. Since this contribution will be negligible it is possible to set $c_i = c_i(\omega)$ and $c_1(\omega) \to c_2(\omega) = 1$ at high frequencies.

The energy can be expressed as

$$E = \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_n \omega_n(k) \tag{4.16}$$

Where $\omega_n(k)$ are the zeros of $g(\omega, k)$. As $g(\omega, k)$ has no poles use the argument principle to evaluate this sum.

It states that for an analytic function $h(z)$ with no poles inside the contour C and a meromorphic function $f(z)$ with no poles or zeros on C. Then

$$\frac{1}{2\pi i} \oint_C dz\, h(z) \frac{f'(z)}{f(z)} = \sum_n m_n h(z_n^0) - k_n h(z_n^p) \tag{4.17}$$

where z_n^0 are zeros, and z_n^p are poles of $f(z)$ inside the contour and m_n, k_n are their respective multiplicity.

Use $h(z) = z$ and an analytic continuation of $g(\omega, k)$ from equation (4.15) to get a sum over the zeros ω_n

$$\sum_n \omega_n(k) = \frac{1}{2\pi i} \oint_C d\omega\, \omega \frac{g'(\omega, k)}{g(\omega, k)} \tag{4.18}$$

Using this the energy is given by

$$E = \frac{\hbar}{8\pi^2 i} \int_{-\infty}^{\infty} dk \oint_C d\omega\, \omega \frac{g'(\omega, k)}{g(\omega, k)} \tag{4.19}$$

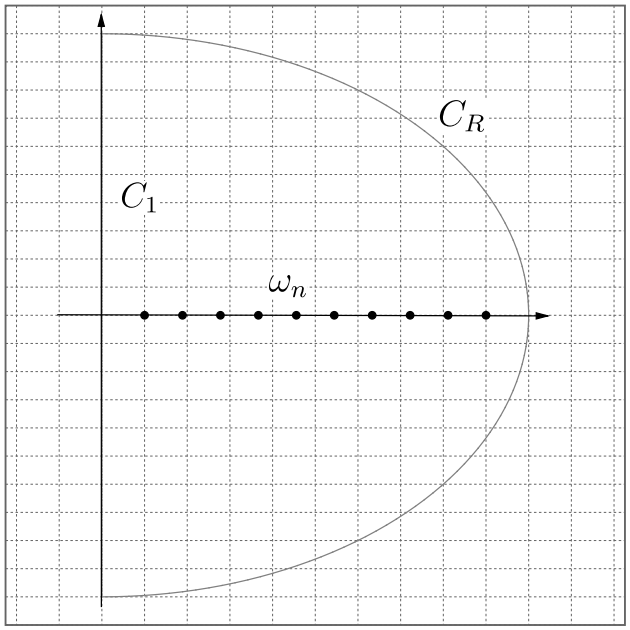

Figure 4.1: The integration contour $C = C_1 \cup C_R$ used in the argument principle for the parallel plates.

Where the contour is given in figure 4.1

$$E = \frac{\hbar}{8\pi^2 i} \int_{-\infty}^{\infty} dk \lim_{R\to\infty} \left(\int_{iR}^{-iR} d\omega\, \omega \frac{g'(\omega,k)}{g(\omega,k)} + \int_{C_R} d\omega\, \omega \frac{g'(\omega,k)}{g(\omega,k)} \right) \tag{4.20}$$

Observe that when for large ω

$$\begin{aligned} g(\omega,k) &\approx (c_1^2 q_1 + c_2^2 q_2)^2 e^{(q_1-q_2)a} \approx -(c_1+c_2)^2 \omega^2 e^{i(\frac{1}{c_1}-\frac{1}{c_2})\omega a} \to -4\omega^2 \\ g'(\omega,k) &\to -8\omega \end{aligned} \tag{4.21}$$

Thus $\omega g'(\omega,k)/g(\omega,k) \to 2$ along C_∞. The energy will then be given by

$$E = \frac{\hbar}{8\pi^2 i} \int_{-\infty}^{\infty} dk \left(-\lim_{R\to\infty} \int_{-iR}^{iR} d\omega\, \omega \frac{d}{d\omega} \log(g(\omega,k)) + 2 \int_{C_\infty} d\omega \right) \tag{4.22}$$

The dominating contribution to the energy for large ω is

$$g_\infty(\omega,k) = (c_1^2 q_1 + c_2^2 q_2)^2 e^{(q_1-q_2)a} \tag{4.23}$$

The energy associated with this contribution is

$$E_\infty = \frac{\hbar}{8\pi^2 i} \int_{-\infty}^{\infty} dk \left(-\lim_{R\to\infty} \int_{-iR}^{iR} d\omega\, \omega \frac{d}{d\omega} \log(g_\infty(\omega,k)) + 2 \int_{C_\infty} d\omega \right) \tag{4.24}$$

Define the regularized Casimir energy as

$$\mathcal{E} = E - E_\infty = -\frac{\hbar}{8\pi^2 i}\int_{-\infty}^{\infty} dk \lim_{R\to\infty}\int_{-iR}^{iR} d\omega\,\omega\frac{d}{d\omega}\log\left(\frac{g(\omega,k)}{g_\infty(\omega,k)}\right) \tag{4.25}$$

And modify this slightly with a partial integration

$$\mathcal{E} = \frac{\hbar}{8\pi^2 i}\int_{-\infty}^{\infty} dk \lim_{R\to\infty}\int_{-iR}^{iR} d\omega\,\log\left(\frac{g(\omega,k)}{g_\infty(\omega,k)}\right) \tag{4.26}$$

The change of variables $\omega = iy$ will result in

$$\mathcal{E} = \frac{\hbar}{8\pi^2}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy\,\log\left(\frac{g(iy,k)}{g_\infty(iy,k)}\right) \tag{4.27}$$

where

$$\frac{g(iy,k)}{g_\infty(iy,k)} = 1 - \frac{c_1^2 q_1 - c_2^2 q_2}{c_1^2 q_1 + c_2^2 q_2} e^{-2q_1 a}\Big|_{\omega=iy} \tag{4.28}$$

Take the limit $c_1 \to 1$ and $c_2 \to 0$ to simulate a perfect conductor. Then

$$\frac{g(iy,k)}{g_\infty(iy,k)} \to 1 - e^{-2q_1 a}\big|_{\omega=iy} = 1 - e^{-2\sqrt{k^2+y^2}a} \tag{4.29}$$

Change to polar coordinates and the energy is given by

$$\begin{aligned}
\mathcal{E} &= \frac{\hbar}{8\pi^2}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy\,\log\left(\frac{g(iy,k)}{g_\infty(iy,k)}\right) \\
&= \frac{\hbar}{2\pi^2}\int_0^{\infty} dk \int_0^{\infty} dy\,\log\left(1 - e^{-2\sqrt{k^2+y^2}a}\right) \\
&= \frac{\hbar}{2\pi^2}\int_0^{\infty} dr \int_0^{\pi/2} d\theta\, r\log\left(1-e^{-2ra}\right) = \frac{\hbar}{4\pi}\int_0^{\infty} dr\, r\log\left(1-e^{-2ra}\right) \\
&= \frac{\hbar}{4\pi}\int_0^{\infty}\frac{du}{2a}\,\frac{u}{2a}\log\left(1-e^{-u}\right) = \frac{\hbar}{16\pi a^2}\int_0^{\infty} du\, u\log\left(1-e^{-u}\right)
\end{aligned} \tag{4.30}$$

For two plates with a separation a the Casimir energy is given by

$$\mathcal{E}(a) = \frac{\hbar}{16\pi a^2}\int_0^{\infty} du\, u\log\left(1-e^{-u}\right) = -\frac{\hbar\zeta(3)}{16\pi a^2} \approx -0.0239142\frac{\hbar}{a^2} \tag{4.31}$$

This coincides with the results of Ambjørn and Wolfram [15](p.4 with d=2).

Using equation 5.9 it is simple to can calculate the pressure from the energy

$$p_1 = \frac{dE(a)}{da} = \frac{\hbar\zeta(3)}{8\pi a^3} \approx 0.0478283\frac{\hbar}{a^3} \tag{4.32}$$

4.2 Concentric circles

Consider the problem of two concentric circles. The two circles have radius r_1 and r_2 such that $r_1 < r_2$. The defining equation in the annulus between the two circles is

$$\begin{aligned} \hat{\varphi}_{tt}(\mathbf{x},t) - c^2\nabla^2\hat{\varphi}(\mathbf{x},t) &= 0 \\ \hat{\varphi}(\mathbf{x},t)|_{Q_j} &= 0 \end{aligned} \tag{4.33}$$

First take a Fourier transform in the time domain.

$$\omega^2\hat{\varphi}(\mathbf{x}) + c^2\nabla^2\hat{\varphi}(\mathbf{x}) = 0 \tag{4.34}$$

In polar coordinates this is

$$\nabla^2 = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_{\theta\theta} = \nabla_r + \frac{1}{r^2}\nabla_\theta \tag{4.35}$$

The new equation is

$$\omega^2\hat{\varphi}(r,\theta) + c^2\nabla_r\hat{\varphi}(r,\theta) + c^2\frac{1}{r^2}\nabla_\theta\hat{\varphi}(r,\theta) = 0 \tag{4.36}$$

This equation is linear and can be solved by defining the solution as

$$\hat{\varphi}(r,\theta) = \sum_{m=-\infty}^{\infty} R_m(r)\Theta_m(\theta) \tag{4.37}$$

Thus

$$\omega^2 R_m(r)\Theta_m(\theta) + c^2\Theta_m(\theta)\nabla_r R_m(r) + c^2\frac{1}{r^2}R_m(r)\nabla_\theta\Theta_m(\theta) = 0 \tag{4.38}$$

Simplify this into

$$-r^2k^2 - r^2\frac{\nabla_r R_m(r)}{R_m(r)} = \frac{\nabla_\theta\Theta_m(\theta)}{\Theta_m(\theta)} = -m^2 \tag{4.39}$$

where $k = \omega/c$ is some constant. These two equations will be solved one by one. First the angular equation

$$\nabla_\theta\Theta_m(\theta) = -m^2\Theta_m(\theta) \tag{4.40}$$

The solution is

$$\Theta_m(\theta) = A_m e^{im\theta} + B_m e^{-im\theta} \tag{4.41}$$

with the symmetric condition that $\hat{\varphi}(r,\theta) = \hat{\varphi}(r,\theta+2\pi)$ the solution is required to only be valid for $m \in \mathbb{Z}$.

The radial equation is given by

$$\nabla_r R_m(r) = \left(\frac{m^2}{r^2} - k^2\right) R_m(r) \tag{4.42}$$

where the solution is

$$R_m(r) = A_m J_m(kr) + B_m Y_m(kr) \tag{4.43}$$

where $J_m(kr)$ is a m^{th} order Bessel function and $Y_m(kr)$ is a m^{th} order Neumann function. The boundary condition $R_m(r_1) = R_m(r_2) = 0$ give

$$R_m(r_1) = A_m J_m(kr_1) + B_m Y_m(kr_1) = 0 \tag{4.44}$$

or

$$B_m = -A_m J_m(kr_1)/Y_m(kr_1) \tag{4.45}$$

Inserted back into the series this gives

$$\hat{\varphi}(r,\theta) = \sum_{m=-\infty}^{\infty} A_m \left(J_m(kr) - Y_m(kr)\frac{J_m(kr_1)}{Y_m(kr_1)} \right) \Theta_m(\theta) \tag{4.46}$$

The second boundary condition $R_m(r_2) = 0$ will give a non trivial solution only if

$$J_m(kr_2)Y_m(kr_1) - J_m(kr_1)Y_m(kr_2) = 0 \tag{4.47}$$

Denote the solution to this equation as k_m for each m. Define a function that is zero on all k_m as

$$f(k) = \prod_m (J_m(kr_2)Y_m(kr_1) - J_m(kr_1)Y_m(kr_2)) \tag{4.48}$$

The energy for this system is defined as

$$E = \sum_m \frac{1}{2}\hbar k_m = \frac{1}{2}\sum_m k_m \tag{4.49}$$

where $\hbar = 1$.

The plan is to sum over all the zeros of $f(k)$, this can be accomplished by using the argument principle from complex analysis.

It states that for an analytic function $h(z)$ with no poles inside the contour C and a meromorphic function $f(z)$ with no poles or zeros on C. Then

$$\frac{1}{2\pi i}\oint_C dz\, h(z)\frac{f'(z)}{f(z)} = \sum_n m_n h(z_n^0) - k_n h(z_n^p) \tag{4.50}$$

where z_n^0 are zeros, and z_n^p are poles of $f(z)$ inside the contour and m_n, k_n are their respective multiplicity.

Use $h(z) = z$ and an analytic continuation of $f(k)$ from equation (4.48) to get a sum over the zeros k_m

$$\sum_m k_m = \frac{1}{2\pi i}\oint_C dk\, k\frac{f'(k)}{f(k)} = \frac{1}{2\pi i}\oint_C dk\, k\frac{d}{dk}\log(f(k)) \tag{4.51}$$

Using this the energy is given by

$$E = \frac{1}{4\pi i}\oint_C dk\, k\frac{d}{dk}\log(f(k)) \tag{4.52}$$

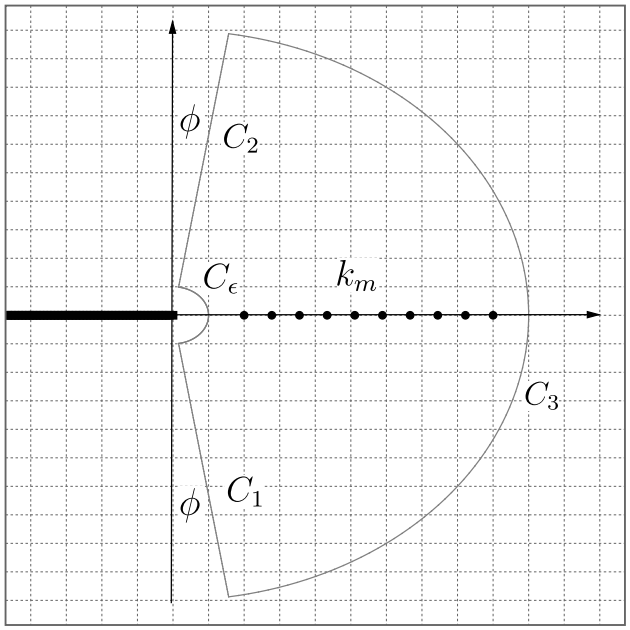

Figure 4.2: The integration contour $C = C_1 \cup C_3 \cup C_2 \cup C_\epsilon$ used in the argument principle for the concentric circles with the branch cut along the negative real axis.

Where the contour is given in figure 4.2. The angle ϕ is there to make the integral converge and the limit $\phi \to 0$ will be added at the end. Let us remove the derivative by performing one partial integration.

$$\begin{aligned} E &= -\frac{1}{4\pi i} \oint_C dk \, \log\left(f(k)\right) \\ &= \frac{1}{4\pi} \int_0^\infty dy \, \left(e^{i\phi} \log\left(f(-iye^{i\phi})\right) + e^{-i\phi} \log\left(f(iye^{-i\phi})\right)\right) \\ &\quad - \frac{1}{4\pi i} \lim_{\rho\to\infty} \int_{C_3} dk \, \log\left(f(k)\right) - \lim_{\epsilon\to 0} \int_{\pi/2-\phi}^{-\pi/2+\phi} \frac{\epsilon e^{i\theta}}{4\pi} d\theta \, \log\left(f(\epsilon e^{i\theta})\right) \end{aligned} \tag{4.53}$$

Take the limit $\phi \to 0$ in the first and last expression to get

$$\begin{aligned} E &= \frac{1}{4\pi} \int_0^\infty dy \, \left(\log\left(f(-iy)\right) + \log\left(f(iy)\right)\right) \\ &\quad - \lim_{\epsilon\to 0} \int_{\pi/2}^{-\pi/2} \frac{\epsilon e^{i\theta}}{4\pi} d\theta \, \log\left(f(\epsilon e^{i\theta})\right) - \frac{1}{4\pi i} \lim_{\substack{\rho\to\infty \\ \phi\to 0}} \int_{C_3} dk \, \log\left(f(k)\right) \end{aligned} \tag{4.54}$$

It is possible to show that $f(-k) = f(k)$, so

$$\begin{aligned} E = \frac{1}{2\pi}\int_0^\infty dy\, \log\left(f(iy)\right) - \lim_{\epsilon\to 0}\int_{\pi/2}^{-\pi/2} \frac{\epsilon e^{i\theta}}{4\pi} d\theta\, \log\left(f(\epsilon e^{i\theta})\right) \\ - \frac{1}{4\pi i}\lim_{\substack{\rho\to\infty \\ \phi\to 0}} \int_{C_3} dk\, \log\left(f(k)\right) \end{aligned} \tag{4.55}$$

Observe that the integrand evaluates $f(k)$ at $k = iy$. A few relations are required to simplify this expression

$$\begin{aligned} J_n(ix) &= i^n I_n(x) \\ K_n(x) &= \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix) \\ H_n^{(1)}(x) &= J_n(x) + iY_n(x) \end{aligned} \tag{4.56}$$

Combining these relations results in

$$Y_n(ix) = -\frac{2}{\pi} i^{-n} K_n(x) + i^{n+1} I_n(x) \tag{4.57}$$

Thus

$$\begin{aligned} f(iy) &= \prod_m (J_m(iyr_2)Y_m(iyr_1) - J_m(iyr_1)Y_m(iyr_2)) \\ &= \prod_m \frac{2}{\pi}\left(I_m(yr_1)K_m(yr_2) - I_m(yr_2)K_m(yr_1)\right) \end{aligned} \tag{4.58}$$

Defining the infinite part of $f(iy)$ as $f_\infty(iy)$, this is the divergent part of the energy evaluated along the imaginary axis. Notice that for $r_2 > r_1$ the integrand will evaluate to $I_m(yr_1)K_m(yr_2) \to 0$ and $I_m(yr_2)K_m(yr_1) \to \infty$ as $y \to \infty$.

Thus for some $y \in \mathbb{R}$

$$f_\infty(iy) \equiv \prod_m -\frac{2}{\pi} I_m(yr_2)K_m(yr_1) \tag{4.59}$$

This is the infinite contribution to the energy along the imaginary axis. If this is rotated out into the complex plane

$$f_\infty(z) = \prod_m -\frac{2}{\pi} I_m(-izr_2)K_m(-izr_1) \tag{4.60}$$

Simplifying this using the Bessel relations above and the Hankel relation

$$H_m^{(1)}(x) = (J_{-m}(x) - e^{-impi}J_m(x))/(i\sin(m\pi)) \tag{4.61}$$

results in

$$\begin{aligned}
f_\infty(z) &= \prod_m -\frac{2}{\pi} I_m(-izr_2)K_m(-izr_1) \\
&= \prod_m -\frac{2}{\pi} i^{-m} J_m(zr_2)K_m(-izr_1) \\
&= \prod_m -iJ_m(zr_2)H_m^{(1)}(zr_1) \\
&= \prod_m -iJ_m(zr_2)(J_{-m}(zr_1) - e^{-impi}J_m(zr_1))/(i\sin(m\pi)) \\
&= \prod_m -2iJ_m(zr_2)J_m(zr_1)(e^{im\pi} - e^{-im\pi})/(e^{im\pi} - e^{-im\pi}) \\
&= \prod_m -2iJ_m(zr_2)J_m(zr_1)
\end{aligned} \tag{4.62}$$

Thus for any complex z it is possible to use the density

$$f_\infty(z) = \prod_m -2iJ_m(zr_2)J_m(zr_1) \tag{4.63}$$

Subtracting the infinite part of the energy will give

$$\begin{aligned}
\mathcal{E} &= E - E_\infty \\
&= \frac{1}{2\pi}\sum_m \int_0^\infty dy\, \log\left(1 - \frac{I_m(yr_1)K_m(yr_2)}{I_m(yr_2)K_m(yr_1)}\right) \\
&\quad - \lim_{\epsilon\to 0} \int_{\pi/2}^{-\pi/2} \frac{\epsilon e^{i\theta}}{4\pi} d\theta\, \log\left(\frac{f(\epsilon e^{i\theta})}{f_\infty(\epsilon e^{i\theta})}\right) \\
&\quad - \frac{1}{4\pi i} \lim_{\substack{\rho\to\infty \\ \phi\to 0}} \int_{C_3} dk\, \log\left(\frac{f(k)}{f_\infty(k)}\right)
\end{aligned} \tag{4.64}$$

Solving the integral over C_ϵ first

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_{C_\epsilon} dz\, \log\left(\frac{f(z)}{f_\infty(z)}\right) \\
&= \lim_{\epsilon\to 0}\sum_m \int_{\pi/2}^{-\pi/2} \epsilon e^{i\theta} d\theta\, \log\left(-\frac{J_m(\epsilon e^{i\theta}r_2)Y_m(\epsilon e^{i\theta}r_1)}{2iJ_m(\epsilon e^{i\theta}r_2)J_m(\epsilon e^{i\theta}r_1)}\right. \\
&\qquad \left. + \frac{J_m(\epsilon e^{i\theta}r_1)Y_m(\epsilon e^{i\theta}r_2)}{2iJ_m(\epsilon e^{i\theta}r_2)J_m(\epsilon e^{i\theta}r_1)}\right) \\
&= \lim_{\epsilon\to 0}\sum_m \int_{\pi/2}^{-\pi/2} \epsilon e^{i\theta} d\theta\, \log\left(i\frac{Y_m(\epsilon e^{i\theta}r_1)}{2J_m(\epsilon e^{i\theta}r_1)} - i\frac{Y_m(\epsilon e^{i\theta}r_2)}{2J_m(\epsilon e^{i\theta}r_2)}\right) \\
&= \lim_{\epsilon\to 0}\sum_m \int_{\pi/2}^{-\pi/2} \epsilon e^{i\theta} d\theta\, \mathrm{Log}\left|\frac{Y_m(\epsilon e^{i\theta}r_1)}{2J_m(\epsilon e^{i\theta}r_1)} - \frac{Y_m(\epsilon e^{i\theta}r_2)}{2J_m(\epsilon e^{i\theta}r_2)}\right|
\end{aligned} \tag{4.65}$$

The asymptotic forms of these functions for $0 < x < \sqrt{m+1}$ are

$$\begin{aligned} Y_m(x) &\approx \begin{cases} \frac{2}{\pi}(\log(x/2)+\gamma) & m=0 \\ -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m & m>0 \end{cases} \\ J_m(x) &\approx \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m \end{aligned} \tag{4.66}$$

Thus for $m = 0$

$$\begin{aligned} \left|\frac{Y_0(zr_1)}{2J_0(zr_1)} - \frac{Y_0(zr_2)}{2J_0(zr_2)}\right| &\approx \frac{1}{2}\left|\frac{2}{\pi}(\log(r_1 z/2)+\gamma) - \frac{2}{\pi}(\log(r_2 z/2)+\gamma)\right| \\ &= \frac{1}{\pi}\log(r_1/r_2) \end{aligned} \tag{4.67}$$

and for $m > 0$

$$\begin{aligned} &\left|\frac{Y_m(zr_1)}{2J_m(zr_1)} - \frac{Y_m(zr_2)}{2J_m(zr_2)}\right| \approx \frac{1}{2}\left|\frac{-\frac{\Gamma(m)}{\pi}\left(\frac{2}{zr_1}\right)^m}{\frac{1}{\Gamma(m+1)}\left(\frac{zr_1}{2}\right)^m} - \frac{-\frac{\Gamma(zr_2)}{\pi}\left(\frac{2}{zr_2}\right)^m}{\frac{1}{\Gamma(m+1)}\left(\frac{zr_2}{2}\right)^m}\right| \\ &= \frac{1}{2}\left|-\frac{\Gamma(m)\Gamma(m+1)}{\pi}\left(\frac{2}{zr_1}\right)^{2m} + \frac{\Gamma(m)\Gamma(m+1)}{\pi}\left(\frac{2}{zr_2}\right)^{2m}\right| \\ &= \frac{\Gamma(m)\Gamma(m+1)}{2\pi}\left(\frac{2}{|z|}\right)^{2m}\left|-\left(\frac{1}{r_1}\right)^{2m} + \left(\frac{1}{r_2}\right)^{2m}\right| \end{aligned} \tag{4.68}$$

For all m the integrand will be a limit of the type ϵ or $\epsilon\log\epsilon$. As $\epsilon \to 0$ the integrals will therefore go to zero

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} dz\, \log\left(\frac{f(z)}{f_\infty(z)}\right) = 0 \tag{4.69}$$

The integral over C_3 is next

$$\begin{aligned} &\lim_{\rho\to\infty}\int_{C_3} dk\, \log\left(\frac{f(k)}{f_\infty(k)}\right) \\ &= \lim_{\rho\to\infty}\sum_m \int_{-\pi/2}^{\pi/2} \rho e^{i\theta} d\theta\, \log\left(-\frac{J_m(\rho e^{i\theta}r_2)Y_m(\rho e^{i\theta}r_1)}{2iJ_m(\rho e^{i\theta}r_2)J_m(\rho e^{i\theta}r_1)}\right. \\ &\qquad \left. + \frac{J_m(\rho e^{i\theta}r_1)Y_m(\rho e^{i\theta}r_2)}{2iJ_m(\rho e^{i\theta}r_2)J_m(\rho e^{i\theta}r_1)}\right) \\ &= \lim_{\rho\to\infty}\sum_m \int_{-\pi/2}^{\pi/2} \rho e^{i\theta} d\theta\, \log\left(-\frac{Y_m(\rho e^{i\theta}r_1)}{2iJ_m(\rho e^{i\theta}r_1)} + \frac{Y_m(\rho e^{i\theta}r_2)}{2iJ_m(\rho e^{i\theta}r_2)}\right) \end{aligned} \tag{4.70}$$

The asymptotic forms of these functions for $x > |m^2 - 1/4|$ are

$$\begin{aligned} Y_m(x) &\approx \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \\ J_m(x) &\approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \end{aligned} \tag{4.71}$$

The integrand will be

$$\begin{aligned}
&\left(\frac{Y_m(zr_1)}{-2iJ_m(zr_1)} - \frac{Y_m(zr_2)}{-2iJ_m(zr_2)}\right) \\
&\approx \frac{1}{-2i}\left(\tan\left(zr_1 - \frac{m\pi}{2} - \frac{\pi}{4}\right) - \tan\left(zr_2 - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right) \\
&= \frac{1}{-2i}\left(\tan\left(\alpha_1 + i\beta_1\right) - \tan\left(\alpha_2 + i\beta_2\right)\right) \\
&= \frac{1}{-2i}\left(\frac{e^{i\alpha_1-\beta_1} - e^{-i\alpha_1+\beta_1}}{i(e^{i\alpha_1-\beta_1} + e^{-i\alpha_1+\beta_1})} - \frac{e^{i\alpha_2-\beta_2} - e^{-i\alpha_2+\beta_2}}{i(e^{i\alpha_2-\beta_2} + e^{-i\alpha_2+\beta_2})}\right) \\
&= \frac{1}{2}\left(\frac{e^{i\alpha_1-\beta_1} - e^{-i\alpha_1+\beta_1}}{e^{i\alpha_1-\beta_1} + e^{-i\alpha_1+\beta_1}} - \frac{e^{i\alpha_2-\beta_2} - e^{-i\alpha_2+\beta_2}}{e^{i\alpha_2-\beta_2} + e^{-i\alpha_2+\beta_2}}\right)
\end{aligned} \tag{4.72}$$

Where $\alpha_i = \rho\cos(\theta)r_i - m\pi/2 - \pi/4$ and $\beta_i = \rho\sin(\theta)r_i$, but this integrand does not converge. Thus this method does not fix the divergence, it might be possible if a cutoff function or another approach altogether is used.

Thus

$$\begin{aligned}
\mathcal{E} = \frac{1}{2\pi}\sum_m \int_0^\infty dy\, \log\left(1 - \frac{I_m(yr_1)K_m(yr_2)}{I_m(yr_2)K_m(yr_1)}\right) \\
- \frac{1}{4\pi i}\lim_{\substack{\rho\to\infty \\ \phi\to 0}} \int_{C_3} dk\, \log\left(\frac{f(k)}{f_\infty(k)}\right)
\end{aligned} \tag{4.73}$$

The same regularization technique was used in appendix C on the parallel plates. Using this regularization in this chapter resulted i a similar divergent integral in the energy. But the correct energy was still recovered from the result. While the lack of cancelation above is troublesome the comparisons in Figure 6.7 and 6.9 agrees with a energy given by

$$\mathcal{E} = \frac{1}{2\pi}\sum_m \int_0^\infty dy\, \log\left(1 - \frac{I_m(yr_1)K_m(yr_2)}{I_m(yr_2)K_m(yr_1)}\right) \tag{4.74}$$

This energy is comparable to the results of F.D. Mazzitelli, D.A.R. Dalvit and F.C. Lombardo [9] (p. 9).

The pressure on the inner and outer circle can be calculated from the energy by equation (5.16) and (5.17)

$$\begin{aligned}
p_1 &= -\frac{1}{2\pi r_1}\frac{\partial E}{\partial r_1} \\
p_2 &= \frac{1}{2\pi r_2}\frac{\partial E}{\partial r_2}
\end{aligned} \tag{4.75}$$

Chapter 5

Virtual Work

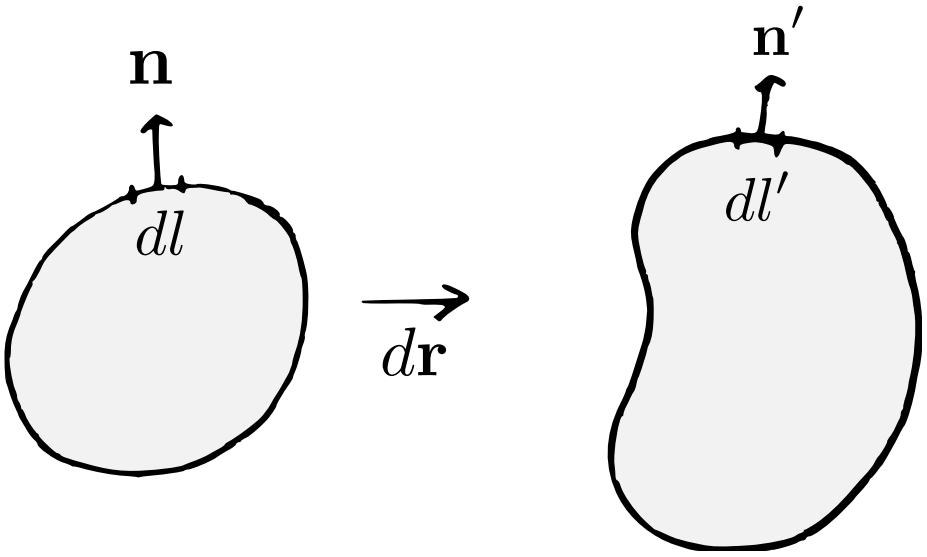

Figure 5.1: Illustration of the tranformation $d\mathbf{r}$ on each object.

The total energy in a system is a function of several parameters. For instance in the case of two concentric circles the energy is of the form $E(r_1, r_2)$ as such a function of each radii r_1 and r_2. In general there are n arguments $E(r_1, \ldots, r_n)$ that control the energy. Let $\gamma(s) = (r_1(s), \ldots, r_n(s))$ be an one parameter curve through the argument space. Then the change in the energy $E(\gamma(s))$ will relate to the parameter s as

$$dE(\gamma(s)) = \nabla E \cdot \gamma'(s) ds \tag{5.1}$$

Given a set of compact objects and the union of the surfaces $Q = \cup_j Q_j$ it is possible to define the force on a surface element dl on each object as $\mathbf{F}_j = \mathbf{n} p_j dl$, where $\mathbf{n}$ is directed into each compact object, or equivalently with an outwards pointing normals and define the force as $\mathbf{F}_j = -\mathbf{n} p_j dl$.

Figure 5.1 shows an example of a deformation of an object. The change in total energy related to the deformation field, $d\mathbf{r}$, is the integral of the force along

the deformation.

$$dE = \oint_Q \mathbf{F} \cdot d\mathbf{r} = -\oint_Q dl\, p\mathbf{n} \cdot d\mathbf{r} = -\sum_j \oint_{Q_j} dl\, p_j \mathbf{n} \cdot d\mathbf{r}_j \tag{5.2}$$

Thus the energy is related to the pressure by

$$-\nabla E \cdot \gamma'(s) ds = \oint_Q dl\, p\mathbf{n} \cdot d\mathbf{r} \tag{5.3}$$

Example 1. *Consider two parallel one dimensional plates with a distance a. Move each plate separately. How will the pressure on the surface relate to the energy change of the system?*

Answer. *Starting with the parameter change. Each plate is determined by its position along the x-axis. Let the left plate have position x_1 and the right position x_2 such that $x_1 < x_2$.*

Lets look at the left plate. Let the parameters be determined by $\gamma(s) = (x_1 \pm s, x_2)$, then the change in energy will be

$$dE(\gamma(s)) = \nabla E \cdot \gamma'(s) ds = \left(\frac{\partial E}{\partial x_1}, \frac{\partial E}{\partial x_2} \right) \cdot (\pm 1, 0) ds = \pm \frac{\partial E}{\partial x_1} ds \tag{5.4}$$

First parametrize the plates to describe the deformation. Let $\boldsymbol{r}_1(s,t) = (x_1 \pm s, t)$ describe the deformations of the surface. The normal is given by: $\boldsymbol{n}_1 = (1,0)$. Then the pressure is related to the energy change by

$$\begin{aligned} dE(\gamma(s)) &= -\oint_{Q_1} dl\, p_1 \boldsymbol{n}_1 \cdot (\pm ds, 0) \\ &= \mp ds \oint_{Q_1} dl\, p_1 = \mp ds L p_1 \end{aligned} \tag{5.5}$$

The pressure on the surface is constant, as can be seen from the symmetries of the problem. If each plate has length L, then the energy per unit of length is given by

$$\pm \frac{\partial E}{\partial x_1} ds = \mp ds\, p_a \tag{5.6}$$

So

$$p_1 = -\frac{\partial E}{\partial x_1} \tag{5.7}$$

For the right plate the sign will be reversed due to the normals having reversed direction.

$$p_2 = \frac{\partial E}{\partial x_2} \tag{5.8}$$

Or using the relation $a = x_2 - x_1$ to relate this to the distance between the plates. The chain rule gives

$$p_1 = p_2 = \frac{\partial E}{\partial a} \tag{5.9}$$

With the energy from equation (B.11) that was calculated from the boundary integral method

$$E(a) = -\frac{\pi}{24a} \tag{5.10}$$

Then the pressure on each surface is

$$p_1 = p_2 = \frac{\pi}{24a^2} \tag{5.11}$$

There is an attractive force between the plates and it is in agreement with the force calculated by the boundary integral method in (A.13).

Example 2. *Consider two concentric circles of radii $R_1 < R_2$. Change the radius of each circle separately. How will the pressure on the surface relate to the energy change of the system?*

Answer. *Each circle is fully described by its radius. The parameter change for the inner circle is given by $\gamma(s) = (R_1 \pm s, R_2)$, and thus*

$$dE = \nabla E(R_1, R_2) \cdot (\pm 1, 0) ds = \pm \frac{\partial E}{\partial R_1} ds \tag{5.12}$$

The parametrization of each circle is given by $\boldsymbol{r}_1(s,t) = (R_1 \pm s)(\cos(t), \sin(t)) = (R_1 \pm s)\boldsymbol{n}_1$. The differential change is given by $\boldsymbol{dr}_1 = (\pm ds)\, \boldsymbol{n}_1$, thus the force is given by

$$\begin{aligned} dE(\gamma(s)) &= -\oint_{Q_1} dl\, p_1 \boldsymbol{n}_1 \cdot (\pm ds) \boldsymbol{n}_1 \\ &= \mp ds \oint_{Q_1} dl\, p_1 \end{aligned} \tag{5.13}$$

By symmetry considerations it is clear that the pressure on each circle is constant and thus

$$dE(\gamma(s)) = \mp 2\pi p_1 R_1 ds \tag{5.14}$$

Combining these to get

$$\pm \frac{\partial E}{\partial R_1} ds = \mp 2\pi p_1 R_1 ds \tag{5.15}$$

or for the pressure

$$p_1 = -\frac{1}{2\pi R_1} \frac{\partial E}{\partial R_1} \tag{5.16}$$

and for the outer circle there is a change in sign from the change in direction of the normals

$$p_2 = \frac{1}{2\pi R_2} \frac{\partial E}{\partial R_2} \tag{5.17}$$

Example 3. *Consider two adjacent circles that are centered in $\boldsymbol{o}_1 = (x_1, 0)$ and $\boldsymbol{o}_2 = (x_2, 0)$ with radius r and minimal separation $a = x_2 - x_1 - 2r > 0$. Change the position of each circle separately. How will the force relate to the energy change of the system?*

Answer. *Move the left circle along the x-axis and the energy will change as*

$$dE = \nabla E(x_1, x_2) \cdot (\pm ds, 0) = \pm ds \frac{\partial E}{\partial x_1} \tag{5.18}$$

The parametrization of the left circle is $\boldsymbol{r}_1(s,t) = (x_1 \pm s, 0) + r(\cos(t), \sin(t))$. *The differential change is given by* $\boldsymbol{dr}_1(s,t) = (\pm ds, 0)$. *Thus*

$$\begin{aligned} dE(\gamma(s)) &= -\oint_{Q_1} dl\, p_1 \boldsymbol{n}_1 \cdot (\pm ds, 0) \\ &= \mp ds \oint_{Q_1} dl\, p_1\, \boldsymbol{n}_{1,x} = \mp ds\, \boldsymbol{F}_{1,x} \end{aligned} \tag{5.19}$$

and the force is given by

$$\boldsymbol{F}_{1,x} = -\frac{\partial E}{\partial x_1} \tag{5.20}$$

The same result holds for the right circle and it is clear that the total force will be oriented along the x-axis.

Chapter 6

Comparison

6.1 Geometries

There are three situations where the methods will be tested. The first two are symmetric and have exact answers that the results can be compared to. The final situation is more complicated and will be used to compare the boundary element and functional integral methods.

6.1.1 Parallel plates

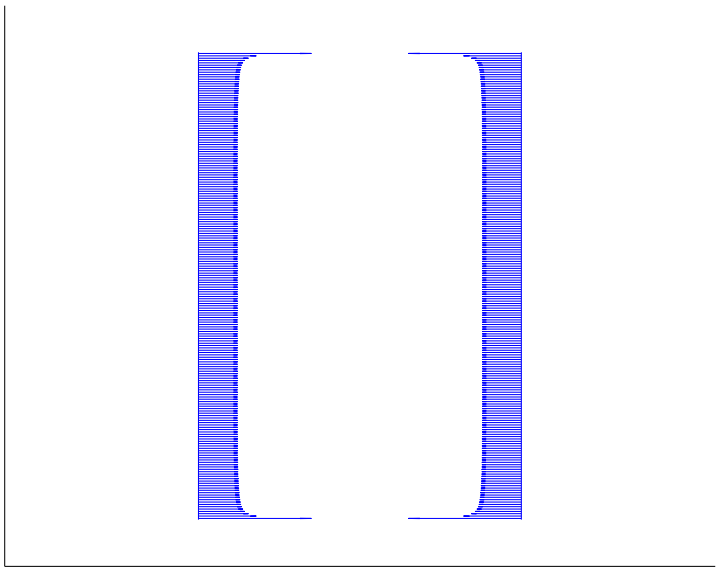

Figure 6.1: Illustration of the parallel plates with the forces on each line segment. Note boundary effects on the edges.

Take two 2D plates located at $x = \pm a/2$ of length L with normals pointing form each plate to the other. Each of these are discretized into N linear line segments and can easily by integrated into the above program. This case is special, since it is comprised of two non-compact objects. Figure 6.1 shows the two plates and the force on each line segment of the plates. Notice that the boundary effect are minimal.

Since the objects have to be of finite length in our program some boundary effects are expected. The plates should be long enough so that the boundary effects are small and constrained to the edges.

Observe that for this case the subtracted self stress is zero, first observe that for $\mathbf{s}_{k'} \in Q_i$ and $\mathbf{s}_{k''} \in Q_j, j \neq i$, thus equation (1.114) gives

$$y^{ij}_{k'k''} = -\partial_{\mathbf{n}'} D_0(\mathbf{s}_{k'}, \mathbf{s}_{k''}) = \frac{\omega}{2\pi} \frac{\mathbf{n}' \cdot (\mathbf{s}_{k'} - \mathbf{s}_{k''})}{||\mathbf{s}_{k'} - \mathbf{s}_{k''}||} K_1(\omega||\mathbf{s}_{k'} - \mathbf{s}_{k''}||) \tag{6.1}$$

When $\mathbf{s}_{k'}, \mathbf{s}_{k''} \in Q_i$ the objects will always have $\mathbf{s}_{k'} - \mathbf{s}_{k''}$ at a right angle to $\mathbf{n}'$. Thus

$$\partial_{\mathbf{n}'} D_0(\mathbf{s}_{k'}, \mathbf{s}_{k''}) = 0 \tag{6.2}$$

When $\mathbf{s}_{k'} \to \mathbf{s}_{k''}$ it is clear that in equation (1.115) the diagonal element will be zero.

This shows that in equation (1.96) the self stress in $B^{ii} = 0$.

6.1.2 Concentric circles

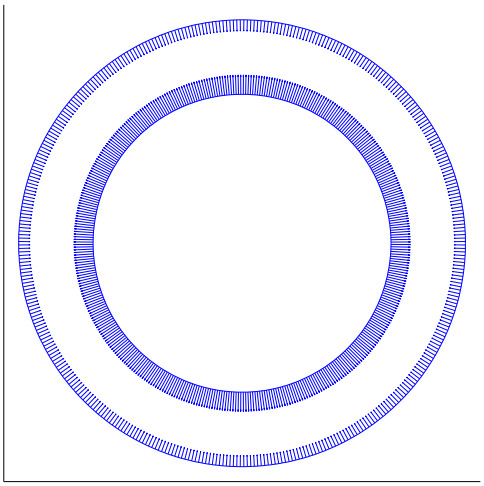

Figure 6.2: Illustration of the concentric circles with the forces on each line segment.

Two concentric circles of radii R_1 and R_2 where $R_2 > R_1$. The normals should be pointing from one circle to the other. Figure (6.2) shows the circles and the force on each line segment of the circle. The total force is zero on each circle

6.1.3 Adjacent circles

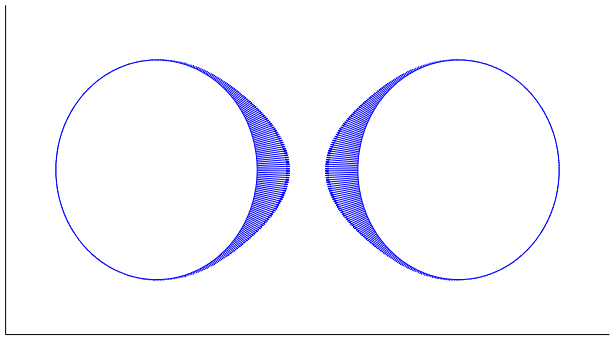

Figure 6.3: Illustration of the two adjacent circles with the forces on each line segment.

Two adjacent circles with radius R that are separated with a minimum distance of a, where all the normals should be pointing out of the circles. There is no exact solution for this case, but it will show that the methods can be used for non-symmetrical situations. Figure 6.3 shows the circles and the force on each line segment.

6.2 Results

To calculate the force with the functional integral method[1] the first order central difference will be used with parameter variation $h = 0.0001$. To be consistent all objects will use $N = 400$ discretization points and the integrals $\int d\omega$ will be solved with an accuracy of 10^{-6} for both methods. Thus the resolution of each object will be the same, the only error should be associated with the solvers themselves.

6.2.1 Parallel plates

For the comparison of the methods the lengths of the plates are set to $L = 30$ and the distance between them will vary in the range $a = 1 \ldots 3$. The methods will be compared with the exact formula from equation (4.31). The results are presented in figure 6.4 and figure 6.5 shows the relative error.

It is surprising to see that all the methods are in agreement. The parallel plate geometry is non-compact and as such outside of the scope of both the boundary element and the functional integral method. But as seen from the results, both methods are accurate even for non-compact objects. The functional integral method shows an increasing error for larger lengths, this is probably due to the constant length of the plates. To verify this suspicion the computations for the functional integral method are repeated in the same range $a = 1 \ldots 3$ but with the plate lengths $L = 30a$. This will keep the ration L/a constant during the computations. Figure 6.6 shows the new error compared to the old. The new lengths will reduce the resolution in each step and should slightly increase the error, but this is seen to be of little effect. The new error profile in figure 6.6 is very different from before, the primary difference is that it is no longer increasing with the separation of the plates.

6.2.2 Concentric circles

For this case there is an exact solution given by equation (4.74). The radius of the outer circle will be constant $R_2 = 15$ and the radius of the inner circle will vary in the range $R_1 = 10 \ldots 14$. Because of the constant discretization of $N = 400$ points on each circle the resolution of the inner circle will change with the radius. Figure 6.7 and 6.9 compares the three solutions and figure 6.8 and 6.10 shows their difference. All three methods give the same result, with little deviation.

6.2.3 Adjacent circles

Two circles of radius $R = 1$ and minimal distance between circles in the range $a = 1 \ldots 3$ will be used to compare the boundary integral method and the functional integral method. Figure 6.11 shows the results and figure 6.12 shows the difference in the solutions.

The break in figure 6.12 reflects that both methods have integrals $\int d\omega$ that of the same accuracy at these distances (10^{-6}). Since there is no exact solution it is not possible to say that either method is more accurate. As the results show, both methods give approximately the same answer.

[1] The resulting force is multiplied by two so it's possible to compare it to the other methods.

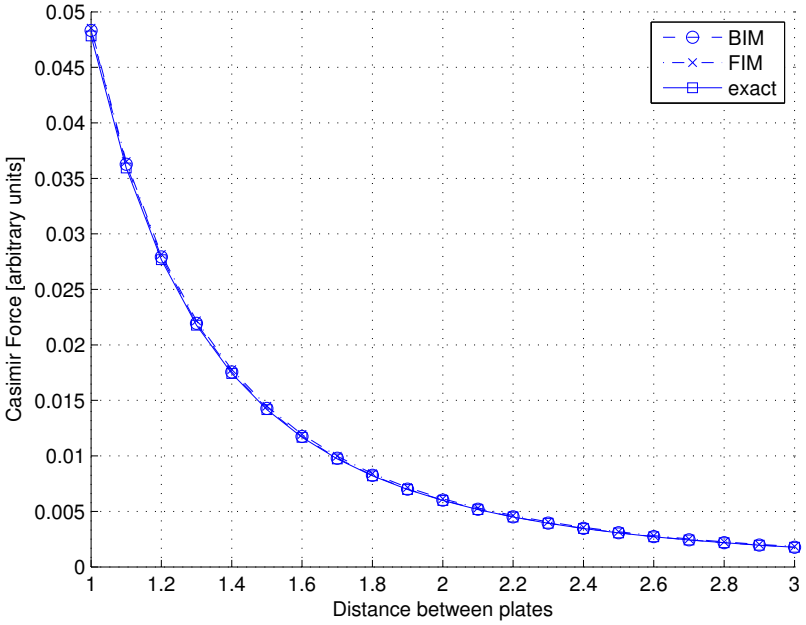

Figure 6.4: Results from the boundary element[1] and functional integral method with the exact solution to the parallel plates problem.

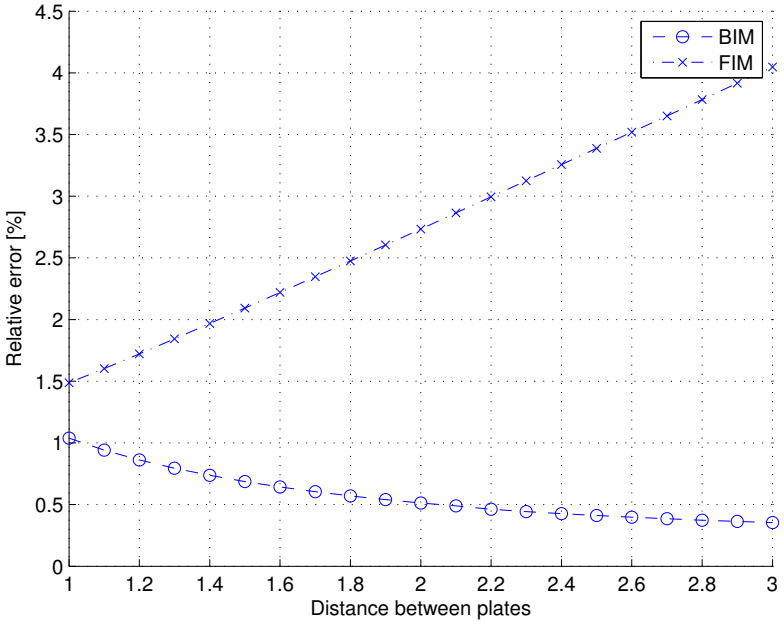

Figure 6.5: Relative error between the numerical solutions and the exact solution for the parallel plates in Figure 6.4.

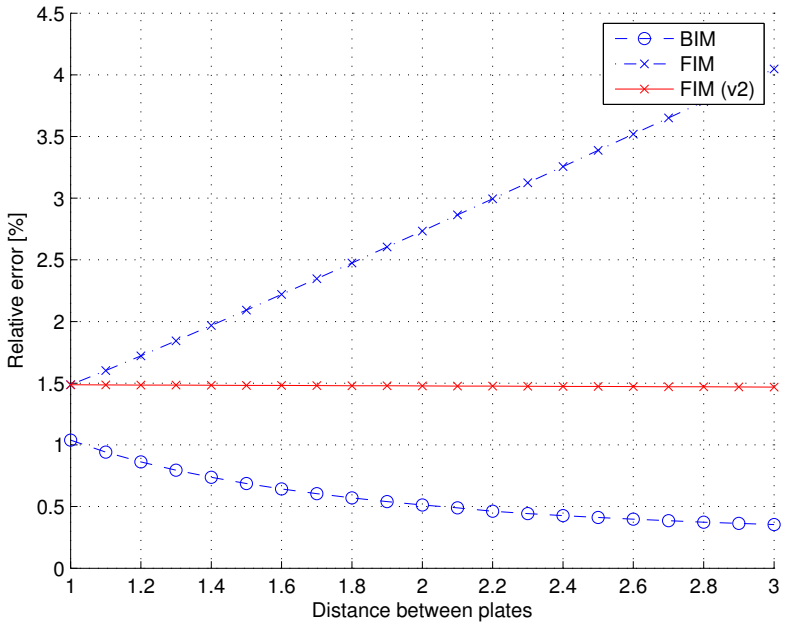

Figure 6.6: Relative error for the parallel plates from figure 6.5 compared to a new calculation with the functional integral method using a variating length $L = 30a$.

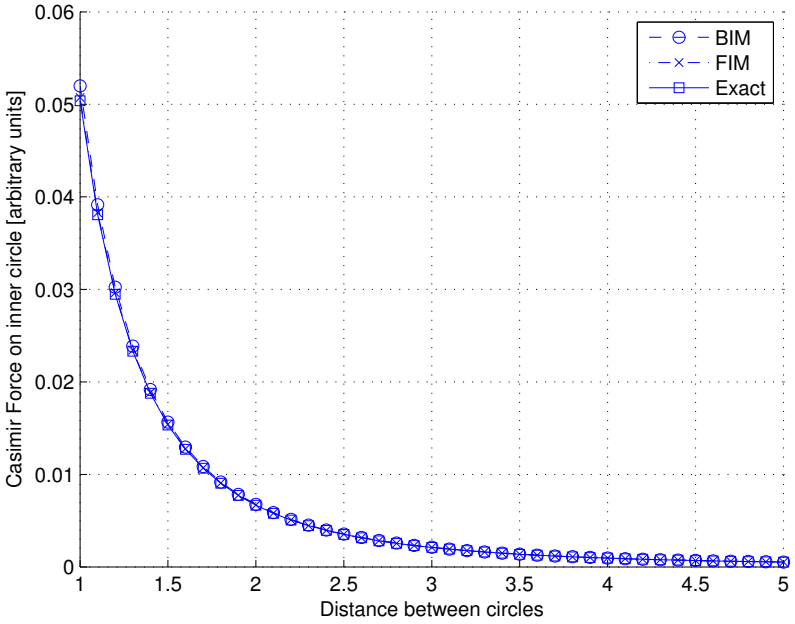

Figure 6.7: Results from the boundary element[1] and functional integral method with the exact solution to the concentric circles problem on the inner circle.

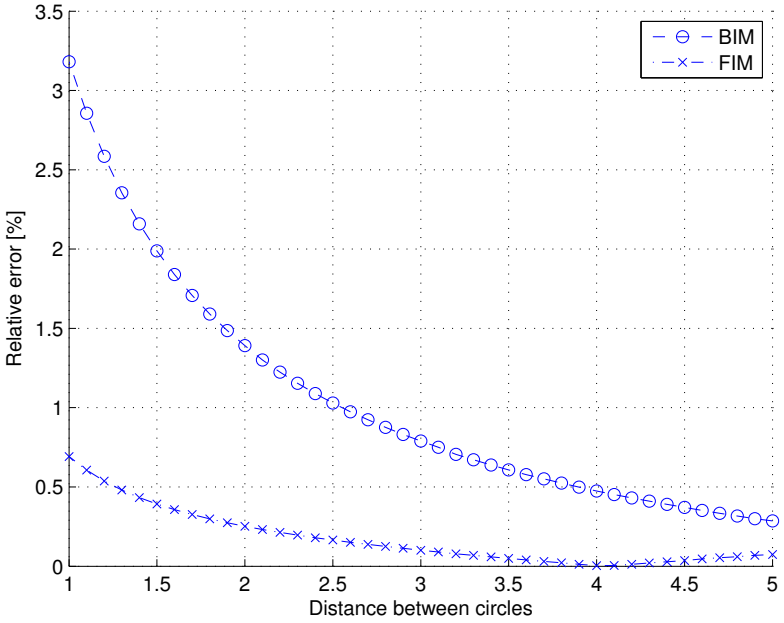

Figure 6.8: Difference between the numerical solutions and the exact solution for the concentric circles on the inner circle in Figure 6.7.

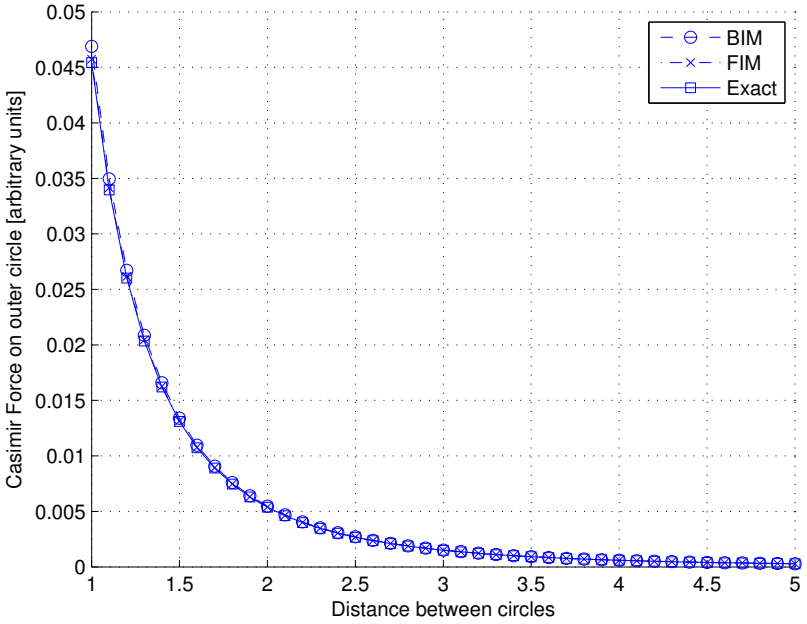

Figure 6.9: Results from the boundary element[1] and functional integral method with the exact solution to the concentric circles problem on the outer circle.

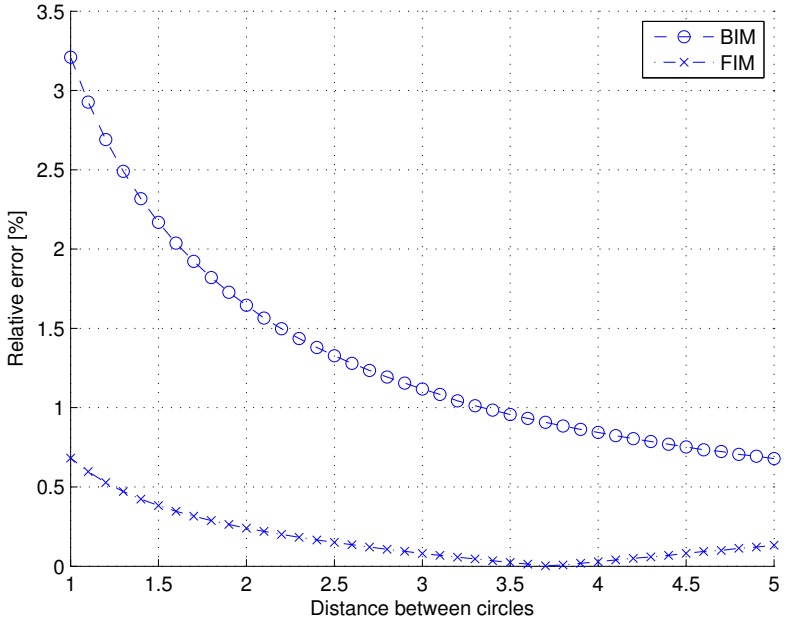

Figure 6.10: Difference between the numerical solutions and the exact solution for the concentric circles on the outer circle in Figure 6.9.

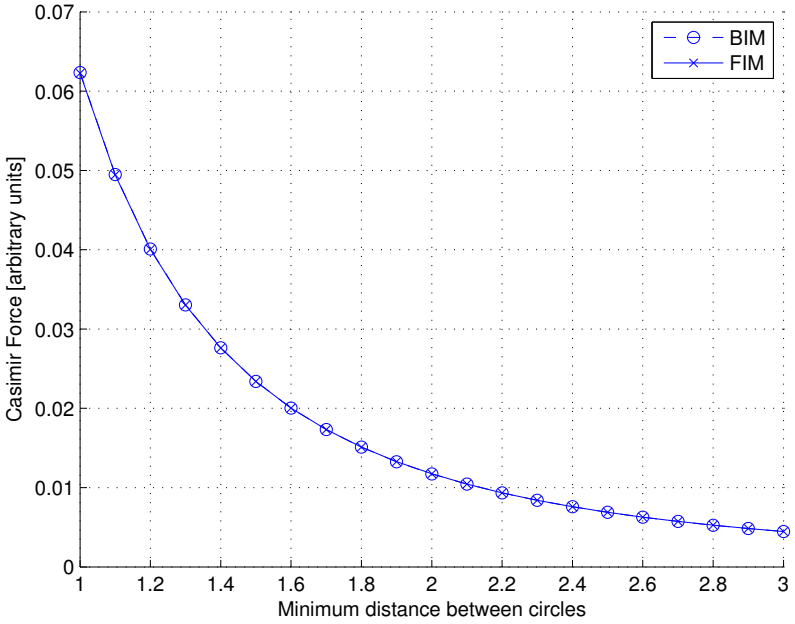

Figure 6.11: Comparison of numerical results from the boundary element[1] and functional integral method with the exact solution to the adjacent circles.

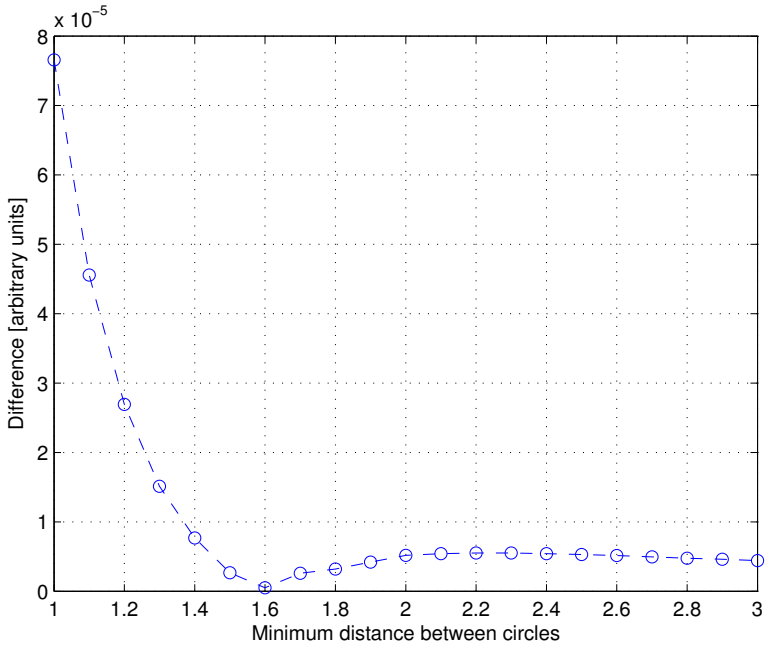

Figure 6.12: Difference between the boundary element[1] and functional integral solutions for the adjacent circles in Figure 6.11.

Chapter 7

Conclusion

In all situations the resulting force produced by the boundary integral method is off by a factor of two. Aside from this factor, the method correctly predicts the geometry dependence of each test problem. The location of this error can be somewhat predicted. In order to generate an exact solution for this problem the equations were modified in section 1.8. Using this minimal modification the exact solution could be predicted without modifying the solver to any considerable degree. This would eliminate any error related to the boundary integral method itself. The results were that the solver was very close to the exact solution, and converged for higher resolutions. While this test did not cover the subtraction of the self stress, this is only a minor addition to the solver. Without this restriction it was not possible to compare the results from the solver to an exact solution. This test shows that the factor of two is located in the theory or calculations before equation (1.52).

The factor of two could be related to the dimension, this is supported by the result in appendix A. The method correctly calculates the force on the two parallel plates in one dimension, but is missing a factor of two when the calculations are repeated in two dimensions. This could be verified if the method was implemented in three dimensions and the result was off by a factor of two (or four).

The renormalization of the equations are twofold: First a limiting process and then the subtraction of the self stress. If the limit was not taken along the surface curves, the right side of the equations would diverge. This appears to be the only possible way to perform this limit. The subtraction of the self energies is natural since the equations decouple at higher frequencies. This is equivalent (at least for these equations) to separating all the objects at infinite distances. The only remaining equation is the self stress equation (1.84) that describes the Casimir effects interaction from each object onto itself. The Casimir force is renormalized by subtracting the self interaction from each object, the resulting force is the interaction Casimir force between the objects.

For the parallel plates and concentric circles the numerical solution is in good agreement with the exact solution in figure 6.4, 6.7 and 6.9. The error plots in figures 6.5, 6.8 and 6.10 show that the maximum relative error of the boundary integral method is less than or approximately equal to 1% for these cases.

There is no exact solution to the Casimir force of the adjacent circles, but the force calculated from the boundary element- and the functional integral

methods are very similar as seen in figure 6.11. Figure 6.12 shows that the difference between the two solutions is less than 10^{-4} for all distances.

The errors in all solutions are low, but they increase as the objects get closer. This is natural because the resolution is constant during the tests. The increasing error from the functional integral method in figure 6.5 is probably an artifact from the non-compact geometry. A simple experiment where the lengths of the plates are increased with the separation was used to test the theory. Figure 6.6 shows that the relative error in the boundary integral method will remain low ($\approx$1.5%) if the plate lengths are increased. As figure 6.5 also shows, there is no similar increasing error for the boundary integral method. This suggest that the method might be less sensitive to edge effects than the functional integral method. Thus for a mixed configuration of both compact and non-compact objects the boundary integral method might give a more accurate result.

We conclude that the boundary integral method, with the given renormalization, correctly calculates the geometry dependence of the force in each problem. A very important next step is to compare the theory to experiments. To apply this method to any physical problems it would first have to be expanded to cover electromagnetic fields.

Appendix A

Boundary integral method for zero dimensional parallel plates

This chapter will cover a less complicated situation where the space is one dimensional space and the parallel plates are two points separated by a distance a. This will both serve to demonstrate the method and provide some comparison. This chapter follows the calculations of the two dimensional boundary integral method completed earlier. Figure A.1 gives a clear illustration of the situation with two zero dimensional parallel plates on the real line.

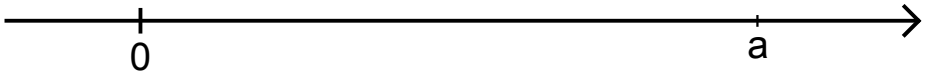

Figure A.1: Illustration of the interval with two zero dimensional parallel plates.

There are two options here, use the final results from the chapter 1 or calculate all the theory from scratch. Both options will lead to the same result.

Using results from main chapters

This will be based on the matrix system obtained as the final result in equation (1.96)

$$\begin{bmatrix} a^{11}_{k''k} & a^{12}_{k''k} \\ a^{21}_{k''k} & a^{22}_{k''k} \end{bmatrix} \begin{bmatrix} x^{11}_{kk'} \\ x^{21}_{kk'} \end{bmatrix} = \begin{bmatrix} 0 \\ y^{12}_{k'k''} - \sum_k a^{12}_{kk''} b^{11}_{kk'} \end{bmatrix} \tag{A.1}$$

and the self stress equation is

$$\sum_k a^{11}_{kk''} b^{11}_{kk'} = y^{11}_{k'k''} \tag{A.2}$$

Use the solution $x^{11}_{kk'}$ along with equation (1.52) to find the force

$$\mathbf{F}_1 = \oint_{Q_1} dl_{\mathbf{x}} \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \, \mathbf{n}(\mathbf{x}) x^{11}(\mathbf{x}, \mathbf{x}, \omega) \tag{A.3}$$

Since our surface is a single point it is natural to insert a delta function into each integral. This will evaluate the Green's function on the surface. The free 1d-Green's function is given by

$$D_0(x, x'', \omega) = -\frac{1}{2|\omega|} e^{-|\omega||x-x''|} \tag{A.4}$$

To calculate the primary matrix the Green's function will be evaluated on each object. Thus

$$\begin{aligned} a^{11} &= -\frac{1}{2|\omega|} \\ a^{12} &= -\frac{1}{2|\omega|} e^{-|\omega|a} \\ a^{21} &= -\frac{1}{2|\omega|} e^{-|\omega|a} \\ a^{22} &= -\frac{1}{2|\omega|} \end{aligned} \tag{A.5}$$

The normal derivative of the free Green's function is

$$\partial_{x'} D_0(x', x'', \omega) = \begin{cases} \frac{1}{2} e^{-|\omega|(x'-x'')} & x' > x'' \\ -\frac{1}{2} e^{-|\omega|(x''-x')} & x' < x'' \end{cases} \tag{A.6}$$

The y^{1j} matrix is (note that by convention it is necessary to first have to let $x'' \to Q_i$ and then $x' \to Q_j$)

$$\begin{aligned} y^{11} &= -n_1 \left(\frac{1}{2} \right) = -\frac{n_1}{2} \\ y^{12} &= -n_1 \left(-\frac{1}{2} e^{-|\omega|a} \right) = \frac{n_1}{2} e^{-|\omega|a} \end{aligned} \tag{A.7}$$

The self stress equation for object 1 is

$$\begin{aligned} a^{11} b^{11} &= y^{11} \\ \left(-\frac{1}{2|\omega|} \right) b^{11} &= -\frac{n_1}{2} \\ b^{11} &= |\omega| n_1 \end{aligned} \tag{A.8}$$

Thus the matrix system is given by

$$\begin{aligned} \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix} \begin{bmatrix} x^{11} \\ x^{12} \end{bmatrix} &= \begin{bmatrix} 0 \\ y^{12} - a^{21} b^{11} \end{bmatrix} \\ \begin{bmatrix} -\frac{1}{2|\omega|} & -\frac{1}{2|\omega|} e^{-|\omega|a} \\ -\frac{1}{2|\omega|} e^{-|\omega|a} & -\frac{1}{2|\omega|} \end{bmatrix} \begin{bmatrix} x^{11} \\ x^{12} \end{bmatrix} &= \begin{bmatrix} 0 \\ n_1 e^{-|\omega|a} \end{bmatrix} \end{aligned} \tag{A.9}$$

This is simplified into

$$\begin{bmatrix} 1 & e^{-|\omega|a} \\ e^{-|\omega|a} & 1 \end{bmatrix} \begin{bmatrix} x^{11} \\ x^{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -2|\omega| n_1 e^{-|\omega|a} \end{bmatrix} \tag{A.10}$$

The solution is

$$x^{11} = n_1 \frac{2|\omega|}{e^{2|\omega|a} - 1} \tag{A.11}$$

and from this the force is given by

$$\begin{aligned}\mathbf{F}_1 &= \oint_{Q_1} dl_{\mathbf{x}} \frac{n_1^2}{4\pi} \int_{-\infty}^{\infty} d\omega \, \frac{2|\omega|}{e^{2|\omega|a} - 1} = \oint_{Q_1} dl_{\mathbf{x}} \frac{n_1^2}{2\pi} \int_0^{\infty} d\omega \, \frac{2\omega}{e^{2\omega a} - 1} \\ &= \oint_{Q_1} dl_{\mathbf{x}} \frac{1}{4\pi a^2} \int_0^{\infty} du \, \frac{u}{e^u - 1} \\ &= \oint_{Q_1} dl_{\mathbf{x}} \frac{1}{4\pi a^2} \int_0^{\infty} du \, \frac{\pi^2}{6}\end{aligned} \tag{A.12}$$

Thus the density is given by

$$p_1 = \frac{\pi}{24a^2} \tag{A.13}$$

The plates are attracted and this agrees with the results of Milton [16] (p.9 with d=0) and also Ambjørn and Wolfram [15](p.4 with d=1).

From scratch

Alternatively it is possible to find this solution without using the previous results. The Green's function satisfies the equation (1.26) in one spatial dimension.

$$\begin{gathered}\frac{d}{dx^2}\mathcal{D}(x, x', \omega) - \omega^2 \mathcal{D}(x, x', \omega) = \delta(x - x') \\ \mathcal{D}(x, x', \omega)|_{Q_j} = 0\end{gathered} \tag{A.14}$$

The *Lagrangian* for this dimension is now given as

$$\mathcal{L} = \frac{1}{2}\varphi_t^2 - \frac{1}{2}\varphi_x \tag{A.15}$$

The stress energy tensor is found to be

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L} \tag{A.16}$$

These are calculated as

$$\begin{aligned} T^{00} &= \frac{1}{2}\varphi_t^2 + \frac{1}{2}\varphi_x^2 & T^{01} &= \varphi_t \varphi_x \\ T^{11} &= -\frac{1}{2}\varphi_t^2 - \frac{1}{2}\varphi_x^2 & T^{10} &= -\varphi_x \varphi_t \end{aligned} \tag{A.17}$$

The conservation equation are given by

$$\partial_t T^{0\nu} + \partial_x T^{1\nu} = 0 \tag{A.18}$$

When $\nu = 1$ the momentum equation appears

$$\partial_t(\varphi_t \varphi_x) + \partial_x(-\frac{1}{2}\varphi_t^2 - \frac{1}{2}\varphi_x^2) = 0 \tag{A.19}$$

or

$$\partial_t \rho + \partial_x S = 0 \tag{A.20}$$

where ρ is the momentum density and S is the momentum flux. The momentum flux is

$$S(x,t) = -\frac{1}{2}\varphi_x^2 - \frac{1}{2}\varphi_t^2 \tag{A.21}$$

This will define the quantum stress tensor through point splitting.

$$S_q(x,t) = \lim_{\substack{x'\to x \\ t'\to t}} \left(-\frac{1}{2}\partial_x\partial_{x'} - \frac{1}{2}\partial_t\partial_{t'}\right)\mathcal{D}(x,t,x',t') \tag{A.22}$$

Using the same procedure as before we set: $t = -iu$, $t' = -iu'$ and $s = u - u'$

$$S_q(x) = \lim_{\substack{x'\to x \\ s\to 0}} \left(-\frac{1}{2}\partial_x\partial_{x'} - \frac{1}{2}\partial_{ss}\right)\mathcal{D}(x,x',s) \tag{A.23}$$

And a Fourier transform in time gives

$$S_q(x,\omega) = \lim_{\substack{x'\to x \\ s\to 0}} \left(-\frac{1}{2}\partial_x\partial_{x'} + \omega^2\frac{1}{2}\right)\mathcal{D}(x,x',\omega) \tag{A.24}$$

The boundary conditions state that $\mathcal{D}(x,x',\omega) = 0$ on the boundary

$$S_q(x,\omega) = \lim_{\substack{x'\to x \\ s\to 0}} -\frac{1}{2}\partial_x\partial_{x'}\mathcal{D}(x,x',\omega) \tag{A.25}$$

where

$$S_q(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_q(x,\omega) \tag{A.26}$$

The force on the interval can now be calculated as

$$\begin{aligned} F = \frac{\partial P}{\partial t} &= \partial_t \int dV\, \rho(x,t) = -\int dV\, \partial_x S_q \\ &= -S_q|_0^a = S_q(0) - S_q(a) \end{aligned} \tag{A.27}$$

Thus the force from the left endpoint to the interval is given by

$$\begin{aligned} F_{\text{left}} = S_q(0) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S_q(0,\omega) \\ &= -\lim_{x'\to x} \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega\, D_{xx'}(x,x',\omega)|_{x=0} \end{aligned} \tag{A.28}$$

To calculate the force *on* the left plate the sign must be reversed. After the limit

$$F_{\text{left}} = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega\, D_{xx}(0,0,\omega) \tag{A.29}$$

For the right endpoint the result will be the same answer except a change in sign due to the normal vector $n_a = +1$.

To make a boundary element formulation for equation (A.14) it is natural to start with the integral relation for the opperator $L = d^2/dx^2 - \omega^2$

$$\begin{aligned}
&\int dx \, (\mathrm{L}\Phi\Psi - \Phi\mathrm{L}\Psi) \\
&= \int dx \, \left(\left(\frac{d^2\Phi}{dx^2} - \omega^2\Phi \right)\Psi - \Phi\left(\frac{d^2\Psi}{dx^2} - \omega^2\Psi \right) \right) \\
&= \int dx \, \left(\frac{d^2\Phi}{dx^2}\Psi - \Phi\frac{d^2\Psi}{dx^2} \right) = \int dx \, \frac{d}{dx}\left(\frac{d\Phi}{dx}\Psi - \Phi\frac{d\Psi}{dx} \right) \\
&= \left(\frac{d\Phi}{dx}\Psi - \Phi\frac{d\Psi}{dx} \right)\Big|_0^a
\end{aligned} \tag{A.30}$$

The free space Green's function satisfies

$$LD_0(x, x'', \omega) = \partial_{xx} D_0(x, x'', \omega) - \omega^2 D_0(x, x'', \omega) = \delta(x - x'') \tag{A.31}$$

where $D_0(x, x'', \omega) = -\frac{1}{2|\omega|}\exp(-|\omega||x - x''|)$. Define $\mathcal{E}(x, x') = \partial_{x'}\mathcal{D}(x, x')$ with the equation.

$$L\mathcal{E}(x, x', \omega) = \partial_{xx}\mathcal{E}(x, x', \omega) - \omega^2\mathcal{E}(x, x', \omega) = \partial_{x'}\delta(x - x') \tag{A.32}$$

Use the integral relation above to write out

$$\begin{aligned}
&\int dx \, (\mathrm{L}D_0(x, x'')\mathcal{E}(x, x') - D_0(x, x'')\mathrm{L}\mathcal{E}(x, x')) \\
&= \int dx \, (\delta(x - x'')\mathcal{E}(x, x') - D_0(x, x'')\partial_{x'}\delta(x - x')) \\
&= \mathcal{E}(x'', x') - \partial_{x'} D_0(x', x'')
\end{aligned} \tag{A.33}$$

The left side of the relation is also equal to

$$\begin{aligned}
&\int dx \, (\mathrm{L}D_0(x, x'')\mathcal{E}(x, x') - D_0(x, x'')\mathrm{L}\mathcal{E}(x, x')) \\
&= (\partial_x D_0(x, x'')\mathcal{E}(x, x') - D_0(x, x'')\partial_x\mathcal{E}(x, x'))|_0^a
\end{aligned} \tag{A.34}$$

Thus

$$\begin{aligned}
&\mathcal{E}(x'', x') - \partial_{x'} D_0(x', x'') \\
&= (\partial_x D_0(x, x'')\mathcal{E}(x, x') - D_0(x, x'')\partial_x\mathcal{E}(x, x'))|_0^a \\
&= -D_0(a, x'')\partial_x\mathcal{E}(a, x') + D_0(0, x'')\partial_x\mathcal{E}(0, x')
\end{aligned} \tag{A.35}$$

When $x'' \to \{0^+, a^-\}$ it is obvious from the boundary conditions that $\mathcal{E}(x'', x') = 0$. Thus

$$\partial_{x'} D_0(x', x'') = D_0(a, x'')\partial_x\mathcal{E}(a, x') - D_0(0, x'')\partial_x\mathcal{E}(0, x') \tag{A.36}$$

For $x'' \in \{0, a\}$

Note that for high frequencies $\omega >> 0$ the equations will decouple and for $x' \to \{0^+, a^-\}$ this results in

$$\begin{aligned}
\partial_{x'} D_0(0^+, 0) &= -D_0(0^+, 0)\partial_x\mathcal{E}(0^+, 0) \\
\partial_{x'} D_0(a^-, a) &= D_0(a^-, a)\partial_x\mathcal{E}(a^-, a)
\end{aligned} \tag{A.37}$$

These equations are parallel to the self stress equations that were found in equation (1.84). It is possible to solve these equation directly and the solution will be defined as $\partial_{xx'} D_j(x, x')$

$$\begin{aligned} \partial_x \mathcal{E}(0,0) \equiv \partial_{xx'} \mathcal{D}_1(0,0) &= \lim_{x' \to 0^+} -\partial_{x'} D_0(x', 0)/D_0(x', 0) \\ \partial_x \mathcal{E}(a,a) \equiv \partial_{xx'} \mathcal{D}_2(a,a) &= \lim_{x' \to a^-} \partial_{x'} D_0(x', a)/D_0(x', a) \end{aligned} \tag{A.38}$$

where

$$\partial_{x'} D_0(x', x'', \omega) = \begin{cases} \frac{1}{2} e^{-|\omega|(x'-x'')} & x' > x'' \\ -\frac{1}{2} e^{-|\omega|(x''-x')} & x' < x'' \end{cases} \tag{A.39}$$

and in the limits

$$\begin{aligned} \lim_{x' \to 0^+} \partial_{x'} D_0(x', 0, \omega) &= -\frac{1}{2} \\ \lim_{x' \to a^-} \partial_{x'} D_0(x', a, \omega) &= -\frac{1}{2} \end{aligned} \tag{A.40}$$

Thus from equation (A.38)

$$\begin{aligned} \partial_{xx'} \mathcal{D}_1(0,0) &= |\omega| \\ \partial_{xx'} \mathcal{D}_2(a,a) &= |\omega| \end{aligned} \tag{A.41}$$

Use these to regularize the force calculations by subtracting the high frequency contribution and redefine equation (A.29) appropriately.

$$\Delta_j(x, x') = \partial_{xx'} \mathcal{D}(x, x') - \partial_{xx'} \mathcal{D}_j(x, x') \tag{A.42}$$

Insert this back into equation (A.36) to get the following system of equations parallel to eq (1.86) and (1.87)

$$\begin{aligned} 0 &= D_0(0,0^+)\Delta_1(0,0^+) - D_0(a,0^+)\partial_{xx'}\mathcal{D}(a,0^+) \quad x', x'' \to 0 \\ 0 &= D_0(a,a^-)\Delta_2(a,a^-) - D_0(0,a^-)\partial_{xx'}\mathcal{D}(0,a^-) \quad x', x'' \to a \end{aligned} \tag{A.43}$$

and

$$\begin{aligned} &\partial_{x'} D_0(0^+, a^-) + D_0(0, a^-)\partial_{xx'}\mathcal{D}_1(0,0^+) \\ &= D_0(a, a^-)\partial_{xx'}\mathcal{D}(a,0^+) - D_0(0,a^-)\Delta_1(0,0^+) \quad \substack{x' \to 0^+ \\ x'' \to a^-} \\ &\partial_{x'} D_0(a^-, 0^+) - D_0(a, 0^+)\partial_{xx'}\mathcal{D}_2(a,a^-) \\ &= D_0(a, 0^+)\Delta_2(a,a^-) - D_0(0,0^+)\partial_{xx'}\mathcal{D}(0,a^-) \quad \substack{x' \to a^- \\ x'' \to 0^+} \end{aligned} \tag{A.44}$$

Insert the known limits into this system and organize to get a better view of the two systems

$$\begin{aligned} 0 &= -\Delta_1(0,0) + e^{-|\omega|a}\partial_{xx'}\mathcal{D}(a,0) \qquad x', x'' \to 0 \\ -2|\omega|e^{-|\omega|a} &= e^{-|\omega|a}\Delta_1(0,0) - \partial_{xx'}\mathcal{D}(a,0) \qquad \substack{x' \to 0 \\ x'' \to a} \end{aligned} \tag{A.45}$$

and for the second object

$$\begin{aligned} 0 &= -\Delta_2(a,a) + e^{-|\omega|a}\partial_{xx'}\mathcal{D}(0,a) \qquad x', x'' \to a \\ 2|\omega|e^{-|\omega|a} &= -e^{-|\omega|a}\Delta_2(a,a) + \partial_{xx'}\mathcal{D}(0,a) \qquad \substack{x' \to a \\ x'' \to 0} \end{aligned} \tag{A.46}$$

These can be organized as matrix equations

$$\begin{bmatrix} -1 & e^{-|\omega|a} \\ e^{-|\omega|a} & -1 \end{bmatrix} \begin{bmatrix} \Delta_1(0,0) \\ \partial_{xx'}\mathcal{D}(a,0) \end{bmatrix} = \begin{bmatrix} 0 \\ -2|\omega|e^{-|\omega|a} \end{bmatrix} \tag{A.47}$$

and

$$\begin{bmatrix} -1 & e^{-|\omega|a} \\ -e^{-|\omega|a} & +1 \end{bmatrix} \begin{bmatrix} \Delta_2(a,a) \\ \partial_{xx'}\mathcal{D}(0,a) \end{bmatrix} = \begin{bmatrix} 0 \\ 2|\omega|e^{-|\omega|a} \end{bmatrix} \tag{A.48}$$

The solutions are

$$\begin{aligned} \Delta_1(0,0,\omega) &= -\frac{2|\omega|}{1-e^{2|\omega|a}} \\ \Delta_2(a,a,\omega) &= -\frac{2|\omega|}{1-e^{2|\omega|a}} \end{aligned} \tag{A.49}$$

Insert this back into equation (A.29)

$$\begin{aligned} F_{\text{left}} &= \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega\, \Delta_1(0,0,\omega) \\ &= -\frac{1}{\pi}\int_{0}^{\infty} d\omega\, \frac{\omega}{1-e^{2\omega a}} \\ &= -\frac{1}{\pi}(-\frac{\pi^2}{24a^2}) \\ &= \frac{\pi}{24a^2} \end{aligned} \tag{A.50}$$

The force on the left endpoint is

$$F_{\text{left}} = \frac{\pi}{24a^2} \tag{A.51}$$

This force is directed into the interval and the only difference for the force on the other endpoint is a change in sign. Thus the plates will be attracted as was found above in equation (A.13).

Appendix B

Functional integral method method for zero dimensional parallel plates

Consider two zero dimensional plates separated by a distance a along the real line. This will both serve to demonstrate the method and provide some comparison for later. This chapter is a step by step walkthrough of chapter 2.8 for the functional integral method.

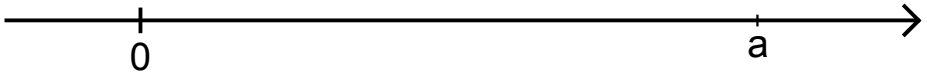

Figure B.1: Illustration of the interval with two zero dimensional parallel plates.

Figure B.1 illustrates the situation of two zero dimensional "parallel plates" along the real axis. Each point has a origo $\mathcal{O}^1 = 0$ and $\mathcal{O}^2 = a$, the basis functions for each point is $1^1 = a^1_{p_1} = a^2_{p_2} = 1^2 = 1$ in the respective coordinates systems. There is only one basis function in each coordinates system so the index set is $p_1 = p_2 = 1$.

The Green's function for *Helmhotz* equation in each coordinate system is given by equation (2.58) and for one dimension it is

$$\begin{aligned} &G^1(x_\alpha, x_{\alpha'}) = G^2(x_\alpha, x_{\alpha'}) = G(x_\alpha - x_{\alpha'}) \\ &= \frac{i}{2k} e^{ik|x_\alpha - x_{\alpha'}|}|_{k=i\kappa} = \frac{1}{2\kappa} e^{-\kappa|x_\alpha - x_{\alpha'}|} \end{aligned} \tag{B.1}$$

for $\kappa \geq 0$

The first goal is to decompose the Green's function into each basis set. To find this decomposition one has to evaluate the Green's function on the surface of each object, use the formula $G^{\alpha\beta}(x_\alpha, x_{\beta'}) = G^\beta(x_\beta(x_\alpha), x_{\beta'})$

$$\begin{aligned} G^{11}(x_1, x_{\beta'}) &= G^1(x_1(x_1), x_{\beta'}) = G^1(0,0) = G(0) = \tfrac{1}{2\kappa} \\ G^{22}(x_2, x_{\beta'}) &= G^2(x_2(x_2), x_{\beta'}) = G^2(0,0) = G(0) = \tfrac{1}{2\kappa} \\ G^{12}(x_1, x_{\beta'}) &= G^2(x_2(x_1), x_{\beta'}) = G^2(-a,0) = G(-a) = \tfrac{1}{2\kappa} e^{-\kappa a} \\ G^{21}(x_2, x_{\beta'}) &= G^1(x_1(x_2), x_{\beta'}) = G^1(a,0) = G(a) = \tfrac{1}{2\kappa} e^{-\kappa a} \end{aligned} \tag{B.2}$$

The decomposition is now found directly from the above calculations since $G^{\alpha\beta}(x_\alpha, x_{\beta'}) = \sum_{p_\alpha} G^{\alpha\beta}_{p_\alpha}(x_{\beta'}) a^\alpha_{p_\alpha}(x_\alpha)$.

The surface integrals are now

$$\begin{aligned} G^{\alpha\beta}_{p_\alpha q_\beta} &= \int_{Q_\beta} G^{\alpha\beta}_{p_\alpha}(x_{\beta'}) a^\beta_{q_\beta}(x_{\beta'})\, dA_{x_{\beta'}} \\ &= G^{\alpha\beta}_{p_\alpha}(x_\beta) a^\beta_{q_\beta}(x_\beta) = \begin{cases} \frac{1}{2\kappa} & \alpha = \beta \\ \frac{1}{2\kappa} e^{-\kappa a} & \alpha \neq \beta \end{cases} \end{aligned} \tag{B.3}$$

and

$$D^\alpha_{p_\alpha q_\beta} = \int_{Q_\alpha} (a^\alpha_{q_\alpha}(x_\alpha))^* a^\alpha_{p_\alpha}(x_\alpha)\, dA_{x_\alpha} = 1 \tag{B.4}$$

The matrices are calcualted as

$$\begin{aligned} H^1 &= \tfrac{1}{2\kappa} & H^2 &= \tfrac{1}{2\kappa} \\ K^{12} &= \tfrac{1}{2\kappa} e^{-\kappa a} & K^{21} &= \tfrac{1}{2\kappa} e^{-\kappa a} \end{aligned} \tag{B.5}$$

and

$$\begin{aligned} T^1 &= \tfrac{1}{\kappa} & T^2 &= \tfrac{1}{\kappa} \\ U^{12} &= \tfrac{1}{\kappa} e^{-\kappa a} & U^{21} &= \tfrac{1}{\kappa} e^{-\kappa a} \end{aligned} \tag{B.6}$$

The final expression of the *Casimir energy* is found from equation (2.103)

$$\mathcal{E}(a) = \frac{\hbar}{2\pi} \int_0^\infty \ln\left(\det M_Q(i\kappa)\right) d\kappa \tag{B.7}$$

Where the matrix M_Q is given by

$$\begin{bmatrix} \mathbb{1} & \cdots & (T^1)^{-1} U^{1r} \\ \vdots & \ddots & \vdots \\ (T^r)^{-1} U^{r1} & \cdots & \mathbb{1} \end{bmatrix} \tag{B.8}$$

Since there are only two objects the energy is reduced to

$$\mathcal{E}(a) = \frac{\hbar}{2\pi} \int_0^\infty \ln\left(\det\left(1 - (T^1)^{-1} U^{12} (T^2)^{-1} U^{21}\right)\right)|_{k=i\,\kappa}\, d\kappa \tag{B.9}$$

After multiplication the energy is given by

$$\mathcal{E}(a) = \frac{\hbar}{2\pi} \int_0^\infty \ln\left(1 - e^{-2ka}\right) d\kappa \tag{B.10}$$

This can be solved as

$$\mathcal{E}(a) = \frac{\hbar}{2\pi}\left(-\frac{\pi^2}{12a}\right) = -\frac{\hbar\pi}{24a} \tag{B.11}$$

This is the Casimir energy for two zero dimensional "parallel plates" along a line. This coincides with the results found of V.G. Kiselev, Y.M. Shinir and A.Ya Tregubovich [17] (p.102 and 106) and also Ambjørn and Wolfram [15] (p.4 with d=1).

Appendix C

Mode summation for parallel plates

Consider two parallel plates with separation a. The defining equation for a massless scalar field is

$$\varphi_{tt}(\mathbf{x},t) - c^2\nabla^2\varphi(\mathbf{x},t) = 0 \tag{C.1}$$

where $c = 1$ and the field $\varphi(\mathbf{x},t)$ satisfies $\varphi(\mathbf{x},t) = 0$ when evaluated on either plate.

Starting with a Fourier transform in the time domain.

$$\omega^2\varphi(\mathbf{x}) + \nabla^2\varphi(\mathbf{x}) = 0 \tag{C.2}$$

and then another Fourier transform of y with wavenumber k will yield

$$-\varphi''(x) + \left(k^2 - \omega^2\right)\varphi(x) = 0 \tag{C.3}$$

or

$$-\varphi''(x) + q^2\varphi(x) = 0 \tag{C.4}$$

where $q = \sqrt{k^2 - \omega^2}$. The solution to this equation is

$$\varphi(x) = Ae^{qx} + Be^{-qx} \tag{C.5}$$

The boundary conditions are

$$\begin{aligned} \varphi(0) = A + B &= 0 \\ \varphi(a) = Ae^{qa} + Be^{-qa} &= 0 \end{aligned} \tag{C.6}$$

or as a matrix system

$$\begin{bmatrix} 1 & 1 \\ e^{qa} & e^{-qa} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{C.7}$$

The only possibility for a non-trivial solution is if the determinant is zero

$$e^{-qa} - e^{qa} = 0 \tag{C.8}$$

The imposes a restriction on the possible values of ω. Note that for real q there is only one solution to the above equation: $q = 0$. This implies that $\omega^2 = k^2$. The other possible solutions are for $\omega^2 > k^2$, this will give

$$q = \sqrt{k^2 - \omega^2} = i\sqrt{\omega^2 - k^2} = ip \tag{C.9}$$

Thus for $\omega^2 \geq k^2$ equation (C.8) will be

$$g(\omega, k) = e^{-ipa} - e^{ipa} = -2i\sin(pa) = 0 \tag{C.10}$$

Note that for all $\omega^2 < k^2$ this function is non-zero.

The energy can be expressed as

$$E = \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_n \omega_n(k) = \frac{\hbar}{2\pi} \int_0^{\infty} dk \sum_n \omega_n(k) \tag{C.11}$$

Where $\omega_n(k)$ are the zeros of $g(\omega, k)$ and units are chosen such that $\hbar = 1$. As $g(\omega, k)$ has no poles it is possible to use the argument principle in order to evaluate this sum.

It states that for an analytic function $h(z)$ with no poles inside the contour C and a meromorphic function $f(z)$ with no poles or zeros on C. Then

$$\frac{1}{2\pi i} \oint_C dz\, h(z) \frac{f'(z)}{f(z)} = \sum_n m_n h(z_n^0) - k_n h(z_n^p) \tag{C.12}$$

where z_n^0 are zeros, and z_n^p are poles of $f(z)$ inside the contour and m_n, k_n are their respective multiplicity.

Use $h(z) = z$ and an analytic continuation of $g(\omega, k)$ from equation (C.10) to get a sum over the zeros ω_n

$$\sum_n \omega_n(k) = \frac{1}{2\pi i} \oint_C d\omega\, \omega \frac{g'(\omega, k)}{g(\omega, k)} \tag{C.13}$$

Insert this back into the energy to get

$$E = \frac{1}{4\pi^2 i} \int_0^{\infty} dk \oint_C d\omega\, \omega \frac{g'(\omega, k)}{g(\omega, k)} \tag{C.14}$$

With a partial integration this will be

$$E = -\frac{1}{4\pi^2 i} \int_0^{\infty} dk \oint_C d\omega\, \log(g(\omega, k)) \tag{C.15}$$

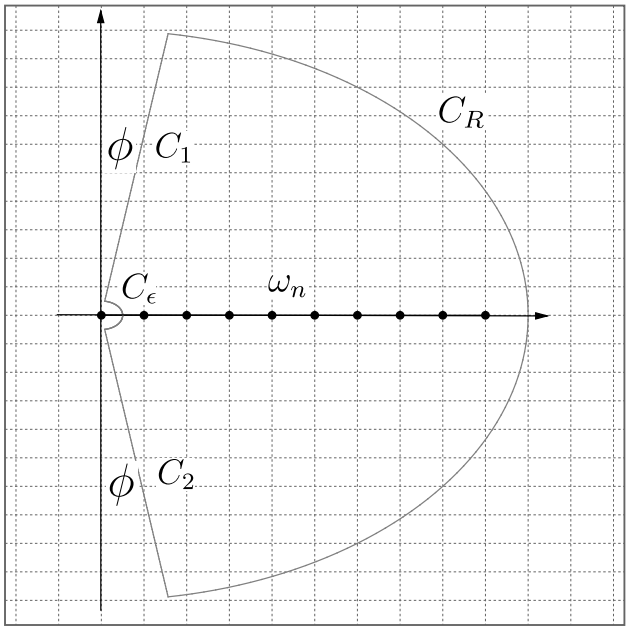

Figure C.1: The integration contour used in the argument principle for the parallel plates.

where the contour C is given in Figure C.1. The angle ϕ will help with the convergence

$$E = -\frac{1}{4\pi^2 i} \lim_{\substack{R\to\infty \\ \epsilon\to 0 \\ \phi\to 0}} \int_0^\infty dk \left(\int_{C_1} d\omega \, \log\left(g(\omega,k)\right) + \int_{C_2} d\omega \, \log\left(g(\omega,k)\right) + \int_{C_\epsilon} d\omega \, \log\left(g(\omega,k)\right) + \int_{C_R} d\omega \, \log\left(g(\omega,k)\right) \right) \tag{C.16}$$

The first two integrals give

$$\begin{aligned} E &= -\frac{1}{4\pi^2 i} \lim_{\substack{R\to\infty \\ \epsilon\to 0 \\ \phi\to 0}} \int_0^\infty dk \left(\int_{C_1} d\omega \, \log\left(g(\omega,k)\right) + \int_{C_2} d\omega \, \log\left(g(\omega,k)\right) \right) \\ &= -\frac{1}{4\pi^2 i} \lim_{\substack{R\to\infty \\ \epsilon\to 0 \\ \phi\to 0}} \int_0^\infty dk \left(\int_R^\epsilon i e^{-i\phi} dy \, \log\left(g(iye^{-i\phi},k)\right) + \int_\epsilon^R (-ie^{i\phi} dy) \, \log\left(g(-iye^{i\phi},k)\right) \right) \\ &= \frac{1}{2\pi^2} \int_0^\infty dk \int_0^\infty dy \, \log\left(g(iy,k)\right) \end{aligned} \tag{C.17}$$

Note that the integrand, $g(\omega, k)$, is evaluated on the imaginary axis. This will result in

$$g(iy,k) = e^{-i\sqrt{(iy)^2-k^2}a} - e^{i\sqrt{(iy)^2-k^2}a} = e^{\sqrt{y^2+k^2}a} - e^{-\sqrt{y^2+k^2}a} \tag{C.18}$$

It is convenient to define the divergent part as

$$g_\infty(iy) = e^{\sqrt{y^2+k^2}a} \tag{C.19}$$

This can be rotated into the complex plane with $\omega = iy$

$$g_\infty(\omega) = e^{i\sqrt{\omega^2-k^2}a} \tag{C.20}$$

This is the divergent part of the energy. Thus it is convenient to redefine the Casimir energy by subtracting this

$$\begin{aligned}
\mathcal{E} &= E - E_\infty \\
&= -\frac{1}{4\pi^2 i} \lim_{\substack{R\to\infty \\ \epsilon\to 0 \\ \phi\to 0}} \int_0^\infty dk \left(-2i \int_0^\infty dy \, \log\left(\frac{g(iy,k)}{g_\infty(iy,k)}\right) \right. \\
&\quad \left. + \int_{C_\epsilon} d\omega \, \log\left(\frac{g(\omega,k)}{g_\infty(\omega,k)}\right) + \int_{C_R} d\omega \, \log\left(\frac{g(\omega,k)}{g_\infty(\omega,k)}\right) \right)
\end{aligned} \tag{C.21}$$

Consider the two final integrals by using the parametrization $\omega = \rho e^{i\theta}$

$$\begin{aligned}
&\int_{C_\rho} d\omega \, \log\left(\frac{g(\omega,k)}{g_\infty(\omega,k)}\right) \\
&= \int_{C_\rho} \rho e^{i\theta} d\theta \, \log\left(\frac{g(\rho e^{i\theta},k)}{g_\infty(\rho e^{i\theta},k)}\right) \\
&= \int_{C_\rho} \rho e^{i\theta} d\theta \, \log\left(\frac{e^{-i\sqrt{(\rho e^{i\theta})^2-k^2}a} - e^{i\sqrt{(\rho e^{i\theta})^2-k^2}a}}{e^{i\sqrt{(\rho e^{i\theta})^2-k^2}a}}\right) \\
&= \int_{C_\rho} \rho e^{i\theta} d\theta \, \log\left(e^{-2i\sqrt{\rho^2 e^{2i\theta}-k^2}a} - 1\right)
\end{aligned} \tag{C.22}$$

On the contour C_ϵ the radius $\rho = \epsilon \to 0$ and thus

$$\begin{aligned}
\int_{C_\epsilon} d\omega \, \log\left(\frac{g(\omega,k)}{g_\infty(\omega,k)}\right) &= \int_{C_\epsilon} \epsilon e^{i\theta} d\theta \, \log\left(e^{-2i\sqrt{\epsilon^2 e^{2i\theta}-k^2}a} - 1\right) \\
&\to 0
\end{aligned} \tag{C.23}$$

And on the contour C_R the radius $\rho = R \to \infty$ and $\theta \in [-\pi/2 + \phi, \pi/2 - \phi]$

$$\begin{aligned}
\int_{C_R} d\omega \, \log\left(\frac{g(\omega,k)}{g_\infty(\omega,k)}\right) &\approx \int_{C_R} R e^{i\theta} d\theta \, \log\left(e^{-2iRe^{i\theta}a} - 1\right) \\
&= \int_{C_R} R e^{i\theta} d\theta \, \log\left(e^{-2iRa\cos(\theta)} e^{2Ra\sin(\theta)} - 1\right) \\
&\to \infty
\end{aligned} \tag{C.24}$$

If this divergence is ignored the Casimir energy will be given by

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2\pi^2}\int_0^\infty dk \int_0^\infty dy\, \log\left(\frac{g(iy,k)}{g_\infty(iy,k)}\right) \\
&= \frac{1}{2\pi^2}\int_0^\infty dk \int_0^\infty dy\, \log\left(1-e^{-2\sqrt{y^2+k^2}a}\right) \\
&= \frac{1}{2\pi^2}\int_0^\infty dr \int_0^{\pi/2} d\theta\, r\log\left(1-e^{-2ra}\right) = \frac{1}{4\pi}\int_0^\infty dr\, r\log\left(1-e^{-2ra}\right) \\
&= \frac{1}{4\pi}\int_0^\infty \frac{du}{2a}\,\frac{u}{2a}\log\left(1-e^{-u}\right) = \frac{1}{16\pi a^2}\int_0^\infty du\, u\log\left(1-e^{-u}\right)
\end{aligned} \tag{C.25}$$

For two plates with a separation a the Casimir energy is given by

$$\mathcal{E}(a) = \frac{\hbar}{16\pi a^2}\int_0^\infty du\, u\log\left(1-e^{-u}\right) = -\frac{\hbar\zeta(3)}{16\pi a^2} \approx -0.0239142\frac{\hbar}{a^2} \tag{C.26}$$

This coincides with equation (4.31) in section 4.1.

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